國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

廣義相對論中的正質量定理

The Positive Mass Theorem in General Relativity

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中華民國 99 年 6 月

June, 2010

Acknowledgment

I have to thank to my advisor, Professor Shu-Cheng Chang, and Professor Jih-Hsin Cheng and Professor Ai-Nung Wang. They taught me very much. I also have to thank to my senior, Po-Yao Chang, because he always teaches me and answers my questions carefully and patiently. Finally, I have to express my deep gratitude to my parents for their supporting in my life.

摘要

此文章主要介紹廣義相對論中的正質量定理, 共分三部分, 前兩節為介紹此定理及預備知識, 第三節為邱成桐及孫理查在 1979 論文的證明, 第四節為他們在 1981 年的文章的證明大概.

關鍵字: 正質量定理, 廣義相對論, 微分幾何, 邱成桐, 孫理查.

Abstract

This survey article introduces the Positive Mass Theorem in General Relativity. It includes three parts. The first two sections are the introduction and the preliminaries. The third section is the proof of the paper of S. T. Yau and R. Schoen in 1979. In the fourth section, we outline the proof of the paper in 1981.

Keyword: General Relativity, Positive Mass Theorem, differential geometry, S. T. Yau, R. Schoen.

Contents

The Positive Mass Theorem in General Relativity

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June, 2010

Abstract The positive mass theorem states that for an isolated system, the total mass is nonnegative. This assertion was discussed by many physicians and mathematicians and finally completely proved by R. Schoen and S.T. Yau. This survey article includes three parts. In the first two sections, we introduce the theorem and some preliminaries. In the third section, we discuss the proof in the first paper of Yau and Schoen [**?**]. In the fourth section, we outline the proof of the second paper of Yau and Schoen [**?**].

1 Introduction

In this survey article, we will discuss the Positive Mass Theorem in General Relativity. In general relativity, the positive mass theorem states that, assuming the dominant energy condition, the mass of an asymptotically flat spacetime is non-negative; furthermore, the mass is zero only for Minkowski spacetime. The problem of mass in gravitational field has been a difficult problem in general relativity. One of the aims in the early research was to understand how the energy of the gravitational fields is distributed. The final conclusion is that the energy of the gravitational field is nonlocalized. Therefore, the goal of trying to find the localized energy in the gravitational fields is impossible to attain. However, we are certain that the gravitational field has energy in it. Therefore, the problem is that how much we can know about the energy in the gravitational field. The mathematicians finally came to the conclusion that the total energy in an isolated system is well-defined. The definition of this energy was proposed by Richard Arnowitt, Stanley Deser and Charles Misner in 1960's. It is called ADM mass and is the mass in the positive mass theorem.

First, we give the definition of spacetime.

Definition 1. *1. The spacetime is a connected, 4-dimensional Lorentz manifold* $(M, \tilde{g})$ *together with the Levi-Civita connection. It must satisfy the Einstein equation* $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$ *where* T_{ab} *is the energy-momentum tensor describing matters.*
2. A nonzero tangent vector v *is timelike (or spacelike,nonspacelike respectively) if* $\tilde{g}(v, v) < 0$ *(or* $> 0, \leqslant 0$*).*

Note The difference between a Lorentz manifold and a Riemannian manifold is that $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$ has signature $(-, +, +, +)$ in a Lorentz manifold.
When we want to define the mass in an isolated system, we must understand what an isolated system is. In physics, an isolated system is a system far from other systems. We define it by letting the metric g_{ij} tends to the Minkowski metric δ_{ij} in an appropriate way. Since in physics the metric should tend to be flat when we are far from an isolate system, the spacetime of an isolated system is defined to be asymptotically flat. People usually define an asymptotically flat spacetime to satisfy the following conditions

$$g_{ij} = \delta_{ij} + O(1/r), \partial_k g_{ij} = O(1/r^2).$$

2 Preliminaries

2.1 Initial Data Set

When we have the definition of isolated systems, we now discuss the total mass of an isolated system. Since the mass is a physical quantities to describe the dynamical behavior of systems, we have to understand the dynamics in general relativity before defining the mass. As we know, Einstein equation is $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$. The left hand side is the metric g_{ab} and its first and second derivatives while the right hand side is the energy momentum tensor T_{ab} which describes the distribution of matter. The form of the equation leads to a misunderstanding that solving the Einstein equation means that to find the g_{ab} when we are given T_{ab}. However, it is not true since the spacetime manifold is 4-dimensional and when we are given T_{ab}, we have known the matter distribution as a function of time. But if we have known the motion of the matters as a function of time, we don't need to solve the equation at all.

Hence, when solving the Einstein equation, we are given the initial data set rather than T_{ab}. First, we need to decompose the spacetime M into $N \times \mathbb{R}$ where N is a 3-dimensional spacelike hypersurface. This is called ADM decomposition. Second, since Einstein equation is a second-order PDE, we need to know the data of some dynamical variable and its time derivatives. In fact, we choose g to be the dynamical variable and hence its induced metric g_{ij} in N is the initial value and the second fundamental form of N is its derivative. Hence, we have the following definition.

Definition 2. *Let M be a 4-dimensional spacetime, $N \subset M$ be a spacelike hypersurface, g be the induced metric on N and h be the second fundamental form. Then (N, g, h) is called an initial data set.*

Hence, solving Einstein equation means that when we are given an initial data set, we try to solve the metric $\tilde{g}$ on the whole spacetime M. This is similar to Cauchy problem.If we are concerned with an isolated system, we need that N is an asymptotically flat hypersurface. Next, we introduce the dominant energy condition.

There are some natural conditions from a physical point of view on the stress-energy tensor T_{ab}. First, it is symmetric. It must satisfy the **dominant energy condition**:

For any timelike vector V , $T_{ab}V^aV^b \geq 0$, and $T_{ab}V^b$ is nonspacelike.

$T_{ab}V^aV^b \geq 0$ means that the energy density observed by any observer is nonnegative.

When we use the Lorentz basis and let $V = (1, 0, 0, 0)$, then

$$T_{ab}V^aV^b = T_{00} \geq 0.$$

Further, $T_{ab}V^b = (T_{00}, T_{01}, T_{02}, T_{03})$ is nonspacelike.

It means $-(T_{00})^2 + (T_{01})^2 + (T_{02})^2 + (T_{03})^2 \leqslant 0$. Hence $T_{00} \geq \sqrt{\sum_{i=1}^{3} T_{0i}^2}$.

2.2 Constraint equations

We can combine the Einstein equation and the Gauss and Codazzi equations to give some constraint equations on the initial data set (N, g, h). Given a basis (E_0, E_1, E_2, E_3) such that E_0 is a unit timelike vector field normal to N and E_1, E_2, E_3 are tangent to N. Let h_{ij} be the second fundamental form, i.e, $(D_{E_i}E_j)^\perp = h_{ij}E_0$. The Gauss equation says that

$$\mathcal{R}_{ijkl} = R_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, 1 \leq i, j, k, l \leq 3.$$

where $\mathcal{R}$ is the curvature on M and R is the curvature on N. Note that the signs differ from the usual ones in Riemannian geometry since M is a Lorentz manifold. Taking trace in the equation we obtain

$$\sum_{i,j} \mathcal{R}_{ijij} = R + (trh)^2 - (trh^2).$$

Using Einstein equation we obtain $\sum_{i,j} \mathcal{R}_{ijij} = 16\pi T_{00}$ where T_{00} is the observed mass density. Hence the equation becomes

$$R + (trh)^2 - (trh^2) = 16\pi T_{00}. \tag{1}$$

This is the first constraint equation. Similarly, from the Codazzi equation we can obtain

$$\nabla_i (h^i_k - (trh)\delta_{ik}) = -8\pi T_{0k}. \tag{2}$$

This is the second constraint equation.

Remark 1. *In the paper of Yau and Schoen, they assume that* $trh = 0$ *,i.e,* N *is a minimal surface. The assumption means that there exists an asymptotically flat maximal surface.*

Since Yau and Schoen used rigorous math to prove this positive mass theorem, they needed to express the dominant condition as conditions on N. This can be done by the first constraint equation. Since the dominant condition asks that $T_{00} \geq \sqrt{\sum_{i=1}^{3} T_{0i}^2}$, the first constraint equation becomes

$$R + (trh)^2 - (trh^2) = 16\pi T_{00} \geq 0.$$

$$\text{Since } \; trh = 0, tr(h^2) \geq 0, \quad \text{we have } R \geq 0.$$

Hence, from the dominant energy condition, we know that $R \geq 0$ on N.

2.3 Asymptotical flatness and ADM mass

The following are the two definitions of asymptotically flat initial data set. The former is the definition in the beginning and the latter is the one which people started to use after a period of time.

Definition 3. *Asymptotically flat initial data set*
An initial data set (N, g, h) *is said to be asymptotically flat if* $\exists$ *compact subset* K *of* N *s.t.* $N \setminus K = \cup_{k=1}^{r} N_k$*, where each* N_k *,which is called an end of* N*, is diffeomorphic to* $\mathbb{R}^3 \setminus B_\rho(0)$ *and in each* N_k*,* g *and* h *satisfy*

$$1.g_{ij} = \delta_{ij}+O(\frac{1}{r}), \partial_k g_{ij} = O(\frac{1}{r^2}), \partial_l\partial_k g_{ij} = O(\frac{1}{r^3}), h_{ij} = O(\frac{1}{r^2}), \partial_k h_{ij} = O(\frac{1}{r^3}).$$

$$2.g_{ij} = (1+\frac{M_k}{2r})^4\delta_{ij} + h_{ij}, |h_{ij}| \leqslant \frac{k_1}{1+r^2}, |\partial h_{ij}| \leqslant \frac{k_2}{1+r^3}, |\partial\partial h_{ij}| \leqslant \frac{k^3}{1+r^4}. \tag{3}$$

for some constant $k_1, k_2, k_3 \gneqq 0$.

The 1 in $1+r^2, 1+r^3, 1+r^4$ are added in order to bound the value when r is small.

Now we give the definition of the ADM mass.

Definition 4. *ADM mass*
If (N, g_{ij}, h_{ij}) *is an asymptotically flat initial data set, then the ADM mass of it is*

$$m = \frac{1}{16\pi}\lim_{r\to\infty}\oint_{S_r}(\partial_i g_{ij} - \partial_j g_{ii})dS_j.$$

In the following theorem, we only consider that N consists of one end.

Theorem 2.1. *(Bartnik, [**?**])*
If (N, g, h) *is an asymptotically flat initial data set(in definition (1)), then the ADM mass* M *is independent of the choice of asymptotically flat coordinates.*

Remark 2. *1. We can see that (2)* $\Rightarrow$ *(1), but the* g_{ij} *satisfying (1) cannot be written in the form of (2). Hence, definition (2) seems to be just a special case of definition (1). However, we can use a coordinate transformation to let the metric* g *satisfying (1) becomes the form of (2).*
2. Yau and Shoen proved that according to definition (2), the ADM mass are nonnegative. However, under the definition (1), the positivity of ADM mass has not jet been proved untill the paper of Bartnik in 1986. Since the general case in definition (1) can be transformed to definition (2) under a coordinate transformation and the ADM mass under the two coordinate systems are the same. Hence, the papers of Yau and Schoen and Bartnik ensures that the ADM mass of the definition (1) is nonnegative.
3. The proof of positive mass theorem uses only the definition of asymptotically flatness and $R \geqslant 0$. *Hence we only need that* T_{ab} *satisfies the dominant energy condition and don't need to solve the Einstein equation.*

3 Positive mass theorem

In this section we discuss the proof of the positive mass theorem. It comes from the paper of S.T.Yau and R. Schoen [**?**].

Assumption There exists a maximal spacelike hypersurface N in the spacetime M and the dominant energy condition holds.

Theorem 3.1. *Let ds^2 be an asymptotically flat metric on the hypersurface N. If $R \geqslant 0$ on N, the the total mass of each end is nonnegative.*

proof. In the proof, we work on a fixed end N_k. Suppose that x^1, x^2, x^3 are asymptotically flat coordinates describing N_k on $\mathbb{R}^3 \setminus B_{\sigma 0}(0)$. We denote the total mass of N_k by M. We suppose that $M < 0$ and will reach a contradiction.

Step1. $\exists$ a.f. metric $d\tilde{S}^2$ conformally equivalent to ds^2. Under the new metric, $\tilde{R} \geq 0$ on N , $\tilde{R} > 0$ outside a compact subset of N_k. $\tilde{M} < 0$.

proof. We want to find a function φ such that the new metric $d\tilde{s}^4 = \varphi^4 ds^2$ satisfies the requirement.

$$\because d\tilde{s}^4 = \varphi^4 ds^2 \quad \therefore \tilde{g}_{ij} = \varphi^4 g_{ij} = \varphi^4[(1+\frac{M}{2r})^4\delta_{ij} + O(\frac{1}{r^2})]$$

To let $d\tilde{s}^2$ be a.f and $\tilde{M} < 0$ also, we let $\varphi = 1 - \frac{M}{4r}$ when r is large. Then

$$\tilde{g}_{ij} = (1-\frac{M}{4r})^4(1+\frac{1}{2r})^4\delta_{ij} + O(\frac{1}{r^2}) = (1+\frac{M}{4r})^4\delta_{ij} + O(\frac{1}{r^2}).$$

Therefore, $d\tilde{s}^2$ is a.f and $\tilde{M} = \frac{M}{2} < 0$. Furthermore, we note that

$$d\tilde{s}^2 = \varphi^4 ds^2 \Rightarrow \tilde{R} = \varphi^{-5}[-8\Delta\varphi + R\varphi].$$

$$\text{and } \Delta\frac{1}{r} = \sum_{i=1}^{3}\frac{\partial}{\partial x^i}((1+\frac{M}{2r})^2\frac{\partial}{\partial x^i}(\frac{1}{r})) + O(\frac{1}{r^5}) = \frac{M}{r^4} + O(\frac{1}{r^5}).$$

$$\therefore \exists \sigma > \sigma_0 \ \ s.t. \ \ \Delta\frac{1}{r} < 0 \text{ for } r \geq \sigma.$$

$$\Rightarrow \Delta\varphi = \frac{-M}{4}\Delta\frac{1}{r} < 0 \text{ when r is large}$$

$\therefore \tilde{R} > 0$ when r is large. To let $\tilde{R} \geq 0$ on N , we need that $\Delta\varphi \leq 0$ on N. To define the precise φ , we first let $t_0 = -\frac{M}{8\sigma_0}$ and define $\zeta(t)$ to be a C^5 function which satisfies

$$\zeta(t) = \begin{cases} t & \text{for } \ t < t_0 \\ \frac{3t_0}{2} & \text{for } \ t > 2t_0, \end{cases}$$

$$\zeta'(t) \geqslant 0, \zeta''(t) \leqslant 0 \text{ for } t \in (0, \infty).$$

Define $\varphi : N \to \mathbb{R}$ be a C^5 function by

$$\varphi = 1 + \frac{3t_0}{2} \text{ on } N \setminus N_k$$
$$\varphi(x) = 1 + \zeta(-\frac{M}{4r}) \text{ on } \mathbb{R}^3 \setminus B_{\sigma_0}(0) = N_k.$$

Then on N_k, we have

$$\varphi(x) = \begin{cases} 1 - \frac{M}{4r} & \text{when } r > -\frac{M}{4t_0} \\ 1 + \frac{3t_0}{2} & \text{when } r < -\frac{M}{8t_0}. \end{cases}$$

$\therefore \Delta\varphi \leq 0$ on N and $\Delta\varphi < 0$ for $r > 2\sigma$. This completes the proof of step1.

Step2 $\exists$ a complete area minimizing(relative to ds^2) surface S properly imbedded in N s.t. $S \cap (N \setminus N_k)$ is compact, and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-spaces defined by x^1, x^2, x^3.

proof. Let $\sigma > 2\sigma_0$, C_σ be the circle of Euclidean radius σ centered at 0 in $x^1 - x^2$ plane. Let S_σ be the smooth imbedded oriented surface of least ds^2 area with boundary C_σ. We want to find $\sigma_i \to \infty$ s.t. S_{σ_i} converges to the required surface S.

(1)**Claim $\exists$ compact subset** K_0 s.t.

$$S_\sigma \cap (N \setminus N_k) \subset K_0 \ \forall \sigma > 2\sigma_0. \tag{4}$$

proof. Let $N_{k'}$ be another end with a.f coordinates y^1, y^2, y^3 in $\mathbb{R}^3 \setminus B_{\tau_0}(0)$, $ds^2 = \sum_{i,j=1}^{3} g'_{ij} dy^i dy^j$, g'_{ij} satisfying (3). Calculate the hessian of $|y|^2$

$$D_{ij}|y|^2 = \frac{\partial^2 |y|^2}{\partial y^i \partial y^j} - D_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}(|y|^2) = 2\delta_{ij} + O(\frac{1}{|y|}).$$

$\therefore \exists \tau_1 > \tau_0$ s.t. $D_{ij}|y|^2$ is positive definite, i.e, $|y|^2$ is a convex function.
Now see $|y|^2$ on S_σ.

$$\because \partial S_\sigma = C_\sigma \subset N_k \therefore \text{by maximum principle,} |y|^2 \text{ is bounded on } S_\sigma \cap N_{k'}$$

$$\Longrightarrow S_\sigma \cap N_{k'} \subset B_{\tau_1}(0)$$

$\because N_{k'} \neq N_k$ is arbitrary $\therefore$ (4) holds.

(2)Now analyze the behavior of $S_\sigma \cap N_k$. In fact, we bound the height of $S_\sigma \cap N_k$ in the x^3 direction.

Claim

$$\exists h > \sigma_0 \text{ s.t. } N_k \cap S_\sigma \subset E_h \ \forall \sigma > 2\sigma_0 \tag{5}$$

proof.For any $h > 0$, we define

$$E_h = \left\{ x \in \mathbb{R}^3 : |x^3| \leq h \right\}.$$

To accomplish this, use the maximum principle for x^3 on $S_\sigma \cap N_k$. First compute the asymptotic behavior of the covariant hessian of x^3 on N_k. Then

$$D_{ij}x^3 = \frac{Mx^j}{r^3}\delta_{i3} + \frac{Mx^i}{r^3}\delta_{j3} - \frac{Mx^3}{r^3}\delta_{ij} + O(\frac{1}{r^3}). \tag{6}$$

Let $\bar{h}$ be the maximum for x^3 on $S_\sigma \cap N_k$. Suppose $\bar{h} > \sigma_0$ is very large, we can use (6) to obtain a contradiction. Similarly, let $\underline{h}$ be the minimum of x^3 on $S_\sigma \cap N_k$. If $\underline{h} < -\sigma_o$ is very small, then we can obtain a contradiction. Hence the claim is proved.

Now, let $\rho > 2\sigma_0$ and define

$$A_\rho = (N \setminus N_k) \cup \left\{ x : |x| \geq \sigma_0, (x^1)^2 + (x^2)^2 \leq \rho^2 \right\}.$$

For any $\sigma > \rho$, (4) and (5) implies

$$S_\sigma \cap A_\rho \subset (K_0 \cup E_h) \cap A_\rho \tag{7}$$

which is a compact subset of N. We now quote a local interior regularity estimate for area minimizing surfaces.

Regularity estimate Let $U_r(x)$ denote the geodesic ball of redius r about $x \in N$. Then $\exists r_0 > 0$ s.t. for any $x_0 \in S_\sigma$ with $U_{r_0}(x_0) \cap C_\sigma = \varnothing$, $S_\sigma \cap U_{r_0}(x_0)$ can be written as the graph of a C^3 function f_σ over the tangent plane to S_σ in a normal coordinate system on $U_{r_0}(x_0)$. Moreover, there is a constant c_1 depending only on (N, ds^2) which bound all the derivatives of f_σ up to order three in $U_{r_0}(x_0)$.
By (7) and the Regularity Estimate we can choose a sequence $\sigma_i^{(\rho)} \to \infty$ s.t. $S_{\sigma^\rho_{\sigma_i}} \cap A_\rho$ converges in C^2 topology. Since this can be done for any $\rho > 2\sigma_0$, we can take a sequence $\rho_j \to \infty$ and by extracting the diagonal sequence we find a sequence $\sigma_i \to \infty$ s.t. $S_{\sigma_i} \to S$, an imbedded C^2-surface, uniformly

in C^2 norm on compact subsets of N. The surface S is properly imbedded by (7), and area-minimizing on any compact subset of N. From (4) we have $S \cap (N \setminus N_k) \subset K_0$ and hence $S \cap (N \setminus N_k)$ is compact. From (5), $S \cap N_k \subset E_h$ which is a region between two parallel 2-planes in $\mathbb{R}^3$.

Step3. $\int_S K > 0$
proof. We use the second variation inequality for S. This expresses the fact that up to second order S has smallest area in a one-parameter compactly supported deformation of S.
Let e_1, e_2, e_3 be orthonormal(with respect to ds^2) vector fields defined locally on N. Let

$$K_{ij} = \text{ sectional curvature of the section } \{e_i, e_j\}$$

The Ricci tensor can be written

$$Ric(e_i) = \sum_{j=1}^{3} K_{ij}$$

where we let $K_{ii} = 0$. The scalar curvature is

$$R = K_{12} + K_{13} + K_{23}$$

Let ν be the unit normal vector field of S, and choose a frame $e_1, e_2, e_3 = \nu$ on S. Let A be the second fundamental form of S, i.e, the matrix in terms of e_1, e_2 is

$$h_{ij} = \langle D_{e_i}\nu, e_j \rangle$$

Let $\|A\|^2 = \sum_{i,j=1}^{2} h_{ij}^2$.

$$\because S \text{ is a minimal surface } \quad \therefore Trace(A) = h_{11} + h_{22} = 0 \tag{8}$$

The second variation inequality for S is

$$\int_S f[\Delta f + (Ric(\nu) + \|A\|^2)f] \leq 0$$

for any C^2 function f with compact support on S. By integration by parts, we get

$$\int_S (Ric(\nu) + \|A\|^2)f^2 \leq \int_S \|\nabla f\|^2. \tag{9}$$

By approximation, the inequality folds for any Lipschitz function f with compact support on S. The Gauss equation expresses the Gauss curvature K of S as

$$K = K_{12} + h_{11}h_{22} - h_{12}^2$$

Applying (8) and the symmetry of A gives

$$\frac{1}{2}\|A\|^2 = K_{12} - K$$

Putting this into (9) gives

$$\int_S (Ric(\nu) + K_{12} - K + \frac{1}{2}\|A\|^2)f^2 \leq \int_S \|\nabla f\|^2.$$

$$\because Ric(\nu) + K_{12} = K_{13} + K_{23} + K_{12} = R$$

$$\therefore \int_S (R - K + \frac{1}{2}\|A\|^2)f^2 \leq \int_S \|\nabla f\|^2.$$

Choose appropriate f and we can get

$$\int_S (R - K + \frac{1}{2}\|A\|^2) \leq 0.$$

Since $R \geq 0$, and $R > 0$ outside a compact subset of S, we get

$$\int_S K > 0.$$

Remark Since S is not compact, we cannot use the Gauss-Bonnet theorem. Instead, the Cohn-Vossen inequality is suitable for this case. It says that if S is a complete, open, 2-dim. surface and K is absolutely integrable on it, then $\int_S K \leq 2\pi\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. Hence, we can get that S is homeomorphic to $\mathbb{R}^2$.

Step 4. $\int_S K \leq 0$
proof. By a result of A. Huber [**?**] and the previous remark, we can know that S is conformally equivalent to the complex plane.That is,

$$\exists \text{comformal diffeomorphism } F : \mathbb{C} \to S.$$

Let C_σ be the circle of radius σ. For $i = 1, 2, ...$, let $L_i = length(F(C_i))$ and $A_i = Area(F(D_i))$. By the results of R. Finn [**?**] and A. Huber [**?**], we have

$$\int_S K = 2\pi - \lim_{i\to\infty} \frac{L_i^2}{2A_i}$$

Thus to show $\int_S K \leq 0$, it suffices to show that

$$\lim_{i\to\infty} \frac{L_i^2}{4\pi A_i} \geq 1 \tag{10}$$

For large i, let $\tilde{L}_i$ be the Euclidean length of $F(C_i)$. By (3),

$$\tilde{L}_i \leq (1+o(1))L_i^2 \quad \text{as} \quad i \to \infty. \tag{11}$$

For the immersed disk Σ_i of least Euclidean area with boundary curve $F(C_i)$, from [?] we have the isoperimetric inequality

$$\tilde{A}(\Sigma_i) \leq \frac{\tilde{L}_i^2}{4\pi} \tag{12}$$

where $\tilde{A}()$ is Euclidean area. Let $\tilde{\Sigma}_i$ be an oriented surface of least Euclidean area among all the surfaces of boundary $F(C_i)$ regardless of topological type.

$$\because \tilde{A}(\tilde{\Sigma}_i) \leq \tilde{A}(\Sigma_i) \therefore \tilde{A}(\tilde{\Sigma}_i) \leq \frac{\tilde{L}_i^2}{4\pi} \tag{13}$$

We want to compare the ds^2-area of $\tilde{\Sigma}_i$ with the Euclidean area, but we cannot do it directly since $\tilde{\Sigma}_i \cap B_{\sigma_0}(0)$ may be nonempty, and ds^2 is not defined in it. Hence we modify $\tilde{\Sigma}_i$ near 0 in the following way.
Let $\bar{\sigma} \in [\sigma_0, \sigma_0 + 1]$ be such that $\tilde{\Sigma}_i \cap \partial B_{\bar{\sigma}} \neq \varnothing$. Then find a domain Ω_i on $\partial B_{\bar{\sigma}}(0)$ s.t.

$$\partial\Omega_i = \tilde{\Sigma}_i \cap \partial B_{\bar{\sigma}}(0).$$

Now define

$$\hat{\Sigma}_i = (\tilde{\Sigma}_i \setminus B_{\bar{\sigma}}(0)) \cup \Omega_i.$$

Let $\tilde{A}_i = \tilde{A}(\hat{\Sigma}_i)$. Then

$$\tilde{A}_i \leq (1+o(1))\tilde{A}(\tilde{\Sigma}_i).$$

which combines with (2.22) to gives

$$\tilde{A}_i \leq (1+o(1))\frac{\tilde{L}_i^2}{4\pi} \tag{14}$$

By asymptotically flatness, we have

$$A(\hat{\Sigma}_i) \leq (1+o(1))\tilde{A}_i \text{ as } i \to \infty \tag{15}$$

Using the area minimizing property of S and the above inequality, we have

$$A_i \leq A(\hat{\Sigma}_i) \leq (1+o(1))\tilde{A}_i \leq (1+o(1))\frac{\tilde{L}_i^2}{4\pi} \leq (1+o(1))\frac{L_i^2}{4\pi} \text{ as } i \to \infty \tag{16}$$

$$\therefore \lim_{i\to\infty} \frac{L_i^2}{4\pi A_i} \geq 1$$

This completes the proof.

$\square$

4 General Theorem

In the previous section, we state that Schoen and Yau proved the positive mass theorem under the assumption that N is a maximal slice in M. However, they proved that this assumption can be removed in the paper of [**?**].

Theorem 4.1. *Let (N, g_{ij}, p_{ij}) be an initial data set. Assume the dominant energy condition holds on N. Then $M_k \geq 0$ for every k.*

Sketch of proof. In section 2.2, we see that the assumption that N is a minimal surface means that $R \geq 0$ on N. Hence, the removement of the assumption that N is a minimal surface means the removement of the condition $R \geq 0$ on N.
However, we still need the dominant energy condition. In the preliminary, we say that is $T_{00} \geq \sqrt{\sum_{i=1}^{3} T_{0i}^2}$ when we use the Lorentz fram. By the two constraint equations, it can be written as

$$\mu \geq (\sum_i J^i J_i)$$

where

$$\mu = \frac{1}{2}[R - \sum_{i,j} p^{ij} p_{ij} + (\sum_i p_i^i)^2]$$

$$J^i = \sum_j D_j [p^{ij} - (\sum_k p_k^k) g^{ij}].$$

We deform the metric g_{ij} and p_{ij} in two steps. First, we consider the product manifold $N \times R$ with the product metric and extend p_{ij} trivially to be a tensor defined on $N \times R$. We want to find a hypersurface $\bar{N}$ in $N \times R$ which projects one to one onto N and whose mean curvature is the same as the trace of p_{ij} on $\bar{N}$. Second, if such a hypersurface exists, then the induced metric on this hypersurface can be deformed conformally to one with $R = 0$. Moreover, the positivity of the mass of $\bar{N}$ can imply the positivity of the mass of N. Since $R \geq 0$ on $\bar{N}$, we know the positivity of the mass of $\bar{N}$. Hence, the positivity of the mass of N is proved.
Step1. Consider the product manifold $N \times R$. Suppose $\Sigma \subset N \times R$ is a hypersurface. Consider the case that Σ is the graph of a function f defined on N. Now we want Σ satisfies the equation

$$H = \sum_i p_{ii} \tag{17}$$

More precisely, it is the equation

$$(1+Df^2)^{-\frac{1}{2}}\sum_{i,j}\bar{g}^{ij}D_iD_jf=\sum_{i,j}\bar{g}^{ij}p_{ij} \tag{18}$$

where $\bar{g}_{ij}$ is the induced metric on Σ

$$\begin{aligned}
\bar{g}_{ij} &= g_{ij}+f_{x^i}f_{x^j} \\
\bar{g}^{ij} &= g^{ij}-\frac{f^if^j}{1+|Df|^2} \\
f^i &= \sum_j g^{ij}f_{x^j}
\end{aligned}$$

Now we study a slightly more general equation than (18). Let $F(x)$ be a given C^2 function on N and suppose μ_1,μ_2,μ_3 are constants s.t.

$$\sup_N |F|\le\mu_1, \sup_N |DF|\le\mu_2, \sup_N |DDF|\le\mu_3 \tag{19}$$

Suppose f is a given C^3 solution of

$$\sum_{i,j=1}^{3}(g^{ij}-\frac{f^if^j}{1+|Df|^2})(\frac{D_iD_jf}{1+(|Df|^2)^{\frac{1}{2}}}-p_{ij})=F \tag{20}$$

We add some assumption on F.

$$F(x)=tf(x)+G(x) \quad \text{on } N \tag{21}$$

$$|G(x)|\le\mu_4(1+r^3)^{-1}, |\partial G(x)|\le\mu_5(1+r^4)^{-1} \quad \text{on } N_k \tag{22}$$

We have the following proposition.

Proposition Suppose f is a C^3 solution of (20) with fuction F satisfying (19),(21),(22). Suppose also that $\lim_{x\to\infty}f(x)=0$ for each N_k. For any $\beta\in(0.1)$, there is a constant $c_{33}=C_{33}(\beta)$ depending only on β, the initial data (N,g_{ij},p_{ij}), and the constants $\mu_1,\mu_2,\mu_3,\mu_4,\mu_5$ s.t.

$$|f(x)|+|x||\partial f(x)|+|x|^2|\partial\partial f(x)|+|x|^3|\partial\partial\partial f(x)|\le c_{33}(\beta)|x|^{-\beta}$$

for any $x\in N_k$, any k.

Remark Since Σ is the graph of f, this means that $\bar{g}_{ij}$ on Σ satisfies the asymptotically flat condition (3).

$$\bar{g}_{ij}=g_{ij}+f_{x^i}f_{x^j}=(1+\frac{M}{2r})^4\delta_{ij}+h_{ij}+f_{x^i}f_{x^j}, |h_{ij}|\le\frac{k_1}{1+r^2}$$

$$\Rightarrow |h_{ij} + f_{x^i} f_{x^j}| \leq |h_{ij}| + |f_{x^i}||f_{x^j}| \leq \frac{k_1}{1+r^2} + \frac{c_{33}(\beta)^2}{r^{2+2\beta}} \leq \frac{k_1'}{1+r^2}$$

Step2 In this step, we prove the existence of solutions of (18), asymptotic to zero at infinity, and defined on the exterior of a finite family of apparent horizons.
To solve (18), we want to introduce an auxiliary equation for $s \in [0,1], t \in [0,1]$.

$$H(f) - sP(f) = tf \tag{23}$$

where $H(f), P(f)$ are given by

$$\begin{aligned} H(f) &= \sum_{i,j} (g^{i,j} - \frac{f^i f^j}{1+|Df|^2}) \frac{D_i D_j f}{\sqrt{1+|Df|^2}} \\ P(f) &= \sum_{i,j} (g^{ij} - \frac{f^i f^j}{1+|Df|^2}) p_{ij} \end{aligned}$$

We want to find the solution for $s = 1, t = 0$. We first solve it for $t > 0$. Then by the continuity method, we can prove that it has a solution f for $s = 1$. We now study the limit of the solutions as $t \to 0$. Although it is not generally true that the solutions of the perturbed equation converge as t approaches 0, we have the following proposition.

Proposition There is a sequence $\{t_i\}$ converges to zero and open sets $\Omega_+, \Omega_-, \Omega_0$ s.t. if f_i satisfies $H(f_i) - P(f_i) = t_i f_i$, we have
(1) The sequence f_i converges uniformly to $+\infty$ (respectively, $-\infty$) on Ω_+ (respectively, Ω_-), and f_i converges to a smooth function f_0 on Ω_0 satisfying (18).
(2) The sets Ω_+ and Ω_- have compact closure, and $N = \bar{\Omega_+} \cup \bar{\Omega_-} \cup \bar{\Omega_0}$. Each boundary component Σ of Ω_+ (respectively Ω_-) is a smooth embedded two-sphere.
(3) The graphs G_i of f_i converge smoothly to a properly embedded limit submanifold $M_0 \subset N \times \mathbb{R}$. Each connected component of M_0 is either a component of the graph of f_0 or the cylinder $\Sigma \times \mathbb{R} \subset N \times \mathbb{R}$ over a boundary component Σ of Ω_+ or Ω_-.

Step3 We use the function f_0 to prove the theorem. We want to prove that $M_k \geq 0$, so we consider only that component of Ω_0 which contains N_k. For simplicity we also denote that component of G_0 as G_0. We now remove the infinities of G_0 except that asymptotic to N_k. This can be done by a conformal change of metric. Let $G_0^l = G_0 \cup (N_l \times \mathbb{R})$. For each $l \neq k$, let ψ_l be a positive solution of $\Delta - \frac{1}{8}\bar{R} = 0$ on N_l where $\bar{R}$ is the scalar curvature of G_0 to $d\bar{s}^2$, the induced metric on G_0. Then let ψ be a positive smooth function on G_0 satisfying

$$\psi = \begin{cases} 1 & \text{on } G_0^k \\ \psi_l & \text{on } G_0^l, l \neq k \\ \psi' & \text{on } G_0 \cup (\Sigma \times \mathbb{R}) \end{cases}$$

where ψ' is some special function satisfying $\Delta - \frac{1}{8}\bar{R} = 0$. Now define a new metric $ds_0^2 = \psi^4 \bar{ds}^2$. The new manifold (N_0, ds_0^2) has $R_0 = \psi^{-5}(\bar{R}\psi - 8\Delta\psi)$. ψ' is chosen s.t. the new manifold has no infinity. Note that $R_0 = 0$ outside G_0^k. Using the dominant energy condition, we can find these solutions satisfying the requirements.

Now choose a positive function u on N_0 satisfying $\Delta u - \frac{1}{8}R_0 u = 0$. It has the form

$$u = 1 + \frac{A_k}{r} + O(r^{-2})$$

on N_0^k where $A_k < 0$. The proof of the existence uses the dominant energy condition. Then the metric $u^4 ds_0^2$ on N_0 has zero scalar curvature, and is asymptotically flat. We have $M_k^0 = M_k + 2A_k$. By the result of Section 3, we have $M_k^0 \geq 0$. Hence $M_k > 0$. This completes the proof.

$\square$

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