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碩士論文

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Master Thesis

奇異積分的二次平均以及最加加權上界

Representing singular kernel as dyadic average and sharp A2 bound

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兩年的碩士生涯一轉眼即將結束，回顧這兩年自己離成爲一個數學家的目標又往前了好多步。

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洪智捷一零九年七月十一日於台灣大學

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中文摘要

在調和分析中一個重要的核心問題是研究奇異積分算子的最佳加權上界問題，而此問題相當於研究奇異積分算子在 L2 加權的有界性。

在 2000 年，S. Petermichl 使用哈爾小波平均來表示希爾伯特轉換的核，此方法後來被發現是研究此問題的重大突破，爾後里斯轉換 (Riesz transform) 的核，甚至一般奇異積分算子的核也被找出類似的表示方法。在此基礎之上，S.Petermichl 於 2007 解決希爾伯特轉換的最佳加權上界問題，T. Hytonen 則於 2012 解決一般奇異積分最佳加權上界問題。

本篇論文會先介紹如何使用哈爾小波平均來表示希爾伯特轉換的核 (2000, S. Petermichl)，此方法雖然簡單卻隱含對希爾伯特轉換深刻的觀察。接著我們會介紹如何使用哈爾小波平均來表示里斯轉換的核 (2002, S. Petermichl, S. Treil and A. Volberg)，這不單單只是推廣希爾伯特轉換的結果到高維度，而是將前方法作一個統整與重新表示，找出一個推廣到高維度的方式，而這證明過程中，出現一個特殊積分不等於零的假設，雖然最後作者提出另一條路徑解決，但原本特殊積分不等於 0 的問題在維度大於 2 還是未知的，本篇論文中我們解決積分非零的問題在維度等於 3 的時候。最後我們介紹如何表示一般的奇異積分算子，並解決最佳加權上界的問題 (2012, T. Hytonen)。

關鍵字: 卡德隆-吉格曼算子, 希爾伯特轉換, 里斯轉換, 二次平均, 最佳加權上界, 哈爾偏移算子.

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Abstract A central research problem in the area of Harmonic analysis is to prove the sharp weighted bound for singular integrals. In 2000 S.Petermichl used dyadic averages of Haar shifts to represent the kernel of Hilbert transform which in turn enabled her to obtain the sharp A2 bound for Hilbert transform. Shortly after, the kernels of Riesz transforms were also obtained via averages of Haar shifts and finally the full generality was made by T. Hytonen who solved the longstanding A2 conjecture for singular integrals. In this dissertation, we first introduce how to use the averages of Haar shifts to represent the kernel of Hilbert transform (2000, S.Petermichl). Second, we will introduce how to represent the kernels of Riesz transforms via dyadic averages of Haar shifts (2002,S. Petermichl, S. Treil and A. Volberg). This result not only extends Petermichl's ideas to higher dimensions, but also explicitly constructs the Haar shifts for Riesz transforms. However in order to make the result nondegenerate an integral that arises in the process of averaging Haar shifts must be nonzero. S. Petermichl, S. Treil and A. Volberg provided a proof to show the integral is nonzero in dimension two but for other dimensions the problem remains unknown. A new part of this dissertation is to prove the integral is nonzero in dimension three. Finally we also discuss the breakthrough work of T. Hytonen in 2012 that solves the A2 conjecture for singular integrals.

key words: Calderón-Zygmund operator , Hilbert transform, Riesz transform, dyadic average, sharp A_2 bound, Harr shift operator.

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Contents

1 Introduction

One of the important milestones that appears in the area of Harmonic analysis in the past decades is the appearance of dyadic Haar shifts. It not only connects the continuous singular integrals with dyadic operators but also enables people to resolve the longstanding A_2 conjecture concerning with the sharp weighted bound for Calderón-Zygmund singular integrals. More precisely, the breakthrough work of Petermichl [12] showed that the kernel of the Hilbert transform is actually an average of some certain dyadic operators:

$$\frac{c_0}{t-x} = \lim_{L\to\infty} \frac{1}{2\log L} \int_{\frac{1}{L}}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \sum_{I\in D^{\alpha,r}} h_I(t)(h_{I-}(x)-h_{I+}(x))d\alpha dr. \quad (1.1)$$

Therefore the sharp weighted bound for Hilbert transform can be reduced to proving a uniform sharp weighted bound for above dyadic operators which are called dyadic Haar shifts. Such representation of dyadic average for Hilbert transform kernel later was generalized by Stefanie Petermichl, Sergei Treil, Alexander Volberg, [13] to a slightly wider class of kernels but still restricted on one dimensional singular integrals. Finally the full generality was made by Hytönen in [6] who showed that any Calderón-Zygmund operator is a simple variant of dyadic averages, and part of the work was built on a previous result obtained by Hytönen, Perez, Treil and Volberg [5].

Moreover as shown in the work of S. Petermichi that an explicit dyadic Haar shift can actually be given for Hilbert transform. As a result it may be also expected that some explicit dyadic Haar shifts can also be given for Riesz transforms.

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Indeed, it was shown in the work of Petermichl, Treil, Volberg [13] that each component of the kernel of Riesz transforms can be explicitly represented by an average of dyadic shift.

In this dissertation, we will go through the history of dyadic averages of Haar shifts for singular integrals and give a proof to a question posed in [13]. More precisely, in section 2 we will demonstrate how to use Haar shifts to represent the kernel of Hilbert transform. In section 3, we also illustrate another way to represent the kernel of Hilbert transform and extend the method to Riesz transforms that are vector singular integrals in higher dimensions. In section 4, we prove a new result that shows an integral arising from averages of Haar shifts for Riesz transforms is nonzero in dimension three. In last section, we discuss the work of T. Hytonen who showed that any Cardelon-Zygmund operator is a simple variant of averages of haar shifts and gave the solution of the longstanding A_2 conjecture.

2 Hilbert transform

Hilbert transform as dyadic operator This part mainly comes from [12]. It connects the discrete Haar shift with continuous singular kernel, $\frac{1}{x}$. We first introduce a variety of dyadic grids in $\mathbb{R}$. The basic dyadic grid, starting at 0 with intervals of length $1 \cdot 2^n$, will be denoted by $\mathcal{D}_0^1$ i.e.

$$\mathcal{D}_0^1 := \{2^k([0,1) + m) : k \in \mathbb{Z}, m \in \mathbb{Z}\}.$$

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h_J is the Haar function for $J \in \mathcal{D}_0^1$, namely

$$h_J := \frac{1}{\sqrt{|J|}}(\chi_{J-} - \chi_{J+}),$$

where $J-$ is the left half of J and $J+$ is the right half of J.

We obtain a variation of $\mathcal{D}_0^1$ by first shifting the starting point 0 to $\alpha \in \mathbb{R}$ and secondly choosing intervals of length $r \cdot 2^n$ for a positive r. The resulting grid is called $\mathcal{D}^{1,\alpha}$, and the corresponding Haar functions h_J are chosen so that they are still normalized in L^2.

Since Haar functions forms a basis in $L^2(\mathbb{R})$, for $f \in L^2(\mathbb{R})$ we have

$$f(x) = \sum_{I \in \mathcal{D}^{\alpha,r}} \langle f, h_I \rangle h_I(x), \quad \forall \alpha \in \mathbb{R}, r > 0.$$

We define for such α, r a dyadic shift operator $S^{\alpha,r}$ by

$$(S^{\alpha,r} f)(x) = \Sigma_{I \in \mathcal{D}^{\alpha,r}} \langle f, h_I \rangle \left(h_{I_-}(x) - h_{I_+}(x)\right).$$

It's L^2 operator norm is $\sqrt{2}$ and its representing kernel is

$$K^{\alpha,r}(t,x) = \sum_{I \in \mathcal{D}^{\alpha,r}} h_I(t) \left(h_{I_-}(x) - h_{I_+}(x)\right). \tag{2.1}$$

Lemma 1. The convergence of sum (see above) is uniform for $|x - t| \geq \delta$ for every $\delta > 0$. For $x \neq t$ let

$$K(t,x) = \lim_{L \to \infty} \frac{1}{2 \log L} \int_{1/L}^{L} \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x) d\alpha \frac{dr}{r}.$$

The limits exist pointwise and the convergence is bounded $|x-t| \geq \delta$ for every $\delta > 0$ and $K(t,x) = \frac{c_0}{t-x}$ for some $c_0 > 0$.

Proof. It is easy to see that $\sum_{I \in \mathcal{D}^{\alpha,r}} |h_I(t) \left(h_{I-}(x) - h_{I+}(x)\right)| \leq 2\sqrt{2}/|t - x|, \forall \alpha \in \mathbb{R}$ and $\forall r > 0.$ 0. In particular, the sum converges absolutely and

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uniformly $|x-t| \geq \delta$ for every $\delta > 0. > 0$. The existence of the limits is due to fact that summands repeat for different dyadic grids. The main point is to show $|K(t,x) = c_0/(t-x)|$ with $c_0 \neq 0.\ 0$. It is enough to prove the following properties of $K(t,x)$:

1 translation invariance, i.e., $K(t,x) = K(t+c, x+c), \forall c \in \mathbb{R}$, so $K(t,x) = K(t-x)$;

2 antisymmetry, i.e, $K(t,x) = -K(-t,-x)$, so $K(x-t) = -K(t-x)$;

3 dilation invariance, i.e., $K(t,x) = \lambda K(\lambda t, \lambda x), \forall \lambda > 0$;

4 $K(1) = c_0 > 0$.

In order to check the first three properties we observe the following simple relationships between the Haar functions of different dyadic grids for translations, reflections and dilations:

For any interval $I \in \mathcal{D}^{\alpha,r}$ there exists an interval of the same length in $\mathcal{D}^{\alpha-c,r}$ so that $h_I^{\alpha,r}(t+c) = h_I^{\alpha-c,r}(t)$. In a similar sense $h_I^{\alpha,r}(-t) = -h_I^{-\alpha,r}(t)$ when changing grids from $\mathcal{D}^{\alpha,r}$ to $\mathcal{D}^{-\alpha,r}$ and $h_I^{\alpha,r}(\lambda t) = \lambda^{-1/2} h_I^{\alpha/\lambda,r/\lambda}(t)$ when changing from $\mathcal{D}^{\alpha,r}$ to $\mathcal{D}^{\alpha/\lambda,r\lambda}$.

Using these facts, the proof of the first three properties are simple computations, mainly involving changes of integration variables. Note that these properties show that $K(t,x) = \frac{c_0}{t-x}$, we turn to the essential part to show that $c_) \neq 0$.

The product $h_I(t)\left(h_{I-}(x) - h_{I+}(x)\right) \neq 0$ if and only if the point (t,x) lies in this square $I \times I$. Its value is $\pm\sqrt{2}/|I|$, where the correct sign is indicated inside

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the smaller rectangles. Let us first compute

$$K_n^r(t,x) := \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \sum_{\substack{I\in\mathcal{D}^{\alpha,r}\\ |I|=r2^n}} h_I(t)\left(h_{I-}(x) - h_{I+}(x)\right) d\alpha, \tag{2.2}$$

for fixed $r > 0$ and $n \in \mathbb{Z}$ and assuming $t > x$. Due to the averaging process in α, this is only going to depend on $t - x$. If:

$t - x = 0$, then $K_n^r(t,x) = 0$ and similarly;

$t - x = |I|/4$, then $K_n^r(t,x) = 3/4 \cdot \sqrt{2}/|I|$;

$t - x = |I|/2$, then $K_n^r(t,x) = 0$;

$t - x = 3|I|/4$, then $K_n^r(t,x) = -1/4 \cdot \sqrt{2}/|I|$;

$t - x \geq |I|$, then $K_n^r(t,x) = 0$.

Now we compute

$$\begin{aligned} K_n^r(t,x) &:= \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x) d\alpha \\ &= \sum_{n\in\mathbb{Z}} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \sum_{\substack{I\in\mathcal{D}^{\alpha,r}\\ |I|=r2^n}} h_I(t)\left(h_{I-}(x) - h_{I+}(x)\right) d\alpha. \end{aligned}$$

So we compute $K^r(t,x)$ using $K_n^r(t,x)$ for different values of n and summing over $n \in \mathbb{Z}$. It suffices to compute $K^r(t,x)$ for values $t - x = 3/4 \cdot r2^n$ and$t - x = \cdot r2^n$:

$$K^r(\frac{3}{4}r2^n) = -\frac{1}{4}\frac{\sqrt{2}}{r2^n} + \frac{3}{16}\frac{\sqrt{2}}{r2^n} + \frac{9}{64}\frac{\sqrt{2}}{r2^n}\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right) = \frac{\sqrt{2}}{8r2^n}, \tag{2.3}$$

$$K^r(r2^n) = \frac{3}{16}\frac{\sqrt{2}}{r2^n}\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right) = \frac{\sqrt{2}}{4r2^n}. \tag{2.4}$$

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The above equations imply that

$$\frac{3\sqrt{2}}{32(t-x)} \leq K^r(t-x) \leq \frac{\sqrt{2}}{4(t-x)} \quad \forall r > 0. \tag{2.5}$$

From above, it is clear that $c_0 > 0$. □

3 Riesz transform

The "simplest" operator whose average is the Hilbert transform This part comes from [13]. It uses average technique to generalize the method in section 1 to the n-dimensional Riesz kernels. Let $\mathcal{L}$ denote a dyadic lattice in $\mathbb{R}$. By $\mathcal{L}(k)$ we understand the dyadic grid of intervals from $\mathcal{L}$ having length 2^{-k}, $k \in \mathbb{Z}$. For the convenience we would like to use the notations $\mathcal{D} =: \mathcal{L}(0)$. We consider first such a dyadic lattice that the grid $\mathcal{D}$ has the point 0 as one of the end-points of its intervals. To emphasize that we write D0. Later we will have $\mathcal{D}_t$ —the point t plays the role of 0.

Let us consider the following linear operation

$$f \to \phi(x) := \sum_{I \in \mathcal{D}_0} \langle f, h_I \rangle \chi_I(x).$$

Here h_I denotes the Haar function of the interval I, that is

$$h_I(x) = \begin{cases} \frac{-1}{|I|^{1/2}} & , \text{for } x \in I_- \\ \frac{1}{|I|^{1/2}} & , \text{for} x \in I_+, \end{cases}$$

and I_-, I_+ are left and right halves of the interval I correspondingly. Symbol χ_I as usual stands for the characteristic function of the interval I.

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This linear operation will be our main building block, so it deserves a name $\mathbb{P}$. Actually, we will call it $\mathbb{P}_0$, thus $\phi_0(x) := \mathbb{P}_0 f := \sum_{I\in\mathcal{D}_0}\langle f, h_I\rangle \chi_I(x)$. Index 0 indicates the end-point of one of the intervals from $\mathcal{D}_0$. So similarly we consider

$$\phi_t(x) := \mathbb{P}_t f$$

defined exactly as before, but with respect to the grid $\mathcal{D}_\sqcup$ of unit intervals such that the end-point of one of them is in $t \in \mathbb{R}$.

Notice that the family of grids $\mathcal{D}_t, \quad t \in \mathbb{R}$, can be naturally provided with the structure of probability space. This space is $(\mathbb{R}/\mathbb{Z}, dt) = ((-1, 0], dt)$. As usual we can use the letter ω for a point from $(-1, 0]$, and $dP(\omega)$ denotes the probability —in this case just Lebesgue measure on the interval $(-1, 0]$. We want to fix $x \in \mathbb{R}$ and to write a nice formula for

$$\mathbb{E}\left(\phi_\omega(x) dP(\omega)\right).$$

So we want to average operators $\mathbb{P}_\omega$. It can be noticed immediately that $\mathbb{E}\mathbb{P}_\omega$ is a convolution operator. In fact, let us denote by La the shift operator: $L_a(f)(x) = f(x + a)$. Then obviously

$$\mathbb{P}_{t-a} L_a = L_a \mathbb{P}_t.$$

Applying averaging (and the fact that our $dP(\omega)$ is invariant with respect to the natural shift on $\mathbb{R}/\mathbb{Z}$ induced by the shift on $\mathbb{R}$) we immediately get

$$\mathbb{E}\mathbb{P}_\omega L_a = L_a \mathbb{E}\mathbb{P}_\omega. \tag{3.1}$$

So the average operator $\mathbb{E}\mathbb{P}_\omega$ is a convolution operator, we will write this as follows

$$\mathbb{E}\left(\phi_\omega(x) dP(\omega)\right) = \mathbb{E}\mathbb{P}_\omega(x) = F_0 * f(x). \tag{3.2}$$

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It is easy to compute F_0. By the definition of $\phi_t(x)$ one can write

$$\phi_t(x) = \int f(s)h^{t-\frac{1}{2}}(s)ds, \quad x - \frac{1}{2} < t - \frac{1}{2} < x + \frac{1}{2}, \tag{3.3}$$

where

$$h^t(s) = \begin{cases} -1 & \text{, for } s \in (t - \frac{1}{2}, t) \\ +1 & \text{, for} s \in (t, t + \frac{1}{2}). \end{cases}$$

But $h^t(s) = k_0(t - x)$, where

$$k_0(s) = \begin{cases} +1 & \text{, for } s \in (-\frac{1}{2}, 0) \\ -1 & \text{, for} s \in (0, \frac{1}{2}). \end{cases}$$

So (3.3) can be rewritten as follows

$$\phi_{t+\frac{1}{2}}(x) = \int f(s)k_0(t - s)ds, \quad x - \frac{1}{2} < t < x + \frac{1}{2}. \tag{3.4}$$

Thus comparing this with (3.2)) (and using again the shift invariance of $dP(\omega)$) we get

$$\begin{aligned} F_0 * f(x) &= \mathbb{E}\left(\phi_\omega(x)dP(\omega)\right) \\ &= \mathbb{E}\left(\phi_{\omega+\frac{1}{2}}(x)dP(\omega)\right) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \left(\int f(x)k_0(t - x)ds\right) dt. \end{aligned}$$

From which we get the formula for F_0:

$$F_0(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} k_0(t)dt = k_0 * \chi_0(x), \tag{3.5}$$

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where χ_0 is the characteristic function of the unit interval $(-1/2, 1/2)$.

Let us start over the beginning of this section with one slight difference — we rescale all our operators, and now $\mathbb{P}_t^\rho, \phi_t^\rho, F_0^\rho, k_0^\rho$ are precisely as above, but when the unit length intervals are replaced by intervals of length $\rho > 0$. We just change the scale —nothing else. In particular,

$$\phi_0^\rho(x) := \mathbb{P}_0^\rho f := \sum_{I \in \mathcal{D}_0^\rho} \langle f, h_I \rangle \chi_I(x)/\sqrt{\rho}$$

where $\mathcal{D}_0^\rho$ is the grid of intervals of length ρ such that 0 is the end-point of two intervals from this grid. We want to remind that h_I here is always normalized in L^2.

Again we have a natural probability space of all grids of intervals of size ρ: $\left(\mathbb{R}/\rho\mathbb{Z}; \frac{1}{\rho}dt|(-\rho, 0]\right)$.

$$\phi_t^\rho(x) := \mathbb{P}_t^\rho f := \sum_{I \in \mathcal{D}_t^\rho} \langle f, h_I \rangle \chi_I(x)/\sqrt{\rho}.$$

Averaging over all grids of intervals of size ρ makes $\mathbb{P}_t^\rho$ a convolution operator —there is no difference with our reasoning above. It is easy to see that this is the convolution operator with the kernel

$$F_t^\rho(x) := \frac{1}{\rho} \int_{x-\frac{\rho}{2}}^{x+\frac{rho}{2}} \frac{1}{\rho} k_0\left(\frac{t}{\rho}\right) dt = \rho F_0\left(\frac{x}{\rho}\right). \tag{3.6}$$

The first $\frac{1}{\rho}$is because of the form our probability has. The second $\frac{1}{\rho}$because we should average a function normalized in L^1.

Let us now consider all convolution operators with kernels F_0^ρ. Let us fix $r \in$

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$[1, 2)$ and let us take a look at the convolution operator with kernel

$$F_r = \sum_{n=-\infty}^{\infty} F_0^{2^n r}. \tag{3.7}$$

The grids $\mathcal{D}_t^{2^n r}$ (t is fixed) can be united into a "dyadic" lattice $\mathcal{L}_t^r$. Here t means the reference point —one of the end-point of intervals from our lattice, and r means the length of one of the intervals of the lattice—let us call r the calibre of the lattice. Obviously the convolution operator with the kernel Fr is the averaging over all "dyadic" lattices (not grids!) $\mathcal{L}_t^r$ of fixed calibre r of the operators given by

$$\mathcal{P}_{\mathcal{L}_t^r} f = \sum_{I \in \mathcal{L}_t^r} \langle f, h_I \rangle \chi_I(x) / \sqrt{|I|}$$

$$F_r * f = \mathbb{E}\mathcal{P}_{\mathcal{L}_t^r} f.$$

This is just because the kernel F_r is the sum of kernels, each of which appeared as averaging of the grid opearators assigned to grids of size $2^n r$, $n = 0, \pm 1, \pm 2, \ldots$, where we summed up over the grids, and the lattice of calibre r is the union of such grids.

Now let us finally average over $r \in [1, 2)$:

$$F(x) := \int_1^2 F_r(x) \frac{dr}{r}.$$

Now we have from one side

$$F * f = (Average \mathcal{P}_{\mathcal{L}})\, f, \tag{3.8}$$

where averaging is performed over all lattices $\mathcal{L}_t^r$.

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where averaging is performed over all lattices Φ.

$$F(x) = \int_1^2 F_r(x)\frac{dr}{r} \tag{3.9}$$

$$= \int_1^2 \sum_{n=-\infty}^{\infty} F_0^{2^n r}\frac{dr}{r} = \int_0^\infty F_0^\rho \frac{d\rho}{\rho} = \int_0^\infty F_0\left(\frac{x}{\rho}\right)\frac{d\rho}{\rho^2}. \tag{3.10}$$

We used (3.6) here. Finally we have

$$F(x) = -\frac{1}{x}\int_0^\infty F_0(t)dt = \frac{1}{4}\frac{1}{x}. \tag{3.11}$$

Theorem 3.1. Averaging of operators $\mathcal{P}_{\mathcal{L}_t^r}$ over both parameters t and r is equal to one quarter of kernel of the Hilbert transform.

We have a good thing:

The Hilbert transform is the averaging over the family of lattices of very simple operators **What is the dyadic shift?** The function that generated everything in the first section was function F_0 —the kernel of the convolution operator which is the averaging of grid operators $\mathbb{P}_t$. It is easy to see that $F_0(x \pm 1)$ are also kernels of the convolution operators which are the averagings of some grid operators. Given f, let us consider $\phi_t(x)$ as above and also

$$\phi_t(x+1) = \sum_{I\in\mathcal{D}_t}\langle f, h_{I-1}\rangle\chi_I(x) = \sum_{I\in\mathcal{D}_t}\langle f, h_I\rangle\chi_{I+1}(x) =: \mathbb{P}_t^+(f)$$

$$\phi_t(x-1) = \sum_{I\in\mathcal{D}_t}\langle f, h_{I+1}\rangle\chi_I(x) = \sum_{I\in\mathcal{D}_t}\langle f, h_I\rangle\chi_{I-1}(x) =: \mathbb{P}_t^-(f)$$

So we test f on h_I and put the result on $I \pm 1$. What if we average these operators? Repeating (3.2) we get

$$\left(\int_0^1 \mathbb{P}_t^{\pm}dt\right) f = F_0(x \mp 1) * f. \tag{3.12}$$

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Consider

$$S(x) := F_0(x) - \frac{1}{2}\left[F_0(x+1) + F_0(x-1)\right]. \tag{3.13}$$

Supposedly S is a kernel of a convolution operator corresponding to averaging over grids of a certain grid operator (we will show which one). If we build S_ρ as before for all calibres, we can consider again $S_r := \sum_{n=-\infty}^{\infty} S^{2^n r}$. Operators S_r are averagings over all lattices of calibre r of the operators which are sums of our hypothetical grid operators. Averaging over $r \in [1,2)$ with respect to the measure dr/r, we will get the operator with kernel

$$\int_1^2 S_r(x)\frac{dr}{r} = \int_0^\infty S(\frac{x}{\rho})\frac{d\rho}{\rho^2} = \frac{1}{x}\int_0^\infty S(t)dt = \frac{1}{4}\frac{1}{x}. \tag{3.14}$$

So we are left to invent a simple "grid" operator, whose average will give us $S(x)$.

Theorem 3.2. Let $\mathcal{D}_t^{(2)}$ be a grid of intervals of length 2 such that t is the end-point. Consider operators

$$f \to \sum_{J\in\mathcal{D}_t^{(2)}} \langle f, h_{J-}\rangle \chi_{J+}$$

$$f \to \sum_{J\in\mathcal{D}_t^{(2)}} \langle f, h_{J+}\rangle \chi_{J-}$$

$$f \to \sum_{J\in\mathcal{D}_t^{(2)}} \langle f, h_{J-}\rangle \chi_{J-}$$

$$f \to \sum_{J\in\mathcal{D}_t^{(2)}} \langle f, h_{J+}\rangle \chi_{J+}$$

The averaging over t of the first operator gives a convolution with kernel $\frac{1}{2}F_0(x-1)$, the averaging over t of the second operator gives a convolution with kernel

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$\frac{1}{2}F_0(x+1)$, and the averaging over t of the third and the fourth operator gives a convolution with kernel $\frac{1}{2}F_0(x)$ each.

Proof. Let us call the first operator H_t, and let us average $\mathbb{E}H_t$ it over its probability space ,$\big(\mathbb{R}/\mathbb{Z}; \frac{1}{2}dt|(-2,0]\big)$. Instead of considering the grid of intervals of length 2 let us consider the grid of intervals of length 1 —we call it $\mathcal{D}_t^1$. Consider operators

$$A_t :\to \sum_{I \text{ is odd}, I\in\mathcal{D}_t^1} \langle f, h_I\rangle \chi_{I+1}$$

$$B_t :\to \sum_{I \text{ is even}, I\in\mathcal{D}_t^1} \langle f, h_I\rangle \chi_{I+1}.$$

Clearly $A_{t+1} = B_t$. Also it is clear that $A_t + B_t = \mathbb{P}_t^+$, where the last operator is our grid operator from the beginning of this Section.

$$\mathbb{E}H_t = \frac{1}{2}\int_0^1 (A_t + A_{t+1}) = \frac{1}{2}\int_0^1 (A_t + B_t) = \frac{1}{2}\int_0^1 \mathbb{P}_t^+ dt.$$

From (3.12) we get that

$$\mathbb{E}H_t = \frac{1}{2}F_0(x-1) * .$$

Similarly, if we call the second operator G_t we get from (3.12)

$$\mathbb{E}G_t = \frac{1}{2}F_0(x+1) * .$$

Using (3.2) and (3.5) we show that averagings of the third and the fourth operators give us convolution operator with kernel $\frac{1}{2}F_0$. The theorem is proved.

□

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Theorem 3.3. Let us consider the following grid operator

$$f \to \sum_{J \in \mathcal{D}_t^{(2)}} \langle f, h_J \rangle,$$

where

$$t \in \left(\mathbb{R}/\mathbb{Z}; \frac{1}{2} dt | (-2, 0] \right).$$

Then its averaging is the convolution operator with kernel $\frac{1}{\sqrt{2}} S(x)$.

Proof. We weite h_J as $\frac{1}{\sqrt{2}}(-\chi_{J-} + \chi_{J+})$. Then it is an obvious algebraic remark that

$$\begin{aligned} \sqrt{2} \textit{ our operator} = & \textit{ third operator of Theorem 3.2} \\ & + \textit{third operator of Theorem 3.2} \\ & - \textit{third operator of Theorem 3.2} \\ & - \textit{third operator of Theorem 3.2} \end{aligned}$$

Averaging this and using Theorem 3.2 finishes the proof. □

As in the previous section, given the lattice $\mathcal{L} = \mathcal{L}_t^r$, we can consider the lattice operator

$$\mathcal{K}^{\mathcal{L}} f := \sum_{J \in \mathcal{L}} \langle f, h_{J+} - h_{J-} \rangle h_J$$

amalgamated from the grid operators of Theorem 3.2.

This operator is called the dyadic shift. It has been proved that averaging of dyadic shifts over all lattices gives us operator which is proportional to the Hilbert transform (we certainly mean that coefficient of proportionality is not zero).

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Let us reproduce this result. Fixing r and averaging over lattices with fixed calibre r (we leave for the reader to invent the natural probability space of all lattices with fixed calibre r) we get the convolution operator with the kernel

$$\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\frac{1}{2^n r}S\left(\frac{x}{2^n r}\right)=:\frac{1}{\sqrt{2}}S_r(x).$$

Averaging convolution operators with kernels $\frac{1}{\sqrt{2}}S_r$ over $\left([1,2);\frac{dr}{r}\right)$, we get the operator with the kernel $\frac{1}{4}\frac{1}{\sqrt{2}}\frac{1}{x}$. So we get averaging of the shift operators over all lattices of all calibres $=\frac{1}{4\sqrt{2}}$ kernel of the Hilbert transform.

Planar case We can and will reason by analogy. We have lattices $\mathcal{L}_t^\rho$ of squares, where t now is in $\Omega^\rho := \mathbb{R}^2/\rho\mathbb{Z}$ with normalized Lebesgue measure (Lebesgue measure on the torus Ω^ρ divided by ρ^2). We have the main grid operator

$$\mathbb{P}_t f := \sum_{Q\in\mathcal{D}_t}\langle f, h_Q\rangle\chi_Q$$

where $\mathcal{D}_t$ is a grid of unit squares such that $t\in\mathbb{R}^2$ is a vertex for 4 of them, where

$$h_Q(x)=\begin{cases}\frac{-1}{|Q|^{1/2}} & \text{, for } x\in Q_l\\ \frac{1}{|Q|^{1/2}} & \text{, for} x\in Q_r,\\ 0, & otherwise\end{cases}$$

Here Ql, Q_r are left and right halves of Q, function h_Q is normalized in L^2. We consider the same type of grid operators for grids $\mathcal{D}_t^\rho$ of squares of side ρ —the only change is that we divide χ_Q by ρ to make it normalized in L^2.

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Let us denote by k_0 the function $-h_{Q_0}$, where Q_0 is the unit square centered at 0. Also χ_0 denotes the characteristic function of this square. Consider

$$\Phi_0 := \chi_0 * k_0,$$

$$\Phi_0^\rho(x) := \frac{1}{\rho^2}\frac{1}{\rho^2}\chi_0\left(\frac{\cdot}{\rho}\right) * k_0\left(\frac{\cdot}{\rho}\right) = \frac{1}{\rho^2}\Phi_0\left(\frac{x}{\rho}\right).$$

Exactly as before (in one dimensional case) function Φ_0 is the kernel of the convolution operator, which appears as averaging of $\mathbb{P}_t$ over Ω^1. Function Φ_0^ρ is the kernel of the convolution operator, which appears as averaging of $\mathbb{P}_t^\rho$ over Ω^ρ.

Again, we can consider kernel

$$k(x) := \int_0^\infty \Phi_0^\rho(x)\frac{d\rho}{\rho} = \frac{\omega\left(\frac{x}{|x|}\right)}{|x|^2}.$$

And it is very easy to see that ω is an odd non-zero function on the unit circle. Literally as before we can see that k is the convolution operator which is the average with respect to measure $\frac{dr}{r}|[1,2)$ of the convolution operators with kernels

$$k_r(x) := \sum_{n=-\infty}^{\infty} \Phi_0^{r\cdot 2^n}(x).$$

In its turn, k_r is the average of the lattice operators which are sums of corresponding grid operators, here are those lattice operators:

$$\mathcal{P}_{\mathcal{L}^r} := \sum_{Q\in\mathcal{L}^r} \langle f, h_Q\rangle \chi_Q/\sqrt{|Q|}.$$

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Here r is fixed and denotes the calibre of the lattice. The averaging over the lattices of this fixed calibre gives us the convolution operator with kernel k_r. So the averaging over the calibres ($= \int_1^2 \dots \frac{dr}{r}$) gives us the averaging over all lattices, over all calibres. As a result we get the convolution operator with kernel $k = \frac{\omega\left(\frac{x}{|x|}\right)}{|x|^2}$.

Again we would like to repeat all this but with slightly different lattice operators —just because there are nicer ones and because $\mathcal{P}_{\mathcal{L}^r}$ are not L^2 bounded. Another problem we face now is that k is not necessarily a kernel of a Riesz transform. So we will need to work a bit more than in the one-dimensional case to obtain the Riesz transform kernel.

For a square Q consider its partition to 4 equal squares and let us call them $Q^{nw}, Q^{ne}, Q^{sw}, Q^{se}$ according to northwest, northeast,.... Let us consider the following grid operator

$$f \to \sum_{Q \in \mathcal{D}_t^{(2)}} \langle f, h_{Q^{ne}} + h_{Q^{se}} - h_{Q^{nw}} - h_{Q^{sw}} \rangle h_Q,$$
$$t \in \Omega^{(2)} := \left(\mathbb{R}^2/2\mathbb{Z}^2; \frac{1}{4} \textit{ Lebesgue measure} \right).$$

Consider also the function ($x = (x_1, x_2)$)

$$\begin{aligned} S(x_1, x_2) = \Phi_0(x_1, x_2) &- \frac{1}{2}\Phi_0(x_1+1, x_2) - \frac{1}{2}\Phi_0(x_1-1, x_2) \\ &+ \frac{1}{2}\Phi_0(x_1, x_2+1) - \frac{1}{4}\Phi_0(x_1+1, x_2+1) - \frac{1}{4}\Phi_0(x_1-1, x_2) \\ &+ \frac{1}{2}\Phi_0(x_1, x_2-1) - \frac{1}{4}\Phi_0(x_1+1, x_2-1) - \frac{1}{4}\Phi_0(x_1-1, x_2-1). \end{aligned} \tag{3.15}$$

Theorem 3.4. The averaging of the grid operator above over $\Omega^{(2)}$ gives the convolution operator with kernel $\frac{1}{2}S(x)$.

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The proof is literally the same as the proof of Theorem 3.3.

Let us start with one observation about (3.15). Function Φ_0 is the convolution $\chi_0 * k_0$. But both functions χ_0 and k_0 are products of functions of one variable $\Phi_0(x_1, x_2) = f_0(x_2) \cdot F_0(x_1)$. Moreover, function f_0 is nonnegative. Actually $f_0(x_2)$ is a convolution square of the characteristic function of the unit interval centered at 0. Formula (3.15) now looks like

$$S(x_1, x_2) = \left(f_0(x_2) + \frac{1}{2} f_0(x_2 + 1) + \frac{1}{2} f_0(x_2 - 1) \right) \times \left(F_0(x_1) - \frac{1}{2} F_0(x_1 + 1) - \frac{1}{2} F_0(x_1 - 1) \right).$$

For the future purposes we can say what happens in $n > 2$ case easily. We get $S_n(x) = S_n(x_1, x_2, \ldots, x_n)$ and

$$S_n(x) = \left(f_0(x_2) + \frac{1}{2} f_0(x_2 + 1) + \frac{1}{2} f_0(x_2 - 1) \right) \times \prod_{i=2}^{n} \left(F_0(x_1) - \frac{1}{2} F_0(x_1 + 1) - \frac{1}{2} F_0(x_1 - 1) \right). \tag{3.16}$$

As in the previous section this S generates kernel s by formula

$$s(x) = \int_0^\infty \frac{1}{\rho^n} S\left(\frac{x}{\rho}\right) \frac{d\rho}{\rho} = \frac{\xi(\frac{x}{|x|})}{|x|^n}.$$

And it is very easy to see that ξ_n is an odd non-zero function on the unit sphere. We will show it below. Literally as before we can see that s is the convolution operator which is the average with respect to measure $\frac{dr}{r}|[1, 2)$ of the convolution operators with kernels

$$s_r(x) := \sum_{n=-\infty}^{\infty} S_0^{r \cdot 2^n}(x).$$

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In its turn, s_r is the average of the lattice operators which are sums of corresponding grid operators, here are those lattice operators:

$$S_{\mathcal{L}^r} := \sum_{Q\in\mathcal{L}^r} \langle f, h_{Q^{ne}} + h_{Q^{se}} - h_{Q^{nw}} - h_{Q^{sw}}\rangle h_Q. \tag{3.17}$$

Here r is fixed and denotes the calibre of the lattice. The averaging over the lattices of this fixed calibre gives us the convolution operator with kernel s_r. So the averaging over the calibres ($= \int_1^2 \dots \frac{dr}{r}$) gives us the averaging over all lattices, over all calibres. As a result we get the convolution operator with kernel $s = \frac{\xi_n\left(\frac{x}{|x|}\right)}{|x|^n}$.

Let S^{n-1} denote as always the boundary sphere of the n-dimensional unit ball. Denote by S^{n-1}_+ the right half sphere —the half that lies in $\{x \in \mathbb{R} : x_1 > 0\}$. Let e_1 be a unit vector in the direction of coordinate axis x_1. Let σ denote Lebesgue measure of S^{n-1}. It would be important to prove

$$\int_{S^{n-1}_+} \xi_n(\omega)\langle\omega, e_1\rangle d\sigma(\omega) < 0. \tag{3.18}$$

For $n = 2$ we can just prove that $\xi_2(\omega) < 0$ for any $\omega \in S^1_+$. Then (3.18) follows immediately. To do this we use formula (3.16) and notice that $f_0(x) + \frac{1}{2}f_0(x+1) + \frac{1}{2}f_0(x-1) = (1-\frac{1}{2}x)_+$. Then the fact that $\xi_2(\omega) < 0$ follows from the following lemma.

Lemma 2. For any $k \in [0, \infty)$ we have

$$\int_0^2 \left(1 - \frac{1}{2}kx\right)_+ \left(F_0(x) - \frac{1}{2}F_0(x-1)\right) x dx < 0.$$

Proof. If $k \geq 2$ then the first factor vanishes everywhere where the second factor is positive. So we are done for such k. For $0 \leq k \leq 1$, we have $(1 - \frac{1}{2}kx)_+ =$

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$(1-\frac{1}{2}kx)$ on [0,2], and we can make an easy calculation of the integral. For the range $1 < k < 2$ the calculation becomes unpleasant, but still straightforward, we skip it just to avoid direct and simple calculations. $\square$

For $n = 2$, ω can be identified with a point of $[-\pi, \pi)$. Under this identification the kernel ξ_2 becomes an even function skew symmetric on $[0, \pi]$ with respect to the point $\pi/2$. Rotation of the kernel $\xi_2(\omega)$ means just the new kernel $\xi_2(\omega - \phi)$. Then

$$\begin{aligned}(\xi_2 * \cos)(\phi) &= \cos\phi \cdot \left(\int_{-\pi}^{\pi} \xi_2(s)\cos s ds\right) \\ &= \cos\phi \cdot \left(\int_{S_1} \xi_2(\omega)\langle\omega, e_1\rangle d\sigma(\omega)\right) = c_2\cos\phi,\end{aligned} \tag{3.19}$$

and constant $A_2 := \frac{\int_{-\pi}^{\pi}|\cos s|ds}{|c_2|}$. Notice that rotation of kernel ξ_2 corresponds to rotation of dyadic lattices on the plane. We have just proved the following theorem.

Theorem 3.5. The Riesz transform $\frac{x_1}{|x|^3}*$ is the operator integral $c_2^{-1}\int \cos\psi \frac{\xi_2\left(U_\psi \frac{x}{|x|}\right)}{|x|^3} * d\psi$. In particular, this means that operator with the kernel $A_2^{-1}\frac{x_1}{|x|^3}$ lies in the closed convex hull (in the weak operator topology) of the planar dyadic shifts. Thus, uniform boundedness of dyadic shift operators in any Banach space implies the boundedness of the Riesz transform in the same space.

For the case $n > 2$ we again start with (3.18). Let us average ξ_n with respect to all rotations that leave e_1 fixed. We get a new function $\eta_n(\omega) = f(\langle\omega, e_1\rangle)$. Obviously,

$$\int_{S_+^{n-1}} f(\langle\omega, e_1\rangle)\langle\omega, e_1\rangle d\sigma(\omega) < 0. \tag{3.20}$$

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Let SO is the group of orthogonal rotations of S^{n-1}.

Let us calculate $c_n = \int_{SO} f(\langle Ue_1, e_1\rangle)\langle Ue_1, e_1\rangle dU$. Obviously,

$$c_n = \int_{S^{n-1}} f(\langle \omega, e_1\rangle)\langle \omega, e_1\rangle d\sigma(\omega) \neq 0,$$

because of (3.20). Now let us consider the rotated functions $f(\langle Ue_1, e_1\rangle)$. Consider

$$g(\omega) = \int_{SO} f(\langle U\omega, e_1\rangle)\langle Ue_1, e_1\rangle dU.$$

Then it is clear that $g(R\omega) = g(\omega)$ for every $R \in SO$ that fixes e_1. In fact,

$$\begin{aligned} g(R\omega) &= \int_{SO} f(\langle UR\omega, e_1\rangle)\langle Ue_1, e_1\rangle dU \\ &= \int_{SO} f(\langle V\omega, e_1\rangle)\langle VR^* e_1, e_1\rangle dV \\ &= \int_{SO} f(\langle V\omega, e_1\rangle)\langle Ve_1, e_1\rangle dV = g(\omega). \end{aligned}$$

On the other hand, it easy to see that

$$g(\omega) = \int_{S^{n-1}} f(\langle \omega, \xi\rangle)\langle \xi, e_1\rangle d\sigma(\omega). \tag{3.21}$$

Such a function (as we saw) depends only on $\langle \omega, e_1\rangle$. But moreover, it can be written as $\int_{S^{n-1}} f(\langle e_1, \xi\rangle)\langle \xi, \omega\rangle d\sigma(\omega)$. This is a restriction of a linear polynomial onto the sphere. This linear polynomial depends on $\langle \omega, e_1\rangle$ only, and, thus, is $c \cdot \langle \omega, e_1\rangle$. The constant c is just our c_n. One can see that by plugging $\omega = e_1$ into our formula (3.21) for $g(\omega)$.

Consider $A_n := \frac{\int_{SO} |\langle Ue_1, e_1\rangle| dU}{||c_n}$. Notice that rotation of kernel ξ_n corresponds to rotation of dyadic lattices on the plane. We have just proved the following theorem.

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Theorem 3.6. The Riesz transform $\frac{x_1}{|x|^{n+1}}*$ is the operator integral

$$c_n^{-1} \int_{SO} \langle Ue_1, e_1 \rangle \frac{\eta_n \left(U \frac{x}{|x|} \right)}{|x|^{n+1}} * dU.$$

In particular, this means that operator with the kernel $A_n^{-1} \frac{x_1}{|x|^{n+1}}$ lies in the closed convex hull (in the weak operator topology) of the planar dyadic shifts. Thus, uniform boundedness of dyadic shift operators in any Banach space implies the boundedness of the Riesz transform in the same space.

4 An integral arising from dyadic average of Riesz transforms

Introduction The question that was risen in their work [13] is whether the following integral is zero or not (the detail definitions of some notations in this integral are given in next section):

$$\int_{S_+^{n-1}} < \omega, e_1 > \xi_n(\omega) d\sigma(\omega). \tag{4.1}$$

They were able to show the integral is nonzero (in fact it is negative) when dimension $n = 2$ but for dimension $n \geq 3$ the problem remains unsolved.

Therefore the purpose of this section is to show the above integral when dimension $n = 3$ is also negative. This was done via a careful and an efficient decomposition for the integral. For some terms in our decomposition we are able to show explicitly that their values are negative. For some other terms we are able to prove an upper bound. Combining all the estimates shows the integral is indeed negative. Now let us mention the difficulties of the integral

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for dimension $n = 3$. First, the integrand functions in the integral are piecewise defined on some compact intervals, and the range of the integration is only half-sphere. Secondly, after we carefully analyse the integrand functions, one of the main difficulties then arises due to the mutual overlaps of their supports. More precisely, after using the sphere coordinates in the integral, the supports of the functions will create several difficulties since the behaviors of the points in these supports will now depend on the values of some complicated trigonometry functions. For these difficulties, it requires us to very carefully distinguish the range of the integrals in our decomposition. Finally for several integrals in our decomposition, we are able to show that their exact definite integrals can be computed. For the other integrals, we are not able to find their exact definite integrals, but we are able to find their upper bounds whose values can be explicitly estimated.

Preliminary In this section we first introduce some notations that will be used

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frequently in this paper. Let F_0, F, f_0, and f be defined as followings.

$$F_0(x) = \begin{cases} \frac{1}{2} - |x + \frac{1}{2}| & -1 \leq x < 0 \\ -(\frac{1}{2} - |x - \frac{1}{2}|) & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases};$$

$$F(x) = F_0(x) - \frac{1}{2}F_0(x+1) - \frac{1}{2}F_0(x-1);$$

$$f_0(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases};$$

$$f(x) = f_0(x) + \frac{1}{2}f_0(x+1) + \frac{1}{2}f_0(x-1).$$

Note that F is an odd function so that we may only describe F on $x \geq 0$, i.e.

$$F(x) = \begin{cases} \frac{-3}{4} + \frac{3}{2}|x - \frac{1}{2}| & if \quad |x - \frac{1}{2}| \leq \frac{1}{2} \\ \frac{1}{4} - \frac{1}{2}|x - \frac{3}{2}| & if \quad |x - \frac{3}{2}| \leq \frac{1}{2} \\ 0 & x \geq 2 \end{cases}$$

and $f(x) = 1 - \frac{1}{2}|x|$ if $|x| \leq 2$ and 0 otherwise. See below for their graphs.

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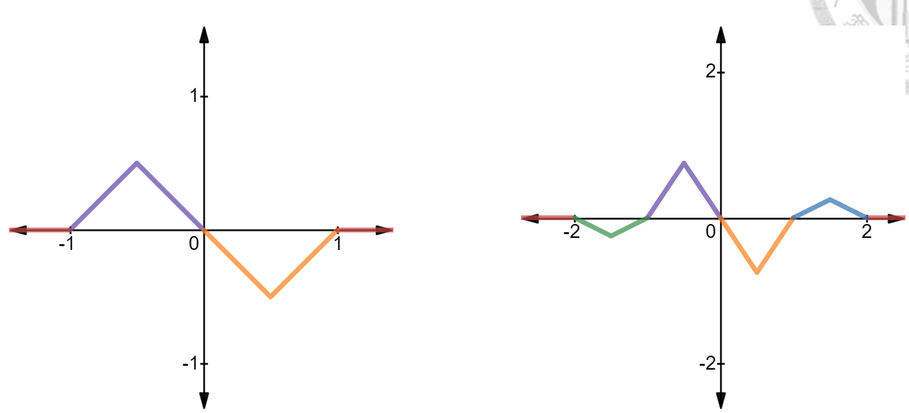
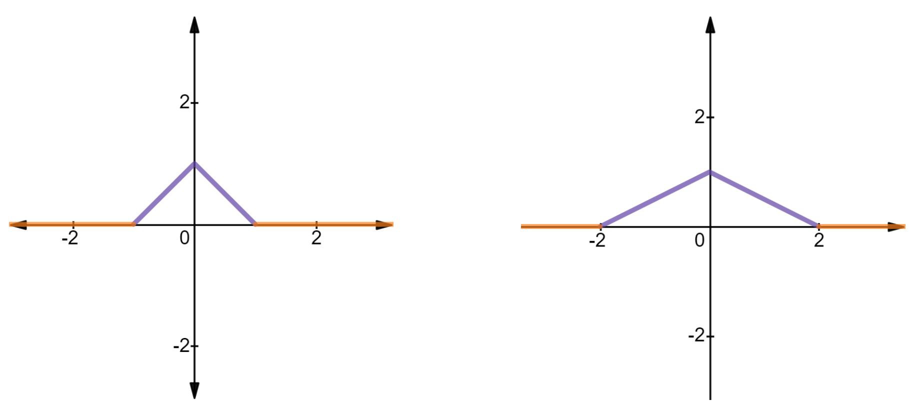


(a) F_0 (b) F

Figure 1: F, and F_0



(a) f_0 (b) f

Figure 2: f, and f_0

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For all $n \geq 2$, and given $x = (x_1, ..., x_n) \neq 0$, we define

$$K_n(x) = F(x_1) \times \prod_{i=2}^{n} f(x_i)$$

and

$$\xi_n(\frac{x}{|x|}) := |x|^n \int_0^\infty \frac{1}{\rho^n} K_n(\frac{x}{\rho}) \frac{d\rho}{\rho}.$$

Let $\rho = \frac{|x|}{t}$, the above formula $\xi_n(\frac{x}{|x|})$ now becomes

$$\xi_n(\omega) := \xi_n(\frac{x}{|x|}) = \int_0^\infty t^{n-1} K_n(\frac{tx}{|x|}) dt,$$

where $\omega \in S^{n-1}$. Recall that the main result that we want to prove is to show the following integral is nonzero (negative),

$$\int_{S_+^{n-1}} < \omega, e_1 > \xi_n(\omega) d\sigma(\omega),$$

where $S_+^{n-1} = \{\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}; \omega_1 > 0\}$. Thus putting it together, our goal is to show for $n = 3$ the following integral is negative.

$$\int_{S_+^2} < \omega, e_1 > \int_0^\infty t^2 F(t\omega_1) f^{\omega_2}(t) f(t\omega_3) dt d\sigma(\omega). \tag{4.2}$$

Before we proceed to give our decompositions for this integral. The diagram in next page gives a big picture how the integral is decomposed and how each term in our decomposition is estimated.

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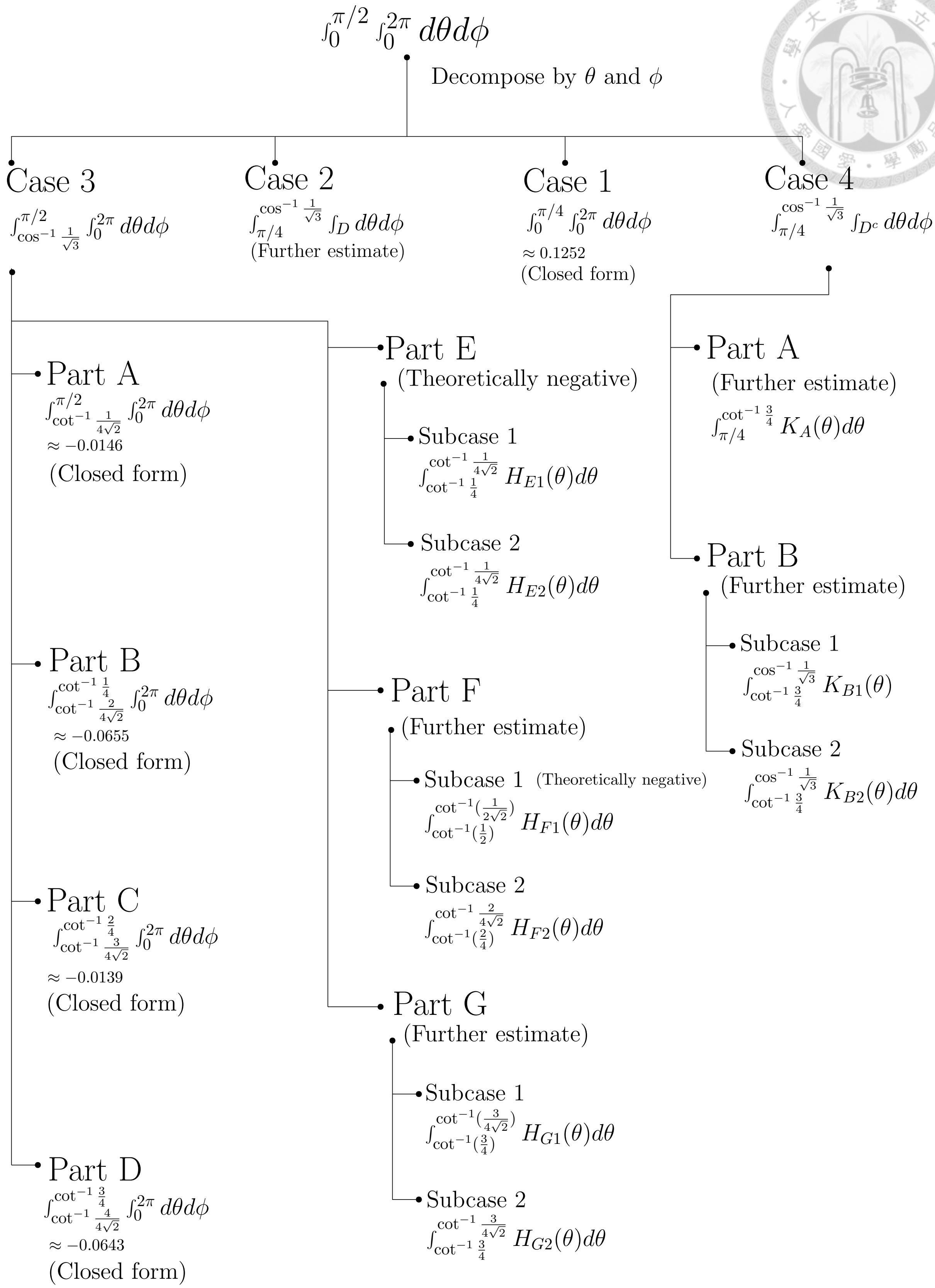

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Decompositions

In order to integrate with spherical measure appearing in (4.2), we use spherical coordinate system to represent ω_1, ω_2, and ω_3 i.e.

$$\omega_1 = \cos\theta,$$

$$\omega_2 = \sin\theta\cos\phi,$$

$$\omega_3 = \sin\theta\sin\phi,$$

where $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$ because we only integrate on the half sphere S^2_+. Putting in these new variables and using change of variables formula, the integral (4.2) which we want to estimate becomes

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos\theta \int_0^{\infty} t^2 F(t\cos\theta) f(t\sin\theta\cos\phi) f(t\sin\theta\sin\phi) dt \sin\theta d\phi d\theta, \quad (4.3)$$

where the factor $\sin\theta$ is due to Jacobian, and the $\cos\theta$ is from $< \omega, e_1 >$. Since the integral range is $0 \leq \theta \leq \frac{\pi}{2}$, we only need to consider $F^{\omega_1}(t) = F(t\cos\theta)$ with $\omega_1 \geq 0$. In order to estimate $\xi_3(\omega)$, we break $F^{\omega_1}(t)$ into 4 linear mutually disjoint support functions with respect to t that is $F^{\omega_1}(t) := F^{\omega_1}_{11}(t) + F^{\omega_1}_{12}(t) + F^{\omega_1}_{13}(t) + F^{\omega_1}_{14}(t)$, where

$$F^{\omega_1}_{11}(t) = \begin{cases} -\frac{3}{2}t\omega_1 & 0 \leq t \leq \frac{1}{2\omega_1} \\ 0 & \text{otherwise} \end{cases} \quad (4.4)$$

$$F^{\omega_1}_{12}(t) = \begin{cases} \frac{3}{2}(t\omega_1 - 1) & \frac{1}{2\omega_1} \leq x \leq \frac{2}{2\omega_1} \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

$$F^{\omega_1}_{13}(t) = \begin{cases} \frac{1}{2}(t\omega_1 - 1) & \frac{2}{2\omega_1} \leq t \leq \frac{3}{2\omega_1} \\ 0 & \text{otherwise} \end{cases} \quad (4.6)$$

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$$F_{14}^{\omega_1}(t) = \begin{cases} -\frac{1}{2}(t\omega_1 - 2) & \frac{3}{2\omega_1} \leq t \leq \frac{4}{2\omega_1} \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

$$f^{\omega_2}(t) = \begin{cases} (1 - \frac{|t\omega_2|}{2}) & 0 \leq |t| \leq \frac{2}{|\omega_2|} \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

$$f^{\omega_3}(t) = \begin{cases} (1 - \frac{|t\omega_3|}{2}) & 0 \leq |t| \leq \frac{2}{|\omega_|} \\ 0 & \text{otherwise} \end{cases} . \tag{4.9}$$

Let

$$S_i(a,\theta,\phi) = \int_0^a t^2 F_{1i}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt = \int_0^a t^2 F_{1i}^{\omega_1}(t) f(t\sin\theta\cos\phi) f(t\cos\theta\sin\phi)dt,$$

for $i = 1, 2, 3, 4$. Then we have $\xi_3(\omega) = \sum_{i=1}^4 S_i(\infty, \theta, \phi)$.

Remark 1. For a fixed ω_1, the support of $F^{\omega_1}(t)$ is $0 \leq t \leq \frac{2}{|\omega_1|}$, and the support of f^{ω_2} and f^{ω_3} are $0 \leq t \leq \frac{2}{|\omega_2|}$ and $0 \leq t \leq \frac{2}{|\omega_3|}$ respectively. Hence the integral range for S_i will be simultaneously determined by the supports of the functions of F^{ω_1}, f^{ω_2}, and f^{ω_3}. This observation leads us to decompose the integral in terms of the supports of these functions. More precisely, we divide the integral into 4 cases depending on which function vanishes first. We now give details in the following sections.

Criteria of decomposition

As just mentioned before, we can reduce the integral range of S_i to $\left[0, \max\{\frac{2}{|\omega_1|}, \frac{2}{|\omega_2|}, \frac{2}{|\omega_3|}\}\right]$. Therefore the integral range of t now depends on the variables θ and ϕ since the variables ω_i depend on θ, ϕ and this also explains why estimating the integral is

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complicated and difficult. Therefore we will have 4 cases that depend on which function F^{ω_1} f^{ω_2} or f^{ω_3} vanishes before the others.

Remark 2. Throughout this paper, given two functions f, g and assuming the supports of f and g are $[0, a]$ and $[0, b]$ respectively, then we say f vanishes before g or g vanishes after f if $a \leq b$.

First part vanishes before the others. Assuming $F^{\omega_1}(t)$ vanishes before the others i.e. $|\omega_1| \geq |\omega_2|$ and $|\omega_1| \geq |\omega_3|$. Notice that $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. If $|\omega_1| = |\cos\theta| \geq \frac{1}{\sqrt{2}}$, then for all ϕ, $|\omega_1|$ is always the largest one. If $|\omega_1| \in [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}]$, then $|\omega_1| \geq |\omega_2|$ for some ϕ. If $|\omega_1| \leq \frac{1}{\sqrt{3}}$, then by pigeonhole principle one of $|\omega_2|, |\omega_3|$ must be larger than $\frac{1}{\sqrt{3}}$ which is larger than $|\omega_1|$. Thus θ must be in $[0, \cos^{-1}\frac{1}{\sqrt{3}}]$. There are two different situations we need to separately deal with.

1 When $\theta \in [0, \pi/4]$, then we will have ω_1 is larger than $|\omega_2|$ and $|\omega_3|$ for all ϕ.

2 When $\theta \in [\pi/4, \cos^{-1}\frac{1}{\sqrt{3}}]$, then we will have ω_1 is larger than $|\omega_2|$ and $|\omega_3|$ for some ϕ.

Case 1 $0 \leq \theta < \pi/4$ and for all $0 \leq \phi < 2\pi$.**(Closed form)**

Since $F^{\omega_1}(t)$ vanishes before the others and $\omega_1 = \cos\theta$, $\omega_2 = \sin\theta\cos\phi$ and $\omega_3 = \sin\theta\sin\phi$. Thus we see that when $\theta \in [0, \pi/4]$, we will have

$$|\omega_1| = |\cos\theta| \geq |\sin\theta\cos\phi| = |\omega_2|,$$

and

$$|\omega_1| = |\cos\theta| \geq |\sin\theta\sin\phi| = |\omega_3|,$$

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for all $0 \le \phi < 2\pi$. Now

$$\xi_3(\omega) = \int_0^{\frac{2}{\omega_1}} t^2 F(t\cos\theta) f(t\sin\theta\cos\phi) f(t\sin\theta\sin\phi) dt = \sum_{i=1}^{4} S_i(\frac{i}{2\omega_1}, \theta, \phi) - S_i(\frac{i-1}{2\omega_1}, \theta, \phi)$$

let

$$\begin{aligned}
h_1(\theta,\phi) &:= S_1(\frac{1}{2\cos\theta}, \theta, \phi), \\
h_2(\theta,\phi) &:= S_2(\frac{1}{\cos\theta}, \theta, \phi) - S_2(\frac{1}{2\cos\theta}, \theta, \phi), \\
h_3(\theta,\phi) &:= S_3(\frac{3}{2\cos\theta}, \theta, \phi) - S_3(\frac{1}{\cos\theta}, \theta, \phi), \\
h_4(\theta,\phi) &:= S_4(\frac{2}{\cos\theta}, \theta, \phi) - S_4(\frac{3}{2\cos\theta}, \theta, \phi).
\end{aligned}$$

Since $t^2 F_{1i}(t\omega_1) f^{\omega_2}(t) f^{\omega_3}(t)$ is a polynomial of degree 5 on this integral range, and h_1, h_2, h_3, and h_4 all have closed forms, therefore we obtain

$$\begin{aligned}
\sum_{i=1}^{4} h_i(\theta,\phi) = &-(720\cos\theta\sin\theta\cos\phi - 680\cos^2\theta - 629\cos\phi\sin\phi \\
&+ 720\cos\theta\sin\theta\sin\phi + 629\cos^2\theta\cos\phi\sin\phi)(3840\cos^5\theta)^{-1},
\end{aligned}$$

$$\int_0^{2\pi} \sin\theta\cos\theta \sum_{i=1}^{4} h_i(\theta,\phi) d\phi = \frac{4(\cos^2\theta(\frac{17\pi}{192} - \frac{629}{7680}) - \frac{3\cos\theta\sin\theta}{8} + \frac{629}{7680})}{\cos^5\theta}.$$

Therefore, we see that the integral below has a closed form so that we can estimate it accurately

$$\int_0^{\pi/4}\int_0^{2\pi} \sin\theta\cos\theta(h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + h_4(\theta,\phi)) d\phi d\theta \approx 0.1252.$$

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Remark 3. This is the only term that has positive value.

Case 2 $\pi/4 \le \theta \le \cos^{-1}\frac{1}{\sqrt{3}}$, and $\phi \in D$. **(further estimate)**
In this case $\pi/4 \le \theta \le \cos^{-1}\frac{1}{\sqrt{3}}$ and since we want $\omega_1 \ge |\omega_2|$, and $\omega_1 \ge |\omega_3|$. Thus we have $\cot\theta \ge |\sin\phi|$ and $\cot\theta \ge |\cos\phi|$ in spherical coordinates. As a result, the range of ϕ will be restricted on D which is

On the first quadrant: $\cos^{-1}(\cot\theta) \le \phi \le \sin^{-1}(\cot\theta)$

On the second quadrant: $\frac{\pi}{2} + \cos^{-1}(\cot\theta) \le \phi \le \frac{\pi}{2} + \sin^{-1}(\cot\theta)$

On the third quadrant: $\pi + \cos^{-1}(\cot\theta) \le \phi \le \pi + \sin^{-1}(\cot\theta)$

On the fourth quadrant: $\frac{3\pi}{2} + \cos^{-1}(\cot\theta) \le \phi \le \frac{3\pi}{2} + \sin^{-1}(\cot\theta)$.

Observe that since f is even so that for all ϕ we have

$$
\begin{aligned}
F^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t) &= F(t\cos\theta)f(t\sin\theta\cos\phi)f(t\sin\theta\sin\phi) \\
&= F(t\cos\theta)f(t\sin\theta\cos(\phi+\pi/2))f(t\sin\theta\sin(\phi+\pi/2)) \\
&= F(t\cos\theta)f(t\sin\theta\sin\phi)f(t\sin\theta\cos\phi) \\
&= F^{\omega_1}(t)f^{\omega_3}(t)f^{\omega_2}(t).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{\cos^{-1}(\cot\theta)}^{\sin^{-1}(\cot\theta)} (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi \\
=&\int_{\cos^{-1}(\cot\theta)+\frac{\pi}{2}}^{\sin^{-1}(\cot\theta)+\frac{\pi}{2}} (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi \\
=&\int_{\cos^{-1}(\cot\theta)+\frac{2\pi}{2}}^{\sin^{-1}(\cot\theta)+\frac{2\pi}{2}} (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi \\
=&\int_{\cos^{-1}(\cot\theta)+\frac{3\pi}{2}}^{\sin^{-1}(\cot\theta)+\frac{3\pi}{2}} (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi.
\end{aligned}
$$

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Hence the integral that we want to estimate is equal to

$$\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \int_D (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi \sin\theta d\theta \tag{4.10}$$
$$= \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} 4\cos\theta \int_{\cos^{-1}(\cot\theta)}^{\sin^{-1}(\cot\theta)} (h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+h_4(\theta,\phi))d\phi \sin\theta d\theta.$$

Integrating with respect to ϕ is a closed form as above. However, when integrating with respect to θ, we are unable to find its closed form. The reason is that the range is from $\cos^{-1}(\cot\theta)$ to $\sin^{-1}(\cot\theta)$, and after integrating the variable ϕ these upper and lower limits make the integrand in the variable θ extremely complicated. Therefore this case will be further estimated in the final section.

Second or third part vanishes before the others.

In this case it suffices to consider f^{ω_2} vanishes before the others. The reasons are the followings. First, we observe that

$$\begin{aligned} F^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t) &= F(t\cos\theta)f(t\sin\theta\cos\phi)f(t\sin\theta\sin\phi) \\ &= F(t\cos\theta)f(t\sin\theta\cos(\phi+\pi/2))f(t\sin\theta\sin(\phi+\pi/2)), \end{aligned}$$

and notice that the if f^{ω_3} vanishes before the others, the integral range for ϕ in this case is only different from the integral range for ϕ by rotating $\frac{\pi}{2}$ in the case that f^{ω_2} vanishes before the others. Now we see that for f^{ω_2} vanishes before the others, we must have $\phi \in [-\pi/4, \pi/4]$ and $[-3\pi/4, 5\pi/4]$. Again since f is even, it suffices to only consider the range $[-\pi/4, \pi/4]$. Thus

$$\begin{aligned} \int_0^{2\pi} \xi_3(\omega)d\phi &= \int_0^{2\pi}\int_0^\infty t^2 F^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dt \\ &= 4\int_{-\pi/4}^{\pi/4}\int_0^\infty t^2F^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi. \end{aligned}$$

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We now determine the range of θ. Assume f^{ω_2} vanishes before the others i.e. $|\omega_2| \geq |\omega_1|$ and $|\omega_2| \geq |\omega_3|$. Notice that $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. If $|\omega_2| = |\sin\theta\cos\phi| \geq \frac{1}{\sqrt{2}}$, then for all ϕ, $|\omega_2|$ is always the largest one. If $|\omega_2| \in [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}]$, then $|\omega_2| \geq |\omega_1|$ for some ϕ. If $|\omega_2| \leq \frac{1}{\sqrt{3}}$, then by pigeonhole principle one of $|\omega_1|, |\omega_3|$ is larger than $\frac{1}{\sqrt{3}}$ which is larger than $|\omega_2|$. Thus θ must be in $[\pi/4, \pi/2]$. There are two different situations we need to separately deal with

1 When $\theta \in [\cos^{-1}\frac{1}{\sqrt{3}}, \pi/2]$, $|\omega_2|$ is larger than $|\omega_1|$ and $|\omega_3|$ for all $\phi \in [-\pi/4, \pi/4]$.

2 When $\theta \in [\pi/4, \cos^{-1}\frac{1}{\sqrt{3}}]$, $|\omega_2|$ is larger than $|\omega_1|$ and $|\omega_3|$ for some $\phi \in [-\pi/4, \pi/4]$.

Case 3 $\cos^{-1}(\frac{1}{\sqrt{3}}) \leq \theta \leq \pi/2$.

We also break F^{ω_1} into 4 pieces as before. And the decompositions in this term are the most complicated one since we need to decide which of the 5 pieces $F_{11}^{\omega_1}$, $F_{11}^{\omega_1} + F_{12}^{\omega_1}$, $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$, $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1} + F_{14}^{\omega_1}$, and f^{ω_2} vanishes before the others according to θ and ϕ. More precisely, since $f^{\omega_2}(t)$ vanishes before the others and $F(t\omega_1) = F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1} + F_{14}^{\omega_1}$. Therefore there are 4 possibilities that f^{ω_2} vanishes before (1) $F_{11}^{\omega_1}$ (2) $F_{11}^{\omega_1} + F_{12}^{\omega_1}$, (3) $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$ and (4) $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1} + F_{14}^{\omega_1}$. We now further decompose these 4 possibilities in terms of the range of ϕ.

1: These 4 possibilities hold for all $\phi \in [-\pi/4, \pi/4]$**.**

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Part A f^{ω_2} vanishes before $F_{11}^{\omega_1}$. It gives

$$\xi_3(\omega) = \int_0^{\frac{2}{|\omega_2|}} t^2 F^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) = \int_0^{\frac{2}{|\omega_2|}} t^2 F_{11}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t).$$

Part B f^{ω_2} vanishes after $F_{11}^{\omega_1}$ and before $F_{11}^{\omega_1} + F_{12}^{\omega_1}$. It gives

$$\xi_3(\omega) = \int_0^{\frac{2}{|\omega_2|}} t^2 (F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t)) f^{\omega_2}(t) f^{\omega_3}(t).$$

Part C f^{ω_2} vanishes after $F_{11}^{\omega_1} + F_{12}^{\omega_1}$ and before $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$. It gives

$$\xi_3(\omega) = \int_0^{\frac{2}{|\omega_2|}} t^2 (F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t) + F_{13}^{\omega_1}(t)) f^{\omega_2}(t) f^{\omega_3}(t).$$

Part D f^{ω_2} vanishes after $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$ and before $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1} + F_{14}^{\omega_1}$.

It gives

$$\xi_3(\omega) = \int_0^{\frac{2}{|\omega_2|}} t^2 (F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t) + F_{13}^{\omega_1}(t) + F_{14}^{\omega_1}(t)) f^{\omega_2}(t) f^{\omega_3}(t).$$

Now we define some notations to simplify our expressions. Let

$$\begin{aligned}
g_1(\theta,\phi) &:= S_1(\frac{2}{|\omega_2|},\theta,\phi) = S_1(\frac{2}{\sin\theta\cos\phi},\theta,\phi),\\
g_2(\theta,\phi) &:= S_2(\frac{2}{|\omega_2|},\theta,\phi) - S_2(\frac{1}{2\cos\theta},\theta,\phi) = S_2(\frac{2}{\sin\theta\cos\phi},\theta,\phi) - S_2(\frac{1}{2\cos\theta},\theta,\phi),\\
g_3(\theta,\phi) &:= S_3(\frac{2}{|\omega_2|},\theta,\phi) - S_3(\frac{2}{2\cos\theta},\theta,\phi) = S_3(\frac{2}{\sin\theta\cos\phi},\theta,\phi) - S_3(\frac{2}{2\cos\theta},\theta,\phi),\\
g_4(\theta,\phi) &:= S_4\left(\frac{2}{|\omega_2|},\theta,\phi\right) - S_4(\frac{3}{2\cos\theta},\theta,\phi) = S_4\left(\frac{2}{\sin\theta\cos\phi},\theta,\phi\right) - S_4(\frac{3}{2\cos\theta},\theta,\phi).
\end{aligned}$$

Therefore

Part A $\xi_3(\omega) = g_1(\theta,\phi)$.

Part B $\xi_3(\omega) = h_1(\theta,\phi) + g_2(\theta,\phi)$.

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Part C $\xi_3(\omega) = h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)$.

Part D $\xi_3(\omega) = h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)$.

Since we have further decomposed the integral into these 4 possibilities for all $\phi \in [-\pi/4, \pi/4]$, we need to determine the range of θ.

Part A (Closed form)

Since we now have $0 \le \frac{2}{|\omega_2|} \le \frac{1}{2|\omega_1|}$ or $0 \le \frac{2}{|\omega_3|} \le \frac{1}{2|\omega_1|}$ for all $\phi \in [-\pi/4, \pi/4]$. In order to satisfy the condition $\max\{\frac{2}{|\omega_2|}, \frac{2}{|\omega_3|}\} \le \frac{1}{2|\omega_1|}$, thus we have $\max\{\frac{2}{|\omega_2|}, \frac{2}{|\omega_3|}\} \le \frac{2\sqrt{2}}{\sin\theta} \le \frac{1}{2\cos\theta} = \frac{1}{2\omega_1}$, which in turn gives that

$$\pi/2 \ge \theta \ge \cot^{-1}\frac{1}{4\sqrt{2}}.$$

Therefore we get the following integral:

$$H_A(\theta) := 4\sin\theta\cos\theta \int_{-\pi/4}^{\pi/4} g_1(\theta,\phi)d\phi = \frac{8\sin^2\theta - 1}{\sin^3\theta}.$$

Moreover the integral of $H_A(\theta)$ with respect to θ is a closed form. Finally we can explicitly compute the vaule

$$\int_{\cot^{-1}\frac{1}{4\sqrt{2}}}^{\pi/2} H_A(\theta)d\theta \approx -0.0146.$$

Part B (Closed form)

Since we have $\frac{1}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{2}{2\omega_1}$ for all ϕ. In other words we have $\frac{2\sqrt{2}}{\sin\theta} \le \frac{2}{2\cos\theta}$, and $\frac{1}{2\cos\theta} \le \frac{2}{\sin\theta}$. Hence we get range of θ

$$\cot^{-1}\frac{1}{4} \ge \theta \ge \cot^{-1}\frac{2}{4\sqrt{2}}.$$

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Therefore

$$
\begin{aligned}
H_B(\theta) :&= 4\sin\theta\cos\theta \int_{-\pi/4}^{\pi/4} h_1(\theta,\phi) + g_2(\theta,\phi)d\phi \\
&= -(24\cos\theta\sin^5\theta - \sin^6\theta - 1024\cos^6\theta + 2048\cos^5\theta\sin\theta - 40\pi\sin^4\theta + 40\pi\sin^6 \\
&+ 5120\log(\sqrt{2}+1)\cos^5\theta\sin\theta + 1024\sqrt{2}\cos^5\theta\sin\theta)/(1280\cos^4\theta\sin^3\theta).
\end{aligned}
$$

Similarly, the integral of $H_B(\theta)$ is a closed form and finally we have

$$
\int_{\cot^{-1}\frac{2}{4\sqrt{2}}}^{\cot^{-1}\frac{1}{4}} H_B(\theta)d\theta \approx -0.0655.
$$

Part C (Closed form)

Since we have: $\frac{2}{2\omega_1} \leq \frac{2}{|\omega_2|} \leq \frac{3}{2\omega_1}$ for all $\phi \in [-\pi/4, \pi/4]$. As above, we will have that the range of θ is

$$
\cot^{-1}\frac{2}{4} \geq \theta \geq \cot^{-1}\frac{3}{4\sqrt{2}}.
$$

Therefore

$$
\begin{aligned}
H_C(\theta) :&= 4\sin\theta\cos\theta \int_{\pi/4}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)d\phi \\
&= -[61\sin\theta - 696\cos\theta + 2088\cos^3\theta + \cos^5\theta(5120\log(\sqrt{2}+1) + 1024\sqrt{2} - 40) \\
&- \cos^7\theta(5120\log(\sqrt{2}+1) + 1024\sqrt{2} + 1352) + (520\pi - 183)(\sin\theta - \sin^3\theta) \\
&- \cos^4\theta\sin\theta(1040\pi - 183) + \cos^6\theta\sin\theta(520\pi - 10301)/(3840\cos^4\theta\sin^4\theta)],
\end{aligned}
$$

and finally we have

$$
\int_{\cot^{-1}\frac{3}{4\sqrt{2}}}^{\cot^{-1}\frac{2}{4}} H_C(\theta)d\theta \approx -0.0139.
$$

Part D (Closed form)

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Since we have: $\frac{3}{2\omega_1} \leq \frac{2}{|\omega_2|} \leq \frac{4}{2\omega_1}$ for all $\phi \in [-\pi/4, \pi/4]$. As above, we will have that the range of θ is

$$\cot^{-1}\frac{3}{4} \geq \theta \geq \cot^{-1}\frac{4}{4\sqrt{2}}.$$

Therefore

$$\begin{aligned}
H_D(\theta) :&= 4\sin\theta\cos\theta\int_{\pi/4}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi \\
&= -(395\sin\theta - 3264\cos\theta + 9792\cos^3\theta + \cos^7\theta(5120\log(\sqrt{2}+1) + 1024\sqrt{2} + 53 \\
&- \cos^5\theta(5120\log(\sqrt{2}+1) + 1024\sqrt{2} + 11840) + (1880\pi - 1185)(\sin\theta - \sin^3\theta) \\
&- \cos^4\theta\sin\theta(3760\pi - 1185) + cos^6\theta\sin\theta(1880\pi + 4725))/(1920\cos^4\theta\sin^4\theta)),
\end{aligned}$$

and finally we have

$$\int_{\cot^{-1}\frac{4}{4\sqrt{2}}}^{\cot^{-1}\frac{3}{4}} H_D(\theta)d\theta \approx -0.0634$$

However there are still some ranges of θ that we have not dealt with in parts A, B, C, D above and the ranges are

$$[\cot^{-1}\frac{1}{4}, \cot^{-1}\frac{1}{4\sqrt{2}}], \quad [\cot^{-1}\frac{2}{4}, \cot^{-1}\frac{2}{4\sqrt{2}}], \text{ and } \quad \left[\cot^{-1}\frac{3}{4}, \cot^{-1}\frac{3}{4\sqrt{2}}\right].$$

For these 3 ranges, we need to further estimate the integrals.

Remark 4. Those 3 parts are more complicated than above 4 parts since in each cases, $\sup\{suppf^{\omega_2}\} = \frac{2}{|\omega_2|}$ is not contained in one of $[0, \frac{1}{2\omega_1}]$, $[\frac{1}{2\omega_1}, \frac{2}{2\omega_1}]$, $[\frac{2}{2\omega_1}, \frac{3}{2\omega_1}]$, and $[\frac{3}{2\omega_1}, \frac{4}{2\omega_1}]$ for all $\phi \in [-\pi/4, \pi/4]$. In fact it depend on ϕ. Therefore those three parts are not part A,B,C and D. Fortunately, it is contained in one of $[0, \frac{2}{2\omega_1}]$, $[\frac{1}{2\omega_1}, \frac{3}{2\omega_1}]$, and $[\frac{2}{2\omega_1}, \frac{4}{2\omega_1}]$. That is why in each of following cases, we need to split the integrals into two subcases according to ϕ.

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Part E $\theta \in [\cot^{-1}\frac{1}{4}, \cot^{-1}\frac{1}{4\sqrt{2}}]$. **(negative value)**

When $\theta \in [\cot^{-1}\frac{1}{4}, \cot^{-1}\frac{1}{4\sqrt{2}}]$ the range $[-\pi/4, \pi/4]$ of ϕ will be split into two cases. One of the ranges will give $\xi_3(\omega) = g_1(\theta,\phi)$, and the other range will give $\xi_3(\phi) = g_2(\theta,\phi)$. More precisely, when $\theta \in [\cot^{-1}\frac{1}{4}, \cot^{-1}\frac{1}{4\sqrt{2}}]$, the variable ϕ will have two possibilities. One possibility is that f^{ω_2} vanishes before $F_{11}^{\omega_1}$ i.e. we will have $0 \le \frac{2}{|\omega_2|} \le \frac{1}{|2\omega_1|}$. The other possibility is that f^{ω_2} vanishes before $F_{12}^{\omega_1}$ and after $F_{11}^{\omega_1}$, which gives $\frac{1}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{2}{2\omega_1}$. As a result, we split the range of ϕ according to which above possibility occurs.

1 $0 \le \frac{2}{|\omega_2|} \le \frac{1}{2\omega_1}$ $(\xi_3(\omega) = g_1(\theta,\phi))$.

Since $0 \le \frac{2}{|\omega_2|} \le \frac{1}{2\omega_1}$ which is $\frac{2}{\sin\theta\cos\phi} \le \frac{1}{2\cos\theta}$. Thus

$$-\cos^{-1} 4\cot\theta \le \phi \le \cos^{-1} 4\cot\theta.$$

Therefore we can estimate the integral below

$$H_{E1}(\theta) := 4\sin\theta\cos\theta \int_{-\cos^{-1}(4\cot\theta)}^{\cos^{-1}(4\cot\theta)} g_1(\theta,\phi)d\phi,$$

which is a closed form in variable ϕ. However after plugging in the upper and lower limits, we are unable to show whether the integral $\int H_{E1}(\theta)d\theta$ has a closed form. But it is easy to show that its value is negative. Notice that $g_1(\theta,\phi) = \int_0^{4/2\sin\theta\cos\phi} t^2 F_{11}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt$, and for all t, t^2, $f^{\omega_2}(t)$, and $f^{\omega_2}(t)$ are positive but $F_{11}^{\omega_1}(t)$ is negative so that $g_1(\theta,\phi) \le 0$ for all θ and ϕ. Hence

$$\int_{\cot^{-1}\frac{1}{4}}^{\cot^{-1}\frac{1}{4\sqrt{2}}} H_{E1}(\theta)d\theta \le 0.$$

2 $\frac{1}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{2}{2\omega_1}$ $(\xi_3(\omega) = h_1 + g_2)$.

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The integral range of ϕ is just the complement of the range in case **1** above. Hence we have

$$H_{E2}(\theta) := 4\sin\theta\cos\theta\left[\int_{\cos^{-1}(4\cot\theta)}^{\pi/4} h_1(\theta,\phi)+g_2(\theta,\phi)d\phi + \int_{-\pi/4}^{-\cos^{-1}(4\cot\theta)} h_1(\theta,\phi)+g_2(\theta,\phi)\right.$$

Again we can explicitly compute $H_{E2}(\theta)$ because the above integrals are closed forms in variable ϕ. However after plugging in the upper and lower limits, the integral $\int H_{E2}(\theta)d\theta$ is difficult to see if it has a closed form. But it is easy to show that its value is negative. Notice that $h_1(\theta,\phi)+g_2(\theta,\phi) = \int_0^{4/2\sin\theta\cos\phi} t^2\left[F_{11}^{\omega_1}(t)+F_{12}^{\omega_1}(t)\right]f^{\omega_2}(t)f^{\omega_3}(t)dt$, and for all t, t^2, $f^{\omega_2}(t)$, and $f^{\omega_2}(t)$ are positive but $F_{11}^{\omega_1}(t)$, and $F_{12}^{\omega_1}(t)$ are negative so that $h_1(\theta,\phi)+g_2(\theta,\phi)\leq 0$ for all θ and ϕ. Hence

$$\int_{\cot^{-1}\frac{1}{4}}^{\cot^{-1}\frac{1}{4\sqrt{2}}} H_{E2}(\theta)d\theta \leq 0.$$

Part F $\quad\theta\in[\cot^{-1}\frac{2}{4},\cot^{-1}\frac{2}{4\sqrt{2}}]$**(further estimate)**

As in part E above, there will be two cases when $\theta\in[\cot^{-1}\frac{2}{4},\cot^{-1}\frac{2}{4\sqrt{2}}]$. One case is that $\frac{1}{2\omega_1}\leq\frac{2}{|\omega_2|}\leq\frac{2}{2\omega_1}$, and the other case is $\frac{2}{2\omega_1}\leq\frac{2}{|\omega_2|}\leq\frac{3}{2\omega_1}$.

1 $\quad\frac{1}{2\omega_1}\leq\frac{2}{|\omega_2|}\leq\frac{2}{2\omega_1}$ $(\xi_3(\omega)=h_1+g_2)$.

Since $\frac{1}{2\omega_1}\leq\frac{2}{|\omega_2|}\leq\frac{2}{2\omega_1}$, it gives that $0\leq\phi\leq\cos^{-1}(2\cot\theta)$. Hence

$$H_{F1}(\theta) := 4\sin\theta\cos\theta\int_{-\cos^{-1}(2\cot\theta)}^{\cos^{-1}(2\cot\theta)} h_1(\theta,\phi)+g_2(\theta,\phi)d\phi.$$

$H_{F1}(\theta)$ can be explicitly computed, but integral $\int H_{F1}(\theta)d\theta$ is difficult to show if it has a closed form. Therefore this case will be further estimated in the final section.

2 $\frac{2}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{3}{2\omega_1}$ $(\xi_3(\omega) = h_1 + h_2 + g_3)$

Since $\frac{2}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{3}{2\omega_1}$. It gives us that

$$\begin{aligned} H_{F2}(\theta) := 4\sin\theta\cos\theta\Big(&\int_{\cos^{-1}(2\cot\theta)}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)d\phi \\ &+ \int_{-\pi/4}^{-\cos^{-1}(2\cot\theta)} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)d\phi\Big). \end{aligned}$$

Again $H_{F2}(\theta)$ can be explicitly computed, but integral $\int H_{F2}(\theta)d\theta$ is difficult to show if it has a closed form. Therefore this case will be further estimated in the final section.

Part G $\theta \in [\cot^{-1}\frac{3}{4}, \cot^{-1}\frac{3}{4\sqrt{2}}]$ **(further estimate)**

1 $\frac{2}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{3}{2\omega_1}$ $(\xi_3(\omega) = h_1 + h_2 + g_3)$.

Since $\frac{2}{2\omega_1} \le \frac{2}{|\omega_2|} \le \frac{3}{2\omega_1}$, it gives us that

$$H_{G1}(\theta) := 4\sin\theta\cos\theta\int_{-\cos^{-1}(\frac{3}{4}\cot\theta)}^{\cos^{-1}(\frac{3}{4}\cot\theta)} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)d\phi.$$

$H_{G1}(\theta)$ can be explicitly computed, but integral $\int H_{G1}(\theta)$ is difficult to show if it has a closed form. Therefore this case will be further estimated in the final section.

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2 $\frac{3}{2\omega_1} \leq \frac{2}{|\omega_2|} \leq \frac{4}{2\omega_1}$ $(\xi_3(\omega) = h_1 + h_2 + h_3 + g_4)$ Since $\frac{3}{2\omega_1} \leq \frac{2}{|\omega_2|} \leq \frac{4}{2\omega_1}$, it gives us that

$$H_{G2}(\theta) := 4\sin\theta\cos\theta(\int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi$$
$$+ \int_{-\pi/4}^{-\cos^{-1}(\frac{3}{4}\cot\theta)} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi)$$

$$H_{G2}(\theta) := \int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi.$$

$H_{G2}(\theta)$ can be explicitly computed, but integral $\int H_{G2}(\theta)$ is difficult to show if it has a closed form. Therefore this case will be further estimated in the final section.

Finally the remaining case is below. **Case 4**: $\frac{\pi}{4} \leq \theta \leq \cos^{-1}(\frac{1}{\sqrt{3}})$ **(further estimate)**

For this case $\frac{\pi}{4} \leq \theta \leq \cos^{-1}(\frac{1}{\sqrt{3}})$, the range of ϕ is actually the complement of the range in case 2 above. In other words, the integral range of ϕ is D^c (see page 8 for the definition of D). Just as what we observe in case 2 the integral that we want to estimate can be reduced to

$$\begin{aligned}
&\int_{\frac{\pi}{4}}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \int_0^{2\pi} t^2 F^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)\,dt\,d\phi \sin\theta\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \int_{D^c} t^2 F^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)\,dt\,d\phi \sin\theta\, d\theta \qquad (4.11)\\
&= \int_{\frac{\pi}{4}}^{\cos^{-1}\frac{1}{\sqrt{3}}} 4\cos\theta \int_{-\cos^{-1}\cot\theta}^{\cos^{-1}\cot\theta} t^2 F^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)\,dt\,d\phi \sin\theta\, d\theta.
\end{aligned}$$

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Now here is the key observations that since $\phi \in [-\cos^{-1}\cot\theta, \cos^{-1}\cot\theta]$ so that

$$\omega_2 = \sin\theta\cos\phi \in [\cos\theta, \sin\theta],$$

Hence we have

$$\frac{2}{|\sin\theta|} \leq \frac{2}{|sin\theta\cos\phi|} \leq \frac{2}{|\cos\theta|} = \frac{2}{|\omega_1|}.$$

Therefore we will only have f^{ω_2} vanishes before $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1} + F_{14}^{\omega_1}$ and after $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$ for all $\phi \in [-\cos^{-1}\cot\theta, \cos^{-1}\cot\theta]$. This gives us that $\frac{2}{|\sin\theta|} \geq \frac{3}{2|\omega_1|}$ and hence we have

$$\frac{\pi}{4} \leq \theta \leq \cot^{-1}\frac{3}{4}.$$

Therefore the other part is

$$\cot^{-1}\frac{3}{4} \leq \theta \leq \cos^{-1}\frac{1}{\sqrt{3}}.$$

Part A' : $\frac{\pi}{4} \leq \theta \leq \cot^{-1}\frac{3}{4}$

In this case we have

$$K_A(\theta) := 4\cos\theta\sin\theta\int_{-\cos^{-1}\cot\theta}^{\cos^{-1}\cot\theta} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi,$$

which shows we need to estimate the integral below for this case.

$$\int_{\pi/4}^{\cot^{-1}\frac{3}{4}} K_A(\theta)d\theta.$$

Still, we are not able to find its closed form and hence this case will be further estimated in the final section.

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Part B' : $\cot^{-1}\frac{3}{4} \leq \theta \leq \cos^{-1}\frac{1}{\sqrt{3}}$

When $\cot^{-1}\frac{3}{4} \leq \theta \leq \cos^{-1}\frac{1}{\sqrt{3}}$, there will be two cases

(1) $f^{\omega_2}(t)$ vanishes after $F_{11}^{\omega_1} + F_{12}^{\omega_1} + F_{13}^{\omega_1}$, which gives $\frac{2}{\omega_2} \geq \frac{3}{2\omega_1}$;

(2) $f^{\omega_2}(t)$ vanishes before $F_{11}^{\omega_1}+F_{12}^{\omega_1}+F_{13}^{\omega_1}$ after $F_{11}^{\omega_1}+F_{12}^{\omega_1}$, which gives $\frac{2}{|\omega_2|} \leq \frac{3}{2\omega_1}$.

1 $\frac{2}{|\omega_2|} \geq \frac{3}{2\omega_1}$ $(\xi_3(\omega) = h_1 + h_2 + h_3 + g_4)$.

From this condition, we get the integral range of ϕ is

$$\cos^{-1}\frac{4}{3}\cot\theta \leq \phi \leq \cos^{-1}\cot\theta.$$

Therefore we have

$$\begin{aligned} K_{B1}(\theta) := 4\cos\theta\sin\theta(&\int_{\cos^{-1}\frac{4}{3}\cot\theta}^{\cos^{-1}\cot\theta} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi \\ &+ \int_{-\cos^{-1}\cot\theta}^{-\cos^{-1}\frac{4}{3}\cot\theta} h_1(\theta,\phi) + h_2(\theta,\phi) + h_3(\theta,\phi) + g_4(\theta,\phi)d\phi. \end{aligned}$$

K_{B1} can be explicitly computed, but integral $\int K_{B1}(\theta)$ is difficult to see if it has a closed form. Therefore this case will be further estimated in the final section.

2 $\frac{2}{2\omega_1} \leq \frac{2}{|\omega_2|} \leq \frac{3}{2\omega_1}$ $(\xi_3(\omega) = h_1 + h_2 + g_3)$.

From this condition, we get the integral range of ϕ is

$$0 \leq \phi \leq \cos^{-1}\frac{4}{3}\cot\theta.$$

$$K_{B2}(\theta) := 4\cos\theta\sin\theta\int_{-\cos^{-1}\frac{4}{3}\cot\theta}^{\cos^{-1}\frac{4}{3}\cot\theta} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi)d\phi.$$

$K_{B2}(\theta)$ can be explicitly computed, but integral $\int K_{B2}$ is difficult to see if it has a closed form. Therefore this case will be further estimated in the final section.

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Some Remarks (1) We use the program of Matlab [14] to find the closed forms for some of above integrals. It can be also directly checked that all the indefinite integrals are correct. (2) The variable-precision floating-point arithmetic (VPA) that we use in the program of Matlab is 32 digits, thus the precision of the values for closed forms is accurate up to 10^{-32} error which would not effect our final value.

Upper bounds for all the further estimate cases Recall that for all the closed forms above, their values add up to be negative. In addition, part E is proved to be negative. Therefore our goal in the section is to give upper bounds for all the further estimate cases above and show that the values of the upper bounds are all negative which after all shows the integral (4.2) is negative.

Note that $\xi_3(\omega) = \int_0^\infty t^2 F(t\omega_1) f(t\omega_2) f(t\omega_3) dt$ and $F(t\omega_1) = F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t) + F_{13}^{\omega_1}(t) + F_{14}^{\omega_1}(t)$, also recall that for all t, ω, $F_{11}^{\omega_1}(t), F_{12}^{\omega_1}(t)$ are negative, and $F_{13}^{\omega_1}(t), F_{14}^{\omega_1}(t), f(t\omega_2), f(t\omega_3)$ are positive. In previous sections, we have shown that for those further estimate cases, it is difficult to see if they have closed forms. Therefore the ideas for these remaining cases are to combine the integrals and split the combined integrals into positive and negative integrals. Finally, we are able to find upper bounds for these negative and positive integrals and show that these upper bounds have closed forms. **Case 2 and case 4** Recall

that in case 2, the integral (4.10) is

$$\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_{\phi\in D} \xi_3(\omega) d\phi d\theta.$$

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In case 4, the integral (4.11) is

$$\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_{\phi \in D^c} \xi_3(\omega) d\phi d\theta.$$

We now separate $\xi_3(\omega)$ into negative part and positive part i.e.

$$\textit{(Negative)} \quad \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_0^{2\pi} \int_0^{2\sqrt{3}} t^2 \left[F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t)\right] f^{\omega_2}(t) f^{\omega_3}(t) dt d\phi d\theta,$$

$$\textit{(Positive)} \quad \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_0^{2\pi} \int_0^{2\sqrt{3}} t^2 \left[F_{13}^{\omega_1}(t) + F_{14}^{\omega_1}(t)\right] f^{\omega_2}(t) f^{\omega_3}(t) dt d\phi d\theta,$$

where negative part indicates that $t^2 \left[F_{11}^{\omega_1}(t) + F_{12}^{\omega_1}(t)\right] f^{\omega_2}(t) f^{\omega_3}(t) \leq 0$ for all t and positive part indicates $t^2 \left[F_{13}^{\omega_1}(t) + F_{14}^{\omega_1}(t)\right] f^{\omega_2}(t) f^{\omega_3}(t) \geq 0$ for all t.

Negative part

Since in case 2 and case 4 the three functions $F, f^{\omega_2}, f^{\omega_3}$ all vanish after $F_{12}^{\omega_1}$ so that the negative part becomes

$$\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_0^{2\pi} h_1(\theta, \phi) + h_2(\theta, \phi) d\phi d\theta.$$

Also this integral has closed form

$$\int \cos\theta \sin\theta \int_0^{2\pi} h_1(\theta, \phi) + h_2(\theta, \phi) d\phi d\theta$$
$$= \left[\frac{7\pi}{32} - \frac{9}{64}\tan\frac{\theta}{2} - \frac{280\pi - 31}{640}\tan^2\frac{\theta}{2} + \frac{7\pi}{32}\tan^4\frac{\theta}{2} + \frac{9}{64}\tan^5\frac{\theta}{2} - \frac{31}{1920}\right] / (\tan^2\frac{\theta}{2} - 1)^3$$
$$- \frac{9\tanh^{-1}(\tan\frac{\theta}{2})}{64}.$$

Plugging in the exact integral range, we thus obtain

$$\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta \sin\theta \int_0^{2\pi} h_1(\theta, \phi) + h_2(\theta, \phi) d\phi d\theta \approx -0.0607.$$

Thus the negative part of case 2 +case 4 is about -0.0607.

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Positive part For positive part, we are unable to get the exact value. Instead, we will find an upper bound for the positive part and show that the upper bound has a closed form. Recall that in case 2, and part A' and part B'1 of case 4, the function $t^2F(t\omega_1)f(t\omega_2)f(t\omega_3)$ vanishes after $F_{13}^{\omega_1}$ and they are

$$\textit{Case 2} \qquad \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta\sin\theta \int_D h_3(\theta,\phi) + h_4(\theta,\phi)d\phi d\theta,$$

$$\textit{Case 4,A'} \qquad \int_{\pi/4}^{\cot^{-1}\frac{3}{4}} \cos\theta\sin\theta \int_{D^c} h_3(\theta,\phi) + g_4(\theta,\phi)d\phi d\theta,$$

$$\textit{Case 4,B'1} \qquad \int_{\cot^{-1}\frac{3}{4}}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta\sin\theta \int_R h_3(\theta,\phi) + g_4(\theta,\phi)d\phi d\theta,$$

where R is the integral range of ϕ in B'1 of case 4.

Notice that in part B'2 of case 4, the function $t^2F(t\omega_1)f(t\omega_2)f(t\omega_3)$ vanishes before $F_{14}^{\omega_1}$ and the integral is

$$\textit{Case 4,B'2} \qquad \int_{\cot^{-1}\frac{3}{4}}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta\sin\theta \int_{R^c} g_3(\theta,\phi)d\phi d\theta.$$

To obtain an upper bound, we introduce a function which is a linear extension of $F_{13}^{\omega_1}$.

$$\tilde{F}_{13}^{\omega_1}(t) = \frac{1}{2}(t\omega_1 - 1).$$

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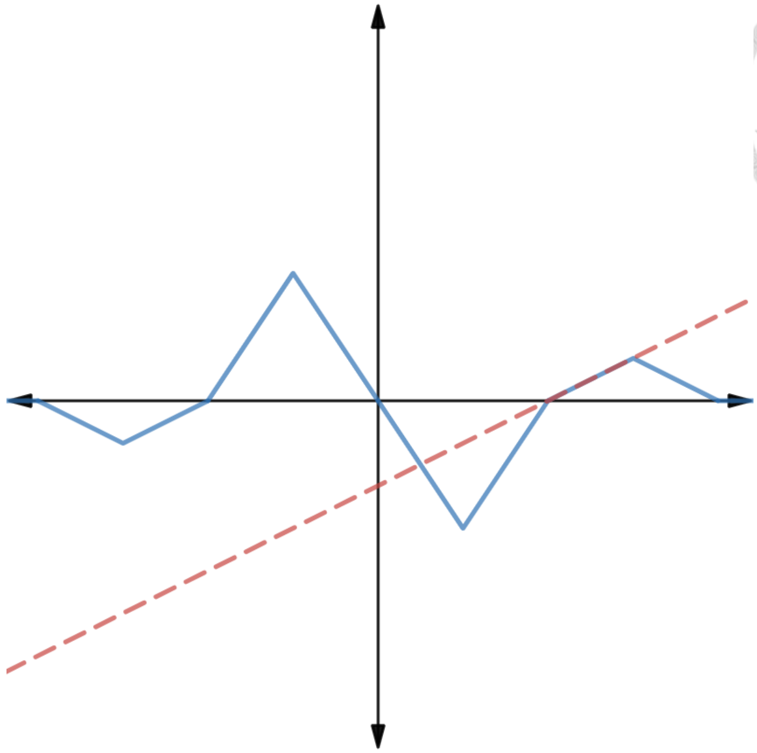

Where the dotted line is $\tilde{F}_{13}^{\omega_1}(t) = \frac{1}{2}(t\omega_1 - 1)$.

Now for case 2

$$
\begin{aligned}
h_3(\theta,\phi) + h_4(\theta,\phi) &= \int_{2/2\cos\theta}^{3/2\cos\theta} t^2 F_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) dt + \int_{3/2\cos\theta}^{4/2\cos\theta} t^2 F_{14}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) dt \\
&\leq \int_{2/2\cos\theta}^{3/2\cos\theta} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) dt + \int_{3/2\cos\theta}^{4/2\cos\theta} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) dt \\
&\leq \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t) dt.
\end{aligned}
$$

The last inequality comes from that fact that F vanishes before f^{ω_2} in case 2, i.e.

$$
\frac{4}{2\cos\theta} \leq \frac{4}{2\sin\theta\cos\phi}.
$$

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For case 4A and 4B'1

$$
\begin{aligned}
h_3(\theta,\phi)+g_4(\theta,\phi) &= \int_{2/2\cos\theta}^{3/2\cos\theta} t^2 F_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt + \int_{3/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 F_{14}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt \\
&\le \int_{2/2\cos\theta}^{3/2\cos\theta} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt + \int_{3/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt \\
&= \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dt.
\end{aligned}
$$

For the case 4B'2, we note that

$$
\begin{aligned}
g_3(\theta,\phi) &= \int_{2/2\cos\theta}^{3/2\sin\theta\cos\phi} t^2 F_{13}^{\omega_1}(t) f(t\omega_2) f(t\omega_3)dt \\
&\le \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f(t\omega_2) f(t\omega_3)dt.
\end{aligned}
$$

Therefore combining all the inequalities above to get an upper bound which is

$$
\begin{aligned}
&\int \cos\theta\sin\theta \int_0^{2\pi} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dtd\phi d\theta \\
=&(\tan^6\frac{\theta}{2}\frac{\pi+2}{6} - \tan^4\frac{\theta}{2}\frac{10\pi+29}{30} + \tan^2\frac{\theta}{2}\frac{15\pi+89}{90} + \tan\frac{\theta}{2}(\frac{2\log(\sqrt{2}+1)}{3} + \frac{2\sqrt{2}+4}{15}) \\
&+ \tan^5\frac{\theta}{2}(2\log(\sqrt{2}+1) + \frac{2\sqrt{2}+4}{5}) - \tan^3\frac{\theta}{2}(2\log(\sqrt{2}+1) + \frac{4\sqrt{2}+9}{10}) \\
&- \tan^7\frac{\theta}{2}(\frac{2\log(\sqrt{2}+1)}{3} + \frac{2\sqrt{2}}{15} + \frac{1}{6}) - \frac{1}{3})/(\tan^2\frac{\theta}{2} - 3\tan^4\frac{\theta}{2} + 3\tan^6\frac{\theta}{2} - \tan^8\frac{\theta}{2}) \\
&- (4\log(\tan\frac{\theta}{2}))/3 - \tanh^{-1}(\frac{6391}{9\tan\frac{\theta}{2}+240)} - \frac{80}{3})/10 + \tan\frac{\theta}{2}(\frac{2\log(\sqrt{2}+1)}{3} + \frac{2\sqrt{2}+4}{15} \\
&+ \frac{\tan\frac{\theta}{2}^2}{3}.
\end{aligned}
$$

Plugging in the exact integral range, we thus obtain

$$
\int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \cos\theta\sin\theta \int_0^{2\pi} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2 \tilde{F}_{13}^{\omega_1}(t) f^{\omega_2}(t) f^{\omega_3}(t)dtd\phi d\theta \approx 0.08718.
$$

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Thus the positive part of case 2 +case 4 is bounded by 0.08717. **Part F in case 3** Part F is

$$\int_{\cot^{-1}\frac{2}{4}}^{\cot^{-1}\frac{2}{4\sqrt{2}}} 4\sin\theta\cos\theta \int_{-\cos^{-1}(2\cot\theta)}^{\cos^{-1}(2\cot\theta)} h_1(\theta,\phi) + g_2(\theta,\phi) d\phi d\theta$$
$$\int_{\cot^{-1}\frac{2}{4}}^{\cot^{-1}\frac{2}{4\sqrt{2}}} 4\sin\theta\cos\theta(\int_{\cos^{-1}(2\cot\theta)}^{\pi/4} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi) d\phi d\theta$$
$$+ \int_{\cot^{-1}\frac{2}{4}}^{\cot^{-1}\frac{2}{4\sqrt{2}}} \int_{-\pi/4}^{-\cos^{-1}(2\cot\theta)} h_1(\theta,\phi) + h_2(\theta,\phi) + g_3(\theta,\phi) d\phi) d\theta,$$

where h_1, h_2, g_2 represent the negative part and g_3 represents the positive part.

Negative part of F We first notice that the negative part of F is bounded by

$$\int 4\cos\theta\sin\theta \int_{-\pi/4}^{\pi/4} h_1(\theta,\phi) d\phi d\theta$$
$$= \left[\frac{3\pi}{64} - \frac{3\tan\frac{\theta}{2}}{160} - \tan^2\frac{\theta}{2}(\frac{3\pi}{32} - \frac{1}{256}) + \frac{3\pi\tan^4\frac{\theta}{2}}{64} + \frac{3\tan^5\frac{\theta}{2}}{160} - \frac{1}{768}\right] /(\tan^2\frac{\theta}{2} - 1)^3 - \frac{3\tanh^{-1}(\tan\frac{\theta}{2})}{160}.$$

Plugging in the exact integral range, we thus obtain

$$\int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\pi/4}^{\pi/4} h_1(\theta,\phi) d\phi d\theta$$
$$\approx -0.026.$$

Thus the negative part of F is bounded by -0.026. **Positive part of F** The positive part of F is

$$\int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\sin\theta\cos\theta \left(\int_{\cos^{-1}(2\cot\theta)}^{\pi/4} g_3(\theta,\phi) d\phi + \int_{-\pi/4}^{-\cos^{-1}(2\cot\theta)} g_3(\theta,\phi) d\phi \right) d\theta \tag{4.12}$$

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$$\leq \int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\sin\theta\cos\theta\Big(\int_{\cos^{-1}(2\cot\theta)}^{\pi/4}\int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi$$
$$+\int_{-\pi/4}^{-\cos^{-1}(2\cot\theta)}\int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi\Big)d\theta. \tag{4.13}$$

But we observe that

$$\int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\sin\theta\cos\theta\int_{-\cos^{-1}(2\cot\theta)}^{\cos^{-1}(2\cot\theta)}\int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi d\theta \geq 0, \tag{4.14}$$

because when $\theta \in [\cot^{-1}(\frac{1}{2}), \cot^{-1}(\frac{1}{2\sqrt{2}})]$, $\phi \in [-\cos^{-1}(2\cot\theta), \cos^{-1}(2\cot\theta)]$,

$$2/2\cos\theta > 4/2\sin\theta\cos\phi,$$

and $\tilde{F}_{13}^{\omega_1}(t) \leq 0$ when $t \leq 2/2\cos\theta$.

Hence (4.12) is bounded by (4.13)+(4.14) which is

$$\int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\sin\theta\cos\theta\int_{-\pi/4}^{\pi/4}\int_{2/2\cos\theta}^{4/2\cos\theta\sin\theta} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi d\theta.$$

It has closed form

$$\int 4\sin\theta\cos\theta\int_{-\pi/4}^{\pi/4}\int_{2/2\cos\theta}^{4/2\cos\theta\sin\theta} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi d\theta$$
$$=8\tan\frac{\theta}{2}\frac{\log(\sqrt{2}+1)+\sqrt{2}}{3} - \frac{32\log(\tan\frac{\theta}{2})}{3}$$
$$+\left[\tan^2\frac{\theta}{2}\frac{\pi+16}{6} + 8(\tan\frac{\theta}{2} - \tan^3\frac{\theta}{2})\frac{\log(\sqrt{2}+1)+\sqrt{2}}{3} - \frac{8}{3}\right])/(\tan^2\frac{\theta}{2} - \tan^4\frac{\theta}{2})$$
$$+\frac{8}{3}\tan^2\frac{\theta}{2}.$$

Plugging in the exact integral range, we thus obtain

$$\int_{\cot^{-1}(\frac{1}{2})}^{\cot^{-1}(\frac{1}{2\sqrt{2}})} 4\sin\theta\cos\theta\int_{-\pi/4}^{\pi/4}\int_{2/2\cos\theta}^{4/2\cos\theta\sin\theta} t^2\tilde{F}_{13}^{\omega_1}(t)dtd\phi d\theta$$
$$\approx 0.0064.$$

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Thus the positive part of F is bounded by 0.0064. **part G in case 3** Part G is

$$\int_{\cot^{-1}\frac{3}{4}}^{\cot^{-1}\frac{3}{4\sqrt{2}}} 4\sin\theta\cos\theta \int_{-\cos(\frac{3}{4}\cot\theta)}^{\cos(\frac{3}{4}\cot\theta)} h_1(\theta,\phi)+h_2(\theta,\phi)+g_3(\theta,\phi)d\theta$$

$$\int_{\cot^{-1}\frac{3}{4}}^{\cot^{-1}\frac{3}{4\sqrt{2}}} 4\sin\theta\cos\theta(\int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+g_4(\theta,\phi)d\phi$$

$$+\int_{-\pi/4}^{-\cos^{-1}(\frac{3}{4}\cot\theta)} h_1(\theta,\phi)+h_2(\theta,\phi)+h_3(\theta,\phi)+g_4(\theta,\phi)d\phi)d\theta,$$

where h_1, h_2 are negative and h_3, g_3, g_4 are positive. **Negative part of G** The negative part of G is

$$\int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\pi/4}^{\pi/4} h_1(\theta,\phi)+h_2(\theta,\phi)d\phi d\theta$$

$$=\left(\frac{7\pi}{32}-\frac{9\tan\frac{\theta}{2}}{64}-\tan^2\frac{\theta}{2}(\frac{7\pi}{16}-\frac{31}{640})+\frac{7\pi\tan^4\frac{\theta}{2}}{32}+\frac{9\tan^5\frac{\theta}{2}}{64}-\frac{31}{1920}\right)/(\tan^2\frac{\theta}{2}-1)^3$$

$$-\frac{9\tanh^{-1}(\tan\frac{\theta}{2})}{64}$$

$$\approx -0.0694.$$

Thus the negative part of G is bounded by -0.0694. **Positive part of G** The positive part of G can be split into two terms.

$$\textit{(G1)} \int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\cos^{-1}(\frac{3}{4}\cot\theta)}^{\cos^{-1}(\frac{3}{4}\cot\theta)} g_3(\theta,\phi)d\phi d\theta$$

$$+\textit{(G2)} \int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta(\int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} h_3(\theta,\phi)+g_4(\theta,\phi)d\phi+$$

$$+\int_{-\pi/4}^{-\cos^{-1}(\frac{3}{4}\cot\theta)} h_3(\theta,\phi)+g_4(\theta,\phi)d\phi)d\theta.$$

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Now

$$\textit{(G1)} \int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\cos^{-1}(\frac{3}{4}\cot\theta)}^{\cos^{-1}(\frac{3}{4}\cot\theta)} g_3(\theta,\phi)d\phi$$

$$\leq \int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\cos^{-1}(\frac{3}{4}\cot\theta)}^{\cos^{-1}(\frac{3}{4}\cot\theta)} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi, \tag{4.15}$$

and

$$\textit{(G2)} \int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta(\int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} h_3(\theta,\phi) + g_4(\theta,\phi)d\phi$$

$$+ \int_{-\pi/4}^{-\cos^{-1}(\frac{3}{4}\cot\theta)} h_3(\theta,\phi) + g_4(\theta,\phi)d\phi)$$

is bounded by

$$\int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta(\int_{\cos^{-1}(\frac{3}{4}\cot\theta)}^{\pi/4} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi$$

$$+ \int_{-\pi/4}^{-\cos^{-1}(\frac{3}{4}\cot\theta)} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi) \tag{4.16}$$

Hence the positive part of G is bounded by (4.15)+(4.16), which has closed form

$$\int 4\cos\theta\sin\theta \int_{-\pi/4}^{\pi/4} \int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi d\theta$$

$$=[\tan^6\frac{\theta}{2}\frac{\pi+2}{6} - \tan^4\frac{\theta}{2}\frac{10\pi+29}{30} + \tan^2\frac{\theta}{2}\frac{15\pi+89}{90} + \tan\frac{\theta}{2}\frac{10\log(\sqrt{2}+1)+2\sqrt{2}+4}{15}$$

$$+\tan^5\frac{\theta}{2}(2\log(\sqrt{2}+1) + \frac{2\sqrt{2}+4}{5}) - \tan^3\frac{\theta}{2}(2\log(\sqrt{2}+1) + \frac{+4\sqrt{2}+9}{10})$$

$$-\tan^7\frac{\theta}{2}(\frac{2\log(\sqrt{2}+1)}{3} + \frac{2\sqrt{2}}{15} + \frac{1}{6}) - \frac{1}{3}]/\left[\tan^2\frac{\theta}{2} - 3\tan^4\frac{\theta}{2} + 3\tan^6\frac{\theta}{2} - \tan^8\frac{\theta}{2}\right]$$

$$-\frac{4\log(\tan\frac{\theta}{2})}{3} - \frac{\tanh\left(\frac{6391}{9\tan\frac{\theta}{2}+240} - \frac{80}{3}\right)}{10} + \tan\frac{\theta}{2}\left(\frac{2\log(\sqrt{2}+1)}{3} + \frac{2\sqrt{2}+4}{15}\right) + \frac{\tan^2\frac{\theta}{2}}{3}.$$

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Plugging in the exact integral range, we thus obtain

$$\int_{\cot^{-1}(\frac{3}{4})}^{\cot^{-1}(\frac{3}{4\sqrt{2}})} 4\cos\theta\sin\theta \int_{-\pi/4}^{\pi/4}\int_{2/2\cos\theta}^{4/2\sin\theta\cos\phi} t^2\tilde{F}_{13}^{\omega_1}(t)f^{\omega_2}(t)f^{\omega_3}(t)dtd\phi d\theta$$
$$\approx 0.0139.$$

Therefore the positive part of G is bounded by 0.0139. **Altogether** The sum of all further estimate cases is

$$\textit{(Negative part of case 2+4)} + \textit{(Positive part of case 2+4)} + \textit{(Negative part of F)}$$
$$+ \textit{(Positive part of F)} + \textit{(Negative part of G)} + \textit{(Positive part of G)}$$
$$\leq -0.0607 + 0.08718 + (-0.026)$$
$$+ 0.0064 + (-0.0694) + 0.0139 < 0.$$

Remark 5. It is very likely to extend the ideas to all dimensions $n \geq 4$ and show the correspondent integral is negative.

5 General Calderon-Zygmund operators and sharp A2 bound

Introduction

Theorem 5.1. For any Calderó-Zygmund operator T on $\mathbb{R}^d$, any $\omega \in A_2$, and $f \in L^2(\omega)$, we have

$$\|Tf\|_{L^2(\omega)} \leq C_T[\omega]_{A_2}\|f\|_{L^2(\omega)}.$$

The proof will proceed via the following steps, in the same order:

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- Reduction to dyadic shift operators: every Calder*ó*n-Zygmund operator T has a representation in terms of these simpler operators, and hence it suffices to prove a similar claim for every dyadic shift S in place of T.

- Reduction to testing conditions: in order to have full norm inequality

$$\|Sf\|_{L^2(\omega)} \leq C_S[\omega]_{A_2} \|f\|_{L^2(\omega)},$$

  it suffices to have such an inequality for special testing functions only:

$$\left\|S(1_Q\omega^{-1})\right\|_{L^2(\omega)} \leq C_S[\omega]_{A_2} \left\|1_Q\omega^{-1}\right\|_{L^2(\omega)},$$

$$\|S^*(1_Q\omega)\|_{L^2(\omega^{-1})} \leq C_{S^*}[\omega]_{A_2} \|1_Q\omega\|_{L^2(\omega^{-1})}.$$

- Verification of the testing conditions for S.

In the original proof of this theorem, in Summer 2010, the two reductions were done in different order: the (quite complicated) reduction to testing functions was obtain for general Caler*ó*n-Zygmund operators by P*é*rez-Treil-Volberg [1]; Hyt*ö*nen's completion of proof [6]

Preliminaries The standard (or reference) system of dyadic cubes is

$$\mathscr{D}^0 := \{2^{-k}([0,1)^d + m) :\in \mathbb{Z}, m \in \mathbb{Z}^d\}.$$

We will need several dyadic systems, obtained by translating the reference system as follows. Let $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}}$ and

$$I \dot{+} \omega := I + \Sigma_{j:2^{-j}<\ell(I)} 2^{-j}\omega_j.$$

Then

$$\mathscr{D}^\omega := \{I \dot{+} \omega : I \in \mathscr{D}^0\},$$

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and it is straightforward to check that $\mathscr{D}^\omega$ inherits the important nestedness property of $\mathscr{D}^0$: if $I, J \in \mathscr{D}^\omega$, then $I \cap J \in \{I, J, \emptyset\}$. When the particular ω is unimportant, then notation $\mathscr{D}$ is sometimes used for a generic dyadic system.

Haar function Any given dyadic system $\mathscr{D}$ has a nautral function system associated to do it: the Haar functions. In one dimension, there are two Haar functions associated with an interval I. the non-cancellative $h_I^0 := |I|^{-1/2}1_I$ and the cancellative $h_I^1 := |I|^{-1/2}(1_{I_l} - 1_{I_r})$, where I_l and I_r are the left and right halves of I. In d dimensions, the Haar functions on a cubes $I = I_1 \times \cdots \times I_d$ are formed of all products of the one-dimensional Haar functions:

$$h_I^\eta(x) = h_{I_1 \times \cdots \times I_d}^{(\eta_1, \cdots, \eta_d)}(x_1, \cdots, x_d) := \prod_{i=1}^d h_{I_i}^{\eta_i}(x_i).$$

The non-cancellative $h_I^0 = |I|^{1-/2}1_I$ has the same formula as in $d = 1$. All other $2^d - 1$ Haar functions h_I^η with $\eta \in \{0,1\}^d \backslash \{0\}$ are cancellative, i.e., satisfy $\int h_I^\eta = 0$, since they are cancellative in at least one coordinate direction.

For a fixed $\mathscr{D}$, all the cancellative Haar functions h_I^η, $I \in \mathscr{D}$ and $\eta \in \{0,1\}^d \backslash \{0\}$, form an othonormal basis of $L^2(\mathbb{R}^d)$. Hence any function $f \in L^2(\mathbb{R}^d)$ has the othogonal expression

$$f = \sum_{I \in \mathscr{D}} \sum_{\eta \in \{0,1\}^d \backslash \{0\}} \langle f, h_I^\eta \rangle h_i^\eta.$$

Since the different η's seldom play any major role, this will be often abbreviated (with slight abuse of language) simply as

$$f = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle h_I,$$

and the summation over η is understood implicitly. **Dyadic shift** A dyadic shift with parameters $i, j \in \mathbb{N}$ is an operator of the form where h_I is a Haar

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function on I (similarly h_J), and the A_{IJK} are coefficients with

$$|a_{IJK}| \leq \frac{\sqrt{|I||J|}}{|K|}.$$

It is also required that all subshifts

$$S_{\mathscr{Q}} = \sum_{K \in \mathscr{Q}}, \ \mathscr{Q} \subset \mathscr{D},$$

maps $S_{\mathscr{Q}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with norm at most one.

The shift is called cancellative, if all the h_I and h_J are cancellative; otherwise, it is called non-cancellative.

The notation A_K indicates an "average operator" on K. Indeed, from the normalization of the Haar functions it follows that

$$|A_K f| \leq 1_K \int_K |f|$$

pointwise.

For cancellative shifts, the L^2 boundedness is automatic from the other conditions. This is a consequence of following facts:

- The pointwise bound for each A_K implies that $\|A_K f\|_{L^p} \leq \|f\|_{L^p}$ for all $p \in [1, \infty]$; in particular, thes components of S are uniformly bounded on L^2 with norm one. (This first point is true even in the non-cancellative case.)
- Let $\mathbb{D}_K^i$ denote the othogonal projection of L^2 onto $\text{span}\{h_I : I \subset K, \ell(I) = 2^{-i}\ell(K)\}$. When i is fixed, it follows readily that any two $\mathbb{D}_K^i$ are othogonal to each other. (This depend on the use of cancellative h_I.) Moreover, we

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have $A_K = \mathbb{D}^j_K A_K \mathbb{D}^i_K$. Then the boundedness of S follows from two applications of Pythagoras's theorem with the uniformly boundedness of the A_K in between.

A prime example of a non-cancellative shift (and the only one we need) is the dyadic paraproduct

$$\prod_b f = \sum_{K\in\mathscr{D}} \langle b, h_K\rangle \langle f\rangle_K h_K = \sum_{K\in\mathscr{D}} |K|^{-1/2}\langle b, k_K\rangle \cdot \langle f, h^0_K\rangle h_K,$$

where $b \in \mathrm{BMO}_d$ (the dyadic BMO space) and h_K is a cancellative Haar funciton. This is a dyadic shift with parameter $(i,j) = (0,0)$, where $a_{IJK} = |K|^{-1/2}\langle b, h_K\rangle$ for $I = J = K$. The L^2 boundedness of the paraproduct, if and only if $b \in \mathrm{BMO}_d$, is part of the classical theroy. Actually, to ensure the normalization condition of the shift, it should be further require that $\|b\|_{\mathrm{BMO}_d} \le 1$.

Random dyadic systems; good and bad cubes We obtain a notion of random dyadic systems by equipping the parameter set $\Omega := (\{0,1\}^d)^{\mathbf{Z}}$ with the natural probability measure: each components are independent of each other.

Let $\phi : [0,1] \to [0,1]$ be a fixed modulus of continuity: a strictly increasing function with $\phi(0) = 0$, $\phi(1) = 1$, and $t \mapsto \frac{\phi(t)}{t}$ decreasing ($\frac{\phi(1)}{1} = 1$ hence $\phi(t) \ge t$ for all $t \in [0,1]$) with $\lim_{t\to 0} \phi(t)/t = \infty$. We further require the Dini condition

$$\int_0^1 \phi(t)\frac{dt}{t} < \infty.$$

Main examples include $\phi(t) = t^\gamma$ with $\gamma \in (0,1)$ and

$$\phi(t) = (1 + \frac{1}{\gamma}\log\frac{1}{t})^{-\gamma}, \gamma > 1.$$

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We also fix a (large) parameter $r \in \mathbb{N}$. (How large, will be specified shortly.) A cube $I \in \mathcal{D}_\omega$ is called nad if there exists $J \in \mathcal{D}_\omega$ such that $\ell(J) \geq 2^r \ell(I)$ and

$$\operatorname{dist}(I, \partial J) \leq \phi(\frac{\ell(I)}{\ell(J)})\ell(J) :$$

roughly, I is relatively close to the boundary of a much bigger cube.

Remark 6. This definition of good cubes goes back to Nazarov-Treil-Volberg in the context of singular integrals with respect to non-doubling measures. They used the modulus of continuity $\phi(t) = t^\gamma$, where γ was chosen to depend on the dimension and the Hölder exponent of the Calderón-Zygmund kernel via

$$\gamma = \frac{\alpha}{2(d+\alpha)}.$$

This choice become "canonical" in the subsequent literature, including the original proof of the A_2 theorem. However, other choices can also be made, as we do here.

We make some basic probablistic observations related to badness. Let $I \in \mathscr{D}^0$ be a refenece interval. The position of the translated interval

$$I \dot{+} \omega = I \dot{+} \sum_{j: 2^{-j} < \ell(I)} 2^{-j} \omega_j,$$

by definition, depends only on ω_j for $2^{-j} < \ell(I)$. On the other hand, the badness of $I \dot{+} \omega$ depends on its relative position with respect to the bigger intervals

$$J \dot{+} \omega = J + \sum_{j: 2^{-j} < \ell(I)} 2^{-j} \omega_j + \sum_{j: \ell(I) \leq 2^{-j} < \ell(I)} 2^{-j} \omega_j.$$

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The same translation component $\sum_{j:2^{-j}\ell(I)} 2^{-j}\omega_j$ appears in both $I\dot{+}\omega$ and $J\dot{+}\omega$, and so does not affect the relative position on these intervals. Thus this relative position, and hence the badness of I, depends only on ω_j for $2^{-j} \geq \ell(I)$. In particular:

Lemma 3. For $I \in \mathscr{D}^0$, the position and the badness of $I\dot{+}\omega$ are independent random variables.

Another observation is the following: by symmetry and the fact that the condition of badness only involves relative position and size of different cubes, it readily follows that the probability of a particular cube $I\dot{+}\omega$ being bad is equal for all cubes $I \in \mathscr{D}^0$:

$$\mathbb{P}_\omega(I\dot{+}\omega\text{bad}) = \pi_{\text{bad}} = \pi_{\text{bad}}(r, d, \phi).$$

The final observation concerns the value of this probability:

Lemma 4. We have

$$\pi_{\text{bad}} \leq 8d \int_0^{2^{-r}} \phi(t)\frac{dt}{t};$$

in particular, $\pi_{\text{bad}} < 1$ if $r = r(d, \phi)$ chosen large enough.

With $r = r(d, \phi)$ chosen like this, we then have $\pi_{\text{good}} := 1 - \pi_{\text{bad}} > 0$, namely, good situations have positive probability.

Proof. Observe that in the definition of badness, we only need to consider those J with $I \subset J$. Namely, if I is closed to the boundary of some bigger J, we can always find another dyadic J' of the same size as J which contains I, and

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then I will also be close to the boundary of J'. Hence we need to consider the relative position of I with respect to each $J \supset I$ with $\ell(J) = 2^k \ell(I)$ and $k = r, r+1, \ldots$. For a fixed k, this relative position is determined by

$$\sum_{j:\ell(I) \leq 2^{-j} < 2^k \ell(I)} 2^{-j} \omega_j,$$

which has 2^{kd} different values with equal probability. These correspond to the subcubes of I of size $\ell(I)$.

Now bad position of I are those which are within distance $\phi(\ell(I)/\ell(J)) \cdot \ell(J)$ from the boundary. Since the possible position of the subcubes are discrete, being integer multiples of $\ell(I)$, , the effective bad boundary region has depth

$$\begin{aligned}\left[\phi\left(\frac{\ell(I)}{\ell(J)}\right)\frac{\ell(J)}{\ell(I)}\right]\ell(I) &\leq \left(\phi\left(\frac{\ell(I)}{\ell(J)}\right)\frac{\ell(J)}{\ell(I)} + 1\right)\ell(I) \\ &= \ell(J)\left(\phi\left(\frac{\ell(I)}{\ell(J)}\right) + \frac{\ell(I)}{\ell(J)}\right) \leq 2\ell(J)\phi\left(\frac{\ell(I)}{\ell(J)}\right),\end{aligned}$$

by using that $t \leq \phi(t)$.

The good region is the cube inside J, whose side-length is $\ell(J)$ minus twice the depth of the bad boundary region:

$$\ell(J) - 2\left[\phi\left(\frac{\ell(I)}{\ell(J)}\right)\frac{\ell(J)}{\ell(I)}\right]\ell(I) \geq \ell(J) - 4\ell(J)\phi\left(\frac{\ell(I)}{\ell(J)}\right).$$

Hence the volume of the bad region is

$$\begin{aligned}|J| - \left(\ell(J) - 2\left[\phi\left(\frac{\ell(I)}{\ell(J)}\right)\frac{\ell(J)}{\ell(I)}\right]\ell(I)\right)^d &\leq |J|\left(1 - \left(1 - 4\phi\left(\frac{\ell(I)}{\ell(J)}\right)\right)^d\right) \\ &\leq |J| \cdot 4d\phi\left(\frac{\ell(I)}{\ell(J)}\right)\end{aligned}$$

by the elementary inequality $(1-\alpha)^d \geq 1 - \alpha d$ for $\alpha \in [0,1]$. (We assume that r is at least so large that $4\phi(2^{-r}) \leq 1$.)

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So the fraction of the bad region of the total volume is at most $4d\phi(\ell(I)/\ell(J)) = 4d\phi(2^{-k})$ for a fixed $k = r, r+1, \ldots$. This gives the final estimate

$$\begin{aligned}\mathbb{P}_\omega(I\dot{+}\omega \text{ bad}) &\le \sum_{k=r}^{\infty} 4d\phi(2^{-k}) = \sum_{k=r}^{\infty} 8d\frac{\phi(2^{-k})}{2^{-k}}2^{-k-1} \\ &\le \sum_{k=r}^{\infty} 8d\int_{2^{-k-1}}^{2^{-k}} \frac{\phi(t)}{t}dt = 8d\int_0^{2^{-r}} \frac{\phi(t)}{t}dt,\end{aligned}$$

where we used that $\phi(t)/t$ is decreasing in the last inequality. □

The dyadic representation theorem Let T be a Calderón-Zygmund operator on $\mathbb{R}^d$. That is, it acts on a suitable dense subspace of functions in $L^2(\mathbb{R}^d)$ (for the present purposes, this class should at least contain the indicators of cubes in $\mathbb{R}^d$) and has the kernel representation

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \qquad x \notin \operatorname{supp} f.$$

Moreover, the kernel should satisfy the standard estimates, which we here assume in a slightly more general form than usual, involving another modulus of continuity ψ, like the one considered above:

$$\begin{aligned}|K(x,y)| &\le \frac{C_0}{|x-y|^d}, \\ |K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| &\le \frac{C_\psi}{|x-y|^d}\psi\left(\frac{|x-x'|}{|x-y|}\right)\end{aligned}$$

for all $x, x', y, y' \in \mathbb{R}^d$ with $|x-y| > 2|x-x'|$. Let us denote the smallest admissible constants C_0 and C_ψ by $\|K\|_{CZ_0}$ and $\|K\|_{CZ_\psi}$. The classical standard estimates correspond to the choice $\psi(t) = t^\alpha$, $\alpha \in (0,1]$, in which case we write $\|K\|_{CZ_\alpha}$ for $\|K\|_{CZ_\psi}$.

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We say that T is a bounded Calderón-Zygmund operator, if in addition T :$L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, and we denote its operator norm by $\|T\|_{L^2\to L^2}$.

Let us agree that $||$ stands for the ℓ^∞ norm on $\mathbb{R}^d$, i.e., $|x| := \max_{1\le i\le d}|x_i|$. While the choice of the norm is not particularly important, this choice is slightly more convenient than the usual Euclidean norm when dealing with cubes as we will: e.g., the diameter of a cube in the ℓ^∞ norm is equal to its sidelength $\ell(Q)$.

Let us first formulate the dyadic representation theorem for general moduli of continuity, and then specialize it to the usual standard estimates. Define the following coefficients for $i, j \in \mathbb{N}$:

$$\tau(i,j) := \phi(2^{-\max\{i,j\}})^{-d}\psi(2^{-\max\{i,j\}}\phi(2^{-\max\{i,j\}})^{-1}),$$

if $\min\{i,j\}$=0.

We assume that ϕ and ψ are such, that

$$\sum_{i,j=0}^{\infty}\tau(i,j) \simeq \int_0^1 \frac{1}{\phi(t)^d}\psi\left(\frac{t}{\phi(t)}\right)\frac{dt}{t} + \int_0^1 \Psi\left(\frac{t}{\phi(t)}\right)\frac{dt}{t} < \infty. \tag{5.1}$$

This is the case, in particular, when $\psi(t) = t^\alpha$ (usual standard estimates) and $\phi(t) = (1 + a^{-1}\log t^{-1})^{-\gamma}$; then one checks that

$$\tau(i,j) \lesssim P(\max\{i,j\})2^{-\alpha\max\{i,j\}},$$

$$P(j) = (1+j)^{\gamma(d+\alpha)},$$

which ch clearly satisfies the required convergence. However, it is also possible to treat weaker forms of the standard estimates with a logarithmic modulus $\psi(t) = (1 + a^{-1}\log t^{-1})^{-\alpha}$. . This might be of some interest for applications, but we do not pursue this line any further here.

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Theorem 5.2. Let T T be a bounded Calderón—Zygmund operator with modulus of continuity satisfying the above assumption. Then it has an expansion, say $f, g \in C_c^1(\mathbb{R}^d)$,

$$\langle g\rangle Tf = c \cdot \left(\|T\|_{L^2 \to L^2} + \|K\|_{CZ_\psi}\right) \cdot \mathbb{E}_\omega \sum_{i,j=0}^{\infty} \tau(i,j) \langle g\rangle S_\omega^{ij} f,$$

where c is a dimensional constant and S_ω^{ij} is a dyadic shift of parameters (i, j) on the dyadic system $\mathcal{D}^\omega$; ω; all of them except possibly S_ω^{00} are cancellative.

The first version of this theorem appeared in [6], and another one in [5].]. The present proof is yet another variant of the same argument. It is slightly simpler in terms of the probabilistic tools that are used: no conditional probabilities are needed, although they were important for the original arguments.

In proving this theorem, we do not actually need to employ the full strength of the assumption that $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$; rather it suffices to have the kernel conditions plus the following conditions of the $T1$ theorem of David-Journé:
$|\langle 1_Q\rangle T 1_Q| \leq C_{WBP}|Q|$ (weak boundedness property),
$T1 \in BMO(\mathbb{R}^d)$, and $T^*1 \in BMO(\mathbb{R}^d)$.

Let us denote the smallest C_{WBP} by $\|T\|_{WBP}$. Then we have the following more precise version of the representation:

Theorem 5.3. Let T be a Calderón—Zygmund operator with modulus of continuity satisfying the above assumption. Then it has an expansion, say for

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$f, g \in C_c^1(\mathbb{R}^d)$,

$$\langle g\rangle Tf = c \cdot \Big(\|K\|_{CZ_0} + \|K\|_{CZ_\phi}\Big)\, \mathbb{E}_\omega \sum_{\substack{i,j=0\\ \max\{i,j\}>0}}^{\infty} \tau(i,j)\langle g\rangle S_\omega^{i,j} f \tag{5.2}$$

$$+ c \cdot \left(\|K\|_{CZ_0} + \|T\|_{WBP}\right) \mathbb{E}_\omega \langle g\rangle S_\omega^{00} f + \mathbb{E}_\omega \langle g\rangle \prod_{T1}^{\omega} f + \mathbb{E}_\omega \langle g\rangle (\prod_{T^*1}^{\omega})^* f,$$

where $S_\omega^{i,j}$ is a cancellative dyadic shift of parameters (i,j)) on the dyadic system $\mathcal{D}^\omega$, and $\prod_b^\omega$ is a dyadic paraproduct on the dyadic system $\mathcal{D}^\omega$ associated with the BMO-function $b \in \{T1, T^*1\}$.

Remark 7. Note that $\prod_b^\omega = \|b\|_{BMO} \cdot S_b^\omega$, where $S_b^\omega = \prod_b^\omega / \|b\|_{BMO}$ O is a shift with the correct normalization. Hence, writing everything in terms of normalized shifts, as in Theorem 5.2, we get the factor $\|T1\|_{BMO} \lesssim \|T\|_{L^2 \to L^2} + \|K\|_{CZ\psi}$ in the second-to-last term, and $\langle T^*1,\rangle\, BMO \lesssim \langle T\rangle_{L^2\to L^2} + \langle K\rangle_{CZ_\psi}$ in the last one. The proof will also show that both occurrences of the factor $\langle K\rangle_{CZ_0}$ could be replaced by $\langle T\rangle_{L^2\to L^2}$, giving the statement of Theorem 5.2 (since trivially $\langle T\rangle_{WBP} \le \langle T\rangle_{L^2\to L^2}$).

As a by-product, Theorem 5.2 delivers a proof of the $T1$ theorem: under the above assumptions, the operator T is already bounded on $L^2(\mathbb{R}^d)$.). Namely, all the dyadic shifts S_ω^{ij} are uniformly bounded on $L^2(\mathbb{R}^d)$) by definition, and the convergence condition (5.1) ensures that so is their average representing the operator T. This by-product proof of the $T1$ theorem is not a coincidence, since the proof of Theorem 5.2 and (5.3) was actually inspired by the proof of the $T1$ Theorem for non-doubling measures due to Nazarov-Treil-Volberg[2] and its vector-valued extension [3].

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A key to the proof of the dyadic representation is a random expansion T in terms of Haar functions h_I, where the bad cubes are avoided:

Proposition 1.

$$\langle g, Tf\rangle = \frac{1}{\pi_{\text{good}}}\mathbb{E}_\omega \sum_{I,J\in\mathscr{D}^\omega} 1_{\text{good}}(\text{smaller}\{I,J\})\cdot\langle g, h_J\rangle\langle h_J, Th_I\rangle\langle h_I, f\rangle,$$

where

$$\text{smaller}\{I,J\} := \begin{cases} I \quad if \quad \ell(I)\leq \ell(J) \\ J \quad if \quad \ell(I) > \ell(J) \end{cases}$$

Proof. Recall that Haar functions form a basis

$$f = \sum_{I\in\mathscr{D}^0} \langle f, h_{I\dot{+}\omega}\rangle h_{I\dot{+}\omega}$$

for any fixed $\omega\in\Omega$; and we can also take expectation $\mathbb{E}_\omega$ of both sides of this identity.

Let

$$1_{\text{good}}(I\dot{+}\omega) := \begin{cases} 1, & \text{if } I\dot{+}\omega \text{ is good,} \\ 0, & \text{else} \end{cases}$$

We make use of the above random Haar expansion of f, multiply and divide by

$$\pi_{\text{good}} = \mathbb{P}_\omega(I\dot{+}\omega \text{ good}) = \mathbb{E}_\omega 1_{\text{good}}(I\dot{+}\omega),$$

and use the independence from above Lemma to get:

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$$
\begin{aligned}
\langle g, Tf\rangle &= \mathbb{E}_\omega \sum_I \langle g, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle \\
&= \frac{1}{\pi_{\text{good}}}\sum_I \mathbb{E}_\omega[1_{\text{good}}(I\dotplus\omega)]\mathbb{E}_\omega[\langle g, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle] \\
&= \frac{1}{\pi_{\text{good}}}\mathbb{E}_\omega \sum_I 1_{\text{good}}(I\dotplus\omega)\langle g, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle \\
&= \frac{1}{\pi_{\text{good}}}\mathbb{E}_\omega \sum_{I,J} 1_{\text{good}}(I\dotplus\omega)\langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle
\end{aligned}
$$

On the other hand, using independence again in half of this double sum, we have

$$
\frac{1}{\pi_{\text{good}}}\sum_{\ell(I)>\ell(J)} \mathbb{E}_\omega[1_{\text{good}}(I\dotplus\omega)\langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle]
$$

$$
= \frac{1}{\pi_{\text{good}}}\sum_{\ell(I)>\ell(J)} \mathbb{E}_\omega[1_{\text{good}}(I\dotplus\omega)]\mathbb{E}_\omega[\langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle]
$$

$$
= \mathbb{E}_\omega \sum_{\ell(I)>\ell(J)} \langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle
$$

and hence

$$
\begin{aligned}
\langle g, Tf\rangle =& \frac{1}{\pi_{good}}\mathbb{E}_\omega \sum_{\ell(I)\le\ell(J)} 1_{good}(I\dotplus\omega)\langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle \\
&+ \mathbb{E}_\omega \sum_{\ell(I)>\ell(J)} \langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle.
\end{aligned}
$$

Comparison with the basic identity

$$
\langle g, Tf\rangle = \mathbb{E}_\omega \sum_{I,J} \langle g, h_{J\dotplus\omega}\rangle\langle h_{J\dotplus\omega}, Th_{I\dotplus\omega}\rangle\langle h_{I\dotplus\omega}, f\rangle \tag{5.3}
$$

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shows that

$$\mathbb{E}_\omega \sum_{\ell(I)\leq\ell(J)} \langle g, h_{J\dot{+}\omega}\rangle\langle h_{J\dot{+}\omega}, Th_{I\dot{+}\omega}\rangle\langle h_{I\dot{+}\omega}, f\rangle$$
$$\frac{1}{\pi_{good}}\mathbb{E}_\omega \sum_{\ell(I)\leq\ell(J)} 1_{good}(I\dot{+}\omega)\langle g, h_{J\dot{+}\omega}\rangle\langle h_{J\dot{+}\omega}, Th_{I\dot{+}\omega}\rangle\langle h_{I\dot{+}\omega}, f\rangle$$

Symmetrically, we also have

$$\mathbb{E}_\omega \sum_{\ell(I)>\ell(J)} \langle g, h_{J\dot{+}\omega}\rangle\langle h_{J\dot{+}\omega}, Th_{I\dot{+}\omega}\rangle\langle h_{I\dot{+}\omega}, f\rangle$$
$$\frac{1}{\pi_{good}}\mathbb{E}_\omega \sum_{\ell(I)>\ell(J)} 1_{good}(I\dot{+}\omega)\langle g, h_{J\dot{+}\omega}\rangle\langle h_{J\dot{+}\omega}, Th_{I\dot{+}\omega}\rangle\langle h_{I\dot{+}\omega}, f\rangle,$$

and this completes the proof. $\square$

This is essentially the end of probability in this proof. Henceforth, we can simply concentrate on the summation inside $\mathbb{E}_\omega$, for a fixed value of $\omega \in \Omega$, and manipulate it into the required form. Moreover, we will concentrate on the half of the sum with $\ell(J) \geq \ell(I)$, the other half being handled symmetrically. We further divide this sum into the following parts:

$$\begin{aligned}\sum_{\ell(I)\leq\ell(J)} =&\sigma_{dist>\ell(J)\phi(\ell(I)/\ell(J))} + \sum_{I\subsetneq J} + \sum_{I=J} + \sum_{\substack{dist\leq\ell(J)\phi(\ell(I)/\ell(J))\\ I\cap J=\emptyset}}\\ =:&\sigma_{out} + \sigma_{in} + \sigma_{=} + \sigma_{near}\end{aligned}$$

In order to recognize these series as sums of dyadic shifts, we need to locate, for each pair (I, J) appearing here, a common dyadic ancestor which contains both of them. The existence of such containing cubes, with control on their size, is provided by the following:

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Lemma 5. If $I \in \mathcal{D}$ is good and $J \in \mathcal{D}$ is a disjoint cube with $\ell(J) \geq \ell(I)$, then there exists $K \supset I \cup J$ which satisfies

$$\ell(K) \leq 2^r \ell(I), \text{ if } \operatorname{dist}(I, J) \leq \ell(J)\phi(\frac{\ell(I)}{\ell(J)})$$

$$\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) \leq 2\operatorname{dist}(I, J), \text{ if } \operatorname{dist}(I, J) > \ell(J)\phi(\frac{\ell(I)}{\ell(J)})$$

We need to find the bound of $\ell(K)$.

Proof. Firstly, we have to show that $I \cup J \subset K$ rather than estimate by $\ell(I) + \ell(J) + \operatorname{dist}(I, J) \leq \ell(K)$ since we don't know the location of I, J and K. Let us start with the following initial observation: if $K \in \mathscr{D}$ satisfies $I \subset K$, and $J \subset K^c$, and $\ell(K) \geq 2^r \ell(I)$, then

$$\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) < \operatorname{dist}(I, \partial K) = \operatorname{dist}(I, K^c) \leq \operatorname{dist}(I, J).$$

Case 1: $\operatorname{dist}(I, J) \leq \ell(J)\phi(\frac{\ell(I)}{\ell(J)})$

Choose any K with $\ell(K) \geq 2^r \ell(I)$, and $I \subset K$. Since I is good, we have $\operatorname{dist}(I, K) > \ell(K)\phi(\frac{\ell(I)}{\ell(K)})$, and $\ell(J) < 2^r \ell(I)$. Assume for contradiction that $J \subset K^c$. Then

$$\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) < \operatorname{dist}(I, \partial K) \leq \operatorname{dist}(I, J) \leq \ell(J)\phi(\frac{\ell(I)}{\ell(J)}).$$

Dividing both sides by $\ell(I)$ and recalling that $\frac{\phi(t)}{t}$ is decreasing, this implies that $\ell(K) < \ell(J)$, a contradiction with $\ell(K) \geq 2^r \ell(I) > \ell(J)$. Hence $J \not\subset K^c$, and since $\ell(J) < \ell(K)$, this implies that $J \subset K$. Since for any K with $\ell(K) \geq 2^r \ell(I)$ can contains $I \cup J$, the minimality of K is $\ell(K) \leq 2^r \ell(I)$.

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Case 2: $\mathrm{dist}(I,J) > \ell(J)\phi(\frac{\ell(I)}{\ell(J)})$.

Consider the minimal $K \supset I$ with $\ell(K) \geq 2^r \ell(I)$ and $\mathrm{dist}(I,J) \leq \ell(K)\phi(\frac{\ell(I)}{\ell(K)})$. (Since $\phi(t)/t \to \infty$ as $t \to 0$, this bound hold for all large enough K.) Then (since $\phi(t)/t$ is decreasing) $\ell(K) > \ell(J)$, and by initial observation, $J \not\subset K^c$. (If $J \subset K^c$, then $\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) < \mathrm{dist}(I,\partial K) \leq \mathrm{dist}(I,J) \leq \ell(K)\phi(\frac{\ell(I)}{\ell(K)})$, which is contradiction.) Hence $J \subset K$. By the minimality of K, there holds

$$\frac{1}{2}\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) < \ell(K)/2\phi(\frac{\ell(I)}{\ell(K)/2}) \leq \mathrm{dist}(I,J),$$

and it implies that

$$\ell(K)\phi(\frac{\ell(I)}{\ell(K)}) < \ell(K\phi(\frac{\ell(I)}{\ell(K)/2})) < 2\mathrm{dist}(I,J)$$

,

so the required bound is true in each case.

□

We denote that minimal such K by $I \vee J$, thus

$$I \vee J := \bigcap_{K \supset I \vee J} K$$

Separated cubes,sigma out We reorganize the sum σ_{out} with respect to the new summation variable $K = I \vee J$, as well as the relative size of I and J with respect to K:

$$\sigma_{\mathrm{out}} = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\sum_{K} \sum_{\substack{\mathrm{dist}(I,J) > \ell(J)\phi(\frac{\ell(I)}{\ell(J)}) \\ I \vee J = K \\ \ell(I) = 2^{-i}\ell(K), \ell(J) = 2^{-j}\ell(K)}} .$$

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Note that we can start the summation from 1 instead of 0, since the disjointness of I and J implies that $K = I \vee J$ must be strictly larger than either of I and J. The goal is to identify the quantity in parentheses as a decaying factor times a cancellative averaging operator with parameters (i, j).

Lemma 6. For I and J appearing in σ_{out}, we have

$$|\langle h_J, Th_I\rangle| \lesssim \|K\|_{CZ_\psi} \frac{\sqrt{|I||J|}}{|K|}\phi\left(\frac{\ell(I)}{\ell(K)}\right)^{-d}\psi\left(\frac{\ell(I)}{\ell(K)}\phi\left(\frac{\ell(I)}{\ell(K)}\right)^{-1}\right), \quad K = I \vee J$$

Proof. Using the cancellation of h_I, standard estimates, and Lemma, and lemma

5

$$\begin{aligned}
|\langle h_J, Th_I\rangle| &= |\iint h_J(x)K(x,y)h_I(y)dydx| \\
&= |\iint h_J(x)\left[K(x,y) - K(x,y_I)\right]h_I(y)dydx| \\
&\lesssim \|K\|_{CZ_\psi} \iint |h_J(x)| \frac{1}{dist(I,J)^d}\psi\left(\frac{\ell(I)}{dist(I,J)}\right)|h_I(y)|dydx \\
&= \|K\|_{CZ_\psi} \frac{1}{dist(I,J)^d}\psi\left(\frac{\ell(I)}{dist(I,J)}\right)\|h_J\|_1 \|h_I\|_1 \\
&\lesssim \|K\|_{CZ_\psi} \frac{1}{\ell(K)^d}\phi\left(\frac{\ell(I)}{\ell(K)}\right)^{-d}\psi\left(\frac{\ell(I)}{\ell(K)}\phi\left(\frac{\ell(I)}{\ell(K)}\right)^{-1}\right)\sqrt{|J|}\sqrt{|I|}.
\end{aligned}$$

□

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Lemma 7.

$$\sum_{\substack{\operatorname{dist}(I,J)>\ell(J)\phi(\frac{\ell(I)}{\ell(J)})\\ I\vee J=K\\ \ell(I)=2^{-i}\ell(K),\ell(J)=2^{-j}\ell(K)}} 1_{good}(I)\cdot\langle g,h_J\rangle\langle h_J,Th_I\rangle\langle h_I,f\rangle$$
$$=\|K\|_{CZ_\psi}\,\phi(2^{-i})^{-d}\psi(2^{-i}\phi(2^{-i})^{-1})\langle g,A^{ij}f\rangle,$$

where A^{ij} is a cancellative averaging operator with parameters (i,j).

Proof. By the previous lemma, substituting $\ell(I)/\ell(K)=2^{-i}$,

$$|\langle h_J,Th_I\rangle|\lesssim\|K\|_{CZ_\psi}\frac{\sqrt{|I||J|}}{|K|}\phi\left(2^{-i}\right)^{-d}\psi\left(2^{-i}\phi\left(2^{-i}\right)^{-1}\right),$$

and the first factor is precisely the required size of the coefficients of A^{ij}_K. □

Summarizing, we have

$$\sigma_{out}=\|K\|_{CZ_\psi}\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\phi(2^{-1})^{-d}\psi(2^{-1}\phi(2^{-i})^{-1})\langle g,S^{ij}\rangle.$$

Containe subcubes, sigma in When $I\subsetneq J$, then I is contained in some subcubes of J, which we denote by J_I.

$$\begin{aligned}\langle h_J,Th_I\rangle&=\langle 1_{J_I^c}h_J,Th_I\rangle+\langle 1_{J_I}h_J,Th_I\rangle\\&=\langle 1_{J_I^c}h_J,Th_I\rangle+\langle h_J\rangle_{J_I}\langle 1_{J_I},Th_I\rangle\\&=\langle 1_{J_I^c}(h_J-\langle h_J\rangle_{J_I}),Th_I\rangle+\langle 1,Th_I\rangle,\end{aligned}$$

where we notice that h_J is constant on $J_I\supset I$.

Lemma 8.

$$|\langle 1_{J_I^c}(h_J-\langle h_J\rangle_{J_I}),Th_I\rangle|\lesssim\left(\|K\|_{CZ_\psi}+\|K\|_{CZ_0}\right)\left(\frac{|I|}{|J|}\right)^{1/2}\Psi\left(\frac{\ell(I)}{\ell(J)}\phi\left(\frac{\ell(I)}{\ell(J)}\right)^{-1}\right),$$

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where

$$\Psi(r) := \int_0^r \psi(t) \frac{dt}{t},$$

and $\|K\|_{CZ_0}$ could be alternatively replaced by $\|T\|_{L^2 \to L^2}$.

Proof.

$$|\langle 1_{J_I^c}(h_J - \langle h_J \rangle_{J_I}), Th_I \rangle| \leq 2 \|h_J\|_\infty \int_{J_I^c} |Th_I(x)| dx,$$

where $\|h_J\|_\infty = |J|^{-1/2}$.

Case $\ell(I) \geq 2^{-r}\ell(J)$. We have

$$\begin{aligned}
\int_{J_I^c} |Th_I(x)| dx \leq & \int_{3I \setminus I} | \int K(x,y) h_I(y) dy | dx \\
& + \int_{(3I)^c} | \int \left(K(x,y) - k(x,y_I)\right) h_I(y) dy | dx \\
\lesssim & \|K\|_{CZ_0} \int_{3I \setminus I} \int_I \frac{1}{|x-y|^d} dy dx \, \|h_I\|_\infty \\
& + \|K\|_{CZ_\psi} \int_{(3I)^c} \frac{1}{dist(x,I)^d} \psi \left(\frac{\ell(I)}{dist(x,I)} \right) \|h_I\|_1 \, dx \\
\lesssim & \|K\|_{CZ_0} |I| \, \|h_I\|_\infty + \|K\|_{CZ_\psi} \int_{\ell(I)}^\infty \frac{1}{r^d} \psi \left(\frac{\ell(I)}{r} \right) r^{d-1} dr \, \|h_I\|_1 \\
= & \|K\|_{CZ_0} |I|^{1/2} + \|K\|_{CZ_\psi} \int_0^1 \psi(t) \frac{dt}{t} |I|^{1/2} \\
\lesssim & (\|K\|_{CZ_0} + \|K\|_{CZ_\psi}) |I|^{1/2},
\end{aligned}$$

by Dini condition in the last step.

Alternatively, the part giving the factor $\|K\|_{CZ_0}$ could have been estimated by

$$\int_{3I \setminus I} | \int K(x,y) h_I(y) dy | dx \leq |3I \setminus I|^{1/2} \|Th_I\|_2 \lesssim |I|^{1/2} \|T\|_{L^2 \to L^2}.$$

Case $\ell(I) < 2^{-r}\ell(J)$. Since $I \subset J_I$ is good, we have

$$dist(I, J_I^c) > \ell(J_I) \phi(\frac{\ell(I)}{\ell(J_I)}) \gtrsim \ell(J) \phi(\frac{\ell(I)}{\ell(J)})$$

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and hence

$$\begin{aligned}\int_{J_I^c}|Th_I(x)|dx &\lesssim \|K\|_{CZ_\phi}\int_{J_I^c}\frac{1}{d(x,I)}\psi\left(\frac{\ell(I)}{dist(x,I)}\right)\|h_I\|_1\,dx\\ &\lesssim \|K\|_{CZ_\phi}\int_{\ell(J)\phi(\ell(I)/\ell(J))}\frac{1}{r^d}\psi\left(\frac{\ell(I)}{r}\right)r^{d-1}dr\cdot\|h_I\|_1\\ &= \|K\|_{CZ_\phi}\int_0^{\ell(I)\backslash\ell(J)\cdot\phi(\ell(I)/\ell(J))^{-1}}\psi(t)\frac{dt}{t}\cdot|I|^{1/2}.\end{aligned}$$

□

Now we can organize

$$\sigma'_{in} := \sum_J\sum_{I\subsetneq J}\langle g\rangle h_J\langle 1_{J_I^c}(h_J-\langle h_{J,\rangle}\,J_I)\rangle Th_I\langle h_I\rangle f = \sum_{i=1}^{\infty}\sum_J\sum_{\substack{I\subset J\\ \ell(I=2^{-i}\ell(J))}},$$

and the inner sum is recongnized as

$$(\|K\|_{CZ_0}+\|K\|_{CZ_\phi})\Psi(2^{-i}\phi(2^{-i})^{-1})\langle g\rangle A_J^{i0}f,$$

or with $\|T\|_{L^2\to L^2}$ in place of $\|K\|_{CZ_0}$, for a cancellative e averaging operator of type $(i,0)$.

On the other hand,

$$\begin{aligned}\sigma''_{in} :&= \sum_J\sum_{I\subsetneq J}\langle g\rangle h_J\langle h_{J,\rangle}\,I\langle 1\rangle Th_I\langle h_I\rangle f\\ &= \sum_I\langle\sum_{J\supsetneq I}\langle g\rangle h_Jh_{J,\rangle}\,I\langle 1\rangle Th_I\langle h_I\rangle f\\ &= \sum_I\langle g_{,\rangle}\,I\langle T^*1\rangle h_I\langle h_I\rangle f\\ &= \langle\sum_I\langle g_{,\rangle}\,I\langle T^*1\rangle h_Ih_I\rangle f =: \langle\prod_{T^*1}g\rangle f = \langle g\rangle\prod_{T^*1}^{*}f.\end{aligned}$$

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Here $\prod_{T^*1}$ is the paraproduct, a non-cancellative shift composed of the non-cancellative averaging operators

$$A_I g = \langle T^*1 \rangle h_I \langle g,_{\rangle} I h_I = |I|^{-1/2} \langle T^*1 \rangle h_I \cdot \langle g \rangle h_I^0 h_I$$

of type $(0,0)$.

Summarizing, we have

$$\begin{aligned}\sigma_{in} &= \sigma'_{in} + \sigma''_{in} \\ &= (\|K\|_{CZ_0} + \|K\|_{CZ_\phi}) \sum_{i=1}^{\infty} \Psi(2^{-i}\phi(2^{-i})^{-1}) \langle g \rangle S_J^{i0} f + \langle \prod_{T^*1} g \rangle f,\end{aligned}$$

where $\Psi(t) = \int_0^t \psi(s)\frac{ds}{s}$, and $\|K\|_{CZ_0}$ could be replaced by $\|T\|_{L^2 \to L^2}$. Note that if we wanted to write $\prod_{T^*1}$ in terms of a shift with correct normalization, we should divide and multiply it by $\|T^*1\|_{BMO}$, thus getting a shift times the factor $\|T^*1\|_{BMO} \lesssim \|T\|_{L^2} + \|K\|_{CZ_\psi}$. **Near-by cubes, sigma in and sigma near.** We are left with the sums $\sigma_=$ of equal cubes $I = J$, as well as σ_{near} of disjoint near-by cubes with $dist(I,J) \leq \ell(J)\phi\left(\ell(I)/\ell(J)\right)$. Since I is good, this necessarily implies that $\ell(I) > 2^{-r}\ell(J)$. Then, for a given J, there are only boundedly many related I in this sum.

Lemma 9.

$$|\langle h_J \rangle T h_I| \lesssim \|K\|_{CZ_0} + \delta_{IJ} \|T\|_{WBP}.$$

Note that if we used the L^2-boundedness of T instead of the CZ_0 and WBP condition (as is done in Theorem 5.2), we could also estimate simply

$$|\langle h_J, T h_I \rangle| \lesssim \langle h_J \rangle_2 \langle T \rangle_{L^2 \to L^2} \langle h_I \rangle_2 = \langle T \rangle_{L^2 \to L^2}.$$

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Proof. For disjoint cubes, we estimate directly

$$\begin{aligned}
|\langle h_J, Th_I\rangle| \lesssim & \langle K\rangle_{CZ_0} \int_J \int_I \frac{1}{|x-y|^d} dy dx \langle h_J\rangle_\infty \langle h_I\rangle_\infty \\
\leq & \langle K\rangle_{CZ_0} \int_J \int_{3J\setminus J} \frac{1}{|x-y|^d} dy dx |J|^{-1/2}|I|^{-1/2} \\
\lesssim & \langle K\rangle_{CZ_0} |J||J|^{-1/2}|I|^{-1/2} = \langle K\rangle_{CZ_0},
\end{aligned}$$

since $|I| \simeq |J|$.

For $J = I$, let I_i be its dyadic children. Then

$$\begin{aligned}
|\langle h_J, Th_I\rangle| \leq & \sum_{i,j=1}^{2^d} |\langle h_I\rangle_{I_i} \langle h_I\rangle_{I_j} \langle 1_{I_i}, T1_{I_i}\rangle| \\
\lesssim & \langle K\rangle_{CZ_0} \sum_{j\neq i} |I|^{-1} \int_{I_i}\int_{I_j} \frac{1}{|x-y|^d} dx dy + \sum_i |I|^{-1} |\langle 1_{I_i}, T1_{I_i}\rangle| \lesssim \ \langle K\rangle_{CZ_0} + \langle T\rangle
\end{aligned}$$

by the same estimate as earlier for the first term, and the weak boundedness property for the second. □

With this lemma, the sum $\sigma_=$ is recognized as a cancellative dyadic shift of type (0, 0) as such:

$$\begin{aligned}
\sigma_= =& \sum_{I\in\mathscr{D}} 1_{good}(I) \cdot \langle g, h_I\rangle \langle h_I, Th_J\rangle \langle h_J, f\rangle \\
=& \left(\langle K\rangle_{CZ_0} + \langle T\rangle_{WBP}\right) \langle g, S^{00} f\rangle,
\end{aligned}$$

where the factor in front could also be replaced by $\langle T\rangle_{L^2\to L^2}$.

For I and J participating in σ_{near}, we conclude from Lemma 5 that $K := I \vee J$

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satisfies $\ell(K) \leq 2^r \ell(I)$, and hence we may organize

$$\sigma_{near} = \sum_{i=1}^{r} \sum_{j=1}^{i} \sum_{K} \sum_{\substack{I,J: I \vee J = K \\ dist(I,J) \leq \ell(J)\phi(\ell(I)/\ell(J)) \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}},$$

and the innermost sum is recognized as $\langle K \rangle_{CZ_0} \langle g, A_K^{ij} f \rangle$ for some cancellative averaging operator of type (i,j).

Summarizing, we have

$$\sigma_{near} + \sigma_{=} = (\langle K \rangle_{CZ_0} + \langle T \rangle_{WBP}) \langle g, S^{00} f \rangle + \langle K \rangle_{CZ_0} \sum_{j=1}^{r} \sum_{i=j}^{r} \langle g, S^{ij} f \rangle,$$

where S^{00} and S^{ij} are cancellative dyadic shifts, and the factor $(\langle K \rangle_{CZ_0} + \langle T \rangle_{WBP})$ could also be replaced by $\langle T \rangle_{L^2 \to L^2}$.

Synthesis. We have checked that

$$\begin{aligned} &\sum_{\ell(I) \leq \ell(J)} 1_{good} \langle g \rangle h_J \langle h_J \rangle T h_I \langle h_I \rangle f \\ &= \big(\langle K \rangle_{CZ_0} + \langle K \rangle_{CZ_\phi}\big) \Big(\sum_{1 \leq j \leq i < \infty} \phi(2^{-i})^{-d} \psi(2^{-i}\phi(2^{-i})^{-1}) \langle g, S^{ij} f \rangle \\ &+ \sum_{1 \leq i < \infty \Psi(2^{-i}\phi(2^{-i})^{-1}) \langle g, S^{i0} f \rangle} \\ &+ (\langle K \rangle_{CZ_0} + \langle T \rangle_{WBP}) \langle g, S^{00} f \rangle + \langle g, \prod_{T^*1}^{*} f \rangle \end{aligned}$$

where $\Psi = \int_0^t \psi(s) ds$, $\prod_{T^*1}$ is a paraproduct—a non-cancellative shift of $(0,0)$, and all other S^{ij} is a cancellative dyadic shifts of type (i,j).

By symmetry (just observing that the cubes of equal size contributed precisely to the presence of the cancellative shifts of type (i,j), and that the dual of a

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shift of type (i,j) is a shift of type (j,i), it follows that

$$\sum_{\ell(I)>\ell(J)} 1_{good}\langle g\rangle h_J\langle h_J\rangle T h_I\langle h_I\rangle f$$

$$= \left(\langle K\rangle_{CZ_0} + \langle K\rangle_{CZ_\phi}\right)\big(\sum_{1\le i<j<\infty} \phi(2^{-j})^{-d}\psi(2^{-j}\phi(2^{-j})^{-1})\langle g, S^{ij} f\rangle$$

$$+ \sum_{1\le j<\infty \Psi(2^{-j}\phi(2^{-j})^{-1})\langle g, S^{0i} f\rangle}$$

$$+ \langle g, \prod_{T1} f\rangle$$

so that altogether

$$\sum_{I,J} 1_{good}(\min\{I,J\})\langle g\rangle h_J\langle h_J\rangle T h_I\langle h_I\rangle f$$

$$= \left(\langle K\rangle_{CZ_0} + \langle K\rangle_{CZ_\phi}\right)\big(\sum_{i,j=1}^{\infty} \phi(2^{-\max(i,j)})^{-d}\psi(2^{-\max(i,j)}\phi(2^{-\max(i,j)})^{-1})\langle g, S^{ij} f\rangle$$

$$+ \sum_{i=1} \Psi(2^{-i}\phi(2^{-i})^{-1})\left(\langle g, S^{i0} f\rangle + \langle g, S^{0i} f\rangle\right)\big)$$

$$+ \left(\langle K\rangle_{CZ_0} + \langle T\rangle_{WBP}\right)\langle g, S^{00} f\rangle + \langle g, \prod_{T^*1}^{*} f\rangle + \langle g, \prod_{T1} f\rangle,$$

and this completes the proof of Theorem 5.2.

Two-weight theory for dyadic shifts

Before proceeding further, it is convenient to introduce a useful trick due to E. Sawyer. Let σ be an everywhere positive, finitely-valued function. Then $f \in L^p(\omega)$ if and only if $\phi = f\backslash\sigma \in L^p(\sigma^p\omega)$, and they have equal norms in the respective spaces. Hence an inequality

$$\|Tf\|_{L^p(\omega)} \le N\,\|f\|_{L^p(\omega)} \quad \forall f \in L^p(\omega) \tag{5.4}$$

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is equivalent to

$$\|T(\phi\sigma)\|_{L^p(\omega)} \le N\|\phi\sigma\|_{L^p(\omega)} = N\|\phi\|_{L^p(\sigma^p\omega)} \quad \forall\phi \in L^p(\sigma^p\omega). \tag{5.5}$$

This is true for any σ, and we now choose it in such a way that $\sigma^p\omega = \sigma$, i.e., $\sigma = \omega^{-1/(p-1)} = \omega^{1-p'}$, where p' is the dual exponent. So finally (5.4) is equivalent to

$$\|T(\phi\sigma)\|_{L^p(\omega)} \le N\|\phi\|_{L^p(\sigma)} \quad \forall\phi \in L^p(\sigma).$$

This formulation has the advantage that the norm on the right and the operator

$$T(\phi\sigma)(x) = \int K(x,y)\phi(y)\cdot\sigma(y)dy$$

involve integration with respect to the same measure σ. In particular, the A_2 theorem is equivalent to

$$\|T(f\sigma)\|_{L^2(\omega)} \le c_T[\omega]_{A_2}\|f\|_{L^2(\sigma)}$$

for all $f \in L^2(\omega)$, for all $\omega \in A_2$ and $\sigma = \omega^{-1}$. But once we know this, we can also study this two-weight inequality on its own right, for two general measures ω and σ, which need not be related by the pointwise relation $\sigma(x) = 1/\omega(x)$.

Theorem 5.4. Let σ and ω be two locally finite measures with

$$[\omega,\sigma]_{A_2} := \sup_Q \frac{\omega(Q)\sigma(Q)}{|Q|^2} < \infty.$$

Then a dyadic shift S of type (i,j) satisfies $S(\sigma\cdot) : L^2(\sigma) \to L^2(\omega)$ if and only if

$$\mathfrak{S} := \sup_Q \frac{\|1_Q S(\sigma 1_Q)\|\ L^2(\omega)}{\sigma(Q)^{1/2}}, \quad \mathfrak{S}^* := \sup_Q \frac{\|1_Q S^*(\omega 1_Q)\|_{L^2(\sigma)}}{\omega(Q)^{1/2}}$$

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are finite, and in this case

$$\|S(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\omega)} \lesssim (1+\kappa)(\mathfrak{S}+\mathfrak{S}^*) + (1+\kappa)^2[\omega,\sigma]_{A_2}^{1/2},$$

where $\kappa = \max\{i,j\}$.

This result from my work with Pérez, Treil, and Volberg [5] was preceded by an analogous qualitative version due to Nazarov, Treil, and Volberg [4].

The proof depends on decomposing functions in the spaces $L^2(\omega) \to L^2(\sigma)$ in terms of expansions similar to the Haar expansion in $L^2(\mathbb{R}^d)$. Let $\mathbb{D}_I^\sigma$ be the orthogonal projection of $L^2(\sigma)$ onto its subspace of functions supported on I, constant on the subcubes of I, and with vanishing integral with respect to $d\sigma$. Then any two $\mathbb{D}_I^\sigma$ are orthogonal to each other. Under the additional assumption that the σ measure of quadrants of $\mathbb{R}^d$ is finite, we have the expansion

$$f = \sum_{Q\in\mathscr{D}} \mathbb{D}_Q^\sigma f$$

for all $f \in L^2(\sigma)$, and Pythagoras' theorem says that

$$\|f\|_{L^2(\sigma)} = \left(\sum_{\mathscr{Q}} \left\|\mathbb{D}_Q^\sigma f\right\|_{L^2(\sigma)}^2\right)^{1/2}.$$

(These formulae needs a slight adjustment if the σ measure of quadrants is finite; Theorem 5.4 remains true without this extra assumption.) Let us also write

$$\mathbb{D}_K^{\sigma,i} := \sum_{\substack{I\subset K\\ \ell(I)=2^{-i}\ell(K)}} \mathbb{D}_I^\sigma.$$

For a fixed $i \in \mathbb{N}$, these are also orthogonal to each other, and the above formulae generalize to

$$f = \sum_{Q\in\mathscr{D}} \mathbb{D}_Q^{\sigma,i} f, \quad \|f\|_{L^2(\sigma)} = \left(\sum_{\mathscr{Q}} \left\|\mathbb{D}_Q^{\sigma,i} f\right\|_{L^2(\sigma)}^2\right)^{1/2}.$$

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The proof is in fact very similar in spirit to that of Theorem 5.2; it is another $T1$ argument, but now with respect to the measures σ and ω in place of the Lebesgue measure. We hence expand

$$\langle g, S(\sigma f)\rangle_\omega = \sum_{Q,R\in\mathscr{D}} \langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega, \quad f \in L^2(\sigma), g \in L^2(\omega),$$

and estimate the matrix coefficients

$$\begin{aligned}\langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega &= \sum_K \langle \mathbb{D}_R^\omega g, A_k(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega \\ &= \sum_K \sum_{I,J\subset K} a_{IJK} \langle \mathbb{D}_R^\omega g, h_J\rangle_\omega \langle h_I, \mathbb{D}_Q^\sigma f\rangle_\sigma.\end{aligned} \tag{5.6}$$

For $\langle h_I, \mathbb{D}_Q^\sigma f\rangle_\sigma \neq 0$, there must hold $I \cap Q \neq \emptyset$, thus $I \subset Q$ or $Q \subsetneq I$. But in the latter case h_I is constant on Q , while $\int \mathbb{D}_Q^\sigma f \cdot \sigma = 0$, so the pairing vanishes even in this case. Thus the only nonzero contributions come from $I \subseteq Q$, and similarly from $J \subseteq R$. Since $I, J \subseteq K$, there holds

$$(I \subseteq Q \subsetneq K \quad or \quad K \subseteq Q) \quad and \quad (J \subseteq\subsetneq K \quad or \quad K \subseteq R).$$

Disjoint cubes. Suppose now that $Q \cap R = \emptyset$, and let K K be among those cubes for which A_K K gives a nontrivial contribution above. Then it cannot be that $K \subset Q$, since this would imply that $Q \cap R \supseteq K \cap J = J \neq \emptyset$, and similarly it cannot be that $K \subseteq R$. Thus $Q, R \subsetneq K$, and hence

$$Q \vee Q \subseteq K.$$

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Then

$$
\begin{aligned}
|\langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega| &\le \sum_{K \supseteq Q \vee R} |\langle \mathbb{D}_R^\omega g, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega| \\
&\lesssim \sum_{K \supseteq Q \vee R} \frac{\|\mathbb{D}_R^\omega g\|_{L^1(\omega)} \left\|\mathbb{D}_Q^\sigma f\right\|_{L^1(\sigma)}}{|K|} \\
&\lesssim \frac{\|\mathbb{D}_R^\omega g\|_{L^1(\omega)} \left\|\mathbb{D}_Q^\sigma f\right\|_{L^1(\sigma)}}{|Q \vee R|}.
\end{aligned}
$$

On the other hand, we have $Q \supseteq I, R \supseteq J$ for some $I, J \subseteq K$ with $\ell(I) = 2^{-i}\ell(K)$ and $\ell(J) = 2^{-j}\ell(K)$. Hence $2^{-i}\ell(K) \le \ell(Q)$ and $2^{-j}\ell(K) \le \ell(R)$, and thus

$$
Q \vee R \subseteq K \subseteq Q^{(i) \cap R^{(j)}}.
$$

Now it is possible to estimate the total contribution of the part of the matrix with $Q \cap R = \emptyset$. Let $P := Q \vee R$ R be a new auxiliary summation variable. Then $Q, R \subset P$, and $\ell(Q) = 2^{-a}\ell(P), \ell(R) = 2^{-b}\ell(P)$ where $a = 1, \ldots, i$

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$b = 1, \ldots, j$. Thus

$$
\begin{aligned}
&\sum_{\substack{Q,R\in\mathscr{D}\\ Q\cap R=\emptyset}} |\langle \mathbb{D}_R^\omega g\rangle S(\sigma \mathbb{D}_Q^\omega f)_\omega| \\
&\lesssim \sum_{a=1}^{i}\sum_{b=1}^{j}\sum_{P\in\mathscr{D}} \frac{1}{|P|} \sum_{\substack{Q,R\in\mathscr{D}:Q\vee R=P\\ \ell(Q)=2^{-a}\ell(P)\\ \ell(R)=2^{-b}\ell(P)}} \|\mathbb{D}_R^\omega g\|_{L^1(\sigma)} \left\|\mathbb{D}_Q^\sigma f\right\|_{L^1(\omega)} \\
&\le \sum_{a,b=1}^{i,j}\sum_{p\in\mathscr{D}} \frac{1}{|P|} \sum_{\substack{R\subseteq P\\ \ell(R)=2^{-b}\ell(P)}} \|\mathbb{D}_R^\omega g\|_{L^1(\sigma)} \sum_{\substack{Q\subseteq P\\ \ell(Q)=2^{-a}\ell(P)}} \left\|\mathbb{D}_Q^\sigma f\right\|_{L^1(\omega)} \\
&= \sum_{a,b=1}^{i,j}\sum_{p\in\mathscr{D}} \frac{1}{|P|} \left\| \sum_{\substack{R\subseteq P\\ \ell(R)=2^{-b}\ell(P)}} \mathbb{D}_R^\omega g \right\|_{L^1(\sigma)} \left\| \sum_{\substack{Q\subseteq P\\ \ell(Q)=2^{-a}\ell(P)}} \mathbb{D}_Q^\sigma f \right\|_{L^1(\omega)} \\
&= \sum_{a,b=1}^{i,j}\sum_{p\in\mathscr{D}} \frac{1}{|P|} \left\|\mathbb{D}_P^{\omega,j} g\right\|_{L^1(\sigma)} \left\|\mathbb{D}_P^{\sigma,i} f\right\|_{L^1(\omega)} \\
&\le \sum_{a,b=1}^{i,j}\sum_{p\in\mathscr{D}} \frac{\sigma(P)^{1/2}\omega(P)^{1/2}}{|P|} \left\|\mathbb{D}_P^{\omega,j} g\right\|_{L^2(\omega)} \left\|\mathbb{D}_P^{\sigma,i} f\right\|_{L^2(\sigma)} \\
&\le \sum_{a,b=1}^{i,j} [\omega,\sigma]_{A_2}^{1/2} \left(\sum_{P\in\mathscr{D}} \left\|\mathbb{D}_P^{\omega,j} g\right\|_{L^2(\omega)}^2\right)^{1/2} \left(\sum_{P\in\mathscr{D}} \left\|\mathbb{D}_P^{\sigma,i} f\right\|_{L^2(\sigma)}^2\right)^{1/2} \\
&\le ij[\omega,\sigma]_{A_2}^{1/2} \|g\|_{L^2(\omega)} \|f\|_{L^2(\sigma)}.
\end{aligned}
$$

Deeply contained cubes. Consider now the part of sum with $Q \subset R$ and $\ell(Q) < 2^{-i}\ell(R)$. (The part with $R \subset Qand\ell(R) < 2^{-J}\ell(Q)$ would be handled in a symmetrical manner.)

Lemma 10. For all $Q \subset R$ with $\ell(Q) < 2^{-i}\ell(R)$, we have

$$\langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega = \langle \mathbb{D}_R^\omega g\rangle_{Q^{(i)}} \langle S^*(\omega 1_{Q^{(i)}}), \mathbb{D}_Q^\sigma f\rangle_\sigma,$$

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where further

$$\mathbb{D}_Q^\sigma S^*(\omega 1_{Q^{(i)}}) = \mathbb{D}_Q^\sigma S^*(\omega 1_P) \qquad \text{for any } P \supseteq Q^{(i)}.$$

Recall that $\mathbb{D}_Q^\sigma = (\mathbb{D}_Q^\sigma)^2 = (\mathbb{D}_Q^\sigma)^*$ is an orthogonal projection on $L^2(\sigma)$, so it can be moved to either or both sides of $\langle , \rangle_\sigma$.

Proof. Recall formula (5.6). If $\langle h_I, \mathbb{D}_Q^\sigma f \rangle_\sigma$ is nonzero, then $I \subseteq Q$, and hence

$$J \subseteq K = I^{(i)} \subseteq Q^{(i)} \subsetneq R$$

for all J participating in the same A_K as I. Thus $\mathbb{D}_R^\omega g$ is constant in $Q^{(i)}$, hence

$$\begin{aligned}
\langle \mathbb{D}_R^\omega g, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega &= \langle 1_{Q^{(i)}\mathbb{D}_R^\omega g}, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega \\
&= \langle \mathbb{D}_R^\omega g\rangle_{Q^{(i)}}^\omega \langle 1_Q^{(i)}, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega \\
&= \langle \mathbb{D}_R^\omega g\rangle_{Q^{(i)}}^\omega \langle A_K^*(\omega 1_{Q^{(i)}}), \mathbb{D}_Q^\sigma f\rangle.
\end{aligned}$$

Moreover, for any $P \supseteq Q^{(i)} \supseteq K$,

$$\begin{aligned}
\langle \mathbb{D}_Q^\sigma A_K^*(\omega 1_{Q^{(i)}}), f\rangle_\sigma &= \langle 1_{Q^{(i)}}, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega \\
&= \int A_K(\sigma \mathbb{D}_Q^\sigma f)\omega \\
&= \langle 1_P, A_K(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega = \langle \mathbb{D}_Q^\sigma A_K^*(\omega 1_P), f\rangle_\sigma.
\end{aligned}$$

Summing these equalities over all relevant K, and using $S = \sum_K A_K$, giving the claim. □

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By the lemma, we can then manipulate

$$\sum_{\substack{Q,R:Q\subset R\\ \ell(Q)<2^{-i}\ell(R)}} \langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega$$

$$= \sum_Q \left(\sum_{R\supsetneq Q^{(i)}} \langle \mathbb{D}_R^\omega g\rangle_{Q^{(i)}}^\omega \right) \langle S^*(\omega 1_{Q^{(i)}}), \mathbb{D}_Q^\sigma f\rangle_\sigma$$

$$= \sum_Q \langle g\rangle_{Q^{(i)}}^\omega \langle S^*(\omega 1_{Q^{(i)}}), \mathbb{D}_Q^\sigma f\rangle_\sigma$$

$$= \sum_R \langle g\rangle_R^\omega \langle S^*(\omega 1_R), \sum_{\substack{Q\subseteq R\\ \ell(Q)=2^{-i}\ell(R)}} \mathbb{D}_Q^\sigma f\rangle_\sigma$$

$$= \sum_R \langle g\rangle_R^\omega \langle S^*(\omega 1_R), \mathbb{D}_R^{\sigma,i} f\rangle_\sigma,$$

where $\langle g\rangle_R^\omega := \omega(R)^{-1}\int_R g\cdot\omega$ is the average of g on R with respect to the ω measure.

By using the properties of the pairwise orthogonal projections $\mathbb{D}_R^{\sigma,i}$ on $L^2(\sigma)$, the above series may be estimated as follows:

$$|\sum_{Q,R:Q\subset R} \langle \mathbb{D}_R^\omega g, S(\sigma \mathbb{D}_Q^\sigma f)\rangle_\omega|$$

$$\leq \sum_R |\langle g\rangle_R^\omega| \left\|\mathbb{D}_R^{\sigma,i} S^*(\omega 1_R)\right\|_{L^2(\sigma)} \left\|\mathbb{D}_R^{\sigma,i} f\right\|_{L^2(\sigma)}$$

$$\leq \left(\sum_R |\langle g\rangle_R^\omega| \left\|\mathbb{D}_R^{\sigma,i} S^*(\omega 1_R)\right\|_{L^2(\sigma)}^2\right) \left(\sum_R \left\|\mathbb{D}_R^{\sigma,i} f\right\|_{L^2(\sigma)}^2\right),$$

where the last factor is equal to $\|f\|_{L^2(\omega)}$.

The first factor on the right is handled by the dyadic Carleson embedding theorem: It follows from the second equality of Lemma

We firstly prove a lemma 10, namely $\mathbb{D}_Q^\sigma S^*(\omega 1_{Q^{(i)}}) = \mathbb{D}_Q^\sigma S^*(\omega 1_P)$ for all $P \supset$

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$Q^{(i)}$, that $\mathbb{D}_R^{\sigma,i} S^*(\omega 1_R) = \mathbb{D}_Q^{\sigma} S^*(\omega 1_P)$ for all $P \subseteq R$. Hence, we have

$$\sum_{R \subset P} \left\| \mathbb{D}_R^{\sigma,i} S^*(\omega 1_R) \right\|_{L^2(\sigma)}^2 = \sum_{R \subset P} \left\| \mathbb{D}_R^{\sigma,i} (1_P S^*(\omega 1_P)) \right\|_{L^2(\sigma)}^2$$
$$\leq \| 1_P S^*(\omega 1_P) \|_{L^2(\sigma)}^2 \lesssim \mathfrak{S}_*^2 \sigma(P)$$

by the (dual) testing estimate for the dyadic shifts. By the Carleson embedding theorem, it then follows that

$$\left(\sum_R |\langle g \rangle_R^{\omega}|^2 \left\| \mathbb{D}_R^{\sigma,i} S^*(\omega 1_R) \right\|_{L^2(\sigma)}^2 \right)^{1/2} \lesssim \mathfrak{S}_* \|g\|_{L^2(\sigma)},$$

and the estimation of the deeply contained cubes is finished. **Contained cubes of comparable size.** It remains to estimate

$$\sum_{\substack{Q,R: Q \subseteq R \\ \ell(Q) \geq 2^{-i} \ell(R)}} \langle \mathbb{D}_R^{\omega} g, S(\sigma \mathbb{D}_Q^{\sigma} f) \rangle_{\omega};$$

the sum over $R \subseteq Q$ with $\ell(R) \geq 2^{-j} \ell(Q)$ would be handled in a symmetric manner. The sum of interest may be written as

$$\sum_{a=0}^{i} \sum_R \sum_{\substack{Q \subseteq R \\ \ell(Q) = 2^{-a} \ell(R)}} \langle \mathbb{D}_R^{\omega} g, S(\sigma \mathbb{D}_Q^{\sigma} f) \rangle_{\omega} = \sum_{a=0}^{i} \sum_R \langle \mathbb{D}_R^{\omega} g, S(\sigma \mathbb{D}_Q^{\sigma,i} f) \rangle_{\omega},$$

and

$$\langle \mathbb{D}_R^{\omega} g, S(\sigma \mathbb{D}_Q^{\sigma,i} f) \rangle_{\omega} = \sum_{k=1}^{2^d} \langle \mathbb{D}_R^{\omega} g \rangle_{R_k} \langle S^*(\omega 1_{R_k}), \mathbb{D}_R^{\sigma,i} f \rangle_{\sigma}$$

where the R_k are the 2^d dyadic children of R, and $\langle \mathbb{D}_R^{\omega} g \rangle_{R_k}$ is the constant valued of $\mathbb{D}_R^{\omega} g$ on R_k. Now

$$\langle S^*(\omega 1_{R_k}), \mathbb{D}_R^{\sigma,i} f \rangle_{\sigma} = \langle 1_{R_k} S^*(\omega 1_{R_k}), \mathbb{D}_R^{\sigma,i} f \rangle_{\sigma} + \langle S^*(\omega 1_{R_k}), 1_{R_k^c} \mathbb{D}_R^{\sigma,i} f \rangle_{\sigma},$$

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where

$$|\langle 1_{R_k} S^*(\omega 1_{R_k}), \mathbb{D}_R^{\sigma,i} f\rangle_\sigma| \le \mathfrak{S}_* \omega(R_k)^{1/2} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^2(\simeq)}$$

and, observing that only those A_K^* where K intersects both R_k and R_k^c contribute to the second part,

$$\begin{aligned}
|\langle S^*(\omega 1_{R_k}), 1_{R_k^c} \mathbb{D}_R^{\sigma,i} f\rangle_\sigma| &= |\sum_{K \supsetneq R_k} \langle A_K^*(\omega 1_{R_k}), 1_{R_k^c} \mathbb{D}_R^{\sigma,i} f\rangle_\sigma| \\
&\lesssim \sum_{K \supseteq R} \frac{1}{|K|} \omega(R_k) \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)} \\
&\lesssim \frac{1}{|R|} \omega(R_k) \sigma(R)^{1/2} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)} \\
&\le \frac{\omega(R^{1/2} \sigma(R)^{1/2}}{|R|} \omega(R_k)^{1/2} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)} \\
&\le [\omega, \sigma]_{A_2} \omega(R_k)^{1/2} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)}.
\end{aligned}$$

It follows that

$$|\langle S^*(\omega 1_{R_k}), \mathbb{D}_R^{\sigma,i} f\rangle_\sigma| \lesssim (\mathfrak{S}_* + [\omega,\sigma]_{A_2}) \omega(R_k)^{1/2} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)}$$

and hence

$$|\langle \omega 1_{R_k}, S(\mathbb{D}_R^{\sigma,i} f)\rangle_\omega| \lesssim (\mathfrak{S}_* + [\omega,\sigma]_{A_2}) \left\| \mathbb{D}_R^{\omega} g \right\|_{L^2(\omega)} \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)}.$$

Finally,

$$\begin{aligned}
&\sum_{a=0}^{i} \sum_R |\langle \mathbb{D}_R^{\omega} g, S(\sigma \mathbb{D}_R^{\sigma,i} f)\rangle_\omega| \\
&\lesssim (\mathfrak{S}_* + [\omega,\sigma]_{A_2}) \sum_{a=0}^{i} \left(\sum_R \left\| \mathbb{D}_R^{\omega} g \right\|_{L^2(\omega)}^2 \right)^{1/2} \left(\sum_R \left\| \mathbb{D}_R^{\sigma,i} f \right\|_{L^1(\sigma)}^2 \right)^{1/2} \\
&\le (1+i)(\mathfrak{S}_* + [\omega,\sigma]_{A_2}) \left\| g \right\|_{L^2(\omega)} \left\| f \right\|_{L^2(\sigma)}.
\end{aligned}$$

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The symmetric case with $R \subset Q$ with $\ell(R) \geq 2^{-j}\ell(Q)$ similarly yields the factor $(1+j)(\mathfrak{S} + [\omega,\sigma]_{A_2})$. This completes the proof of Theorem (5.4). **Final decompositions: verification of the testing conditions** We now turn to the estimation of the testing constant

$$\mathfrak{S} := \sup_{Q\in\mathscr{D}} \frac{\|1_Q S(\sigma 1_Q)\|_{L^2(\omega)}}{\sigma(Q)^{1/2}}.$$

Bounding $\mathfrak{S}_*$ is analogous by exchanging the roles of ω and σ. **Several splittings.** First observe that

$$1_Q S(\sigma 1_Q) = 1_Q \sum_{K:K\cap Q\neq\emptyset} A_K(\sigma 1_Q) + 1_Q \sum_{K\supsetneq Q} A_K(\sigma 1_Q).$$

The second part is immediate to estimate even pointwise by

$$|1_Q A_K(\sigma 1_Q)| \leq 1_Q \frac{\sigma(Q)}{|K|}, \quad \sum_{K\supsetneq} \frac{1}{|K|} \leq \frac{1}{|Q|},$$

and hence its $L^2(\omega)$ norm is bounded by

$$\left\| 1_Q \frac{\sigma(Q)}{|Q|} \right\|_{L^2(\omega)} = \frac{\omega(Q)^{1/2}\sigma(Q)}{|Q|} \leq [\omega,\sigma]_{A_2}\sigma(Q)^{1/2}.$$

So it remains to concentrate on $K \supseteq Q$, and we perform several consecutive splittings of this collection of cubes. First, we separate scales by introducing the splitting according to the $\kappa+1$ possible values of $\log_2 \ell(K) \mod (\kappa+1)$. We denote a generic choice of such a collection by

$$\mathscr{K} = \mathscr{K} := \{K \supseteq Q : \log_2 \ell(K) \equiv \kappa \mod (\kappa+1)\},$$

where κ is arbitrary but fixed. (We will drop the subscript k, since its value plays no role in the subsequent argument.) Next, we freeze the A2 characteristic

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by setting

$$\mathscr{H} := \{K \in \mathscr{H} : 2^{a-1} < \frac{\omega(K)\sigma(K)}{|K|} \leq 2^a\}, \quad a \in \mathbb{Z}, \quad a \leq \lceil \log_2[\omega, \simeq]_{A_2} \rceil,$$

where $\lceil\rceil$ means rounding up to the next integer.

In the next step, we choose the principal cubes $P \in \mathscr{P} \supseteq \mathscr{H}^a$. This construction was first introduced by B. Muckenhoupt and R. Wheeden [8], and it has been influential ever since. Let $\mathscr{P}_0^a$ consist of all maximal cubes in $\mathscr{H}^a$, and inductively $\mathscr{P}_{p+1}^a$ consist of maximal $P' \in \mathscr{H}^a$ such that

$$P' \subset P \in \mathscr{H}_p^a, \quad \frac{\sigma(P')}{|P'|} > 2\frac{\sigma(P)}{|P|}.$$

Finally, let $\mathscr{P}^a : \cup_{p=0}^{\infty} \mathscr{P}_p^a$. For each $K \in \mathscr{H}^a$, let $\prod^a(K)$ denote the minimal $P \in \mathscr{P}^a$ such that $K \subseteq P$. Then we set

$$\mathscr{H}^a(P) := \{K \in \mathscr{H}^a : \prod^a(K) = P\}, \quad P \in \mathscr{P}^a.$$

Note that $\sigma(K)/|K| \leq 2\sigma(P)/|P|$ for all $K \in \mathscr{H}^a(P)$, , which allows us to freeze the $\sigma-$to-Lebesgue measure ratio by the final subcollections

$$\mathscr{H}_b^a := \{K \in \mathscr{H}^a(P) : 2^{-b} < \frac{\sigma(K)}{|K|}\frac{|P|}{\sigma(P)}\} \leq 2^{1-b}, \quad b \in \mathbb{N}.$$

We have

$$\{K \in \mathscr{D} : K \subseteq Q\} = \cup_{k=0}^{\kappa} \mathscr{H}_k, \quad \mathscr{H}_k = \mathscr{H} = \cup_{a \leq \lceil \log_2[\omega,\sigma]_{A_2} \rceil} \mathscr{H}^a,$$
$$\mathscr{H}^a = \cup_{P \in \mathscr{P}^a} \mathscr{H}^a(P), \quad \mathscr{H}^a(P) = \cup_{b=0}^{\infty} \mathscr{H}_b^a(P),$$

where all unions are disjoint. Note that we drop the reference to the separation-ofscales parameter k, since this plays no role in the forthcoming arguments.

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Recalling the notation for subshifts $S_{\mathscr{Q}} = \sum_{K \in \mathscr{Q}} A_K$, this splitting of collections of cubes leads to the splitting of the function

$$\sum_{K \subseteq Q} A_K(\sigma 1_Q) = \sum_{k=0}^{\kappa} \sum_{a \le \lceil \log_2 [\omega,\sigma]_{A_2} \rceil} \sum_{P \in \mathscr{P}^a} \sum_{b=0}^{\infty} S_{\mathscr{H}_b^a(P)}(\sigma 1_Q).$$

On the level of the function, we split one more time to write

$$S_{\mathscr{H}_b^a(P)}(\sigma 1_Q) = \sum_{n=0}^{\infty} 1_{E_b^a(P,n)} S_{\mathscr{H}_b^a(P)}(\sigma 1_Q),$$

$$E_b^a(P,n) := \{x \in \mathbb{R}^d : n2^{-b}\langle \sigma \rangle_P < |S_{\mathscr{H}_b^a(P)}(\sigma 1_Q(x))| \le (n+1)2^{-b}\langle \sigma \rangle_P\}.$$

This final splitting, from [7], is not strictly ‘necessary’ in that it was not part of the original argument in [6], nor its predecessor in [10], which made instead more careful use of the cubes where $S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)$ stays constant; however, it now seems that this splitting provides another simplification of the argument.

Now all relevant cancellation is inside the functions $S_{\mathscr{H}_b^a}(\sigma 1_Q)$, so that we can simply estimate by the triangle inequality:

$$\begin{aligned} &|\sum_{K \subseteq} A_K(\sigma 1_Q)| \\ &\le \sum_{k=0}^{\kappa} \sum_{a \le \lceil \log_2 [\omega,\sigma]_{A_2} \rceil} \sum_{P \in \mathscr{P}^a} \sum_{b=0}^{\infty} \sum_{n=0}^{\infty} (1+n) 2^{-b} \langle \sigma \rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle \sigma \rangle_P\}}, \end{aligned}$$

and

$$\left\| \sum_{K \subseteq Q} A_K(\sigma 1_Q) \right\|_{L^2(\omega)} \le \sum_{k=0}^{\kappa} \sum_{a} \sum_{b=0}^{\infty} 2^{-b} \sum_{n=0}^{\infty} (1+n) \left\| \sum_{P \in \mathscr{P}^a} \langle \sigma \rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle \sigma \rangle_P\}} \right\|_{L^2(\omega)}$$

Obviously, we will need good estimates to be able to sum up these infinite series.

Write the last norm as

$$\left(\int \left[\sum_{P \in \mathscr{P}} \langle \sigma \rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle \sigma \rangle_P\}}(x) \right]^2 d\omega(x) \right)^{1/2},$$

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observe that

$$\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle\sigma\rangle_P\} \subseteq P,$$

and look at the integrand at a fixed point $x \in \mathbb{R}^d$. At this point we sum over a subset of those values of $\langle\sigma\rangle_P$ where the principal cube $P \ni x$. Let P_0 be the smallest cube such that $|\mathcal{S}_{\mathscr{H}_b^a}(P)| > n2^{-b}\langle\sigma\rangle_P$, let P_1 be the next smallest, and so on. Then $\langle\sigma\rangle_{P_m} < 2^{-1}\langle\sigma\rangle_{P_{m-1}} < \cdots < 2^{-m}\langle\sigma\rangle_{P_0}$ by the construction of the principal cubes, and hence

$$\begin{aligned}\left[\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)|>n2^{-b}\langle\sigma\rangle_P\}}(x)\right]^2 &= \left[\sum_{m=0}^{\infty}\langle\sigma\rangle_{P_m}\right]^2 \\ &\leq \left[\sum_{m=0}^{\infty}2^{-m}\langle\sigma\rangle_{P_0}\right]^2 = 4\langle\sigma\rangle_{P_0}^2 \\ &\leq 4\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P^2 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)|>n2^{-b}\langle\sigma\rangle_P\}}(x).\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)|>n2^{-b}\langle\sigma\rangle_P\}}\right\|_{L^2(\omega)} \\ &\leq \left(\int\left(4\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P^2 1_{\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)|>n2^{-b}\langle\sigma\rangle_P\}}\right)\omega\right)^{1/2} \\ &= 2\left(\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P^2\omega(\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle\sigma\rangle_P\})\right)^{1/2},\end{aligned}$$

and it remains to obtain good estimates for the measure of the level sets

$$\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > n2^{-b}\langle\sigma\rangle_P\}.$$

Weak-type and John—Nirenberg-style estimates. We still need to estimate the sets above. Recall that $S_{\mathscr{H}_b^a(P)}$ is a subshift of S, which in particular

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has its scales separated so that $\log_2 \ell(K) \equiv \kappa \mod (\kappa+1)$ for all K for which A_K participating in $S_{\mathscr{H}_b^a(P)}$ is nonzero and $k \in \{0, 1, \ldots, \kappa := \max\{i, j\}\}$ is fixed, S being of type (i, j). The following estimate deals with such subshifts, which we simply denote by S.

Proposition 2. Let S be a dyadic shift of type (i,j) with scales separated. Then

$$|\{|Sf| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1, \quad \forall \lambda > 0,$$

where C depends only on the dimension.

Proof. The proof uses the classical Calderón—Zygmund decomposition:

$$f = g + h, \quad b := \sum_{L \in \mathscr{B}} b_L := \sum_{L \in \mathscr{B}} 1_B \langle f - \langle f \rangle_L \rangle,$$

where $L \in \mathscr{B}$ are the maximal dyadic cubes with $\langle |f| \rangle_L > \lambda$: hence $\langle |f| \rangle_L \leq 2^d \lambda$. As usual,

$$g = f - b = 1_{(\cup \mathscr{B})^c} f + \sum_{L \in \mathscr{B}} \langle f \rangle_L$$

satisfies $\|g\|_\infty \leq 2^d \lambda$ and $\|g\|_1 \leq \|f\|_1$, hence $\|g\|_2^2 \leq \|g\|_\infty \|g\|_1 \leq 2^d \lambda \|f\|_1$, and thus

$$|\{|Sg| > \frac{1}{2}\lambda\}| \leq \frac{4}{\lambda^2} \|Sg\|_2^2 \leq 4 \cdot 2^d \frac{1}{\lambda} \|f\|_1 .$$

It remains to estimate $\{|Sb| > \frac{1}{2}\lambda\}$. First observe that

$$Sb = \sum_{K \in \mathscr{D}} \sum_{L \in \mathscr{B}} A_K b_L = \sum_{L \in \mathscr{B}} \left(\sum_{K \subseteq L} A_K b_L + \sum_{K \supsetneq L} A_K b_L \right),$$

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since $A_K b_L \neq 0$ only if $K \cap L \neq \emptyset$. Now

$$\begin{aligned}|\{|Sb| > \frac{1}{2}\lambda\}| &\leq |\{|\sum_{L\in\mathscr{B}}\sum_{K\subseteq L} A_K b_L| > 0\}| + |\{|\sum_{L\in\mathscr{B}}\sum_{K\supsetneq L} A_K b_L| > \frac{1}{2}\lambda\}| \\ &\leq \sum_{L\in\mathscr{B}} |L| + \frac{2}{\lambda}\left\|\sum_{L\in\mathscr{B}}\sum_{K\supsetneq L} A_K b_L\right\|_1 \\ &\frac{1}{\lambda}\|f\|_1 + \frac{2}{\lambda}\sum_{L\in\mathscr{B}}\sum_{K\supsetneq L}\|A_K b_L\|_1,\end{aligned}$$

where we used the elementary properties of the Calder$\acute{o}$n—Zygmund decomposition to estimate the first term.

For the remaining double sum, we still need some observations. Recall that

$$A_K b_L = \sum_{\substack{I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} a_{IJK} h_I \langle h_J, b_L\rangle.$$

Now, if $\ell(K) > 2^{\kappa}\ell(L) \geq 2^{j}\ell(L)$, then $\ell(J > \ell(L))$, and hence h_J is constant on L. But the integral of b_L vanishes, hence $\langle h_J, b_L\rangle = 0$ for all relevant J, and thus $A_K b_L = 0$ whenever $\ell(K) > 2^{\kappa}\ell(L)$.

Thus, in the inner sum, the only possible nonzero terms are $A_K b_L$ for $K = L^{(m)}$ for $m = 1, \ldots, \kappa$. By the separation of scales, at most one of these terms is nonzero, and we write $\tilde{L}$ for the corresponding unique K. So in fact

$$\frac{2}{\lambda}\sum_{L\in\mathscr{B}}\sum_{K\supsetneq L}\|A_K b_L\|_1 = \frac{2}{\lambda}\sum_{L\in\mathscr{B}}\|A_{\tilde{L}} b_L\|_1 \leq \frac{2}{\lambda}\sum_{L\in\mathscr{B}}\|b_L\|_1 \leq \frac{2}{\lambda}\cdot 2\|f\|_1 = \frac{4}{\lambda}\|f\|_1$$

by using the normalized boundedness of the averaging operator $A_{\tilde{L}}$ on $L^1(\mathbb{R}^d)$, and an elementary estimate for the bad part of the Calder$\acute{o}$n—Zygmund decomposition.

Altogether, we obtain the claim with $C = 4 \cdot 2^d + 5$. □

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For the special subshift $S_{\mathscr{H}_b^a(P)}$, we can improve the weak-type (1,1) estimate to an exponential decay:

Proposition 3. Let $S_{\mathscr{H}_b^a(P)}$ be the subshift of S as constructed earlier. Then the following estimate holds when nu is either the Lebesgue measure or ω:

$$\nu\left(\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > C2^{-b}\langle\sigma\rangle_P \cdot t\}\right) \lesssim C2^{-t}\nu(P), \quad t \geq 0,$$

where C is a constant.

Proof. Let $\lambda := C2^{-b}\langle\sigma\rangle_P$, where C is a large constant, and $n \in \mathbb{Z}_+$. Let $x \in \mathbb{R}^d$ be a point where

$$|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)(x)| > n\lambda. \tag{5.7}$$

Then for all small enough $L \in \mathscr{H}_b^a(P)$ with $L \ni x$, there holds

$$|\sum_{K\in\mathscr{H}_b^a(P) K\supseteq L} A_K(\sigma 1_Q)(x)| > n\lambda.$$

Since $\sum_{K\in\mathscr{H}_b^a(P) K\supseteq L} A_K(\sigma 1_Q)$ is constant on L (thanks to separation of scales), and

$$\|A_L(\sigma 1_Q)\|_\infty \lesssim \frac{\sigma(L)}{|L|} \leq 2^{1-b}\frac{\sigma(P)}{|P|}, \tag{5.8}$$

it follows that

$$|\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\supsetneq L}} A_K(\sigma 1_Q)| > (n-\frac{2}{3})\lambda \quad \text{on } L. \tag{5.9}$$

Let $\mathscr{L} \subseteq \mathscr{H}_b^a(P)$ be the collection of maximal cubes with the above property. Thus all $L \in \mathscr{L}$ are disjoint, and all x with (5.7) belong to some L. By

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maximality of L, the minimal $L^* \in \mathscr{H}_b^a(S)$ with $L^* \supsetneq L$ satisfies

$$|\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\supsetneq L^*}} A_K(\sigma 1_Q)| > (n-\frac{2}{3})\lambda \quad on\ L^*.$$

By an estimate similar to (5.8), with L^* in place of L, it follows that

$$|\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\supsetneq L}} A_K(\sigma 1_Q)| > (n-\frac{1}{3})\lambda \quad on\ L.$$

Thus, if x satisfies (5.7) and $x \in L \in \mathscr{L}$, then necessarily

$$|S_{K\in\mathscr{H}_b^a(P):K\subseteq L}(\sigma 1_Q)(x)| = |\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\subsetneq L}} A_K(\sigma 1_Q)(x)| > \frac{1}{3}\lambda$$

□

Using the weak-type L^1 estimate to the shift $S_{K\in\mathscr{H}_b^a(P):K\subseteq L}$ of type (i,j) with scales separated, noting that $A_K(\sigma 1_Q) = A_K(\sigma 1_L)$ for $K \subseteq L$, it follows that

$$\begin{aligned}|\{|\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\subsetneq L}} A_K(\sigma 1_Q)(x)| > \frac{1}{3}\lambda\}| &\le \frac{C}{\lambda}\sigma(L)\\ &\le \frac{C}{\lambda}2^{1-b}\frac{\sigma(S\cap Q)}{|S|}|L| \le \frac{1}{3}|L|,\end{aligned}$$

provided that the constant in the definition of λ was chosen large enough. Recalling (5.9), there holds

$$\begin{aligned}|\sum_{K\in\mathscr{H}_b^a}(P)A_K(\sigma 1_Q)| \ge |\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\supsetneq L}} A_K(\sigma 1_Q)| - |\sum_{\substack{K\in\mathscr{H}_b^a(P)\\ K\subseteq L}} A_K(\sigma 1_Q)|\\ > (n-\frac{2}{3})\lambda - \frac{1}{3}\lambda = (n-1)\lambda \quad on\ \tilde{L}\subset L\ with\ |\tilde{L}| \ge \frac{2}{3}|L|.\end{aligned}$$

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Thus

$$
\begin{aligned}
|\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)}| > n\lambda\}| &\leq \sum_{L \in \mathscr{L}} |L \cap \{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| \\
&\leq \sum_{L \in \mathscr{L}} |\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > \frac{1}{3}\lambda\}| \\
&\leq \sum_{L \in \mathscr{L}} \frac{1}{3}|L| \leq \sum_{L \in \mathscr{L}} \frac{1}{3} \cdot \frac{3}{2}|\tilde{L}| \\
&\leq \frac{1}{2} \sum_{L \in \mathscr{L}} |L \cap \{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > (n-1)\lambda\}| \\
&\leq \frac{1}{2}|\{|S_{\mathscr{H}_b^a(P)}(\sigma 1_Q)| > (n-1)\lambda\}|.
\end{aligned}
$$

By induction it follows that

$$
\begin{aligned}
\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)}| > n\lambda\}| &\leq 2^{-n}\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)}| > 0\}| \\
&\leq 2^{-n} \sum_{M \in \mathscr{M}} |M| \leq 2^{-n}|P|,
\end{aligned}
$$

where $\mathscr{M}$ is the collection of maximal cubes in $\mathscr{H}_b^a(S)$.

Recalling that we defined $\lambda := C2^{-b}\langle\sigma\rangle_P$ in the beginning of the proof, the previous display gives precisely the claim of the Proposition in the case that ν is the Lebesgue measure. We still need to consider the case that $\nu = \omega$. To this end, selected intermediate steps of the above computation, as well as the definition of $\mathscr{H}_b^a(P)$, will be exploited. Recall that $K \in \mathscr{H}^a$ means that $2^{a-1} < \langle\omega\rangle_K\langle\sigma\rangle_K \leq 2^a$, while $K \in \mathscr{H}_b^a(P)$ means that in addition $2^{-b} < \langle\sigma\rangle_K/\langle\sigma\rangle_P \leq 2^{1-b}$. Put together, this says that

$$
2^{a+b-2}\langle\sigma\rangle_P < \frac{\omega(K)}{|K|} < 2^{a+b}\langle\sigma\rangle_P \quad \forall K \in \mathscr{H}_b^a(P).
$$

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Hence, using the collections $\mathscr{L}, \mathscr{M} \subseteq \mathscr{H}_b^a(P)$ as above,

$$\begin{aligned}
\omega(\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)}| > n\lambda\}) &\leq \sum_{L \in \mathscr{L}} \omega(L) \leq \sum_{L\in\mathscr{L}} 2^{a+b} \langle \sigma \rangle_P |L| \\
&\leq 2^{a+b}\langle\sigma\rangle_P |\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)}| > (n-1)\lambda\}| \\
&\leq 2^{a+b}\langle\sigma\rangle_P \cdot 2^{-n} \sum_{M\in\mathscr{M}} |M| \\
&\leq 4\cdot 2^{-n} \sum_{M\in\mathscr{M}} \omega(M) \leq 4 \cdot 2^{-n}\omega(S).
\end{aligned}$$

Conclusion of the estimation of the testing conditions. Recall that

$$\begin{aligned}
&\left\| \sum_{K\subseteq Q} A_K(\sigma 1_Q)\right\|_{L^2(\omega)} \\
&\leq \sum_{k=0}^{\kappa} \sum_a \sum_{b=0}^{\infty} 2^{-b} \sum_{n=0}^{\infty} (1+n) \left\| \sum_{P\in\mathscr{P}^a} \langle\sigma\rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}| > n2^{-b}\langle\sigma\rangle_P\}} \right\|_{L^2(\omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \sum_{P\in\mathscr{P}^a} \langle\sigma\rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)}| > n2^{-b}\langle\sigma\rangle_P\}} \right\|_{L^2(\omega)} \\
&\leq 2 \left(\sum_{P\in\mathscr{P}^a} \langle\sigma\rangle_P^2 \omega(\{|S_{\mathscr{H}_b^a(P)}| > n2^{-b}\langle\sigma\rangle_P\}) \right)^{1/2} \\
&\leq C \left(\sum_{P\in\mathscr{P}^a} \langle\sigma\rangle_P^2 2^{n/C} \omega(P) \right)^{1/2} \\
&= C2^{-cn} \left(\sum_{P\in\mathscr{P}^a} \frac{\sigma(P)\omega(P)}{|P|^2} \sigma(P) \right)^{1/2} \\
&\leq C2^{-cn} \left(2^a \sum_{P\in\mathscr{P}^a} \sigma(P) \right)^{1/2},
\end{aligned}$$

recalling the freezing of the A_2 characteristic between 2^{a-1} and 2^a for cubes in $\mathscr{H}^a \supseteq \mathscr{P}^a$.

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For the summation over the principal cubes, we observe that

$$\sum_{P\in\mathscr{P}}\sigma(P)=\sum_{P\in\mathscr{P}}\langle\sigma\rangle_P|P|=\int_Q\sum_{P\in\mathscr{P}^a}\langle\sigma\rangle_P 1_P(x)dx.$$

At any given x, if $P_0\supsetneq P_1\subsetneq\cdots\subseteq Q$ are the principal cubes containing it, we have

$$\sum_{P\in\mathscr{P}^a}\langle\sigma\rangle_P 1_P(x)=\sum_{m=0}^{\infty}\langle\sigma\rangle_{P_m}\leq\sum_{m=0}^{\infty}2^{-m}\langle\sigma\rangle_{P_0}=2\langle\sigma\rangle_{P_0}\leq 2M(\sigma 1_Q)(x),$$

where M is the dyadic maximal operator. Hence

$$\sum_{P\in\mathscr{P}^a}\sigma(P)\leq 2\int_Q M(\sigma 1_Q)dx\leq 2[\sigma]_{A_\infty}\sigma(Q),$$

where we use the following notion of the A_∞ characteristic:

$$[\sigma]_{A_\infty}:=\sup_Q\frac{1}{\sigma(Q)}\int_Q M(\sigma 1Q)dx;$$

this was implicit already in the work of Fujii ([11]) and it was taken as an explicit definition by the author and C. Pérez ([9]). Substituting back, we have

$$\begin{aligned}
&\left\|\sum_{K\subseteq Q}A_K(\sigma 1Q)\right\|_{L^2(\omega)}\\
&\leq\sum_{k=0}^{\kappa}\sum_a\sum_{b=0}^{\infty}2^{-b}\sum_{n=0}^{\infty}(1+n)\left\|\sum_{P\in\mathscr{P}^a}\langle\sigma\rangle_P 1_{\{|S_{\mathscr{H}_b^a(P)(\sigma 1_Q)|>n2^{-b}\langle\sigma\rangle_P}}\right\|_{L^2(\omega)}\\
&\leq\leq\sum_{k=0}^{\kappa}\sum_a\sum_{b=0}^{\infty}2^{-b}\sum_{n=0}^{\infty}(1+n)\cdot C2^{-cn}\left(2^a\sum_{P\in\mathscr{P}}\sigma(P)\right)^{1/2}\\
&\leq\leq\sum_{k=0}^{\kappa}\sum_a\sum_{b=0}^{\infty}2^{-b}\sum_{n=0}^{\infty}(1+n)\cdot C2^{-cn}\left(2^a[\sigma]_{A_\infty}\right)^{1/2}\\
&=C\cdot[\sigma]_{A_\infty}^{1/2}\sum_{k=0}^{\kappa}\left(\sum_{a\leq\lceil\log_2[\omega,\sigma]_{A_2}\rceil}2^{a/2}\right)\left(\sum_{b=0}^{\infty}2^{-b}\right)\left(\sum_{n=0}^{\infty}(1+n)\cdot 2^{-cn}\right)\\
&\leq C\cdot[\sigma]_{A_\infty}^{1/2}\cdot(1+\kappa)\cdot[\omega,\sigma]_{A_2}^{1/2},
\end{aligned}$$

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and thus the testing constant $\mathscr{S}$ is estimated by

$$\mathscr{S} \leq C \cdot (1+\kappa) \cdot [\omega, \sigma]_{A_2}^{1/2} \cdot [\sigma]_{A_\infty}^{1/2}.$$

By symmetry, exchanging the roles of ω and σ, we also have the analogous result for $\mathscr{S}^*$, and so we have completed the proof of the following

Theorem 5.5. Let $\sigma, \omega \in A_\infty$ be functions which satisfy the joint A_2 condition

$$[\omega, \sigma]_{A_2} := \sup_Q \frac{\omega(Q\sigma(Q))}{|Q|^2} < \infty.$$

Then the testing constant $\mathscr{S}$ and $\mathscr{S}^*$ associated with a dyadic shift S of type (i, j) satisfy the following bounds, where $\kappa := \max\{i, j\}$:

$$\mathfrak{S} \leq C \cdot (1+\kappa) \cdot [\omega, \sigma]_{A_2}^{1/2} \cdot [\sigma]_{A_\infty}^{1/2},$$
$$\mathfrak{S}^* \leq C \cdot (1+\kappa) \cdot [\omega, \sigma]_{A_2}^{1/2} \cdot [\sigma]_{A_\infty}^{1/2}.$$

CONCLUSIONS

In this section we simply collect the fruits of the hard work done above. A combination of Theorem 5.4 and 5.5 gives the following two-weight inequality, whose qualitative version was pointed out by Lacey, Petermichl and Reguer [10]. In the precise form as stated, this result and its consequences below were obtained by Pérez and Hytönen [9], although originally formulated only in the case that $\sigma^{-1} = \omega$ is dual weight.

Theorem 5.6. Let $\sigma, \omega \in A_\infty$ be functions which satisfy the joint A_2 condition

$$[\omega, \sigma]_{A_2} := \sup_Q \frac{\omega(Q)\sigma(Q)}{|Q|^2} < \infty.$$

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Then a dyadic shift S of type (i,j) satisfies $S(\sigma\cdot) : L^2(\sigma) \to L^2(\omega)$, and more precisely

$$\|S(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\omega)} \lesssim (1+\kappa)^2[\omega,\sigma]_{A_2}^{1/2}([\omega]_{A_\infty}^{1/2}+[\sigma]_{A_\infty}^{1/2}),$$

where $\kappa = \max\{i,j\}$.

The quantitative bound as stated, including the polynomial dependence on κ, allows to sum up these estimates in the Dyadic Representation Theorem to deduce:

Theorem 5.7. Let $\sigma, \omega \in A_\infty$ be functions which satisfy the joint A_2 condition. Then any L^2 bounded Calderón—Zygmund operator T whose kernel K has Hölder type modulus of continuity $\psi(t) = t^\alpha$, $\alpha \in (0,1)$, satisfies

$$\|T(\sigma\cdot)\|_{L^2\to L^2(\omega)} \lesssim \left(\|T\|_{L^2\to L^2}+\|K\|_{CZ_\alpha}\right)[\omega,\sigma]_{A_2}^{1/2}\left([\omega]_{A_\infty}^{1/2}+[\sigma]_{A_\infty}^{1/2}\right).$$

Recalling the dual weight trick and specializing to the one-weight situation with $\sigma = \omega^{-1}$, this in turn gives:

Theorem 5.8. Let $\omega \in A_2$. Then any L^2 bounded Calderón—Zygmund operator T whose kernel K has Hölder type modulus of continuity $\psi(t) = t^\alpha$, $\alpha \in (0,1)$, satisfies

$$\|T\|_{L^2\to L^2} \lesssim \left(\|T\|_{L^2\to L^2}+\|K\|_{CZ_\alpha}\right)[\omega,\sigma]_{A_2}^{1/2}\left([\omega]_{A_\infty}^{1/2}+[\sigma]_{A_\infty}^{1/2}\right)$$
$$\left(\|T\|_{L^2\to L^2}+\|K\|_{CZ_\alpha}\right)[\omega,\sigma]_{A_2}.$$

The second displayed line is the origin A_2 theorem [6], , and it follows from the first line by $[\omega]_{A_\infty} \lesssim [\omega]_{A_2}$ and $[\omega^{-1}]_{A_\infty} \lesssim [\omega^{-1}]_{A_2} = [\omega]_{A_2}$.

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