國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

狄克技巧

The Deturck’ s tricks

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中華民國 99 年 6 月

June, 2010

Contents

摘要:

這份報告中，主要是專注於瑞奇流的局部解以及給定瑞奇曲率，求解黎曼度量的局部解。這兩個主題之證明，主要是判定他們是否分別為拋物型與橢圓型方程，進而使用偏微分方程的已知理論求解。不過就是由於他們並非能恰巧滿足已知理論條件, 所以我們必須去修改方程使其能滿足條件。

這篇報告之所有內容都是 D.M. Deturck 所得到，故詳細內容可參閱其所著相關論文。

關鍵詞: 瑞奇流、瑞奇曲率、拋物型方程、橢圓型方程、局部可解性

Abstract

In this review, we would concentrate on two main results "Short-time existence of Ricci flow" and "Local existence of metrics with prescribed Ricci curvature". All of these materials could be found in the original papers by D.M. Deturck.([2],[3]) Both proofs depended on whether they're strictly parabolic and elliptic, resp. Because $\frac{\partial g}{\partial t} = -2Ricc(g)$ and $Ricc'(g)h$ are not strictly parabolic and elliptic, we must modify the equations (i.e. adding some terms) to make them to satisfy the requirements. Without complete proofs we would just point out the key steps in Chapter 3 after giving some preliminaries.

Keywords: Ricci flow, Ricci curvature, strictly parabolic, elliptic, local solvability

1 Introduction

In 1982, R.S. Hamilton proved that if M^3 is a compact three-dimensional Riemannian manifold which admits a Riemannian metric with strictly positive Ricci curvature, then M also admits a metric of constant positive sectional curvature. Hamilton used the Nash-Moser implicit-function theorem to prove the local existence of Ricci flow.([4]) Later, in 1983, D.M. Deturck found out that this local existence cound be deduced from the classical existence and uniqueness theorems for initial-value problems for quasilinear parabolic systems and for systems of ODE. So Deturck improved the proof of Hamilton in [3].

In [2], Deturck showed that local existence of metrics with prescribed Ricci curvature. He used the similar method to show the local existence. (i.e. adding some terms to the original equation such that the new one becomes an elliptic equation) And then, by performing the usual contracting mapping iteration to prove the modified equation, it could give a solution to the original one.

2 Preliminaries

2.1 Some equalities

In this section, we would derive some equalities which will be useful in Chapter 3. The method of proof of the propositions is mostly deduced by local coordinate method due to [1],[3]. So we provide another path. (Main proofs here followed [7])

Let (M, g) be a Riemannian manifold, $g = g(t) \in \Gamma(S^2T^*M)$ defined on an open interval in $\mathbf{R}$ and $h := \frac{\partial g}{\partial t}$.

Notations:

$$G(T) := T - \frac{1}{2}tr(T)g = T_{ij} - \frac{1}{2}g_{ij}(g^{st}T_{st}) \ \ (Gravitation\ \ operator)$$

for $T \in \Gamma(S^2T^*M)$.

$$(\delta h)_i := (div h)_i := -g^{st}\nabla_s h_{ti} := -g^{st}h_{si;t}, (div^* v)_{ij} := \frac{1}{2}(v_{i;j} + v_{j;i})$$

for $h \in \Gamma(S^2T^*M), v \in \Gamma(T^*M)$.

$$\begin{aligned}(\Delta_L h)(X,W) :&= (\Delta h)(X,W) + 2trh(R(X,\cdot)W,\cdot) - h(X, Ricc(W)) - h(W, Ricc(X)) \\ &= h_{ij;s}{}^{s} + 2R_{isjt}h^{st} - R_{is}h^s_j - R_{js}h^s_i \ \ (Lichnerowicz - Laplacian)\end{aligned}$$

for $h \in \Gamma(S^2T^*M), Ricc(W) := (Ricc(W,\cdot))^\sharp$.

Firstly, we give the linearization of Ricci curvature :

Proposition 1. *(Variation of Ricci formula)* $\frac{\partial}{\partial t}Ricc_g = -\frac{1}{2}[\Delta_L h + L_{(\delta G(h))^\sharp}g]$.

We need some lemmas as follows:

Lemma 1. $\langle \Pi(X,Y), Z\rangle = \frac{1}{2}[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)]$ *where* $\Pi(X,Y) := \frac{\partial}{\partial t}(\nabla_X Y)$, $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$.

proof.

$$\begin{aligned}\langle \Pi(X,Y), Z\rangle &= \frac{\partial}{\partial t}\langle \nabla_X Y, Z\rangle - h(\nabla_X Y, Z) \\ &= \frac{\partial}{\partial t}[X\langle Y,Z\rangle - \langle Y, \nabla_X Z\rangle] - h(\nabla_X Y, Z) \\ &= [Xh(Y,Z) - h(Y,\nabla_X Z) - g(Y, \frac{\partial}{\partial t}\nabla_X Z)] - h(\nabla_X Y, Z) \\ &= (\nabla_X h)(Y,Z) - \langle \Pi(Z,X), Y\rangle.\end{aligned}$$

By this identity, we have

$$\begin{aligned}\langle \Pi(X,Y), Z\rangle &= (\nabla_X h)(Y,Z) - [(\nabla_Z h)(X,Y) - \langle \Pi(Y,Z), X\rangle] \\ &= (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y) + (\nabla_Y h)(Z,X) - \langle \Pi(X,Y), Z\rangle \\ &\Longrightarrow \langle \Pi(X,Y), Z\rangle = \frac{1}{2}[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)].\end{aligned}$$

□

Lemma 2. $\frac{\partial}{\partial t}R(X,Y)W = (\nabla_Y \Pi)(X,W) - (\nabla_X \Pi)(Y,W)$.

proof. By Lemma 1, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}R(X,Y)W &= \frac{\partial}{\partial t}(\nabla_Y\nabla_X W - \nabla_X\nabla_Y W + \nabla_{[X,Y]}W) \\
&= [\Pi(Y,\nabla_X W) + \nabla_Y(\Pi(X,W))] - [\Pi(X,\nabla_Y W) + \nabla_X(\Pi(Y,W))] \\
&+ \Pi([X,Y],W) \\
&= (\nabla_Y \Pi)(X,W) - (\nabla_X\Pi)(Y,W) + \Pi(\overline{T}(X,Y),W) \\
&= (\nabla_Y \Pi)(X,W) - (\nabla_X\Pi)(Y,W) \ \ where\ \ \overline{T} : torsion\ \ in\ \ (M,g).
\end{aligned}
$$

□

Lemma 3. $\frac{\partial}{\partial t}Rm(X,Y,W,Z) = \frac{1}{2}[h(R(X,Y)W,Z) - h(R(X,Y)Z,W) + \nabla^2_{Y,W}h(X,Z) - \nabla^2_{X,W}h(Y,Z) + \nabla^2_{X,Z}h(Y,W) - \nabla^2_{Y,Z}h(X,W)]$.

proof. WLOG, may assume $\nabla X = 0 = \nabla Y = \nabla Z = \nabla W$ at a time t, at $p \in M$.
By Lemma 2,

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle R(X,Y)W,Z\rangle &= h(R(X,Y)W,Z) + \langle\frac{\partial}{\partial t}\langle R(X,Y)W,Z\rangle \\
&= h(R(X,Y)W,Z) + \langle(\nabla_Y\Pi)(X,W) - \nabla_X\Pi)(Y,W),Z\rangle.
\end{aligned}
$$

By Lemma 1,

$$
\begin{aligned}
\langle(\nabla_Y\Pi)(X,W),Z\rangle &= \langle\nabla_Y(\Pi(X,W)),Z\rangle \\
&= \frac{1}{2}Y[(\nabla_W h)(X,Z) + (\nabla_X h)(W,Z) - (\nabla_Z h)(X,W)] \\
&= \frac{1}{2}[(\nabla_Y\nabla_W h)(X,Z) + (\nabla_Y\nabla_X h)(W,Z) - (\nabla_Y\nabla_Z h)(X,W)] \\
&= \frac{1}{2}[(\nabla^2_{Y,W} h)(X,Z) + (\nabla^2_{Y,X} h)(W,Z) - (\nabla^2_{Y,Z} h)(X,W)].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial}{\partial t}Rm(X,Y,W,Z) &= h(R(X,Y)W,Z) \\
&+ \frac{1}{2}[(\nabla^2_{Y,W}h)(X,Z) - (\nabla^2_{X,W}h)(Y,Z) + (\nabla^2_{Y,X}h)(W,Z) \\
&- (\nabla^2_{X,Y}h)(W,Z) - (\nabla^2_{Y,Z}h)(X,W) + (\nabla^2_{X,Z}h)(Y,W)].
\end{aligned}
$$

By Ricci identity

$$-(\nabla^2_{X,Y}h)(W,Z)+(\nabla^2_{Y,X}h)(W,Z)=-[h(R(X,Y)W,Z)+h(R(X,Y)Z,W)],$$

we obtain the result. □

Lemma 4. $\frac{\partial}{\partial t}(tr\alpha)=-\langle h,\alpha\rangle+tr(\frac{\partial\alpha}{\partial t})$ *where* $\alpha(t)\in\Gamma(\otimes^2T^*M)$*.*

proof. Using the local coordinate, let

$$\alpha:=\alpha_{ij}dx^i\otimes dx^j.$$

So

$$\frac{\partial}{\partial t}(tr\alpha)=\frac{\partial}{\partial t}(g^{ij}\alpha_{ij})=-h^{ij}\alpha_{ij}+g^{ij}\frac{\partial\alpha_{ij}}{\partial t}=-\langle h,\alpha\rangle+tr(\frac{\partial\alpha}{\partial t}).$$

□

Let us complete the proof of Proposition 1 as follows:

proof. By Lemma 4,

$$\frac{\partial}{\partial t}Ricc(X,W)=-\langle Rm(X,\cdot,W,\cdot),h\rangle+tr[\frac{\partial}{\partial t}Rm(X,\cdot,W,\cdot)].$$

By Lemma 3 and Ricci identity, we have

$$\begin{aligned}\frac{\partial}{\partial t}Rm(X,Y,W,Z)=&\frac{1}{2}[h(R(X,Y)W,Z)-h(R(X,Y)Z,W)+h(R(Y,W)X,Z)\\&+h(R(Y,W)Z,X)]+\frac{1}{2}[(\nabla^2_{W,Y}h)(X,Z)-(\nabla^2_{X,W}h)(Y,Z)\\&+(\nabla^2_{X,Z}h)(Y,W)-(\nabla^2_{Y,Z}h)(X,W)].\end{aligned}$$

We could observe that

$$tr(\nabla^2_{X,\cdot}h(\cdot,W))=-(\nabla\delta h)(X,W),\ \ tr(\nabla^2_{X,W}h(\cdot,\cdot))=\nabla^2_{X,W}(trh)=Hess(trh)(X,W)$$

and

$$tr(\nabla^2_{\cdot,\cdot}h(X,W))=(\Delta h)(X,W).$$

Substituting these into the preceding equation, and use

$$tr(h(R(W,\cdot)\cdot,X)) = -h(X,RicW),\ \ tr(h(R(X,\cdot)W,\cdot)) = \langle Rm(X,\cdot,W,\cdot),h\rangle.$$

We have

$$\begin{aligned}\frac{\partial}{\partial t}Ricc(X,W) = &-\frac{1}{2}tr[h(R(X,\cdot)W,\cdot)+h(R(X,\cdot)\cdot,W)\\ &+h(R(W,\cdot)X,\cdot)+h(R(W,\cdot)\cdot,X)]\\ &-\frac{1}{2}[(\nabla\delta h)(X,W)+Hess(trh)(X,W)\\ &+(\nabla\delta h)(W,X)+(\Delta h)(X,W)].\end{aligned}$$

Because

$$L_{(\omega^\sharp)}g(X,W) = \nabla\omega(X,W)+\nabla\omega(W,X)$$

and

$$L_{(\delta G(h))^\sharp}g = L_{(\delta h)^\sharp}g + Hess(trh),$$

by $L_{(df)^\sharp}g = L_{(\nabla f)}g = 2Hess(f)$, we obtain the proof! □

And then it's sufficient to obtain the linearization of $Bian(g,R)$ which will be used in Chapter 3.

Proposition 2. *If $T\in\Gamma(S^2T^*M)$ is independent of t, then* $(\frac{\partial}{\partial t}\delta G(T))Z = -T((\delta G(h))^\sharp,Z)+\langle h,\nabla T(\cdot,\cdot,Z)-\frac{1}{2}\nabla_Z T\rangle$.

Before doing the linearization, we also need some Lemmas.

Lemma 5. $\frac{\partial}{\partial t}R = -\langle Ricc,h\rangle + \delta^2 h - \Delta(trh)$.

proof. By Lemma 4,

$$\frac{\partial R}{\partial t} = -\langle h,Ricc\rangle + tr(\frac{\partial}{\partial t}Ricc).$$

Then, by Proposition 1, we have

$$tr(\frac{\partial}{\partial t}Ricc) = -\frac{1}{2}tr[\Delta_L h + L_{(\delta h)^\sharp}g + Hess(trh)].$$

By

$$h(X, Ricc(W)) = -tr(h(R(W,\cdot)\cdot, X)) = \langle h(X,\cdot), Ricc(W,\cdot)\rangle$$

which could be proved by orthonormal frame, and

$$tr(h(R(X,\cdot)W,\cdot)) = \langle Rm(X,\cdot,W,\cdot), h\rangle,$$

then

$$tr(\Delta_L h) = \Delta(trh).$$

We also know $tr(L_{(\delta h)^\sharp}g) = -2\delta^2 h \ and \ tr(Hess) = \Delta$, so

$$\frac{\partial R}{\partial t} = -\langle h, Ricc\rangle + \delta^2 h - \Delta(trh).$$

□

Lemma 6. *If* $\omega(t) \in \Gamma(T^*M)$*, then* $\frac{\partial}{\partial t}(\delta\omega) = \delta(\frac{\partial\omega}{\partial t}) + \langle h, \nabla\omega\rangle - \langle \delta G(h), \omega\rangle$.

proof. Because $(\delta\alpha)dV = \pm d(*\alpha) \ and \ \delta(f\alpha) = -\langle df, \alpha\rangle + f(\delta\alpha)$ for $f : (M, g) \longrightarrow \mathbf{R}$,

$$\int \langle df, \alpha\rangle dV = \int f(\delta\alpha)dV.$$

So

$$\int (\frac{\partial}{\partial t}(\delta\omega))fdV = -\int h(df,\omega)dV + \int \langle df, \frac{\partial\omega}{\partial t}\rangle dV - \int [(\delta\omega)f - \langle df,\omega\rangle]\frac{1}{2}(trh)dV.$$

The last term on the RHS is derived from Lemma 7.
We have

$$\int h(\omega, df)dV = \int \langle df, h(\omega,\cdot)\rangle dV = \int f\delta(h(\omega,\cdot))dV = \int [\langle \delta h, \omega\rangle - \langle h, \nabla\omega\rangle]fdV.$$

The last equality of integrand is proved by local coordinate.
Let $\alpha = \omega$ in $\delta(f\alpha) = -\langle df, \alpha\rangle + f(\delta\alpha)$, $\alpha = \frac{\partial\omega}{\partial t}$ in $\int \langle df, \alpha\rangle dV = \int f(\delta\alpha)dV$ and $\alpha = f\omega, \overline{f} = \frac{trh}{2}$ in $\int \langle d\overline{f}, \alpha\rangle dV = \int \overline{f}(\delta\alpha)dV$.
So it follows that

$$\int [\frac{\partial}{\partial t}(\delta\omega) + \langle \delta h, \omega\rangle - \langle h, \nabla\omega\rangle - \delta\frac{\partial\omega}{\partial t} + \langle d(\frac{trh}{2}), \omega\rangle]fdV = 0.$$

Therefore, we have

$$\frac{\partial}{\partial t}(\delta\omega) = \delta\frac{\partial\omega}{\partial t} + \langle h, \nabla\omega\rangle - \langle\delta G(h), \omega\rangle.$$

□

Lemma 7. $\frac{\partial}{\partial t}dV = \frac{1}{2}(trh)dV$.

proof. This follows from

$$\frac{d}{dt}[log\ det(g_{ij}(t))] = tr[(g_{ij}(t))^{-1}\frac{d(g_{ij}(t))}{dt}]$$

and

$$\frac{\partial}{\partial t}\sqrt{det(g_{ij})} = \frac{\partial}{\partial t}(e^{\frac{1}{2}log(det(g_{ij}))}).$$

□

Now we can complete the proof of Proposition 2 as follows:

proof. Firstly, we have

$$(\delta S)Z = \delta(S(\cdot, Z)) + \frac{1}{2}\langle S, L_Z g\rangle \tag{1}$$

for $S \in \Gamma(S^2T^*M)$, proved by local coordinate. And $\frac{\partial}{\partial t}(L_Z g)(X, Y) = h(\nabla_X Z, Y) + h(X, \nabla_Y Z) + (\nabla_Z h)(X, Y)$ by Lemma 1. By (1), set $S = G(T)$,

$$(\frac{\partial}{\partial t}\delta G(T))Z = \frac{\partial}{\partial t}\delta(G(T)(\cdot, Z)) + \frac{1}{2}\frac{\partial}{\partial t}\langle G(T), L_Z g\rangle.$$

Let us deal with the first term on the RHS.
Because $G(T) := T - \frac{1}{2}(trT)g$, by Lemma 4,

$$\frac{\partial G(T)}{\partial t} = \frac{1}{2}[\langle h, T\rangle g - (trT)h]. \tag{2}$$

Then, by Lemma 6, $\delta(f\alpha) = -\langle df, \alpha\rangle + f(\delta\alpha)$ for α 1-form,
and $L_{(\delta G(h))^\sharp} g = L_{(\delta h)^\sharp} g + Hess(trh)$,

$$\begin{aligned}
\frac{\partial}{\partial t}\delta(G(T)(\cdot, Z)) &= \delta\frac{\partial}{\partial t}(G(T)(\cdot, Z)) + \langle h, \nabla(G(T)(\cdot, Z))\rangle - \langle \delta G(h), G(T)(\cdot, Z)\rangle \\
&= \delta[\frac{1}{2}(\langle h, T\rangle g - (trT)h)(\cdot, Z)] + \langle h, (\nabla G(T))(\cdot, \cdot, Z)\rangle \\
&+ \langle h, G(T)(\cdot, \nabla.Z)\rangle - \langle \delta G(h), T(\cdot, Z) - \frac{1}{2}(trT)g(\cdot, Z)\rangle \\
&= -\frac{1}{2}Z\langle h, T\rangle + \frac{1}{2}\langle h, T\rangle\delta(g(\cdot, Z)) + h(\nabla(\frac{trT}{2}), Z) - \frac{1}{2}(trT)\delta(h(\cdot, Z)) \\
&+ \langle h, \nabla T(\cdot, \cdot, Z)\rangle + \langle h, -\frac{1}{2}d(trT)\otimes g(\cdot, Z)\rangle \\
&+ \langle h, G(T)(\cdot, \nabla.Z)\rangle - T((\delta G(h))^\sharp, Z) + \frac{1}{2}(trT)(\delta G(h))Z.
\end{aligned}$$

Using (1) with $S = g, h$, we then have

$$\begin{aligned}
\frac{\partial}{\partial t}\delta(G(T)(\cdot, Z)) &= -\frac{1}{2}\langle \nabla_Z h, T\rangle - \frac{1}{2}\langle \nabla_Z T, h\rangle - \frac{1}{4}\langle h, T\rangle tr(L_Z g) \\
&- \frac{1}{2}(trT)[(\delta h)Z - \frac{1}{2}\langle h, L_Z g\rangle] + \langle h, \nabla T(\cdot, \cdot, Z)\rangle + \langle h, G(T)(\cdot, \nabla.Z)\rangle \\
&- T((\delta G(h))^\sharp, Z) + \frac{1}{2}(trT)(\delta h)Z + \frac{1}{4}(trT)Z(trh) \\
&= [-T(\delta G(h))^\sharp, Z) + \langle h, \nabla T(\cdot, \cdot, Z)\rangle - \frac{1}{2}\langle h, \nabla_Z T\rangle] - \frac{1}{2}\langle \nabla_Z h, G(T)\rangle \\
&- \frac{1}{4}\langle h, T\rangle tr(L_Z g) + \frac{1}{4}(trT)\langle h, L_Z g\rangle + \langle h, G(T)(\cdot, \nabla.Z)\rangle.
\end{aligned}$$

Using the similar method in Lemma 4, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\langle G(T), L_Z g\rangle &= \frac{1}{2}\langle \frac{\partial}{\partial t}G(T), L_Z g\rangle + \frac{1}{2}\langle G(T), \frac{\partial}{\partial t}L_Z g\rangle \\
&- \langle h, G(T)(\cdot, \nabla.Z)\rangle - \langle G(T), h(\cdot, \nabla.Z)\rangle,
\end{aligned}$$

then, by (2) and $\frac{\partial}{\partial t}(L_Z g) = h(\nabla_X Z, Y) + h(\nabla_Y Z, X) + (\nabla_Z h)(X, Y)$,

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\langle G(T), L_Z g\rangle &= \frac{1}{2}\langle(\frac{1}{2}\langle h, T\rangle g - \frac{1}{2}(trT)h), L_Z g\rangle + \frac{1}{2}\langle G(T), (2h(\nabla.Z, \cdot) + \nabla_Z h)\rangle \\
&- \langle h, G(T)(\cdot, \nabla.Z)\rangle - \langle G(T), h(\cdot, \nabla.Z)\rangle \\
&= \frac{1}{4}\langle h, T\rangle tr(L_Z g) - \frac{1}{4}(trT)\langle h, L_Z g\rangle \\
&+ \frac{1}{2}\langle G(T), \nabla_Z h\rangle - \langle h, G(T)(\cdot, \nabla.Z)\rangle.
\end{aligned}$$

Combining two formulas, we conclude that

$$(\frac{\partial}{\partial t}\delta G(T))Z = -T((\delta G(h))^\sharp, Z) + \langle h, \nabla T(\cdot,\cdot,Z)\rangle - \frac{1}{2}\langle h, \nabla_Z T\rangle.$$

□

2.2 The principal symbol on vector bundle

In this section, we give the generalization of notion of principle symbol in PDE. So we also have similar results, such as existence and uniqueness of solution of strictly parabolic equations. Then applying these to the problems: Short-time existence of Ricci flow.

Definition 1. E: vector bundle over closed manifold M, $v := v^\alpha e_\alpha \in \Gamma(E)$ for local frame $\{e_\alpha\}$ on E,

$$\frac{\partial v}{\partial t} = L(v)$$

where L is a linear second order differential operator. (i.e.

$$\begin{aligned} L : \Gamma(E) &\longrightarrow \Gamma(E) \\ v &\longmapsto [a^{ij}_{\alpha\beta}\partial_i\partial_j v^\beta + b^i_{\alpha\beta}\partial_i v^\beta + c_{\alpha\beta}v^\beta]e_\alpha. \end{aligned}$$

in local coordinates $\{x^i\}$ and local frames $\{e_\alpha\}$ on E)

$$\begin{aligned} \sigma(L) : \Pi^{-1}(E) &\longrightarrow \Pi^{-1}(E) \\ (x,\xi)v &\longmapsto \sigma(L)(x,\xi)v := (a^{ij}_{\alpha\beta}\xi_i\xi_j v^\beta)e_\alpha \end{aligned}$$

where $\Pi : T^*M \longrightarrow M$, is called the principal symbol on vector bundle E over M. The definition is equivalent to the condition: $\forall (x,\xi) \in T^*M, v \in \Gamma(E), \phi : M \longrightarrow \mathbf{R}$ *with* $d\phi(x) = \xi$,

$$\sigma(L)(x,\xi)v = \lim_{s\to\infty} s^{-2}e^{-s\phi(x)}L(e^{s\phi}v)(x).$$

Definition 2. $\frac{\partial v}{\partial t} = L(v)$ is called strictly parabolic if there exists $\lambda > 0$ s.t.

$$\langle\sigma(L)(x,\xi)v, v\rangle \geq \lambda|\xi|^2|v|^2$$

for all $(x,\xi) \in T^*M, v \in \Gamma(E)$.

Definition 3. Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a quasilinear second order differential operator.

$$\frac{\partial v}{\partial t} = P(v)$$

is called parabolic at $w \in \Gamma(E)$ if $\frac{\partial v}{\partial t} = [DP(w)]v$ is parabolic.

Example: $\sigma(\Delta)(x,\xi) = |\xi|^2 id$.

Remark: From PDE, we know that if $\frac{\partial v}{\partial t} = P(v)$ is strictly parabolic at w, then there exists $\epsilon > 0, v(t) \in \Gamma(E)$ for $t \in [0,\epsilon]$ s.t. $\frac{\partial v}{\partial t} = P(v), v(0) = w$.

2.3 Local solvability

In the last section, we have introduced some convection of strictly parabolic equations on manifold which would be applied in the first topic (Theorem 1). And then we developed some notion of elliptic equation to solve the second topic as follows (Theorem 2).

Consider

$$F_j(x, D^\alpha u) = 0 \qquad (*)$$

$$for\ \ j \in I_p := \{1, 2...p\}, |\alpha| \leq r, u = (u^1(x), ..., u^q(x)), x \in \mathbf{R}^n.$$

where $F_j \in C_x^{m+\sigma} \cap C_{D^\alpha u}^\infty$.

Definition 4. $(*)$ is called elliptic at x_0 for u_0 if

$$L_j w := \sum_{|\beta|=r, k\leq q} \frac{\partial F_j}{\partial D^\beta u^k}(x_0, D^\alpha u_0) D^\beta w^k := \sum_{|\beta|=r,k\leq q} c_{j\beta k} D^\beta w^k \ \ for\ \ j \in I_p$$

is elliptic. (This means that $\forall \xi \in \mathbf{R}^n \backslash \{0\}, [\sigma_{jk}^L] : principal\ \ symbol\ \ of\ \ \{L_j\}$ has maximal rank, where $\sigma_{jk}^L := i^r \sum_{|\beta|=r} c_{j\beta k}\xi^\beta, j \in I_p, k \in I_q$)

Definition 5. $(*)$ is called determined/overdetermined/underdetermined elliptic if its principal symbol is bijective/injective/surjective. $u_0(x)$ is called an infinitesimal solution of $(*)$ at x_0 if

$$F_j(x, D^\alpha u_0)|_{x=x_0} = 0 \ \ \forall j \in I_p.$$

To note the later description of preceding definition, this just means the infinitesimal solution of $(*)$ at x_0 is "very local" solution.

Lemma 8. *(Local solvability) If u_0 is an infinitesimal solution of a determined or underdetermined elliptic system $(*)$ at x_0, then for ρ sufficiently small, there exists $u \in C^{m+r+\sigma}$ which is a solution of $(*)$ for $|x - x_0| < \rho$.*

Remark: We would modify the context of the following proof to achieve proof of Theorem 2 !

proof. Firstly, assume $F_j(x, D^\alpha u)$ is determined. WLOG., may assume $u_0 = 0, x_0 = 0$. Let $v(y)$ be a function on $B_1(0)$, $\rho \in \mathbf{R}$,

$$\Phi : \mathbf{R} \times C^{m+r+\sigma}_{B_1(0)}(\mathbf{R}^q) \longrightarrow C^{m+\sigma}_{B_1(0)}(\mathbf{R}^p)$$

$$(\rho, v) \longmapsto F(\rho y, \rho^{r-|\alpha|} D^\alpha_y v).$$

We just claim: $\Phi(\rho, v) = 0$ for some $\rho > 0$. Because

$$u(x) := \rho^r v(\frac{x}{\rho})$$

on $B_\rho(0)$ gives a solution of $F_j(x, D^\alpha u) = 0$.

$$\frac{\partial \Phi}{\partial v}(0,0) := \Phi_2(0,0) := L_j = \sum_{|\beta|=r, k\leq q} c_{j\beta k} D^\beta \quad for \;\; j \in I_p.$$

As such it admits a continuous linear right inverse(see[5,Lemma 9.5]):

$$S : C^{m+\sigma}_{B_1(0)}(\mathbf{R}^p) \longrightarrow C^{m+r+\sigma}_{B_1(0)}(\mathbf{R}^q).$$

By the implicit function theorem(see[6,Theorem 6.1.1]), we know that for ρ sufficiently small,

$$v \longmapsto v - S(\Phi(\rho, v))$$

is a strict contraction for v near zero. The fixed point of this mapping is what we want. Secondly, for $F_j(x, D^\alpha u)$: underdetermined. Notice that the LL^* : determined elliptic, so the above proof could be applied to $F_j(x, D^\alpha L^* u)$. $\square$

2.4 Banach submanifold of solutions of $F_j(x, D^\alpha u) = 0$

We adapted notations in [8] and assumed L is underdetermined. By some deduction, Φ (given in the preceding Lemma) is really a submersion, so we can write

$$\Phi : \mathbf{R} \times ker\Phi_2(0,0) \times Im(L^*S) \longrightarrow C^{m+\sigma}_{B_1(0)}(\mathbf{R}^p)$$

(intersection with $C^{m+r+\sigma}_{B_1(0)}(\mathbf{R}^q)$ is understood on the left), and then by the implicit function theorem in [6], there is

$$\phi : [0, \epsilon) \times ker\Phi_2(0,0) \longrightarrow Im(L^*S) \in C^\infty$$

that yields solutions of $F_j(x, D^\alpha u) = 0$.

The submersion mapping ϕ satisfies $F(x, D^\alpha_x \rho^r[k(\frac{x}{\rho}) + \phi(\rho, k)(\frac{x}{\rho})]) = 0$ for $x \in B_\rho(0), k \in ker(\Phi_2(0,0))$ *near* 0.

Conclude it as follows:

Lemma 9. *If $L := \Phi_2(0,0)$ is the highest-order constant coefficient part of the underdetermined elliptic system $(*)$ at x_0 and the infinitesimal solution u_0, then for ρ sufficiently small, there is a Banach submanifold of solutions of $F_j(x, D^\alpha u) = 0$, parametrized by functions in $ker(\Phi_2(0,0))$.* $\diamond$

Lemma 10. *If R is nonsingular, then Bian(g,R) is an underdetermined elliptic operator.*

Notation:

$$Bian(g, R) := -div(G(R)).$$

proof.

$$Because\ \ Bian'(g, R)h = R^s_m(div(G(h)))_s - T^{qs}_m h_{qs}$$
$$where\ \ T^{qs}_m := g^{qk}g^{sl}[\frac{\partial R_{lm}}{\partial x^k} - \frac{1}{2}\frac{R_{kl}}{\partial x^m} - \Gamma^i_{kl}R_{im}]\ \ for\ \ h \in S^2T^*M,$$

we only prove the principal symbol of $div(G\cdot)$ is surjective. (i.e. $\forall \xi \in T^*M, v \in T^*M$, we should solve

$$g^{st}(\xi_s p_{tm} - \frac{1}{2}\xi_m p_{st}) = v_m$$

$for\ \ p$) So we just choose

$$p_{kl} = \frac{\xi_k v_l + \xi_l v_k}{g^{st}\xi_s\xi_t}.$$

□

We would assume two facts as follows: (Because these two facts are not the main results in this review, we skip their proofs, but their deduction could be referred to [2])

Fact 1. *If R is nonsingular, then the infinitesimal solution of $Bian(g, R) = 0$ exists.*

So we have the following lemma by Lemma 9.

Lemma 11. *If $R^{-1}(0)$ exists, then for sufficiently small $\rho > 0$, the solutions of Bian(g,R)=0 on $B_\rho(0)$ near a given infinitesimal solution g_0 form a submanifold of the Banach manifold of metrics on $B_\rho(0)$.* ⋄

3 Proofs

This chapter is dedicated to the two theorems as promised in Introduction.

3.1 Short-time existence of Ricci flow

First, by some calculation and Chapter 2.1, we know $\frac{\partial g}{\partial t} = Q(g) := -2Ricc(g)$ on $E := S^2T^*M$ isn't strictly parabolic. For if, we have

$$\sigma(L)(x,\xi)h = |\xi|^2 h - \xi \otimes h(\xi^\sharp, \cdot) - h(\xi^\sharp, \cdot) \otimes \xi + (\xi \otimes \xi)trh$$

Let $h := \xi \otimes \xi$, $\Rightarrow \sigma(L)(x,\xi)h = 0$ *where* $\frac{\partial h}{\partial t} = Lh := [DQ(g)]h = \Delta_L h + L_{(\delta G(h))^\sharp} g$.

Theorem 1. *(Short-time existence)If g_0 is a smooth metric on a closed Riemannian manifold M, then there exists a smooth solution $g(t)$ to the Ricci flow defined on some small time interval with $g(0) = g_0$. (i.e. $\exists \epsilon > 0$, $g(t)$ on $[0, \epsilon)$, s.t. $\frac{\partial g}{\partial t} = -2Ricc(g), g(0) = g_0$ on $[0, \epsilon)$)*

Remark: We just focus on the existence rather than uniqueness, so the proof of uniqueness can be referred to [1,P.113∼P.116].

proof. Let $T \in \Gamma(S^2T^*M)$ be fixed, positve definite. Denote by T the invertible map $\Gamma(T^*M) \longrightarrow \Gamma(T^*M)$ which is induced by T. Let

$$P(g) := -2Ricc(g) + L_{(T^{-1}\delta G(T))^\sharp}g.$$

By some calculations, we have

$$\frac{\partial}{\partial t}L_{(T^{-1}\delta G(T))^\sharp}g = -L_{(\delta G(T))^\sharp}g + A(h, \nabla h)$$

where $h := \frac{\partial g}{\partial t}$. (That's why we choose $P(g)$!)

$$[DP(g)]h = \Delta h + \overline{A}(h, \nabla h) \implies \sigma(DP(g))(x,\xi)h = |\xi|^2 h$$
$$\implies \frac{\partial g}{\partial t} = P(g) : strictly\ \ parabolic.$$

There exists a family of diffeomorphisms $\psi_t : M \longrightarrow M$ corresponding to $(-T^{-1}\delta G(T))^\sharp$.

Set

$$\overline{g}(t) := (\psi_t^* g).$$

We have that for all g_0: smooth, $\exists \epsilon > 0, \exists\, \overline{g}(t)$: solution of $\frac{\partial \overline{g}}{\partial t} = -2Ricc(\overline{g}), \overline{g}(0) = g_0$. □

3.2 Local existence of metrics with prescribed Ricci curvature

In this section, we may omit several steps of proofs of lemmas or even not give their proofs. (It will be better to grip the main idea without tedious deduction or details)

By observing

$$Ricc'(g)h = -[\frac{1}{2}\Delta_L h + div^*(div(G(h)))],$$

$$Bian'(g,R)h = R_m^s(div(G(h)))_s - T_m^{qs}h_{qs}$$

$$where\ T^{qs}_m := g^{qk}g^{sl}[\frac{\partial R_{lm}}{\partial x^k} - \frac{1}{2}\frac{R_{kl}}{\partial x^m} - \Gamma^i_{kl}R_{im}] \quad for\ \ h \in S^2T^*M,$$

we'll consider the equation

$$Ricc(g) + div^*(R^{-1}Bian(g,R)) = R. \qquad (**)$$

It's elliptic !

Fact 2. *If R is invertible s.t. R(0) is diagonal and all first partial derivatives of R vanish at 0, then we can choose a metric* g_0 *of form* $(g_0)_{ij} = \delta_{ij} + O(x^2)$ *s.t.* $Ricc(g_0)|_{x=0} = R(0), Bian(g_0,R)|_{x=0} = 0,$ *and* $\partial_i Bian(g_0,R)|_{x=0} = 0$ *for all* $i \in I_n$.

This result would reduce some tedious calculations in latter work.

Theorem 2. *If* R_{ij} *is a* $C^{m+\sigma}$*(resp.* C^∞, C^ω*)* $tensor\ field(m > 2)$ *in a neighborhood of p on* M^n $(n \geq 3)$ *and* $R^{-1}(p)$ *exists, then there exists a metric g with prescribed R as its Ricci curvature tensor locally. (More precisely, there exists* $g \in C^{m+\sigma}$*(resp.* C^∞, C^ω*) : Riemann metric s.t.* $Ricc(g) = R$ *in some neighborhood of p)*

proof. Because this proof is more complicated than previous ones, we divide it into several steps and lemmas.

Outlines of the proof

Considering

$$Bian(g_0 + h, R) = 0,$$

we set X to be the submanifold of solutions of $Bian(g_0 + h, R) = 0$. By the Lemma 11, we know that X is parametrized by ρ, $k(\in$ kernel of the highest-order constant-coefficient part of the linearization of the Bianchi identity about g_0 at 0). Denote the constant-coefficient operator by

$$div_0 G_0.$$

Let $h_0 := \phi(\rho, 0)$ be the point of X corresponding to our chosen small value of ρ where ϕ is defined in section 2.4 .

Step1: Given h_j for $j \in \mathbf{N}' := \mathbf{N} \bigcup \{0\}$, perform the contracting iteration to form $\overline{h}_j$.

Step2: Let $h_{j+1} \in X$ be the projection of $\overline{h}_j$ onto X by ϕ and the decomposition

$$C^{m+2+\sigma}(S^2T^*M) = ker(div_0 G_0) \oplus Im(div_0^* S).$$

Firstly, we pick the special continuous linear right inverses of Bianchi operator $Bian(g, R)$ and $(**)$ operator.

Let

$$F(x, D^\alpha h) := Bian(g_0 + h, R)$$

for $h \in S^2T^*M$ (defined near 0).

Let

$$\Phi(\rho, v) := F(\rho y, \rho^{1-|\alpha|} D_y^\alpha v) \ \ on \ \ B_1(0),$$

we obtain

$$\Phi_2(0,0)w = R(0) div_0 G_0(w).$$

Choose S s.t.

$$\frac{1}{2} S(R(0)\cdot)$$

solves the Dirichlet problem for $\Delta_1 := -\sum_{j \in I_n} \partial_j^2$ as a right inverse of

$$\Phi_2(0,0) div_0^*.$$

Notice that why we not select S as a right inverse of $\Phi_2(0,0)\Phi_2(0,0)^*$, it's due to

$$R(0) div_0 G_0 div_0^*(v) = \frac{1}{2} R(0) \Delta_1(v).$$

By the implicit function theorem, there exists

$$\phi : \mathbf{R} \times ker\Phi_2(0,0) \longrightarrow Im(\Phi_2(0,0)^* S)$$

s.t. $\Phi(\rho, k + \phi(\rho, k)) = 0$ for $\rho > 0$, $k \in ker\Phi_2(0,0)$: sufficiently small.

We have the property of ϕ :

Lemma 12. $\phi(0,0) = 0 = \phi_1(0,0) = \phi_2(0,0)$. $\diamond$

Its proof could be deduced from $\Phi(\rho, k + \phi(\rho, k)) = 0$.

Because

$$C^{m+\sigma}(S^2T^*M) = ker(div_0 G_0) \oplus Im(div_0^* S),$$

we have that both equations

$$P_1 : C^{m+\sigma}(S^2T^*M) \longrightarrow ker(div_0 G_0)$$
$$P_2 : C^{m+\sigma}(S^2T^*M) \longrightarrow Im(div_0^* S)$$

are canonical projections.

So the above discussion could be concluded with a sequence :

$$C^{m+1+\sigma}(T^*M) \overset{div_0^*}{\to} C^{m+\sigma}(S^2T^*M) \overset{\Phi(\rho,-),R(0)div_0 G_0}{\longrightarrow} C^{m-1+\sigma}(T^*M) \overset{S}{\to} C^{m+1+\sigma}(T^*M).$$

From definition, we have

$$\Phi_2(0,0) : Im(div_0^* S) \overset{\simeq}{\to} C^{m-1+\sigma}(T^*M).$$

So these imply

$$P_2 h = div_0^* S(R(0) div_0 G_0(h)) \;\; for \;\; h \in C^{m+\sigma}(S^2T^*M). \tag{3}$$

Let

$$H(x, D^\alpha h) := Ricc(g_0 + h) + div^* R^{-1}(Bian(g_0 + h, R)) - R$$

and

$$\Psi(\rho, \upsilon) := H(\rho y, \rho^{2-|\alpha|} D_y^\alpha \upsilon) \;\; on \;\; B_1(0).$$

We obtain

$$\Psi_2(0,0)h = \frac{1}{2}\Delta_2 h$$

where Δ_2 is the standard Laplacian operating componentwise on $h \in S^2T^*M$.

Let T be a right inverse of $\Psi_2(0,0)$ chosen as follows :

Lemma 13. *If $m \geq 1$, then for any continuous linear right inverse S of $\Phi_2(0,0)div_0^*$, there exists $T : C^{m+\sigma}(S^2T^*M) \longrightarrow C^{m+2+\sigma}(S^2T^*M)$ is a bounded linear mapping s.t. for $h \in C^{m+\sigma}(S^2T^*M)$,*

$$
\begin{aligned}
div_0 G_0 T(h) &= S(R(0)div_0 G_0(h)) \\
P_1(T(h)) &= T(P_1(h)) \\
\frac{1}{2}\Delta_2(T(h)) &= h. \quad \diamond
\end{aligned}
$$

proof. (Sketch of proof of Lemma 13)
1. Let $h \in Im(div_0^* S)$. We set $Th := div_0^* S(R(0)S(\Phi_2(0,0)h))$.

2. Let $h \in ker(div_0 G_0)$. Let N be the fundamental solution right inverse of $\frac{1}{2}\Delta_2$. This means that

$$N(h)_{ij} := \int_{B_1(0)} \frac{2h_{ij}(\xi)}{r^{n-2}} dV_\xi$$

where $r := |x - \xi|$. Let N_i be the inverse for the $(n-1)$-variable scalar Laplacians on $B_1(0) \bigcap (x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n)$.

Set

$$
\begin{aligned}
&Q : C^{m+2+\sigma}_{B_1(0)}(T^*M) \longrightarrow C^{m+2+\sigma}_{B_1(0)}(S^2T^*M) \\
&\quad v \longmapsto Q(v)(x^1, \dots, x^n) := diag\{-\int_0^{x^i} v_i dx^i + N_i(D_i v_i)(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n)\}.
\end{aligned}
$$

So we set

$$Th := N(h) - G_0^{-1} Q(div_0 G_0 N(h))$$

where $G_0^{-1}(h) = h - \frac{tr(h)}{n-2} g_0$.

3. Combine 1 and 2 , we have that T is well-defined on $C^{m+\sigma}(S^2T^*M)$. $\square$

For all $\rho > 0, h \in C^{m+\sigma}_{B_1(0)}(S^2T^*M)$, we define

$$B_h(x, D^\alpha R) := -div(G(R))$$

where divergence and gravitational operators are those of metric

$$g_0(x) + \rho^2 h(\frac{x}{\rho})$$

for $x \in B_\rho(0)$, and let

$$\eta_\rho(h)(r) := B_h(\rho y, \rho^{1-|\alpha|} D_y^\alpha r)$$

for $r \in C^{m+\sigma}_{B_1(0)}(S^2T^*M)$.
Note that

$$\|\eta_\rho(h) + div_0 G_0\| \longrightarrow 0 \;\; as \;\; \rho \to 0, \|h\|_{C^{m+\sigma}_{B_1(0)}(S^2T^*M)} \to 0.$$

and

$$if \;\; Bian(g, R) = 0 \;\; for \;\; \rho > 0 \;\; , \;\; g(x) = g_0(x) + \rho^2 h(\frac{x}{\rho}), \;\; then \;\; \eta_\rho(h)(\Psi(\rho, h)) = 0.$$

Let

$$\overline{\phi}(\rho, k) := \frac{1}{\rho}\phi(\rho, \rho k) \;\; for \;\; \rho > 0$$

and $\overline{X}$ be the submanifold of $\mathbf{R} \times C^{m+\sigma}_{B_1(0)}(S^2T^*M)$ consisting of points of the form

$$(\rho, k + \overline{\phi}(\rho, k))$$

for $\rho > 0, k \in ker(\Phi_2(0,0))$. We use this smooth submanifold $\overline{X}$ instead of using the submanifold X in the outlines of the proof. And we set $\overline{X}_\rho := \overline{X}|_{\{\rho\}\times C^{m+\sigma}_{B_1(0)}(S^2T^*M)}$.

Let us complete the issues of convergence of iterates and verification that the limit is what we want !

Because we are looking for a solution of $\Psi(\rho, v) = 0$ for some $\rho > 0$, that lies on $\overline{X}$, all of the iterates will be required to lie in $\overline{X}_\rho$.

Set

$$k_0 := 0,$$
$$k_{i+1} := N_\rho(k_i) := k_i - P_1(T\Psi(\rho, k_i + \overline{\phi}(\rho, k_i))).$$

It's clear that $\{k_i\} \subseteq ImT \bigcap kerdiv_0G_0$.

Let $\widehat{B}_\rho(0)$ be the ball of radius ρ centered at 0 in $ker(div_0G_0) \bigcap Im\, T$.

Secondly, we show that the convergence of $\{k_i\}$ is hold.

Choose $\epsilon < 1$ s.t.

$$\epsilon\|\Psi_2(0,0)\|\|T\|\|P_1\| < \frac{1}{6} \;\; and \;\; \epsilon\|T\|\|P_1\| < \frac{1}{6}.$$

We obtain that

$$|\Psi_2(0,0)(u-v) - \Psi(\rho,u) + \Psi(\rho,v)| < \epsilon|u-v| \;\; if \;\; u,v \in C^{m+\sigma}_{B_1(0)}(S^2T^*M)$$

with $|u|, |v| < \delta'$ for sufficiently small $\rho, \delta' > 0$ by MVT. ([6])

Because

$$\lim_{\rho\to 0}\overline{\phi}(\rho,k) = 0 \;\; and \;\; \phi_2(0,0) = 0$$

by Lemma 12, we have

$$|k + \overline{\phi}(\rho,k)| < \delta' \;\; and \;\; \|\overline{\phi}_2(\rho,k)\| < \epsilon$$

if $k \in ker(div_0 G_0), |k| < \delta$ for ρ, δ sufficiently small.

These imply the following equations

$$|\overline{\phi}(\rho,k) - \overline{\phi}(\rho,l)| < \epsilon|k-l| \;\; for \;\; k,l \in \widehat{B}_\delta(0),$$

$$N_\rho(k) - N_\rho(l) = P_1T[\Psi_2(0,0)(\overline{k}-\overline{l}) - \Psi(\rho,\overline{k}) + \Psi(\rho,\overline{l})] - P_1[T\Psi_2(0,0)(\overline{\phi}(\rho,k) - \overline{\phi}(\rho,l))]$$

where $k, l \in \widehat{B}_\delta(0), \overline{k} := k + \overline{\phi}(\rho,k), \overline{l} := l + \overline{\phi}(\rho,l)$.

So we have $|N_\rho(k) - N_\rho(l)| \leq \frac{1}{2}|k-l|$.
May decrease ρ s.t.

$$|T(P_1(\Psi(\rho,\overline{\phi}(\rho,0))))| < \frac{\delta}{2}.$$

We obtain the proof of $\{k_i\}_{i\in\mathbf{N}}$: converges in $\widehat{B}_\rho(0)$ for ρ sufficiently small.

Finally, we want to show that, for ρ, δ sufficiently small, if $k \in \widehat{B}_\delta(0)$, and $N_\rho(k) = k$, then $\Psi(\rho,\overline{k}) = 0$.

Because

$$T \;\; is \;\; a \;\; bounded \;\; isomorphism$$

and

$$\|\eta_\rho(\overline{k}) + div_0 G_0\| \longrightarrow 0 \;\; as \;\; \rho \searrow 0, \;\; \|\overline{k}\|_{C^{m+\sigma}_{B_1(0)}(S^2T^*M)} \searrow 0,$$

so

$$|R(0)|^2 \cdot |div_0^* SS(\eta_\rho(\overline{k}) + div_0 G_0)(k)| \le \frac{1}{2}|Tk|$$

for all $k, |\overline{k}| < \delta$, for ρ, δ sufficiently small.

This follows from that $|Tk| > \lambda|k|$ for some λ, so if ρ, δ sufficiently small, then

$$\|\eta_\rho(\overline{k}) + div_0 G_0\| \le \frac{\lambda}{2\|div_0^* SS\| \cdot |R(0)|^2}$$

for $|\overline{k}| < \delta$.

It's clear that

$$P_2 T(\Psi(\rho, \overline{k})) = T(\Psi(\rho, \overline{k})).$$

By (3), Lemma 13 and the paragraph below it,

$$\begin{aligned} |P_2 T(\Psi(\rho, \overline{k}))| &= |div_0^* S(R(0) div_0 G_0 T \Psi(\rho, \overline{k}))| \\ &= |div_0^* SSR(0)^2 [\eta_\rho(\overline{k}) + div_0 G_0] \Psi(\rho, \overline{k})| \\ &\le \frac{1}{2} |T\Psi(\rho, \overline{k})|. \end{aligned}$$

So

$$T\Psi(\rho, \overline{k}) = 0 \implies \Psi(\rho, \overline{k}) = 0.$$

Hence we complete the proof of Theorem 2. □

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