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多項式調變不變奇異積分算子

Polynomial Modulation Invariant Singular Integral Operator

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摘要

本文針對多項式卡爾松算子高維推廣在勒貝格空間下的有界性作深入探討。相比於Victor Lie與Pavel Zorin-Kranich之前的工作，該文章的主要貢獻包含：以具體的構造法來確認細節論證、用稀疏算子的語言來重新詮釋部分證明、及提供一個具教學啟發性的完整說明。

關鍵詞：時頻分析、多重解析度分析、CZ算子、稀疏壓制、TT*-T*T方法

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Polynomial Modulation Invariant Singular Integral Operators

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Abstract

We deeply study the L^p boundedness of the generalization of Polynomial Carleson Operator. Our main contributions, comparing to previous works done by Victor Lie and by Pavel Zorin-Kranich, are to verify details with explicit constructions, modify some part with language of Sparse Dominance, and provide a heuristic interpretation about the whole treatment in general.

Keywords—Time-Frequency Analysis, Multi-Resolution Analysis, CZO, Sparse Dominance, TT* method

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1 Introduction

There are three major themes in Harmonic Analysis that ordinary tools in Real Analysis are weak against:

$$\begin{cases} \textbf{Singular} & \Rightarrow \text{Singular Integral Operator} \\ \textbf{Maximal} & \Rightarrow \text{Hardy-Littlewood Maximal Operator} \\ \textbf{Oscillatory} & \Rightarrow \text{Fourier Integral Operator.} \end{cases}$$

Still, mathematicians have developed **tools for individual** class of operators and have gained fruitful understanding. Before becoming overly optimistic, however, what if there is an instance where the **three themes combine** together?

Definition 1.0.1 (Carleson Operator)**.**

$$Cf(\cdot) := \sup_{N\in\mathbb{R}} \left| p.v. \int \frac{e^{iNy}}{\cdot - y} f(y)dy \right|.$$

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Indeed, we see that there are:

- **Singularity** in the integral kernel $\frac{1}{\cdot - y}$.
- **Pointwise maximal** in the evaluation.
- **Oscillation** within the integral.

Naturally, we can not expect the tools designed for one particular theme to be effective against such operator. Maybe, we just need to **combine all the tools** in a smart ways. Additionally, we better do so in a way that **separate different features** from different themes so that each individual tools can shine. In hindsight, the missing glue to **stick** all the tools together is **Time-Frequency Analysis**. While, the participation of **sparse dominance** is a pleasant surprise.

Of course, this operator is not something mathematicians conjure up just for fun. To convince the reader that such type of operators arises naturally, we first introduce some notions.

1.1 Basic Notions

As a preparation for stating the main result, we introduce some definitions and notations. Throughout this thesis, we only work under Euclidean setting ($\mathbb{R}^D$).

Definition 1.1.1.

$$\mathcal{Q}_d := \{q \in \mathbb{R}[x_1][x_2]\cdots[x_D] \mid \deg q \leq d\}$$

Definition 1.1.2 (Standard Kernel)**.**
Given $K : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{C}$, we say K is a ***Standard Kernel*** *if given $x, y \in \mathbb{R}^D$, we have "**Size Control**":*

$$|K(x,y)| \lesssim \|x-y\|^{-D}.$$

*Furthermore, there's $\tau \in (0,1]$ such that for $\Delta \in \mathbb{R}^D$ satisfying $\frac{\|\Delta\|}{\|x-y\|} \leq \frac{1}{2}$, we also have "**$\tau$-Hölder Type Control**":*

$$|K(x+\cdot,y)\Big|_0^\Delta| + |K(x,y+\cdot)\Big|_0^\Delta| \lesssim \frac{(\|\Delta\|/\|x-y\|)^\tau}{\|x-y\|^D}.$$

Definition 1.1.3 (Calderon-Zygmund Operator)**.**
Given $T \in \mathcal{BL}(L^2, L^2)$, we say T is a Calderon-Zygmund Operator (CZO) if it's ***associated*** *to a standard kernel K in the following sense:*

$$\forall f, g \in C_c^\infty,\ \operatorname{supp} f \cap \operatorname{supp} g = \varnothing \Rightarrow \langle Tf, g\rangle = \int K(x,y) f(y)\overline{g(x)}dxdy.$$

Remark. *Kernel determines a CZO up to a difference of Multiplication Operator. That is: Given $T, S \in \mathcal{BL}(L^2, L^2)$ be a pair of CZOs, if T, S are associated to the same kernel, then*

$$\exists m \in L^\infty \ s.t.\ \forall f \in L^2,\ Tf - Sf = mf.$$

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For the rest of the thesis, we fix $T \in \mathcal{BL}(L^2, L^2)$ a CZO, denote the corresponding kenerl as $K(\cdot,\cdot)$, and use $f \in C_c^\infty$ to denote a generic function. Now, we introduce some related operators.

Definition 1.1.4 (Singular Integral Operator)**.**
If the kernel satisfies additional regularity condition:

$$\forall' x \in \mathbb{R}^D, \ \lim_{\epsilon \to 0^+} \int_{\epsilon \le \|x-y\| \le 1} K(x,y)dy \ \ \textit{exists},$$

the following limit:

$$\lim_{\epsilon \to 0^+} \int_{\epsilon \le \|\cdot - y\|} K(\cdot, y) f(y) dy$$

actually defines a CZO associated to K. We call this particular type of CZO Singular Integral Operator.

Definition 1.1.5 (Maximal Truncated CZO)**.**

$$T_* f(\cdot) := \sup_{r<R} \left| \int_{r \le \|\cdot - y\| < R} K(\cdot, y) f(y) dy \right|.$$

Definition 1.1.6 (Maximal Operator)**.**

$$M_r f(\cdot) := \sup_{B \ni \cdot} |f|_{B,r}$$

where B denotes a cube and $|f|_{B,r} := \left(⨍_B |f|^r d\mu \right)^{1/r}$ with $r \in [1, \infty)$ and μ the Lebesgue measure. Notice that ***Hardy-Littlewood Maximal Operator*** *is essentially the case when $r = 1$. For convenience, we write:*

$$Mf := M_1 f \ \ \textit{and} \ \ |f|_B := |f|_{B,1}.$$

Definition 1.1.7 (Polynomial Modulation Invariant CZO)**.**

$$C_d f(\cdot) := \sup_{q \in \mathcal{Q}_d} \left| T(e^{iq} f)(\cdot) \right|$$

Definition 1.1.8 (Maximal Truncated Polynomial Modulation Invariant CZO)**.**

$$C_{d*} f(\cdot) := \sup_{q \in \mathcal{Q}_d} T_*(e^{iq} f)(\cdot)$$

Observation. *Due to a version of Cotlar's Inequality([Duo+01]Lemma 5.15), we always have:*

$$T_* f \lesssim MTf + Mf,$$

and thus,

$$C_{d*} f \lesssim MC_d f + Mf.$$

As a result, boundedness of C_d implies boundedness of C_{d*}.

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1.2 Motivation

We provide some instances where considering such type of operators are relevant.

- **Pointwise a.e. Convergence of Fourier Series**: In 1915, Luzin conjectured that the **Foruier series** of a L^2 function converges **almost everywhere** to the function itself. The result is proved fifty years afterward.

  Theorem 1.2.1 (Carleson's Theorem)**.**
  Qualitative statement:(Lennart Carleson, 1966 [Car66])

  $$\textit{The Fourier Series of } L^2 \textit{ function converge a.e. to itself.}$$

  Quantitative statement:(Charles Fefferman, 1973 [Fef73])

  $$T \textit{ be Hilbert Transform on } \mathbb{T},\ \|C_1 f\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})}.$$

  The original proof was quite complicated. It was not until 1973 that Fefferman gave a much elegant proof on the quantitative equivalence based on **Stein Maximal Principle** and ideas of **Time-Frequency Analysis**.

- **Constant Coefficient PDE**: We provide the most elementary case: **Heat equation** to illustrate the idea.

  $$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = f(x,t), & t > 0 \\ u(x,0) = u_0(x), \end{cases}$$

  Due to the **linearity** of the equation, we reduce to solve the following two sets of equation:

  $$\begin{cases} \begin{cases} u_t(x,t) - \Delta_x u(x,t) = 0, & t > 0 \\ u(x,0) = u_0(x) \end{cases} & \textbf{homogeneous} \\ \\ \begin{cases} u_t(x,t) - \Delta_x u(x,t) = f(x,t), & t > 0 \\ u(x,0) = 0 \end{cases} & \textbf{non-homogeneous}. \end{cases}$$

  Suppose we have understood how the **regularity** of the **initial data** u_0 affects the **regularity** of the **solution** u of the **homogeneous** equation. We now proceed to investigate how the **non-homogeneous term** f affects the **regularity** of the **solution** u in the sense of **Sobolev space** language. To do so, we first assume the following **stronger condition**: Given $u(\cdot,\cdot), f(\cdot,\cdot) \in S(\mathbb{R}^D \times \mathbb{R})$ that vanishes for $t \leq \epsilon$ with some $\epsilon > 0$,

  $$u_t(x,t) - \Delta_x u(x,t) = f(x,t)$$
  $$\overset{\text{Fourier}}{\Longleftrightarrow} \left(2\pi i\tau + 4\pi^2|\xi|^2\right) \widehat{u}(\xi,\tau) = \widehat{f}(\xi,\tau)$$

  By defining $m(\xi,\tau) := \frac{2\pi i\tau}{2\pi i\tau + 4\pi^2|\xi|^2}$ and setting

  $$L_t f := \mathfrak{F}^{-1}\left(m\mathfrak{F}(f)\right),$$

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we expect $L_t f$ to **solve** u_t. Notice that $m(\lambda\xi, \lambda^2\tau) = m(\xi, \tau)$, thus by setting $K := \mathfrak{F}^{-1}(m)$, we have:

$$K(\lambda x, \lambda^2 t) = \lambda^{-D-2} K(x, t).$$

As we expand $L_t f$:

$$\begin{aligned}
&L_t f(\cdot) := K * f(\cdot) \\
&= \int_{\mathbb{R}_+} \rho^{D+2} \int_{S^D} K(\rho x, \rho^2 t) f(\cdot - (\rho x, \rho^2 t)) J(x,t) d(x,t) \frac{d\rho}{\rho} \\
&= \int_{S^D} K(x,t) J(x,t) \underline{\int_{\mathbb{R}_+} f(\cdot - (\rho x, \rho^2 t)) \frac{d\rho}{\rho}} d(x,t),
\end{aligned}$$

we reduce to control the following operator:

Definition 1.2.2 (Hilbert Transform Along Paraboloid)**.**

$$H_{(y,s)} f(x,t) := p.v. \int_{\mathbb{R}} f((x,t) - (\rho y, \rho^2 s)) \frac{d\rho}{\rho}$$

Denoting **Fourier on** $(\tilde{x}, t) := (x_2, x_3, \cdots, x_D, t)$" **as** $\tilde{\mathfrak{F}}$, we deduce:

$$\tilde{\mathfrak{F}} \left(H_{(y,s)} f \right) (\cdot, \tilde{\xi}, \tau) = p.v. \int_{\mathbb{R}} e^{-2\pi i \left(\rho^2 \tau s + \rho \tilde{\xi} \cdot \tilde{y} \right)} \tilde{\mathfrak{F}} f(\cdot - \rho y_1, \tilde{\xi}, \tau) \frac{d\rho}{\rho},$$

which can be controlled by C_2 with T be Hilbert Transform:

$$\left| \tilde{\mathfrak{F}} \left(H_{(y,s)} f \right) (\cdot, \tilde{\xi}, \tau) \right| \lesssim C_2 \tilde{\mathfrak{F}} f(\cdot, \tilde{\xi}, \tau).$$

If we have $\|C_2 f\|_{L^2} \lesssim \|f\|_{L^2}$, then using the **tensor product** structure of the **product measure** and **Plancherel theorem**, we have:

$$\begin{aligned}
\because \left\| \tilde{\mathfrak{F}} \left(H_{(y,s)} f \right) (\cdot, \tilde{\xi}, \tau) \right\|_{L^2} &\lesssim \left\| \tilde{\mathfrak{F}} f(\cdot, \tilde{\xi}, \tau) \right\|_{L^2} \\
\therefore \left\| H_{(y,s)} f \right\|_{L^2} = \left\| \tilde{\mathfrak{F}} \left(H_{(y,s)} f \right) \right\|_{L^2} &\lesssim \left\| \tilde{\mathfrak{F}} f \right\|_{L^2} = \|f\|_{L^2}.
\end{aligned}$$

This implies that $\|L_t f\|_{L^2} \lesssim \|f\|_{L^2}$. (There is an analogous statement for $\Delta_x u$.) As a result, we can use **density argument** to infer that:

$\forall f \in L^2(\mathbb{R}^D \times \mathbb{R}_+)$, $\exists u$ **solving** the equation $s.t.$ u_t, $\Delta_x u \in L^2(\mathbb{R}^D \times \mathbb{R}_+)$,

which can be easily translated to **Sobolev space** language.

Remark. *If $D = 1$, the **linear term** in the **modulation** vanishes. This case is covered by Stein and Wainger's result in [SW01]*

- **Modulation Symmetries**: An operator may possess certain symmetry. One such instance is **polynomial modulation symmetry**. We expect that understanding C_d and C_{d*} paves the way to the understanding of some more complicated operators.

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- Explicit Polynomial Modulation Invariance: (**Hard but have result on $L^p \to L^p$ boundedness.**)

$$q \in \mathcal{Q}_d \Longrightarrow \begin{cases} C_d\left(e^{iq}f\right) & = C_d\left(f\right) \\ C_{d*}\left(e^{iq}f\right) & = C_{d*}\left(f\right) \end{cases}$$

- Implicit Polynomial Modulation Symmetry: (**No good result on the boundedness of the operator for** $n > 2$)

$$H_{\vec{\alpha}}\left(f_j\right)_{j=1}^n(\cdot) := p.v. \int_{\mathbb{R}} \prod_{j=1}^n f_j(\cdot - \alpha_j t)\frac{dt}{t}$$

$$\sum_{j=1}^n q_j(\cdot - \alpha_j t) = q(\cdot) \Longrightarrow H_{\vec{\alpha}}\left(e^{iq_j}f_j\right)_{j=1}^n = e^{iq}H_{\vec{\alpha}}\left(f_j\right)_{j=1}^n$$

Indeed, inspired by Fefferman's proof on the boundedness of C_1, Thiele and Lacey came up with a much **elegant argument** using the **same philosophy** to prove the boundedness of $H_{(1,-1)}$:

Theorem 1.2.3 (Christoph Thiele & Michael Lacey, 1997 [LT97]). *$\forall p, q, r \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$,*

$$|\langle H_{(1,-1)}(f, g), h\rangle| \underset{p,q,r}{\lesssim} \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

Later on, they notice the similarity (**similar modulation symmetry**) between C_1 and $H_{(1,-1)}$ and use their method to prove:

Theorem 1.2.4 (Christoph Thiele & Michael Lacey, 2000 [LT00]).

$$\|C_1 f\|_{L^{2,\infty}} \lesssim \|f\|_{L^2}, \quad \textit{where } T \textit{ is Hilbert Transform.}$$

It is tempting to believe that there is an implicit **correspondence**:

$$C_d f,\ C_{d*}f \ \overset{\Longleftarrow}{\Longrightarrow}\ H_{\vec{\alpha}}\left(f_j\right)_{j=1}^n.$$

However, there must be some **missing links** between the two scenarios. To elaborate, we present some of the differences:

($\Longrightarrow$) We need to find a way to convert the **multilinear** nature of the operator into **products** of **linear** structures. Additionally, we better **extract** the **implicit modulation symmetry** into the form of **explicit modulation invariance**.

($\Longleftarrow$) The conversion of C_1 into $H_{(1,-1)}$-like operator, relies on the **Fourier correspondence** between **linear modulation** and **translation**. There is no good notion for **polynomial modulation**.

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- **Detection of the Singularity**: It is an idea from one of my colleagues. Let us compare $\frac{1}{\cdot}$ and $\frac{1}{|\cdot|}$ and its corresponding operators:

$$\begin{cases} Hf(\cdot) := p.v \int \frac{1}{\cdot - y} f(y)dy \\ Xf(\cdot) := p.v \int \frac{1}{|\cdot - y|} f(y)dy. \end{cases}$$

Some easy verification shows that:

$$\begin{cases} \|Hf\|_{L^p} \lesssim \|f\|_{L^p} \\ \|Xf\|_{L^p} \not\lesssim \|f\|_{L^p} \end{cases}, \quad \forall p \in (1,\infty).$$

As we put in **modulation**: Fixing $\mathcal{Q} \subset C^\infty(\mathbb{R},\mathbb{R})$, we define:

$$Qf(\cdot) := \sup_{\phi\in\mathcal{Q}} \left| p.v. \int \frac{1}{\cdot - y} e^{i\phi(y)} f(y)dy \right|,$$

we see that the behavior of Q is **morally governed by** the two cases: H and X. That is, if $\mathcal{Q}$ is too large, we can expect the modulation **recovers** the **absolute value** that is:

$$|Xf| \le |Qf| \quad \text{and, thus,} \quad \|Qf\|_{L^p} \not\lesssim \|f\|_{L^p}, \quad \forall p \in (1,\infty).$$

Otherwise, we have for example: $\mathcal{Q} := \mathcal{Q}_1$ and $T := H$,

$$C_1 f = Qf \quad \text{and, thus,} \quad \|Qf\|_{L^p} \lesssim \|f\|_{L^p}, \quad \forall p \in (1,\infty).$$

The interesting part is to find the **borderline** between the two cases:

Definition 1.2.5 (Detection of Singularity).
*Given $T \in \mathcal{BL}(L^2, L^2)$ a **CZO**, we say $\mathcal{Q} \subset C^\infty(\mathbb{R}^D,\mathbb{R})$ **detects the singularity** at $p \in (1,\infty)$ if the operator defined as:*

$$Qf(\cdot) := \sup_{\phi\in\mathcal{Q}} \left| T\left(e^{i\phi} f\right)(\cdot) \right|$$

is not bounded at p. That is, $\|Qf\|_{L^p} \not\lesssim \|f\|_{L^p}$.

In other words, $\mathcal{Q}_1$ does **not detect** the singularity of **Hilbert transform**. We think a **non-trivial** example of $\mathcal{Q}$ that **detects** the singularity at **specific** p would give us new light on the understanding of the **singularity** of an operator.

1.3 Main Result

Stein conjectured that C_d is bounded for suitable $K(\cdot,\cdot)$. In his joint work with Wainger [SW01], a **restricted case** (**excluding linear modulation**) is resolved through the technique of **stationary phase formula** and TT^*-T^*T arguments. While, Lie, after proving the **weak**$(2,2)$ bound of C_2 with T being **Hilber transform** on $\mathbb{T}$, proved the **Stein conjecture** for the following case:

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Theorem 1.3.1 (Victor Lie, 2020 Annals of Mathematics [Lie20]).
T be Hilbert Transform on $\mathbb{T}$,

$$\|C_d f\|_{L^p(\mathbb{T})} \underset{p,d}{\lesssim} \|f\|_{L^p(\mathbb{T})}, \quad \forall p \in (1, \infty)$$

Inspired by the proof, Zorin-Kranich extended the result and resolved the full **Stein conjecture**:

Theorem 1.3.2 (Pavel Zorin-Kranich, 2019 [Zor19]).
For arbitrary D, T,

$$\|C_{d*} f\|_{L^p} \underset{T,D,d,p}{\lesssim} \|f\|_{L^p}, \quad \forall p \in (1, \infty).$$

Remark. *The precise condition for **Theorem** 1.3.2 is actually weaker:*

$$\|T_* f\|_{L^p} \underset{p}{\lesssim} \|f\|_{L^p}, \quad \forall p \in (1, \infty).$$

*That is, even if there is no **C.Z.O** associated to the kernel $K(\cdot, \cdot)$, the condition is still valid. Alternatively, it infers that polynomials with **bouneded degree** can**not detect** the **singularity of the kernel** if T^* is bounded.*

By previous observation, it's tempting to think C_d a more fundamental object and try proving its boundedness first. Naturally, we would come up with our first guess:

Theorem 1.3.3.
If T is a Singular Integral Operator, we always have:

$$\|C_d f\|_{L^p} \underset{T,D,d,p}{\lesssim} \|f\|_{L^p}, \ \forall p \in (1, \infty)$$

However, in hindsight, we actually treat **Theorem** 1.3.3 as a direct corollary of **Theorem** 1.3.2. Notice that it's quite different from the treatment in [Lie20]. The author proves **Theorem** 1.3.3 for T being Hilbert Transform directly. We will address what causes the difference in 3.4.

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2 Mathematical Jigsaw Puzzle

In this section, we give a heuristic explanation about how we'll use Time-Frequency Analysis to proceed with the proof of **Theorem** 1.3.2.

2.1 Cut out the Pieces

The idea is to linearize C_{d*}:

$$C_{d*}f(\cdot) \rightsquigarrow \int_{r_{(\cdot)} \leq \|\cdot - y\| < R_{(\cdot)}} K(\cdot, y) e^{iq_{(\cdot)}(y)} f(y) dy =: \tilde{C}_{d*} f(\cdot)$$

so that the time-frequency information of $f(\cdot)$ gets transferred to the operator itself. Since $q_{(\cdot)}$ is encoded with the **sheet music–time-frequency portrait** of $f(\cdot)$, Time-Frequency Analysis would be done on $\tilde{C}_{d*}$ instead of f.

Next, we break $\tilde{C}_{d*}$ into tiny pieces and treat them as mathematical jigsaw puzzles. Our goal is to fit those pieces into a "**bounded**" box. To do so, we do the following decomposition:

- **Scale**($s \in \mathbb{Z}$): We break $K(\cdot, \cdot)$ according to **scales** so that each piece mimics the behavior of a **wavelet**. As a result, the **s-scale** piece of the operator extracts **2^s-resolution** features only. In short, we have

$$K(x, y) \sim \sum_{s \in \mathbb{Z}} \mathbf{wavelet}_s(x - y) \ \wedge \ T_* f(\cdot) \sim \sup_{\underline{s} < \overline{s}} \left| \left(\sum_{s = \underline{s}}^{\overline{s}} \mathbf{wavelet}_s \right) * f(\cdot) \right|.$$

- **Temporal block**($I \subset \mathbb{R}^D$): With a fixed scale, we decompose the piece to separate the support into different **temporal position** with **block-size** matching the **scale**.

- **Spectral block**($\omega \subset \mathcal{Q}_d$): Fixing scale and temporal position, we decompose the piece again so that $q_{(\cdot)}$ fall in distinct **spectral position** with **block-size** respecting some kind of **Uncertainty Principle**.

That is, a generic piece satisfies:

$$2^s \sim \text{diameter of } I \sim \text{diameter of } \omega^{-1},$$

where s is the **natural scaling** of $I \times \omega$ and is denoted by $s_{I \times \omega}$. In short,

$$\tilde{C}_{d*} f(\cdot) \sim \sum \mathbf{piece}_{I \times \omega} f(\cdot),$$

where

$$\mathbf{piece}_{I \times \omega} f(\cdot) \sim \mathbf{wavelet}_{s_{I \times \omega}} * \left(e^{iq_{(\cdot)}} f \right)(\cdot) \chi_{E_{I \times \omega}}(\cdot)$$

with

$$E_{I \times \omega} := \left\{ x \in I \mid q_x \in \omega \ \wedge \ r_x \leq 2^{s_{I \times \omega}} \leq R_x \right\}.$$

Naturally, this comes with good properties. For instance, all the pieces have similar sizes in $\mathcal{BL}(L^p, L^p)$. However, we need finer estimation, and we do so by tracking the following attributes for each piece:

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- **Scale** : This corresponds to the resolution of features the operator detects/takes in.
- **Tile position** : This refers to the position of **tile** $P := I_P \times \omega_P$ on the time-frequency phase plane.
- **Density** : This measures how large portion of I_P gets sent through $q_{(\cdot)}$ to ω_P within the **acceptable scale range**. That is, $\mathcal{A}(P) := \frac{|E_P|}{|I_P|}$.

As an immediate result,

$$\|\mathbf{piece}_P f\|_{L^p} \lesssim \mathcal{A}(P)^{1/p} \|f\|_{L^p}.$$

This provides us with some intuition. By classifying the pieces according to their density (i.e. $\mathcal{A}(P) \approx 2^{-n}$), we just need to remember extracting the 2^{-n}-factor from our arguments. Namely, we shall focus on $\mathbb{P}_n := \{P \mid \mathcal{A}(P) \approx 2^{-n}\}$. (Details would be made precise in 4.)

2.2 Find Good Configurations

Up till now, we've reduced the puzzle to $\mathbb{P}_n$ sub-puzzle. To proceed, we need to know how well pieces can be packed together in $\mathcal{BL}(L^p, L^p)$. Naturally, a good starting point would be $\mathcal{BL}(L^2, L^2)$. This way, we can use **Orthogonality** to help us organize our pieces. As expected, $\exists \epsilon > 0,\ s.t.\ \forall P_j \in \mathbb{P}_n$

$$\begin{cases} \langle \mathbf{piece}_{P_0} f, \mathbf{piece}_{P_1} f \rangle = 0 & \Longleftarrow P_0 \cap P_1 = \varnothing \\ \left| \langle \mathbf{piece}^*_{P_0} f, \mathbf{piece}^*_{P_1} f \rangle \right| & \lesssim 2^{-n} \left(1 + \mathbf{distance}_{P_0, P_1}\right)^{-\epsilon}. \end{cases} \tag{7.1.3}$$

Alternatively, if $\mathbb{P} \subset \mathbb{P}_n$ **cluster** at a spot $(\xi, \eta) \in \mathbb{R}^D \times \mathcal{Q}_d$, the cluster $\mathbf{cluster}_{\mathbb{P}} f := \sum_{P \in \mathbb{P}} \mathbf{piece}_P f$ will extract **distinct** 2^{s_P}**-resolution** features of f near (ξ, η). Therefore, provided that $\begin{cases} \{s_P\}_{P \in \mathbb{P}} = \{s \in \mathbb{Z} \mid \underline{s} \le s \le \overline{s}\} \\ \forall P \in \mathbb{P},\ \mathbf{distance}_{P,(\xi,\eta)} \ll 1 \end{cases}$, we have $q_x \sim \eta$ as long as $x \in \bigcup_{P \in \mathbb{P}} E_P$ is around ξ, and **Multi-Resolution Analysis** yields

$$\begin{aligned} |\mathbf{cluster}_{\mathbb{P}} f| &\sim \left| \left(\sum_{s=\underline{s}}^{\overline{s}} \mathbf{wavelet}_s \right) * (e^{i\eta} f) \right| \chi_{\substack{2^{-n}\textbf{-dense set} \\ \textbf{around } \xi}} \\ &\lesssim T_*(e^{i\eta} f) \chi_{\substack{2^{-n}\textbf{-dense set} \\ \textbf{around } \xi}}. \end{aligned} \tag{8.2.3}$$

Moreover, by viewing cluster of tiles as a whole, we have analogue of previous two **Orthogonality** relation: for $\mathbb{P}^j \subset \mathbb{P}_n$ cluster at $\mathfrak{p}_j \in \mathbb{R}^D \times \mathcal{Q}_d$, we have

$$\begin{cases} \langle \mathbf{cluster}_{\mathbb{P}^0} f, \mathbf{cluster}_{\mathbb{P}^1} f \rangle = 0 & \Longleftarrow \bigcup \mathbb{P}^0 \cap \bigcup \mathbb{P}^1 = \varnothing \\ |\langle \mathbf{cluster}^*_{\mathbb{P}^0} f, \mathbf{cluster}^*_{\mathbb{P}^1} f \rangle| & \lesssim 2^{-n} \left(1 + \mathbf{distance}_{\mathfrak{p}_0, \mathfrak{p}_1}\right)^{-\epsilon}. \end{cases} \tag{8.3.1}$$

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Combining what have been learned, a reasonable strategy to solve the puzzle would be to organize $\mathbb{P}_n$ into the following two "**good**" configurations:

- **Sparse Parts**: $\mathbb{P} \subset \mathbb{P}_n$ has few overlaps on $\mathbb{R}^D \times \mathcal{Q}_d$, and **Orthogonality** gives strong enough control. (Details are presented in 7)

- **Cluster Parts**: $\mathbb{P} \subset \mathbb{P}_n$ consists of multiple clusters but clusters are 2^{Cn} apart on $\mathbb{R}^D \times \mathcal{Q}_d$ with $C \gg 1$. By combining both **Orthogonality** and **Multi-Resolution Analysis**, we can apply **Cotlar-Stein Lemma** and arrive at a suitable control. (Details are presented in 8).

2.3 Combinatorial Wizardry and Analytic Magecraft

Now, to systematically extract those good configurations from $\mathbb{P}_n$, we follow both [Lie20] and [Zor19] , which follow Charles Fefferman's idea in [Fef73]. To elaborate, we equip $\mathbb{P}_n$ with an "**order-like**" relation to reflect their "**incidental properties**". Consequently, both sparse parts and cluster parts have alternate interpretations:

- **Sparse Parts**: Collections of **Anti-Chains**

- **Cluster Parts**: Collections of **Convex Sets**

Therefore, through some **Combinatorial** methods devised by Fefferman, we can extract the desired configurations. (Details in 5.3.)

Still, the original argument in [Fef73] has no control over how "**high**" clusters stack. The author isolates those who stack too high and proves that they have "**small supports**", which is why "**Exceptional Sets**" arise in [Fef73]. This prevents us from finer estimate and direct $L^2 \to L^2$ bound.

One of the innovation in [Lie20] is the clever use of **John-Nirenberg inequality**. The arguments guarantee that "**higher clusters**" has "**smaller supports**". That is, instead of **stacking like Jenga**, the clusters **stack like Eiffel Tower**. Consequently, Lie eliminated the use of Exceptional Sets and derived $L^2 \to L^2$ bound directly. (Details in 6.3.)

On the other hand, Zorin-Kranich simplified the argument and put additional steps to make the system more compatible with certain "**temporal dilation**". (Details in 6.4.)

Finally, to acquire full $L^p \to L^p$ bound, we modify Lie's argument on sparse parts with the language in [LN15] and adopt Zorin-Kranich's treatment on cluster parts. To be more specific, we first derive p-bounds insensitive to density:

- **Sparse Parts**: We resort to pointwise sparse dominance on sparse parts.

- **Cluster Parts**: We use the **Multi-Resolution Analysis** on clusters to derive "localized estimate" and the extrapolation method adopted by Bateman in [BT13] to acquire $L^{p,1} \to L^{p,\infty}$ bound. (Detail in 8.7.)

To complete the argument, we interpolate to spread the 2^{-n} factor to $L^{p_\theta} \to L^{p_\theta}$ bound and use the geometric decay on density to sum everything up.

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3 Tools and Facts

In this section, we establish some tools and some useful facts without proof.

For starters, we borrow part of the setting and language in [Zor19] and [SW01] to quantify the effect of polynomial phases on behavior of oscillatory integrals.

Next, we follow the setting in [LN15] and sum up some useful facts about sparse systems.

At the end of the section, we introduce our modified settings and explain how it relates to the original settings and why the change of the formulation in [Zor19] may be necessary to generalize the result in [Lie20].

Remark. *Throughout this thesis, we will sometimes suppress the dependence on κ, κ^*, D, d within the $\lesssim, \ll, \subsetsim$ relation.*

3.1 Local Oscillation of Polynomial

To apply **Cotlar-Stein Lemma**, we expect the need for an estimate as the following:

$$q \in \mathcal{Q}_d,\ \psi \in L^0 \text{ (measurable function)} \Longrightarrow \left| \int e^{iq} \psi d\mu \right| \underset{\substack{D,d \\ \text{Oscillation of } \psi, q \\ \text{on } \operatorname{supp}\psi}}{\lesssim} \quad ?$$

Indeed, when $d = 1$, **Riemann–Lebesgue Lemma** gives us qualitative description: the higher the oscillation, the greater the cancellation. This motivates the need to quantify the oscillation of q within the support of ψ. However, to simplify the matters, we model the support as **cubes**, and we, therefore, need some related terminology:

Definition 3.1.1 (Attributes of a cube $I \subset \mathbb{R}^D$)**.**

- *$c_I \in \mathbb{R}^D$ denotes the center of mass of I.*
- *ℓ_I denotes the side-length of I.*
- *$|I| := {\ell_I}^D$ denotes the D-volume of I.*

In short, $I := c_I + \ell_I[-1/2, 1/2)^D = c_I + [-\ell_I/2, \ell_I/2)^D$.

Definition 3.1.2 (Temporal Dilation)**.**

$$\forall C \in \mathbb{R}_+,\ CI := c_I + C\ell_I[-1/2, 1/2)^D = c_I + \left[-\frac{C\ell_I}{2}, \frac{C\ell_I}{2}\right)^D.$$

Now, we define a weaker form of "$\subset$". Given $I, J \subset \mathbb{R}^D$ be cubes,

Definition 3.1.3 (Roughly Contain)**.**

$$I \subsetsim J \iff \exists C \in \mathbb{R}_+ \textit{ prescribed, s.t. } I \subset CJ$$

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Finally, we characterize the local oscillation of $q \in \mathcal{Q}_d$ on cube.

Definition 3.1.4 (Seminorm on $\mathcal{Q}_d$ [Zor19](**4.1.5.**)**).**

$$\|q\|_I := \sup_{x,y\in I} |q(x) - q(y)| \,.$$

As an immediate result, since $\mathcal{Q}_d$ is a finite dimensional vector space, all non-trivial(vanishing only on constant) seminorms are equivalent. Therefore, we may unambiguously assign a topology generated by seminorm on $\mathcal{Q}_d$. Still, for our purpose, we need quantitative controls:

Properties 3.1.5 (Embedding Inequality [Zor19]**Lemma 4.1.6.**).

$$I \subsetneq J \Longrightarrow \frac{\ell_J}{\ell_I}\|q\|_I \underset{D,d}{\lesssim} \|q\|_J \underset{D,d}{\lesssim} \left(\frac{\ell_J}{\ell_I}\right)^d \|q\|_I,$$

Such estimate would become important as we do Multi-Resolution Analysis.

3.2 Van der Corput Estimate

Continuing previous settings,

Properties 3.2.1 ([Zor19]**Lemma 4.6.1.** [SW01]**Proposition 2.1.**).

$$\begin{aligned}\forall \psi \in L^0,\ \operatorname{supp}\psi \subset I \Longrightarrow \left|\int e^{iq}\psi d\mu\right| &\underset{D,d}{\lesssim} \sup_{\frac{\|\Delta\|}{\ell_I} < \langle \|q\|_I\rangle^{1/d}} \|\psi - \tau_\Delta \psi\|_{L^1} \\ &\lesssim \sup_{\frac{\|\Delta\|}{\ell_I} < \langle \|q\|_I\rangle^{1/d}} \|\psi - \tau_\Delta \psi\|_{L^\infty} |I|.\end{aligned}$$

where $\langle\cdot\rangle := \frac{1}{1+|\cdot|}$ *and* $\tau_\Delta\psi(\cdot) := \psi(\cdot - \Delta)$.

As a immediate corollary, we have a version designed for partition of unity: For generic $\psi \in L^0,\ \delta > 1,\ I \subset \mathbb{R}^D$ be cube, we consider a fragment of partition of unity located around I. That is,

$$\chi \in C_c^\infty \ s.t. \begin{cases} |\chi| \underset{\delta}{\lesssim} \chi_{\delta I} \\ \|\nabla\chi\| \underset{\delta}{\lesssim} \chi_{\delta I}/\ell_I, \end{cases}$$

and we have

Corollary 3.2.1.1.

$$\left|\int \chi e^{iq}\psi d\mu\right| \underset{D,d,\delta}{\lesssim} |I| \begin{cases} \langle\|q\|_I\rangle^{1/d} \|\psi\|_{L^\infty((1+2\delta)I)} & \textit{Height of } \psi \\ + & \\ \displaystyle\sup_{\frac{|\Delta|}{\ell_I} < \langle\|q\|_I\rangle^{1/d}} \|\psi - \tau_\Delta\psi\|_{L^\infty(\delta I)} & \textit{Oscillation of } \psi. \end{cases}$$

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3.3 Sparse Language and Ambient System

The Sparse System we refer to is a sub-system of a 2^κ**-adic** System satisfying certain properties. For our purpose, we do not work under usual **Dyadic** System. Yet, all the language in [LN15] can be easily converted. For starters, we construct our **ambient system**:

Definition 3.3.1 (Standard 2^κ-adic System $\langle \mathbb{D}, \subset \rangle$)**.**

$$\mathbb{D} := \bigsqcup_{s\in\mathbb{Z}} \mathbb{D}_s, \quad where \;\; \mathbb{D}_s := \left\{ 2^{s\kappa}\left(\zeta + [0,1)^D\right) \;\, be\ cube \mid \zeta \in \mathbb{Z}^D \right\}.$$

We equip $\mathbb{D}$ *with* $\subset$ *as* ***partial order*** *and, for* $\mathbb{I} \subset \mathbb{D}$*, define:*

$$\begin{cases} M\mathbb{I} := \{I \in \mathbb{I} \mid \nexists J \in \mathbb{I} \ s.t.\ I \subsetneq J\} & \textbf{\textit{maximal elements}} \\ \mathbb{I}^{\subset} := \{I \in \mathbb{D} \mid \exists J \in \mathbb{I} \ s.t.\ I \subset J\} & \textbf{\textit{downward envelope}}. \end{cases}$$

Also, we denote ***the parent****(immediate predecessor) of* $I \in \mathbb{D}$ *as* $\widehat{I} \in \mathbb{D}$*.*

Now, given $\mathbb{S} \subset \mathbb{D}$, $1 \le \Lambda$, we call $\mathbb{S}$ a Sparse System if it satisfies either of the following equivalent([LN15] **6.1.**) conditions:

Definition 3.3.2 (Λ-Carleson Condition [LN15] **Definition 6.2.**)**.**

$$\mathbb{S} \ is\ \Lambda\text{-}Carleson \iff \forall J \in \mathbb{S}(or\ equivalently,\ \mathbb{D}), \sum_{I\in\mathbb{S},\ I\subset J} |I| \le \Lambda|J|.$$

Definition 3.3.3 (Λ^{-1}-Sparse Condition [LN15] **Definition 6.1.**)**.**

$$\mathbb{S} \ is\ \Lambda^{-1}\text{-}Sparse \iff \forall I \in \mathbb{S},\ \exists E_I \subset I \ measurable\ s.t. \begin{cases} |I| \le \Lambda|E_I| \\ E_I s\ are\ disjoint \end{cases}.$$

With basic terminology established, we provide the following two constructions. Given $\mathbb{D} \overset{\omega_{(\cdot)}}{\to} \mathbb{R}_+$, $\mathbb{S}$ Λ-Carleson, we construct $\begin{cases} M_\omega(\cdot) := \sup\limits_{\cdot\in I\in\mathbb{D}} \omega_I \\ S_{\mathbb{S},\omega}(\cdot) := \sum\limits_{I\in\mathbb{S}} \omega_I \chi_I(\cdot) \end{cases}$.

Through **Definition 3.3.3.**, we relate the two constructions:

Lemma 3.3.4 (Sparse-Maximal Dominance)**.**

$$\begin{aligned} &|\langle S_{\mathbb{S},\omega}, f\rangle| \le \sum_{I\in\mathbb{S}} \omega_I \left|\langle \chi_I, f\rangle\right| \\ \le &\sum_{I\in\mathbb{S}} |I|\omega_I \fint_I |f| d\mu \le \sum_{I\in\mathbb{S}} \Lambda|E_I|\omega_I \fint_I |f| d\mu \\ \le &\Lambda \sum_{I\in\mathbb{S}} \int_{E_I} M_\omega M f d\mu \le \Lambda\langle M_\omega, Mf\rangle \\ \Longrightarrow &\|S_{\mathbb{S},\omega}\|_{L^p} \underset{p}{\lesssim} \Lambda\|M_\omega\|_{L^p}. \end{aligned}$$

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3.4 Modified Settings

We introduce a **smoothed-out** but **scale-discretized** version of T_* and C_{d*}, which would become major tools later on. For our purpose, we

1. Prescribe $n_D := \lceil 2\sqrt{D}+1 \rceil \in \mathbb{N},\ \kappa \underset{D,d}{\gg} 1,\ \delta \underset{D,d}{\ll} 2^{-\kappa}$, where the values of $2^\kappa \in \mathbb{N},\ \delta \in \mathbb{R}_+$ would be made clear in the subsequent sections.

2. Fix $\chi \in C_c^\infty$ satisfying:

$$\chi_{(n_D+\delta)[-1,1]^D} \leq \chi \leq \chi_{(n_D+2^{-\kappa}-\delta)[-1,1]^D}.$$

3. Define $\phi(\cdot) := \chi(2^{-\kappa}\cdot) - \chi(\cdot) \in C_c^\infty$. Note that:

$$\mathrm{supp}\phi \subset (-n_D 2^\kappa - 1, n_D 2^\kappa + 1)^D \backslash [-n_D, n_D]^D.$$

As a result, certain shifts $\mathbb{S}\mathrm{h} := \{z \in \mathbb{Z} \mid n_D \leq |z| \leq n_D 2^\kappa + 1\}^D$ yield

$$x \in [0,1)^D \implies \begin{cases} \mathrm{supp}\phi(x-\cdot) \\ \mathrm{supp}\phi(\cdot - x) \end{cases} \subset \bigsqcup_{\xi \in \mathbb{S}\mathrm{h}} \xi + [0,1)^D,$$

and, by our constructions,

$$x, x' \in [0,1)^D \quad \wedge \quad y \in \bigsqcup_{\xi \in \mathbb{S}\mathrm{h}} \xi + [0,1)^D \implies \frac{\|x-x'\|}{\|x-y\|} \leq \frac{\sqrt{D}}{n_D - 1} \leq 1/2,$$

which is exactly the **condition** for τ-Hölder Type Control of K. For convenience, we also define for $I \in \mathbb{D}$ the following collection and set:

$$\mathbb{S}h_I := \{\ell_I \xi + I \in \mathbb{D} \mid \xi \in \mathbb{S}\mathrm{h}\} \quad \text{and} \quad I^* := \bigsqcup \mathbb{S}h_I.$$

4. Decompose K into **wavelet-like** pieces:

$$K = \sum_{s \in \mathbb{Z}} K_s$$

where $\forall x, y \in \mathbb{R}^D\ s.t.\ x \neq y$

$$K_s(x,y) := \phi\left(2^{-s\kappa}(x-y)\right) K(x,y)$$

Since K_s inherits the standard kernel properties of K and the support constraint on ϕ, translation and dilation yield the following three properties:

Properties 3.4.1 (L^0\Support Control)**.**

$$x \in I \in \mathbb{D}_s \implies \begin{cases} \mathrm{supp}K_s(x,\cdot) \\ \mathrm{supp}K_s(\cdot,x) \end{cases} \subset I^*.$$

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Properties 3.4.2 ($L^\infty\backslash$Size Control)**.**

$$|K_s| \underset{D,d}{\lesssim} 2^{-sD\kappa}.$$

Properties 3.4.3 (τ-Hölder Regularity)**.**

$$x, x' \in I \in \mathbb{D}_s \implies \begin{cases} |K_s(x,\cdot) - K_s(x',\cdot)| \\ |K_s(\cdot,x) - K_s(\cdot,x')| \end{cases} \underset{D,d}{\lesssim} \left(\frac{\|x-x'\|}{\ell_I}\right)^\tau |I|^{-1}\chi_{I^*}(\cdot).$$

Corollary 3.4.3.1 (Locally τ-Hölder Continuity)**.**

$$|x-x'| \lesssim 2^{s\kappa} \implies \begin{cases} |K_s(x,\cdot) - K_s(x',\cdot)| \\ |K_s(\cdot,x) - K_s(\cdot,x')| \end{cases} \underset{D,d}{\lesssim} (2^{-s\kappa}\|x-x'\|)^\tau 2^{-sD\kappa}.$$

Proof. *Given $|x-x'| \lesssim 2^{s\kappa}$, we can always find $\lesssim 1$ cubes $I_j \in \mathbb{D}_s$ covering the straight line joining x and x' with $x_j \in I_j$ on the line, where $x = x_0$ and $x' = x_n$, such that:*

$$\begin{aligned} |K_s(x,\cdot) - K_s(x',\cdot)| &\leq \sum_{j=1}^{n} |K_s(x_k,\cdot) - K_s(x_{k-1},\cdot)| \\ &\underset{D,d}{\lesssim} \sum_{j=1}^{n} \left(\frac{\|x_k - x_{k-1}\|}{\ell_{I_j}}\right)^\tau |I_j|^{-1} \lesssim \left(2^{-s\kappa}\|x-x'\|\right)^\tau 2^{-sD\kappa}. \end{aligned}$$

The dual notion holds similarly. □

With such **scale** decomposition, we may define:

Definition 3.4.4 (Modified Truncated Maximal CZO)**.**

$$\mathfrak{T}_* f(\cdot) := \sup_{\underline{s}<\overline{s}} \left| \int \sum_{s=\underline{s}}^{\overline{s}} K_s(\cdot,y)f(y)dy \right|.$$

By tinkering with $(\underline{s},\overline{s},r,R) \in \mathbb{Z}^2 \times \mathbb{R}_+^2$ so that $\begin{cases} n_D 2^{\underline{s}\kappa} \approx r \\ n_D 2^{\overline{s}\kappa} \approx R \end{cases}$, we have:

$$\left| \int_{r\leq\|\cdot-y\|<R} K(\cdot,y)f(y)dy - \int \sum_{s=\underline{s}}^{\overline{s}} K_s(\cdot,y)f(y)dy \right| \underset{D,d}{\lesssim} Mf(\cdot).$$

As a result,

Properties 3.4.5.

$$|T_* f - \mathfrak{T}_* f| \underset{D,d}{\lesssim} Mf.$$

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Therefore, the $L^p \to L^p$ behaviors of T_* and $\mathfrak{T}_*$ are identical. Consequently, it is relevant to consider:

Definition 3.4.6.
$$\mathfrak{C}_{d*}f(\cdot) := \sup_{q\in\mathcal{Q}_d} \mathfrak{T}_*(e^{iq}f)(\cdot),$$

and immediately, we have:

Corollary 3.4.6.1.
$$|C_{d*}f - \mathfrak{C}_{d*}f| \underset{D,d}{\lesssim} Mf.$$

Eventually, $L^p \to L^p$ behavior of C_{d*} is governed by $\mathfrak{C}_{d*}$, and the main result **Theorem** 1.3.2 can be reduced to proving:

Theorem 3.4.7.
$$\|\mathfrak{C}_{d*}f\|_{L^p} \underset{D,d,p}{\lesssim} \|f\|_{L^p}, \ \forall p \in (1,\infty)$$

On the other hand, the main result **Theorem** 1.3.3 for Singular Integral type operator cannot be derived directly through such method, since, in general:

$$Tf(\cdot) := \lim_{\epsilon\to 0^+} \int_{\epsilon\le\|\cdot-y\|} K(\cdot,y)f(y)dy \neq \sum_{s\in\mathbb{Z}} \int K_s(\cdot,y)f(y)dy,$$

even if:
$$\forall' x\in\mathbb{R}^D, \ \lim_{\epsilon\to 0^+} \int_{\epsilon\le\|x-y\|\le 1} K(x,y)dy \ \text{ exists.}$$

Unless, K is, for example, **Anti-Symmetric**: in Lie's works [Lie08], [Lie20], $D=1, \ K(x,y)=\frac{1}{x-y}$. If we choose $\chi\in C_c^\infty$ even, we have:

$$\forall s\in\mathbb{Z}, \ \int K_s(\cdot,y)dy = 0.$$

As a result, by using **M.V.T.** and **D.C.T.**, we have:

$$\begin{aligned}
&\sum_{\underline{s}<s\le\overline{s}} \int K_s(\cdot,y)f(y)dy \\
&= \int \sum_{\underline{s}<s\le\overline{s}} K_s(\cdot,y)\left(f(y)-f(\cdot)\right)dy \\
&\xrightarrow[\underline{s}\searrow-\infty]{\overline{s}\nearrow\infty} \int K(\cdot,y)\left(f(y)-f(\cdot)\right)dy \\
&= p.v.\int K(\cdot,y)f(y)dy = Hf(\cdot) = Tf(\cdot).
\end{aligned}$$

In conclusion, for general standard kernel K, we should adopt Zorin-Kranich's approach in [Zor19].

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4 Decomposition of the Operator

In the section, we provide the rigorous version of the following decomposition:

$$\tilde{C}_{d*}f(\cdot) \sim \sum_P \mathbf{wavelet}_{s_P} * \left(e^{iq_{(\cdot)}}f\right)(\cdot)\chi_{E_P}(\cdot).$$

To be more specific, since we've established that

$$\|C_{d*}f\|_{L^p} \lesssim \|f\|_{L^p} \Longleftarrow \|\mathfrak{C}_{d*}f\| \lesssim \|f\|_{L^p},$$

we may shift our focus to $\mathfrak{C}_{d*}$ for the rest of the arguments. Our goal is to reduce $\mathfrak{C}_{d*}f$ into sum and maximum over **finite elements**, to **linearize** the operator, and to do the **tile decomposition**.

4.1 Reduction and Linearization

For starters, we notice that

Observation. *$\mathcal{Q}_d$ is separable.*

That is, by explicitly enumerating rational coefficient polynomials:

$$\{q \in \mathbb{Q}[x_1][x_2]\cdots[x_D] \mid \deg q \le d\} =: \{q_n\}_{n\in\mathbb{N}},$$

Fatou's Lemma and some limiting arguments yield:

$$\begin{aligned}
\mathfrak{C}_{d*}f(\cdot) &= \sup_{n\in\mathbb{N}} \mathfrak{T}_*(e^{iq_n}f)(\cdot) \\
&= \sup_{\substack{n\in\mathbb{N}\\ \underline{s}<\overline{s}}} \left| \int \sum_{s=\underline{s}}^{\overline{s}} K_s(\cdot,y)e^{iq_n(y)}f(y)dy \right| \\
&\underset{\text{as } N\to\infty}{\longleftarrow} \max_{\substack{n\le N\\ -N\le\underline{s}<\overline{s}\le N}} \left| \int \sum_{s=\underline{s}}^{\overline{s}} K_s(\cdot,y)e^{iq_n(y)}f(y)dy \right| =: \mathfrak{C}_{d*,N}f(\cdot)
\end{aligned}$$

Finally, by **M.C.T.**,

$$\|\mathfrak{C}_{d*}f\|_{L^p} = \sup_{N\in\mathbb{N}} \|\mathfrak{C}_{d*,N}f\|_{L^p}.$$

Consequently, we only need to acquire bounds on $\mathfrak{C}_{d*,N}f$ independent of N. Indeed, $\mathfrak{C}_{d*,N}f$ is a sum and a maximum over **finite elements**. As a result, we can do an elementary **stopping time** argument to linearize $\mathfrak{C}_{d*,N}f$:

$$\forall N\in\mathbb{N},\ \exists \begin{cases} \mathbb{R}^D \xrightarrow{\underline{s}_{(\cdot)}} \{-N,-N+1,\cdots,N-1\} \\ \mathbb{R}^D \xrightarrow{\overline{s}_{(\cdot)}} \{-N+1,\cdots,N-1,N\} \\ \mathbb{R}^D \xrightarrow{q_{(\cdot)}} \{q_n\}_{n=1}^N \end{cases} \quad \textbf{simple and measurable}$$

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such that

$$\mathfrak{C}_{d*,N}f(\cdot) = \left|\int \sum_{s=\underline{s}_{(\cdot)}}^{\overline{s}_{(\cdot)}} K_s(\cdot, y)e^{iq_{(\cdot)}(y)}f(y)dy\right|$$

That is, regardless of the choice of $N \in \mathbb{N}$, the problem reduces to analyze the following form of **linear** operator:

$$\mathfrak{L}f(\cdot) := \sum_{s=\underline{s}_{(\cdot)}}^{\overline{s}_{(\cdot)}} \int K_s(\cdot, y)e^{iq_{(\cdot)}(y)}f(y)dy,$$

where $\underline{s}_{(\cdot)}$, $\overline{s}_{(\cdot)}$, $q_{(\cdot)}$ are simple measurable functions.

4.2 Tile Decomposition and Trivial Estimate

To proceed with our 3-step decomposition schemes, we first need to refine the following relation:

$$2^s \sim \text{diameter of } I \sim \text{diameter of } \omega^{-1}.$$

For our purpose, we adjust the above statement to our **modified settings**:

- $2^{s\kappa}$ is the actual scaling that works well with our analysis.
- $I \subset \mathbb{R}^D$ is an element chosen from $\mathbb{D}_s$(Standard 2^κ-adic System) to **match** the $2^{s\kappa}$-scale.
- $\omega \subset \mathcal{Q}_d$ will be chosen from $\mathbb{D}_I^*$, a $\mathcal{Q}_d$-tiling (assumed to exist) that respects the oscillation of polynomials on I and the **Uncertainty Principle**:

$$q, q' \in \omega \implies \|q - q'\|_I \lesssim 1$$

Notice that, by the definition of $\mathbb{D}_s$ and the Embedding Inequality, **dimensional analysis** yields

$$\begin{cases} 2^{s\kappa} = \ell_I \underset{D}{\approx} \text{diameter of } I \\ I\text{'s and } \omega\text{'s "diameters" are } \textbf{scale-reversed} \end{cases}$$

Naturally, we follow our convention and denote the **natural scaling** s as $s_{I\times\omega}$. For now, we shall postpone the construction of $\mathbb{D}_I^*$ and complete the decomposition first:

$$\mathfrak{L}f(\cdot) = \sum_{s=\underline{s}}^{\overline{s}} \sum_{I\in\mathbb{D}_s} \sum_{\omega\in\mathbb{D}_I^*} \int K_s(\cdot, y)e^{iq_{(\cdot)}(y)}f(y)dy \cdot \chi_{E_{I\times\omega}}(\cdot),$$

where $\begin{cases} \overline{s} := \max\limits_{x\in\mathbb{R}^D} \overline{s}_x \\ \underline{s} := \min\limits_{x\in\mathbb{R}^D} \underline{s}_x \end{cases}$ and $E_{I\times\omega} := \{x \in I \mid q_x \in \omega \ \wedge \ \underline{s}_x \leq s_{I\times\omega} \leq \overline{s}_x\}$.

To further simplify the notation, we shall organize the I-ω parings and define:

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Definition 4.2.1 (Tile System)**.**

$$\tilde{\mathbb{D}} := \bigsqcup_{s=\underline{s}}^{\overline{s}} \bigsqcup_{I \in \mathbb{D}_s} \{I \times \omega \subset \mathbb{R}^D \times \mathcal{Q}_d \mid \omega \in \mathbb{D}_I^*\}$$

Definition 4.2.2 (A **piece** associated to $P \in \tilde{\mathbb{D}}$)**.**

$$\mathfrak{L}_P f(\cdot) := \int K_{s_P}(\cdot, y) e^{iq_{(\cdot)}(y)} f(y) dy \cdot \chi_{E_P}(\cdot)$$

Immediately, **support** and **size** controls yield:

Properties 4.2.3 (Single tile estimate)**.**

$$\begin{cases} |\mathfrak{L}_P f| & \underset{D,d}{\lesssim} 2^{\kappa D} |f|_{\tilde{I}_P} \chi_{E_P} \\ |\mathfrak{L}_P^* f| & \underset{D,d}{\lesssim} \frac{\|f\|_{L^1(E_P)}}{|I_P|} \chi_{\tilde{I}_P} \end{cases}, \quad \textit{where} \quad \tilde{I} := \left(n_D 2^{\kappa+1} + 3\right) I \supset I^*.$$

Through direct computation, we also have:

$$\begin{cases} \|\mathfrak{L}_P f\|_{L^1} & \lesssim |f|_{\tilde{I}_P} |E_P| & \lesssim \mathcal{A}(P) \|f\|_{L^1(\tilde{I}_P)} \\ \|\mathfrak{L}_P f\|_{L^\infty} & \lesssim |f|_{\tilde{I}_P} & \leq \|f\|_{L^\infty(\tilde{I}_P)}, \end{cases}$$

(where $\mathcal{A}(P) := \frac{|E_P|}{|I_P|}$) and, through interpolation:

Corollary 4.2.3.1 (Trivial Estimate)**.**

$$\|\mathfrak{L}_P f\|_{L^p} \underset{\kappa, D, d, p}{\lesssim} \mathcal{A}(P)^{1/p} \|f\|_{L^p(\tilde{I}_P)}$$

On the other hand, given $\mathbb{P} \subset \tilde{\mathbb{D}}$, we set:

Definition 4.2.4.

$$\mathfrak{L}_{\mathbb{P}} f := \sum_{P \in \mathbb{P}} \mathfrak{L}_P f$$

Eventually, we have the succinct expression:

$$\mathfrak{L} f = \mathfrak{L}_{\tilde{\mathbb{D}}} f := \sum_{P \in \tilde{\mathbb{D}}} \mathfrak{L}_P f$$

with each piece behaving "nicely". Moreover, since

- $f \in C_c^\infty$ has compact support,
- $\underline{s}_{(\cdot)}, \ \overline{s}_{(\cdot)}, \ q_{(\cdot)}$ have finite ranges,

the sum only consists of **finitely many non-zero terms**. As a result, we may freely rearrange and reorganize the sum.

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4.3 Adaptive Christ Grid Construction

Before we construct $\mathbb{D}_I^*$, let us list what we expect from the construction:

- $\mathbb{D}_I^*$ tiles $\mathcal{Q}_d$, and, when viewed as in $\langle \mathcal{Q}_d, \|\cdot\|_I \rangle$, every piece in $\mathbb{D}_I^*$ contains and is contained in a ball with radius≈ 1.
- Given $J \subset I$, $(\omega, \omega') \in \mathbb{D}_I^* \times \mathbb{D}_J^*$, we have either $\omega \cap \omega' = \varnothing$ or $\omega \subset \omega'$.

In short, we would like to have a **hyper-adic** system on $\mathcal{Q}_d$. To do so, Zorin-Kranich follows Michael Christ's idea on constructing dyadic system on space of homogeneous type. However, the construction would be much easier since we only need to consider $I \in \mathbb{D}_s$, where $\underline{s} \le s \le \overline{s}$. Essentially, we can work our ways down from the top scale $\overline{s}$. By constructing the **finest** layer first, the rest of the arguments become finding the correct ways to **group** the pieces together. For starters, we prescribe $\kappa^* \underset{D,d}{\gg} 1$ and, by using the Embedding Inequality, find $\kappa \underset{D,d}{\gg} 1$ such that, given $J \subset I$ be cubes and $q \in \mathcal{Q}_d$, we have:

$$\ell_J \le 2^{-\kappa} \ell_I \implies \|q\|_J \le 2^{-\kappa^*} \|q\|_I.$$

We now set $\varsigma := \frac{1}{2^{\kappa^*}-1}$ and proceed inductively as follows:

$(s = \overline{s} - 0)$: For all $I \in \mathbb{D}_s$,

(a) we select a **maximal** collection of polynomials $\mathcal{Q}_I \subset \mathcal{Q}_d$ such that

$$\forall q, q' \in \mathcal{Q}_I,\ q \ne q' \implies \|q - q'\|_I \ge 1.$$

Due to **maximality**,

$$\begin{cases} \mathcal{Q}_d \subset \bigcup_{q \in \mathcal{Q}_I} B_I(q,1) \\ \forall q, q' \in \mathcal{Q}_I,\ q \ne q' \implies B_I(q, 1/2) \cap B_I(q', 1/2) = \varnothing, \end{cases}$$

where $B_I(c, r) := \{q \in \mathcal{Q}_d \mid \|q - c\|_I < r\}$.

(b) we construct the $\mathcal{Q}_d$-tiling $\mathbb{D}_I^*$ inductively with each piece assigned a center. That is, $\exists \mathbb{D}_I^* \underset{\omega_{(\cdot)}}{\overset{c_{(\cdot)}}{\rightleftarrows}} \mathcal{Q}_I$ such that, for all $\omega \in \mathbb{D}_I^*$,

$$B_I(c_\omega, 1/2 - \varsigma) \subset B_I(c_\omega, 1/2) \subset \omega \subset B_I(c_\omega, 1) \subset B_I(c_\omega, 1 + \varsigma).$$

$(s > \overline{s} - k)$: Suppose the construction be completed so that:

(a) for all $I \in \mathbb{D}_s$, we have a $\mathcal{Q}_d$-tiling $\mathbb{D}_I^*$.

(b) we assign for each piece in $\mathbb{D}_I^*$ a unique center: $\exists \mathbb{D}_I^* \underset{\omega_{(\cdot)}}{\overset{c_{(\cdot)}}{\rightleftarrows}} \mathcal{Q}_I$, where

$$\omega \in \mathbb{D}_I^* \implies B_I(c_\omega, 1/2 - \varsigma) \subset \omega \subset B_I(c_\omega, 1 + \varsigma).$$

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($s = \overline{s} - k$): Given $I \in \mathbb{D}_{s+1}$, for all $J \in \mathbb{D}_s \cap 2^I$,

(a) we select a **maximal** collection of polynomials $\mathcal{Q}_J \subset \mathcal{Q}_I$ such that

$$\forall q, q' \in \mathcal{Q}_J,\ q \neq q' \implies \|q - q'\|_J \geq 1.$$

Due to **maximality**,

$$\begin{cases} \mathcal{Q}_I \subset \bigcup_{q \in \mathcal{Q}_J} B_J(q,1) \\ \forall q, q' \in \mathcal{Q}_J,\ q \neq q' \implies B_J(q,1/2) \cap B_J(q',1/2) = \varnothing, \end{cases}$$

(b) we construct inductively a partition on $\mathcal{Q}_I$ indexed by $\mathcal{Q}_J$:

$\{\mathrm{Ch}_q\}_{q \in \mathcal{Q}_J}$ *where* $\forall q \in \mathcal{Q}_J,\ B_J\,(q,1/2) \cap \mathcal{Q}_I \subset \mathrm{Ch}_q \subset B_J\,(q,1) \cap \mathcal{Q}_I$.

(c) we define $\omega_{(\cdot)}$, by setting:

$$\mathbb{D}_J^* := \{\omega_q\}_{q \in \mathcal{Q}_J}, \quad \text{where} \ \ \omega_q := \bigsqcup_{q' \in \mathrm{Ch}_q} \omega_{q'},$$

with $\mathbb{D}_J^* \overset{c_{(\cdot)}}{\to} \mathcal{Q}_J$ defined naturally. Essentially, $\forall q \in \mathcal{Q}_J,\ \{\omega_{q'}\}_{q' \in \mathrm{Ch}_q}$ is the collection of **children** of ω_q.

(d) we characterize the size of each piece in $\mathbb{D}_J^*$: pick $q \in \mathcal{Q}_J$,

- Exterior:

$$\begin{aligned} \omega_q := \bigsqcup_{q' \in \mathrm{Ch}_q} \omega_{q'} &\subset \bigcup_{q' \in \mathrm{Ch}_q} B_I(q', 1+\varsigma) \\ &\subset \bigcup_{q' \in B_J(q,1)} B_J\left(q', \cancelto{\varsigma}{2^{-\kappa^*}(1+\varsigma)}\right) \subset B_J\,(q, 1+\varsigma) \end{aligned}$$

- Interior:

$$\begin{aligned} &\forall q' \in B_J(q, 1/2 - \varsigma),\ \exists! \omega' \in \mathbb{D}_I^* \ s.t.\ q' \in \omega' \\ &\implies \|c_{\omega'} - q\|_J \leq \|c_{\omega'} - q'\|_J + \|q' - q\|_J \\ &\qquad < 2^{-\kappa^*} \|c_{\omega'} - q'\|_I + 1/2 - \varsigma \\ &\qquad < \cancelto{\varsigma}{2^{-\kappa^*}(1+\varsigma)} + 1/2 - \varsigma = 1/2 \\ &\implies c_{\omega'} \in \mathrm{Ch}_q \implies q' \in \omega' \subset \omega_q \implies B_J(q, 1/2 - \varsigma) \subset \omega_q \end{aligned}$$

($\underline{s} \leq s \leq \overline{s}$): In conclusion, we have:

- for every $I \in \mathbb{D}_s$, $\mathbb{D}_I^*$ tiles $\mathcal{Q}_d$ (that is, $\bigsqcup \mathbb{D}_I^* = \mathcal{Q}_d$) and

$$\omega \in \mathbb{D}_I^* \implies B_I\,(c_\omega, 1/2 - \varsigma) \subset \omega \subset B_I\,(c_\omega, 1+\varsigma)\,.$$

- for all $I, J \in \bigsqcup_{s=\underline{s}}^{\overline{s}} \mathbb{D}_s$, if $J \subset I$, then, for any $(\omega, \omega') \in \mathbb{D}_I^* \times \mathbb{D}_J^*$, we, by our **grouping** construction, have either $\omega \cap \omega' = \varnothing$ or $\omega \subset \omega'$.

Notice that, by setting $\kappa^* \underset{D,d}{\gg} 1$, we have $0 < \varsigma \underset{D,d}{\ll} 1$.

This completes the construction.

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5 From Incidental Geometry to Order Theory and Combinatorics

Organizing tiles is essentially an **incidental geometric** problem. However, due to the **hyper-adic** properties of $\tilde{\mathbb{D}}$, we can equip $\tilde{\mathbb{D}}$ an **order** structure to suitably represent its **incidental** behavior. As a result, we can treat the **order theoretical** counterpart with some **combinatorial** tricks.

5.1 Conversion and Basic Operations

We start with some observations: given $I, J \in \mathbb{D}$,

- either $I \cap J = \varnothing$
- or $I \subset J \vee I \supset J$ and, thus, for any $(\omega, \omega') \in \mathbb{D}_I^* \times \mathbb{D}_J^*$,
  - either $\omega \cap \omega' = \varnothing$
  - or $\omega \supset \omega' \vee \omega \subset \omega'$ respectively.

This motivates the following definition:

Definition 5.1.1 ($\left\langle \tilde{\mathbb{D}}, \trianglelefteq \right\rangle$)**.**

$$\forall P, P' \in \tilde{\mathbb{D}},\ P \trianglelefteq P' \iff I_P \subset I_{P'} \wedge \omega_P \supset \omega_{P'}.$$

For strict inequality, we write $\triangleleft$ instead.

We see that $\trianglelefteq$ indeed defines a partial order on $\tilde{\mathbb{D}}$. Moreover, it reflects the incidental properties precisely:

$$\forall P, P' \in \tilde{\mathbb{D}},\ E_P \cap E_{P'} = \varnothing \impliedby P \cap P' = \varnothing \iff P, P' \text{ are } \trianglelefteq\text{-incomparable.}$$

As a result, to extract sparse parts(**$\trianglelefteq$-anti-chains**), we heavily rely on the following operations:

Definition 5.1.2 (Maximal and minimal elements)**.**

$$\forall \mathbb{P} \subset \tilde{\mathbb{D}},\ \begin{cases} M\mathbb{P} := \{P \in \mathbb{P} \quad | \nexists P' \in \mathbb{P} \text{ } s.t. \text{ } P \triangleleft P'\} \\ m\mathbb{P} := \{P \in \mathbb{P} \quad | \nexists P' \in \mathbb{P} \text{ } s.t. \text{ } P' \triangleleft P\}. \end{cases}$$

We also define the iterated versions:

$$\forall k \in \mathbb{N},\ \begin{cases} M_{k+1}\mathbb{P} := M\left(\mathbb{P} \setminus M_k\mathbb{P}\right) \\ m_{k+1}\mathbb{P} := m\left(\mathbb{P} \setminus m_k\mathbb{P}\right). \end{cases}$$

Notice that, by construction, both $M_k\mathbb{P}$ and $m_k\mathbb{P}$ are $\trianglelefteq$-anti-chains.

On the other hand, for cluster parts, we shall define:

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Definition 5.1.3 (Convexity)**.**
$\mathbb{P} \subset \tilde{\mathbb{D}}$ *is* $(\trianglelefteq -)$*convex, if and only if:* $\forall P_j \in \mathbb{P},\ P \in \tilde{\mathbb{D}}$

$$P_0 \trianglelefteq P \trianglelefteq P_1 \ (\textit{or equivalently, } P_0 \triangleleft P \triangleleft P_1) \implies P \in \mathbb{P}$$

However, due to the nature of **Fefferman's Trick**, it is necessary to extend our settings to include **spectral** dilation: given scales $\lambda,\ \lambda_j,\ \Lambda_j \in \mathbb{R}_+$ and tiles $P := I \times \omega,\ P_j := I_j \times \omega_j \in \tilde{\mathbb{D}}$, we define:

Definition 5.1.4 (Spectral dilation)**.**

$$\lambda P := I \times \lambda\omega, \quad \textit{where} \ \ \lambda\omega := \{\lambda(q - c_\omega) + c_\omega \in \mathcal{Q}_d \mid q \in \omega\}.$$

Since dilation destroy the hyper-adic structure, there are two variant analogues of $\trianglelefteq$ under such setting:

Definition 5.1.5 (**Order** and **order-like** relations on dilated tiles)**.**

$$\begin{cases} \lambda_0 P_0 \trianglelefteq \lambda_1 P_1 & \iff I_0 \subset I_1 \ \wedge \ \lambda_0\omega_0 \supset \lambda_1\omega_1 \\ \lambda_0 P_0 \leq \lambda_1 P_1 & \iff I_0 \subset I_1 \ \wedge \ \lambda_0\omega_0 \cap \lambda_1\omega_1 \neq \varnothing. \end{cases}$$

If, additionally, $I_0 \subsetneq I_1$*, we write* $\triangleleft$ *and* $<$ *instead. Also, we denote:*

$$\lambda_0 P_0 \sim \lambda_1 P_1 \iff (\lambda_0 P_0 \leq \lambda_1 P_1 \ \wedge \ \lambda_0 P_0 \geq \lambda_1 P_1)$$

Since $\leq$ does not satisfy **associative law**, some order construction will not work as we expected. Still, it reflects the incidental properties of dilated tiles:

$$\lambda_0 P_0 \cap \lambda_1 P_1 = \varnothing \iff \lambda_0 P_0,\ \lambda_1 P_1 \text{ are } \leq\text{-incomparable.}$$

Moreover, $\leq$ is only **a dilation away** from $\trianglelefteq$: by setting $\rho := \frac{1+\varsigma}{1/2-\varsigma}$, we have:

Lemma 5.1.6 (Order Upgrade Lemma)**.**
*Suppose the following **upgrade** condition is satisfied:*

$$(0 <) \ \frac{\Lambda_1 + \lambda_1}{\Lambda_0/\rho - \lambda_0} \leq 2^{\kappa^*\left(s_{P_1} - s_{P_0}\right)},$$

*we have the following upgrade from **order-like** relation to **true partial order**:*

$$\lambda_0 P_0 \leq \lambda_1 P_1 \implies \Lambda_0 P_0 \trianglelefteq \Lambda_1 P_1.$$

Proof. *Assume the upgrade condition, we see that:*

$$\lambda_0 P_0 \leq \lambda_1 P_1 \implies \exists q \in \lambda_0\omega_0 \cap \lambda_1\omega_1.$$

*Triangle inequality and **Embedding Inequality** yield:*

$$\begin{aligned} q_1 \in \Lambda_1\omega_1 \implies \|q_1 - c_{\omega_0}\|_{I_0} &\leq (\|q_1 - c_{\omega_1}\|_{I_0} + \|c_{\omega_1} - q\|_{I_0}) + \|q - c_{\omega_0}\|_{I_0} \\ &\leq 2^{-\kappa^*\left(s_{P_1} - s_{P_0}\right)} (\|q_1 - c_{\omega_1}\|_{I_1} + \|c_{\omega_1} - q\|_{I_1}) + \lambda_0(1+\varsigma) \\ &\leq 2^{-\kappa^*\left(s_{P_1} - s_{P_0}\right)} (\Lambda_1 + \lambda_1)(1+\varsigma) + \lambda_0(1+\varsigma) \\ &\leq \left(\Lambda_0/\rho - \cancel{\lambda_0}\right)(1+\varsigma) + \cancel{\lambda_0(1+\varsigma)} \leq \Lambda_0(1/2 - \varsigma). \end{aligned}$$

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Eventually, we have:

$$\Lambda_1\omega_1 \subset B_{I_0}\left(c_{\omega_0}, \Lambda_0(1/2-\varsigma)\right) \subset \Lambda_0\omega_0 \quad i.e. \quad \Lambda_0 P_0 \trianglelefteq \Lambda_1 P_1.$$

$\square$

Remark. *The Order Upgrade Lemma is especially useful when we are allowed to **tinker** with the size of κ^* (,by tuning κ). This is the main reason we, instead of a standard dyadic system, choose to work under a 2^κ-adic system.*

$(I_0 \subsetneq I_1)$: *Since $\rho \searrow 2$ as $\kappa^* \nearrow \infty$, we can always choose **large enough** κ^* to fulfill the **upgrade condition** as long as the dilation ratio of P_0 is slightly larger than 2. That is, given:*

$$\frac{\Lambda_0}{\lambda_0} > 2,$$

we always have:

$$\kappa^* \underset{\Lambda_j,\lambda_j}{\gg} 1 \implies \left(\frac{\Lambda_0}{\lambda_0} > \rho > 2 \ \wedge \ \frac{\Lambda_1+\lambda_1}{\Lambda_0/\rho-\lambda_0} \leq 2^{\kappa^*}\right).$$

$(I_0 = I_1)$: *Since 2^{κ^*}-factor on the **RHS** of the **upgrade condition** disappears, we require Λ_0 to be larger to fulfill the condition:*

$$\frac{\Lambda_0}{\lambda_0+\Lambda_1+\lambda_1} > 2$$

Then, tuning κ^* yields:

$$\kappa^* \underset{\Lambda_j,\lambda_j}{\gg} 1 \implies \frac{\Lambda_0}{\lambda_0+\Lambda_1+\lambda_1} \geq \rho > 2 \iff \frac{\Lambda_1+\lambda_1}{\Lambda_0/\rho-\lambda_0} \leq 1.$$

*Essentially, as long as we only do **finitely** many upgrades during the rest of the arguments, we only need to check* $\begin{cases} \frac{\Lambda_0}{\lambda_0} > 2 & (< \rightsquigarrow \triangleleft) \\ \frac{\Lambda_0}{\lambda_0+\Lambda_1+\lambda_1} > 2 & (\leq \rightsquigarrow \trianglelefteq) \end{cases}$ *without worrying about the size condition on κ^*.*

5.2 Geometric and Analytic Interaction

We explicitly define a way to measure the **distance** between a pair of tiles:

Definition 5.2.1 (distance$_{P_0,P_1}$ factor).

$$\Delta(P_0, P_1) := \inf_{q_j \in \omega_j} \|q_0 - q_1\|_{\tilde{I}_0 \cap \tilde{I}_1},$$

where we set $\|\cdot\|_\varnothing := \infty$ and $\tilde{I} := \left(n_D 2^{\kappa+1}+3\right) I$

This **quantify** the incidental properties on $\tilde{\mathbb{D}}$ in the following sense:

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Properties 5.2.2 (Proximity)**.**

$$(s_P \le s_{P'} \ \wedge \ \Delta(P,P') \lesssim \eta) \implies \|c_\omega - c_{\omega'}\|_I \underset{\kappa,D,d}{\lesssim} 1+\eta.$$

Proof. *For starters, we note that* $I \underset{\kappa,D}{\subsetneq} \tilde{I} \cap \tilde{I}' \underset{\kappa,D}{\subsetneq} I'$. *Therefore, by assumption,*

$$\exists\,(q,q') \in \omega \times \omega' \ s.t. \ \begin{cases} \|q-q'\|_I & \underset{\kappa,D,d}{\lesssim} \|q-q'\|_{\tilde{I}\cap\tilde{I}'} \lesssim \eta \\ \|q'-c_{\omega'}\|_I & \underset{\kappa,D,d}{\lesssim} \|q'-c_{\omega'}\|_{I'} < 1+\varsigma \\ \|q-c_\omega\|_I & < \ 1+\varsigma. \end{cases}$$

In conclusion, triangle inequality implies:

$$\|c_\omega - c_{\omega'}\|_I \underset{\kappa,D,d}{\lesssim} \eta + 2(1+\varsigma) \lesssim 1+\eta.$$

$\square$

Corollary 5.2.2.1 (Spectral packing constraint)**.**
Given $P' \in \tilde{\mathbb{D}}$ *and* $I \in \mathbb{D}_s$ *with* $s \le s_{P'}$ *and* $\tilde{I} \cap \tilde{I}_{P'} \ne \varnothing$, *we have:*

$$\#\left\{P \in \tilde{\mathbb{D}} \mid I_P = I \ \wedge \ \Delta(P,P') \lesssim \eta\right\} \underset{\kappa,D,d}{\lesssim} (1+\eta)^{dD},$$

where $dD := \frac{(D+d)!}{D!d!} - 1$.

Proof. *We first observe that the* ***LHS*** *equals:*

$$\#\left\{\omega \in \mathbb{D}_I^* \mid \Delta(I \times \omega, P') \lesssim \eta\right\}.$$

By ***Proximity*** *properties, we have: For some* $\lambda \underset{\kappa,D,d}{\lesssim} 1+\eta$,

$$\begin{aligned} &\left\{\omega \in \mathbb{D}_I^* \mid \Delta(I\times\omega, P') \lesssim 1\right\} \\ \subset &\left\{\omega \in \mathbb{D}_I^* \mid \|c_\omega - c_{\omega'}\|_I < \lambda\right\} \\ \subset &\left\{\omega \in \mathbb{D}_I^* \mid B_I(c_\omega, 1/2-\varsigma) \subset B_I(c_{\omega'}, \lambda + 1/2 - \varsigma)\right\}. \end{aligned}$$

The problem becomes measuring ***packing number****: the number of* ***disjoint*** *small balls* ***packed inside*** *a larger ball. Yet, due to the homogeneity of* $\|\cdot\|_{(\cdot)}$,

$$\begin{aligned} &B_I(c_\omega, 1/2-\varsigma) \subset B_I(c_{\omega'}, \Lambda + 1/2 - \varsigma) \\ &\rightsquigarrow B_{[0,1)^D}(c,1) \subset B_{[0,1)^D}(0,\Lambda), \quad \text{where } \ \Lambda = 1 + \frac{\lambda}{1/2-\varsigma}. \end{aligned}$$

Since the ***packing dimension*** *equals* $\dim \mathcal{Q}_d/_{\mathbb{R}} = dD$, *we have:*

$$\text{the } \textbf{packing number} \underset{D,d}{\lesssim} \Lambda^{dD} \underset{\kappa,D,d}{\lesssim} (1+\eta)^{dD}.$$

Thus, the result. $\square$

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Properties 5.2.3 (Almost comparability)**.**
For any $\gamma > 6$, we can take $\kappa^* \underset{\gamma}{\gg} 1$ such that:

$$(I_0 \subset I_1 \ \wedge \ \Delta(P_0, P_1) \lesssim 1) \implies \gamma P_0 \trianglelefteq P_1.$$

If, additionally, $I_0 \subsetneq I_1$, we only require $\gamma > 2$.

Proof. *Given $q \in \omega_1$ and $q_j \in \omega_j$, triangle inequality yields:*

$$\|q - c_{\omega_0}\|_{I_0} \leq \|q - q_1\|_{I_0} + \|q_1 - q_0\|_{I_0} + \|q_0 - c_{\omega_0}\|_{I_0}.$$

*Through **Embedding Inequality**, we have: for any $\epsilon > 0$,*

$$\kappa^* \underset{\epsilon}{\gg} 1 \implies \begin{cases} \|q - q_1\|_{I_0} & \leq \begin{cases} \cancelto{1+\varsigma}{\|q - c_{\omega_1}\|_{I_1}} + \cancelto{1+\varsigma}{\|q_1 - c_{\omega_1}\|_{I_1}} & I_0 \subset I_1 \\ 2^{-\kappa^*} \|q - q_1\|_{I_1} < \epsilon & I_0 \subsetneq I_1 \end{cases} \\ \|q_1 - q_0\|_{I_0} & \leq 2^{-\kappa^*} \|q_1 - q_0\|_{\tilde{I}_0} \searrow 2^{-\kappa^*} \Delta(P_0, P_1) < \epsilon \\ \|q_0 - c_{\omega_0}\|_{I_0} & < 1 + \varsigma. \end{cases}$$

As a result,

$$\|q - c_{\omega_0}\|_{I_0} < \begin{cases} 3 + 3\varsigma + \epsilon & I_0 \subset I_1 \\ 1 + \varsigma + 2\epsilon & I_0 \subsetneq I_1 \end{cases}$$

Therefore,

$$\omega_1 \subset \begin{cases} B_{I_0}\left(c_{\omega_0}, 3 + 3\varsigma + \epsilon\right) \subset \frac{3+3\varsigma+\epsilon}{1/2-\varsigma} \omega_0 & I_0 \subset I_1 \\ B_{I_0}\left(c_{\omega_0}, 1 + \varsigma + 2\epsilon\right) \subset \frac{1+\varsigma+2\epsilon}{1/2-\varsigma} \omega_0 & I_0 \subsetneq I_1 \end{cases}$$

Some fine tuning of $0 < \epsilon \underset{\gamma}{\ll} 1$ and $\kappa^* \underset{\gamma,\epsilon}{\gg} 1$ yields:

$$\begin{cases} 6 < \frac{3+3\varsigma+\epsilon}{1/2-\varsigma} \leq \gamma & I_0 \subset I_1 \\ 2 < \frac{1+\varsigma+2\epsilon}{1/2-\varsigma} \leq \gamma & I_0 \subsetneq I_1 \end{cases}$$

and, thus, $\gamma P_0 \triangleleft P_1$. $\square$

Moreover, we see that the geometric characterization interacts well with our partial order structure:

Properties 5.2.4 (Δ-monotonicity)**.**
By construction, we have:

$$P_0 \trianglelefteq P_1 \implies \Delta\left(P_0, P\right) \leq \Delta\left(P_1, P\right)$$

*Specifically, **Embedding Inequality** yields:*

$$(P_0 \trianglelefteq P_1 \ \wedge \ I_1 \subset I) \implies \Delta\left(P_0, P\right) \leq 2^{-\kappa^*\left(s_{P_1} - s_{P_0}\right)} \Delta\left(P_1, P\right)$$

Remark. *Essential, the **distance** factor, though itself does not satisfy **triangle inequality**, quantifies the following concepts:*

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- *The **incidental** relation between I_0 and I_1.*
- *The **spectral distance** between ω_0 and ω_1 measured through **smaller** scale of the two.*

The last piece of ingredients for Fefferman's Trick is to incorporate the **geometric structure** into the measurement of **density**. Given a **reference** of measurement $\Pi \subset \tilde{\mathbb{D}}$ and a prescribed **small** constant $\epsilon \in \mathbb{R}_+$, we consider:

Definition 5.2.5 (Π-relative density)**.**

$$\mathcal{A}_\Pi (P) := \sup_{\substack{\pi\in\Pi \\ I_P \subset I_\pi}} \mathcal{A}(\pi) \langle \Delta (P, \pi) \rangle^{\epsilon},$$

where we use the convention: $\sup \varnothing = 0$ *and* $\langle \cdot \rangle := \frac{1}{1+|\cdot|}$.

The **distance** factor reflects how far **off** the measurement is to the targeted tiles. Therefore, if we have good control on it, $\mathcal{A}_\Pi$ should behavior almost like $\mathcal{A}$. For instance, we may formulate the control in the following way:

Definition 5.2.6 (P-relevant Π-collection)**.**

$$\Pi_P := \{\pi \in \Pi \mid I_\pi \subsetneq I_P \ \wedge \ P \trianglelefteq \pi\}.$$

Properties 5.2.7 (Density recovery)**.**

$$P \subset \bigcup \Pi_P \implies \mathcal{A}(P) \underset{\kappa,D,d}{\lesssim} \mathcal{A}_\Pi(P).$$

Proof. *By construction, since:*

$$P \trianglelefteq \pi \implies \Delta(P,\pi) = 0 \quad \textit{and} \quad \#\{J \in \mathbb{D} \mid I_P \subset J \subsetneq I_P\} \underset{D}{\lesssim} 1,$$

spectral packing constraint *and inclusion implies:*

$$\#\Pi_P \underset{\kappa,D,d}{\lesssim} 1 \quad \textit{and} \quad |I_P| \underset{D}{\sim} |I_\pi|.$$

As a result,

$$\begin{aligned} P \subset \bigcup \Pi_P &\implies E_P \subset \bigcup_{\pi\in\Pi_P} E_\pi \\ &\implies |E_P| \le \sum_{\pi\in\Pi_P} |E_\pi| \le \#\Pi_P \max_{\pi\in\Pi_P} |E_\pi| \\ &\implies \mathcal{A}(P) \underset{\kappa,D,d}{\lesssim} \max_{\pi\in\Pi_P} \frac{|E_\pi|}{|I_\pi|} \le \mathcal{A}_\Pi(P). \end{aligned}$$

□

On the other hand, the corresponding monotonicity (with a flip of direction) follows directly from the construction and the **Δ-monotonicity**.

Properties 5.2.8 ($\mathcal{A}_\Pi$-monotonicity)**.**

$$P_0 \trianglelefteq P_1 \implies \mathcal{A}_\Pi(P_0) \ge \mathcal{A}_\Pi(P_1).$$

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5.3 Feffermann's Trick

Continuing previous settings, we now state Fefferman's Trick:

$$\mathbb{P} \rightsquigarrow \begin{cases} D\mathbb{P} = \bigsqcup_{k \lesssim m} D_k\mathbb{P}, \quad D_k\mathbb{P} \rightsquigarrow \begin{cases} L_k\mathbb{P} & 1 \lesssim \textbf{-apart clusters} \\ H_k\mathbb{P} & \textbf{high anti-chain} \end{cases} \\ E\mathbb{P} = \bigsqcup_{k \lesssim n} E_k\mathbb{P}, \quad \lesssim n \text{ layers } \textbf{of anti-chains} \end{cases}$$

1. Start with $\mathbb{P} \subset \tilde{\mathbb{D}}$ **convex**. Due to $\mathcal{A}_\Pi$**-monotonicity**, we can **isolate** tiles with a range of (Π-relative) density without disturbing the convexity and thus we may WLOG assume:

$$P \in \mathbb{P} \implies \textbf{upper bound} \geq \mathcal{A}_\Pi(P) > \textbf{lower bound} \geq 2^{-n}.$$

2. Organize tiles into layers of **anti-chains**:

$$\mathbb{P} = \bigsqcup_{k \in \mathbb{N}} M_k\mathbb{P}.$$

By construction, $\forall P \in M_{k+1}\mathbb{P}$, $\exists P_j \in M_j\mathbb{P}$ for $j \leq k$ such that:

$$P \lhd P_k \lhd P_{k-1} \lhd \cdots \lhd P_2 \lhd P_1,$$

and, by definition, $\exists \pi_1 \in \Pi$ such that:

$$I_1 \subset I_{\pi_1} \ \wedge \ \cancel{\mathcal{A}(\pi_1)} \langle \Delta\left(P_1, \pi_1\right)\rangle^\epsilon > 2^{-n}.$$

Focusing on the distance factor, Δ**-monotonicity** yields:

$$\Delta\left(P, \pi_1\right) \leq 2^{-\kappa^* k} \Delta\left(P_1, \pi_1\right) < 2^{-\kappa^* k}(2^{n/\epsilon} - 1) < 2^{n/\epsilon - \kappa^* k}.$$

As long as $k \gtrsim n$ and suitable $\kappa^* \gtrsim \epsilon^{-1}$, we always have: $\Delta(P, \pi_1) \lesssim 1$.

3. Fixing $\lambda > 2$, **Almost comparability** yields:

$$\begin{aligned} P \in \bigsqcup_{k \gtrsim n} M_k\mathbb{P} &\implies \exists \pi_1 \in \Pi \ s.t. \ \lambda P \lhd \pi_1 \\ &\iff \exists \pi \in M\Pi \ s.t. \ \lambda P \lhd \pi. \end{aligned}$$

We, therefore, can safely extract those Π**-comparable** tiles:

$$D\mathbb{P} := \{P \in \mathbb{P} \mid \exists \pi \in M\Pi \ s.t. \ \lambda P \lhd \pi\},$$

and the rest become $\lesssim n$**-layers of anti-chains**:

$$E\mathbb{P} = \bigsqcup_{k \lesssim n} E_k\mathbb{P}, \quad \text{where} \quad E_k\mathbb{P} := M_k\mathbb{P} \cap \mathbb{P} \setminus D\mathbb{P}.$$

Notice that, by definition, $D\mathbb{P}$ is still **convex**.

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4. Viewing $M\Pi$ as a counter, we keep track of the following values:
$$B(P) := \#\{\pi \in M\Pi \mid \lambda P \lhd \pi\}.$$
Given any $P \in D\mathbb{P}$ and $C \lesssim 1$ fixed, we have **qualitative** bound:
$$1 \le B(P) \le \left\| \sum_{\pi \in M\Pi} \chi_{I_\pi} \right\|_{L^\infty} \le \textbf{upper bound} \le Cm2^m \lesssim \overline{s} - \underline{s}.$$
Decompose $D\mathbb{P}$ accordingly, we have:
$$D\mathbb{P} = \bigsqcup_{k \lesssim m} D_k\mathbb{P}, \quad \text{where} \quad D_k\mathbb{P} := \left\{P \in D\mathbb{P} \mid B(P) \in [2^{k-1}, 2^k)\right\}.$$

5. Fixing $k \in \mathbb{N}$, we aim to extract $1 \lesssim$**-apart clusters** from $D_k\mathbb{P}$, where the two terminologies are explained as the followings:

Definition 5.3.1 (**Cluster** or **Tree** in [Lie20], [Fef73], and [Zor19])**.**
$\mathfrak{P} \subset \tilde{\mathbb{D}}$ be a cluster at $\mathfrak{p} \in \tilde{\mathbb{D}}$ if:

- *$\mathfrak{P}$ is **convex**.*
- $P \in \mathfrak{P} \implies \lambda P \lhd \mathfrak{p}$.

Definition 5.3.2 (Λ**-apartness**)**.**
*Given $\mathfrak{P}_j \subset \tilde{\mathbb{D}}$ associated with $\mathfrak{p}_j \in \tilde{\mathbb{D}}$, we say $\mathfrak{P}_0$ and $\mathfrak{P}_1$ are Λ-**apart** if:*
$$\{j,k\} = \{0,1\} \implies \forall P_j \in \mathfrak{P}_j,\ (I_j \subset I_{\mathfrak{p}_k} \implies \Delta(P_j, \mathfrak{p}_k) \ge \Lambda).$$

Now, for simplicity, we **suppress** the notation $\mathbb{P}$: $(\cdot)_k\mathbb{P} \rightsquigarrow (\cdot)_k$.

(a) Collect **maximal elements** under the **dilated** relation on D_k: given $P, P' \in D_k$, we write:
$$P\ \mathbf{Rel}_\lambda\ P' \iff \lambda P\ \mathbf{Rel}\ \lambda P'$$
as a shorthand for previously introduced order-like relations. We now collect $\le_\lambda$**-maximal elements** in the following sense:
$$D_k^\lambda := \{P \in D_k \mid \nexists P' \in D_k\ s.t.\ P <_\lambda P'\}.$$

(b) Extract **high part** from **low part**: fixing $\gamma > 2$,
$$\begin{cases} L_k & := \left\{P \in D_k \mid \exists P' \in D_k^\lambda\ s.t.\ \gamma P \lhd P'\right\} \\ H_k & := D_k \setminus L_k. \end{cases}$$
Notice that, by setting $\lambda > 2\gamma$, **Order Upgrade Lemma** yields:
$$\gamma P_0 \lhd P_1 \implies P_0 \lhd_\lambda P_1 \implies P_0 <_\lambda P_1,$$
and, thus, $L_k \cap D_k^\lambda = \varnothing$. That is, $D_k^\lambda \subset H_k$. For now, we can safely discard **unused** elements in D_k^λ:
$$T_k := \left\{P' \in D_k^\lambda \mid \exists P \in L_k\ s.t.\ \gamma P \lhd P'\right\}.$$

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(c) Check H_k is an **anti-chain**: given $P, P' \in H_k$,

$(P' \in D_k^\lambda)$: Since $\gamma > 2$, **Order Upgrade Lemma** yields:

$$P \lhd P' \implies \gamma P \lhd P'.$$

$(P' \notin D_k^\lambda)$: There is a chain $\{P_j\}_{j=1}^l \subset D_k$ such that:

$$P' <_\lambda P_l <_\lambda \cdots <_\lambda P_2 <_\lambda P_1 \in D_k^\lambda.$$

Order Upgrade Lemma yields:

$$P \lhd P' \implies \gamma P \lhd \gamma\lambda P' \lhd \gamma\lambda P_l \lhd \cdots \lhd \gamma\lambda P_2 \lhd \gamma\lambda P_1 \trianglelefteq P_1.$$

In both cases, $P \lhd P' \implies P \in L_k \Rightarrow\Leftarrow P \in H_k$. As a result, H_k must be an **anti-chain**.

(d) Augment **closeness** into relation on T_k: given $P_j' \in T_k$,

$$\begin{aligned} P_0' \prec P_1' &\overset{\textbf{def}}{\iff} \exists P_0 \in L_k \ s.t.\ \gamma P_0 \lhd P_0' \ \wedge\ \gamma P_0 \trianglelefteq P_1' \\ &\iff \exists P_0 \in L_k \ s.t.\ \gamma P_0 \lhd P_0' \ \wedge\ \gamma P_0 \lhd P_1' \end{aligned}$$

The latter **temporal equality** will never hold. **Otherwise**, we have:

$$P_1' \leq \gamma P_0 \lhd P_0'.$$

By setting $\frac{\lambda}{2\gamma+1} > 2$, **Order Upgrade Lemma** yields:

$$\lambda P_1' \trianglelefteq \gamma P_0 \lhd P_0' \sim \lambda P_0', \quad i.e.\ \ P_1' <_\lambda P_0',$$

which contradicts $P_1' \in T_k \subset D_k^\lambda$.

(e) **Closeness** implies **comparability** on T_k:

$$P_0' \prec P_1' \implies P_0' \sim_\lambda P_1'.$$

The reason is that $\frac{\lambda}{\gamma} > 2$ and **Order Upgrade Lemma** imply:

$$\gamma P_0 \lhd P_j' \implies P_0 \lhd_\lambda P_j'.$$

If $P_0' \nsim_\lambda P_1'$, then, since $P_j' := I_j' \times \omega_j' \in T_k \subset D_k^\lambda$, they must be $\leq_\lambda$**-incomparable**$(\lambda P_0' \cap \lambda P_1' = \varnothing)$. As a result,

$$\because I_0 \subset I_0' \cap I_1' \quad \therefore \lambda\omega_0' \cap \lambda\omega_1' = \varnothing.$$

However, a **combinatorial trick** yields a contradiction:

$$2^k > B(P_0) \geq B(P_0') + B(P_1') \geq 2 \cdot 2^{k-1} \Rightarrow\Leftarrow P_0, P_j' \in D_k.$$

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(f) $\sim_\lambda$ is an **equivalence** relation on T_k. **Reflexivity** and **Symmetry** are trivial, we check for **Transitivity**: Suppose $P'_j, P' \in T_k$, and

$$P' \sim_\lambda P'_j.$$

By definition, $\exists P \in L_k$ such that $\gamma P \lhd P'$, but, fixing $\Lambda > 6\lambda$, **Order Upgrade Lemma** yields:

$$\lambda P \lhd \Lambda P' \unlhd \lambda P'_j, \quad i.e. \quad P \lhd_\lambda P'_j.$$

Through previous **combinatorial trick**, we have $P'_0 \sim_\lambda P'_1$. We, therefore, mod out $\sim_\lambda$ and denote $\tau, \tau_j \in T_k^\lambda := T_k/_{\sim_\lambda}$.

(g) Verify **cluster** properties of the T_k^λ-indexed configuration:

$$\mathfrak{P}_\tau := \{P \in L_k \mid \exists P' \in \tau \ s.t. \ \gamma P \lhd P'\}.$$

- Check **convexity**: for $P_j \in \mathfrak{P}_\tau$, we consider:

$$P \in \tilde{\mathbb{D}} \ s.t. \ P_0 \lhd P \lhd P_1$$

$(P \in D)$: Since D is **convex**,

$$\because P_j \in \mathfrak{P}_\tau \subset D_k \subset D \quad \therefore P \in D.$$

$(P \in D_k)$: **Order Upgrade Lemma** implies:

$$\because P_0 \lhd_\lambda P \lhd_\lambda P_1 \quad \therefore 2^k > B(P_0) \geq B(P) \geq B(P_1) \geq 2^{k-1}$$

$(P \in \mathfrak{P}_\tau)$: **Order Upgrade Lemma** implies:

$$\exists P' \in \tau \ s.t. \ \gamma P_0 \lhd \gamma P \lhd \gamma P_1 \lhd P'$$

Therefore, $P \in \mathfrak{P}_\tau$, which means that $\mathfrak{P}_\tau$ is **convex**.

- Mark the **position** of $\mathfrak{P}_\tau$ with an arbitrary **cover** $\mathfrak{p}_\tau \in \tau$: **Order Upgrade Lemma** implies:

$$\forall P' \in \tau, \ \Lambda \mathfrak{p}_\tau \unlhd P'.$$

As a result,

$$\forall P \in \mathfrak{P}_\tau, \ \gamma P < \Lambda \mathfrak{p}_\tau.$$

We use **Order Upgrade Lemma** again:

$$\forall P \in \mathfrak{P}_\tau, \ \lambda P \lhd \mathfrak{p}_\tau.$$

In conclusion, $\mathfrak{P}_\tau$ is a **cluster** at $\mathfrak{p}_\tau$.

(h) Identify **cross-cluster** separation:

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- Check **disjointness**: Given any $P_j \in \mathfrak{P}_j := \mathfrak{P}_{\tau_j}$, we see that:

  $$P_0 = P_1 \implies \exists P_j' \in \tau_j \ s.t.\ \gamma P_0 = \gamma P_1 \lhd P_j' \ \ i.e.\ \ P_0' \prec P_1'.$$

  Therefore, if $\mathfrak{P}_0 \cap \mathfrak{P}_1 \neq \varnothing$, due to (e) and (f), we have:

  $$\exists P_j' \in \tau_j \ s.t.\ P_0' \sim_\lambda P_1' \implies \tau_0 = \tau_1 \implies \mathfrak{P}_0 = \mathfrak{P}_1.$$

- Verify **incomparability**: **Order Upgrade Lemma** yields:

  $$P_0 \trianglelefteq P_1 \iff P_0 \lhd P_1 \implies \exists P_1' \in \tau_1 \ s.t.\ \gamma P_0 \lhd \gamma P_1 \lhd P_1'.$$

  Again, using definition (d) and properties (e), (f), we have:

  $$P_0 \trianglelefteq P_1 \implies \tau_0 = \tau_1 \implies \mathfrak{P}_0 = \mathfrak{P}_1.$$

- Prove 1 $\lesssim$**-apartness**: Given any $(P_0, P_1') \in \mathfrak{P}_0 \times \tau_1$ such that $I_0 \subset I_1'$, **Almost comparability** implies:
  - If $I_0 \subsetneq I_1'$, since $\gamma > 2$, we have:

    $$\Delta\left(P_0, P_1'\right) \lesssim 1 \implies \gamma P_0 \lhd P_1' \implies \exists P_0' \in \tau_0 \ s.t.\ P_0' \prec P_1'.$$

    By (e) and (f), we see that $\tau_0 = \tau_1$. That is, $\mathfrak{P}_0 = \mathfrak{P}_1$.
  - If $I_0 = I_1'$, since $3\gamma > 6$, we have:

    $$\Delta\left(P_0, P_1'\right) \lesssim 1 \implies P_0 \trianglerighteq 3\gamma P_1'.$$

    By setting $\frac{\lambda}{4\gamma+1} > 2$, **Order Upgrade Lemma** implies:

    $$\exists P_0' \in \tau_0 \ s.t.\ \lambda P_1' \trianglelefteq \gamma P_0 \lhd P_0' \sim \lambda P_0' \Rightarrow\Leftarrow P_1' \in \tau_1 \subset T_k \subset D_k^\lambda$$

  In conclusion, distinct $\mathfrak{P}_j$ are 1 $\lesssim$**-apart**: Given that $\mathfrak{P}_0 \neq \mathfrak{P}_1$,

  $$\forall\left(P_0, P_1'\right) \in \mathfrak{P}_0 \times \tau_1,\ \left(I_0 \subset I_1' \implies \Delta\left(P_0, P_1'\right) \gtrsim 1\right).$$

(i) **Stack** the **covers**: for any $P' \in \tau$, we see that:

$$\because P' \sim_\lambda \mathfrak{p}_\tau \quad \therefore I' = I_{\mathfrak{p}_\tau} =: I_\tau.$$

We count how high I_τs stack via counting comparable πs in $M\Pi$:

$$B_\tau := \#\left\{\pi \in M\Pi \mid \exists P' \in \tau \ s.t.\ \lambda P' \lhd \pi\right\} \geq B(\mathfrak{p}_\tau) \geq 2^{k-1}.$$

By modding out $\sim_\lambda$, we prevent double counting and acquire:

$$2^{k-1} \sum_{\tau \in T_k^\lambda} \chi_{I_\tau} \leq \sum_{\tau \in T_k^\lambda} B_\tau \chi_{I_\tau} \leq \sum_{\pi \in M\Pi} \chi_{I_\pi} \leq Cm2^m.$$

In conclusion, the counting function satisfies the following control:

$$\sum_{\tau \in T_k^\lambda} \chi_{I_\tau} \leq Cm2^{m+1-k},$$

which measures the **height** the covers can **stack temporally**.

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(j) **Subdivide** collections with covers stack **too high**: This step is not mandatory. Still, it makes estimation cleaner. First off, we have:

$$k \geq 1 + \log_2 C + \log_2 m \implies \sum_{\tau \in T_k^\lambda} \chi_{I_\tau} \leq 2^m.$$

For $k < 1 + \log C + \log_2 m$, with careful selection, we can partition T_k^λ into $m_k := \lceil Cm2^{1-k} \rceil$ collections:

$$T_k^\lambda = \bigsqcup_{j=1}^{m_k} T_{k,j}^\lambda, \quad \text{where} \quad \sum_{\tau \in T_{j,k}^\lambda} \chi_{I_\tau} \leq 2^m, \quad \forall j.$$

We now reorganize the corresponding clusters:

$$L_{k,j} := \bigsqcup_{\tau \in T_{k,j}^\lambda} \mathfrak{P}_\tau$$

By moving cluster as a whole, we do not destroy any previously established structure. Therefore, $L_{k,j}$ still contains 1 $\lesssim$**-apart** clusters. Lastly, We count the number of layers:

$$\log_2 \left(\cancel{Cm} 2^m \right) - \cancel{\log_2 C - \log_2 m} + \sum_{k < 1 + \log_2 C + \log_2 m} m_k \lesssim m$$

As a result, since the number stays morally the same, we might as well renumber the index: $(\cdot)_{k,j} \rightsquigarrow (\cdot)_k$ and thus:

$$\sum_{\tau \in T_k^\lambda} \chi_{I_\tau} \leq 2^m, \quad \forall k \lesssim m$$

Eventually, we summarize that $L_k\mathbb{P}$ has the following structure:

Definition 5.3.3 (**Λ-apart Ξ-stack** or L^∞ **Forest** in [Lie20] or **Fefferman forest** in [Zor19])**.**
*$\mathbb{P} \subset \tilde{\mathbb{D}}$ is a **Λ-apart Ξ-stack** if it is a collection of **clusters**:*

$$\mathbb{P} = \bigsqcup_j \mathfrak{P}_j \quad \wedge \quad \forall j, (P \in \mathfrak{P}_j \implies \lambda P \lhd \mathfrak{p}_j),$$

which satisfies the following properties:

- ***Height Control**: $\sum_j \chi_{I_{\mathfrak{p}_j}} \leq \Xi$.*
- ***Cross-Cluster Separation**: Given distinct $\mathfrak{P}_j$ and $\mathfrak{P}_k$,*
  - *$P_j \in \mathfrak{P}_j$ and $P_k \in \mathfrak{P}_k$ are **$\trianglelefteq$-incomparable**.*
  - *$\mathfrak{P}_j$ and $\mathfrak{P}_k$ are **Λ-apart**.*

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We formulate our current progress as the following lemma:

Lemma 5.3.4 (Fefferman's Trick)**.**
Given $\lambda > 18$, $\mathbb{P} \subset \tilde{\mathbb{D}}$ convex, and $\Pi \subset \tilde{\mathbb{D}}$ such that:

- ***Lower bound** on Π-**relative density** of $\mathbb{P}$:*

$$P \in \mathbb{P} \implies \textbf{\textit{Const.}} \geq \mathcal{A}_\Pi(P) > 2^{-n}$$

- ***Upper bound** on **temporal overlap** of $M\Pi$:*

$$M_\Pi := \sum_{\pi \in M\Pi} \chi_{I_\pi} \lesssim m2^m,$$

we may choose $\kappa^* \underset{\lambda}{\gg} 1$ such that $\mathbb{P}$ can be decomposed into:

- *$\lesssim n + m$ layers of **anti-chains**: $\{E_k\mathbb{P}\}_{k \lesssim n}$ and $\{H_k\mathbb{P}\}_{k \lesssim m}$*
- *$\lesssim m$ layers of $1 \lesssim$-**apart** 2^m-**stacks**: $\{L_k\mathbb{P}\}_{k \lesssim m}$*

5.4 Boundary Removal

To exclude bad behaviors when tiles get **temporally dilated** (as in the Trivial Estimate) while doing the TT^*-T^*T argument, we need careful treatment on the following configurations:

Definition 5.4.1 (**Interior** and **Boundary**)**.**
Fixing $\varpi \gg 1$ as a buffer, given $\mathfrak{P} \subset \tilde{\mathbb{D}}$ a cluster at $\mathfrak{p} \in \tilde{\mathbb{D}}$, we set:

$$\mathfrak{P}^\circ := \left\{ P \in \mathfrak{P} \mid \varpi \tilde{I}_P \subset I_\mathfrak{p} \right\} \quad \textit{and} \quad \partial\mathfrak{P} := \mathfrak{P} \setminus \mathfrak{P}^\circ.$$

*Notice that both $\mathfrak{P}^\circ$ and $\partial\mathfrak{P}$ are cluster at $\mathfrak{p}$ since the **temporal** operation preserves **convexity** and **location**. As a result, we say:*

$$\begin{cases} \mathfrak{P} \textit{ is an } \textbf{\textit{open}} \textit{ cluster} & \textit{if } \mathfrak{P} = \mathfrak{P}^\circ \\ \mathfrak{P} \textit{ is an } \textbf{\textit{boundary}} \textit{ cluster} & \textit{if } \mathfrak{P} = \partial\mathfrak{P}. \end{cases}$$

*We also extend the terminology to **collections** of clusters: Given $\mathbb{P} \subset \tilde{\mathbb{D}}$ a collection of clusters:*

$$\mathbb{P} = \bigsqcup_j \mathfrak{P}_j \quad \wedge \quad \forall j, (P \in \mathfrak{P}_j \implies \lambda P \lhd \mathfrak{p}_j),$$

we set:

$$\mathbb{P}^\circ := \bigsqcup_j \mathfrak{P}_j^\circ \quad \textit{and} \quad \partial\mathbb{P} := \bigsqcup_j \partial\mathfrak{P} = \mathbb{P} \setminus \mathbb{P}^\circ.$$

*Similarly, we say $\mathbb{P}$ is **open** if $\mathbb{P} = \mathbb{P}^\circ$.*

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For convenience, we introduce the following notion to focus on the **temporal aspect** of the structure.

Definition 5.4.2 (Temporal projection).
*Given $\mathbb{P} \subset \tilde{\mathbb{D}}$, we define its **temporal projection** as:*

$$\mathbb{I}_{\mathbb{P}} := \{I_P \in \mathbb{D} \mid P \in \mathbb{P}\} \quad \textit{and} \quad \mathbb{I}_{\mathbb{P},s} := \mathbb{I}_{\mathbb{P}} \cap \mathbb{D}_s.$$

Boundary cluster is the culprit we need to deal with. Yet, an easy verification shows the following temporal properties:

Properties 5.4.3 (Tooth configuration).
*Given a **boundary** cluster $\mathfrak{P}$, there is $s_\Delta \underset{\varpi,D}{\approx} 1$ such that:*

$$s' - s \geq s_\Delta \implies \forall J \in \mathbb{D}_{s'}, \sum_{\substack{I \in \mathbb{I}_{\mathfrak{P},s} \\ I \subset J}} |I| \leq 2^{-\kappa}|J|.$$

Remark. *The name chosen is because of the shape it formed ($D = 1$) when drawing $\mathbb{I}_{\mathfrak{P},s}$ horizontally and stacking $\mathbb{I}_{\mathfrak{P},s}$s vertically.*

We see that the **tooth configuration** almost screams **sparsity**. As a result, we shall expect the following configurations:

Definition 5.4.4 (**Λ-decay stack** or **Sparse L^∞ Forest** in [Lie20]).
*$\mathbb{P} \subset \tilde{\mathbb{D}}$ is a **Λ-decay stack** if:*

$$s' - s \geq \Lambda \implies \forall J \in \mathbb{D}_{s'}, \sum_{\substack{I \in \mathbb{I}_{\mathbb{P},s} \\ I \subset J}} |I| \leq 2^{-\kappa}|J|.$$

*A direct computation shows that $\mathbb{I}_{\mathbb{P}}$ is also $\lesssim$ **Λ-carleson**.*

Putting things in action, we have:

Lemma 5.4.5 (Boundary removal).
*A **Λ-apart Ξ-stack** $\mathbb{P}$ can be decomposed into:*

$$\mathbb{P} \rightsquigarrow \begin{cases} \partial\mathbb{P} & \underset{\varpi,D}{\lesssim} 1 + \frac{\log_2 \Xi}{\kappa}\textbf{\textit{-decay stack}} \\ \mathbb{P}^\circ & \textbf{\textit{open }}\Lambda\textbf{\textit{-apart }}\Xi\textbf{\textit{-stack}}. \end{cases}$$

Proof. *For starters, we notice that the **temporal operation** does **not affect** the **spectral behaviors** of the clusters **nor** the **covers**' configurations. That is, trivially, $\mathbb{P}^\circ$ is an **open Λ-apart Ξ-stack**. We now check the **temporal property** of $\partial\mathbb{P}$. Fixing $N := s_\Delta \left\lceil 1 + \frac{\log_2 \Xi}{\kappa} \right\rceil \underset{\varpi,D}{\lesssim} 1 + \frac{\log_2 \Xi}{\kappa}$, an easy computation shows that: Given $s' - s \geq N$, due to **height control** and properties of **tooth configuration**, we have:*

$$\forall J \in \mathbb{D}_{s'}, \sum_{\substack{I \in \mathbb{I}_{\partial\mathbb{P},s} \\ I \subset J}} |I| \leq \overset{\textbf{\textit{up to }}\Xi\textbf{\textit{ overlaps}}}{\cancel{\sum_j}} \sum_{\substack{I \in \mathbb{I}_{\partial\mathfrak{P}_j,s} \\ I \subset J}} |I| \leq \Xi 2^{-N\kappa}|J| \leq 2^{-\kappa}|J|.$$

□

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5.5 Separation Upgrade

To compensate for the **height** the covers stack, we expect to gain enough **decay** from **orthogonality** when **clusters** are mutually **far apart**. To achieve this, we present the following lemma:

Lemma 5.5.1 (Separation upgrade)**.**
*A Λ-**apart** Ξ-**tack** $\mathbb{P}$ can be decomposed into:*

$$\mathbb{P} \rightsquigarrow \begin{cases} m\mathbb{P} & \textbf{\textit{anti-chain}} \\ \mathbb{P} \setminus m\mathbb{P} & 2^{\kappa^*}\Lambda\textbf{\textit{-apart}}\ \Xi\textbf{\textit{-Stack}} \end{cases}$$

Proof. *Trivially, $m\mathbb{P}$ is, by construction, an **anti-chain**. On the other hand,*

$$\overbrace{\mathbb{P} \setminus m\mathbb{P}}^{\mathbb{P}':=} = \bigsqcup_j (\overbrace{\mathfrak{P}_j \setminus m\mathfrak{P}_j}^{\mathfrak{P}'_j:=}).$$

*Excluding possible empty clusters, we notice that the operation does not affect the location (**cover** $\mathfrak{p}_j$) of the clusters, **Incomparablity** ,and **Height Control**. Therefore, we only need to verify the **Apartness**: Since*

$$\forall P'_j \in \mathfrak{P}'_j,\ \exists P_j \in m\mathfrak{P}_j \ s.t.\ P_j \lhd P'_j,$$

*by Δ-**monotonicity**, we have: for any $k \neq j$ such that $I'_j \subset I_{\mathfrak{p}_k}$,*

$$\because I_j \subsetneq I'_j \subset I_{\mathfrak{p}_k} \quad \therefore \Delta\left(P'_j, \mathfrak{p}_k\right) \geq 2^{\kappa^*}\Delta\left(P_j, \mathfrak{p}_k\right) \geq 2^{\kappa^*}\Lambda.$$

$\square$

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6 Search for Good Trades

With the tools established in previous sections, we can organize **tiles** into several **well-behaved** configurations. Yet, to put things together, we need to:

- **Choose** suitable collection of $\mathbb{P}$s and Πs to start with.
- **Combine** all the tools smartly.
- **Balance** the **trade-off** among different aspects of the control.
- **Sum** up all the contributions.

In this section, we first demonstrate the delicate phenomenon among the **trade-offs** and mention a problem encountered in Fefferman's original treatment [Fef73]. Next, we provide the insight of Lie's solution in [Lie20] and Zorin-Kranich's modification in [Zor19]. Lastly, we construct **explicitly** the collection of Πs through an **elementary model**.

6.1 Trade-off: Polynomial v.s. Exponential

Let us start from the following assumptions: $\mathbb{P} \subset \tilde{\mathbb{D}}$ convex and $\Pi \subset \tilde{\mathbb{D}}$,

- **Π-relative density**: $P \in \mathbb{P} \implies \mathcal{A}_\Pi(P) \in \left(2^{-n}, 2^{1-n}\right]$.
- **Temporal overlap**: $M_\Pi := \sum_{\pi \in M\Pi} \chi_{I_\pi} \lesssim m2^m$.

We combine the three lemmas:

- **Fefferman's Trick**: with $\kappa \underset{\lambda}{\gg} 1$,

$$\mathbb{P} \rightsquigarrow \begin{cases} \bigsqcup_{k \lesssim n} E_k\mathbb{P} \sqcup \bigsqcup_{k \lesssim m} H_k\mathbb{P} & \lesssim n+m \text{ layers of } \textbf{anti-chains} \\ \qquad \bigsqcup_{k \lesssim m} L_k\mathbb{P} & \lesssim m \text{ layers of } 1 \lesssim \textbf{-apart } 2^m\textbf{-stacks} \end{cases}$$

- **Boundary removal**: for all $L_k\mathbb{P}$,

$$L_k\mathbb{P} \rightsquigarrow \begin{cases} \partial L_k\mathbb{P} & \underset{D}{\lesssim} 1 + \frac{m}{\kappa}\textbf{-decay stack} \\ L_k\mathbb{P}^\circ & \textbf{open } 1 \lesssim \textbf{-apart } 2^m\textbf{-stack} \end{cases}$$

- **Separation upgrade**: for all $L_k\mathbb{P}^\circ$ (apply iteratively),

$$L_k\mathbb{P}^\circ \rightsquigarrow \begin{cases} \bigsqcup_{j \leq l} m_j L_k\mathbb{P}^\circ & l \text{ layers of } \textbf{anti-chain} \\ \textbf{Else} =: L_k^l\mathbb{P}^\circ & \textbf{open } 2^{l\kappa^*} \lesssim \textbf{-apart } 2^m\textbf{-stack} \end{cases}$$

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As a result, we have the **decomposition scheme** on $(\mathbb{P}, \Pi)$:

- **Sparse Parts**:
  - $\lesssim n + m(l+1)$ layers of **anti-chains**:

$$\{E_k\mathbb{P}\}_{k\lesssim n}, \quad \{H_k\mathbb{P}\}_{k\lesssim m}, \quad \text{and} \quad \{m_j L_k \mathbb{P}^\circ\}_{j\le l,\ k\lesssim m}.$$

  - $\lesssim m$ layers of $\underset{D}{\lesssim} 1 + \frac{m}{\kappa}$**-decay stacks**: $\{\partial L_k\mathbb{P}\}_{k\lesssim m}$.

- **Cluster Parts**:
  - $\lesssim m$ layers of **open** $2^{l\kappa}$ $\lesssim$**-apart** 2^m**-stacks**: $\left\{L_k^l\mathbb{P}^\circ\right\}_{k\lesssim m}$.

A natural strategy is to:

- Extract **exponential decay** of the **density factor**$\eqsim 2^{-n}$ out of all the estimation (as in Trivial Estimate) to **absorb polynomial growth** of the number of **layers** and **sparsity factor**.
- Use large **separation** to compensate for high **temporal overlaps** when using T^*T-TT^* argument.

In summary, we should aim for $m \ll l \lesssim n$. Indeed,

$$\textbf{Polynomial} \leftrightsquigarrow \textbf{Exponential}$$

$$\begin{cases} \#\textbf{Layers}(Bad) \\ \textbf{Sparsity}(Bad) \end{cases} \leftrightsquigarrow \begin{cases} \#\textbf{Overlaps}(Bad) \\ \textbf{Density}(Good) \\ \textbf{Separation}(Good) \end{cases}$$

It is, for the most part, a good trade:

$$\textbf{Decomposition} \implies \textbf{Polynomial Growth} \times \textbf{Exponential Decay}.$$

However, the assumption itself hides a **counteracting theme**. To find suitable $(\mathbb{P}, \Pi)$, we need to find balance within the following conflicts:

$$\textbf{Temporal overlap} \Rightarrow\Leftarrow 2^{-n}\textbf{-dense collection}$$
$$\downarrow M_\Pi \Rightarrow\Leftarrow \#\mathbb{P}\uparrow.$$

Our first attempt might start with **discarding irrelevant** $\pi \in \Pi$. In fact, since the **distance factor** within the definition of $\mathcal{A}_\Pi$ only provides decay, it follows that we have:

Properties 6.1.1 (Equivalent reference)**.**
Given $\mathbb{P}, \Pi \subset \tilde{\mathbb{D}}$ *such that* $P \in \mathbb{P} \implies \mathcal{A}_\Pi(P) > \eta$, *we have:*

$$\forall P \in \mathbb{P},\ \mathcal{A}_\Pi(P) = \mathcal{A}_{\Pi_\eta}(P) \quad \textit{with} \quad \Pi_\eta := \{\pi \in \Pi \mid \mathcal{A}(\pi) > \eta\}.$$

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In our case, we derive a natural assumption:

$$\pi \in \Pi \implies \mathcal{A}(\pi) > 2^{-n}.$$

Still, we need to check if trimming down Π would actually make M_Π smaller:

Properties 6.1.2 (M_Π-monotonicity)**.**

$$\Pi_0 \subset \Pi_1 \subset \tilde{\mathbb{D}} \implies M_{\Pi_0} \leq M_{\Pi_1}.$$

Proof. *Suppose the otherwise:*

$$\exists x \in \mathbb{R}^D \text{ s.t. } M_{\Pi_0}(x) > M_{\Pi_1}(x).$$

By ***Pigeon-hole principle****, there must be* **distinct** $\pi_0, \pi_0' \in M\Pi_0$ *such that:*

$$x \in I_{\pi_0} \cap I_{\pi_0'} \neq \varnothing \ \wedge \ \exists \pi_1 \in M\Pi_1, \text{ s.t. } \pi_0, \pi_0' \trianglelefteq \pi_1.$$

However, since π_0, π_0' *are* ***$\trianglelefteq$-incomparable****, we must have:*

$$c_{\omega_{\pi_1}} \in \omega_{\pi_1} \subset \omega_{\pi_0} \cap \omega_{\pi_0'} = \varnothing$$

which is a contradiction. □

Meanwhile, we can **locate** all the **high overlaps**:

$$E := \left\{ x \in \mathbb{R}^D \mid M_\Pi(x) \gtrsim m2^m \right\}.$$

Naturally, references **temporally** in E are those who cause the **overshoot**, and we shall exclude them: $\Pi^+ \subset \Pi \setminus \overbrace{\{\pi \in \Pi \mid I_\pi \subset E\}}^{\textbf{bad references}}$. By **$M_\Pi$-monotonicity**, we can control the **temporal overlap**:

$$M_{\Pi^+} \leq M_{\Pi \setminus \{\pi \in \Pi \mid I_\pi \subset E\}} = \sum_{\substack{\pi \in M\Pi \\ I_\pi \not\subset E}} \chi_{I_\pi} \lesssim m2^m.$$

However, **discarding** the **bad references** would result in a **decay** of density when tiles being measured. Therefore, we should modify $\mathbb{P}$:

$$\mathbb{P}^+ \subset \left\{ P \in \mathbb{P} \mid \mathcal{A}_{\Pi^+}(P) > 2^{-n} \right\} \setminus \{P \in \mathbb{P} \mid I_P \subset E\} \quad \text{and is } \textbf{convex}.$$

By construction, $(\mathbb{P}^+, \Pi^+)$ satisfies our assumptions and can be treated with our **decomposition scheme**. The rest is to derive control on $\mathbb{P}^- := \mathbb{P} \setminus \mathbb{P}^+$. In general, we expect that $P \in \mathbb{P}^-$ has **low relative density** or is **temporally contained** in E. Thus, a good control on E would always be helpful. Yet, as we trace back its construction: $E \rightsquigarrow M_\Pi \rightsquigarrow M\Pi$, we see that a deeper understanding of the structure of $M\Pi$ is needed. For instance, our natural assumption, with **double counting** taken into consideration, actually implies the **2^{-n}-sparse** condition:

$$\begin{cases} |I_\pi| \leq 2^n |E_\pi| & \forall \pi \in M\Pi \\ \{E_\pi\}_{\pi \in M\Pi} & \text{are disjoint,} \end{cases}$$

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or, equivalently, the 2^n**-carleson** condition:

$$\forall I \in \mathbb{D}, \sum_{\substack{\pi \in M\Pi \\ I_\pi \subset I}} |I_\pi| \leq 2^n |I|.$$

This implicitly gives us structures on E:

$$2^n\textbf{-carleson} \rightsquigarrow \textbf{control on } M_\Pi \rightsquigarrow \textbf{control on } E$$

and may shade some light on the treatment for $\mathbb{P}^-$.

6.2 Charles Fefferman's Exceptional Set

Using language established, we explain Fefferman's idea. In [Fef73], Fefferman analyzed **Carleson operator** under **torus** $\mathbb{T} \simeq [0,1)$ settings. We first organized tiles according to $\tilde{\mathbb{D}}$**-relative density**:

$$\tilde{\mathbb{D}} = \bigsqcup_{n \in \mathbb{N}} \mathbb{P}_n, \quad \text{where } \mathbb{P}_n := \left\{ P \in \tilde{\mathbb{D}} \mid \mathcal{A}_{\tilde{\mathbb{D}}}(P) \in \left(2^{-n}, 2^{1-n}\right] \right\}.$$

Using $\mathbb{P}_n$'s equivalent reference:

$$\Pi_n := \left\{ \pi \in \tilde{\mathbb{D}} \mid \mathcal{A}(\pi) > 2^{-n} \right\}$$

paired with previous discussion:

$$\|M_{\Pi_n}\|_{L^1} = \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset \mathbb{T}}} |I_\pi| \leq 2^n,$$

we may apply **Markov's inequality** to derive: $\mu\left(M_{\Pi_n}^{-1}(\eta, \infty]\right) \leq 2^n/\eta$. Therefore, if we choose $\eta = m2^m$ and define the **Exceptional Set** as:

$$E_{n,m} := M_{\Pi_n}^{-1}(m2^m, \infty],$$

we can exclude $\Pi_{n,m}^- := \{\pi \in \Pi_n \mid I_\pi \subset E_{n,m}\}$, all tiles causing the overshoot, from Π_n and verify that:

$$M_{\Pi_{n,m}^+} \leq m2^m, \quad \text{where } \Pi_{n,m}^+ := \Pi_n \setminus \Pi_{n,m}^-.$$

In conclusion, we have **height** control on I_πs not contained in $E_{n,m}$ and **support** control on $E_{n,m}$. Therefore, as we modify $\mathbb{P}_n$ accordingly:

$$\mathbb{P}_{n,m}^- := \{P \in \mathbb{P}_n \mid I_P \subset E_{n,m}\} \quad \text{and} \quad \mathbb{P}_{n,m}^+ := \mathbb{P}_n \setminus \mathbb{P}_{n,m}^-,$$

we must have:

$$P \in \mathbb{P}_{n,m}^+ \implies \mathcal{A}_{\Pi_{n,m}^+}(P) = \mathcal{A}_{\Pi_n}(P) \in \left(2^{-n}, 2^{1-n}\right].$$

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We can apply the **decomposition scheme** on $(\mathbb{P}^+_{n,m}, \Pi^+_{n,m})$s and expect that:

$$\left\|\mathfrak{L}_{\mathbb{P}^+_{n,m}} f\right\|_{L^2} \lesssim p(n,m)2^{-\epsilon n} \|f\|_{L^2},$$

for some small $\epsilon \in \mathbb{R}_+$ and a polynomial $p(\cdot,\cdot)$. On the other hand,

$$\mu\Big(\overbrace{\mathrm{supp}\mathfrak{L}_{\mathbb{P}^-_{n,m}} f}^{\subset E_{n,m}}\Big) \leq m^{-1}2^{n-m}.$$

Combining both in the form of distributional estimate, we get:

$$\begin{aligned}\mu\left(\left|\mathfrak{L}_{\mathbb{P}_n} f\right|^{-1}(\eta,\infty]\right) \leq& \mu\left(\left|\mathfrak{L}_{\mathbb{P}^+_{n,m}} f\right|^{-1}(\eta,\infty]\right) + \mu\left(\left|\mathfrak{L}_{\mathbb{P}^-_{n,m}} f\right|^{-1}(0,\infty]\right)\\ \lesssim& \left(p(n,m)2^{-\epsilon n}\frac{\|f\|_{L^2}}{\eta}\right)^2 + m^{-1}2^{n-m}.\end{aligned}$$

Unfortunately, through minimizing the **RHS**, we can only derive $L^2 \to L^{2-\epsilon}$ bound. To make matters worse, we rely on the **finite measure** structure on $\mathbb{T}$ to control the **exceptional set**. This prevents us an easy adaptation from $\mathbb{T}$ settings to $\mathbb{R}^D$ settings. Alternatively, this shows that a possible path to tackle the issue is to **localize** the analysis on the level set. That is, we aim for good control on:

$$I \cap M^{-1}_{\Pi_n}(m2^m,\infty], \quad \text{for various} \quad I \in \mathbb{D}.$$

6.3 Victor Lie's Stopping Collection

Continuing previous discussion, our goal is to do finer estimate on the level set. In [Lie20], Lie's innovation is the use of the **John-Nirenberg inequality** on his inductive construction. We give our interpretation of his treatments. For starters, we observe that:

Observation. ***Carlson packing*** *condition implies the boundedness of* 2^κ***-adic BMO norm*** *of the corresponding* ***counting function****.*

Using similar settings: Given $\mathbb{P}_n \subset \tilde{\mathbb{D}}$ convex and $\Pi_n \subset \tilde{\mathbb{D}}$ such that

- $P \in \mathbb{P}_n \implies \mathcal{A}_{\Pi_n}(P) \in \left(2^{-n}, 2^{1-n}\right]$,
- $\pi \in \Pi_n \implies \mathcal{A}(\pi) > 2^{-n}$,

we see that $\{I_\pi\}_{\pi \in M\Pi_n}$ is 2^n**-carleson** (counting with multiplicity), and thus,

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for any $I \in \mathbb{D}$, we have:

$$
\begin{aligned}
\chi_I \left(M_{\Pi_n} - |M_{\Pi_n}|_I \right) = & \cancel{\sum_{\substack{\pi \in M\Pi_n \\ I_\pi \supsetneq I}} \chi_{I \cap \not{I}_\pi} - \fint_I \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \supsetneq I}} \chi_{I \cap \not{I}_\pi} d\mu \cdot \chi_I} \\
& + \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{\not{I} \cap I_\pi} - \fint_I \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{\not{I} \cap I_\pi} d\mu \cdot \chi_I \\
& = \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{I_\pi} - |I|^{-1} \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} |I_\pi| \cdot \chi_I .
\end{aligned}
$$

Therefore, doing another average, we have:

$$
|M_{\Pi_n} - |M_{\Pi_n}|_I|_I \lesssim |I|^{-1} \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} |I_\pi| \leq 2^n .
$$

That is, we conclude that: $\|M_{\Pi_n}\|_{BMO_\Delta} \lesssim 2^n$. Now, we may apply **John-Nirenberg inequality**: For some $c_j \underset{\kappa, D}{\lesssim} 1$,

$$
\begin{aligned}
& \left| \left\{ x \in I \mid \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{I_\pi} > \eta \right\} \right| \\
\leq & \left| \left\{ x \in I \mid \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{I_\pi} - |I|^{-1} \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} |I_\pi| > \eta - 2^n \right\} \right| \\
\leq & \left| \left\{ x \in I \mid |M_{\Pi_n}(x) - |M_{\Pi_n}|_I| > \eta - 2^n \right\} \right| \\
\leq & e^{c_0 - \frac{\eta - 2^n}{c_1 2^n}} |I| \leq e^{c_0 + 1/c_1 - 2^{-n}\eta / c_1} |I| .
\end{aligned}
$$

Consequently, for any $C \gg c_1(c_0+1)$, there is $\Lambda \underset{\kappa, D}{\gtrsim} C$ such that:

$$
\left| \left\{ x \in I \mid \sum_{\substack{\pi \in M\Pi_n \\ I_\pi \subset I}} \chi_{I_\pi} > Cn2^n \right\} \right| \leq e^{-\Lambda n} |I| .
$$

In particular, if $(\pi, I) \in \Pi_n \times \mathbb{A}$, either $I_\pi \subset I$ or $I_\pi \cap I = \varnothing$, for example:

$$
\mathbb{A} := M \left\{ I_\pi \in \mathbb{D} \mid \pi \in \Pi_n \right\},
$$

we always have:

$$
|I \cap E| \leq e^{-\Lambda n} |I|, \quad \text{where} \quad E := M_{\Pi_n}^{-1} \left(Cn2^n, \infty \right].
$$

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In short, **John-Nirenberg inequality** yields a much stronger decay and more localized control than **Markov's inequality** does. With that in mind, we now modify $(\mathbb{P}_n, \Pi_n)$ accordingly:

$$\begin{cases} \Pi_n^- := \{\pi \in \Pi_n \mid I_\pi \subset E\} & \text{and } \ \Pi_n^+ := \Pi_n \setminus \Pi_n^- \\ \mathbb{P}_n^- := \{P \in \mathbb{P}_n \mid I_P \subset E\} & \text{and } \ \mathbb{P}_n^+ := \mathbb{P}_n \setminus \mathbb{P}_n^-. \end{cases}$$

In conclusion,

- $(\mathbb{P}_n^+, \Pi_n^+)$ can be treated with the **decomposition scheme**.
- E can be decomposed into a disjoint collection of 2^κ**-adic** cubes:

$$\mathbb{A}^- := M\left\{I \in \mathbb{D} \mid I \subset E\right\}.$$

- both $\mathbb{P}_n^+$ and $\mathbb{P}_n^-$ have support control: $\begin{cases} P \in \mathbb{P}_n^+ & \implies I_P \in \mathbb{A}^{\subset} \setminus \mathbb{A}^{-\subset} \\ P \in \mathbb{P}_n^- & \implies I_P \in \mathbb{A}^{-\subset} \end{cases}.$
  Therefore, if our estimate preserves the structure of the **support control** we might be able to benefit from its **decay**.
- we shall analyze $\mathbb{P}_n^-$ with compatible references: Π_n^-, but some **decay of density** might happen. Thus, we need further treatment so that we can apply our arguments iteratively.

Still, with some tweaking, we can inductively build up the collection of $(\mathbb{P}, \Pi)$s. The following is a sketch of the method in [Lie20]:

1. Starting with $n = 1$, we first collect 2^{-1}**-dense** tiles:

$$\mathbb{P}_{res(0)} := \left\{P \in \tilde{\mathbb{D}} \mid \mathcal{A}_{\tilde{\mathbb{D}}}(P) \in \left(2^{-1}, 1\right]\right\},$$

   equivalent references, and **default cubes**:

$$\Pi_{res(0)} := \left\{\pi \in \tilde{\mathbb{D}} \mid \mathcal{A}(\pi) > 2^{-1}\right\} \quad \text{and} \quad \mathbb{A}_0 := \mathbb{D}_{\overline{s}}.$$

2. Define inductively $(\mathbb{P}_k, \Pi_k, \mathbb{A}_k) := \left(\mathbb{P}^+_{res(k-1)}, \Pi^+_{res(k-1)}, \mathbb{A}^-_{k-1}\right)$.

3. Due to the **decay** of density, $\left(\mathbb{P}^-_{res(k-1)}, \Pi^-_{res(k-1)}\right)$ might not satisfy the assumption for Lie's arguments. We modify as such:

$$\mathbb{P}_{res(k)} := \left\{P \in \mathbb{P}^-_{res(k-1)} \mid \mathcal{A}_{\Pi^-_{res(k-1)}}(P) > 2^{-1}\right\}$$

   and $\Pi_{res(k)} := \Pi^-_{res(k-1)}$ untouched. What remains are those affected by the **decay** of density: $\mathbb{P}_{decay(k)} := \mathbb{P}^-_{res(k-1)} \setminus \mathbb{P}_{res(k)}$.

4. By construction, we have for all k,

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- $(\mathbb{P}_k, \Pi_k)$ can be treated with the **decomposition scheme**.
- $\mathbb{A}_k$s posses a **cell** structure: $\mathbb{I}_k := \mathbb{A}_{k-1}^{\subseteq} \setminus \mathbb{A}_k^{\subseteq}$
- Tiles in $\mathbb{P}_k$ are **temporally** controlled by $\mathbb{I}_k$: $P \in \mathbb{P}_k \implies I_P \in \mathbb{I}_k$.
- Size of $\mathbb{A}_k$ **locally** posses **exponential decay**:

$$J \in \mathbb{A}_{k-1} \implies \mu\left(J \cap \bigsqcup \mathbb{A}_k\right) = \sum_{\substack{I \in \mathbb{A}_k \\ I \subset J}} |I| \leq e^{-\Lambda}|J|,$$

  which also screams **sparsity**.
- With **temporally restricted** references: $\mathbb{P}_{decay(k)}$ has **decayed density** less than 2^{-1}.

5. To deal with $\mathbb{P}_{decay(k)}$s, we preserve the $\mathbb{I}_k$**-cell** structure when collecting the 2^{-2}**-dense** tiles.

$$\mathbb{P}_{res(k,0)} := \left\{P \in \tilde{\mathbb{D}} \mid I_P \in \mathbb{I}_k \ \wedge \ \mathcal{A}_{\Pi_{res(k,0)}}(P) \in \left(2^{-2}, 2^{-1}\right]\right\},$$

   where:

$$\Pi_{res(k,0)} := \left\{\pi \in \tilde{\mathbb{D}} \mid I_\pi \in \mathbb{I}_k \ \wedge \ \mathcal{A}(\pi) > 2^{-2}\right\} \quad \text{and} \quad \mathbb{A}_{k,0} := \mathbb{A}_{k-1}.$$

   The rest is to pass the arguments into every **cells** and **inductively** create **finer** cells to **compensate** for the **decay of density**.

In short, there are natural ways to build **nested cells** from the **level set** so that, within each cell, we have good control on M_Πs and $\mathbb{P}$s. Yet, the argument looks daunting due to the **complicated process** and **indexes**.

6.4 Pavel Zorin-Kranich's Modifications

Inspired by Lie's arguments, Zorin-Kranich simplified the arguments. Through **combining** level set estimates from **different densities**, he first constructed the **cell** structure fitting for **all densities** and then classified tiles according to the relative density **localized** within the cell. This prevents the problem arising from **decayed densities** since all the **decay** happens within the **cell** and the measurement is done **after** the decay. Additionally, his **cell** structure interacts well with **temporal dilation**. This allows him to verify some **localized estimates** to apply the **extrapolation arguments** from [BT13]. We present his arguments as two parts: **Cell estimate** and **Mollification**. Before the discussion, we first introduce some terminologies:

Definition 6.4.1 (Carpet: collection of disjoint cubes)**.**
Consider the following collection:

$$\mathbb{X} := \left\{\mathbb{A} \in 2^{\mathbb{D}} \mid \overbrace{\forall I, J \in \mathbb{A}, (I \cap J \neq \varnothing \implies I = J)}^{\textit{Collection of disjoint cubes}}\right\}.$$

*We call an element $\mathbb{A} \in \mathbb{X}$ a **carpet**.*

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Definition 6.4.2 (Covering relation)**.**
We equip $\mathbb{X}$ *a partial order relation* $\prec$*:*

$$\forall \mathbb{A}, \mathbb{B} \in \mathbb{X}, \left(\mathbb{A} \prec \mathbb{B} \iff \mathbb{A} \subset \mathbb{B}^{\subset}\right)$$

Additionally, we define the δ***-covering relation*** *as such:*

$$\mathbb{A} \underset{\delta}{\prec} \mathbb{B} \iff \left(\mathbb{A} \prec \mathbb{B} \ \wedge \ \forall J \in \mathbb{B}, \sum_{\substack{I \in \mathbb{A} \\ I \subset J}} |I| \leq \delta |J|\right).$$

Typically, we only consider $\delta \in (0, 1)$*.*

Definition 6.4.3 (Smooth carpet)**.**
$\mathbb{A} \in \mathbb{X}$ *is* ***smooth*** *if: Given* $(I, J) \in \mathbb{D} \times \mathbb{A}$*,*

$$\left(\ell_I \leq 2^{-\kappa} \ell_J \ \wedge \ \tilde{I} \cap J \neq \varnothing\right) \implies \exists J' \in \mathbb{A} \ s.t. \ \left(2^{-\kappa} \ell_J \leq \ell_{J'} \ \wedge \ I \subset J'\right).$$

We denote the collection of ***smooth carpets*** *as* $\mathbb{X}^{\infty}$*.*

Remark. *Another way to view smoothness is the following: If* $\mathbb{A} \in \mathbb{X}^{\infty}$*,*

$$I \notin \mathbb{A}^{\subset} \implies \forall J \in \mathbb{A}, \left(\tilde{I} \cap J \neq \varnothing \implies \ell_I \geq \ell_J\right).$$

Heuristically speaking, the ***size/scale*** *of cubes in a* ***smooth carpet*** *must* ***varies smoothly****. Thus,* ***dilated*** *cubes share* ***similar incidental properties*** *with its* ***non-dilated*** *counterpart.*

We now present the core estimate:

Lemma 6.4.4 (Cell estimate)**.**
Given $\delta \in (0, 1)$ *and* $\mathbb{A} \in \mathbb{X}$*, there is* $C \underset{\delta, \kappa, D}{\gg} 1$ *such that we can find* $\mathbb{A} \underset{\delta}{\succ} \mathbb{A}^{-} \in \mathbb{X}$ *locating all the* ***bad references****. That is, by removing all references* ***temporally*** *located in* $\mathbb{A}^{-\subset}$*:*

$$\Pi^{+} := \left\{\pi \in \tilde{\mathbb{D}} \mid I_\pi \in \mathbb{A}^{\subset} \setminus \mathbb{A}^{-\subset}\right\},$$

its 2^{-n}***-dense*** *equivalent reference:* $\Pi_n^{+} := \{\pi \in \Pi^{+} \mid \mathcal{A}(\pi) > 2^{-n}\}$ *follows:*

$$\forall n \in \mathbb{N}, \ M_{\Pi_n^{+}} \leq C n 2^{n}.$$

Remark. *It essentially states that: within certain* ***temporal location****, the* 2^{-n} ***dense equivalent reference*** *follows our desired control.*

Proof. *Considering the* ***temporally*** *localized* ***references****:*

$$\Pi_n := \left\{\pi \in \Pi \mid \mathcal{A}(\pi) > 2^{-n}\right\}, \quad \textit{where} \ \ \Pi := \left\{\pi \in \tilde{\mathbb{D}} \mid I_\pi \in \mathbb{A}^{\subset}\right\},$$

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*we can locate the **high overlaps** across all **different densities**:*

$$E := \bigcup_{n\in\mathbb{N}} E_n, \quad \text{where} \;\; E_n := M^{-1}_{\Pi_n}\left(Cn2^n, \infty\right].$$

*Applying **John-Nirenberg inequality**, we have: Given $I \in \mathbb{A}$,*

$$\mu\left(I \cap E\right) \le \sum_{n\in\mathbb{N}} \mu\left(I \cap E_n\right) \le \sum_{n\in\mathbb{N}} e^{-\Lambda n}|I| \le \frac{|I|}{e^{\Lambda}-1}.$$

We now decompose E into disjoint cubes: $\mathbb{A}^- := M\left\{I \in \mathbb{D} \mid I \subset E\right\}$ and take large enough $C \underset{\kappa,D}{\gtrsim} \Lambda \underset{\delta}{\gg} 1$ to verify $\mathbb{A}^- \underset{\delta}{\prec} \mathbb{A}$:

$$\forall J \in \mathbb{A}, \; \sum_{\substack{I\in\mathbb{A}^-\\ I\subset J}} |I| = \mu\left(J \cap E\right) \le \frac{|J|}{e^{\Lambda}-1} \le \delta|J|.$$

*Meanwhile, we isolate **bad reference**: $\Pi^- := \{\pi \in \Pi \mid I_\pi \in \mathbb{A}^{-\subset}\}$ and define:*

$$\Pi_n^+ := \Pi_n \setminus \Pi^- \;\; \text{so that, by construction,} \;\; M_{\Pi_n^+} \le Cn2^n.$$

$\square$

To this stage, we have established methods to adapt $(\mathbb{P},\Pi)$s to the **cell** structure: $\mathbb{A}^{\subset} \setminus \mathbb{A}^{-\subset}$. Yet, before doing so, Zorin-Kranich put additional steps to equip the cells with **Smooth structure**:

Lemma 6.4.5 (Mollification)**.**
Given $(\mathbb{A},\mathbb{B}) \in \mathbb{X} \times \mathbb{X}^\infty$, if $\mathbb{A} \underset{\delta}{\prec} \mathbb{B}$, we can construct $\beta\mathbb{A} \in \mathbb{X}^\infty$ satisfying:

$$\mathbb{A} \prec \beta\mathbb{A} \underset{\delta'}{\prec} \mathbb{B}, \quad \text{where} \;\; \delta' \underset{\kappa,D}{\gtrsim} \delta.$$

We **postpone** the proof and see how the two lemmas help us construct the **cell structure** and $(\mathbb{P},\Pi)$s. We recall the comparison between **Jenga** and **Eiffel Tower** and state our desired result in the following lemma.

Lemma 6.4.6 (**Eiffel Tower** construction)**.**
*Given $\delta \in (0,1)$, we can construct a **chain** of **smooth carpets**:*

$$\left\{\mathbb{A}_\alpha\right\}_{\alpha\in\mathbb{N}} \subset \mathbb{X}^\infty \;\; \text{and defaut} \;\; \mathbb{A}_0 := \mathbb{D}_{\overline{s}} \in \mathbb{X}^\infty$$

such that we have the following:

- ***δ-covering relation**:*

$$\cdots \underset{\delta}{\prec} \mathbb{A}_\alpha \underset{\delta}{\prec} \mathbb{A}_{\alpha-1} \underset{\delta}{\prec} \cdots \underset{\delta}{\prec} \mathbb{A}_1 \underset{\delta}{\prec} \mathbb{A}_0.$$

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- ***Cell structure:***

$$\mathbb{D}_{\bar{s}}^{\subsetneq} = \bigsqcup_{\alpha\in\mathbb{N}} \mathbb{I}_\alpha, \quad \textit{where} \;\; \mathbb{I}_\alpha := \mathbb{A}_{\alpha-1}^{\subsetneq} \setminus \mathbb{A}_\alpha^{\subsetneq}.$$

Accordingly, we also have:

$$\tilde{\mathbb{D}} = \bigsqcup_{\alpha\in\mathbb{N}} \mathbb{P}_{*,\alpha}, \quad \textit{where} \;\; \mathbb{P}_{*,\alpha} := \Pi_\alpha := \left\{\pi \in \tilde{\mathbb{D}} \mid I_\pi \in \mathbb{I}_\alpha\right\}.$$

- ***Relative density partition:***

$$\mathbb{P}_{*,\alpha} = \bigsqcup_{n\in\mathbb{N}} \mathbb{P}_{n,\alpha}, \quad \textit{where} \;\; \mathbb{P}_{n,\alpha} := \left\{P \in \mathbb{P}_{*,\alpha} \mid \mathcal{A}_{\Pi_\alpha}(P) \in \left(2^{-n}, 2^{1-n}\right]\right\}.$$

- ***Temporal overlap control:***

$$M_{\Pi_{n,\alpha}} \underset{\delta,\kappa,D}{\lesssim} n2^n, \quad \textit{where} \;\; \overset{2^{-n}\textit{-dense equivalent reference}}{\Pi_{n,\alpha}} := \left\{\pi \in \Pi_\alpha \mid \mathcal{A}(\pi) > 2^{-n}\right\}.$$

Proof.

1. *Starting with* $\mathbb{A}_0 := \mathbb{D}_{\bar{s}} \in \mathbb{X}^\infty$*, we assume* $\mathbb{A}_{\alpha-1} \in \mathbb{X}^\infty$ *constructed.*
2. *Through* **cell estimate***, we have:* $\mathbb{A}_{\alpha-1}^- \underset{\delta}{\prec} \mathbb{A}_{\alpha-1}$ *and set, accordingly,*

$$\Pi_{n,\alpha}^+ := \left\{\pi \in \tilde{\mathbb{D}} \mid \mathcal{A}(\pi) > 2^{-n} \;\wedge\; I_\pi \in \mathbb{A}_{\alpha-1}^{\subsetneq} \setminus \mathbb{A}_{\alpha-1}^{-\subsetneq}\right\}$$

3. *Since* $\mathbb{A}_{\alpha-1} \in \mathbb{X}^\infty$*, we may apply* ***mollification****, set* $\mathbb{A}_\alpha := \beta\mathbb{A}_{\alpha-1}^- \in \mathbb{X}^\infty$*, and yield a chain of relations:*

$$\cdots \underset{\delta'}{\prec} \mathbb{A}_\alpha \underset{\delta'}{\prec} \mathbb{A}_{\alpha-1} \underset{\delta'}{\prec} \cdots \underset{\delta'}{\prec} \mathbb{A}_1 \underset{\delta'}{\prec} \mathbb{A}_0.$$

As a result, with a renaming of variable $\delta' \rightsquigarrow \delta$*, we have* δ***-covering relation****. Additionally,* **cell structure** *and* **relative density partition** *follow directly from construction. The rest is to verify the* ***temporal overlap control****. This follows from* **cell estimate***. Since* $\mathbb{I}_\alpha \subset \mathbb{A}_{\alpha-1}^{\subsetneq} \setminus \mathbb{A}_{\alpha-1}^{-\subsetneq}$*, we have:*

$$\Pi_{n,\alpha} \subset \Pi_{n,\alpha}^+ \;\; \textit{and, thus,} \;\; M_{\Pi_{n,\alpha}} \le M_{\Pi_{n,\alpha}^+} \underset{\delta,\kappa,D}{\lesssim} n2^n.$$

□

Remark. *The result matches our settings for* ***decomposition scheme*** *with* $n = m$*. Moreover, both* $\mathbb{P}_{n,\alpha}$ *and* $\Pi_{n,\alpha}$ *are* ***temporally localized*** *inside* $\mathbb{A}_{\alpha-1}$ *but outside* $\mathbb{A}_\alpha$*. Due to the* ***nested structure****, as long as our analysis* ***reflect*** *these temporal properties, we can benefit from the* δ***-covering*** *and the* ***smoothness*** *of carpet when treating the operator and its* ***adjoint****.*

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Lastly, with a change of perspective, we can organize the collection as such:

$$\tilde{\mathbb{D}} = \bigsqcup_{n\in\mathbb{N}} \mathbb{P}_n, \quad \text{where} \quad \mathbb{P}_n := \bigsqcup_{\alpha\in\mathbb{N}} \mathbb{P}_{n,\alpha}$$

so that, by our construction, we also have:

$$\because \mathbb{P}_{n,\alpha} \subset \Pi_\alpha \quad \therefore P \in \mathbb{P}_n \implies \mathcal{A}(P) \le \mathcal{A}_{\Pi_\alpha}(P) \approx 2^{-n}.$$

6.5 Explicit Construction of Smooth Carpet

We resume to prove the **mollification lemma** 6.4.5. In the original literature [Zor19], Zorin-Kranich neither gave an explicit construction nor verified the **δ-covering relation**. For the sake of completeness, we present our arguments with **explicit construction**. A reasonable starting point is to first consider the following question: What is the **simplest non-trivial** smooth carpet? A direct guess leads us to the next definition:

Definition 6.5.1 (The Ink-bleeding)**.**
*Given $A \in \mathbb{D}$, we define the **Ink-bleeding** of A:*

$$\beta_A \in m\{\mathbb{A} \in \mathbb{X}^\infty \mid \{A\} \prec \mathbb{A}\}$$

*as the **$\prec$-minimal smooth carpet** that **covers** the one cube carpet $\{A\}$ constructed through the following process:*

1. *For some $s \in \mathbb{Z}$, $A \in \mathbb{D}_s$. We set $\mathbb{A}_0 := \{A\} \in \mathbb{X}$ at our initial stage.*
2. *Suppose we have $\mathbb{A}_{k-1} \in \mathbb{X}$ at $k-1$th stage, we build $\mathbb{A}_k \in \mathbb{X}$ as such:*

$$\begin{aligned}\mathbb{A}_k :=& M\left(\mathbb{A}_{k-1} \cup \bigcup_{J\in\mathbb{A}_{k-1}} \left\{I \in \mathbb{D} \mid \ell_I \le 2^{-\kappa}\ell_J \ \wedge\ \tilde{I} \cap J \ne \varnothing\right\}\right)\\ =& \mathbb{A}_{k-1} \sqcup \left\{I \in \mathbb{D}_{s-k} \setminus \mathbb{A}_{k-1}^{\subset} \mid \tilde{I} \cap \bigsqcup \mathbb{A}_{k-1} \ne \varnothing\right\}.\end{aligned}$$

*Essentially, we attempt to use **greedy algorithm** by adding the bare requirement for it to be **smoother**. Incidentally, the process adds **barely smaller** layer of cubes on the **edge** of the carpet.*

We define $\beta_A := \bigcup_{k\in\mathbb{N}} \mathbb{A}_k$. It is easy to check that $\{A\} \prec \beta_A \in \mathbb{X}$:

$$\{A\} \subset \mathbb{A}_0 \subset \mathbb{A}_1 \subset \cdots \subset \mathbb{A}_k \subset \cdots \subset \beta_A \in \mathbb{X}.$$

By construction, $\beta_A \in \mathbb{X}^\infty$ since, given $(I, J) \in \mathbb{D} \times \mathbb{A}_{k-1}$, we have:

$$\left(\ell_I \le 2^{-\kappa}\ell_J \ \wedge\ \tilde{I} \cap J \ne \varnothing\right) \implies \exists J' \in \mathbb{A}_k \ s.t. \left(2^{-\kappa}\ell_J \le \ell_{J'} \ \wedge\ I \subset J'\right).$$

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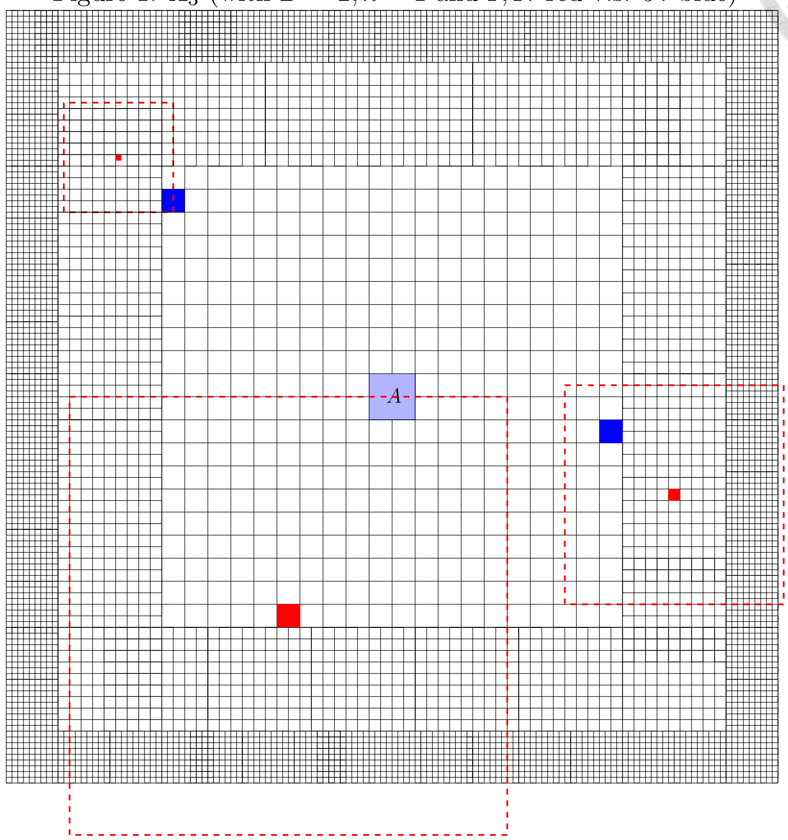
Figure 1: $\mathbb{A}_3$ (with $D = 2, \kappa = 1$ and $I, \tilde{I}$: red v.s. J: blue)



Also, ***minimality*** *is guaranteed by the* ***greedy algorithm****. Lastly, we give some* ***quantitative description****:*

$$\begin{aligned}\bigsqcup \beta_I &= \left(1 + 2\left(n_D 2^\kappa + 1\right) \sum_{k \in \mathbb{N}} 2^{-\kappa k}\right) A \\ &= \frac{\left(2n_D + 1\right) 2^\kappa + 1}{2^\kappa - 1} A \subset C_D A, \quad \textit{where} \ \ C_D := 4n_D + 3.\end{aligned}$$

With building blocks constructed, we still need ways to sew things together:

Properties 6.5.2 (Sewing)**.**
Given $\mathbb{Y} \subset \mathbb{X}^\infty$ *and* $\mathbb{B} \in \mathbb{X}$*, we have:*

$$(\forall \mathbb{A} \in \mathbb{Y},\ \mathbb{A} \prec \mathbb{B}) \implies \mathbb{B} \succ \bigvee \mathbb{Y} := M \bigcup \mathbb{Y} \in \mathbb{X}^\infty.$$

Proof. *By construction, we only need to verify the smoothness. Given* $I \in \mathbb{D}$

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and $J \in \bigvee \mathbb{Y}$*, since there is* $\mathbb{A} \in \mathbb{Y}$ *such that* $J \in \mathbb{A}$*, we have:*

$$\begin{aligned} &\left(\ell_I \leq 2^{-\kappa}\ell_J \ \wedge\ \tilde{I} \cap J \neq \varnothing\right) \\ \Longrightarrow\ & \exists J' \in \mathbb{A} \ s.t. \left(2^{-\kappa}\ell_J \leq \ell_{J'} \ \wedge\ I \subset J'\right) \\ \Longrightarrow\ & \exists J'' \in \bigvee \mathbb{Y} \ s.t. \left(2^{-\kappa}\ell_J \leq \ell_{J'} \leq \ell_{J''} \ \wedge\ I \subset J' \subset J''\right). \end{aligned}$$

As a result, $\mathbb{B} \succ \bigvee \mathbb{Y} \in \mathbb{X}^\infty$*.*

Now we are ready to prove the **mollification lemma**:

Proof (**Lemma** 6.4.5)**.** *Through* ***sewing Ink-bleedings****, we immediately have:*

$$\because \forall A \in \mathbb{A},\ \{A\} \prec \beta_A \prec \mathbb{B} \ \therefore \mathbb{A} \prec \beta\mathbb{A} \prec \mathbb{B}, \quad \textit{where} \ \ \beta\mathbb{A} := \bigvee_{A \in \mathbb{A}} \beta_A \in \mathbb{X}^\infty.$$

On the other hand, since $\mathbb{A} \underset{\delta}{\prec} \mathbb{B}$ *with* $\delta \in (0,1)$*, we must have:*

$$\forall (A,B) \in \mathbb{A} \times \mathbb{B}, \left(A \subset B \implies \ell_A \leq 2^{-\kappa}\ell_B\right).$$

Consequently, given $(A,B) \in \mathbb{A} \times \mathbb{B}$*, we have* $C_{\kappa,D} := 1 + 2^{-\kappa}C_D$ *such that:*

$$\exists I \in \beta_A \ s.t. \ I \subset B \implies B \cap C_D A \neq \varnothing \implies A \subset C_{\kappa,D} B.$$

We now verify the ***quantitative covering relation****. Given* $B \in \mathbb{B}$*, since* $\mathbb{B} \in \mathbb{X}^\infty$ *(****scale*** *of cubes varies* ***smoothly*** *in* $\mathbb{B}$*), previous estimate yields:*

$$\begin{aligned} \sum_{\substack{I \in \beta\mathbb{A} \\ I \subset B}} |I| \leq \sum_{\substack{A \in \mathbb{A} \\ A \subset C_{\kappa,D} B}} \sum_{I \in \beta_A} |I| &\leq \sum_{\substack{B' \in \mathbb{B} \\ B' \cap C_{\kappa,D} B \neq \varnothing}} \sum_{\substack{A \in \mathbb{A} \\ A \subset B'}} \mu\left(\bigsqcup \beta_A\right) \\ &\underset{D}{\lesssim} \sum_{\substack{B' \in \mathbb{B} \\ B' \cap C_{\kappa,D} B \neq \varnothing}} \sum_{\substack{A \in \mathbb{A} \\ A \subset B'}} |A| \leq \sum_{\substack{B' \in \mathbb{B} \\ B' \cap C_{\kappa,D} B \neq \varnothing}} \delta|B'| \underset{\kappa,D}{\lesssim} \delta|B|. \end{aligned}$$

□

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7 Sparse Domination of Sparse Parts

With **Eiffel Tower construction**, we have set up for our **decomposition scheme**. The rest of the work is to provide good control on both **sparse parts** and **cluster parts**. Here, we choose the the **setting**: $l \lesssim m = n$ to do the decomposition and present the argument for **sparse parts** in the form of **sparse form dominance** and **pointwise sparse dominance**.

7.1 Reductions

Definition 7.1.1.

$$C_{\mathbb{P}} := \sum_{P \in \mathbb{P}} \chi_{E_P}, \quad \textit{where} \ \ \mathbb{P} \subset \tilde{\mathbb{D}}.$$

Definition 7.1.2 (Spectral η-control)**.**
Given $P_j \in \tilde{\mathbb{D}}$, we define:

$$\begin{cases} P_0 \lesssim_{<} P_1 & \iff s_{P_0} \leq s_{P_1} \ \wedge \ \Delta(P_0, P_1) < \eta \\ P_0 \lesssim_{\geq} P_1 & \iff s_{P_0} \leq s_{P_1} \ \wedge \ \Delta(P_0, P_1) \in [\eta, \infty) . \end{cases}$$

Notice that either relation implies $\tilde{I}_{P_0} \cap \tilde{I}_{P_1} \neq \varnothing$ and thus, $I_{P_0} \subset 2\tilde{I}_{P_1}$. Additionally, given $\mathbb{P} \subset \tilde{\mathbb{D}}$ and $P \in \tilde{\mathbb{D}}$, we define:

$$\begin{cases} \mathbb{P}_{P,<} := \{P' \in \mathbb{P} \mid P' \lesssim_{<} P\} \\ \mathbb{P}_{P,\geq} := \{P' \in \mathbb{P} \mid P' \lesssim_{\geq} P\} \end{cases}$$

Lemma 7.1.3 (Tile-tile interaction)**.**
Given $P_j \in \tilde{\mathbb{D}}$, we have:

$$\begin{cases} \left|\mathfrak{L}_{P_1}^{*}\mathfrak{L}_{P_0} f\right| = 0 & \Longleftarrow \ \ P_0, P_1 \ \ \textit{are} \ \ \trianglelefteq \textbf{\textit{-incomparable}} \\ \left|\mathfrak{L}_{P_1}\mathfrak{L}_{P_0}^{*} f\right| & \underset{\kappa, D, d}{\lesssim} \langle \Delta\left(P_0, P_1\right)\rangle^{\tau/d} \frac{|\tilde{I}_{P_0} \cap \tilde{I}_{P_1}|}{|I_{P_0}| \cdot |I_{P_1}|} \|f\|_{L^1(E_{P_0})} \chi_{E_{P_1}} . \end{cases}$$

Proof. *The first relation is trivial since:*

$$P_0, P_1 \ \ \trianglelefteq \textbf{\textit{-incomparable}} \implies E_{P_0} \cap E_{P_1} = \varnothing.$$

The second relation follows from estimating the ***kernel****:*

$$\mathfrak{L}_{P_1}\mathfrak{L}_{P_0}^{*} f(\cdot) = \int K_{P_0,P_1}(\cdot, y) f(y) dy,$$

where the explicit form of K_{P_0,P_1} is:

$$K_{P_0,P_1}(x, y) = \int_{\tilde{I}_{P_0} \cap \tilde{I}_{P_1}} e^{i(q_x - q_y)(z)} K_{s_{P_1}}(x, z) \overline{K_{s_{P_0}}(y, z)} dz \cdot \chi_{E_{P_1}}(x) \chi_{E_{P_0}}(y).$$

Considering $J := \tilde{I}_{P_0} \cap \tilde{I}_{P_1} \neq \varnothing$ and $(x, y) \in E_{P_1} \times E_{P_0}$, we have:

$$\because (q_x, q_y) \in \omega_{P_1} \times \omega_{P_0} \quad \therefore \|q_x - q_y\|_{\tilde{I}_{P_0} \cap \tilde{I}_{P_1}} \geq \Delta\left(P_0, P_1\right).$$

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*To apply **Van der Corput estimate**, we need a way to measures the **Oscillation** of $\psi_{P_0,P_1}(\cdot) := K_{s_{P_1}}(x,\cdot)\overline{K_{s_{P_0}}(y,\cdot)}$. Using kernel's properties: L^∞**Size Control** and **Locally τ-Hölder Continuity**(3.4.3.1), we have:*

$$\|\Delta\| \lesssim \ell_J \implies |\psi_{P_0,P_1} - \tau_\Delta \psi_{P_0,P_1}| \underset{\kappa,D,d}{\lesssim} (\|\Delta\|/\ell_J)^\tau |I_{P_0}|^{-1} |I_{P_1}|^{-1}.$$

Plugging everything into the estimate yields:

$$\begin{aligned}|K_{P_0,P_1}(x,y)| &\underset{D,d}{\lesssim} \sup_{\frac{\|\Delta\|}{\ell_J} < \langle \|q_x - q_y\|_J\rangle^{1/d}} \|\psi_{P_0,P_1} - \tau_\Delta \psi_{P_0,P_1}\|_{L^\infty} |J| \\ &\underset{\kappa,D,d}{\lesssim} \left\langle \|q_x - q_y\|_J \right\rangle^{\tau/d} |I_{P_0}|^{-1} |I_{P_1}|^{-1} |J| \\ &\le \langle \Delta(P_0,P_1)\rangle^{\tau/d} \frac{\left|\tilde I_{P_0} \cap \tilde I_{P_1}\right|}{|I_{P_0}| \cdot |I_{P_1}|}.\end{aligned}$$

□

Remark. *Comparing to the single tile estimate, we successfully extract the **distance factor** and keep all other the good estimate.*

Through **single tile estimate** and **tile-tile interaction**, we aim to control the behavior of the **sparse part**. For starters, we first observe that: Given $\mathbb{P} \subset \mathbb{P}_n$ be **sparse parts**, we have two ways to proceed with our control:

- Pointwise Dominance: Using **single tile estimate**, we suspect that:

  $$|\mathfrak{L}_\mathbb{P}| \lesssim \sum_{P\in\mathbb{P}} |f|_{\tilde I_P} \chi_{E_P} \overset{?}{\lesssim} \sum_{I \in \mathbb{S}} |f|_{\Lambda I} \chi_I,$$

  for some large constant $\Lambda \lesssim 1$ and $\mathbb{S} \subset \mathbb{D}$ $p(n)$**-carleson** with $p(\cdot)$ be a prescribed polynomial. By **Sparse-Maximal dominance**, we expect:

  $$\|\mathfrak{L}_\mathbb{P} f\|_{L^p} \lesssim p(n) \|Mf\|_{L^p} \lesssim p(n) \|f\|_{L^p}, \quad \forall p \in (1,\infty).$$

- L^2 control: Expanding the L^2 norm explicitly, we have:

  $$\begin{aligned}\|\mathfrak{L}^*_\mathbb{P} f\|^2_{L^2} = \langle \mathfrak{L}^*_\mathbb{P} f, \mathfrak{L}^*_\mathbb{P} f\rangle &\lesssim \sum_{\substack{P_j \in \mathbb{P} \\ s_{P_0} \le s_{P_1}}} \left|\left\langle \mathfrak{L}^*_{P_0} f, \mathfrak{L}^*_{P_1} f\right\rangle\right| \\ &\le \left\langle \sum_{\substack{P_j \in \mathbb{P} \\ s_{P_0} \le s_{P_1}}} \left|\mathfrak{L}_{P_1}\mathfrak{L}^*_{P_0} f\right|, |f| \right\rangle.\end{aligned}$$

  To control the L^2 norm is to control the first term in the last expression.

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With **Tile-tile interaction**, we have:

$$\begin{aligned}\sum_{\substack{P_j\in\mathbb{P}\\ s_{P_0}\le s_{P_1}}}\left|\mathfrak{L}_{P_1}\mathfrak{L}^*_{P_0}f\right| &\lesssim \sum_{\substack{P_j\in\mathbb{P}\\ P_0\lesssim_{\geq}P_1}}\langle\Delta(P_0,P_1)\rangle^{\tau/d}\left|I_{P_1}\right|^{-1}\|f\|_{L^1(E_{P_0})}\chi_{E_{P_1}}\\ &\quad+\sum_{\substack{P_j\in\mathbb{P}\\ P_0\lesssim_{<}P_1}}\langle\Delta(P_0,P_1)\rangle^{\tau/d}\left|I_{P_1}\right|^{-1}\|f\|_{L^1(E_{P_0})}\chi_{E_{P_1}}\\ &\lesssim\sum_{P'\in\mathbb{P}}\begin{Bmatrix}(1+\eta)^{-\tau/d} & \left|C_{\mathbb{P}_{P',\geq}}f\right|_{2\tilde{I}_{P'}}\\ + & \left|C_{\mathbb{P}_{P',<}}f\right|_{2\tilde{I}_{P'}}\end{Bmatrix}\chi_{E_{P'}}.\end{aligned}$$

Applying **Hölder's inequality**, we get:

$$\sum_{\substack{P_j\in\mathbb{P}\\ s_{P_0}\le s_{P_1}}}\left|\mathfrak{L}_{P_1}\mathfrak{L}^*_{P_0}f\right|\lesssim\sum_{P'\in\mathbb{P}}\begin{Bmatrix}(1+\eta)^{-\tau/d}\left|C_{\mathbb{P}_{P',\geq}}\right|_{2\tilde{I}_{P'},r'}\\ +\left|C_{\mathbb{P}_{P',<}}\right|_{2\tilde{I}_{P'},r'}\end{Bmatrix}|f|_{2\tilde{I}_{P'},r}\,\chi_{E_{P'}}.$$

We wish to extract **density factor** from the $\{\cdots\}$ term. If we can actually do so with $r\in(1,2)$:

$$\sum_{\substack{P_j\in\mathbb{P}\\ s_{P_0}\le s_{P_1}}}\left|\mathfrak{L}_{P_1}\mathfrak{L}^*_{P_0}f\right|\lesssim 2^{-n\epsilon}\sum_{P'\in\mathbb{P}}|f|_{2\tilde{I}_{P'},r}\,\chi_{E_{P'}},$$

the **RHS** is again possible to be dominated by the corresponding **sparse operator** with a $p(n)$**-carlson sparse cubes** $\mathbb{S}'\subset\mathbb{D}$. This in turn can further be norm dominated by M_rf:

$$\begin{aligned}&\sum_{P'\in\mathbb{P}}|f|_{2\tilde{I}_{P'},r}\,\chi_{E_{P'}}\overset{?}{\lesssim}\sum_{I\in\mathbb{S}'}|f|_{\Lambda I,r}\,\chi_I\\ \Longrightarrow&\left\|\sum_{P'\in\mathbb{P}}|f|_{2\tilde{I}_{P'},r}\,\chi_{E_{P'}}\right\|_{L^2}\lesssim p(n)\,\|M_rf\|_{L^2}\underset{r}{\lesssim}p(n)\|f\|_{L^2}.\end{aligned}$$

As a result, through duality, we shall expect:

$$\|\mathfrak{L}_{\mathbb{P}}f\|_{L^2}\lesssim p(n)2^{-n\epsilon/2}\,\|f\|_{L^2}\,.$$

Suppose everything works as intended, we can easily **spread out** the $2^{-n\epsilon/2}$ **decay** in L^2 to all L^p and sum over $n\in\mathbb{N}$ to complete the L^p control:

Theorem 7.1.4 (L^p bound on **sparse parts**).
Given $\mathbb{P}\subset\tilde{\mathbb{D}}$ be the full collection of the ***sparse parts****, we have:*

$$\|\mathfrak{L}_{\mathbb{P}}f\|_{L^p}\underset{p}{\lesssim}\|f\|_{L^p}\,,\quad\forall p\in(0,\infty).$$

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For a more precise analysis, we consider the following configuration:

Definition 7.1.5 (**Sparse tower** or **Sparse Forest** in [Lie20], **Anti-chain and boundary** in [Zor19])**.**
Given $\mathbb{P} \subset \mathbb{P}_n$, we say:

$$\mathbb{P} \text{ is } \begin{cases} \text{an } \textbf{\textit{anti-chain}} \\ \text{a } \lesssim n\textbf{\textit{-decay}} \end{cases} \textbf{\textit{tower}}$$

$$\iff \mathbb{P} \cap \mathbb{P}_{n,\alpha} \text{ is } \begin{cases} \text{an } \textbf{\textit{anti-chain}} \\ \text{a } \lesssim n\textbf{\textit{-decay stack}} \end{cases} \quad \forall \alpha \in \mathbb{N}$$

*In either case, we call $\mathbb{P}$ a **sparse tower**.*

Remark. *In our case, using **decomposition scheme** on **Eiffel Tower construction** with $l \lesssim m = n$ gives us:*

$$\mathbb{P}_n \rightsquigarrow \begin{cases} \lesssim n^2 \ \textbf{\textit{anti-chain towers}} \\ \lesssim n \ \lesssim n\textbf{\textit{-decay towers}} \\ \text{A lot of } \textbf{\textit{clusters}} \end{cases}$$

Therefore, to compensate the **polynomial growth** of the number of **sparse towers**, we shall extract some **exponential decay** from the estimate of a **sparse tower**:

Theorem 7.1.6 (**Sparse tower** estimate)**.**
*Given $\mathbb{P} \subset \mathbb{P}_n$ a **sparse tower**, we have:*

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^2} \lesssim p(n) 2^{-n\eta_2} \|f\|_{L^2},$$

*and we can construct a $p(n)$-**carleson** collection $\mathbb{S} \subset \mathbb{D}$ such that:*

$$|\mathfrak{L}_{\mathbb{P}} f| \lesssim \sum_{I \in \mathbb{S}} |f|_{\tilde{I}} \chi_I.$$

As a result, we have full control:

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^p} \underset{p}{\lesssim} p(n) 2^{-n\eta_p} \|f\|_{L^p}, \quad \text{where } \ \eta_p > 0, \quad \forall p \in (1, \infty).$$

The theorem follows directly from the following two lemmas.

Lemma 7.1.7 (**Sparse dominance**)**.**
*Given $\mathbb{P} \subset \mathbb{P}_n$ be **sparse tower**, we can find $p(n)$-**carleson** $\mathbb{S} \subset \mathbb{D}$ such that:*

$$\sum_{P \in \mathbb{P}} |f|_{\Lambda I_P, r} \chi_{E_P} \leq \sum_{I \in \mathbb{S}} |f|_{\Lambda I, r} \chi_I, \quad \forall r \in [1, \infty).$$

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Lemma 7.1.8 (**Density extraction**).
*Given $\mathbb{P} \subset \mathbb{P}_n$ be **sparse tower**, $P' \in \tilde{\mathbb{D}}$, and $r \in (1, \infty)$, we have:*

$$\begin{cases} \left| C_{\mathbb{P}_{P'},\geq} \right|_{2\tilde{I}_{P'},r'} \lesssim_r p(n) \\ \left| C_{\mathbb{P}_{P'},<} \right|_{2\tilde{I}_{P'},r'} \lesssim_r p(n) 2^{-n/r'} (1+\eta)^{(dD+\epsilon)/r'}. \end{cases}$$

Remark. *To apply **density extraction** to the proof of theorem, we **fine tune** $\eta, \epsilon \in \mathbb{R}_+$ and $r \in (1,2)$ so that:*

$$(1+\eta)^{-\tau/d} + 2^{-n/r'} (1+\eta)^{(dD+\epsilon)/r'} \lesssim 2^{-n\eta_2}.$$

Before we proceed with the proof of the lemmas, we present our plan:

1. Prove the lemmas with $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$ be an **anti-chain**.

2. For any $\lesssim n$-**decay stack** $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$, we can construct a decomposition on $\mathbb{P}$ with respect to a **decomposition** on its **temporal projection** to encode the **decay** property. We first recall that there is $s_\Delta \eqsim n$ such that:

$$s' - s \geq s_\Delta \implies \forall J \in \mathbb{D}_{s'}, \sum_{\substack{I \in \mathbb{I}_{\mathbb{P}^s} \\ I \subset J}} |I| \leq 2^{-\kappa} |J|.$$

We now reorganize the collection by **modding** out s_Δ on the **scaling**:

$$\mathbb{I}_{\mathbb{P}} = \bigsqcup_{j=1}^{s_\Delta} \mathbb{I}_{\mathbb{P}}^j, \quad \text{where} \quad \mathbb{I}_{\mathbb{P}}^j := \bigsqcup_{t \in \mathbb{Z}} \mathbb{I}_{\mathbb{P}, s_\Delta t + j},$$

and do the following **canonical decomposition** into **carpets**:

$$\mathbb{I}_{\mathbb{P}}^j = \bigsqcup_{k \in \mathbb{N}} \mathbb{M}_{\mathbb{P},k}^j, \quad \text{where} \quad \mathbb{M}_{\mathbb{P},k}^j := M\left(\mathbb{I}_{\mathbb{P}}^j \setminus \bigsqcup_{l<k} \mathbb{M}_{\mathbb{P},l}^j \right) \in \mathbb{X}, \quad \forall k \in \mathbb{N}.$$

By construction, if $(I, J) \in \left(\mathbb{D}_s \cap \mathbb{M}_{\mathbb{P},k+1}^j \right) \times \left(\mathbb{D}_{s'} \cap \mathbb{M}_{\mathbb{P},k}^j \right)$, then:

$$I \subset J \implies \frac{s'-s}{s_\Delta} \in \mathbb{N}.$$

Therefore, we can verify the following covering condition:

$$\forall J \in \mathbb{D}_{s'} \cap \mathbb{M}_{\mathbb{P},k}^j, \quad \sum_{\substack{I \in \mathbb{M}_{\mathbb{P},k+1}^j \\ I \subset J}} |I| = \sum_{\substack{s \in \mathbb{Z} \\ \frac{s'-s}{s_\Delta} \in \mathbb{N}}} \sum_{\substack{I \in \mathbb{D}_s \cap \mathbb{M}_{\mathbb{P},k+1}^j \\ I \subset J}} |I|$$

$$\text{by } \textbf{decay property,} \leq \sum_{\substack{s \in \mathbb{Z} \\ \frac{s'-s}{s_\Delta} \in \mathbb{N}}} 2^{-\frac{s'-s}{s_\Delta}\kappa} |J| = \frac{1}{2^\kappa - 1} |J|.$$

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That is, we have:

$$\cdots \underset{\frac{1}{2^{\kappa}-1}}{\prec} \mathbb{M}^j_{\mathbb{P},k} \underset{\frac{1}{2^{\kappa}-1}}{\prec} \mathbb{M}^j_{\mathbb{P},k-1} \underset{\frac{1}{2^{\kappa}-1}}{\prec} \cdots \underset{\frac{1}{2^{\kappa}-1}}{\prec} \mathbb{M}^j_{\mathbb{P},2} \underset{\frac{1}{2^{\kappa}-1}}{\prec} \mathbb{M}^j_{\mathbb{P},1}, \quad \forall j = 1 \sim s_\Delta.$$

As a direct consequence, $\mathbb{I}^j_{\mathbb{P}}$ is $\lesssim 1$**-carleson**. Correspondingly, we define:

$$\mathbb{P} = \bigsqcup_{j=1}^{s_\Delta} \bigsqcup_{k\in\mathbb{N}} \mathbb{P}^j_k \ \text{ with } \ \mathbb{P}^j_k := \left\{ P \in \mathbb{P} \mid I_P \in \mathbb{M}^j_{\mathbb{P},k} \right\}.$$

Notice that $\mathbb{P}^j_k$s are **anti-chains**. With the the $\underset{\frac{1}{2^{\kappa}-1}}{\prec}$**-chain** structure, estimate from **individual anti-chain** $\mathbb{P}^j_k$ can be sum up to similar order.

3. For $\mathbb{P} \subset \mathbb{P}_n$ be an **sparse tower**, we decompose the collection with respect to the **level/cell structure**:

$$\mathbb{P} = \bigsqcup_{\alpha\in\mathbb{N}} \mathbb{P}^{(\alpha)}, \quad \text{where } \ \mathbb{P}^{(\alpha)} := \mathbb{P} \cap \mathbb{P}_{n,\alpha}.$$

δ**-covering relation** among $\mathbb{A}_\alpha$s should allow us to sum everything up.

7.2 Sparse Dominance

Following our plan, we split the proof in three parts:

Claim (Anti-chain sparse dominance)**.**
Given an ***anti-chain*** $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$*, we can construct* $\mathbb{S} \in \mathbb{X}$ *a carpet, which is* 1***-carleson*** *by definition, such that* $\mathbb{S} \subset \mathbb{I}_{\mathbb{P}} \subset \mathbb{I}_\alpha$ *and:*

$$\sum_{P\in\mathbb{P}} |f|_{\Lambda I_P, r}\, \chi_{E_P} \leq \sum_{I\in\mathbb{S}} |f|_{\Lambda I, r}\, \chi_I, \quad \forall r \in [1,\infty).$$

Proof (Anti-chain sparse dominance)**.** *Since* $\trianglelefteq$***-incomparability*** *implies* ***disjointness****,* $\{E_P\}_{P\in\mathbb{P}}$ *are mutually disjoint. We can first collapse all the tiles sharing the same* ***temporal block****:*

$$\sum_{P\in\mathbb{P}} |f|_{\Lambda I_P, r} \chi_{E_P} = \sum_{I\in\mathbb{I}_{\mathbb{P}}} |f|_{\Lambda I, r} \chi_{E_I}, \quad \textit{where } \ E_I := \bigsqcup_{\substack{P\in\mathbb{P}\\ I_P = I}} E_P.$$

Still, since $\{E_I\}_{I\in\mathbb{I}_{\mathbb{P}}}$ *are mutually disjoint, we have:*

$$\sum_{I\in\mathbb{I}_{\mathbb{P}}} |f|_{\Lambda I, r} \chi_{E_I}(x) \leq \sup_{\substack{I\in\mathbb{I}_{\mathbb{P}}\\ x\in E_I}} |f|_{\Lambda I, r} \leq \sup_{\substack{I\in\mathbb{I}_{\mathbb{P}}\\ x\in I}} |f|_{\Lambda I, r}, \quad \forall x \in \mathbb{R}^D$$

Notice that, since $\mathbb{I}_{\mathbb{P}} \subset \mathbb{I}_\alpha \subset \mathbb{D}^{\subsetneq}_{\overline{s}} \setminus \mathbb{D}^{\subsetneq}_{\underline{s}}$*, the* ***supremum*** *is actually just a* ***maximum****. It is now valid to collect all the cubes that* ***reach maximum*** *for every point* $x \in \bigcup \mathbb{I}_{\mathbb{P}}$ *and define:*

$$\mathbb{S} := M\left(\bigcup_{x\in\mathbb{R}^D} \mathbb{S}_x \right), \quad \textit{where } \ \mathbb{S}_x := \left\{ J \in \mathbb{I}_{\mathbb{P}} \mid x \in J \ \wedge \ |f|_{\Lambda J, r} = \max_{\substack{I\in\mathbb{I}_{\mathbb{P}}\\ x\in I}} |f|_{\Lambda I, r} \right\}.$$

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By construction, $\mathbb{S} \in \mathbb{X}$ and $\mathbb{S} \subset \mathbb{I}_{\mathbb{P}} \subset \mathbb{I}_{\alpha}$. Most importantly, we have:

$$\sum_{P \in \mathbb{P}} |f|_{\Lambda I_P, r} \chi_{E_P}(x) \leq \max_{\substack{I \in \mathbb{I}_{\mathbb{P}} \\ x \in I}} |f|_{\Lambda I, r} = \sum_{I \in \mathbb{S}} |f|_{\Lambda I, r} \chi_I(x), \quad \forall x \in \mathbb{R}^D.$$

$\square$

Claim ($\lesssim n$-decay stack sparse dominance)**.**
*Given a $\lesssim n$-**decay stack** $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$, we can construct a $\lesssim n$-**carleson** collection $\mathbb{S} \subset \mathbb{I}_{\mathbb{P}} \subset \mathbb{I}_{\alpha}$ such that:*

$$\sum_{P \in \mathbb{P}} |f|_{\Lambda I_P, r} \, \chi_{E_P} \leq \sum_{I \in \mathbb{S}} |f|_{\Lambda I, r} \, \chi_I, \quad \forall r \in [1, \infty).$$

Proof ($\lesssim n$-decay stack sparse dominance)**.** *Following our plan, we apply **anti-chain sparse dominance** on $\mathbb{P}^j_k$. As a result, we have $\mathbb{S}^j_k \in \mathbb{X}$ satisfying $\mathbb{S}^j_k \subset \mathbb{I}_{\mathbb{P}^j_k} = \mathbb{M}^j_{\mathbb{P},k}$ and the following relation:*

$$\sum_{P \in \mathbb{P}^j_k} |f|_{\Lambda I_P, r} \, \chi_{E_P} \leq \sum_{I \in \mathbb{S}^j_k} |f|_{\Lambda I, r} \, \chi_I, \quad \forall r \in [1, \infty).$$

We now sum over j, k and have:

$$\sum_{P \in \mathbb{P}} |f|_{\Lambda I_P, r} \, \chi_{E_P} \leq \sum_{I \in \mathbb{S}} |f|_{\Lambda I, r} \, \chi_I, \quad \forall r \in [1, \infty), \quad \textit{where} \;\; \mathbb{S} := \bigsqcup_{j,k} \mathbb{S}^j_k.$$

*The rest is an easy verification of the **Carleson packing condition**:*

$$\forall J \in \mathbb{D}, \; \sum_{\substack{I \in \mathbb{S} \\ I \subset J}} |I| \leq \sum_{\substack{I \in \mathbb{I}_{\mathbb{P}} \\ I \subset J}} |I| \lesssim n|J|.$$

$\square$

Proof (**Sparse dominance**)**.** *For the general case, we start by constructing $p(n)$-**carleson** collection $\mathbb{S}_{\alpha} \subset \mathbb{I}_{\alpha}$ such that:*

$$\sum_{P \in \mathbb{P}^{(\alpha)}} |f|_{\Lambda I_P, r} \, \chi_{E_P} \leq \sum_{I \in \mathbb{S}_{\alpha}} |f|_{\Lambda I, r} \, \chi_I, \quad \forall r \in [1, \infty), \;\; \alpha \in \mathbb{N}.$$

Again, summing over $\alpha \in \mathbb{N}$ yields:

$$\sum_{P \in \mathbb{P}} |f|_{\Lambda I_P, r} \, \chi_{E_P} \leq \sum_{I \in \mathbb{S}} |f|_{\Lambda I, r} \, \chi_I, \quad \forall r \in [1, \infty), \quad \textit{where} \;\; \mathbb{S} := \bigsqcup_{\alpha \in \mathbb{N}} \mathbb{S}_{\alpha}.$$

*The rest is to show the **Carleson packing condition**. Given $J \in \mathbb{S}$, we set*

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$\alpha_0 \in \mathbb{N}$ *such that* $J \in \mathbb{S}_{\alpha_0}$. *As we expand the following expression:*

$$\begin{aligned}\sum_{\substack{I\in\mathbb{S}\\ I\subset J}} |I| = \sum_{\substack{I\in\mathbb{S}_{\alpha_0}\\ I\subset J}} |I| + \sum_{\alpha>\alpha_0}\sum_{\substack{I\in\mathbb{S}_\alpha\\ I\subset J}} |I| \\ \lesssim p(n)|J| + \sum_{\alpha>\alpha_0}\sum_{\substack{J'\in\mathbb{A}_\alpha\\ J'\subset J}}\sum_{\substack{I\in\mathbb{S}_\alpha\\ I\subset J}} |I| \\ \lesssim p(n)|J| + \sum_{\alpha>\alpha_0}\sum_{\substack{J'\in\mathbb{A}_\alpha\\ J'\subset J}} p(n)|J'| \\ \text{by } \delta\text{-}\textbf{covering relation,} \quad \leq p(n)\left(|J| + \sum_{\alpha>\alpha_0}\delta^{\alpha-\alpha_0-1}|J|\right) \lesssim p(n)|J|.\end{aligned}$$

□

7.3 Density Extraction

Again we split the proof into three parts:

Claim (Anti-chain density extraction)**.**
Given $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$ *be* ***anti-chain****,* $P' \in \tilde{\mathbb{D}}$*, and* $r \in (1,\infty)$*, we have:*

$$\begin{cases} \left\|C_{\mathbb{P}_{P',\geq}}\right\|_{L^{r'}} \underset{r}{\lesssim} \mu\left(\bigcup \mathbb{I}_{\mathbb{P}_{P',\geq}}\right)^{1/r'} \\ \left\|C_{\mathbb{P}_{P',<}}\right\|_{L^{r'}} \underset{r}{\lesssim} \mu\left(\bigcup \mathbb{I}_{\mathbb{P}_{P',<}}\right)^{1/r'} 2^{-n/r'}(1+\eta)^{(dD+\epsilon)/r'}. \end{cases}$$

Proof (Anti-chain density extraction)**.** *The first relation is obvious since for any* ***anti-chain*** $\mathbb{P}' \subset \tilde{\mathbb{D}}$*, we always have:*

$$\because \bigsqcup_{P\in\mathbb{P}'} E_P \subset \bigcup \mathbb{I}_{\mathbb{P}'} \quad \therefore \begin{cases} \sum_{P\in\mathbb{P}'} \mu(E_P) \leq \mu\left(\bigcup \mathbb{I}_{\mathbb{P}'}\right) \\ \sum_{P\in\mathbb{P}'} \chi_{E_P} \leq \chi_{\bigcup \mathbb{I}_{\mathbb{P}'}}. \end{cases}$$

Interpolation yields:

$$\|C_{\mathbb{P}'}\|_{L^{r'}} \leq \mu\left(\bigcup \mathbb{I}_{\mathbb{P}'}\right)^{1/r'}.$$

Clearly, $\mathbb{P}_{P',\geq}$ *is still an* ***anti-chain*** *and, thus, the result. The harder part is to actually* ***extract density factor*** $2^{-n}(1+\eta)^{dD+\epsilon}$ *from the* L^1 *estimate of* $C_{\mathbb{P}_{P',<}}$*. The rest just follows from interpolation. The idea is to look into the definition of* $\mathcal{A}_{\Pi_\alpha}$ *see what kind of control benefits our purpose:*

$$\forall P \in \mathbb{P},\ \mathcal{A}_{\Pi_\alpha}(P) = \mathcal{A}_{\Pi_{n,\alpha}}(P) := \sup_{\substack{\pi\in\Pi_{n,\alpha}\\ I_P\subset I_\pi}} \mathcal{A}(\pi)\left\langle\Delta(P,\pi)\right\rangle^{\epsilon} \in \left(2^{-n}, 2^{1-n}\right].$$

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Suppose we have good control on $\Delta(P, \pi)$, then we automatically get a collection of ***roughly*** 2^{-n}***-dense*** *tiles from* $\Pi_{n,\alpha}$***-relative references****. If we can further* ***recover*** *E_Ps with E_πs, we can bound the collection with a factor from* ***density*** *and* ***distance****. As a result, we shall analyze $C_{\mathbb{P}_{P',<}}$ with as* ***high temporal resolution*** *as possible, so that we only need the* ***coarsest spectral control*** *to complete the estimate. We start by setting up the* ***resolution*** *we analyze on:*

$$\mathbb{J} := M\left\{J \in \ \mathbb{I}^{\subseteq}_{\mathbb{P}_{P',<}} \mid \forall P \in \mathbb{P}_{P',<},\ I_P \not\subset J\right\} \in \mathbb{X}.$$

Since, by construction, $\bigcup \mathbb{I}_{\mathbb{P}_{P',<}} = \bigsqcup \mathbb{J}$, our goal reduces to the following:

$$\forall J \in \mathbb{J}, \sum_{P \in \mathbb{P}_{P',<}} \mu\left(E_P \cap J\right) = \sum_{\substack{P \in \mathbb{P}_{P',<} \\ J \subsetneq I_P}} \mu\left(E_P \cap J\right) \lesssim ?$$

Observe that, given $J \in \mathbb{J}$ and $P \in \mathbb{P}_{P',<}$ such that $J \subsetneq I_P$, we have:

$$\exists P_J \in \mathbb{P}_{P',<} \ s.t.\ I_{P_J} \subset \widehat{J} \subset I_P, \quad and,\ thus, \quad \widehat{J} \in \mathbb{I}_\alpha.$$

To recover $E_P \cap J$ while ***temporally*** *locked onto $\widehat{J} \in \mathbb{I}_\alpha$, we find:*

$$\exists! \pi_{J,P} \in \Pi_{n,\alpha} \ s.t.\ I_{\pi_{J,P}} = \widehat{J} \ \wedge \ \pi_{J,P} \trianglelefteq P.$$

Indeed, we verify that:

$$\because \omega_{\pi_{J,P}} \supset \omega_P \quad \therefore E_P \cap J \subset E_P \cap \widehat{J} \subset E_{\pi_{J,P}}.$$

Moreover, by Δ***-monotonicity****, we have:*

$$\because \pi_{J,P} \trianglelefteq P \lesssim_< P' \quad \therefore \Delta\left(\pi_{J,P}, P'\right) \leq \Delta(P, P') < \eta,\ i.e.\ \pi_{J,P} \lesssim_< P',$$

which tells us where to ***locate*** *the* ***needed reference*** *with respect to P'. On the other hand, to acquire* ***density control****, we need to* ***quantify*** *the* ***distance*** *between P_J and $\pi_{J,P}$. First, by* ***Embedding Inequality****, we see that:*

$$\Delta(P_J, \pi_{J,P}) \lesssim \|c_{\omega_{P_J}} - c_{\omega_{\pi_{J,P}}}\|_{I_{P_J}} \leq \|c_{\omega_{P_J}} - c_{\omega_{P'}}\|_{I_{P_J}} + \|c_{\omega_{P'}} - c_{\omega_{\pi_{J,P}}}\|_{I_{\pi_{J,P}}}.$$

The ***RHS*** *can be controlled:*

$$\begin{aligned} &\because P_J,\ \pi_{J,P} \lesssim_< P' \quad \therefore \Delta(P_J, P'),\ \Delta(\pi_{J,P}, P') < \eta \\ &\qquad by\ \textbf{\textit{proximity}},\ \ \|c_{\omega_{P_J}} - c_{\omega_{P'}}\|_{I_{P_J}},\ \|c_{\omega_{P'}} - c_{\omega_{\pi_{J,P}}}\|_{I_{\pi_{J,P}}} \lesssim 1 + \eta. \end{aligned}$$

In conclusion, we have: $\Delta(P_J, \pi_{J,P}) \lesssim 1 + \eta$ and, thus,

$$\begin{aligned} \frac{\left|E_{\pi_{J,P}}\right|}{\left|\widehat{J}\right|} = \mathcal{A}(\pi_{J,P}) \lesssim & \mathcal{A}(\pi_{J,P}) \left\langle \Delta\left(P_J, \pi_{J,P}\right)\right\rangle^\epsilon (1+\eta)^\epsilon \\ \leq & \mathcal{A}_{\Pi_{n,\alpha}}(P_J)(1+\eta)^\epsilon \leq 2^{1-n}(1+\eta)^\epsilon. \end{aligned}$$

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Now, we shall sum over $P \in \mathbb{P}_{P',<}$. To do so, we collect the ***needed references:***

$$\Pi_J := \left\{ \pi \in \Pi_{n,\alpha} \mid I_\pi = \widehat{J} \ \wedge \ \exists P \in \mathbb{P}_{P',<} \ s.t. \ \pi \trianglelefteq P \right\}.$$

By ***spectral packing constraint****, we see that:*

$$\because \forall \pi \in \Pi_j, \ \Delta(\pi, P') < \eta \quad \therefore \#\Pi_J \lesssim (1+\eta)^{dD}.$$

As a result, we have:

$$\sum_{\substack{P \in \mathbb{P}_{P',<} \\ J \subsetneq I_P}} \mu(E_P \cap J) \leq \sum_{\pi \in \Pi_J} \mu(E_\pi) \leq \#\Pi_J \cdot \sup_{\pi \in \Pi_J} |E_\pi| \lesssim 2^{-n}(1+\eta)^{dD+\epsilon}|J|.$$

Summing over $J \in \mathbb{J}$ completes the proof. □

Claim ($\lesssim n$-decay stack density extraction)**.**
*Given $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$ be $\lesssim n$-**decay stack**, $P' \in \tilde{\mathbb{D}}$, and $r \in (1,\infty)$, we have:*

$$\begin{cases} \left\| C_{\mathbb{P}_{P',\geq}} \right\|_{L^{r'}} \underset{r}{\lesssim} n\mu\left(\bigcup \mathbb{I}_{\mathbb{P}_{P',\geq}}\right)^{1/r'} \\ \left\| C_{\mathbb{P}_{P',<}} \right\|_{L^{r'}} \underset{r}{\lesssim} n\mu\left(\bigcup \mathbb{I}_{\mathbb{P}_{P',<}}\right)^{1/r'} 2^{-n/r'}(1+\eta)^{(dD+\epsilon)/r'}. \end{cases}$$

Proof ($\lesssim n$-decay stack density extraction)**.** *Since $\mathbb{P}_{P',\geq}$ and $\mathbb{P}_{P',<}$ are still $\lesssim n$-**decay stacks**, we can apply the **canonical decomposition**. By previous claim, we now have:*

$$\forall j,k, \quad \begin{cases} \left\| C_{\mathbb{P}^j_{P',\geq,k}} \right\|_{L^{r'}} \underset{r}{\lesssim} \mu\left(\bigsqcup \mathbb{I}_{\mathbb{P}^j_{P',\geq,k}}\right)^{1/r'} \\ \left\| C_{\mathbb{P}^j_{P',<,k}} \right\|_{L^{r'}} \underset{r}{\lesssim} \mu\left(\bigsqcup \mathbb{I}_{\mathbb{P}^j_{P',<,k}}\right)^{1/r'} 2^{-n/r'}(1+\eta)^{(dD+\epsilon)/r'}. \end{cases}$$

As we sum over j,k, for $\mathbb{P}'$ be $\mathbb{P}_{P',\geq}$ or $\mathbb{P}_{P',<}$, we have:

$$\|C_{\mathbb{P}'}\|_{L^{r'}} \leq \sum_{j,k} \left\| C_{\mathbb{P}'^j_k} \right\|_{L^{r'}}.$$

We only need to check:

$$\sum_{j,k} \mu\left(\bigsqcup \mathbb{I}_{\mathbb{P}'^j_k}\right)^{1/r'} = \sum_{j,k} \mu\left(\bigsqcup \mathbb{M}^j_{\mathbb{P}',k}\right) \overset{?}{\lesssim} n\mu\left(\bigcup \mathbb{I}_{\mathbb{P}'}\right)^{1/r'}.$$

To do so, we first represent $\bigcup \mathbb{I}_{\mathbb{P}'}$ with a ***carpet:***

$$\bigcup \mathbb{I}_{\mathbb{P}'} = \bigsqcup \mathbb{J}, \quad where \ \ \mathbb{J} := M\mathbb{I}_{\mathbb{P}'} \in \mathbb{X}.$$

By the $\underset{\frac{1}{2^\kappa - 1}}{\prec}$ ***-chain*** *structure on $\mathbb{M}^j_{\mathbb{P}',k}$:*

$$\cdots \underset{\frac{1}{2^\kappa-1}}{\prec} \mathbb{M}^j_{\mathbb{P}',k} \underset{\frac{1}{2^\kappa-1}}{\prec} \mathbb{M}^j_{\mathbb{P}',k-1} \underset{\frac{1}{2^\kappa-1}}{\prec} \cdots \underset{\frac{1}{2^\kappa-1}}{\prec} \mathbb{M}^j_{\mathbb{P}',2} \underset{\frac{1}{2^\kappa-1}}{\prec} \mathbb{M}^j_{\mathbb{P}',1} \prec \mathbb{J},$$

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we have:

$$\forall J \in \mathbb{J}, \quad \mu\left(J \cap \bigsqcup \mathbb{M}^j_{\mathbb{P}',k}\right) = \sum_{\substack{I \in \mathbb{M}^j_{\mathbb{P}',k} \\ I \subset J}} |I| \leq (2^\kappa - 1)^{1-k} |J|.$$

After summing over $J \in \mathbb{J}$, a direct computation shows that:

$$\begin{aligned}
\sum_{j,k} \mu\left(\bigsqcup \mathbb{I}_{\mathbb{P}'^j_k}\right)^{1/r'} &= \sum_{j=1}^{s_\Delta} \sum_{k \in \mathbb{N}} \mu\left(\bigsqcup \mathbb{M}^j_{\mathbb{P}',k}\right)^{1/r'} \\
&\leq \sum_{j=1}^{s_\Delta} \sum_{k \in \mathbb{N}} (2^\kappa - 1)^{\frac{1-k}{r'}} \mu(\bigsqcup \mathbb{J})^{1/r'} \\
&\lesssim_r \sum_{j=1}^{s_\Delta} \mu\left(\bigsqcup \mathbb{J}\right)^{1/r'} \lesssim n\mu\left(\bigcup \mathbb{I}_{\mathbb{P}'}\right)^{1/r'},
\end{aligned}$$

which completes the proof. □

Proof (Density extraction). *For the same reason in $\lesssim n$-**decay stack density extraction**, we only need to verify the following sum:*

$$\sum_{\alpha \in \mathbb{N}} \mu\left(\bigcup \mathbb{I}_{\mathbb{P}'(\alpha)}\right)^{1/r'} \overset{?}{\lesssim} \left|2\tilde{I}_{P'}\right|^{1/r'},$$

*where $\mathbb{P}'$ can be $\mathbb{P}_{P',\geq}$ or $\mathbb{P}_{P',<}$. We first recall that $\mathbb{I}_{\mathbb{P}'(\alpha)} \subset \mathbb{I}_\alpha := \mathbb{A}^{\subset}_{\alpha-1} \setminus \mathbb{A}^{\subset}_\alpha$. As we replace every layer with **carpets**:*

$$\mathbb{J}_\alpha := M\mathbb{I}_{\mathbb{P}'(\alpha)} \in \mathbb{X} \quad \text{and} \quad \mathbb{J} := M\left\{I \in \mathbb{D} \mid I \subset 2\tilde{I}_{P'}\right\} \in \mathbb{X},$$

we reduce to show that:

$$\sum_{\alpha \in \mathbb{N}} \mu\left(\bigsqcup \mathbb{J}_\alpha\right)^{1/r'} \leq \sum_{J \in \mathbb{J}} \sum_{\alpha \in \mathbb{N}} \mu\left(J \cap \bigsqcup \mathbb{J}_\alpha\right)^{1/r'} \overset{?}{\lesssim} \sum_{J \in \mathbb{J}} |J|^{1/r'} \lesssim \left|2\tilde{I}_{P'}\right|^{1/r'}.$$

*We now fix $J \in \mathbb{J}$ and find the $\alpha_J \in \mathbb{N}$ such that $J \in \mathbb{I}_{\alpha_J}$. Since $\mathbb{J}_\alpha \prec \mathbb{J}$ and $\mathbb{J}_\alpha \prec \mathbb{A}_{\alpha-1}$ for all $\alpha \in \mathbb{N}$, the δ-**covering relation** on $\mathbb{A}_\alpha$s implies:*

$$\mu\left(J \cap \bigsqcup \mathbb{J}_\alpha\right) = \sum_{\substack{I \in \mathbb{J}_\alpha \\ I \subset J}} |I| \quad \begin{cases} = 0 & \alpha - \alpha_J < 0 \\ \leq |J| & \alpha - \alpha_J = 0 \\ \leq \delta^{\alpha - \alpha_J - 1}|J| & \alpha - \alpha_J > 0. \end{cases}$$

Summing over $\alpha \in \mathbb{N}$ yields:

$$\begin{aligned}
\sum_{\alpha \in \mathbb{N}} \mu\left(J \cap \bigsqcup \mathbb{J}_\alpha\right)^{1/r'} &= \sum_{\alpha \geq \alpha_J} \left(\sum_{\substack{I \in \mathbb{J}_\alpha \\ I \subset J}} |I|\right)^{1/r'} \\
&\leq |J|^{1/r'} + \sum_{\alpha > \alpha_J} \delta^{\frac{\alpha - \alpha_J - 1}{r'}} |J|^{1/r'} \lesssim_r |J|^{1/r'}.
\end{aligned}$$

Summing over $J \in \mathbb{J}$ completes the proof. □

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8 TT* - T*T Arguments for Cluster Parts

We recall our settings: After **Eiffel Tower construction**, we choose $l \lesssim m = n$ for our **decomposition scheme**. In previous section, we have dealt with all the **sparse parts**. The rest is to control all the **cluster parts**.

8.1 Reductions

We build our arguments from **small** structure towards **large** structure, and we do so in a way to exploit both the **Pointwise control** and the **Orthogonality structure** of the L^2 settings. We lay out our plan:

1. Encode **density factor** into the **pointwise control** on **a single cluster**.
2. Control the **continuity/oscillation** of the **adjoint** of **a single cluster**.
3. Extract **apartness** through orthogonality between **a pair of clusters**.
4. Exploit δ**-covering relation** to control **interaction across** $\mathbb{A}_\alpha$s.
5. Organize **clusters** into **open** $2^{l\kappa}$**-apart** 1**-stacks**.
6. TT^*-T^*T **arguments** for L^2 estimate to **extract density factor**.
7. Modify TT^*-T^*T **arguments** for **extrapolation**.

With a plan in mind, we introduce some terminology and basic properties. We start by observe the kernel of a operator. Given $\mathbb{P} \subset \tilde{\mathbb{D}}$, we can **collapse** everything **except oscillation** into the kernel:

$$\mathfrak{L}_{\mathbb{P}} f(\cdot) = \int K_{\mathbb{P}}(\cdot, y) e^{iq_{(\cdot)}(y)} f(y) dy, \quad \text{where} \quad K_{\mathbb{P}}(x, y) := \sum_{P \in \mathbb{P}} K_{s_P}(x, y) \chi_{E_P}(x).$$

For simplicity, we also denote $E_{\mathbb{P}} := \bigcup_{P \in \mathbb{P}} E_P$ the **support** of the operator. Naturally, we expect some structures from $\mathbb{P}$ will be **reflected** in $K_{\mathbb{P}}$:

Properties 8.1.1 (Kernel structure of a **convex set**).
*Given $\mathbb{P} \subset \tilde{\mathbb{D}}$ **convex**, there are **simple measurable** functions $\underline{s}_{\mathbb{P}(\cdot)}$ and $\overline{s}_{\mathbb{P}(\cdot)}$ from $\mathbb{R}^D$ to $\mathbb{Z} \sqcup \{-\infty, \infty\}$ such that we have the following kernel expression:*

$$K_{\mathbb{P}}(x, y) = \chi_{E_{\mathbb{P}}}(x) \cdot \sum_{s = \underline{s}_{\mathbb{P}x}}^{\overline{s}_{\mathbb{P}x}} K_s(x, y) = \sum_{s = \underline{s}_{\mathbb{P}x}}^{\overline{s}_{\mathbb{P}x}} K_s(x, y)$$

Remark. *This is what we have said the **consecutive scaling**. Since a **cluster** is **convex**, it gives us hints to control a **cluster** with $\mathfrak{T}$.*

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Proof. *For fix $x \in E_{\mathbb{P}}$, we first verify the* ***consecutive scaling****. Given $P_j \in \mathbb{P}$ with $s_{P_0} < s_{P_1}$ such that $x \in E_{P_j}$, we have:*

$$\because \begin{cases} q_x \in \omega_{P_1} \subset \omega_{P_0} \\ x \in I_{P_0} \subset I_{P_1} \end{cases} \qquad \therefore P_0 \lhd P_1.$$

For any $s \in \mathbb{Z}$ such that $s_{P_0} < s < s_{P_1}$, we construct a tile as such:

$$\exists! I \in \mathbb{D}_s \text{ s.t. } x \in I \text{ and, } \textbf{then, } \exists! \omega \in \mathbb{D}_I^* \text{ s.t. } q_x \in \omega.$$

We define $P := I \times \omega$ and verify that:

$$\because \begin{cases} x \in E_P \\ I_{P_0} \subsetneq I \subsetneq I_{P_1} \\ \omega_{P_0} \supsetneq \omega \supsetneq \omega_{P_1} \end{cases} \qquad \therefore P_0 \lhd P \lhd P_1$$

By ***convexity****, $P \in \mathbb{P}$ and thus verify the* ***consecutive scaling****. We now explicitly define $\underline{s}_{\mathbb{P}x}$ and $\overline{s}_{\mathbb{P}x}$ for $x \in E_{\mathbb{P}}$:*

$$\begin{cases} \underline{s}_{\mathbb{P}x} := \min \{ s_P \in \mathbb{Z} \mid P \in \mathbb{P} \ \wedge \ x \in E_P \} \\ \overline{s}_{\mathbb{P}x} := \max \{ s_P \in \mathbb{Z} \mid P \in \mathbb{P} \ \wedge \ x \in E_P \}. \end{cases}$$

For $x \notin E_{\mathbb{P}}$, we ***assign*** *$\min \varnothing := \infty$ and $\max \varnothing := -\infty$ as our* ***convention*** *so that the definition conveniently gives us* ***empty sum****. Lastly, since $q_{(\cdot)}$, $\underline{s}_{(\cdot)}$, and $\overline{s}_{(\cdot)}$ are* ***simple measurable****, $\underline{s}_{\mathbb{P}(\cdot)}$ and $\overline{s}_{\mathbb{P}(\cdot)}$ must also be by construction.* $\square$

Now, we demonstrate the benefit we pick **cluster** as our **building block**. Given $\mathfrak{p} \in \tilde{\mathbb{D}}$, we set $q_{\mathfrak{p}} := c_{\omega_{\mathfrak{p}}}$ and decompose the **oscillation** term:

$$e^{iq_x(y)} = e^{i(q_x - q_{\mathfrak{p}})(x)} \begin{Bmatrix} e^{i\,(q_{\mathfrak{p}} - q_x)|_y^x} - 1 \\ +1 \end{Bmatrix} e^{iq_{\mathfrak{p}}(y)}$$

We view the first term as an **error correction** and the second term as the main **oscillation** from $\mathfrak{p}$. To control the error term, we use an elementary inequality:

$$\left| e^{i\textbf{radian}} - 1 \right| = |\textbf{displacement}| \leq |\textbf{radian}|$$

and, then, bound with a **local oscillation on polynomial**. As an important example, we have the following:

Properties 8.1.2 (Error correction of the oscillation**).**
Given $P, \mathfrak{p} \in \tilde{\mathbb{D}}$ such that $\lambda P \lhd \mathfrak{p}$, we have:

$$(x, y) \in E_P \times \Lambda I_P \implies \left| e^{i\,(q_{\mathfrak{p}} - q_x)|_y^x} - 1 \right| \leq \left| (q_{\mathfrak{p}} - q_x)|_y^x \right| \lesssim_{\Lambda, \lambda, \kappa, D, d} \frac{\|x - y\|}{\ell_{I_P}}$$

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Proof. *Fix $(x, y) \in E_P \times \Lambda I_P$, we define $I_{x,y}$ the* ***smallest*** *cube* ***containing*** *x, y.* ***Embedding Inequality*** *implies:*

$$\left|e^{i\,(q_{\mathfrak{p}}-q_x)|_y^x}-1\right| \le \left|(q_{\mathfrak{p}}-q_x)|_y^x\right| \le \|q_{\mathfrak{p}}-q_x\|_{I_{x,y}}$$
$$\lesssim_{D,d} \frac{\ell_{I_{x,y}}}{\ell_{\Lambda I_P}} \|q_{\mathfrak{p}}-q_x\|_{\Lambda I_P} \lesssim_{\Lambda,\kappa,D,d} \frac{\|x-y\|}{\ell_{I_P}} \|q_{\mathfrak{p}}-q_x\|_{I_P}.$$

Since $\lambda P \lhd \mathfrak{p}$, we have:

$$\|q_{\mathfrak{p}}-q_x\|_{I_P} \le \|q_{\mathfrak{p}}-q_P\|_{I_P} + \|q_P-q_x\|_{I_P} \lesssim_{\kappa} \lambda+1,$$

which completes the proof. □

The above-mentioned properties are the rigorous justification for choosing **cluster** as our **building block**. In short, we expect that a **cluster** should:

- Behave like $\mathfrak{T}$.
- **Temporally localized** on $I_{\mathfrak{p}}$.
- **Spectrally modulated** to $e^{iq_{\mathfrak{p}}}$.

From now on, we fix a **(open) cluster** $\mathfrak{P} \in \mathbb{P}_{n,\alpha}$ at $\mathfrak{p} \in \mathbb{P}_{n,\alpha}$ and investigate the inner structure of the corresponding operator.

Definition 8.1.3 (Inner structure of a **cluster**)**.**
We introduce the following notions:

- *Modulation operators:* $\begin{cases} \mu_{\mathfrak{p}} f(y) & := e^{iq_{\mathfrak{p}}(y)} f(y) \\ \mu f(x) & := e^{iq_x(x)} f(x). \end{cases}$
- *Model operators:*

$$\Omega_{\mathfrak{P}} f(x) := \begin{cases} \Phi_{\mathfrak{P}} f(x) := & \int K_{\mathfrak{P}}(x,y) \left(e^{i\,(q_{\mathfrak{p}}-q_x)|_y^x}-1\right) f(y)dy \\ & + \\ \Psi_{\mathfrak{P}} f(x) := & \int K_{\mathfrak{P}}(x,y) f(y)dy. \end{cases}$$

A direct consequence is that: $\mathfrak{L}_{\mathfrak{P}} = \mu\mu_{\mathfrak{p}}^* \Omega_{\mathfrak{P}} \mu_{\mathfrak{p}} = \mu\mu_{\mathfrak{p}}^* \Phi_{\mathfrak{P}} \mu_{\mathfrak{p}} + \mu\mu_{\mathfrak{p}}^* \Psi_{\mathfrak{P}} \mu_{\mathfrak{p}}$.

As a result, the boundedness of $\mathfrak{L}_{\mathfrak{P}}$ is completely governed by $\Phi_{\mathfrak{P}}$ and $\Psi_{\mathfrak{P}}$. One the other hand, the **spectral** behavior of $\mathfrak{L}_{\mathfrak{P}}$ hides inside the **modulation**. We need to consider the **adjoint** to flip it outside and extract the **separation factor**. In the next part, we use **Multi-resolution Analysis** to treat the **pointwise control** of the operator.

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8.2 Pointwise Control on Cluster

The idea is to work under **suitable** resolution and **preserve** the **density** information encoded in E_Ps. To proceed, we consider the following:

Definition 8.2.1 ($\mathfrak{P}$-fine setting)**.**
We construct the following carpet:

$$\mathbb{J}_{\mathfrak{P}} := M \left\{ J \in \mathbb{I}_{\mathfrak{P}}^{\subsetneq} \mid \forall P \in \mathfrak{P},\ I_P \not\subset J \right\} \in \mathbb{X}$$

so that $\bigcup \mathbb{I}_{\mathfrak{P}} = \bigsqcup \mathbb{J}_{\mathfrak{P}}$*. Additionally, for each* $J \in \mathbb{J}_{\mathfrak{P}}$*, we assign* ***references****:*

$$\Pi_J := \left\{ \pi \in \Pi_\alpha \mid I_\pi = \widehat{J}\ \wedge\ \exists P \in \mathfrak{P}\ s.t.\ \pi \trianglelefteq P \right\}$$

and the corresponding set:

$$E_J := J \cap \bigcup_{P \in \mathfrak{P}} E_P.$$

We expect that under suitable assumption, E_Js would carry some properties from E_Ps. Indeed, if we consider a cluster, we have the following:

Properties 8.2.2 (Density preservation)**.**
The $\mathfrak{P}$***-fine*** *setting satisfies:*

$$\frac{|E_J|}{|J|} \lesssim 2^{-n}, \quad \forall J \in \mathbb{J}_{\mathfrak{P}}.$$

Proof. *Fix* $J \in \mathbb{J}_{\mathfrak{P}}$*, we follow mostly* ***anti-chain density extraction****:*

- *By* ***maximality****, there is* $P_J \in \mathfrak{P} \subset \mathbb{P}_{n,\alpha}$ *such that* $I_{P_J} \subset \widehat{J}$*.*
- *For any* $\pi \in \Pi_J$*,* π *should be relatively close to* $\mathfrak{p} \in \mathbb{P}_{n,\alpha}$ *since:*

  $$\exists P \in \mathfrak{P},\ s.t.\ \begin{cases} \pi = P & \textit{and, thus,}\quad \lambda\pi \lhd \mathfrak{p} \\ \pi \lhd P & \textit{and, thus,}\quad \lambda\pi \lhd \lambda P \lhd \mathfrak{p}. \end{cases}$$

  As a result, by ***spectral packing constraint****,* $\#\Pi_J \underset{\lambda,\kappa,D,d}{\lesssim} 1$*.*
- *On the other hand, since* $\lambda P_J \lhd \mathfrak{p}$*,* ***Embedding Inequality*** *and* ***triangle inequality*** *implies:*

  $$\Delta\left(P_J, \pi\right) \leq \|q_{P_J} - q_{\mathfrak{p}}\|_{\tilde{I}_{P_J}} + \|q_{\mathfrak{p}} - q_\pi\|_{\tilde{\widehat{J}}} \underset{\lambda,\kappa,D,d}{\lesssim} 1.$$
- *Through the definition of* $\mathcal{A}_{\Pi_{n,\alpha}}$*, we have* ***density control****:*

  $$\mathcal{A}(\pi) \underset{\epsilon,\lambda,\kappa,D,d}{\lesssim} \mathcal{A}(\pi) \left\langle \Delta(P_J, \pi) \right\rangle^{\epsilon} \leq \mathcal{A}_{\Pi_\alpha}(P_J) = \mathcal{A}_{\Pi_{n,\alpha}}(P_J) \lesssim 2^{-n}.$$

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- Π_J *actually recovers* E_J *in the following sense:*

$$\because E_J = J \cap \bigcup_{\substack{P\in\mathfrak{P} \\ \widehat{J}\subset I_P}} E_P \subset \bigsqcup_{\pi\in\Pi_J} E_\pi$$

$$\therefore \frac{|E_J|}{|J|} \lesssim \sum_{\pi\in\Pi_J} \frac{|E_\pi|}{|\widehat{J}|} \leq \#\Pi_J \cdot \sup_{\pi\in\Pi_J} \mathcal{A}(\pi) \underset{\epsilon,\lambda,\kappa,D,d}{\lesssim} 2^{-n}.$$

□

Now, we proceed to estimate the contribution of **error correction** in $\Phi_{\mathfrak{P}} f$ and bound $\Psi_{\mathfrak{P}} f$ with $\mathfrak{T} f$.

Lemma 8.2.3 **(Cluster estimate).**
Both ***Model Operators*** *have* ***pointwise control****:*

$$\begin{cases} |\Phi_{\mathfrak{P}} f| \lesssim \displaystyle\sum_{J\in\mathbb{J}_{\mathfrak{P}}} \left(\inf_J Mf\right) \chi_{E_J} \\ |\Psi_{\mathfrak{P}} f| \lesssim \displaystyle\sum_{J\in\mathbb{J}_{\mathfrak{P}}} \left(\inf_J \mathfrak{T} f + \inf_J Mf\right) \chi_{E_J}. \end{cases}$$

Consequently, we have:

$$|\mathfrak{L}_{\mathfrak{P}} f| \lesssim \sum_{J\in\mathbb{J}_{\mathfrak{P}}} \left(\inf_J \mathfrak{T}\mu_{\mathfrak{p}} f + \inf_J Mf\right) \chi_{E_J}.$$

Remark. *Since* E_J*s* ***preserve density****, a direct consequence is:*

$$\begin{aligned} \|\mathfrak{L}_{\mathfrak{P}} f\|_{L^p} &\underset{p}{\lesssim} \left(\sum_{J\in\mathbb{J}_{\mathfrak{P}}} \left| \inf_J \mathfrak{T}\mu_{\mathfrak{p}} f + \inf_J Mf \right|^p |E_J| \right)^{1/p} \\ &\underset{p}{\lesssim} \left(2^{-n} \sum_{J\in\mathbb{J}_{\mathfrak{P}}} \left| \inf_J \mathfrak{T}\mu_{\mathfrak{p}} f + \inf_J Mf \right|^p |J| \right)^{1/p} \\ &\leq 2^{-n/p} \|\mathfrak{T}\mu_{\mathfrak{p}} f + Mf\|_{L^p(I_{\mathfrak{p}})}. \end{aligned}$$

Proof. *We verify the control for each* $J \in \mathbb{J}_{\mathfrak{P}}$*. Starting with* $\Phi_{\mathfrak{P}}$*, since* $\mathfrak{P}$ *is a cluster at* $\mathfrak{p}$*, we have* $P \in \mathfrak{P} \implies \lambda P \lhd \mathfrak{p}$*.* ***Error correction*** *of the oscillation implies the following control:*

$$\forall P \in \mathfrak{P}, \left((x,y) \in E_P \times \tilde{I}_P \implies \left| e^{i\,(q_{\mathfrak{p}} - q_x)|_y^x} - 1 \right| \lesssim \frac{\|x-y\|}{\ell_{I_P}} \right).$$

Yet, with a change of perspective, we can choose ***the best bound*** *for each* $x \in E_J$*. That is, we consider the following collection:*

$$\mathfrak{P}_x := \{P \in \mathfrak{P} \mid x \in E_P\}.$$

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Since for each ***scale*** $s \in \mathbb{Z}$ *there is* ***unique*** $P \in \tilde{\mathbb{D}}$ *such that* $(x, q_x) \in I_P \times \omega_P$ *and* $s_P = s$*, we see that:*

$$\underline{s}_{\mathfrak{P}_x} \leq s \leq \overline{s}_{\mathfrak{P}_x} \implies \exists! P_s \in \mathfrak{P}_x \ s.t. \ s_{P_s} = s.$$

We, therefore use the following estimate:

$$\forall y \in \tilde{I}_{P_{\overline{s}_{\mathfrak{P}_x}}}, \ \left| e^{i\,(q_{\mathfrak{p}} - q_x)|_y^x} - 1 \right| \lesssim \frac{\|x-y\|}{\ell_{I_{P_{\overline{s}_{\mathfrak{P}_x}}}}} = 2^{-\overline{s}_{\mathfrak{P}_x}\kappa}\|x-y\|.$$

As a result, since:

$$\forall P \in \mathfrak{P}_x, \ J \subsetneq I_P \subset I_{P_x},$$

the estimate can be used ***universally*** *when dealing with the collection* $\mathfrak{P}_x$*. Combined with* ***support*** *and* ***size*** *control on kernel* K*, we have:*

$$\begin{aligned}
|\Phi_{\mathfrak{P}} f(x)| &\leq \sum_{P \in \mathfrak{P}_x} \int |K_{s_P}(x,y)| \cdot \left| e^{i\,(q_{\mathfrak{p}} - q_x)|_y^x} - 1 \right| \cdot |f(y)| dy \\
&\lesssim \sum_{s=\underline{s}_{\mathfrak{P}_x}}^{\overline{s}_{\mathfrak{P}_x}} \int \frac{\chi_{\tilde{I}_{P_s}}(y)}{|I_{P_s}|} \frac{\cancelto{\ell_{I_{P_s}}}{\|x-y\|}}{\ell_{I_{P_{\overline{s}_{\mathfrak{P}_x}}}}} |f(y)| dy \\
&\lesssim \sum_{s=\underline{s}_{\mathfrak{P}_x}}^{\overline{s}_{\mathfrak{P}_x}} 2^{(s-\overline{s}_{\mathfrak{P}_x})\kappa} \cdot \sup_{\substack{P \in \mathfrak{P} \\ J \subsetneq I_P}} \fint_{\tilde{I}_P} |f| d\mu \lesssim \inf_J Mf.
\end{aligned}$$

This completes the estimate for $\Phi_{\mathfrak{P}}$*. For* $\Psi_{\mathfrak{P}}$*, we use the following principle:*

$$\textbf{\textit{Upper bound on }} J \leq \textbf{\textit{Lower bound on }} J + \textbf{\textit{Oscillation on }} J.$$

Thus, we shall first measures the ***oscillation****: Given arbitrary* $(x, \xi) \in E_J \times J$*,*

$$\begin{aligned}
\left| \Psi_{\mathfrak{P}}(x) - \int \sum_{s=\underline{s}_{\mathfrak{P}_x}}^{\overline{s}_{\mathfrak{P}_x}} K_s(\xi,y) f(y) dy \right| &\leq \sum_{\substack{P \in \mathfrak{P}_x \\ (J \subsetneq I_P)}} \int \left| K_{s_P}(\cdot, y)|_\xi^x \right| \cdot |f(y)| dy \\
\tau\textbf{\textit{-Hölder regularity implies}} \quad &\lesssim \sum_{\substack{P \in \mathfrak{P}_x \\ (J \subsetneq I_P)}} \left(\frac{\|x-\xi\|}{\ell_{I_P}} \right)^\tau \int \frac{\chi_{\tilde{I}_P}(y)}{|I_P|} |f(y)| dy \\
\because \sum_{\substack{P \in \mathfrak{P}_x \\ (J \subsetneq I_P)}} (\ell_J / \ell_{I_P})^\tau \lesssim_\tau 1 \quad \therefore \ &\lesssim_\tau \sup_{\substack{P \in \mathfrak{P} \\ J \subsetneq I_P}} \fint_{\tilde{I}_P} |f| d\mu \lesssim \inf_J Mf.
\end{aligned}$$

On the other hand,

$$\forall \epsilon > 0, \ \exists \xi \in J \ s.t. \ \left| \int \sum_{s=\underline{s}_{\mathfrak{P}_x}}^{\overline{s}_{\mathfrak{P}_x}} K_s(\xi, y) f(y) dy \right| \leq \mathfrak{T} f(\xi) < \inf_J \mathfrak{T} f + \epsilon.$$

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Triangular inequality *yields:*

$$\begin{aligned}|\Psi_{\mathfrak{P}}(x)| \leq & \left|\int \sum_{s=\underline{s}_{\mathfrak{P}x}}^{\overline{s}_{\mathfrak{P}x}} K_s(\xi,y)f(y)dy\right| + \left|\Psi_{\mathfrak{P}}(x) - \int \sum_{s=\underline{s}_{\mathfrak{P}x}}^{\overline{s}_{\mathfrak{P}x}} K_s(\xi,y)f(y)dy\right| \\ & \lesssim \inf_J \mathfrak{T}f + \not{e} + \inf_J Mf,\end{aligned}$$

which completes the proof. □

As we have mentioned earlier, we also need the control on its **adjoint**. To complete the argument, we introduce an operator which **arises naturally** in our analysis on the **adjoint**:

Definition 8.2.4 (Auxiliary maximal operator)**.**
Recall the buffer $\varpi \gg 1$ used in the definition of ***openness*** *of a cluster, we consider the following maximal operator:*

$$M^*_{\mathfrak{P}}f(y) := \sup_{\substack{P\in\mathfrak{P} \\ y\in\varpi\tilde{I}_P}} |I_P|^{-1} \int_{E_P} |f|\, d\mu.$$

Properties 8.2.5.

$$\left\|M^*_{\mathfrak{P}}f\right\|_{L^p} \lesssim \left(\sup_{P\in\mathfrak{P}} \mathcal{A}(P)\right)^{1/p'} \|f\|_{L^p}, \quad \forall p \in (1,\infty].$$

Proof. *It is easy to see that:*

$$\left\|M^*_{\mathfrak{P}}f\right\|_{L^\infty} \leq \sup_{P\in\mathfrak{P}} \mathcal{A}(P)\, \|f\|_{L^\infty}.$$

To verify the full range of the property, we only need to acquire:

$$\left\|M^*_{\mathfrak{P}}f\right\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}$$

and interpolate to finish the proof. For $t \in \mathbb{R}_+$, we consider the following set:

$$\mathfrak{P}_t := \left\{P \in \mathfrak{P} \mid t < |I_P|^{-1} \int_{E_P} |f| d\mu\right\}.$$

By construction, we have:

$$\begin{aligned}&\because\ \left|M^*_{\mathfrak{P}}f\right|^{-1}(t,\infty] = \bigcup_{P\in\mathfrak{P}_t} \varpi\tilde{I}_P = \bigcup_{P\in M\mathfrak{P}_t} \varpi\tilde{I}_P \\ &\therefore\ \mu\left(\left|M^*_{\mathfrak{P}}f\right|^{-1}(t,\infty]\right) \underset{\varpi,\kappa,D}{\lesssim} \sum_{P\in M\mathfrak{P}_t} |I_P| \\ &\leq t^{-1} \sum_{P\in M\mathfrak{P}_t} \int_{E_P} |f| d\mu = t^{-1} \int_{E_{M\mathfrak{P}_t}} |f| d\mu \leq t^{-1} \|f\|_{L^1},\end{aligned}$$

which completes the proof. □

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Remark. *Due to the definition of the* ***open cluster****, we see that if* $\mathfrak{P}$ *is an* ***open cluster*** *at* $\mathfrak{p}$*, we have:*

$$\because \forall P \in \mathfrak{P},\ \varpi \tilde{I}_P \subset I_{\mathfrak{p}} \quad \therefore \mathrm{supp} M^*_{\mathfrak{P}} f \subset I_{\mathfrak{p}}.$$

Additionally, if furthermore $\mathfrak{P} \subset \mathbb{P}_{n,\alpha} \subset \Pi_\alpha$*, we have:*

$$\mathcal{A}(P) = \mathcal{A}(P) \langle \Delta(P,P) \rangle^\epsilon \leq \mathcal{A}_{\Pi_\alpha}(P) = \mathcal{A}_{\Pi_{n,\alpha}}(P) \lesssim 2^{-n}.$$

Therefore, $\left\| M^*_{\mathfrak{P}} f \right\|_{L^p} = \left\| M^*_{\mathfrak{P}} f \right\|_{L^p(I_{\mathfrak{p}})} \lesssim 2^{-n/p'} \left\| f \right\|_{L^p(I_{\mathfrak{p}})}$.

After the necessary setup, we investigate the properties of the **adjoint** operator. We expect that the adjoint should reflect some properties from the kernel, and, indeed, we have the following:

Lemma 8.2.6 (**Adjoint local** τ**-Hölder continuity**).
Given a cube $L \subset \mathbb{R}^D$ *satisfying the following: For any* $P \in \mathfrak{P}$*,*

$$I_P^* \cap L \neq \varnothing \implies \ell_L \underset{\varpi,\kappa,D}{\lesssim} \ell_{I_P} \approx \mathrm{dist}(L, I_P),$$

we then have:

$$\forall y, \eta \in L,\ \left| \Omega^*_{\mathfrak{P}} f \Big|_\eta^y \right| \lesssim \left(\frac{\|y-\eta\|}{\ell_L} \right)^\tau \inf_L M^*_{\mathfrak{P}} f$$

Remark. *The condition on* L *is designed to* ***fully exploit*** *the* ***local*** τ***-Hölder continuity*** *of* $K_s s$*.*

Proof. *Given* $y, \eta \in L$*, we evaluate the difference:*

$$\begin{aligned}
\left| \Omega^*_{\mathfrak{P}} f \Big|_\eta^y \right| &= \left| \sum_{\substack{P \in \mathfrak{P} \\ I_P^* \cap L \neq \varnothing}} \int_{E_P} \left(K_{s_P}(x,\cdot) e^{i\,(q_{\mathfrak{p}} - q_x)|^x_{(\cdot)}} \right) \Big|_\eta^y f(x) dx \right| \\
&\leq \sum_{\substack{P \in \mathfrak{P} \\ I_P^* \cap L \neq \varnothing}} \int_{E_P} \left| K_{s_P}(x,\cdot)|_\eta^y \right| \cdot \left| \cancel{e^{i(q_{\mathfrak{p}} - q_x)(y)}} \right| \cdot |f(x)| dx \\
&+ \sum_{\substack{P \in \mathfrak{P} \\ I_P^* \cap L \neq \varnothing}} \int_{E_P} |K_{s_P}(x,\eta)| \cdot \left| e^{i\,(q_{\mathfrak{p}} - q_x)|^x_{(\cdot)}} \Big|_\eta^y \right| \cdot |f(x)| dx.
\end{aligned}$$

For now, we fix $P \in \mathfrak{P}$ *with* $I_P^* \cap L \neq \varnothing$*. By assumption, we have:*

$$\|y - \eta\| \lesssim \ell_L \lesssim \ell_{I_P} = 2^{s_P \kappa}.$$

As a result, ***local*** τ***-Hölder continuity*** *and* ***size control*** *of* $K_s s$ *implies:*

$$\forall x \in E_P,\ \begin{cases} \left| K_{s_P}(x,\cdot)|_\eta^y \right| & \lesssim |I_P|^{-1} \left(\frac{\|y-\eta\|}{\ell_{I_P}} \right)^\tau \\ |K_{s_P}(x,\eta)| & \lesssim |I_P|^{-1}. \end{cases}$$

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On the other hand, ***error correction*** *on oscillation yields:*

$$\forall x \in E_P,\ \left| e^{i\,(q_{\mathfrak{p}}-q_x)|^{x}_{(\cdot)}} \Big|^{y}_{\eta} \right| = \left| e^{-i\,(q_{\mathfrak{p}}-q_x)|^{y}_{\eta}} - 1 \right| \lesssim \frac{\|y-\eta\|}{\ell_{I_P}}.$$

Combine estimate on ***kernel*** *and* ***oscillation****, we get:*

$$\left| \Omega^*_{\mathfrak{P}} f \Big|^{y}_{\eta} \right| \lesssim \sum_{\substack{P \in \mathfrak{P} \\ I^*_P \cap L \neq \varnothing}} \left[\left(\frac{\|y-\eta\|}{\ell_{I_P}} \right)^{\tau} + \cancel{\frac{\|y-\eta\|}{\ell_{I_P}}} \right] |I_P|^{-1} \int_{E_P} |f| d\mu.$$

To sum over such Ps, we need to make sure that there are only $\lesssim 1$ tiles Ps with the same scales in the sum. This is guaranteed by the assumption:

$$\begin{aligned} &\because\ 2^{s_P \kappa} = \ell_{I_P} \approx dist(L, I_P), \\ &\therefore\ \forall s \in \mathbb{Z},\ \#\{P \in \mathfrak{P} \mid s_P = s\ \wedge\ I^*_P \cap L \neq \varnothing\} \lesssim 1. \end{aligned}$$

Therefore, we can safely sum over those Ps and acquire:

$$\begin{aligned} \left| \Omega^*_{\mathfrak{P}} f \Big|^{y}_{\eta} \right| &\lesssim \sum_{\substack{P \in \mathfrak{P} \\ I^*_P \cap L \neq \varnothing}} \left(\frac{\|y-\eta\|}{\ell_{I_P}} \right)^{\tau} \sup_{\substack{P \in \mathfrak{P} \\ I^*_P \cap L \neq \varnothing}} |I_P|^{-1} \int_{E_P} |f| d\mu \\ &\lesssim \sum_{\substack{s \in \mathbb{Z} \\ \ell_L \lesssim 2^{s\kappa}}} \left(\frac{\|y-\eta\|}{2^{s\kappa}} \right)^{\tau} \sup_{\substack{P \in \mathfrak{P} \\ L \subset \varpi \tilde{I}_P}} |I_P|^{-1} \int_{E_P} |f| d\mu \\ &\lesssim \left(\frac{\|y-\eta\|}{\ell_L} \right)^{\tau} \inf_{L} M^*_{\mathfrak{P}} f. \end{aligned}$$

□

This gives us hint on how **high** the **resolution** we shall analyze on:

Definition 8.2.7 ($\mathfrak{P}$-fine dual setting)**.**
We define the following carpet:

$$\mathbb{L}_{\mathfrak{P}} := M \left\{ L \in \mathbb{S}h^{\subsetneq}_{\mathfrak{P}} \mid \forall I \in \mathbb{S}h_{\mathfrak{P}},\ I \not\subset L \right\} \in \mathbb{X},$$

$$\textit{where}\ \ \mathbb{S}h_{\mathfrak{P}} := \{\ell_{I_P}\xi + I_P \in \mathbb{D} \mid (P, \xi) \in \mathfrak{P} \times \mathbb{S}\mathrm{h}\} = \bigcup_{P \in \mathfrak{P}} \mathbb{S}h_{I_P}$$

so that, by construction, we have $\bigcup \mathbb{S}h_{\mathfrak{P}} = \bigsqcup \mathbb{L}_{\mathfrak{P}}$.

Remark. *This comes from the original construction:*

$$I^* := \bigsqcup \mathbb{S}h_I = \bigsqcup_{\xi \in \mathbb{S}\mathrm{h}} \ell_I \xi + I.$$

By construction, we guarantee that:

$$\forall\,(L, P) \in \mathbb{L}_{\mathfrak{P}} \times \mathfrak{P}, \left(I^*_P \cap L \neq \varnothing \implies \exists \xi \in \mathbb{S}\mathrm{h}\ s.t.\ L \subsetneq \ell_{I_P}\xi + I_P \right).$$

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Yet, recall that $\text{Sh} := \{z \in \mathbb{Z} \mid n_D \leq |z| \leq n_D 2^\kappa + 1\}^D$. *This entails that:*

$$\exists \xi \in \text{Sh} \text{ s.t. } L \subsetneq \ell_{I_P}\xi + I_P \implies \ell_L < \ell_{I_P} \underset{\kappa,D}{\approx} dist(L, I_P),$$

which is exactly our condition for ***Adjoint local*** τ***-Hölder continuity****.*

As a direct consequence, we have:

Lemma 8.2.8 (**Adjoint cluster estimate**).
Adjoint *of the* ***Model Operator*** *has pointwise control:*

$$\left|\Omega^*_{\mathfrak{P}} f\right| \lesssim \sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left(\inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right) \chi_L.$$

Also, we recover that:

$$\left|\mathfrak{L}^*_{\mathfrak{P}} f\right| \lesssim \sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left(\inf_L \left|\mathfrak{L}^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right) \chi_L.$$

Remark. *The* ***density*** *information is packed inside* $\inf_L \left|\Omega^*_{\mathfrak{P}} f\right|$ *and* $\inf_L M^*_{\mathfrak{P}} f$*.*

Proof. *Fixing* $L \in \mathbb{L}_{\mathfrak{P}}$*,* ***adjoint local*** τ***-Hölder continuity*** *implies:*

$$\forall y, \eta \in L, \ \left|\Omega^*_{\mathfrak{P}} f\Big|^y_\eta\right| \lesssim \cancel{\left(\frac{\|y-\eta\|}{\ell_L}\right)^\tau} \inf_L M^*_{\mathfrak{P}} f.$$

On the other hand, we use the same trick:

$$\forall \epsilon > 0, \ \exists \eta \in L \text{ s.t. } \left|\Omega^*_{\mathfrak{P}} f(\eta)\right| < \inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \epsilon.$$

Thus, ***triangle inequality*** *yields:*

$$\left|\Omega^*_{\mathfrak{P}} f(y)\right| \leq \left|\Omega^*_{\mathfrak{P}} f\Big|^y_\eta\right| + \left|\Omega^*_{\mathfrak{P}} f(\eta)\right| \lesssim \inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \cancel{\epsilon} + \inf_L M^*_{\mathfrak{P}} f,$$

which completes the proof. □

8.3 Extraction of Separation Factor

Finally, with adjoint **local** τ**-Hölder continuity**. We are ready to extract the **separation/apartness factor**. We first observe that, given **cluster** $\mathfrak{P}_j \subset \mathbb{P}_{n,\alpha}$ at $\mathfrak{p}_j \in \mathbb{P}_{n,\alpha}$, we can write:

$$\left\langle \mathfrak{L}^*_{\mathfrak{P}_0} f_0, \mathfrak{L}^*_{\mathfrak{P}_1} f_1 \right\rangle = \int e^{i(q_{\mathfrak{p}_0} - q_{\mathfrak{p}_1})} \Omega^*_{\mathfrak{P}_0} f_{\mathfrak{p}_0} \cdot \overline{\Omega^*_{\mathfrak{P}_1} f_{\mathfrak{p}_1}} d\mu,$$

where $f_{\mathfrak{p}_0} := \mu_{\mathfrak{p}_0} \mu^* f$ and $f_{\mathfrak{p}_1} := \mu_{\mathfrak{p}_1} \mu^* f_1$. This is exactly the form of **Van der Corput estimate** if we view $q := q_{\mathfrak{p}_0} - q_{\mathfrak{p}_1}$ and $\psi := \Omega^*_{\mathfrak{P}_0} f_{\mathfrak{p}_0} \cdot \overline{\Omega^*_{\mathfrak{P}_1} f_{\mathfrak{p}_1}}$. Yet, since we only have **adjoint local** τ**-Hölder continuity**, we should apply the version **adapted** to a **partition of unity**. On the other hand, there is always some location where the **local oscillation of polynomial** is **small**. We need to find some balance in our analysis.

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Lemma 8.3.1 (**Apartness** control)**.**
*Given $\mathfrak{P}_j \subset \mathbb{P}_{n,\alpha}$ **open cluster** at $\mathfrak{p}_j \in \mathbb{P}_{n,\alpha}$, if $\mathfrak{P}_0$ and $\mathfrak{P}_1$ are Λ-apart, then:*

$$\left|\left\langle \mathfrak{L}^*_{\mathfrak{P}_0} f_0, \mathfrak{L}^*_{\mathfrak{P}_1} f_1 \right\rangle\right| \lesssim \Lambda^{-\epsilon} 2^{-n} \|f_0\|_{L^2} \|f_1\|_{L^2}.$$

Remark. *The estimate we acquire is actually a little bit **different** compared to the one in [Lie20] or in [Zor19] since we extract the **density factor** and **separation factor simultaneously**. Still, this improvement only indicates that the **separation factor** only need to serve the role to compensate the **temporal overlaps** of the **covers**.*

Notice the estimate is trivial if $\Lambda \lesssim 1$ or $I_{\mathfrak{p}_0} \cap I_{\mathfrak{p}_1} = \varnothing$, thus, we shall assume $\Lambda \gg 1$ and $I_{\mathfrak{p}_0} \subset I_{\mathfrak{p}_1}$. Also, due to the **openness**, we only need to evaluate the integral on $I_{\mathfrak{p}_0}$. Eventually, we reduces to show the following:

Lemma 8.3.2 (Extraction of separation factor)**.**
*Given Λ-apart $\mathfrak{P}, \mathfrak{P}' \subset \tilde{\mathbb{D}}$ **open clusters** at $\mathfrak{p}, \mathfrak{p}' \in \tilde{\mathbb{D}}$ respectively with $I_{\mathfrak{p}} \subset I_{\mathfrak{p}'}$,*

$$\left|\int e^{i(q_{\mathfrak{p}} - q_{\mathfrak{p}'})} \Omega^*_{\mathfrak{P}} f \overline{\Omega^*_{\mathfrak{P}'} g} d\mu\right| \lesssim \Lambda^{-\epsilon_p} \left\| \left|\Omega^*_{\mathfrak{P}} f\right| + M^*_{\mathfrak{P}} f \right\|_{L^p(I_{\mathfrak{p}})} \left\| \left|\Omega^*_{\mathfrak{P}'} g\right| + M^*_{\mathfrak{P}'} g \right\|_{L^{p'}(I_{\mathfrak{p}})}.$$

Proof. *To separate **Major oscillation** from **Noise**, we first set $q := q_{\mathfrak{p}} - q_{\mathfrak{p}'}$, pick $\varrho < 1 < \delta \underset{\kappa,D}{\ll} C \underset{\kappa,D}{\ll} \varpi$, and consider the following collection:*

$$\mathfrak{M} := \left\{P \in \mathfrak{P} \cup \mathfrak{P}' \mid \|q\|_{I_P} \geq \Lambda^{\varrho}\right\} \quad \text{and} \quad \mathfrak{N} := (\mathfrak{P} \cup \mathfrak{P}') \setminus \mathfrak{M}.$$

*We fist notice that, **apartness** implies:*

$$\forall P \in \mathfrak{P}, \quad \because I_P \subset I_{\mathfrak{p}} \subset I_{\mathfrak{p}'} \quad \therefore \Delta(P, \mathfrak{p}') > \Lambda$$

If $\Lambda \underset{\varrho,\lambda}{\gg} 1$, any $P \in \mathfrak{P}$ satisfies:

$$\|q\|_{I_P} \eqsim \|q\|_{\tilde{I}_P} \gtrsim \|q_{\mathfrak{p}'} - q_P\|_{\tilde{I}_P} - \|q_P - q_{\mathfrak{p}}\|_{\tilde{I}_P} \gtrsim \Delta(P, \mathfrak{p}') - \lambda \gtrsim \Lambda.$$

Alternatively, any $P \in \mathfrak{P}'$ with $I_P \subset I_{\mathfrak{p}}$ must also satisfy:

$$\|q\|_{I_P} \eqsim \|q\|_{\tilde{I}_P} \gtrsim \|q_{\mathfrak{p}} - q_P\|_{\tilde{I}_P} - \|q_P - q_{\mathfrak{p}'}\|_{\tilde{I}_P} \gtrsim \Delta(P, \mathfrak{p}) - \lambda \gtrsim \Lambda.$$

*As a direct result of **monotonicity** of the **semi-norm**, another $P' \in \mathfrak{P}'$ with $I_{P'} \supset I_{\mathfrak{p}} \supset I_P$ would also satisfy $\|q\|_{I_{P'}} \geq \|q\|_{I_P} \gtrsim \Lambda$. This poses quite a lot **restriction** on the **configuration** of $\mathfrak{N}$. In short, for large Λ, we always have $\mathfrak{P} \subset \mathfrak{M}$ and the following characterization:*

$$\mathfrak{N} = \mathfrak{P}' \setminus \mathfrak{M} = \{P \in \mathfrak{P}' \mid \|q\|_{I_P} < \Lambda^{\varrho}\} \subset \{P \in \mathfrak{P}' \mid I_P \cap I_{\mathfrak{p}} = \varnothing\}.$$

*For the **Major oscillation** in $\mathfrak{P}'$, we denote $\mathfrak{Q} := \mathfrak{P}' \cap \mathfrak{M}$. We now investigate the properties of the decomposition $\mathfrak{P}' = \mathfrak{Q} \sqcup \mathfrak{N}$. Due to the **semi-norm structure**: Given $P_j \in \mathfrak{Q}$ and $P \in \tilde{\mathbb{D}}$,*

$$P_0 \lhd P \lhd P_1 \implies \left(P \in \mathfrak{P}' \ \wedge\ \Lambda^{\varrho} \leq \|q\|_{I_{P_0}} \leq \|q\|_{I_P}\right) \implies P \in \mathfrak{Q}.$$

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Similarly, for $P_j \in \mathfrak{N}$ and $P \in \tilde{\mathbb{D}}$,

$$P_0 \lhd P \lhd P_1 \implies \left(P \in \mathfrak{P}' \ \wedge\ \|q\|_{I_P} \le \|q\|_{I_{P_1}} < \Lambda^{\varrho}\right) \implies P \in \mathfrak{N}.$$

By preserving the ***convex*** *structure, $\mathfrak{D}$ and $\mathfrak{N}$ are both* ***open clusters*** *at $\mathfrak{p}'$. We now reduce to analyze the two integrals:*

$$\left|\int_{I_{\mathfrak{p}}} e^{iq}\Omega^*_{\mathfrak{P}} f \overline{\Omega^*_{\mathfrak{P}'} g} d\mu\right| \le \begin{cases} \left|\int e^{iq}\Omega^*_{\mathfrak{P}} f \overline{\Omega^*_{\mathfrak{D}} g} d\mu\right| & \textbf{\textit{Major oscillation}} \\ + & \\ \int_{I_{\mathfrak{p}}} \cancel{\left|e^{iq}\right|} \left|\Omega^*_{\mathfrak{P}} f\right| \cdot \left|\Omega^*_{\mathfrak{N}} g\right| d\mu & \textbf{\textit{Noise}}. \end{cases}$$

To locate the different features from ***Major oscillation****, we do a* ***Whitney-like*** *decomposition on $\mathbb{R}^D$:*

$$\mathbb{L} := M\left\{L \in \mathbb{D} \mid \forall (P,\xi) \in \mathfrak{M} \times \mathrm{Sh},\ \ell_{I_P}\xi + I_P \not\subset CL\right\} \in \mathbb{X}$$

so that any element $L \in \mathbb{L}$ satisfies the following:

- *Locate* ***Major oscillation****:*

$$\begin{aligned} &\because\ \exists (P,\xi) \in \mathfrak{M} \times \mathrm{Sh},\ s.t.\ \ell_{I_P}\xi + I_P \subset C\widehat{L}, \\ &\therefore\ \Lambda^{\varrho} \le \|q\|_{I_P} \approx \|q\|_{\tilde{I}_P} \approx \|q\|_{\ell_{I_P}\xi + I_P} \le \|q\|_{CL} \approx \|q\|_L. \end{aligned}$$

- *Condition for* ***Adjoint local τ-Hölder continuity****: Given $P \in \mathfrak{M}$,*

$$3\delta L \cap I_P^* \ne \varnothing \implies \ell_L \lesssim \ell_{I_P} \approx dist(L, I_P).$$

  This follows from the fact that $3\delta L \cap I_P^* \ne \varnothing$ implies:

$$\exists \xi \in \mathrm{Sh},\ s.t.\ 3\delta L \cap \ell_{I_P}\xi + I_P \ne \varnothing.$$

  Yet, if $\ell_{3\delta L} \ge 3\delta \ell_{I_P}$ (or equivalently $\ell_L \ge \ell_{I_P}$), then, by choosing $C \ge 3\delta + 2$, we have the following:

$$\begin{aligned} &\because\ \forall x \in \ell_{I_P}\xi + I_P,\ \|x - c_L\|_\infty \le 3/2\delta\ell_L + \ell_{I_P} \le (3/2\delta + 1)\ell_L, \\ &\therefore\ \ell_{I_P}\xi + I_P \subset CL \Rightarrow\!\Leftarrow L \in \mathbb{L}. \end{aligned}$$

  Therefore, we must have $\ell_{3\delta L} \le 3\delta\ell_{I_P}$, and, additionally, $dist(3\delta L, I_P) \approx dist(\ell_{I_P}\xi + I_P, I_P) \approx \ell_{I_P}$ ***as long as*** *$C \underset{D}{\gg} 1$.*

- ***Slow varying*** *scaling: Given $L' \in \mathbb{L}$, then $\delta L' \cap \delta L \ne \varnothing \implies \ell_L \underset{\kappa}{\approx} \ell_{L'}$. The reason is that $\delta L' \cap \delta L \ne \varnothing$ implies:*

$$\forall x \in C\widehat{L},\ \|x - c_{L'}\|_\infty \le \|x - c_L\|_\infty + \|c_L - c_{L'}\|_\infty \le \frac{(C2^\kappa + \delta)\ell_L + \delta\ell_{L'}}{2}.$$

  If, additionally, we have $\frac{C2^\kappa + \delta}{C - \delta} \ll \frac{\ell_{L'}}{\ell_L}$, then:

$$\because \forall x \in C\widehat{L},\ \|x - c_{L'}\|_\infty \le C\ell_{L'} \quad \therefore C\widehat{L} \subset CL' \Rightarrow\!\Leftarrow L' \in \mathbb{L}.$$

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Due to these properties on $\mathbb{L}$, we can safely construct a ***adaptive partition of unity*** $\{\tilde{\chi}_L\}_{L\in\mathbb{L}}$ *satisfying:*

$$\forall L \in \mathbb{L},\ \tilde{\chi}_L \in C_c^\infty \ s.t. \begin{cases} |\tilde{\chi}_L| \lesssim_\delta \chi_{\delta L} \\ \|\nabla \tilde{\chi}_L\| \lesssim_\delta \chi_{\delta L}/\ell_L. \end{cases}$$

Applying the ***Van der Corput estimate****,* ***adjoint local*** τ***-Hölder continuity****, and the* ***adjoint cluster estimate****, we have:*

$$\begin{aligned} &\left|\int \tilde{\chi}_L e^{iq} \Omega_{\mathfrak{P}}^* f \overline{\Omega_{\mathfrak{Q}}^* g} d\mu\right| \\ \lesssim &\Lambda^{-\varrho\tau/d} \left(\inf_{3\delta L} \left|\Omega_{\mathfrak{P}}^* f\right| + \inf_{3\delta L} M_{\mathfrak{P}}^* f\right) \left(\inf_{3\delta L} |\Omega_{\mathfrak{Q}}^* g| + \inf_{3\delta L} M_{\mathfrak{Q}}^* g\right) |L|. \end{aligned}$$

This is almost the form we want. We only need to replace the $\mathfrak{Q}$ on the ***RHS*** *with $\mathfrak{P}'$. To do so, since $\mathfrak{Q} \subset \mathfrak{P}'$, by the definition of our* ***auxiliary maximal operator****, we can dominate $M_{\mathfrak{Q}}^* g$ with $M_{\mathfrak{P}'}^* g$. The rest is to estimate the loss caused by $\Omega_{\mathfrak{N}}^* g = \Omega_{\mathfrak{P}'}^* g - \Omega_{\mathfrak{Q}}^* g$. We notice that:*

- *We only need to focus on $L \in \mathbb{L}_+ := \{I \in \mathbb{L} \mid 3\delta I \subset I_{\mathfrak{p}}\}$ since, otherwise,*

$$3\delta L \not\subset I_{\mathfrak{p}} \implies 3\delta L \not\subset \bigcup_{P\in\mathfrak{P}} \varpi \tilde{I}_P \implies \inf_{3\delta L} \left|\Omega_{\mathfrak{P}}^* f\right| = \inf_{3\delta L} M_{\mathfrak{P}}^* f = 0.$$

- ***Temporal size constraint*** *on* ***Noise****: Given $(L, P) \in \mathbb{L}_+ \times \mathfrak{R}$,*

$$L \cap I_P^* \neq \varnothing \implies \ell_{I_P} \approx \ell_L.$$

  The reason is that ***Noise*** *must lie* ***temporally outside*** *$I_{\mathfrak{p}}$:*

$$3\delta L \cap I_P \subset I_{\mathfrak{p}} \cap I_P = \varnothing.$$

  Therefore, if I_P is too small:

$$\ell_{I_P} \leq \ell_{\tilde{I}_P} \leq \ell_L \implies \mathit{dist}(L, \tilde{I}_P) > \delta - 1 \implies L \cap I_P^* = \varnothing,$$

  which forces the lower bound on the size. On the other hand, if $L \cap I_P^* \neq \varnothing$ but $\ell_{I_P} \gg \ell_L$, ***Embedding Inequality*** *implies:*

$$\Lambda^\varrho \lesssim \|q\|_L \ll \|q\|_{\tilde{I}_P} \approx \|q\|_{I_P} \Rightarrow\Leftarrow P \in \mathfrak{N}.$$

- *Recall that $\mathfrak{N}$ is still a* ***cluster****, thus, due to* ***spectral packing constraint*** *and* ***temporal size constraint****, we must have:*

$$\forall L \in \mathbb{L}_+,\ \#\{P \in \mathfrak{N} \mid L \cap I_P^* \neq \varnothing\} \lesssim 1.$$

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Fixing $L \in \mathbb{L}_+$ and choosing $\varpi \gg 1$, the above three properties and ***single tile estimate*** *imply the following:*

$$\sup_L |\Omega^*_{\mathfrak{N}} g| \lesssim \sum_{\substack{P\in\mathfrak{N}\\ L\cap I^*_P\neq\varnothing}} |I_P|^{-1}\int_{E_P}|g|d\mu \lesssim \sup_{\substack{P\in\mathfrak{N}\\ L\subset\varpi\tilde{I}_P}} |I_P|^{-1}\int_{E_P}|g|d\mu \lesssim \inf_L M^*_{\mathfrak{N}} g.$$

As a immediate result, we have:

$$\inf_{3\delta L}|\Omega^*_{\mathfrak{Q}} g| \le \inf_L |\Omega^*_{\mathfrak{Q}} g| \le \inf_L \left|\Omega^*_{\mathfrak{P}'} g\right| + \sup_L |\Omega^*_{\mathfrak{N}} g| \lesssim \inf_L \left|\Omega^*_{\mathfrak{P}'} g\right| + \inf_L M^*_{\mathfrak{N}} g.$$

As we dominate $M^*_{\mathfrak{N}} g$ with $M^*_{\mathfrak{P}'} g$ and replace $3\delta L$ with L, we have:

$$\begin{aligned}&\left|\int \tilde{\chi}_L e^{iq}\Omega^*_{\mathfrak{P}} f\overline{\Omega^*_{\mathfrak{Q}} g}d\mu\right|\\ \lesssim&\Lambda^{-\varrho\tau/d}\left(\inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right)\left(\inf_L \left|\Omega^*_{\mathfrak{P}'} g\right| + \inf_L M^*_{\mathfrak{P}'} g\right)|L|.\end{aligned}$$

Summing over $L \in \mathbb{L}_+$, we get:

$$\begin{aligned}\left|\int e^{iq}\Omega^*_{\mathfrak{P}} f\overline{\Omega^*_{\mathfrak{Q}} g}d\mu\right| \lesssim&\Lambda^{-\varrho\tau/d}\int \left(\left|\Omega^*_{\mathfrak{P}} f\right| + M^*_{\mathfrak{P}} f\right)\left(\left|\Omega^*_{\mathfrak{P}'} g\right| + M^*_{\mathfrak{P}'} g\right)d\mu\\ \le&\Lambda^{-\varrho\tau/d}\left\|\left|\Omega^*_{\mathfrak{P}} f\right| + M^*_{\mathfrak{P}} f\right\|_{L^p(I_{\mathfrak{p}})}\left\|\left|\Omega^*_{\mathfrak{P}'} g\right| + M^*_{\mathfrak{P}'} g\right\|_{L^{p'}(I_{\mathfrak{p}})}.\end{aligned}$$

For the ***Noise****, we consider the carpet $\mathbb{L}_{\mathfrak{P}} \in \mathbb{X}$ instead. Due to the construction of $\mathfrak{N}$, the element $L \in \mathbb{L}_{\mathfrak{P}}$ satisfies the following:*

- ***Size control****:*

  $$\forall P \in \mathfrak{N}, \left(L \cap I^*_P \neq \varnothing \implies \Lambda^{\frac{1-\varrho}{d}}\ell_{I_P} \lesssim \ell_L\right).$$

  Otherwise, since $L \in \mathbb{L}_{\mathfrak{P}}$, there is $(P_L, \xi_L) \in \mathfrak{P} \times \text{Sh}$ such that $\ell_{I_{P_L}}\xi_L + I_{P_L} \subset \widehat{L}$, if there is $P \in \mathfrak{N}$ such that $\Lambda^{\frac{1-\varrho}{d}}\ell_{I_P} \gg \ell_L$, we have:

  $$\Lambda \lesssim \|q\|_{\tilde{I}_{P_L}} \approx \|q\|_{\ell_{I_{P_L}}\xi_L + I_{P_L}} \lesssim \|q\|_{\widehat{L}} \ll \|q\|_{\Lambda^{\frac{1-\varrho}{d}}I_P} \lesssim \Lambda^{1-\varrho}\|q\|_{I_P},$$

  which contradict with the condition $\|q\|_{I_P} < \Lambda^{\varrho}$.

- ***Packing constraint****:*

  $$\forall s \in \mathbb{Z},\ \#\{P \in \mathfrak{N} \mid L \cap I^*_P \neq \varnothing\ \wedge\ s_P = s\} \underset{\lambda,\kappa,D,d}{\lesssim} \left(2^{-s\kappa}\ell_L\right)^{D-1}.$$

  This follows from the ***configuration*** *of $\mathfrak{N}$ and the fact that $\mathfrak{P}$ is* ***open****:*

  $$\begin{aligned}&\because\ P \in \mathfrak{N} \implies L \cap I_P \subset I_{\mathfrak{p}} \cap I_P = \varnothing,\\ &\therefore\ \forall P \in \mathfrak{N}, \left(L \cap I^*_P \neq \varnothing \implies \partial L \cap \tilde{I}_P \neq \varnothing\right),\end{aligned}$$

  which forces the ***packing*** *to* ***concentrate*** *on the* ***boundary*** *of L. Also, since $\mathfrak{N}$ is a* ***cluster*** *at $\mathfrak{p}'$,* ***spectral packing constraint*** *only gives a factor of $\underset{\kappa,D,d}{\lesssim} (1+\lambda)^{dD}$ for those sharing the* ***same temporal blocks****.*

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Using ***adjoint cluster estimate*** *on* $\Omega^*_{\mathfrak{P}} f$ *and* ***single tile estimate*** *on* $\Omega^*_{\mathfrak{N}} g$, *we have for all* $L \in \mathbb{L}_{\mathfrak{P}}$,

$$\int_L \left|\Omega^*_{\mathfrak{P}} f\right| \cdot \left|\Omega^*_{\mathfrak{N}} g\right| d\mu \lesssim \left(\inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right) \sum_{\substack{P \in \mathfrak{N} \\ L \cap I^*_P \neq \varnothing}} \frac{|L \cap I^*_P|}{|I_P|} \int_{E_P} |g| d\mu.$$

For the summation part, we notice that:

$$\begin{aligned} \sum_{\substack{P \in \mathfrak{N} \\ L \cap I^*_P \neq \varnothing}} \frac{|L \cap I^*_P|}{|I_P|} \int_{E_P} |g| d\mu \leq & \sum_{\substack{P \in \mathfrak{N} \\ L \cap I^*_P \neq \varnothing}} \int_L \chi_{L \cap I^*_P} M^*_{\mathfrak{N}} g d\mu \\ \textbf{\textit{Hölder's inequality}} \ \leq & \left\|M^*_{\mathfrak{N}} g\right\|_{L^{p'}(L)} \sum_{\substack{P \in \mathfrak{N} \\ L \cap I^*_P \neq \varnothing}} |I_P|^{1/p}. \end{aligned}$$

Using the ***size control*** *and* ***packing constraint****, the* ***summation*** *on the right can be further reduced to:*

$$\begin{aligned} \sum_{\substack{P \in \mathfrak{N} \\ L \cap I^*_P \neq \varnothing}} |I_P|^{1/p} \lesssim & \sum_{\substack{s \in \mathbb{Z} \\ 2^{s\kappa} \lesssim \Lambda^{\frac{\varrho-1}{d}} \ell_L}} \left(2^{-s\kappa} \ell_L\right)^{(D-1)/p} 2^{s\kappa D/p} \\ \leq & \ell_L^{(D-1)/p} \sum_{\substack{s \in \mathbb{Z} \\ 2^{s\kappa} \lesssim \Lambda^{\frac{\varrho-1}{d}} \ell_L}} 2^{s\kappa/p} \lesssim \Lambda^{\frac{\varrho-1}{dp}} |L|^{1/p}. \end{aligned}$$

As we recombine and sum over $L \in \mathbb{L}_{\mathfrak{P}}$, *we have:*

$$\int \left|\Omega^*_{\mathfrak{P}} f\right| \cdot \left|\Omega^*_{\mathfrak{N}} g\right| d\mu \lesssim \Lambda^{\frac{\varrho-1}{dp}} \sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left(\inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right) |L|^{1/p} \left\|M^*_{\mathfrak{N}} g\right\|_{L^{p'}(L)}$$

$$\begin{aligned} \leq & \Lambda^{\frac{\varrho-1}{dp}} \left(\sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left(\inf_L \left|\Omega^*_{\mathfrak{P}} f\right| + \inf_L M^*_{\mathfrak{P}} f\right)^p |L|\right)^{1/p} \left(\sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left\|M^*_{\mathfrak{N}} g\right\|^{p'}_{L^{p'}(L)}\right)^{1/p'} \\ \leq & \Lambda^{\frac{\varrho-1}{dp}} \left\|\left|\Omega^*_{\mathfrak{P}} f\right| + M^*_{\mathfrak{P}} f\right\|_{L^p(I_{\mathfrak{p}})} \left\|M^*_{\mathfrak{N}} g\right\|_{L^{p'}(I_{\mathfrak{p}})} \\ \leq & \Lambda^{\frac{\varrho-1}{dp}} \left\|\left|\Omega^*_{\mathfrak{P}} f\right| + M^*_{\mathfrak{P}} f\right\|_{L^p(I_{\mathfrak{p}})} \left\|\left|\Omega^*_{\mathfrak{P}'} g\right| + M^*_{\mathfrak{P}'} g\right\|_{L^{p'}(I_{\mathfrak{p}})}. \end{aligned}$$

The rest is to ***fine-tune*** $\varrho \in (0,1)$ *so that* $\frac{1-\varrho}{dp} = \frac{\varrho\tau}{d} =: \epsilon_p$. □

8.4 Support Restriction and Cross-Level Decay

We have been working within $\mathbb{P}_{n,\alpha}$ for a while. Let us investigate the interaction across $\mathbb{P}_{n,\alpha}$ and $\mathbb{P}_{n,\beta}$ with $\beta > \alpha$. Given a cluster $\mathfrak{P} \subset \mathbb{P}_{n,\alpha}$ at $\mathfrak{p} \in \mathbb{P}_{n,\alpha}$ and an **open** cluster $\mathfrak{P}' \subset \mathbb{P}_{n,\beta}$ at $\mathfrak{p}' \in \mathbb{P}_{n,\beta}$, we have:

$$\begin{cases} \left|\left\langle \mathfrak{L}_{\mathfrak{P}} f, \mathfrak{L}_{\mathfrak{P}'} g\right\rangle\right| = \left|\left\langle \mathfrak{L}_{\mathfrak{P}} f, \chi_{E_{\mathfrak{p}'}} \mathfrak{L}_{\mathfrak{P}'} g\right\rangle\right| & = \left|\left\langle \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}_{\mathfrak{P}} f, \mathfrak{L}_{\mathfrak{P}'} g\right\rangle\right| \\ \left|\left\langle \mathfrak{L}^*_{\mathfrak{P}} f, \mathfrak{L}^*_{\mathfrak{P}'} g\right\rangle\right| = \left|\left\langle \mathfrak{L}^*_{\mathfrak{P}} f, \chi_{I_{\mathfrak{p}'}} \mathfrak{L}^*_{\mathfrak{P}'} g\right\rangle\right| & = \left|\left\langle \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}^*_{\mathfrak{P}} f, \mathfrak{L}^*_{\mathfrak{P}'} g\right\rangle\right|. \end{cases}$$

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To acquire good control is to understand the behavior of operators restricted to $\bigsqcup \mathbb{A}_{\beta-1}$. As we expend the operator:

$$\begin{cases} \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \left|\mathfrak{L}_{\mathfrak{P}} f\right| & \lesssim \sum_{J \in \mathbb{J}_{\mathfrak{P}}} \left(\inf_J \mathfrak{T}\mu_{\mathfrak{p}} f + \inf_J M f\right) \chi_{E_J \cap \bigsqcup \mathbb{A}_{\beta-1}} \\ \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \left|\mathfrak{L}_{\mathfrak{P}}^* f\right| & \lesssim \sum_{L \in \mathbb{L}_{\mathfrak{P}}} \left(\inf_L \left|\mathfrak{L}_{\mathfrak{P}}^* f\right| + \inf_L M_{\mathfrak{P}}^* f\right) \chi_{L \cap \bigsqcup \mathbb{A}_{\beta-1}}, \end{cases}$$

we immediately spot an **almost trivial** control:

Lemma 8.4.1 (Support restriction control)**.**
*Given $\mathfrak{P}$ be an **open cluster** at $\mathfrak{p}$, and a measurable set $A \subset \mathbb{R}^D$, we have:*

$$\begin{cases} \left\|\chi_A \mathfrak{L}_{\mathfrak{P}} f\right\|_{L^p} \lesssim \left(\sup_{J \in \mathbb{J}_{\mathfrak{P}}} \frac{|E_J \cap A|}{|J|}\right)^{1/p} \left\|\mathfrak{T}\mu_{\mathfrak{p}} f + M f\right\|_{L^p(I_{\mathfrak{p}})} \\ \left\|\chi_A \mathfrak{L}_{\mathfrak{P}}^* f\right\|_{L^p} \lesssim \left(\sup_{L \in \mathbb{L}_{\mathfrak{P}}} \frac{|L \cap A|}{|L|}\right)^{1/p} \left\|\left|\mathfrak{L}_{\mathfrak{P}}^* f\right| + M_{\mathfrak{P}}^* f\right\|_{L^p(I_{\mathfrak{p}})}. \end{cases}$$

Proof. *Using the **(adjoint) cluster estimate**, we have:*

$$\begin{cases} \left\|\chi_A \mathfrak{L}_{\mathfrak{P}} f\right\|_{L^p} & \lesssim \left[\sum_{J \in \mathbb{J}_{\mathfrak{P}}} \frac{|E_J \cap A|}{|J|} \left(\inf_J \mathfrak{T}\mu_{\mathfrak{p}} f + \inf_J M f\right)^p |J|\right]^{1/p} \\ \left\|\chi_A \mathfrak{L}_{\mathfrak{P}}^* f\right\|_{L^p} & \lesssim \left[\sum_{L \in \mathbb{L}_{\mathfrak{P}}} \frac{|L \cap A|}{|L|} \left(\inf_L \left|\mathfrak{L}_{\mathfrak{P}}^* f\right| + \inf_L M_{\mathfrak{P}}^* f\right)^p |L|\right]^{1/p} \end{cases}.$$

*An elementary use of **Hölder's inequality** yields the result.* □

We now can expect the **δ-covering relation** to play an essential role.

Properties 8.4.2 (Cross-Level Decay)**.**
Given $\mathfrak{P} \subset \mathbb{P}_{n,\alpha}$ and $\Delta \in \mathbb{N}$, we have:

$$(J, L) \in \mathbb{J}_{\mathfrak{P}} \times \mathbb{L}_{\mathfrak{P}} \implies \frac{\left|J \cap \bigsqcup \mathbb{A}_{\alpha+\Delta}\right|}{|J|}, \ \frac{\left|L \cap \bigsqcup \mathbb{A}_{\alpha+\Delta}\right|}{|L|} \lesssim \delta^{\Delta}.$$

Through iterative use of the **δ-covering relation** , the above property can be derived from the following claim:

Claim.
Given $\mathfrak{P} \subset \mathbb{P}_{n,\alpha}$, we have:

$$(I, J, L) \in \mathbb{A}_{\alpha} \times \mathbb{J}_{\mathfrak{P}} \times \mathbb{L}_{\mathfrak{P}} \implies \begin{cases} I \subset J \ \vee \ I \cap J = \varnothing \\ I \subset L \ \vee \ I \cap L = \varnothing. \end{cases}$$

Proof. *Fix $(I, J, L) \in \mathbb{A}_{\alpha} \times \mathbb{J}_{\mathfrak{P}} \times \mathbb{L}_{\mathfrak{P}}$.*

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- *Suppose $J \subsetneq I$, then the construction of $\mathbb{J}_{\mathfrak{P}}$ implies:*

$$\exists P \in \mathfrak{P} \ s.t. \ I_P \subset \widehat{J} \subset I \in \mathbb{A}_\alpha \Rightarrow\Leftarrow I_P \in \mathbb{I}_\alpha.$$

- *Suppose $L \subsetneq I$, then the construction of $\mathbb{L}_{\mathfrak{P}}$ implies:*

$$\exists (P, \xi) \in \mathfrak{P} \times \mathrm{Sh} \ s.t. \ \ell_{I_P}\xi + I_P \subset \widehat{L} \subset I \in \mathbb{A}_\alpha.$$

  In other words, we have:

$$\exists P \in \mathfrak{P} \ s.t. \left(\tilde{I}_P \cap I \neq \varnothing \ \wedge \ \ell_{I_P} \leq \ell_I \right).$$

  However, for such $P \in \mathfrak{P}$,

$$\because \mathbb{A}_\alpha \in \mathbb{X}^\infty \quad \therefore \exists I' \in \mathbb{A}_\alpha \ s.t. \ I_P \subset I' \Rightarrow\Leftarrow I_P \in \mathbb{I}_\alpha.$$

$\square$

Combining the two lemmas, we have:

$$\begin{cases} \left\| \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}_{\mathfrak{P}} f \right\|_{L^p} \lesssim \min\left(\delta^{\frac{\beta-\alpha-1}{p}}, 2^{-n/p} \right) \|f\|_{L^p} \\ \left\| \chi_{\bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}^*_{\mathfrak{P}} f \right\|_{L^p} \lesssim \delta^{\frac{\beta-\alpha-1}{p}} 2^{-n/p'} \|f\|_{L^p}. \end{cases}$$

Also, the inner product form:

$$\begin{cases} \left| \langle \mathfrak{L}_{\mathfrak{P}} f, \mathfrak{L}_{\mathfrak{P}'} g \rangle \right| & \lesssim \min\left(\delta^{\frac{\beta-\alpha-1}{2}}, 2^{-n/2} \right) 2^{-n/2} \|f\|_{L^2} \|g\|_{L^2} \\ \left| \left\langle \mathfrak{L}^*_{\mathfrak{P}} f, \mathfrak{L}^*_{\mathfrak{P}'} g \right\rangle \right| & \lesssim \delta^{\frac{\beta-\alpha-1}{2}} 2^{-n} \|f\|_{L^2} \|g\|_{L^2}. \end{cases}$$

8.5 Row Configuration

With **clusters** being thoroughly examined, we build from them a larger structure to exploit the temporal aspect of the control.

Definition 8.5.1 (Row)**.**

- *A **row** is an **open** ∞**-apart** 1**-stack**. That is, $\mathfrak{R}$ is a **row** if:*

$$\mathfrak{R} = \bigsqcup_j \mathfrak{P}_j \ \wedge \ \forall j, \begin{cases} \mathfrak{P}_j & \trianglelefteq\text{-}\textbf{\textit{convex}} \\ P \in \mathfrak{P}_j & \Longrightarrow \begin{cases} \lambda P \lhd \mathfrak{p}_j \\ \tilde{I}_P \subset I_{\mathfrak{p}_j} \end{cases} \\ k \neq j & \Longrightarrow I_{\mathfrak{p}_j} \cap I_{\mathfrak{p}_k} = \varnothing. \end{cases}$$

- *Two **rows** are Λ**-apart** if the collection of **clusters** are Λ**-apart**.*

- *An **open** Λ**-apart** Ξ**-stack** is actually Ξ **rows** that are both mutually Λ**-apart** and mutually $\trianglelefteq$**-incomparable**.*

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Remark. *Due to the* ***disjointness*** *of the* ***supports*** *of the corresponding operators of* ***open clusters*** *in a* ***row****, all the preceding estimates have direct adaptations replacing* ***open clusters*** *with* ***rows****.*

Lemma 8.5.2 (Row estimates)**.**
Given $\mathfrak{R}_\alpha, \mathfrak{R}'_\alpha \subset \mathbb{P}_{n,\alpha}$ and $\mathfrak{R}_\beta \subset \mathbb{P}_{n,\beta}$ three ***rows*** *and a measurable set $A \subset \mathbb{R}^D$, we have the following estimates:*

- ***Single row estimate****:*

$$\|\mathfrak{L}_{\mathfrak{R}_\alpha}\|_{\mathcal{BL}(L^2,L^2)} = \left\|\mathfrak{L}^*_{\mathfrak{R}_\alpha}\right\|_{\mathcal{BL}(L^2,L^2)} \lesssim 2^{-n}$$

- ***In-level interaction****:*

$$\begin{cases} \left\|\mathfrak{L}_{\mathfrak{R}'_\alpha}\mathfrak{L}^*_{\mathfrak{R}_\alpha}\right\|_{\mathcal{BL}(L^2,L^2)} \lesssim \Lambda^{-\epsilon}2^{-n} & \mathfrak{R}_\alpha \textit{ and } \mathfrak{R}_\alpha \textit{ are } \Lambda\textbf{\textit{-apart}} \\ \left\|\mathfrak{L}^*_{\mathfrak{R}'_\alpha}\mathfrak{L}_{\mathfrak{R}_\alpha}\right\|_{\mathcal{BL}(L^2,L^2)} = 0 & \mathfrak{R}_\alpha \textit{ and } \mathfrak{R}_\alpha \textit{ are } \trianglelefteq\textbf{\textit{-incomparable}} \end{cases}$$

- ***Cross-level interaction****:*

$$\begin{cases} \left\|\mathfrak{L}_{\mathfrak{R}_\beta}\mathfrak{L}^*_{\mathfrak{R}_\alpha}\right\|_{\mathcal{BL}(L^2,L^2)} \lesssim \delta^{\frac{\beta-\alpha-1}{2}}2^{-n} & \alpha < \beta \\ \left\|\mathfrak{L}^*_{\mathfrak{R}_\beta}\mathfrak{L}_{\mathfrak{R}_\alpha}\right\|_{\mathcal{BL}(L^2,L^2)} \lesssim \min\left(\delta^{\frac{\beta-\alpha-1}{2}}, 2^{-n/2}\right)2^{-n/2} & \alpha < \beta \end{cases}$$

Proof. *We consider the following* ***natural decomposition****:*

$$\mathfrak{L}_{\mathfrak{R}_\alpha} f = \sum_j \mathfrak{L}_{\mathfrak{P}_{\alpha,j}} f = \sum_j \chi_{E_{\mathfrak{P}_{\alpha,j}}} \mathfrak{L}_{\mathfrak{P}_{\alpha,j}} \chi_{I_{\mathfrak{p}_{\alpha,j}}} f.$$

Since $E_{\mathfrak{P}_{\alpha,j}}$s are disjoint and so are $I_{\mathfrak{p}_{\alpha,j}}$s, we have:

$$\begin{aligned} \|\mathfrak{L}_{\mathfrak{R}_\alpha} f\|^2_{L^2} =& \sum_j \left\|\mathfrak{L}_{\mathfrak{P}_{\alpha,j}}\chi_{I_{\mathfrak{p}_{\alpha,j}}} f\right\|^2_{L^2\left(E_{\mathfrak{P}_{\alpha,j}}\right)} \\ \lesssim& 2^{-n}\sum_j \|f\|^2_{L^2\left(I_{\mathfrak{p}_{\alpha,j}}\right)} \le 2^{-n}\|f\|^2_{L^2}. \end{aligned}$$

This completes the ***single row estimate****. For $\mathfrak{R}_\alpha$ and $\mathfrak{R}'_\alpha$ being* ***Λ-apart****, we*

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extract separation factor. *By setting* $\mathfrak{U}_{\mathfrak{P}} := \left|\mathfrak{L}^*_{\mathfrak{P}}\right| + M^*_{\mathfrak{P}}$*, we have:*

$$\begin{aligned}
&\left|\left\langle \mathfrak{L}^*_{\mathfrak{R}_\alpha} f, \mathfrak{L}^*_{\mathfrak{R}'_\alpha} g\right\rangle\right| \\
\leq &\sum_{j,k}\left|\int_{I_{\mathfrak{p}_{\alpha,j}}\cap I_{\mathfrak{p}'_{\alpha,k}}} \mathfrak{L}^*_{\mathfrak{P}_{\alpha,j}}\chi_{E_{\mathfrak{P}_{\alpha,j}}} f \cdot \overline{\mathfrak{L}^*_{\mathfrak{P}'_{\alpha,k}}\chi_{E_{\mathfrak{P}'_{\alpha,k}}} g} d\mu\right| \\
\lesssim &\Lambda^{-\epsilon_2}\sum_{j,k}\left\|\mathfrak{U}_{\mathfrak{P}_{\alpha,j}}\chi_{E_{\mathfrak{P}_{\alpha,j}}} f\right\|_{L^2\left(I_{\mathfrak{p}_{\alpha,j}}\cap I_{\mathfrak{p}'_{\alpha,k}}\right)}\left\|\mathfrak{U}_{\mathfrak{P}'_{\alpha,k}}\chi_{E_{\mathfrak{P}'_{\alpha,k}}} g\right\|_{L^2\left(I_{\mathfrak{p}_{\alpha,j}}\cap I_{\mathfrak{p}'_{\alpha,k}}\right)} \\
\leq &\Lambda^{-\epsilon_2}\left(\sum_j \left\|\mathfrak{U}_{\mathfrak{P}_{\alpha,j}}\chi_{E_{\mathfrak{P}_{\alpha,j}}} f\right\|^2_{L^2\left(I_{\mathfrak{p}_{\alpha,j}}\right)}\right)^{1/2}\left(\sum_k \left\|\mathfrak{U}_{\mathfrak{P}'_{\alpha,k}}\chi_{E_{\mathfrak{P}'_{\alpha,k}}} g\right\|^2_{L^2\left(I_{\mathfrak{p}'_{\alpha,k}}\right)}\right)^{1/2} \\
\lesssim &\Lambda^{-\epsilon_2}2^{-n}\left(\sum_j \|f\|^2_{L^2\left(E_{\mathfrak{P}_{\alpha,j}}\right)}\right)^{1/2}\left(\sum_k \|g\|^2_{L^2\left(E_{\mathfrak{P}'_{\alpha,k}}\right)}\right)^{1/2} \\
\leq &\Lambda^{-\epsilon_2}2^{-n}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

The other ***in-level interaction*** *is trivial since the support of the operators are disjoint. Lastly,* ***cross-level interaction*** *is reduced to the following:*

$$\begin{cases}\left\|\chi_{\bigsqcup \mathbb{A}_{\beta-1}}\mathfrak{L}_{\mathfrak{R}_\alpha} f\right\|_{L^2} \lesssim \min\left(\delta^{\frac{\beta-\alpha-1}{2}}, 2^{-n/2}\right)\|f\|_{L^2} \\ \left\|\chi_{\bigsqcup \mathbb{A}_{\beta-1}}\mathfrak{L}^*_{\mathfrak{R}_\alpha} f\right\|_{L^2} \lesssim \delta^{\frac{\beta-\alpha-1}{2}}2^{-n/2}\|f\|_{L^2}.\end{cases}$$

Using the ***natural decomposition*** *and an* ***analogue*** *of the* ***single row*** *argument, we can extend the result from* ***clusters*** *to* ***rows***. □

8.6 Almost Orthogonality

We specify our constructions: $l = m = n$ and $\delta = 2^{-4}$,

$$\forall \alpha\in\mathbb{N},\ \mathbb{P}_{n,\alpha} \rightsquigarrow \begin{cases}\text{some } \textbf{sparse parts} \\ \lesssim n \textbf{ open } 2^{n\kappa}\lesssim\textbf{-apart } 2^n\textbf{-stacks}\end{cases}$$

We interpret the **open** $2^{n\kappa}$ $\lesssim$**-apart** 2^n**-stacks** as 2^n **rows** with additional structure. Therefore, we naturally would consider the following configuration:

Definition 8.6.1 (**Cluster tower** or **BMO Forest** in [Lie20])**.**
Given $\mathbb{P}\subset\mathbb{P}_n$*, we say* $\mathbb{P}$ *is a* ***cluster tower*** *if, in every level* $\alpha\in\mathbb{N}$*, we have:*

$$\mathbb{P}\cap\mathbb{P}_{n,\alpha} = \bigsqcup_{j=1}^{2^n}\mathfrak{R}_{\alpha,j}, \quad \textit{where} \quad \{\mathfrak{R}_{\alpha,j}\}_{j=1}^{2^n} \quad \textit{are} \begin{cases}\textbf{\textit{rows}} \\ \trianglelefteq\textbf{\textit{-incomparable}} \\ 2^{n\kappa}\lesssim\textbf{\textit{-apart}}.\end{cases}$$

We see that $\mathbb{P}_n$ consists of $\lesssim n$ **cluster towers**. Therefore, we only need to obtain a bound with an **exponential decay** to compensate the **polynomial growth** of the the number of **cluster tower** in $\mathbb{P}_n$ as we **sum over** $n\in\mathbb{N}$. As we apply the **Cotlar-Stein lemma** (TT^*-T^*T argument), we have:

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Theorem 8.6.2 (**Cluster tower** L^2 estimate)**.**
Given $\mathbb{P} \subset \mathbb{P}_n$ a ***cluster tower****, as long as $\kappa \geq 2/\epsilon_2$, we have:*

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^2} \lesssim n2^{-n/2} \|f\|_{L^2}.$$

Proof. *Decomposing everything into* ***rows****, we have*

$$\mathbb{P} = \bigsqcup_{\alpha\in\mathbb{N}} \mathbb{P} \cap \mathbb{P}_{n,\alpha} = \bigsqcup_{\alpha\in\mathbb{N}} \bigsqcup_{j=1}^{2^n} \mathfrak{R}_{\alpha,j} = \bigsqcup_{\gamma=0}^{2n-1} \bigsqcup_{\alpha\in 2n\mathbb{N}+\gamma} \bigsqcup_{j=1}^{2^n} \mathfrak{R}_{\alpha,j}$$

We verify the condition for ***Cotlar-Stein Lemma*** *(TT^*-T^*T argument): For fixed $\alpha \in 2n\mathbb{N}+\gamma$ and $1 \leq j \leq 2^n$, we have:*

$$\begin{aligned}
&\sum_{\beta\in 2n\mathbb{N}+\gamma} \sum_{k=1}^{2^n} \left\| \mathfrak{L}_{\mathfrak{R}_{\beta,k}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
=& \left\| \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} + \sum_{\substack{1\leq k\leq 2^n\\ k\neq j}} \left\| \mathfrak{L}_{\mathfrak{R}_{\alpha,k}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
+& \sum_{\substack{\beta\in 2n\mathbb{N}+\gamma\\ \beta\neq\alpha}} \sum_{k=1}^{2^n} \left\| \mathfrak{L}_{\mathfrak{R}_{\beta,k}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
\lesssim& 2^{-n/2} + 2^{-n\kappa\epsilon_2/2} 2^{-n/2} \cdot (2^n - 1) + \sum_{\beta-\alpha\in 2n\mathbb{Z}\setminus\{0\}} 2^{1-|\beta-\alpha|} 2^{-n/2} 2^n \\
\lesssim& 2^{-n/2}.
\end{aligned}$$

For the dual estimate we have:

$$\begin{aligned}
&\sum_{\beta\in 2n\mathbb{N}+\gamma} \sum_{k=1}^{2^n} \left\| \mathfrak{L}^*_{\mathfrak{R}_{\beta,k}} \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
=& \left\| \mathfrak{L}^*_{\mathfrak{R}_{\alpha,j}} \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} + \sum_{\substack{1\leq k\leq 2^n\\ k\neq j}} \left\| \mathfrak{L}_{\mathfrak{R}^*_{\alpha,k}} \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
+& \sum_{\substack{\beta\in 2n\mathbb{N}+\gamma\\ \beta\neq\alpha}} \sum_{k=1}^{2^n} \left\| \mathfrak{L}^*_{\mathfrak{R}_{\beta,k}} \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
\lesssim& 2^{-n/2} + 0 + \sum_{\beta-\alpha\in 2n\mathbb{Z}\setminus\{0\}} \min\left(2^{1-|\beta-\alpha|}, 2^{-n/4}\right) 2^{-n/4} 2^n \\
\lesssim& 2^{-n/2}.
\end{aligned}$$

As a result, we have:

$$\|\mathfrak{L}_{\mathbb{P}}\|_{\mathcal{BL}(L^2,L^2)} \leq \sum_{\gamma=0}^{2n-1} \left\| \sum_{\alpha\in 2n\mathbb{N}+\gamma} \sum_{j=1}^{2^n} \mathfrak{L}_{\mathfrak{R}_{\alpha,j}} \right\|_{\mathcal{BL}(L^2,L^2)} \lesssim n2^{-n/2}. \quad \square$$

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Theorem 8.6.3 (L^2 bound on **cluster parts**).
Let $\mathbb{P} \subset \tilde{\mathbb{D}}$ be the full collection of the ***cluster parts****, we have:*

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^2} \lesssim \|f\|_{L^2}.$$

Proof. *We break $\mathbb{P} \cap \mathbb{P}_n$ into $\lesssim n$* ***cluster towers*** *and apply the* ***cluster tower*** *L^2* ***estimate****:*

$$\|\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_n} f\|_{L^2} \lesssim n^2 \cdot 2^{-n/2} \|f\|_{L^2}.$$

As we sum over $n \in \mathbb{N}$, we finally have:

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^2} \lesssim \sum_{n\in\mathbb{N}} n^2 \cdot 2^{-n/2} \|f\|_{L^2} \lesssim \|f\|_{L^2},$$

which completes the full argument. $\square$

8.7 Bateman's Extrapolation Argument

In order to recover full L^p bound for the **cluster parts** while exploiting the **orthogonality** structure of $\mathcal{BL}(L^2, L^2)$, Zorin-Kranich adopted an **extrapolation argument** used in [BT13] by **refining/localizing** the L^2 estimate. Yet, his argument requires a **reorganization** of the full collection of the tiles including the **sparse parts**. We come up with a similar idea without altering the configuration of the **sparse parts**. For starters, we state the extrapolation method matching our L^2 settings:

Lemma 8.7.1 (L^2 Extrapolation).
Fix $p > 2$ and an operator T mapping $L^{p,1}$ ***qualitatively*** *to $L^{p,\infty}$. Suppose for any $G, H \subset \mathbb{R}^D$ measurable we can find measurable subset $G' \subset G$ and $H' \subset H$ such that:*

- ***Error loss control****:*

$$\left(\frac{|G \setminus G'|}{|G|}\right)^{1/p} + \left(\frac{H \setminus H'}{H}\right)^{1/p'} \leq \epsilon < 1,$$

- ***Testing condition****:*

$$\|\chi_{H'} T\left(\chi_{G'} f\right)\|_{L^2} \leq \Lambda \left(\frac{|H|}{|G|}\right)^{1/2-1/p} \|\chi_{G'} f\|_{L^2},$$

we then have the following ***quantitative*** *control:*

$$\|Tf\|_{L^{p,\infty}} \lesssim \frac{\Lambda}{1-\epsilon} \|f\|_{L^{p,1}}.$$

Our goal now is to **extrapolate** a $L^{p,1} \to L^{p,\infty}$ bound that does not necessary have a **exponential decay** for a **cluster tower** in $\mathbb{P}_n$. This is still okay since we can first **extrapolate** a bit **further** and **interpolate** with the L^2

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bound to **spread** the **exponential decay**. Also, for $p \in (1,2)$, we just switch to control the **adjoint**. With that been said, we still need to find a systematic way to choose the subset G', H' for given G, H. Zorin-Kranich made the following observation:

Observation. *Given measurable set $A \subset \mathbb{R}^D$ and $\rho \in (0,1)$, we have:*

$$I \not\subset M\chi_A^{-1}(\rho, \infty] \implies \frac{|I \cap A|}{|I|} \leq \rho.$$

This is equivalent to say:

$$I \subset M\chi_A^{-1}(\rho, \infty] \impliedby \fint_I |\chi_A| \, d\mu = \frac{|I \cap A|}{|I|} > \rho.$$

That reminds us the **support restriction control**. As we explore the idea, we would naturally come up with the following settings:

Definition 8.7.2.
Given measurable $A \subset \mathbb{R}^D$, we set:

$$A_\rho := M\chi_A^{-1}(\rho, \infty], \quad \textit{where} \;\; \rho \in (0,1).$$

For a collection of ***tiles*** $\mathbb{P} \subset \tilde{\mathbb{D}}$, *we set:*

$$\begin{cases} \mathbb{P}_{A,\rho} := \{P \in \mathbb{P} \mid I_P \not\subset A_\rho\} \\ \mathbb{P}^*_{A,\rho} := \left\{P \in \mathbb{P} \mid \tilde{I}_P \not\subset A_\rho\right\}. \end{cases}$$

Due to our construction, we have the following:

Lemma 8.7.3 (Density Manipulation)**.**
Given $\mathbb{P} \subset \tilde{\mathbb{D}}$, a measurable set $A \subset \mathbb{R}^D$, and $\rho \in (0,1)$, we have:

$$I \in \mathbb{J}_{\mathbb{P}_{A,\rho}} \cup \mathbb{L}_{\mathbb{P}_{A,\rho}} \cup \mathbb{J}_{\mathbb{P}^*_{A,\rho}} \cup \mathbb{L}_{\mathbb{P}^*_{A,\rho}} \implies \frac{|I \cap A|}{|I|} \lesssim \rho.$$

Proof. *By construction, we have:*

$$\begin{aligned} & I \in \mathbb{J}_{\mathbb{P}_{A,\rho}} \cup \mathbb{L}_{\mathbb{P}_{A,\rho}} \cup \mathbb{J}_{\mathbb{P}^*_{A,\rho}} \cup \mathbb{L}_{\mathbb{P}^*_{A,\rho}} \\ \implies & \exists P \in \mathbb{P}_{A,\rho} \cup \mathbb{P}^*_{A,\rho} \; s.t. \; \tilde{I}_P \underset{\kappa,D}{\lesssim} I \\ \implies & \exists \Lambda \underset{\kappa,D}{\lesssim} 1 \; s.t. \; \tilde{I}_P \subset \Lambda I \not\subset A_\rho = M\chi_A^{-1}(\rho,\infty] \\ \implies & \frac{|I \cap A|}{|I|} \leq \Lambda \frac{|\Lambda I \cap A|}{|\Lambda I|} \leq \Lambda\rho. \end{aligned}$$

□

From this, we derive:

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Corollary 8.7.3.1 (**In-level localized** estimate)**.**
*For an **open cluster** $\mathfrak{P} \subset \mathbb{P}_{n,\alpha}$ at $\mathfrak{p} \in \mathbb{P}_{n,\alpha}$, we have:*

$$\begin{cases} \left\| \chi_A \mathfrak{L}_{\mathfrak{P}} \chi_{A_\rho^c} f \right\|_{L^p} & \lesssim \min\left(2^{-n/p}, \rho^{1/p}\right) \left\| \chi_{A_\rho^c} f \right\|_{L^p(I_\mathfrak{p})} \\ \left\| \chi_A \mathfrak{L}^*_{\mathfrak{P}} \chi_{A_\rho^c} f \right\|_{L^p} & \lesssim 2^{-n/p'} \rho^{1/p} \left\| \chi_{A_\rho^c} f \right\|_{L^p(I_\mathfrak{p})}. \end{cases}$$

*Similarly, for a **row** $\mathfrak{R} \subset \mathbb{P}_{n,\alpha}$, we have:*

$$\begin{cases} \left\| \chi_A \mathfrak{L}_{\mathfrak{R}} \chi_{A_\rho^c} f \right\|_{L^p} & \lesssim \min\left(2^{-n/p}, \rho^{1/p}\right) \left\| \chi_{A_\rho^c} f \right\|_{L^p} \\ \left\| \chi_A \mathfrak{L}^*_{\mathfrak{R}} \chi_{A_\rho^c} f \right\|_{L^p} & \lesssim 2^{-n/p'} \rho^{1/p} \left\| \chi_{A_\rho^c} f \right\|_{L^p}. \end{cases}$$

*Lastly, for an **open** $2^{n\kappa}$**-apart** 2^n**-stack** $\mathbb{P} \subset \mathbb{P}_{n,\alpha}$, we have:*

$$\left\| \chi_A \mathfrak{L}_{\mathbb{P}} \chi_{A_\rho^c} f \right\|_{L^2}, \ \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}} \chi_{A_\rho^c} f \right\|_{L^2} \lesssim \rho^{1/2} \left\| \chi_{A_\rho^c} f \right\|_{L^2}$$

as long as $\kappa \geq 2/\epsilon_2$.

Proof. *We observe that:*

$$\begin{cases} \chi_A \mathfrak{L}_{\mathfrak{P}} \chi_{A_\rho^c} f & = \chi_{I_\mathfrak{p} \cap A} \mathfrak{L}_{\mathfrak{P}^*_{A,\rho}} \chi_{I_\mathfrak{p} \cap A_\rho^c} f \\ \chi_A \mathfrak{L}^*_{\mathfrak{P}} \chi_{A_\rho^c} f & = \chi_{I_\mathfrak{p} \cap A} \mathfrak{L}^*_{\mathfrak{P}_{A,\rho}} \chi_{I_\mathfrak{p} \cap A_\rho^c} f. \end{cases}$$

*Since both $\mathfrak{P}_{A,\rho}$ and $\mathfrak{P}^*_{A,\rho}$ are **open cluster** at $\mathfrak{p}$, applying **support restriction control** on $\chi_{I_\mathfrak{p} \cap A} \mathfrak{L}_{\mathfrak{P}^*_{A,\rho}}$ and $\chi_{I_\mathfrak{p} \cap A} \mathfrak{L}^*_{\mathfrak{P}_{A,\rho}}$ gives the desired control. As a immediate result, **natural decomposition** yield the estimate for **row** configuration. To control an **open** $2^{n\kappa}$**-apart** 2^n**-stack**, we discard irrelevant tiles:*

$$\begin{cases} \chi_A \mathfrak{L}_{\mathbb{P}} \chi_{A_\rho^c} f & = \chi_A \mathfrak{L}_{\mathbb{P}^*_{A,\rho}} \chi_{A_\rho^c} f \\ \chi_A \mathfrak{L}^*_{\mathbb{P}} \chi_{A_\rho^c} f & = \chi_A \mathfrak{L}^*_{\mathbb{P}_{A,\rho}} \chi_{A_\rho^c} f \end{cases}$$

and proceed in the following two ways:

- *To control $\chi_A \mathfrak{L}_{\mathbb{P}^*_{A,\rho}}$, we exploit the **density manipulation** to improve the **extraction of separation factor**. That is, given an **open cluster** $\mathfrak{P} \subset \mathbb{P}_{n,\alpha}$ at $\mathfrak{p} \in \mathbb{P}_{n,\alpha}$, we have:*

  $$\begin{aligned} & \left\| \left| \mathfrak{L}^*_{\mathfrak{P}^*_{A,\rho}} \chi_A f \right| + M^*_{\mathfrak{P}^*_{A,\rho}} \chi_A f \right\|_{L^2(I_\mathfrak{p})} \\ \lesssim & \left\| \chi_A \mathfrak{L}_{\mathfrak{P}^*_{A,\rho}} \right\|_{\mathcal{BL}(L^2,L^2)} \|f\|_{L^2(I_\mathfrak{p})} + \left(\sup_{P \in \mathfrak{P}^*_{A,\rho}} \frac{|E_P \cap A|}{|I_P|} \right)^{1/2} \|f\|_{L^2(I_\mathfrak{p})} \\ \lesssim & \min\left(2^{-n/2}, \rho^{1/2}\right) \|f\|_{L^2(I_\mathfrak{p})}. \end{aligned}$$

  *As a result, for **open clusters** $\mathfrak{P}, \mathfrak{P}' \subset \mathbb{P}_{n,\alpha}$ at $\mathfrak{p}, \mathfrak{p}' \in \mathbb{P}_{n,\alpha}$ respectively that are Λ**-apart** and $\trianglelefteq$**-incomparable**, we have:*

  $$\left| \left\langle \mathfrak{L}^*_{\mathfrak{P}^*_{A,\rho}} \chi_A f, \mathfrak{L}^*_{\mathfrak{P}'^*_{A,\rho}} \chi_A f \right\rangle \right| \lesssim \Lambda^{-\epsilon_2} \min\left(2^{-n}, \rho\right) \|f\|^2_{L^2(I_\mathfrak{p} \cap I_{\mathfrak{p}'})}.$$

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Therefore, for Λ***-apart rows*** $\mathfrak{R}, \mathfrak{R}' \subset \mathbb{P}_{n,\alpha}$, *we also have:*

$$\left\| \chi_A \mathfrak{L}_{\mathfrak{R}'^*_{A,\rho}} \mathfrak{L}^*_{\mathfrak{R}^*_{A,\rho}} \chi_A \right\|_{\mathcal{BL}(L^2,L^2)} \lesssim \Lambda^{-\epsilon_2/2} \min\left(2^{-n/2}, \rho^{1/2}\right).$$

This gives us the desired control to apply the ***Cotlar-Stein Lemma***(*TT^*-T^*T argument*)*: We first decompose* $\mathbb{P}^*_{A,\rho}$ *into* ***rows*** $\{\mathfrak{R}_j\}_{j=1}^{2^n}$ *and verify the following:*

$$\begin{aligned}
&\sum_{k=1}^{2^n} \left\| \chi_A \mathfrak{L}_{\mathfrak{R}_k} \mathfrak{L}^*_{\mathfrak{R}_j} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
&= \left\| \chi_A \mathfrak{L}_{\mathfrak{R}_j} \mathfrak{L}^*_{\mathfrak{R}_j} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} + \sum_{\substack{1 \le k \le 2^n \\ k \ne j}} \left\| \chi_A \mathfrak{L}_{\mathfrak{R}_k} \mathfrak{L}^*_{\mathfrak{R}_j} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
&\lesssim \min\left(2^{-n/2}, \rho^{1/2}\right) + 2^{-n\kappa\epsilon_2/2} \min\left(2^{-n/2}, \rho^{1/2}\right) \cdot (2^n - 1) \\
&\lesssim \min\left(2^{-n/2}, \rho^{1/2}\right).
\end{aligned}$$

For the dual estimate, $\trianglelefteq$***-incomparability*** *implies:*

$$\begin{aligned}
&\sum_{k=1}^{2^n} \left\| \mathfrak{L}^*_{\mathfrak{R}_k} \chi_A \mathfrak{L}_{\mathfrak{R}_j} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
&= \left\| \mathfrak{L}^*_{\mathfrak{R}_j} \chi_A \chi_A \mathfrak{L}_{\mathfrak{R}_j} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} + \sum_{\substack{1 \le k \le 2^n \\ k \ne j}} \left\| \cancel{\mathfrak{L}^*_{\mathfrak{R}_k} \chi_A \mathfrak{L}_{\mathfrak{R}_j}} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
&\lesssim \min\left(2^{-n/2}, \rho^{1/2}\right).
\end{aligned}$$

Combining the two, we have:

$$\left\| \chi_A \mathfrak{L}_{\mathbb{P}^*_{A,\rho}} f \right\|_{L^2} \lesssim \min\left(2^{-n/2}, \rho^{1/2}\right) \|f\|_{L^2}.$$

- *To control* $\chi_A \mathfrak{L}^*_{\mathbb{P}_{A,\rho}}$, *we use* ***orthogonality*** *directly. After decomposing* $\mathbb{P}_{A,\rho}$ *into* ***rows*** $\{\mathfrak{R}_j\}_{j=1}^{2^n}$, *we can control its* ***adjoint***:

$$\begin{aligned}
\left\| \mathfrak{L}_{\mathbb{P}_{A,\rho}} \chi_A f \right\|^2_{L^2} &= \sum_{j=1}^{2^n} \left\| \mathfrak{L}_{\mathbb{R}_j} \chi_A f \right\|^2_{L^2} \\
&\le \sum_{j=1}^{2^n} \left(\left\| \chi_A \mathfrak{L}^*_{\mathbb{R}_j} \right\|_{\mathcal{BL}(L^2,L^2)} \cdot \|f\|_{L^2} \right)^2 \\
&\lesssim 2^n \cdot \left(2^{-n/2} \rho^{1/2} \|f\|_{L^2} \right)^2 = \rho \|f\|^2_{L^2} \\
\implies \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}_{A,\rho}} f \right\|_{L^2} &\lesssim \rho^{1/2} \|f\|_{L^2}.
\end{aligned}$$

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We now present the analogue for a **cluster tower**:

Lemma 8.7.4 (**Cluster tower localized** L^2 estimate)**.**
*Given $\mathbb{P} \subset \mathbb{P}_n$ a **cluster tower**, as long as $\kappa \geq 2/\epsilon_2$, we have:*

$$\left\|\chi_A \mathfrak{L}_{\mathbb{P}} \chi_{A_\rho^c} f\right\|_{L^2}, \ \left\|\chi_A \mathfrak{L}^*_{\mathbb{P}} \chi_{A_\rho^c} f\right\|_{L^2} \lesssim n\,(1-\log_2 \rho)\,\rho^{1/2} \|\chi_{A_\rho^c} f\|_{L^2}.$$

Proof. *For starters, we take $N := \left\lceil \frac{n}{2}(1-\log_2 \rho)\right\rceil$ and decompose $\mathbb{P}$:*

$$\mathbb{P} = \bigsqcup_{\alpha \in \mathbb{N}} \mathbb{P} \cap \mathbb{P}_{n,\alpha} = \bigsqcup_{\alpha \in \mathbb{N}} \bigsqcup_{j=1}^{2^n} \mathfrak{R}_{\alpha,j} = \bigsqcup_{\gamma=0}^{N-1} \bigsqcup_{\alpha \in N\mathbb{N}+\gamma} \bigsqcup_{j=1}^{2^n} \mathfrak{R}_{\alpha,j}$$

*We again verify the condition for **Cotlar-Stein Lemma** but, this time, view a **stack** as a whole. We start with estimating $\chi_A \mathfrak{L}_{\mathbb{P}} \chi_{A_\rho^c}$. Given $\alpha \in N\mathbb{N}+\gamma$, we have:*

$$\begin{aligned}
&\sum_{\beta \in N\mathbb{N}+\gamma} \left\|\chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c} \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
=&\left\|\chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
+&\sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta \neq \alpha}} \left\|\chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c} \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)} \\
\leq&\left\|\chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)} \\
+&\sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta < \alpha}} \left(\sum_{k=1}^{2^n} \left\|\chi_A \mathfrak{L}_{\mathfrak{R}_{\beta,k}} \chi_{A_\rho^c \cap \bigsqcup \mathbb{A}_{\alpha-1}}\right\|_{\mathcal{BL}(L^2,L^2)} \left\|\chi_{A_\rho^c} \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A\right\|_{\mathcal{BL}(L^2,L^2)}\right)^{1/2} \\
+&\sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta > \alpha}} \left(\left\|\chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)} \sum_{k=1}^{2^n} \left\|\chi_{A_\rho^c \cap \bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,k}} \chi_A\right\|_{\mathcal{BL}(L^2,L^2)}\right)^{1/2} \\
\lesssim&\rho^{1/2} + \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta \neq \alpha}} 2^{1-|\beta-\alpha|} \left(\min(2^{-n/2}, \rho^{1/2}) \cdot 2^n \cdot \rho^{1/2}\right)^{1/2} \lesssim \rho^{1/2}.
\end{aligned}$$

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For the dual condition, we have:

$$
\begin{aligned}
&\sum_{\beta\in N\mathbb{N}+\gamma}\left\|\chi_{A_\rho^c}\mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}}\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}\\
=&\left\|\chi_{A_\rho^c}\mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}\\
&+\sum_{\substack{\beta\in N\mathbb{N}+\gamma\\ \beta\neq\alpha}}\left\|\chi_{A_\rho^c}\mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}}\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}\\
\leq&\left\|\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)}\\
&+\sum_{\substack{\beta\in N\mathbb{N}+\gamma\\ \beta<\alpha}}\left(\sum_{k=1}^{2^n}\left\|\chi_{A_\rho^c}\mathfrak{L}^*_{\mathfrak{R}_{\beta,k}}\chi_{A\cap\bigsqcup\mathbb{A}_{\alpha-1}}\right\|_{\mathcal{BL}(L^2,L^2)}\left\|\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)}\right)^{1/2}\\
&+\sum_{\substack{\beta\in N\mathbb{N}+\gamma\\ \beta>\alpha}}\left(\left\|\chi_{A_\rho^c}\mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}}\chi_A\right\|_{\mathcal{BL}(L^2,L^2)}\sum_{k=1}^{2^n}\left\|\chi_{A\cap\bigsqcup\mathbb{A}_{\beta-1}}\mathfrak{L}_{\mathfrak{R}_{\alpha,k}}\chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)}\right)^{1/2}\\
\lesssim&\rho^{1/2}+\sum_{\substack{\beta\in N\mathbb{N}+\gamma\\ \beta\neq\alpha}}\left(\min\left(2^{2-2|\beta-\alpha|},2^{-n/2},\rho^{1/2}\right)\cdot 2^n\cdot\rho^{1/2}\right)^{1/2}\lesssim\rho^{1/2}.
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
\left\|\chi_A\mathfrak{L}_{\mathbb{P}}\chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)}\leq&\sum_{\gamma=0}^{N}\left\|\sum_{\alpha\in N\mathbb{N}+\gamma}\chi_A\mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}\chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)}\\
\lesssim&N\rho^{1/2}\lesssim n\left(1-\log_2\rho\right)\rho^{1/2}.
\end{aligned}
$$

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To estimate $\chi_A \mathfrak{L}_{\mathbb{P}}^* \chi_{A_\rho^c}$, we follow similar arguments:

$$\sum_{\beta \in N\mathbb{N}+\gamma} \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$= \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta\neq\alpha}} \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$\leq \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta<\alpha}} \left(\sum_{k=1}^{2^n} \left\| \chi_A \mathfrak{L}^*_{\mathfrak{R}_{\beta,k}} \chi_{A_\rho^c \cap \bigsqcup \mathbb{A}_{\alpha-1}} \right\|_{\mathcal{BL}(L^2,L^2)} \left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A \right\|_{\mathcal{BL}(L^2,L^2)} \right)^{1/2}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta>\alpha}} \left(\left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)} \sum_{k=1}^{2^n} \left\| \chi_{A_\rho^c \cap \bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}_{\mathfrak{R}_{\alpha,k}} \chi_A \right\|_{\mathcal{BL}(L^2,L^2)} \right)^{1/2}$$

$$\lesssim \rho^{1/2} + \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta\neq\alpha}} \left(\min\left(2^{2-2|\beta-\alpha|}, 2^{-n/2}\right) \rho^{1/2} \cdot 2^n \cdot \rho^{1/2} \right)^{1/2} \lesssim \rho^{1/2}.$$

For the dual condition, we have:

$$\sum_{\beta \in N\mathbb{N}+\gamma} \left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$= \left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta\neq\alpha}} \left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|^{1/2}_{\mathcal{BL}(L^2,L^2)}$$

$$\leq \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta<\alpha}} \left(\sum_{k=1}^{2^n} \left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathfrak{R}_{\beta,k}} \chi_{A \cap \bigsqcup \mathbb{A}_{\alpha-1}} \right\|_{\mathcal{BL}(L^2,L^2)} \left\| \chi_A \mathfrak{L}^*_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}} \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)} \right)^{1/2}$$

$$+ \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta>\alpha}} \left(\left\| \chi_{A_\rho^c} \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\beta}} \chi_A \right\|_{\mathcal{BL}(L^2,L^2)} \sum_{k=1}^{2^n} \left\| \chi_{A \cap \bigsqcup \mathbb{A}_{\beta-1}} \mathfrak{L}^*_{\mathfrak{R}_{\alpha,k}} \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)} \right)^{1/2}$$

$$\lesssim \rho^{1/2} + \sum_{\substack{\beta \in N\mathbb{N}+\gamma \\ \beta\neq\alpha}} \left(\min\left(2^{2-2|\beta-\alpha|}, \rho^{1/2}\right) \cdot 2^{-n/2} \cdot 2^n \cdot \rho^{1/2} \right)^{1/2} \lesssim \rho^{1/2}.$$

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Therefore, we have:

$$\left\|\chi_A \mathfrak{L}_{\mathbb{P}}^* \chi_{A_\rho^c}\right\|_{\mathcal{BL}(L^2,L^2)} \leq \sum_{\gamma=0}^{N} \left\| \sum_{\alpha \in N\mathbb{N}+\gamma} \chi_A \mathfrak{L}_{\mathbb{P}\cap\mathbb{P}_{n,\alpha}}^* \chi_{A_\rho^c} \right\|_{\mathcal{BL}(L^2,L^2)}$$
$$\lesssim N\rho^{1/2} \lesssim n\left(1-\log_2 \rho\right)\rho^{1/2}.$$

This completes the proof. □

We now use such **localized estimate** to **extrapolate** our estimate:

Theorem 8.7.5 (**Cluster tower weak** estimate)**.**
*Given $\mathbb{P} \subset \mathbb{P}_n$ a **cluster tower**, as long as $\kappa \geq 2/\epsilon_2$, we have:*

$$\left\|\mathfrak{L}_{\mathbb{P}} f\right\|_{L^{p,\infty}}, \ \left\|\mathfrak{L}_{\mathbb{P}}^* f\right\|_{L^{p,\infty}} \underset{p}{\lesssim} n\left\|f\right\|_{L^{p,1}}, \quad \forall p \in (2,\infty).$$

Proof. *Let T denote either $\mathfrak{L}_{\mathbb{P}}$ or $\mathfrak{L}_{\mathbb{P}}^*$. We intend to use L^2 **Extrapolation**.*

- *For measurable sets $G, H \subset \mathbb{R}^D$. We want to find suitable measurable subsets $G' \subset G$ and $H' \subset H$ satisfying both **error loss control** and **testing condition**.*
- *To match the form, we should set: $\rho \approx \frac{|H|}{|G|}$. That is, we will fine tune a constant $C \in \mathbb{R}_+$ and set $\rho := C\frac{|H|}{|G|}$.*
- *We define $G' := G \setminus H_\rho$ and $H' := H$ and verify the **error loss control**:*

  $$\left(\frac{|G\setminus G'|}{|G|}\right)^{1/p} + \left(\frac{|H\setminus H'|}{|H|}\right)^{1/p'} = \left(\frac{|G\cap H_\rho|}{|G|}\right)^{1/p} \leq \left(\frac{|H_\rho|}{|G|}\right)^{1/p}$$
  $$\leq \left(\frac{\rho^{-1}\left\|M\right\|_{L^1\to L^{1,\infty}}|H|}{|G|}\right)^{1/p} = \left(\frac{\left\|M\right\|_{L^1\to L^{1,\infty}}}{C}\right)^{1/p} =: \epsilon < 1$$

  ***as long as** $C > \left\|M\right\|_{L^1\to L^{1,\infty}} \gtrsim C$.*
- *To verify the **testing condition**, we see that:*
  - *If $\rho \gtrsim 1$, we may just apply **cluster tower** L^2 estimate:*

    $$\left\|\chi_{H'} T \chi_{G'} f\right\|_{L^2} \leq \left\|T\chi_{G'} f\right\|_{L^2} \lesssim n2^{-n/2}\left\|\chi_{G'} f\right\|_{L^2}$$
    $$\lesssim n\rho^{1/2-1/p}\left\|\chi_{G'} f\right\|_{L^2} \underset{p}{\lesssim} n\left(\frac{|H|}{|G|}\right)^{1/2-1/p}\left\|\chi_{G'} f\right\|_{L^2}.$$
  - *If $\rho \ll 1$, we use **cluster tower localized** L^2 estimate:*

    $$\left\|\chi_{H'} T \chi_{G'} f\right\|_{L^2} = \left\|\chi_H T \chi_{H_\rho^c} \chi_{G'} f\right\|_{L^2}$$
    $$\lesssim n\left(1-\log_2 \rho\right)\rho^{1/2}\left\|\chi_{H_\rho^c}\chi_{G'} f\right\|_{L^2}$$
    $$\underset{p}{\lesssim} n\rho^{1/2-1/p}\left\|\chi_{G'} f\right\|_{L^2} \underset{p}{\lesssim} n\left(\frac{|H|}{|G|}\right)^{1/2-1/p}\left\|\chi_{G'} f\right\|_{L^2}.$$

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- L^2 ***Extrapolation*** *yields:*

$$\|Tf\|_{L^{p,\infty}} \underset{p}{\lesssim} n \|f\|_{L^{p,1}},$$

which completes the proof. □

As a direct corollary, through **interpolation**, we have:

Corollary 8.7.5.1 (**Cluster tower strong** estimate)**.**
*Given $\mathbb{P} \subset \mathbb{P}_n$ a **cluster tower**, as long as $\kappa \geq 2/\epsilon_2$, we have:*

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^p} \underset{p}{\lesssim} n 2^{-n\eta_p} \|f\|_{L^p}, \quad \textit{where} \ \ \eta_p > 0, \quad \forall p \in (1, \infty).$$

Corollary 8.7.5.2 (L^p bound on **cluster parts**)**.**
*Given the full collection of the **cluster parts** $\mathbb{P} \subset \tilde{\mathbb{D}}$, we have:*

$$\|\mathfrak{L}_{\mathbb{P}} f\|_{L^p} \underset{p}{\lesssim} \|f\|_{L^p}, \quad \forall p \in (1, \infty).$$

Remark. *Through our method, instead of **rearranging** the whole collection as in [Zor19], we recover the result in [Lie20]. That is, the **decomposition** itself is **effective enough** for the $L^p \to L^p$ bound. Still, the formulation in [Lie20] is similar to a **decoupling inequality**, which contains more information about the structure of the L^p estimate.*

As we combine the estimation of **sparse tower** and **cluster tower**, we prove the main result in the following reduced form:

Theorem 8.7.6 (Main theorem for the **linearized** operator)**.**

$$\|\mathfrak{L}_{\mathbb{P}_n} f\|_{L^p} \underset{p}{\lesssim} p(n) 2^{-n\eta_p} \|f\|_{L^p}, \quad \textit{where} \ \ \eta_p > 0, \quad \forall p \in (1, \infty).$$

Summing over $n \in \mathbb{N}$ yields:

$$\|\mathfrak{L} f\|_{L^p} \underset{p}{\lesssim} \|f\|_{L^p}, \quad \forall p \in (1, \infty).$$

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