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碩士論文

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Master Thesis

基於羅森—歐斯曼錐構造的均曲流自相似解

Self-similar solutions to the mean curvature flow based on the Lawson-Osserman cone

李宸寬

Chen-Kuan Lee

指導教授: 蔡忠潤 副教授

Advisor: Chung-Jun Tsai, Associative Professor

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國立臺灣大學碩士學位論文
口試委員會審定書

基於羅森－歐斯曼錐構造的均曲流自相似解
Self-similar solutions to the mean curvature flow based on the Lawson-Osserman cone

本論文係李宸寬君（R09221013）在國立臺灣大學數學系完成之碩士學位論文，於民國 111 年 6 月 10 日承下列考試委員審查通過及口試及格，特此證明

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摘要

在這篇論文中，我們首先得到基於羅森—歐斯曼錐構造的均曲流自相似解必須滿足的等式，並證明了自擴張解的存在性。主要的關鍵是利用羅森—歐斯曼錐的對稱性將偏微分方程轉化為常微分方程組，並研究這種近似於自治系統的常微分方程組。特別地，我們發現從狄利克雷問題的觀點來看，我們構造的自擴張解具唯一性。

關鍵字：幾何分析、高餘維均曲流、自相似解、羅森—歐斯曼錐、狄利克雷問題

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Abstract

In this thesis, we derived the equation of self-similar solutions to mean curvature flow based on the Lawson-Osserman cone and proved the existence of self-expander. The main point is to use the symmetry to transform the PDE into a system of ODEs and analyze such analogous autonomous system. In particular, the self-expander is unique form the viewpoint of Dirichlet problem.

Keywords: Geometric Analysis, Mean Curvature Flow in Higher Codimensions, Self-Similar Solution, Lawson-Osserman Cone, Dirichlet Problem

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1 Introduction

In [10], Lawson and Osserman found the minimal cone of higher codimensions in $\mathbb{R}^{3n+1}$,

$$C_n = \{(\mathbf{x}, \kappa_n \frac{\mathcal{H}(\mathbf{x})}{r}) \big| \, \mathbf{x} \in \mathbb{R}^{2n}, r = \|\mathbf{x}\|\},$$

where $n = 2, 4, 8$, $\kappa_n = \sqrt{\frac{2n+1}{4(n-1)}}$ and $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}^{n+1}$ is the Hopf map. Later, Harvey and Lawson [5] proved that C_2 is in fact coassociative when equipping $\mathbb{R}^7$ a G_2-structure, therefore area-minimizing. Recently, Xu, Yang and Zhang [12] showed that the rest C_4, C_8 are also area-minimizing by using Lawlor's curvature criterion [9].

Ding and Yuan resolved the singularities in [2]. They found that there exists an unique function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a family of minimal graphs

$$G_\mu = \{(\mathbf{x}, \mu^{-1} g(\mu r) \frac{\mathcal{H}(\mathbf{x})}{r^2}) \big| \, \mathbf{x} \in \mathbb{R}^{2n}, r = \|\mathbf{x}\|\}, \mu > 0$$

which is asymptotic to C_n yet smooth at the origin.

On the other hand, geometric flows, especially Ricci flow and the mean curvature flow, have arrested much attention over the past half-century. Hamilton [4] is the first one developing the theory of Ricci flow. In addition, Brakke [1] created the notion of the mean curvature flow from the viewpoint of geometric measure theory. Then Huisken [7] studied it from the classical point of view. There are many important breakthroughs in the mean curvature flow in codimension one; however, the complexity of the quasi-linear PDE system is one of the difficulties for understanding the mean curvature flow in higher codimensions. Without imposing extra condition, such as Lagrangian mean curvature flow, there are still very few ways to deal with the mean curvature flow in higher codimensions.

In this note, we consider the self-similar solutions to the mean curvature flow based on the Lawson-Osserman cone, i.e. the self-similar solutions in higher codimensions. In section 4, we provide a stable curve theorem for an analogous autonomous system. Using it, we are able to show our main result:

Main Theorem. *There exist* $0 < \varepsilon < \kappa_n$, $r_0 > 0$ *and an unique smooth self-expander in* $\mathbb{R}^{3n+1}$ *of the form*

$$\Sigma_n = \{(\mathbf{x}, f(r) \frac{\mathcal{H}(\mathbf{x})}{r^2}) \big| \, \mathbf{x} \in \mathbb{R}^{2n}, r = \|\mathbf{x}\|\}$$

for $n = 2, 4, 8$ *such that*

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$$

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satisfies the condition

$$f(r_0) = \varepsilon r_0.$$

Moreover, f has the following property:

1.
$$0 \leq f(r) < \kappa_n r, \quad 0 \leq f'(r).$$

2. *As $r \to 0$,*
$$f \in O(r^{2-\delta}), \quad f' \in O(r^{1-\delta})$$
for any $\delta > 0$.

3.
$$\lim_{r\to\infty} \frac{f(r)}{r}$$
exists. In other words, Σ_n is asymptotic to a cone
$$\{(\mathbf{x}, L\frac{H(\mathbf{x})}{r}) \,\big|\, \mathbf{x} \in \mathbb{R}^{2n}\},$$
where $L = \lim_{r\to\infty} \frac{f(r)}{r}$, as $r \to \infty$.

2 Background materials

We first recall the definition of the Hopf map

$$\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}^{n+1},$$

where $n = 2, 4, 8$.

Definition. We identify $\mathbb{R}^n$ with the normed algebra, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ for $n = 2, 4, 8$, repectively. Let $\mathbf{x} = (p, q) \in \mathbb{R}^n \times \mathbb{R}^n$, then the Hopf map is defined by

$$\mathcal{H}(\mathbf{x}) = (\|p\|^2 - \|q\|^2, 2q\bar{p}).$$

We also note that the Hopf map $\mathcal{H}$ is equivariant.

Proposition 2.1. *For any fixed $\mathbf{x} \in \mathbb{R}^{2n}$, there is an orthogonal transformation $M \in O(2n)$ and an induced orthogonal transformation $\tilde{M} \in O(2n)$ such that*

$$\mathbf{x} = M(\|\mathbf{x}\|, 0, \cdots, 0)$$

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and

$$\mathcal{H}(M(\mathbf{y})) = \tilde{M}(\mathcal{H}(\mathbf{y}))\,\forall\,\mathbf{y} \in \mathbb{R}^{2n}.$$

[cf. [2], Appendix, Prop. 4.1]

We also recall the definition of mean curvature flow here.

Definition. Let Σ be a smooth submanifold in a Riemannian manifold M. If there exists a family of smooth immersions $F_t : \Sigma \to M$ satisfying

$$\begin{cases} (\dfrac{\partial F_t}{\partial t}(\mathbf{x}))^{\perp} = H_{\Sigma_t}(\mathbf{x}) \\ F_0 = \mathrm{id} \end{cases},$$

where

$$H_{\Sigma_t} \coloneqq (g_t)^{ij} \nabla^{\perp}_{\frac{\partial F_t}{\partial x^i}} \frac{\partial F_t}{\partial x^j}$$

denotes the mean curvature vector of Σ_t and

$$(g_t)_{ij} \coloneqq \langle \frac{\partial F_t}{\partial x^i}, \frac{\partial F_t}{\partial x^j} \rangle_M,$$

then F_t is called a mean curvature flow of Σ.

In geometric flows, such as the mean curvature flow or Ricci flow, singularities are often locally modelled on soliton solutions. For the mean curvature flow case, there are two types of soliton solutions in Euclidean space that are particularly interested. One is the solitons moved by scaling, and the other one is that moved by translation. We now recall the solitons moved by scaling:

Definition. A submanifold Σ in Euclidean space $\mathbb{R}^n$ is called a self-similar solution if

$$H_{\Sigma} \equiv CF^{\perp}$$

on Σ for some constant $C \in \mathbb{R}$, where $F^{\perp}$ denotes the projection of the position vector F in $\mathbb{R}^n$ to the normal bundle $N\Sigma$ of Σ and H_{Σ} is the mean curvature vector of $\Sigma \subset \mathbb{R}^n$. Moreover, it is called a self-shrinker if $C < 0$ and a self-expander if $C > 0$.

Notice that if Σ is a self-similar solution and $F : \Sigma \to \mathbb{R}^n$ is the position vector, then F_t defined by

$$F_t = \sqrt{1 + 2Ct}F$$

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is moved by the mean curvature flow. It follows from Huisken's monotonicity formula [8] that any central blow-up of a finite-time singularity of the mean curvature flow is a self-similar solution. When $C = 0$, the submanifold is minimal.

Remark 2.2. The only crucial part of C is its sign since its absolute value can be shifted to 1 after a scaling of Σ. That is to say, it suffices to consider $C = 1, 0, -1$.

On the other hand, the submanifolds that are moved by translation along the mean curvature flow are of the following form:

Definition. A submanifold Σ in Euclidean space $\mathbb{R}^n$ is called a translating soliton if there exists a constant vector $T \in \mathbb{R}^n$ such that

$$H_\Sigma \equiv T^\perp$$

on Σ.

We also note that if Σ is a translating soliton and $F : \Sigma \to \mathbb{R}^n$ is the position vector, then F_t defined by

$$F_t = F + tT$$

is a mean curvature flow of Σ. Such T is called the translating vector.

Remark 2.3. In this note, we do not consider the translating solitons. Since we are assuming the symmetric condition and considering a submanifold of codimension higher than 1, the non-zero translating vector does not exist in this case.

3 The desired ODE of self-similar solutions

Due to the equivariance of Hopf map,

$$H_{\Sigma_n} = CF^\perp$$

holds if and only if it holds at $\mathbf{x} = F(r, 0, \cdots, 0) \in \Sigma_n$. We now calculate $H_{\Sigma_n}(\mathbf{x})$ and $CF^\perp(\mathbf{x})$.

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Note that at $\mathbf{x}$, $T_{\mathbf{x}}\Sigma_n$ has an induced basis

$$\begin{cases} e_1 = F_*(\frac{\partial}{\partial x^1}) = (1, 0, \cdots, 0, f', 0, \cdots, 0), \\ \quad \text{which values only at the first and the } (2n+1)\text{-th components.} \\ e_i = F_*(\frac{\partial}{\partial x^i}) = (0, \cdots, 0, 1, 0, \cdots, 0), \quad 2 \leq i \leq n, \\ \quad \text{which values only at the } i\text{-th component.} \\ e_j = F_*(\frac{\partial}{\partial x^j}) = (0, \cdots, 0, 1, 0, \cdots, 0, \frac{2f}{r}, 0, \cdots, 0), \quad n+1 \leq j \leq 2n, \\ \quad \text{which values only at the } j\text{-th and the } (n+j+1)\text{-th components.} \end{cases}$$

We first observe that

$$CF^{\perp} = \frac{C}{\sqrt{1+(f')^2}}(-rf' + f)\eta_1,$$

where

$$\eta_1 = \frac{1}{\sqrt{1+(f')^2}}(-f', 0, \cdots, 0, 1, 0, \cdots, 0)$$

is an unit normal vector in $N_{\mathbf{x}}\Sigma_n$.

Moreover, at $\mathbf{x}$, the induced metric g_{ij} is of the form

$$\begin{cases} g_{11} = 1 + (f')^2 \\ g_{ii} = 1, \quad 2 \leq i \leq n \\ g_{jj} = 1 + \frac{4f^2}{r^2}, \quad n+1 \leq j \leq 2n \\ g_{ij} = 0, \quad \forall i \neq j \end{cases}$$

and

$$\begin{cases} \overline{\nabla}_{e_1} e_1 = \frac{\partial^2 F}{(\partial x^1)^2} = (0, \cdots, 0, f'', 0, \cdots, 0), \\ \quad \text{which values only at the first and the } (2n+1)\text{-th components.} \\ \overline{\nabla}_{e_i} e_i = \frac{\partial^2 F}{(\partial x^i)^2} = (0, \cdots, 0, \frac{f'}{r}, 0, \cdots, 0), \quad 2 \leq i \leq n, \\ \quad \text{which values only at the first and the } (2n+1)\text{-th components.} \\ \overline{\nabla}_{e_j} e_j = \frac{\partial^2 F}{(\partial x^j)^2} = (0, \cdots, 0, \frac{rf' - 4f}{r^2}, 0, \cdots, 0), \quad n+1 \leq j \leq 2n, \\ \quad \text{which values only at the first and the } (2n+1)\text{-th components.} \end{cases}$$

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Now,

$$H_{\Sigma_n} = g^{ij}\nabla^{\perp}_{e_i} e_j$$
$$= \frac{1}{\sqrt{1+(f')^2}}\left(\frac{f''}{1+(f')^2} + (n-1)\frac{f'}{r} + \frac{n(rf'-4f)}{r^2+4f^2}\right)\eta_1 \, .$$

Therefore, we obtain a second order ODE

$$\frac{f''}{1+(f')^2} + (n-1)\frac{f'}{r} + \frac{n(rf'-4f)}{r^2+4f^2} = C(-rf'+f)$$

which is equivalent to the equation of self-similar solutions.

4 An analogous autonomous system

We consider the following system of ODEs

$$\begin{cases} X'(t) = -\lambda X(t) + f_1(X(t),Y(t)) + e^{-t} g_1(X(t),Y(t)) \\ Y'(t) = \mu Y(t) + f_2(X(t),Y(t)) + e^{-t} g_2(X(t),Y(t)) \end{cases},$$

where $\mu > 0 > -\lambda$, $\frac{f_i(X,Y)}{\sqrt{X^2+Y^2}} \to 0$ as $(X,Y) \to (0,0)$ and $g_j \in O(\sqrt{X^2+Y^2})$ as $(X,Y) \to (0,0)$ $\forall i,j = 1,2$. In this case, $(0,0)$ is a equilibrium point.

If we omit the exponential term, then it is a classical planar autonomous system. Under that situation, $(0,0)$ is in fact a saddle equilibrium point. There is a stable curve theorem for such case [cf. [6], Chap. 8.3, p.169], which states that we can find an $\varepsilon > 0$ and an unique local stable curve of the form $Y = h(X)$ that is defined for $|X| < \varepsilon$ and satisfies $h(0) = 0$. Moreover, this curve is tangent to the X-axis and all solutions with initial conditions that lies on this curve tend to the origin as $t \to \infty$.

In this section, the goal is to provide a similar stable curve theorem. We first give some notations to clarify the meaning of "local". Let S_ε be the square bounded by $X = \pm\varepsilon$ and $Y = \pm\varepsilon$. Let $E^{\pm}_{\varepsilon}$ be the left and right boundaries ($X = -\varepsilon$ and $X = \varepsilon$ respectively) of S_ε. We also define $R_{M,\varepsilon}$ to be the region given by $|Y| \le M|X|$ inside S_ε.

Now, we state two lemmas about the behavior of the vector field inside $R_{M,\varepsilon}$.

Lemma 4.1. *Given $M > 0$, there exists $\varepsilon > 0$ and $T > 0$ such that $X'(t) < 0$ in $R_{M,\varepsilon} \cap \{X > 0\}$ whenever $t > T$.*

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Proof. Since $\frac{f_1(X,Y)}{\sqrt{X^2+Y^2}} \to 0$ as $(X,Y) \to (0,0)$, we may choose $\varepsilon_1 > 0$ so that

$$|f_1(X,Y)| \le \frac{\lambda}{3\sqrt{M^2+1}}\sqrt{X^2+Y^2} \quad \forall (X,Y) \in S_{\varepsilon_1}.$$

Moreover, since $g_1 \in O(\sqrt{X^2+Y^2})$, there exist $\varepsilon_2 > 0$ such that

$$|g_1(X,Y)| < C\sqrt{X^2+Y^2} \quad \forall (X,Y) \in S_{\varepsilon_2}$$

for some positive constant C. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Now, we set $T > 0$ so that

$$e^{-t} < \frac{\lambda}{3C\sqrt{M^2+1}} \quad \forall t > T.$$

Note that in $R_{M,\varepsilon} \cap \{X > 0\}$, $|Y| \le MX \implies \sqrt{X^2+Y^2} \le \sqrt{M^2+1}X$. Therefore,

$$\begin{aligned}
X'(t) &= -\lambda X + f_1(X,Y) + e^{-t}g_1(X,Y) \\
&\le -\lambda X + |f_1(X,Y)| + e^{-t}|g_1(X,Y)| \\
&\le -\lambda X + \frac{\lambda}{3\sqrt{M^2+1}}\sqrt{X^2+Y^2} + \frac{\lambda}{3C\sqrt{M^2+1}}(C\sqrt{X^2+Y^2}) \\
&\le -\lambda X + \frac{\lambda}{3\sqrt{M^2+1}}\sqrt{M^2+1}X + \frac{\lambda}{3C\sqrt{M^2+1}}(C\sqrt{M^2+1}X) \\
&= \frac{-\lambda}{3}X(t) < 0
\end{aligned}$$

whenever $t > T$. □

Lemma 4.2. *Given $M > 0$, there exists $\varepsilon > 0$ and $T > 0$ such that $Y'(t) > 0$ on $\{(X,Y) \in R_{M,\varepsilon} \big| Y = MX, X > 0\}$ and $Y'(t) < 0$ on $\{(X,Y) \in R_{M,\varepsilon} \big| Y = -MX, X > 0\}$.*

Proof. Since $\frac{f_2(X,Y)}{\sqrt{X^2+Y^2}} \to 0$ as $(X,Y) \to (0,0)$, we may choose $\varepsilon_1 > 0$ so that

$$|f_2(X,Y)| \le \frac{M\mu}{3\sqrt{M^2+1}}\sqrt{X^2+Y^2} \quad \forall (X,Y) \in S_{\varepsilon_1}.$$

Furthermore, since $g_2 \in O(\sqrt{X^2+Y^2})$, there exist $\varepsilon_2 > 0$ such that

$$|g_2(X,Y)| < C\sqrt{X^2+Y^2} \quad \forall (X,Y) \in S_{\varepsilon_2}$$

for some positive constant C. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and choose $T > 0$ so that

$$e^{-t} < \frac{M\mu}{3C\sqrt{M^2+1}} \quad \forall t > T.$$

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Then on $\{(X,Y) \in R_{M,\varepsilon} \big| Y = MX, X > 0\}$,

$$\begin{aligned} Y'(t) &= \mu Y + f_2(X,Y) + e^{-t} g_2(X,Y) \\ &\geq \mu Y - |f_2(X,Y)| - e^{-t}|g_2(X,Y)| \\ &\geq \mu Y - \frac{M\mu}{3\sqrt{M^2+1}}\sqrt{X^2+Y^2} - \frac{M\mu}{3C\sqrt{M^2+1}}(C\sqrt{X^2+Y^2}) \\ &\geq \mu Y - \frac{M\mu}{3\sqrt{M^2+1}}\sqrt{M^{-2}+1}Y - \frac{M\mu}{3C\sqrt{M^2+1}}(C\sqrt{M^{-2}+1}Y) \\ &= \frac{\mu}{3}Y(t) > 0 \end{aligned}$$

whenever $t > T$. Similarly, on $\{(X,Y) \in R_{M,\varepsilon} \big| Y = -MX, X > 0\}$,

$$\begin{aligned} Y'(t) &= \mu Y + f_2(X,Y) + e^{-t} g_2(X,Y) \\ &\leq \mu Y + |f_2(X,Y)| + e^{-t}|g_2(X,Y)| \\ &\leq \mu Y - \frac{M\mu}{3\sqrt{M^2+1}}\sqrt{M^{-2}+1}Y - \frac{M\mu}{3C\sqrt{M^2+1}}(C\sqrt{M^{-2}+1}Y) \\ &= \frac{\mu}{3}Y(t) < 0 \end{aligned}$$

whenever $t > T$. □

With the above two lemmas, we are ready to show the existence of the stable curve for our analogous system; however, unlike the autonomous case, it depends on the initial time.

Note that by Lemma 4.1, the solutions with initial conditions $(X(T), Y(T)) \in R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ strictly decrease in the X-direction when they remain in $R_{M,\varepsilon} \cap \{X > 0\}$. In particular, the solution can remain in $R_{M,\varepsilon} \cap \{X > 0\}$ for all $t > T$ only if it tends to $(0,0)$.

According to Lemma 4.2, there is a set of initial conditions $\{(X(T), Y(T))\} \subset R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ with solutions that eventually exit $R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ to the top. There also exist another set of initial conditions $\{(X(T), Y(T))\} \subset R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ with solutions that eventually exit $R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ to the below. Due to the smooth dependence of initial conditions, these two set are single open intervals. Note that $R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ is connected. We therefore conclude that there exists a nonempty set of initial conditions $\{(X(T), Y(T))\} \subset R_{M,\varepsilon} \cap \{X > 0\} \cap E_\varepsilon^+$ such that the solutions never leave $R_{M,\varepsilon} \cap \{X > 0\}$. That is to say, the solutions tend to $(0,0)$ as $t \to \infty$.

Moreover, since $X'(t) \leq \frac{-\lambda}{3}X(t)$, the Grönwall's inequality shows that $X(t) \leq Ce^{-\frac{\lambda}{3}t}$ for some constant C. That is to say, $X(t) \in O(e^{-\frac{\lambda}{3}t})$ as $t \to \infty$. Since $|Y| \leq M|X|$, we also conclude that $Y(t) \in O(e^{-\frac{\lambda}{3}t})$ as $t \to \infty$.

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Hence, we proved the following theorem.

Theorem 4.3. *Given a system of ODEs*

$$\begin{cases} X'(t) = -\lambda X(t) + f_1(X(t), Y(t)) + e^{-t} g_1(X(t), Y(t)) \\ Y'(t) = \mu Y(t) + f_2(X(t), Y(t)) + e^{-t} g_2(X(t), Y(t)) \end{cases},$$

where $\mu > 0 > -\lambda$, $\frac{f_i(X,Y)}{\sqrt{X^2+Y^2}} \to 0$ *as* $(X, Y) \to (0, 0)$ *and* $g_j \in O(\sqrt{X^2 + Y^2})$ *as* $(X, Y) \to (0, 0)$ $\forall i, j = 1, 2$. *Then for all* $M > 0$, *there is an* $\varepsilon > 0$, *an initial time* $T > 0$ *and a solution curve* $(X(t), Y(t))$ *defined on* $t \in [T, \infty)$ *such that* $X(T) = \varepsilon$, $-M\varepsilon \le Y(T) \le M\varepsilon$ *and* $(X, Y) \to (0, 0)$ *as* $t \to \infty$. *Furthermore,* $X(t), Y(t) \in O(e^{-\frac{\lambda}{3}t})$ *as* $t \to \infty$.

Remark 4.4. Fix any $\delta > 0$. Shrinking ε small enough, we in fact derive

$$X'(t) < -\lambda(1 - \delta)X(t)$$

in Lemma 4.1. Therefore, we can improve the limiting behavior to

$$X(t), Y(t) \in O(e^{-\lambda(1-\delta)t})$$

as $t \to \infty$

5 The existence of self-expander

Given a constant $C = 1$ or -1, the self-similar solution ($C = 1$ is self-expander and $C = -1$ is self-shrinker) is characterized by

$$\frac{f''}{1 + (f')^2} + (n - 1)\frac{f'}{r} + \frac{n(rf' - 4f)}{r^2 + 4f^2} = C(-rf' + f).$$

Define $t = \log r$, $\varphi = \frac{f}{r}$ and $\psi = \varphi_t$. Then $f' = \varphi + \psi$ and $f'' = \frac{1}{r}(\psi_t + \psi) = e^{-t}(\psi_t + \psi)$. We can therefore convert the second order equation to the following system of first order ODEs:

$$\begin{cases} \varphi_t = \psi \\ \psi_t = -\psi - ((n - 1 + \frac{n}{1 + 4\varphi^2} + Ce^{2t})\psi + (n - 1 - \frac{3n}{1 + 4\varphi^2})\varphi)(1 + (\varphi + \psi)^2) \end{cases}.$$

Note that this system has a saddle equilibrium point $(0, 0)$ and a sink equilibrium point $(\kappa_n, 0)$,

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where $\kappa_n = \frac{1}{2}\sqrt{\frac{2n+1}{n-1}}$. At $(0,0)$, the linearized system looks like

$$\begin{pmatrix} 0 & 2n+1 \\ 1 & -2n \end{pmatrix}$$

with eigenvalues $\lambda_1 = 1, \lambda_2 = -2n-1$ and associative eigenvector $V_1 = \begin{pmatrix} 2n+1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Proposition 5.1. *For the self-expander case, i.e., $C = 1$, the closed region Δ enclosed by $\varphi = 0$, $\psi = 0$ and $n\varphi + \psi = n\kappa_n$ is a positive invariant set of the system of ODEs.*

Proof. It suffices to check the following three conditions.

1. $\varphi' \geq 0$ on $\{(0,\psi) | 0 \leq \psi \leq n\kappa_n\}$.
2. $\psi' \geq 0$ on $\{(\varphi, 0) | 0 \leq \varphi \leq \kappa_n\}$.
3. $\langle(\varphi', \psi'), (n, 1)\rangle \leq 0$ on $\{(\varphi, n\kappa_n - n\varphi) | 0 \leq \varphi \leq \kappa_n\}$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^2$.

The first two ones are clear. For the third one, we first notice that $\psi = n(\kappa_n - \varphi) \geq 0$ and $e^{2t} > 0$. Then

$$\begin{aligned}
\langle(\varphi', \psi'), (n, 1)\rangle =& n\psi - \psi \\
& - ((n-1+\frac{n}{1+4\varphi^2} + e^{2t})\psi + (n-1-\frac{3n}{1+4\varphi^2})\varphi)(1+(\varphi+\psi)^2) \\
\leq& (n-1)\psi - ((n-1+\frac{n}{1+4\varphi^2})\psi + (n-1-\frac{3n}{1+4\varphi^2})\varphi)(1+(\varphi+\psi)^2) \\
=& n(n-1)(\kappa_n - \varphi) \\
& - \frac{((n-1)(n\kappa_n - (n-1)\varphi)(1+4\varphi^2) + n^2\kappa_n - 3n\varphi))}{1+4\varphi^2} \\
& \times (1 + (n\kappa_n - (n-1)\varphi)^2)
\end{aligned}.$$

Define

$$\begin{aligned}
h(\varphi) :=& n(n-1)(\kappa_n - \varphi)(1+4\varphi^2)((n-1)(n\kappa_n - (n-1)\varphi)(1+4\varphi^2) + n^2\kappa_n - 3n\varphi)) \\
& \times (1 + (n\kappa_n - (n-1)\varphi)^2)
\end{aligned}$$

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on $[0,\kappa_n]$. Note that

$$\begin{aligned}h'(\varphi)=&-n(n-1)(1+4\varphi^2)+8n(n-1)\varphi(\kappa_n-\varphi)\\&+2(n-1)^2(1+4\varphi^2)(n\kappa_n-(n-1)\varphi)^2+2(n-1)(n^2\kappa_n-3n\varphi)(n\kappa_n-(n-1)\varphi)\\&+((n-1)^2(1+4\varphi^2)-8\varphi(n\kappa_n-(n-1)\varphi)+3n)(1+(n\kappa_n-(n-1)\varphi)^2)\\\geq&\tilde{h}(\varphi)\end{aligned},$$

where

$$\begin{aligned}\tilde{h}(\varphi):=&-n(n-1)(1+4\varphi^2)+8n(n-1)\varphi(\kappa_n-\varphi)\\&+2(n-1)^2(1+4\varphi^2)(n\kappa_n-(n-1)\varphi)^2+2(n-1)(n^2\kappa_n-3n\varphi)(n\kappa_n-(n-1)\varphi)\\&+(n-1)^2(1+4\varphi^2)-8\varphi(n\kappa_n-(n-1)\varphi)+3n\end{aligned}$$

and $\varphi\in[0,\kappa_n]$. We further compute

$$\begin{aligned}\tilde{h}'(\varphi)=&8n(n-1)(\kappa_n-2\varphi)-8(n\kappa_n-(n-1)\varphi)+16(n-1)^2\varphi(n\kappa_n-(n-1)\varphi)^2\\&-4(n-1)^3(1+4\varphi^2)(n\kappa_n-(n-1)\varphi)-6n(n-1)(n\kappa_n-(n-1)\varphi)\\&-2(n-1)^2(n^2\kappa_n-3n\varphi)\end{aligned}$$

and

$$\tilde{h}''(\varphi)=4(n-1)(24(n-1)^3\varphi^2-24(n-1)^2n\kappa_n\varphi+3n^3+n^2-4n+1)\quad.$$

Since $\tilde{h}''(\varphi)=0$ only when $\varphi=\frac{6n(n-1)\kappa_n\pm\sqrt{(n-1)(n^2+8n-2)}}{12(n-1)^2}$, the maximum point of $\tilde{h}'(\varphi)$ can only possibly occur at 0, $\varphi=\frac{6n(n-1)\kappa_n\pm\sqrt{(n-1)(n^2+8n-2)}}{12(n-1)^2}$ or κ_n. Plugging each point and checking its value show that

$$\tilde{h}'(\varphi)<0\quad\forall\,\varphi\in[0,\kappa_n],n=2,4,8.$$

Hence,

$$\tilde{h}(\varphi)\geq\tilde{h}(\kappa_n)\geq 0\quad\forall\,\varphi\in[0,\kappa_n].$$

Recall that

$$h'(\varphi)\geq\tilde{h}'(\varphi)\geq 0$$

by construction. We therefore conclude that

$$h(\varphi)\leq h(\kappa_n)=0$$

and

$$\langle(\varphi',\psi'),(n,1)\rangle\leq\frac{h(\varphi)}{1+4\varphi^2}\leq 0$$

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as desired. $\square$

From now on, we only consider the self-expander case $C = 1$.

Let $X = \frac{2n+1}{2n+2}\hat{\varphi} + \frac{1}{2n+2}\hat{\psi}$, $Y = \frac{1}{2n+2}\hat{\varphi} - \frac{1}{2n+2}\hat{\psi}$, where $\hat{\varphi}(t) = \varphi(-t)$, $\hat{\phi}(t) = \phi(-t)$. Then the system of ODEs changes into the form

$$\begin{cases} X' = -X + 4(\dfrac{nY(X+(n+2)Y)(2X-(n-1)Y)}{1+4(X+Y)^2} + \dfrac{n-1}{n+1}(X-nY)^3) \\ \qquad + \dfrac{1}{2n+2}e^{-2t}(X-(2n+1)Y)(1+4(X-nY)^2) \\ Y' = (2n+1)Y - 4(\dfrac{nY(X+(n+2)Y)(2X-(n-1)Y)}{1+4(X+Y)^2} + \dfrac{n-1}{n+1}(X-nY)^3) \\ \qquad - \dfrac{1}{2n+2}e^{-2t}(X-(2n+1)Y)(1+4(X-nY)^2) \end{cases},$$

which satisfies all the assumptions in the Section 4.

Hence, we simply choose $M = 1$ and apply Theorem 4.3, then there is an $\hat{\varepsilon} > 0$, an initial time $T > 0$ and a solution curve $(X(t), Y(t))$ defined on $t \in [T, \infty)$ such that $X(T) = \hat{\varepsilon}$, $-M\hat{\varepsilon} \leq Y(T) \leq M\hat{\varepsilon}$ and $(X, Y) \to (0, 0)$ as $t \to \infty$. Furthermore, for any $\delta > 0$, $X(t), Y(t) \in O(e^{-(1-\delta)t})$ as $t \to \infty$.

Since $\varphi(t) = X(-t) + Y(-t)$, $\psi(t) = X(-t) - (2n+1)Y(-t)$, we conclude that there is an $\varepsilon > 0$, an initial time $-T < 0$ and a solution curve $(\varphi(t), \psi(t))$ defined on $t \in (-\infty, -T]$ such that $\varphi(-T) = \varepsilon$, $(\varphi(t), \psi(t)) \in \Delta\, \forall t \in (-\infty, -T]$ and $(\varphi(t), \psi(t)) \to (0, 0)$ as $t \to -\infty$. Moreover, for any $\delta > 0$, $\varphi(t), \psi(t) \in O(e^{(1-\delta)t})$ as $t \to -\infty$.

Now, Proposition 5.1 shows that Δ is a compact positive invariant set. It follows that we can actually extend (φ, ψ) to be a global solution [cf. [6], Chap. 7.2, p.146-147]. That is to say, there is a solution curve $(\varphi(t), \psi(t))$ defined on $t \in (-\infty, \infty)$ such that $\varphi(T) = \varepsilon$, $(\varphi(t), \psi(t)) \in \Delta\, \forall t$.

Recall that $f = r\varphi$ and $f' = \varphi + \psi$. Then $f \geq 0$, $f' \geq 0$ follows from $(\varphi, \psi) \in \Delta$. Moreover, for any $\delta > 0$, since $\varphi(t), \psi(t) \in O(e^{-(1-\delta)t})$ as $t \to -\infty$, we also conclude that

$$f(r) \in O(r^{2-\delta})$$
$$f'(r) \in O(r^{1-\delta})$$

as $r \to 0$. Therefore,

$$F(\mathbf{x}) = (\mathbf{x}, f(r)\frac{H(\mathbf{x})}{r^2})$$

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is $\mathcal{C}^1$ near the origin. Applying the classical bootstrapping argument [cf. [11], Theorem 6.8.1 or [3], Theorem 9.13], which follows from elliptic regularity and Sobolev embedding, $F(\mathbf{x})$ is actually smooth near the origin.

Note that $f(r)$ is smooth for all $r > 0$. It follows that $F(\mathbf{x})$ is smooth everywhere.

$$\Sigma_n = \{F(\mathbf{x}) \,\big|\, \mathbf{x} \in \mathbb{R}^{2n}\}$$

is a smooth self-expander.

6 The uniqueness of self-expander from the perspective of Dirichlet problem

Recall that we already have several equivalent expressions of the self-expander equation.

1. The geometric one:
$$H_{\Sigma_n} = F^{\perp}.$$

2. The graphical one:
$$\frac{f''}{1+(f')^2} + (n-1)\frac{f'}{r} + \frac{n(rf'-4f)}{r^2+4f^2} = -rf' + f.$$

3. The analogous autonomous one:
$$\begin{cases} \varphi_t = \psi \\ \psi_t = -\psi - ((n-1+\dfrac{n}{1+4\varphi^2} + e^{2t})\psi + (n-1-\dfrac{3n}{1+4\varphi^2})\varphi)(1+(\varphi+\psi)^2) \end{cases}.$$

Now, fixing the initial conditions $T << 0$ and $\varphi(T) = \varepsilon$ of the analogous autonomous version, we can also interpret the self-expander equation as a Dirichlet problem for the minimal map equation

$$g : B(0; e^T) \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times \mathbb{R}^{n+1}$$

with boundary condition

$$g\big|_{\partial B(0;e^T)} = (\mathrm{id}_{\mathbb{R}^{2n}}, \varepsilon e^{-T}\mathcal{H}),$$

where $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}^{n+1}$ denotes the Hopf map. Here, the word "minimal" is with respect to the conformal metric $e^{\frac{\|x\|^2}{n}}\delta_{ij}$.

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In this regard, we have the following uniqueness property.

Proposition 6.1. *Fix boundary condition $f(r_0) = \varepsilon r_0$, where $r_0 = e^T$ in the discussions above (shrink T such that T is small enough). Then there is only one $f(r)$ defined on $r \in [0, r_0]$ satisfying the ODE*

$$\frac{f''}{1+(f')^2} + (n-1)\frac{f'}{r} + \frac{n(rf'-4f)}{r^2+4f^2} = -rf' + f$$

and $f(0) = 0$.

Proof. Suppose that both f_1, f_0 satisfy the conditions. Let $g(r) = f_1(r) - f_0(r)$. We first notice that g is continuous in $[0, r_0]$. Moreover, it is at least $\mathcal{C}^2$, in fact smooth, in $(0, r_0)$.

Now,

$$\begin{aligned}
\frac{f_1''}{1+(f_1')^2} - \frac{f_0''}{1+(f_0')^2} &= \frac{g''(1+(f_0')^2) - g' f_0''(f_1'+f_0')}{(1+(f_1')^2)(1+(f_0')^2)},\\
(n-1)\frac{f_1'}{r} - (n-1)\frac{f_0'}{r} &= (n-1)\frac{g'}{r},\\
(rf_1' - f_1) - (rf_0' - f_0) &= rg' - g,\\
\frac{n(rf_1'-4f_1)}{r^2+4f_1^2} - \frac{n(rf_0'-4f)}{r^2+4f_0^2} &= \frac{n(g'(r^3+4rf_0^2) - g(4r^2 - 16f_0f_1 + 4rf_0'(f_1+f_0)))}{(r^2+4f_1^2)(r^2+4f_0^2)}.
\end{aligned}$$

We therefore conclude that g satisfies the following ODE

$$\begin{aligned}
&\frac{g''(1+(f_0')^2) - g' f_0''(f_1'+f_0')}{(1+(f_1')^2)(1+(f_0')^2)} + \frac{(n-1)g'}{r} + rg' - g \\
&+ \frac{n(g'(r^3+4rf_0^2) - g(4r^2 - 16f_1f_0 + 4rf_0'(f_1+f_0)))}{(r^2+4f_1^2)(r^2+4f_0^2)} = 0
\end{aligned}.$$

Suppose that $r_1 \in (0, r_0)$ is a local maximum point of g. Then $g'(r_1) = 0$ and $g''(r_1) \le 0$. At r_1, the ODE above becomes

$$\frac{1+(f_0')^2}{(1+(f_1')^2)(1+(f_0')^2)} g'' - \left(\frac{n(4r^2 - 16f_1f_0 + 4rf_0'(f_1+f_0))}{(r^2+4f_1^2)(r^2+4f_0^2)} + 1\right) g = 0.$$

Notice that $f_0, f_1 \in O(r^{2-\delta})$ as $r \to 0$. Then $4r^2 - 16f_1f_0 \ge 0$ if r is small enough, i.e. T is small enough. Combining with $f_0, f_1, f_0', f_1' \ge 0$ and $C > 0$, we conclude that

$$\frac{1+(f_0')^2}{(1+(f_1')^2)(1+(f_0')^2)} \ge 0 \text{ and } \frac{n(4r^2 - 16f_1f_0 + 4rf_0'(f_1+f_0))}{(r^2+4f_1^2)(r^2+4f_0^2)} + 1 \ge 0.$$

It follows from the ODE of g that $g(r_1)$ must not exceed 0.

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If $r_2 \in (0, r_0)$ is a local minimum point of g, then similar argument shows that $g(r_2)$ can not be less than 0. Combining with $g(0) = g(r_0) = 0$, we concludes that $g \equiv 0$. That is to say, $f_1 \equiv f_0$. □

In other words, the property above shows that given the initial conditions $T << 0$ and $\varphi(T) = \varepsilon$, there is only one choice of $\psi(T)$ such that (φ, ψ) is the solution to the analogous autonomous system and satisfies $(\varphi, \psi) \to 0$ as $t \to \infty$. This is equivalent to say that the solution to the Dirichlet problem mentioned above must be unique.

7 The behavior of the self-expander at infinity

In this section, we investigate the behavior of self-expander we construct at infinity. We first back to the system of ODEs

$$\begin{cases} \varphi_t = \psi \\ \psi_t = -\psi - ((n-1+\frac{n}{1+4\varphi^2}+e^{2t})\psi + (n-1-\frac{3n}{1+4\varphi^2})\varphi)(1+(\varphi+\psi)^2) \end{cases}.$$

Note that in the region Δ, we have

$$\psi_t \leq 0$$

or

$$(n-1+\frac{n}{1+4\varphi^2}+e^{2t})\psi + (n-1-\frac{3n}{1+4\varphi^2})\varphi < 0.$$

In other words,

$$e^{2t}\psi < \varphi((2n+1) - 4(n-1)\varphi^2).$$

The critical point of $\varphi((2n+1) - 4(n-1)\varphi^2)$ is $\varphi = \pm\sqrt{\frac{2n+1}{12(n-1)}} = \pm\frac{\kappa_n}{\sqrt{3}}$. Therefore,

$$\varphi((2n+1) - 4(n-1)\varphi^2) \leq \frac{\kappa_n}{\sqrt{3}}((2n+1) - 4(n-1)\frac{\kappa_n^2}{3}) = \frac{(2n+1)\kappa_n}{3\sqrt{3}}.$$

We conclude that $\psi_t > 0$ only if

$$\psi < \tilde{C}e^{-2t},$$

where $\tilde{C} = \frac{(2n+1)\kappa_n}{3\sqrt{3}}$.

Now, we observe that $\lim_{t\to\infty}\varphi$ exists since Δ is compact and $\varphi_t = \psi \geq 0$ in Δ.

Proposition 7.1. $\lim_{t\to\infty}\psi$ *also exists and equal to* 0.

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Proof. We split into two cases.

1. Suppose that $\exists\, T > 0$ such that $\psi(t) \neq \tilde{C}e^{-2t}\ \forall t > T$. Therefore, either

   (a) $\psi(t) > \tilde{C}e^{-2t}\ \forall t > T$ or

   (b) $\psi(t) < \tilde{C}e^{-2t}\ \forall t > T$

   happens.

   For (a), note that $\psi_t(t) < 0\ \forall t > T$. Since Δ is compact, it implies that $\lim_{t\to\infty} \psi$ exists. Moreover, since $\lim_{t\to\infty} \varphi$ exists and $\varphi_t = \psi$, $\lim_{t\to\infty} \psi$ must equal to 0.

   For (b), note that $0 \leq \psi(t) < \tilde{C}e^{-2t}\ \forall t > T$. Then by the squeeze lemma,

$$\lim_{t\to\infty} \psi = 0.$$

2. Suppose that $\forall\, \tilde{T} > 0$, $\exists\, T > \tilde{T}$ such that $\psi(T) = \tilde{C}e^{-2T}$. We first claim that if $\psi(T) = \tilde{C}e^{-2T}$, then $\psi(t) < \tilde{C}e^{-2T}\ \forall t > T$. We argue it by contradiction.

   Assume that the statement is false, say $\exists\, t_1 > T$ such that $\psi(t) > \tilde{C}e^{-2T} > \tilde{C}e^{-2t_1}$. Let

$$g(t) := \psi(t) - \tilde{C}e^{-2t}.$$

   Define

$$S := \{t \in [T, t_1] \,\big|\, g(t) = 0\}.$$

   Since S is bounded, $\sup S$ exists. Moreover, the continuity of F and the fact that $g(t_1) > 0$ show that $t_0 := \sup S < t_1$. Now, by the Intermediate Value Theorem,

$$\psi(t) > \tilde{C}e^{-2t} \quad \forall\, t \in (t_0, t_1].$$

   Furthermore, by the Mean Value Theorem, $\exists\, t_2 \in (t_0, t_1)$ such that

$$\psi_t(t_2) = \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0} > 0,$$

   which contradicts to the fact that $\psi_t > 0$ only if $\psi < \tilde{C}e^{-2t}$.

   Now, according to the claim, we have a sequence of $\{T_i\}_{i=1}^{\infty}$ such that $T_i < T_j\ \forall i < j$, $T_i \to \infty$ as $i \to \infty$ and $\psi(t) < \tilde{C}e^{-2T_i}\ \forall t > T_i$. By the squeeze lemma, we conclude that

$$\lim_{t\to\infty} \psi = 0.$$

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□

Let $L := \lim_{t\to\infty}\varphi$. Recall that $f = r\varphi$, then the aforementioned discussion shows that the self-expander

$$\Sigma_n = \{F(\mathbf{x}) = (\mathbf{x}, f(r)\frac{\mathcal{H}(\mathbf{x})}{r^2}) \,\big|\, \mathbf{x} \in \mathbb{R}^{2n}\}$$

is asymptotic to a cone

$$\{(\mathbf{x}, L\frac{\mathcal{H}(\mathbf{x})}{r}) \,\big|\, \mathbf{x} \in \mathbb{R}^{2n}\}$$

as $r = \|\mathbf{x}\| \to \infty$.

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Appendix — A brief view from power series expansion

In this appendix, we provide some observations about the power series expansion of f, the function we found, near the origin. It gives a glimpse of the smoothness and uniqueness (as a Dirichlet problem).

Recall that f satisfies the following ODE

$$\frac{f''}{1+(f')^2}+\frac{f'}{r}+\frac{2(\frac{f'}{r}-\frac{4f}{r^2})}{1+\frac{4f^2}{r^2}}+rf'-f=0.$$

We consider the power series expansion of f near the origin

$$f=\sum_{i=0}^{\infty}a_i r^i=a_0+a_1 r+a_2 r^2+\cdots.$$

Since $(\varphi:=\frac{f}{r},\psi:=\frac{d\varphi}{dt})\to(0,0)$ as $t:=\log r\to-\infty$, $a_0=a_1=0$. Then

$$\begin{cases} f=\sum_{n\geq 2}a_n r^n \\ f'=\sum_{n\geq 1}(n+1)a_{n+1}r^n \\ f''=\sum_{n\geq 0}(n+2)(n+1)a_{n+2}r^n \end{cases}.$$

Note that given $G=1+\sum_{n\geq 1}b_n r^n$, there is a closed form of the multiplicative inverse of G,

$$G^{-1}=1+\sum_{n\geq 1}(\sum_{\substack{\beta_1,\beta_2,\cdots\\ \sum_{i\geq 1}i\beta_i=n}}(-1)^{\sum_{i\geq 1}\beta_i}\frac{(\sum_{i\geq 1}\beta_i)!}{\prod_{i\geq 1}\beta_i!}\prod_{i\geq 1}b_i^{\beta_i})r^n.$$

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Therefore, when we replace each part of the ODE with power series, it becomes

1.

$$\begin{aligned}\frac{f''}{1+(f')^2} =&\sum_{n\geq 0}(n+2)(n+1)a_{n+2}r^n \\ &+\sum_{\substack{n\geq 0\\ m\geq 2}}(n+2)(n+1)a_{n+2}r^{n+m} \\ &\quad\times\Big(\sum_{\substack{\beta_2,\beta_3,\ldots\\ \sum_{i\geq 2} i\beta_i=n}}(-1)^{\sum_{i\geq 2}\beta_i}\frac{(\sum_{i\geq 2}\beta_i)!}{\prod_{i\geq 2}\beta_i!}\prod_{i\geq 2}\Big(\sum_{\substack{\alpha_1,\alpha_2,\ldots\\ \sum_{j\geq 1} j\alpha_j=i\\ \sum_{j\geq 1}\alpha_j=2}}\frac{2\prod_{j\geq 1}((j+1)a_{j+1})^{\alpha_j})^{\beta_i}}{\prod_{j\geq 1}\alpha_j!}\Big),\end{aligned}$$

2.

$$\frac{f'}{r}=\sum_{n\geq 0}(n+2)a_{n+2}r^n,$$

3.

$$\begin{aligned}\frac{2(\frac{f'}{r}-\frac{4f}{r^2})}{1+\frac{4f^2}{r^2}} =&2\sum_{n\geq 0}(n-2)a_{n+2}r^n \\ &+2\sum_{\substack{n\geq 0\\ m\geq 2}}(n-2)a_{n+2}r^{n+m} \\ &\quad\times\Big(\sum_{\substack{\beta_2,\beta_3,\ldots\\ \sum_{i\geq 2} i\beta_i=n}}(-1)^{\sum_{i\geq 2}\beta_i}\frac{(\sum_{i\geq 2}\beta_i)!}{\prod_{i\geq 2}\beta_i!}\prod_{i\geq 2}\Big(\sum_{\substack{\alpha_1,\alpha_2,\ldots\\ \sum_{j\geq 1} j\alpha_j=i\\ \sum_{j\geq 1}\alpha_j=2}}\frac{2\prod_{j\geq 1}(2a_{j+1})^{\alpha_j})^{\beta_i}}{\prod_{j\geq 1}\alpha_j!}\Big),\end{aligned}$$

4.

$$rf'-f=\sum_{n\geq 0}(n+1)a_{n+2}r^{n+2}.$$

Combining the aforementioned calculations together, we can derived the following two properties.

Proposition A.1. $a_{2k-1}=0\ \forall k\in\mathbb{N}$.

Proof. We argue it by induction. The $k=1$ case has been discussed at the beginning of this section. For $k=3$ case, we only need to consider the coefficient of r, which must satisfies

$$6a_3+3a_3-2a_3=0.$$

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It follows that $a_3 = 0$. Now, we suppose that this property holds for $1, 2, \cdots, k$ and consider the $k+1$ case. Note that we have to see the coefficient of r^{2k-1}. It follows from the induction hypothesis that

$$
\begin{aligned}
0 =& (2k(2k+1) + (2k+1) + 2(2k-3))a_{2k+1} \\
&+ \sum_{m=0}^{k-2}(m+2)(m+1)a_{2m+2} \\
&\times [\sum_{\substack{\beta_2,\beta_3,\cdots \\ \sum_{i\geq 2} i\beta_i = 2k-1-2m}} (-1)^{\sum_{i\geq 2}\beta_i} \frac{(\sum_{i\geq 2}\beta_i)!}{\prod_{i\geq 2}\beta_i!} \prod_{i\geq 2}(\sum_{\substack{\alpha_1,\alpha_2,\cdots \\ \sum_{j\geq 1} j\alpha_j = i \\ \sum_{j\geq 1}\alpha_j = 2}} \frac{2}{\prod_{j\geq 1}\alpha_j!} \prod_{j\geq 1}((j+1)a_{j+1})^{\alpha_j})^{\beta_i}] \\
&+ 2\sum_{m=0}^{k-2}(2m-2)a_{2m+2} \\
&\times [\sum_{\substack{\beta_2,\beta_3,\cdots \\ \sum_{i\geq 2} i\beta_i = 2k-1-2m}} (-1)^{\sum_{i\geq 2}\beta_i} \frac{(\sum_{i\geq 2}\beta_i)!}{\prod_{i\geq 2}\beta_i!} \prod_{i\geq 2}(\sum_{\substack{\alpha_1,\alpha_2,\cdots \\ \sum_{j\geq 1} j\alpha_j = i \\ \sum_{j\geq 1}\alpha_j = 2}} \frac{2}{\prod_{j\geq 1}\alpha_j!} \prod_{j\geq 1}(2a_{j+1})^{\alpha_j})^{\beta_i}]
\end{aligned}
$$

We claim that both of the bracket terms above contain a_{2n-1} for some $n \leq k$. Then the equation becomes

$$(2k(2k+1) + (2k+1) + 2(2k-3))a_{2k+1} = 0,$$

and $a_{2k+1} = 0$ as desired.

Therefore, it suffices to justify the claim now. Since the idea is the same, we only show the case of

$$\sum_{\substack{\beta_2,\beta_3,\cdots \\ \sum_{i\geq 2} i\beta_i = 2k-1-2m}} (-1)^{\sum_{i\geq 2}\beta_i} \frac{(\sum_{i\geq 2}\beta_i)!}{\prod_{i\geq 2}\beta_i!} \prod_{i\geq 2}(\sum_{\substack{\alpha_1,\alpha_2,\cdots \\ \sum_{j\geq 1} j\alpha_j = i \\ \sum_{j\geq 1}\alpha_j = 2}} \frac{2}{\prod_{j\geq 1}\alpha_j!} \prod_{j\geq 1}((j+1)a_{j+1})^{\alpha_j})^{\beta_i}$$

here. Let us argue it by contradiction. Suppose that

$$\sum_{\substack{\alpha_1,\alpha_2,\cdots \\ \sum_{j\geq 1} j\alpha_j = i \\ \sum_{j\geq 1}\alpha_j = 2}} \frac{2}{\prod_{j\geq 1}\alpha_j!} \prod_{j\geq 1}((j+1)a_{j+1})^{\alpha_j}$$

dose not contain $a_{2n-1}\ \forall\, 0 \leq n \leq k$. That is to say, it contains either

$$(j_1+1)a_{j_1+1}(j_2+1)a_{j_2+1}$$

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for some different odd j_1, j_2 or

$$((j_1+1)a_{j_1+1})^2$$

for some odd j_1. For the first case, $i = j_1 + j_2$ is even. For the second case, $i = 2j_1$ is also even. Hence, $\sum_{i\geq 2} i\beta_i$ must be even, which contradicts to

$$\sum_{i\geq 2} i\beta_i = 2k - 1 - 2m.$$

□

Recall that $r = \|\mathbf{x}\|$ in the previous discussions. Therefore, this proposition is the necessary condition for that

$$F(\mathbf{x}) = (\mathbf{x}, f(r)\frac{\mathcal{H}(\mathbf{x})}{r^2})$$

is smooth at the origin.

Proposition A.2. *a_2 completely determines a_{2k} $\forall k \in \mathbb{N}$.*

Proof. Consider the coefficient of r^{2k}, it must satisfy the equation

$$4(k^2+3k)a_{2k+2} + (2k-1)a_{2k} + h(a_2, a_4, \cdots, a_{2k}) = 0,$$

where h is a polynomial. This observation follows from Proposition A.1 that $a_{2k-1} = 0$ for all k. Now, induction on k leads to the ideal result. □

This property shows that $F(\mathbf{x})$ is uniquely determined by a_2, which provides a glimpse that $F(\mathbf{x})$ is unique under some boundary conditions.

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