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Hopf 分歧之探討與研究

A Survey On Hopf bifurcation

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摘要

這篇論文是 Hopf 分歧的文獻探討，Hopf 分歧是關於出現週期解的分歧現象，由於 Hopf 分歧的現象廣泛的出現在許多領域，所以這篇論文主要在陳述 Hopf 分歧定理的性質和證明，而特別的是證明的過程中也給出了計算的方法與公式，從中我們可以得到關於週期解，週期，穩定性的一些性質，最後給出幾個簡單的應用例子。

關鍵字：Hopf，分歧，Floquet 定理，週期解，中央流型。

Abstract

This paper is a survey on Hopf bifurcation, Hopf bifurcation is very important in many areas. In this paper, we focus on the properties and proofs of Hopf bifurcation. The proof in this paper give the bifurcation formula, and we give some examples after we proved the theorem.

Keywords: Hopf, bifurcation, Floquet theorem, center manifold, periodic solution.

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A Survey On Hopf Bifurcation

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1 Introduction

In history, many interesting examples arise in engineering have periodic behaviors, like James Watt's centrifugal governor. Now, periodic phenomena can be observed in many place, like biology and chemistry. The Hopf bifurcation is discussing about this phenomena. In this survey, we focus on the autonomous system with one parameter, and state the properties and proofs of Hopf bifurcation. The Hopf bifurcation can be proved in many ways [3, 8]. The proof in this survey give the bifurcation formula, and we will give examples in the last section.

2 Statement of the Theorem

We investigate the autonomous system of differential equations

$$\frac{dX}{dt} = F(X, \mu) \tag{1}$$

where $X \in \mathbb{R}^n$ and μ is a real parameter on an open interval I.

Theorem. *($C^L - $Hopf Bifurcation)[5] If*

(1) $F(0,\mu) = 0$ *for* μ *in an open interval containing* 0*, and* $0 \in \mathbb{R}^n$ *is an isolated stationary point of F.*

(2) *All partial derivatives of F of order* $\leq L+2$ $(L \geq 2)$ *exist and are continuous in X and* μ *in a neighborhood of* $(0,0)$ *in* $\mathbb{R}^n \times \mathbb{R}^1$

(3) $A(\mu) = D_X F(0,\mu)$ *has a pair of complex conjugate eigenvalue* λ *and* $\overline{\lambda}$ *such that*

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu),$$

where

$$\omega(0) = \omega_0 > 0, \qquad \alpha(0) = 0, \qquad \alpha'(0) \neq 0.$$

(4) *The remaining* $n-2$ *eigenvalues of* $A(0)$ *have strictly negative real parts,*

then the system (1) *has a family of periodic solutions satisfies the following properties:*

(a) *There exist an* $\mathscr{C} > 0$ *and a* C^{L+1} *function* $\mu(\varepsilon)$,

$$\mu(\varepsilon) = \sum_{i=1}^{[\frac{L}{2}]} \mu_{2i}\varepsilon^{2i} + O(\varepsilon^{L+1}) \qquad (0 < \varepsilon < \mathscr{C})$$

such that for each $\varepsilon \in (0,\mathscr{C})$ *there exists a periodic solution* $P_\varepsilon(t)$, *occurring for* $\mu = \mu(\varepsilon)$. *There is a neighborhood* $\mathfrak{N}$ *of* $X = 0$ *and an open interval* I *containing* 0 *such that for any* $\mu \in I$ *the only nonconstant periodic solutions of* (1) *that lie in* $\mathfrak{N}$ *are the members of the periodic solution* $P_\varepsilon(t)$ *which satisfies* $\mu(\varepsilon) = \mu$, $\varepsilon \in (0,\mathscr{C})$

(b) *The period* $T(\varepsilon)$ *of* $P_\varepsilon(t)$ *is a* C^{L+1} *function*

$$T(\varepsilon) = \frac{2\pi}{\omega_0}\left(1 + \sum_{i=1}^{[\frac{L}{2}]} \tau_{2i}\varepsilon^{2i}\right) + O(\varepsilon^{L+1}) \qquad (0 < \varepsilon < \mathscr{C})$$

(c) *The periodic solution* $P_\varepsilon(t)$ *has two exactly Floquet exponents [2, 4, 6] approach* 0 *as* $\varepsilon \to 0$, *one is* 0 *for* $\varepsilon \in (0,\mathscr{C})$ *and the other is a* C^{L+1} *function*

$$\beta(\varepsilon) = \sum_{i=1}^{[\frac{L}{2}]} \beta_{2i}\varepsilon^{2i} + O(\varepsilon^{L+1}) \qquad (0 < \varepsilon < \mathscr{C})$$

the periodic solution $P_\varepsilon(t)$ *is orbitally asymptotically stable with asymptotic phase if* $\beta(\varepsilon) < 0$, *is unstable if* $\beta(\varepsilon) > 0$.

3 Proof of the Theorem

The proof is divided into three main parts:

- **Part1** : The proof of 2×2 systems in Poincaré normal form;
- **Part2** : The reduction of general 2×2 systems to Poincaré normal form;
- **Part3** : Application of the Center Manifold Theorem to reduce general $n \times n$ systems to the 2×2 case on center manifold.

Part1(A) Existence

Assume we have the 2×2 Poincaré normal form[1]:

$$\dot{X} = A(\mu)X + \sum_{j=1}^{[\frac{L}{2}]} B_j(\mu)X|X|^{2j} + O(|X||(X,\mu)|^{L+1}) = F(X,\mu) \tag{2}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad A(\mu) = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix} \qquad \lambda(\mu) = \alpha(\mu) + i\omega(\mu)$$

$$B_j(\mu) = \begin{pmatrix} \mathrm{Re}C_j(\mu) & -\mathrm{Im}C_j(\mu) \\ \mathrm{Im}C_j(\mu) & \mathrm{Re}C_j(\mu) \end{pmatrix} \qquad (1 \le j \le [\frac{L}{2}])$$

and $F(X,\mu)$ is jointly C^{L+2} in X and μ.

If we let $\zeta = x_1 + ix_2$ then (2) can be written as

$$\dot{\zeta} = \lambda(\mu)\zeta + \sum_{j=1}^{[\frac{L}{2}]} c_j(\mu)\zeta|\zeta|^{2j} + O(|\zeta||(\zeta,\mu)|^{L+1}).$$

Let $X = \varepsilon Y$. We have

$$\dot{Y} = A(\mu)Y + \sum_{j=1}^{[\frac{L}{2}]} \varepsilon^{2j} B_j(\mu)Y|Y|^{2j} + O(|Y||(\varepsilon Y,\mu)|^{L+1}), \tag{3}$$

with initial condition $Y(0) = (1,0)^T$, which is $X(0) = (\varepsilon,0)^T$. Since $F(X,\mu)$ is C^{L+2} jointly in X and μ, the R.H.S of (3) is C^{L+1} jointly in Y, μ, ε. We let $Y = Y(t,\varepsilon,\mu)$ denote the solution of (3) satisfies initial condition $(1,0)^T$. Now for $\varepsilon = 0$ and μ small, $Y(t,0,\mu)$ will cross y_1-axis for $t = T_0(\mu)$, where

$$T_0(\mu) = \frac{2\pi}{\omega(\mu)} + O(\mu^{L+1}).$$

Now,

$$Y(T_0(\mu),0,\mu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot [e^{\frac{2\pi\alpha(\mu)}{\omega(\mu)}} + O(\mu^{L+1})],$$

$Y \in C^{L+1}$ jointly in t, ε, μ, and

$$\dot{Y}(T_0(\mu),0,\mu) = \begin{pmatrix} \alpha(\mu) \\ \omega(\mu) \end{pmatrix} \cdot [e^{\frac{2\pi\alpha(\mu)}{\omega(\mu)}} + O(\mu^{L+1})].$$

Since $\omega(\mu) > 0$ for μ in a neighborhood of 0, $y_2(t,\varepsilon,\mu) = 0$ for $t = T(\varepsilon,\mu)$ with $T(0,\mu) = T_0(\mu)$ and $T \in C^{L+1}$ jointly in ε, μ, if we let $I(\varepsilon,\mu) \equiv y_1(T(\varepsilon,\mu),\varepsilon,\mu)$, then $I \in C^{L+1}$ for ε,μ small, and since

$$\begin{aligned}
\frac{\partial I}{\partial \mu}(0,0) &= \lim_{\mu\to 0} \frac{y_1(T(0,\mu),0,\mu) - y_1(\omega_0,0,0)}{\mu} \\
&= \lim_{\mu\to 0} \frac{e^{\frac{2\pi\alpha(\mu)}{\omega(\mu)}} + O(\mu^{L+1}) - e^{\frac{2\pi\alpha_0}{\omega_0}}}{\mu} \\
&= 2\pi\frac{\alpha'(0)}{\omega(0)} \neq 0.
\end{aligned}$$

By the implicit function theorem, we have $\mu = \mu(\varepsilon)$, $\mu \in C^{L+1}$ for $\varepsilon \in [0,\mathscr{C})$ such that $I(\varepsilon,\mu(\varepsilon)) = 1$, so we got a family of periodic solutions for each $\varepsilon \in (0,\mathscr{C})$ that satisfies the system $\dot{X} = F(X,\mu)$ with initial condition $X(0) = (\varepsilon,0)^T$. This completes the prove of the existence of periodic solutions for the Poincaré normal form.

Part1(B) Bifurcation Formula

In Part(A) we have proved the existence of periodic solutions. In this part, we are going to prove the following lemma

Lemma. *If the Poincaré normal form of (1) is*

$$\begin{aligned}\dot{\zeta} &= \lambda(\mu)\zeta + \sum_{j=1}^{[\frac{L}{2}]} c_j(\mu)\zeta|\zeta|^{2j} + O(|\zeta||(\zeta,\mu)|^{L+1}) \\ &\equiv H(\zeta,\overline{\zeta},\mu)\end{aligned} \tag{4}$$

where $H(\zeta,\overline{\zeta},\mu)$ is C^{L+2} jointly in ζ, $\overline{\zeta}$, μ in a neighborhood of $0 \in \mathbb{C}\times\mathbb{C}\times\mathbb{R}^1$, then the periodic solution of period $T(\varepsilon)$ such that $\zeta(0,\mu) = \varepsilon$ of (4) has the form

$$\zeta = \varepsilon e^{\frac{2\pi i t}{T(\varepsilon)}} + O(\varepsilon^{L+2}), \tag{5}$$

where

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \sum_{i=1}^{L}\tau_i\varepsilon^i] + O(\varepsilon^{L+1}), \tag{6}$$

and

$$\mu(\varepsilon) = \sum_{i=1}^{L}\mu_i\varepsilon^i + O(\varepsilon^{L+1}). \tag{7}$$

Proof. From Part(A), we see that $T(\varepsilon)$ and $\mu(\varepsilon)$ are in C^{L+1} with $T(0) = \frac{2\pi}{\omega_0}$, $\mu(0) = 0$. So $T(\varepsilon)$ and $\mu(\varepsilon)$ can be written as (6) and (7). We use the change of variables (τ,η) with

$$\tau = \frac{t}{T(\varepsilon)} \qquad \text{and} \qquad \zeta = \varepsilon e^{2\pi i\tau}\eta.$$

Then because $\frac{d\zeta}{dt} = \varepsilon 2\pi i e^{2\pi i\tau}\eta\frac{d\tau}{dt} + \varepsilon e^{2\pi i\tau}\frac{d\eta}{d\tau}\frac{d\tau}{dt}$ and $\frac{d\tau}{dt} = \frac{1}{T(\varepsilon)}$, so put τ, η in (4) we have

$$2\pi i\eta + \frac{d\eta}{d\tau} = T(\varepsilon)\eta[\lambda(\mu) + \sum_{j=1}^{[\frac{L}{2}]} c_j(\mu)(\eta\bar{\eta})^j\varepsilon^{2j}] + O(\varepsilon^{L+1}). \tag{8}$$

By the smoothness of $H(\zeta,\bar{\zeta},\mu)$ we can write the solution η with initial condition $\eta(0) =$

1 as

$$\eta(\tau) = \sum_{i=0}^{L} \eta_i(\tau)\varepsilon^i + O(\varepsilon^{L+1}), \text{ with } \eta_0(0) = 1 \text{ , } \eta_i(0) = 0 \qquad (1 \le i \le L). \quad (9)$$

We want to show that $\eta_0(\tau) \equiv 1$, and $\eta_i(\tau) \equiv 0$ for $1 \le i \le L$, which means (5) is true. We put (6) and (9) into (8). Then at $O(\varepsilon^0)$, (8) have the relations

$$2\pi i\eta_0 + \frac{d\eta_0}{d\tau} = 2\pi i\eta_0,$$

so $\frac{d\eta_0}{d\tau} = 0$ and we know that $\eta_0(0) = 1$. This implies $\eta_0 \equiv 1$.

At $O(\varepsilon^1)$, (8) have the relations

$$2\pi i\eta_1 + \frac{d\eta_1}{d\tau} = 2\pi i\eta_1 + v_1$$

where $v_1 = \tau_1\eta_0\omega_0 + \frac{2\pi}{\omega_0}\eta_0\mu_1$ is a constant independent of ε. Thus $\eta_1 = v_1\tau + v_2$, where v_2 is a constant. But since $\eta(\tau)$ is a periodic solution with period 1, so does η_1. Hence $v_1 = 0$. On the other hand, we know $\eta_1(0) = 0$. Hence, $v_2 = 0$ which implies $\eta_1 \equiv 0$.

At $O(\varepsilon^2)$,

$$2\pi i\eta_2 + \frac{d\eta_2}{d\tau} = 2\pi i\eta_2 + v_3,$$

where $v_3 = \frac{2\pi}{\omega_0}(\alpha'(0) + i\omega'(0))\tau_1\mu_1 + 2\pi i\tau_2 + \frac{2\pi}{\omega_0}[(\alpha'(0) + i\omega'(0))\mu_2 + \frac{\alpha''(0)+i\omega''(0)}{2}\mu_1^2]$ is a constant. Thus $\eta_2 = v_3\tau + v_4$. Notice that η_2 has period 1 as well. Therefore, $v_3 = 0$ and $v_4 = 0$. By $\eta_2(0) = 0$, we obtain $\eta_2 \equiv 0$. Using the same argument, we have

$$\eta_i = 0 \qquad\qquad (1 \le i \le L).$$

□

Next we want to derive the coefficients in the expansions of $\mu(\varepsilon)$ and $T(\varepsilon)$. To do this, we put (5) into (4). Then we get

$$\varepsilon\frac{2\pi i}{T(\varepsilon)}e^{\frac{2\pi it}{T(\varepsilon)}} = \lambda(\mu)\varepsilon e^{\frac{2\pi it}{T(\varepsilon)}} + \sum_{j=1}^{[\frac{L}{2}]} c_j(\mu)\varepsilon e^{\frac{2\pi it}{T(\varepsilon)}}|\varepsilon e^{\frac{2\pi it}{T(\varepsilon)}}|^{2j} + O(...),$$

which implies

$$\frac{2\pi i}{T(\varepsilon)} = \lambda(\mu) + \sum_{j=1}^{[\frac{L}{2}]} c_j(\mu)\varepsilon^{2j} + O(...). \tag{10}$$

Inferring from (10), we have

$$\frac{2\pi}{T(\varepsilon)} = \mathrm{Im}\lambda(\mu) + \sum_{j=1}^{[\frac{L}{2}]} \mathrm{Im}c_j(\mu)\varepsilon^{2j} + \mathrm{Im}O(...), \tag{11}$$

$$0 = \mathrm{Re}\lambda(\mu) + \sum_{j=1}^{[\frac{L}{2}]} \mathrm{Re}c_j(\mu)\varepsilon^{2j} + \mathrm{Re}O(...) \qquad (,12)$$

where $O(...)$ is a term with order at least ε^{L+1}. Next, we want to derive the coefficients of the expansions of $T(\varepsilon)$ and $\mu(\varepsilon)$ with order smaller then ε^{L+1}. In what follows, we omit the high order term in (11), (12), and (6), (7) in the discussion. Moreover, we assume $\alpha'(0) \neq 0$.

We use (12) to calculate the coefficients of $\mu(\varepsilon)$. Expanding (12) in μ, and noting that $\mathrm{Re}\lambda(\mu) = \alpha(\mu)$, we have

$$\begin{aligned} &\alpha(0) + \alpha'(0)\mu + \frac{\alpha''(0)}{2!}\mu^2 + ... + \mathrm{Re}c_1(0)\varepsilon^2 + \mathrm{Re}c_1'(0)\mu\varepsilon^2 + ... \\ &+\mathrm{Re}c_2(0)\varepsilon^4 + \mathrm{Re}c_2'(0)\mu\varepsilon^4 + ... = 0. \end{aligned} \tag{13}$$

Plugging (7) into (13), by $\alpha(0) = 0$, we have

$$\begin{aligned} &\alpha'(0)(\sum_{j=1}^{L}\mu_j\varepsilon^j) + \frac{\alpha''(0)}{2!}(\sum_{j=1}^{L}\mu_j\varepsilon^j)^2 + ... + \mathrm{Re}c_1(0)\varepsilon^2 + \\ &\mathrm{Re}c_1'(0)(\sum_{j=1}^{L}\mu_j\varepsilon^j)\varepsilon^4 + ... + \mathrm{Re}c_2(0)\varepsilon^4 + ... = 0. \end{aligned} \tag{14}$$

By the above equation, at $O(\varepsilon^1)$, (14) implies

$$\alpha'(0)\mu_1 = 0.$$

Since $\alpha'(0) \neq 0$, we obtain

$$\mu_1 = 0.$$

At $O(\varepsilon^2)$, (14) and $\mu_1 = 0$ imply

$$\alpha'(0)\mu_2 + \mathrm{Re}c_1(0) = 0.$$

We therefore obtain

$$\mu_2 = \frac{-\mathrm{Re}c_1(0)}{\alpha'(0)}.$$

At $O(\varepsilon^3)$, (14) and $\mu_1 = 0$ imply

$$\alpha'(0)\mu_3 = 0.$$

Hence,

$$\mu_3 = 0.$$

At $O(\varepsilon^4)$, (14) and $\mu_1 = \mu_4 = 0$ imply

$$\alpha'(0)\mu_4 + \frac{\alpha''(0)}{2}\mu_2^2 + \mathrm{Re}c_1'(0)\mu_2 + \mathrm{Re}c_2(0) = 0.$$

Thus

$$\mu_4 = \frac{-1}{\alpha'(0)}[\frac{\alpha''(0)}{2}\mu_2^2 + \mathrm{Re}c_1'(0)\mu_2 + \mathrm{Re}c_2(0)],$$

where μ_2 is given as above. Continuing the same process, we get $\mu_i = 0$, when i is odd. We therefore obtain the formula stated in the **Theorem** *property* (a)

$$\mu(\varepsilon) = \sum_{i=1}^{[\frac{L}{2}]} \mu_{2i}\varepsilon^{2i} + O(\varepsilon^{L+1}).$$

By the same argument on (11), and noting that $T(0) = \frac{2\pi}{\omega_0} = \frac{2\pi}{\omega_0}\tau_0$, we see $\tau_0 = 1$. Thus, the L.H.S of (11) can be written as

$$\begin{aligned}\frac{2\pi}{T(\varepsilon)} &= \frac{\omega_0}{\sum_{i=0}^{L}\tau_i\varepsilon^i} = \frac{\omega_0}{1+\sum_{i=1}^{L}\tau_i\varepsilon^i} \\ &= \omega_0 + \omega_0(-\sum_{i=1}^{L}\tau_i\varepsilon^i) + \omega_0(-\sum_{i=1}^{L}\tau_i\varepsilon^i)^2 + ...,\end{aligned}$$

and the terms $\mathrm{Im}\lambda(\mu) = \omega(\mu)$ and $\mathrm{Im}c_j(\mu)$ of R.H.S of (11) can be expended in terms of μ. Hence,

$$\omega_0 + \omega_0(-\sum_{i=1}^{L} \tau_i \varepsilon^i) + \omega_0(-\sum_{i=1}^{L} \tau_i \varepsilon^i)^2 + ... = \omega_0 + \omega'(0)\mu +$$
$$\omega''(0)\mu^2 + ... + \mathrm{Im}c_1(0)\varepsilon^2 + \mathrm{Im}c_1'(0)\mu\varepsilon^2 + ... + \tag{15}$$
$$\mathrm{Im}c_2(0)\varepsilon^4 + \mathrm{Im}c_2'(0)\mu\varepsilon^4 + ...$$

Plugging the formula of μ into (15), we can write (15) as an equation in ε as

$$\omega_0(-\sum_{i=1}^{L} \tau_i \varepsilon^i) + \omega_0(-\sum_{i=1}^{L} \tau_i \varepsilon^i)^2 + ... = \omega'(0)(\mu_2\varepsilon^2 + \mu_4\varepsilon^4 + ...) +$$
$$\omega''(0)(\mu_2\varepsilon^2 + \mu_4\varepsilon^4 + ...)^2 + ... + \mathrm{Im}c_1(0)\varepsilon^2 + \tag{16}$$
$$\mathrm{Im}c_1'(0)(\mu_2\varepsilon^2 + \mu_4\varepsilon^4 + ...)\varepsilon^2 + ... + \mathrm{Im}c_2'(0)\varepsilon^4 + ...$$

Applying the same argument at $O(\varepsilon)$, we have

$$-\omega_0\tau_1 = 0.$$

Since $\omega_0 \neq 0$, we see

$$\tau_1 = 0.$$

At $O(\varepsilon^2)$,

$$-\omega_0\tau_2 = \omega'(0)\mu_2 + \mathrm{Im}c_1(0),$$

so

$$\tau_2 = \frac{-1}{\omega_0}(\omega'(0)\mu_2 + \mathrm{Im}c_1(0)).$$

At $O(\varepsilon^3)$, we have

$$-\omega_0\tau_3 = 0.$$

Hence,

$$\tau_3 = 0.$$

At $O(\varepsilon^4)$, we get

$$-\omega_0\tau_4+\omega_0\tau_2^2=\omega'(0)\mu_4+\frac{\omega''(0)}{2}\mu_2^2+\mathrm{Im}c_1'(0)\mu_2+\mathrm{Im}c_2(0).$$

Thus,

$$\tau_4=\frac{-1}{\omega_0}[\omega'(0)\mu_4+\frac{\omega''(0)}{2}\mu_2^2+\mathrm{Im}c_1'(0)\mu_2+\mathrm{Im}c_2(0)-\omega_0\tau_2^2].$$

Comparing the terms in R.H.S of (16) have even order with respect to ε, we get $\tau_i=0$ if i is odd. This proved *property* (b) in **Theorem**

$$T(\varepsilon)=\frac{2\pi}{\omega_0}\left(1+\sum_{i=1}^{[\frac{L}{2}]}\tau_{2i}\varepsilon^{2i}\right)+O(\varepsilon^{L+1}).$$

Part1(C) Stability

We have already proved *property*(a) and (b) in **Theorem**, in this part, we are going to prove the stability about the periodic solutions, which is quite important. The approach we use is *Floquet theory*, which is stated as follow and can be found in [2, 4, 6]

Theorem. *If A(t) is a continuous, T-periodic matrix, then for all t* $\in\mathbb{R}$ *any fundamental matrix solution for the nonautonomous linear system*

$$\dot{x}=A(t)x \tag{17}$$

can be written in the form

$$\Phi(t)=Q(t)e^{Bt}$$

where Q(t) is a nonsingular, differentiable, T-periodic matrix and B is a constant matrix. Furthermore, if $\Phi(0)=I$ *then* $Q(0)=I$.

The eigenvalues λ_i, $i=1,\ldots.,n$ of the matrix B are called *characteristic exponents*, the eigenvalues $e^{\lambda_i T}$ of e^{BT} are called *characteristic multipliers*.

If $P(t)$ is a nonconstant periodic solution of below equation (17), then $P(t)$ is *asymptotically, orbitally stable with asymptotic phase* if and only if there exists an $\varepsilon>0$ such that if $\varphi(t)$

is any solution of (17) for which $|\varphi(t_0) - P(t_0)| < \varepsilon$ at some t_0, then there exists a constant c, called the *asymptotic phase*, that satisfies

$$\lim_{t \to 0} |\varphi(t) - P(t+c)| = 0.$$

The proof is based on the following Theorem,

Theorem. *If*

(1) P(t) is a nonconstant T-periodic solution of

$$\dot{x} = f(x) \qquad\qquad (f \in C^1(\mathbb{R}^n, \mathbb{R}^n)) \tag{17}$$

(2) The characteristic multiplier 1 of the first variation of (17) *with respect to the periodic solution P, namely of*

$$\frac{dy}{dt} = \frac{\partial f(P(t))}{\partial x} y,$$

is simple,

(3) All other characteristic multipliers of (17) *have moduli less than 1, then P(t) is asymptotically, orbitally stable with asymptotic phase.*

We shall use this theorem to prove the stability if the 2×2 system (2). By Part(B), $\mu(\varepsilon)$ is C^{L+1}. Hence, the nonconstant periodic solution of (2), $P_\varepsilon(t) = \varepsilon y(t, \varepsilon, \mu(\varepsilon))$ is C^{L+1} jointly in t and ε with period $T(\varepsilon)$, and $\dot{P}_\varepsilon(t)$ is a nontrivial $T(\varepsilon)$ period solution of the variational system $\dot{y} = A(t;\varepsilon)y$. This is because $\dot{P}_\varepsilon(t) = F(P_\varepsilon(t), \mu(\varepsilon))$. Thus $\ddot{P}_\varepsilon(t) = D_X F(P_\varepsilon(t), \mu(\varepsilon))\dot{P}_\varepsilon(t)$, where $D_X F(P_\varepsilon(t), \mu(\varepsilon)) = A(t;\varepsilon)$. By Floquet's theory, if $\Phi(t)$ is a fundamental matrix solution of the variational system, then it can be written as $\Phi(t) = Q(t)e^{Bt}$. Thus $\dot{P}_\varepsilon(t)$ can be written as $Q(t)e^{Bt}c$ for some constant vector c. Since $\Phi(t + T(\varepsilon)) = \Phi(t)$, $Q(t + T(\varepsilon)) = Q(t)$, we have $Q(t + T(\varepsilon))e^{B(t+T)}c = Q(t)e^{B(t+T)}c = Q(t)e^{Bt}c$, which means $e^{BT}c = c$, thus e^{BT} has characteristic multiplier 1. Thus characteristic exponent 0. If the other characteristic exponent is $\beta(\varepsilon)$, we use the following theory.

If λ_i $(i = 1, ..., n)$ *are the characteristic exponents of* (17), *then*

$$\sum_{i=1}^{n} \lambda_i = \frac{1}{T} \int_0^T trA(s)ds.$$

where $trA(s) = \sum_{i=1}^{n} A_{ii}(s)$.

We define

$$\beta(\varepsilon) = \frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} trA(s;\varepsilon)ds.$$

Since $T(\varepsilon)$ is C^{L+1}, $A(t;\varepsilon)$ is C^{L+1} jointly in t and ε, $\beta(\varepsilon)$ is C^{L+1} in ε. The system (2) is written as

$$\dot{x}_1 = \alpha x_1 - \omega x_2 + [(\text{Re}c_1)x_1 - (\text{Im}c_1)x_2]r^2 + O(\varepsilon^4) \equiv F_1,$$
$$\dot{x}_2 = \omega x_1 + \alpha x_2 + [(\text{Re}c_1)x_2 + (\text{Im}c_1)x_1]r^2 + O(\varepsilon^4) \equiv F_2.$$

Note that

$$\frac{\partial F_1}{\partial x_1} = \alpha + (\text{Re}c_1)r^2 + 2(\text{Re}c_1)x_1^2 - 2(\text{Im}c_1)x_1x_2,$$
$$\frac{\partial F_2}{\partial x_2} = \alpha + (\text{Re}c_1)r^2 + 2(\text{Re}c_1)x_2^2 + 2(\text{Im}c_1)x_1x_2.$$

Hence,

$$tr\frac{\partial F}{\partial X}(P(t,\mu(\varepsilon))) = 2\alpha(\mu(\varepsilon)) + 4[\text{Re}c_1(\mu(\varepsilon))]\varepsilon^2 + O(\varepsilon^3),$$

where $\mu(\varepsilon) = O(\varepsilon^2)$, and $r^2 = \varepsilon^2 + O(\varepsilon^5)$. Hence,

$$\frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} trA(s;\varepsilon)ds = 2\alpha(\mu(\varepsilon)) + 4[\text{Re}c_1(\mu(\varepsilon))]\varepsilon^2 + O(\varepsilon^3).$$

By

$$\alpha(\mu(\varepsilon)) = \alpha'(0)\mu_2\varepsilon^2 + ... = -\text{Re}c_1(0)\varepsilon^2 + ...,$$

we have

$$\beta(\varepsilon) = 2\text{Re}c_1(0)\varepsilon^2 + O(\varepsilon^3).$$

Thus $\beta(\varepsilon) < 0$ for sufficient small ε if $\mathrm{Re}c_1(0) < 0$.

Part2 Reduction of Two-dimensional Systems to Poincarè Normal Form

In previous part, we have shown the Hopf bifurcation in the Poincarè normal form. In this part, we shall show how general autonomous systems satisfying the hypotheses of the Hopf bifurcation can be transformed into Poincarè normal form. We use single complex equation to replace 2×2 real system in the following proof. Consider

$$\dot{z} = \lambda z + g(z, \bar{z}; \mu), \tag{18}$$

where

$$g(z, \bar{z}; \mu) = \sum_{2 \leq i+j \leq L} g_{ij}(\mu) \frac{z^i \bar{z}^j}{i! j!} + O(|z|^{L+1}) \tag{19}$$

and

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu).$$

Here $z = y_1 + iy_2$, and (18) is equivalent to the 2×2 system

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2; \mu), \\ \dot{y}_2 &= f_2(y_1, y_2; \mu), \end{aligned}$$

with an isolated stationary point at origin and

$$D_y f(0, 0, \mu) = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix}.$$

We use the transformation

$$\begin{aligned} z &= \zeta + \chi(\zeta, \overline{\zeta}; \mu) \\ &= \zeta + \sum_{2 \leq i+j \leq L} \chi_{ij}(\mu) \frac{\zeta^i \overline{\zeta}^j}{i! j!}, \end{aligned} \tag{20}$$

where $\chi_{ij} \equiv 0$ for $i = j+1$ to transform (18) into the Poincarè normal form

$$\dot{\zeta} = \lambda(\mu)\zeta + \sum_{j=1}^{[\frac{L}{2}]} c_j \zeta |\zeta|^{2j} + O(|\zeta||(\zeta,\mu)|^{L+1}) \tag{21}$$

$$= \lambda(\mu)\zeta + \phi(\zeta,\overline{\zeta};\mu).$$

Formally, transformation (19) can take (18) into (20). We then compute the coefficients in the expansion of ϕ. After all, our computation validate the above transformation.

First, we take derivative of (19) with respect to t,

$$\dot{z} = \dot{\zeta} + \chi_\zeta \dot{\zeta} + \chi_{\overline{\zeta}} \dot{\overline{\zeta}}. \tag{22}$$

By (18)-(20), we rewrite (21) as

$$\lambda(\zeta+\chi) + g(\zeta+\chi,\overline{\zeta}+\overline{\chi};\mu) = \lambda\zeta + \phi + \chi_\zeta(\lambda\zeta+\phi) + \chi_{\overline{\zeta}}(\overline{\lambda\zeta}+\overline{\phi}),$$

or

$$\lambda\zeta\chi_\zeta + \overline{\lambda\zeta}\chi_{\overline{\zeta}} - \lambda\chi = g(\zeta+\chi,\overline{\zeta}+\overline{\chi}) - (\phi + \chi_\zeta\phi + \chi_{\overline{\zeta}}\overline{\phi}). \tag{23}$$

Since $\chi_\zeta = \sum_{2\le i+j\le L} i\chi_{ij}\frac{\zeta^{i-1}\overline{\zeta}^j}{i!j!}$, $\zeta\chi_\zeta = \sum_{2\le i+j\le L} i\chi_{ij}\frac{\zeta^i\overline{\zeta}^j}{i!j!}$. Similarly, $\overline{\zeta}\chi_{\overline{\zeta}} = \sum_{2\le i+j\le L} j\chi_{ij}\frac{\zeta^i\overline{\zeta}^j}{i!j!}$. The L.H.S of (22) can be written as

$$\sum_{2\le i+j\le L} (i\lambda + j\overline{\lambda} - \lambda)\chi_{ij}\frac{\zeta^i\overline{\zeta}^j}{i!j!}. \tag{24}$$

By (19) and (24), to express χ_{ij} in terms of g_{ij}. We do the following comparison of $|\zeta|^2$ terms of (23). L.H.S of (23) is

$$\lambda\chi_{20}\frac{\zeta^2}{2} + \overline{\lambda}\chi_{11}\zeta\overline{\zeta} + (2\overline{\lambda}-\lambda)\chi_{02}\frac{\overline{\zeta}^2}{2!}, \tag{25}$$

and the R.H.S of (23) is

$$\frac{g_{20}}{2}\zeta^2 + g_{11}\zeta\overline{\zeta} + \frac{g_{02}}{2}\overline{\zeta}^2. \tag{26}$$

Hence,

$$\chi_{20} = \frac{g_{20}}{\lambda}, \qquad \chi_{11} = \frac{g_{11}}{\overline{\lambda}}, \qquad \chi_{02} = \frac{g_{02}}{2\overline{\lambda} - \lambda}. \tag{27}$$

For the $\zeta^2\overline{\zeta}$ term, the L.H.S of (23) is 0. Since $\chi_{21} \equiv 0$, we have

$$0 = g_{20}\chi_{11} + g_{11}\overline{\chi}_{11} + g_{11}\frac{\chi_{20}}{2} + g_{02}\frac{\overline{\chi}_{02}}{2} + \frac{g_{21}}{2} - c_1(\mu). \tag{28}$$

Plugging (27) into (28), we have

$$\begin{aligned} c_1(\mu) &= \frac{g_{20}g_{11}}{\overline{\lambda}} + \frac{g_{11}\overline{g}_{11}}{\lambda} + \frac{g_{11}g_{20}}{2\lambda} + \frac{g_{02}\overline{g}_{02}}{2(2\lambda - \overline{\lambda})} + \frac{g_{21}}{2} \\ &= \frac{g_{20}g_{11}(2\lambda + \overline{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \overline{\lambda})} + \frac{g_{21}}{2}. \end{aligned} \tag{29}$$

If $\mu = 0$, we have

$$c_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{20}|^2}{3}) + \frac{g_{21}}{2}.$$

Since the possible of transformation into Poincaré normal form is assumed apriori, we have to prove that transformation is valid. More precisely, we use the smoothness, existence of ϕ, and assume the expansion of ϕ in (21) apriori. To prove this. We define

$$\Lambda(\xi, \overline{\zeta}; \mu) = \sum_{2 \le i+j \le L} \frac{\chi_{ij}^{(k)}(0)\zeta^i\overline{\zeta}^j\mu^k}{i!j!k!},$$

where the coefficient $\chi_{ij}(\mu)$ are expressed as in above discussion. Note that the difference between function Λ and χ in (20) is that Λ has better smoothness, and has only finite order terms. Thus, to calculate the normal form, we can replace χ by Λ in (23). Since Λ is a function of ζ, $\overline{\zeta}$, and μ, (23) can be viewed as a function of $\mathrm{Re}\zeta$, $\mathrm{Im}\zeta$, $\mathrm{Re}\phi$, and $\mathrm{Im}\phi$. Express it in the form

$$F(\mathrm{Re}\zeta, \mathrm{Im}\zeta, \mathrm{Re}\phi, \mathrm{Im}\phi; \mu) = 0,$$

where F is at least C^1 and $F(0,0,0,0;0) = 0$. Since $\mathrm{Re}F = c + \mathrm{Re}\phi + \mathrm{Re}\Lambda_\zeta\mathrm{Re}\phi -$

$\mathrm{Im}\Lambda_\zeta\mathrm{Im}\phi+\mathrm{Re}\Lambda_{\overline{\zeta}}\mathrm{Re}\phi+\mathrm{Im}\Lambda_{\overline{\zeta}}\mathrm{Im}\phi$, where c is independent of ϕ, thus $\frac{\partial \mathrm{Re}F}{\partial \mathrm{Re}\phi}=1+\mathrm{Re}\Lambda_\zeta-\mathrm{Im}\Lambda_\zeta$, and $\frac{\partial \mathrm{Re}F}{\partial \mathrm{Im}\phi}=-\mathrm{Im}\Lambda_\zeta+\mathrm{Im}\Lambda_{\overline{\zeta}}$. In the same way, $\frac{\partial \mathrm{Im}F}{\partial \mathrm{Re}\phi}=\mathrm{Im}\Lambda_\zeta+\mathrm{Im}\Lambda_{\overline{\zeta}}$, and $\frac{\partial \mathrm{Im}F}{\partial \mathrm{Im}\phi}=1+\mathrm{Re}\Lambda_\zeta+\mathrm{Re}\Lambda_{\overline{\zeta}}$. Evaluating

$$\begin{pmatrix} \frac{\partial \mathrm{Re}F}{\partial \mathrm{Re}\phi} & \frac{\partial \mathrm{Re}F}{\partial \mathrm{Im}\phi} \\ \frac{\partial \mathrm{Im}F}{\partial \mathrm{Re}\phi} & \frac{\partial \mathrm{Im}F}{\partial \mathrm{Im}\phi} \end{pmatrix}$$

at $(0,0,0,0;0)$, we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, by the implicit function theorem, there exists a unique $\phi=\phi(\zeta,\overline{\zeta};\mu)$ with $\phi(0,0;0)=0$ and for fixed μ both g and Λ are C^{L+2}. Thus ϕ can be written in Taylor expansion as

$$\phi(\zeta,\overline{\zeta};\mu)=\sum_{2\le i+j\le L}\frac{\phi_{ij}(\mu)}{i!j!}\zeta^i\overline{\zeta}^j+O(|\zeta|^{L+2}).$$

By direct computation, we have

$$\phi_{ij}(\mu)=O(|\mu|^{L-i-j+2}), \qquad \text{for } i\neq j+1,\ 2\le i+j\le L+1,$$

and

$$\phi_{j+1,j}(\mu)=(j+1)!j!c_j(\mu)+O(|\mu|^{L-2j+1}), \qquad \text{for } 1\le j\le [\frac{L}{2}].$$

Thus, (18) can be transformed into (21) by means of (20) with the function Λ.

Part3 Reduce the n-dimensional System to the Two-dimensional System by Center Manifold Theorem

For the n-dimensional system

$$\dot{X}=F(X;\mu), \tag{30}$$

where $(X,\mu) \in \mathbb{R}^n \times \mathbb{R}$. We consider the suspended system

$$\begin{aligned} \dot{X} &= F(X;\mu), \\ \dot{\mu} &= 0, \end{aligned} \tag{31}$$

and apply the center manifold theorem to (31) at $(X,\mu) = (0,0)$. The center manifold theorem is stated as

Theorem. (The Center Manifold Theorem) *For the system*

$$\dot{x} = f(x). \tag{*}$$

Let $f \in C^{r+1}(E)$*, where* E *is an open subset of* $\mathbb{R}^n$ *containing the origin and* $r \geq 1$*. Suppose that* $f(0) = 0$ *and that* $Df(0)$ *has* c *eigenvalues with zero real parts and* s *eigenvalues with negative real parts, where* $c+s=n$*. The system then can be written in diagonal form*

$$\begin{aligned} \dot{x} &= Cx + F(x,y) \\ \dot{y} &= Py + G(x,y), \end{aligned}$$

where $(x,y) \in \mathbb{R}^c \times \mathbb{R}^s$*,* C *is a square matrix with* c *eigenvalues having zero real parts,* P *is a square matrix with* s *eigenvalues with negative real parts, and* $F(0) = G(0) = 0$*,* $DF(0) = DG(0) = O$*, furthermore, there exist a neighborhood* V *of* 0 *in* $\mathbb{R}^n$ *and a* C^r *submanifold* M *of* V *of dimension* c*, passing through* 0 *and tangent to the generalized eigenspace of* C *at* 0*, such that*

(a) *(Local Invariance): If to each* x *in* M *the solution* $x(t)$ *of the system (*) with initial condition* $x(0) = x$ *remains in* M *for some interval* $0 \leq t \leq \tau$*, where* $\tau = \tau(x) > 0$*.*

(b) *(Local Attractivity): If* $x(t) \in V$ *for all* $t \geq 0$*,* $x(t)$ *approaches* M *as* $t \to \infty$*.*

Since the linear part of (31) at $(0,0)$ is

$$\begin{pmatrix} F_X(0;0) & 0 \\ 0 & 0 \end{pmatrix}$$

has eigenvalues 0, $i\omega_0$, $-i\omega_0$ and other eigenvalues with negative real part by the hypotheses. Thus by the center manifold theorem, system (31) has locally invariant, locally attractive, three-dimension center manifold $\mathscr{C}$ in $\mathbb{R}^n \times \mathbb{R}$ with $0 \in \mathscr{C}$ and $|\mu| < \mu_0$. Note that, we apply the center manifold theorem on the suspended system (31) instead of (30) because we got a center manifold with $|\mu| < \mu_0$ in (31). If we apply center manifold theorem in (30), we need to assume $\mu = 0$ which is more restricted.

Let $q(\mu)$ be the eigenvector of

$$A(\mu) = F_X(0;\mu),$$

and $q^*(\mu)$ be the eigenvector for A^T, corresponding to the simple eigenvalues

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu) \text{ and } \bar{\lambda}(\mu).$$

We normalize q^* with respect to q, that is

$$\langle q^*, q\rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the Hermitian product

$$\langle u, v\rangle = \sum_{i=1}^{n} \bar{u}_i v_i.$$

Let $P_0 = (\mathrm{Re}q(0), -\mathrm{Im}q(0), e_1, ..., e_{n-2})$ be the $n \times n$ real matrix where e_j, $j = 1, ..., n-2$, satisfies the condition $\langle q^*(0), e_j\rangle = 0$, $j = 1, ..., n-2$. Use the change of variables $X =$

P_0Y the system (31) can be transformed into the canonical form

$$\dot{Y} = \begin{pmatrix} 0 & -\omega_0 & \vdots \\ \omega_0 & 0 & \vdots \\ \cdots & \cdots & D_0 \end{pmatrix} + G(Y;\mu),$$

$$\dot{\mu} = 0,$$

where $\cdots$ and $\vdots$ are all zero, D_0 is real $(n-2)\times(n-2)$ matrix with eigenvalues $\lambda_3(0),...,\lambda_n(0)$, and $G(Y;\mu) = O(|(Y,\mu)|^2)$ is C^{L+2} jointly in Y and μ. The canonical system has a center manifold

$$\mathscr{C} = \{(Y,\mu)|(Y,\mu) = (y_1,y_2,W(y_1,y_2,\mu),\mu), \quad |(y_1,y_2,\mu)| < \delta\},$$

where δ is sufficient small, and

$$W : \mathbb{R}^3 \to \mathbb{R}^{n-2}$$

is C^{L+2} jointly in y_1, y_2, μ, and $W(y_1,y_2,\mu) = O(|(y_1,y_2,\mu)|^2)$. In the X-coordinates,

$$\mathscr{C} = \{(X,\mu)|(X,\mu) = (y_1\mathrm{Re}q(0) - y_2\mathrm{Im}q(0) + \sum_{j=1}^{n-2} e_jW_j,\mu), \quad |(y_1,y_2,\mu)| < \delta\}.$$

To discuss the system (30), we restrict $\mathscr{C}$ for fixed μ, and write it as

$$\mathscr{C}_\mu = \{X|(X,\mu) \in \mathscr{C}\}.$$

Since $\mathscr{C}$ is C^{L+2} in X and μ, thus $\mathscr{C}_\mu$ is C^{L+2} in X.

We define a new coordinate related to the system (31). The system (31) can be written as

$$\dot{X} = A(\mu)X + f(X,\mu), \tag{32}$$

where $f(X,\mu) = F(X,\mu) - A(\mu)X$. If $x(t)$ is a solution of (32), we define

$$z(t) = \langle q^*(\mu), x(t)\rangle, \tag{33}$$

and regard z and $\bar{z}$ as local coordinate in the direction q and $\bar{q}$. Next, we define

$$w(t) = x(t) - z(t)q(\mu) - \bar{z}(t)\bar{q}(\mu). \tag{34}$$

In the variables z and w, we can relate the system (32) with z and w. Since $z = \langle q^*, x\rangle$, $\langle q^*, Ax\rangle = \langle A^* q^*, x\rangle = \langle \bar{\lambda} q^*, x\rangle = \lambda \langle q^*, x\rangle = \lambda z$. We have $\dot{z} = \langle q^*, \dot{x}\rangle = \langle q^*, Ax + f\rangle = \langle q^*, Ax\rangle + \langle q^*, f\rangle$. We can get a differential equation in z, in the same manner in w, we have

$$\begin{aligned} \dot{z} &= \lambda(\mu)z + G(z, \bar{z}, w; \mu) \\ \dot{w} &= A(\mu)w + H(z, \bar{z}, w; \mu), \end{aligned} \tag{35}$$

where

$$\begin{aligned} G(z, \bar{z}, w; \mu) &= \langle q^*, f(w + 2\mathrm{Re}[zq]; \mu)\rangle \\ H(z, \bar{z}, w; \mu) &= f(w + 2\mathrm{Re}[zq]; \mu) - 2\mathrm{Re}[qG]. \end{aligned} \tag{36}$$

Note that, since $\langle q^*, w\rangle = 0$, the orthogonality relations imply two components of w are linear combinations of the other components.

From the center manifold theorem, the restricted manifolds $\mathscr{C}_\mu$ where $|\mu| < \delta$ may be locally represented as a real vector-valued function

$$w = w(z, \bar{z}; \mu),$$

where w is C^{L+1} in z, $\bar{z}$, and μ, and satisfies

$$w_z(0,0;\mu) = w_{\bar{z}}(0,0;\mu) = 0, \text{ and } \langle q^*, w\rangle = 0.$$

Since the information about the periodic solutions we care about is all included in the center manifold, thus we restrict (35) to $\mathscr{C}_\mu$ by setting

$$w(t) = w(z(t), \bar{z}(t); \mu),$$

and the system on $\mathscr{C}_\mu$ is expressed as

$$\dot{z} = \lambda z + g(z, \bar{z}; \mu), \tag{37}$$

where

$$g(z, \bar{z}; \mu) = G(z, \bar{z}, w(z, \bar{z}; \mu); \mu),$$

and for $w(z, \bar{z}; \mu)$, $\dot{w} = w_z \dot{z} + w_{\bar{z}} \dot{\bar{z}} = w_z(\lambda z + g) + w_{\bar{z}}(\bar{\lambda}\bar{z} + \bar{g})$. Using (35), we can determine $w(z, \bar{z}; \mu)$ by

$$\dot{w} = w_z(\lambda z + g) + w_{\bar{z}}(\bar{\lambda}\bar{z} + \bar{g}) = A(\mu)w + H(z, \bar{z}, w; \mu). \tag{38}$$

Note that the right hand side of (37) is C^{L+1} in z, $\bar{z}$, and μ, and g is C^{L+1} in z and $\bar{z}$, for fixed μ, and satisfies

$$g_z(0, 0; \mu) = g_{\bar{z}}(0, 0; \mu) = 0.$$

Thus (37) is the desired form for the two dimension case.

To complete the prove, we need to show the stability in $\mathbb{R}^n$ is the same with the stability in the $\mathbb{R}^2$ case as we discuss in part2 and 1. To do this, we show that $\beta(\varepsilon)$ which we discuss the stability in two dimensional case is also a characteristic exponent for the n-dimensional system (31). The verification is as follows.

Let x_0 denote a point on the orbit of the n-dimensional solution $P_\varepsilon(t)$, write it as $x_0 = P_\varepsilon(t_0)$. We want to use the Poincaré map about the periodic solution $P_\varepsilon(t)$. To define the Poincaré map. Let e_n denote the unit vector in the direction of $\dot{p}_\varepsilon(t_0) = F(x_0; \mu(\varepsilon))$, and denote Λ the hyperplane

$$\Lambda = \{x | (x - x_0) \cdot e_n = 0\}.$$

The hyperplane Λ has $n - 1$ dimension, thus we can find a set of orthogonal unit vector

$\{e_1, e_2, ..., e_{n-1}\}$ with the properties

$$e_j \cdot e_n = 0; \;\text{ for } j = 1, ..., n-1,$$
$$\Lambda = \{x = x_0 + \sum_{j=1}^{n-1} \eta_j e_j, \quad \eta \in \mathbb{R}^{n-1}\}.$$

Using Poincaré map [6], for $x \in \Lambda$ closed enough to x_0, we have a solution $x(t)$ for $t = \tau(\eta)$ nearest to $T(\varepsilon)$ and the solution $x(\tau(\eta)) \in \Lambda$. If we let $\psi = \psi(\eta)$ be the vector such that

$$x(\tau(\eta)) = x_0 + \sum_{j=1}^{n-1} \psi_j(\eta) e_j.$$

Then $\psi : \eta \to \mathbb{R}^{n-1}$ is the corresponding Poincaré map. It has been show that [6] the eigenvalues of the matrix $\frac{\partial \psi}{\partial \eta}(0)$ are precisely $n-1$ of the characteristic multiplier associated with the periodic solution.

For the system $\dot{x} = F(x;\mu)$, $P_\varepsilon(t)$ is a periodic solution lies on the two dimensional invariant manifold $\mathscr{C}_\mu$. Thus the manifold $\mathscr{C}_\mu$ intersects the hyperplane Λ, and the intersection describes a curve Γ in $\mathbb{R}^n$. Since Γ belongs to Λ, it can be parametrized as

$$\Gamma = \{x_s = x_0 + \sum_{j=1}^{n-1} \eta_j(s) e_j, \quad |s| < \sigma\},$$

where $\eta_j(0) = 0$ for $j = 1, ..., n-1$, $\sigma > 0$ is sufficiently small, where s denote the arclength. For each $x_s \in \mathscr{C}$, there are points $z_s \in \mathbb{C}$ and $\zeta_s \in \mathbb{C}$ such that

$$x_s = z_s q + \bar{z}_s \bar{q} + w(z_s, \bar{z}_s; \mu)$$
$$z_s = \zeta_s + \chi(\zeta_s, \bar{\zeta}_s; \mu).$$

Define the tangent vector

$$v = \frac{d\eta}{ds}(0).$$

We claim this vector is an eigenvector of the Jacobian matrix $\frac{\partial \psi}{\partial \eta}(0)$ with eigenvalue $e^{\beta(\varepsilon)T(\varepsilon)}$. To prove this, we denote $\tilde{\eta}(s) = \psi(\eta(s))$, and let $\tilde{x}_s$, $\tilde{z}_s$, $\tilde{\zeta}_s$, correspond to $\tilde{\eta}(s)$.

Since by the above discussion about $\beta(\varepsilon)$ in Part2, we have

$$\lim_{s\to 0}\frac{|\tilde{\zeta}_s-\zeta_0|}{|\zeta_s-\zeta_0|}=e^{\beta(\varepsilon)T(\varepsilon)}.$$

Using the Taylor expansion for $w(z,\bar{z};\mu)$, we have

$$\lim_{s\to 0}\frac{|\tilde{\eta}(s)|}{|\eta(s)|}=\lim_{s\to 0}\frac{|\tilde{z}_s-z_0|}{|z_s-z_0|}=\lim_{s\to 0}\frac{|\tilde{\zeta}_s-\zeta_0|}{|\zeta_s-\zeta_0|}=e^{\beta(\varepsilon)T(\varepsilon)}.$$

If we write $\tilde{\eta}(s)=\eta(\tilde{s}(s))$ for some $\tilde{s}=\tilde{s}(s)$, then

$$\lim_{s\to 0}\frac{\tilde{s}(s)}{s}=\lim_{s\to 0}\frac{|\tilde{\eta}(s)|}{|\eta(s)|}=e^{\beta(\varepsilon)T(\varepsilon)}.$$

Thus, use Taylor expansion of the Poincaré map ψ, we have

$$\tilde{\eta}(s)=\psi(\eta(s))=\frac{\partial\psi}{\partial\eta}(0)\eta(s)+O(|s|^2),$$
$$\frac{\tilde{s}}{s}\frac{\eta(\tilde{s})}{\tilde{s}}=\frac{\partial\psi}{\partial\eta}(0)\frac{\eta(s)}{s}+O(|s|).$$

Thus, let s approach 0, we see

$$e^{\beta(\varepsilon)T(\varepsilon)}v=\frac{\partial\psi}{\partial\eta}(0)v.$$

Hence $e^{\beta(\varepsilon)T(\varepsilon)}$ is an eigenvalue of $\frac{\partial\psi}{\partial\eta}(0)$ as claimed.

The stability about the periodic solution $P_\varepsilon(t)$ is determined by its characteristic multipliers, and we know that the characteristic multipliers are the eigenvalues of $\frac{\partial\psi}{\partial\eta}(0)$. One of these is 1, and from above we know another is $e^{\beta(\varepsilon)T(\varepsilon)}$. The remaining $n-2$ multipliers are

$$\rho_j=e^{\frac{2\pi\lambda_j(0)}{\omega(0)}}+o(1)\qquad\qquad (j=3,\ldots,n)$$

for ε is small, this is due to the continuity at $\varepsilon=0$. Thus, by the Hopf hypotheses, ρ_j have moduli strictly less than 1. The proof is now complete.

4 Examples

The proof we gave in the above has the advantage which tells us how to calculate the parameter $\mu(\varepsilon)$, the period $T(\varepsilon)$, and the characteristic exponent $\beta(\varepsilon)$ i.e., the stability about the periodic solution. Moreover, the periodic solution can also be approximated. Recall that since the coefficient

$$\mu_2 = \frac{-\mathrm{Re}c_1(0)}{\alpha'(0)},$$

$$\tau_2 = \frac{-1}{\omega_0}(\omega'(0)\mu_2 + \mathrm{Im}c_1(0)),$$

$$\beta_2 = 2\mathrm{Re}c_1(0).$$

Hence, to calculate the above coefficients, we only need to evaluate $c_1(0)$. By Part2,

$$c_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{20}|^2}{3}) + \frac{g_{21}}{2}.$$

To get $c_1(0)$, we have to know the coefficients of $g(z,\bar{z};\mu)$ in Taylor expansion with older 2, and g_{21}. By (37) $g(z,\bar{z};\mu) = G(z,\bar{z},w(z,\bar{z};\mu);\mu)$, and $G(z,\bar{z},w(z,\bar{z};\mu);\mu) = G(z,\bar{z},0;\mu) + G_w(z,\bar{z},0;\mu)w + O(|w|^2)$. Note that $w(z,\bar{z};\mu) = O(z^2)$. Thus $g_{20} = G_{zz}(0,0,0;\mu) = G_{20}(\mu)$, $g_{02} = G_{\bar{z}\bar{z}}(0,0,0;\mu) = G_{02}(\mu)$, $g_{11} = G_{11}(\mu)$, and $g_{21} = G_{21}(\mu) + \frac{\partial^2 G}{\partial w \partial z}(0,0,0;\mu)2w_{11}(\mu) + \frac{\partial^2 G}{\partial w \partial \bar{z}}(0,0,0;\mu)w_{20}(\mu)$. To calculate w_{11} and w_{20}, inferring from (38) we have

$$Lw = H(z,\bar{z},w;\mu) - gw_z - \bar{g}w_{\bar{z}}, \tag{39}$$

where

$$L \equiv (\lambda z\frac{\partial}{\partial z} + \bar{\lambda}\bar{z}\frac{\partial}{\partial \bar{z}} - A).$$

Considering the expansion $w(z,\bar{z};\mu) = \sum_{i+j=2}^{L+1} \frac{w_{ij}(\mu)}{i!j!} z^i \bar{z}^j + O(|z|^{L+2})$, the L.H.S of (39) can be written as

$$Lw = \sum_{i+j=2}^{L+1} [(\lambda i + \bar{\lambda} j)I - A]w_{ij}(\mu)\frac{z^i\bar{z}^j}{i!j!} + O(|z|^{L+2}). \tag{40}$$

To obtain w_{11}, and w_{20}, we just write the R.H.S of (39) into the Taylor expansion, and find the corresponding coefficient. For the convenience, we will only show how to compute w_{11}, and w_{20}.

Example 4.1. van der Pol's equation

$$\dot{x} = -y + \mu x + x^3,$$
$$\dot{y} = x.$$

This system has stationary point $(x,y) = (0,0)$, and the system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^3 \\ 0 \end{bmatrix}.$$

The linear part of the system has eigenvalues

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

If $\mu \leq -2$, the eigenvalues are real and negative. While $-2 < \mu < 0$, the eigenvalues are complex conjugate number with negative real part. In case $0 < \mu < 2$, the eigenvalues are complex conjugate number with positive real part, and for $\mu \geq 2$, the eigenvalues are real and positive. As μ is increased past $\mu = 0$, the stationary solution lose stability. Write

$$\lambda_{1,2} = \alpha(\mu) \pm i\omega(\mu),$$

where $\alpha(\mu) = \frac{\mu}{2}$, $\omega(\mu) = \frac{\sqrt{\mu^2-4}}{2}$. Since $\alpha(0) = 0$ and $\alpha'(0) = \frac{1}{2} \neq 0$, we can apply the Hopf's theorem to obtain the existence of periodic solutions bifurcating from $(0,0)$.

From the above, if we want to know μ_2, τ_2, β_2, we need to evaluate $c_1(0)$. For the sake of convenience, we translate the stationary point and the bifurcating point to 0, and use the change of variables to put the Jacobian matrix of the system which evaluate at the stationary point into the canonical form. In this example, the Jacobian matrix of the system is already in real canonical form. We let

$$F^1(x,y;0) = -y - x^3, \qquad\qquad F^2(x,y;0) = x.$$

From the above we know $g_{11}(0) = G_{z\bar{z}}(0,0;0)$. Since $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, we get

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial x^2} + \frac{\partial^2 F^1}{\partial y^2} + i\left(\frac{\partial^2 F^2}{\partial x^2} + \frac{\partial^2 F^2}{\partial y^2}\right)\right].$$

In a similar way we can get g_{02}, g_{20}, and g_{21}. In this example we get

$$g_{11} = g_{02} = g_{20} = 0, \qquad g_{21} = -\frac{3}{4}, \qquad c_1(0) = -\frac{3}{8},$$

thus we have

$$\mu_2 = \frac{3}{4}, \ \ \tau_2 = 0, \ \text{and} \ \ \beta_2 = -\frac{3}{4}.$$

Since $\mu_2 > 0$ and $\beta_2 < 0$, we know the periodic solutions exist for $\mu > 0$ and are stable.

The following example is a three dimensional case.

Example 4.2. Langford's system

$$\dot{x}_1 = (\mu - 1)x_1 - x_2 + x_1x_3,$$

$$\dot{x}_2 = x_1 + (\mu - 1)x_2 + x_2x_3,$$

$$\dot{x}_3 = \mu x_3 - (x_1^2 + x_2^2 + x_3^2).$$

This system has the following two stationary points

$$x_*^0 = (0,0,0)^T \ \ \text{and} \ \ x_*^1 = (0,0,\mu)^T.$$

Linearized the system about x_*^0, the coefficients matrix of the linear part is

$$\begin{pmatrix} \mu-1 & -1 & 0 \\ 1 & \mu-1 & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

and the corresponding eigenvalues are

$$\lambda_1^0 = \mu, \quad \lambda_{2,3}^0 = \mu - 1 \pm i.$$

Thus, for $\mu < 0$, x_*^0 is stable; while for $\mu > 0$, x_*^0 is unstable.

Linearized about x_*^1 we have

$$\begin{pmatrix} 2\mu-1 & -1 & 0 \\ 1 & 2\mu-1 & 0 \\ 0 & 0 & -\mu \end{pmatrix},$$

the eigenvalues are

$$\lambda_{1,2}^1 = 2\mu - 1 \pm i, \quad \lambda_3^1 = -\mu.$$

So x_*^1 is linearly stable for $0 < \mu < \frac{1}{2}$, and is unstable for $\mu < 0$ or $\mu > \frac{1}{2}$. For $\mu = \frac{1}{2}$, we have a pair of eigenvalues with zero real part $\lambda_{1,2}^1(\frac{1}{2}) = \pm i$ and an eigenvalue with negative real part $\lambda_3^1(\frac{1}{2}) = -\frac{1}{2}$. Thus, the Hopf bifurcation theorem can be applied to this case.

Let $x = x_*^1 + y$, the system becomes

$$\begin{aligned} \dot{y}_1 &= (2\mu-1)y_1 - y_2 + y_1 y_3 \equiv F^1, \\ \dot{y}_2 &= y_1 + (2\mu-1)y_2 + y_2 y_3 \equiv F^2, \\ \dot{y}_3 &= -\mu y_3 - (y_1^2 + y_2^2 + y_3^2) \equiv F^3. \end{aligned}$$

At $y = 0$. The system is in real canonical form. To get μ_2, τ_2 and β_2. We apply the formulae as in Example3.1. Since $\frac{\partial^2 F^k}{\partial y_i \partial y_j} = 0$ at $y = 0$, $\mu = \frac{1}{2}$ for $i,j,k = 1,2$. We have

$$g_{11} = g_{02} = g_{20} = G_{21} = 0.$$

Next we need to calculate w_{11} and w_{20}. By (40), we obtain

$$-Aw_{11} = H_{11}, \quad (2i\omega_0 I - A)w_{20} = H_{20}. \tag{41}$$

Since $w_{11} = (0,0,w_{11})^T$ and $H_{11} = (0,0,F_{11}^3)^T$, by (41), $\frac{1}{2}w_{11} = -1$ and $(-1\frac{1}{2})w_{20} = 0$. Therefore, we get

$$w_{11} = -2, \quad w_{20} = 0.$$

By $\frac{\partial^2 G}{\partial w \partial z}(0,0,0,\frac{1}{2}) = \frac{\partial}{\partial y_3}(\frac{1}{2}(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}))(F^1 + iF^2) = 1$, we see $G_{w\bar{z}} = 0$ and

$$g_{21} = -4.$$

To sum up, we obtain

$$c_1(\frac{1}{2}) = -2, \quad \mu_2 = 1, \quad \tau_2 = 0, \quad \text{and } \beta_2 = -4.$$

Since $\mu_2 = 1 > 0$, the periodic solution $P(t;\mu)$ exist for $\mu > \frac{1}{2}$, and is asymptotically orbitally stable.

References

[1] Arnol'd, V. I. "*Supplementary Chapter of the Theory of Ordinary Differential Equations*", Nauka, Moscow (1978), 304 pp. (in Russian)

[2] Coddington, E. A. and Levinson, N. "*Theory of Ordinary Differential Equations*", McGraw-Hill, N. Y. (1955) 429 pp

[3] Crandall, M. G. and Rabinowitz, P. H. *Bifurcation from simple eigenvalues*, J. Funct. Anal. 8 (1971), 321-340

[4] Hale, J. K. "*Ordinary Differential Equations*", Applied Math. Sci. vol. 3, Springer-Verlag, N. Y. (1971)

[5] Hassard, B. D., Kazarinoff, N. D. and Wan Y.-H. *Theory and Applications of Hopf Bifurcation*, Cambridge Univ. Press, Cambridge (1981)

[6] Hartman, P. "*Ordinary Differential Equations*", Second Edition, P. Hartman (1973)

[7] Kelly, A. *The stable, center-stable, center, center-unstable, and stable manifolds,* J. Diff. Eqns. 3 (1967), 546-570

[8] Poore, A. B. *On the theory and application of the Hopf-Friedrichs bifurcation theory*, Arch. Rat. Mech. and Anal. 60 (1975-76), 371-393