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離散曲面理論之探討

A survey on discrete surface theory

吳漢中
Han-Chung Wu

指導教授：崔茂培博士
Advisor: Mao-Pei Tsui, Ph.D.

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離散曲面理論之探討
A survey on discrete surface theory

本論文係吳漢中君（R06221008）在國立臺灣大學數學系完成之碩士學位論文，於民國 109 年 07 月 29 日承下列考試委員審查通過及口試及格，特此證明

口試委員：崔茂培（簽名）
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離散曲面理論之探討

學生：吳漢中　　　　　　　　　　　　　　　　指導教授：崔茂培 博士

國立台灣大學 數學系

摘　　要

在本篇論文，我們主要是探討 M. Kotani, H. Naito and T. Omori ([3]) 所提出的離散曲面理論。我們首先回顧他們論文的總體結果。然後我們討論了斜線四面體的平均曲率流的行為以及離散曲率和高斯-博內定理的收斂問題。

關鍵字：離散曲面；平均曲率；高斯曲率

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A survey on discrete surface theory

Student: Han-Chung Wu　　　　　　Advisor: Dr. Mao-Pei Tsui

Department of Mathematics
National Taiwan University

Abstract

In this thesis, we discuss discrete surface theories developed by M. Kotani, H. Naito and T. Omori in ([3]). We first review the general results from their paper. Then we discuss the behavior of the mean curvature flow of skew line tetrahedron and the issue of the convergence of discrete curvatures and Gauss-Bonnet Theorem.

Keywords: discrete surface; mean curvature; Gauss curvature

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Table of Contents

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List of Figures

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Chapter 1

The classical surface theory in $\mathbb{R}^3$

In this chapter, we briefly review basic facts of the classical surface theory in $\mathbb{R}^3$. We review the definition of first fundamental form, second fundamental form, the Weingarten map, mean curvature and Gauss curvature. See Dierkes et al.(2010) [2] for example for details.

Let $M \subseteq \mathbb{R}^3$ be a regular surface (of class C^2), which is (locally) parameterized by, say, $p = p(u, v) : \Omega \to \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^2$. The tangent plane T_pM at $p = p(u, v)$ is the vector space spanned by the partial derivatives $\partial_u p$ and $\partial_v p$ of p with respect to u and v, respectively. It is equipped with the standard inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^3$.

The *first fundamental form* $\mathrm{I} = \mathrm{I}(u, v)$ of M at $p(u, v)$ is a symmetric 2-tensor defined as

$$\mathrm{I} = dp \cdot dp = \langle \partial_u p, \partial_u p \rangle du \cdot du + 2\langle \partial_u p, \partial_v p \rangle du \cdot dv + \langle \partial_v p, \partial_v p \rangle dv \cdot dv,$$

which is also expressed by the matrix-form:

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \partial_u p, \partial_u p \rangle & \langle \partial_u p, \partial_v p \rangle \\ \langle \partial_v p, \partial_u p \rangle & \langle \partial_v p, \partial_v p \rangle \end{pmatrix}.$$

The matrix $\mathrm{I}(u, v)$ has rank 2 (positive definite) since we assume that M is regular. The *unit normal vector field*

$$n = n(u, v) = \frac{\partial_u p \times \partial_v p}{|\partial_u p \times \partial_v p|}$$

is well-defined at every point $(u, v) \in \Omega$. The *second fundamental form* $\mathrm{II} = \mathrm{II}(u, v)$

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is then defined as

$$\mathrm{II} = -dp \cdot dn = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} -\langle \partial_u p, \partial_u n \rangle & -\langle \partial_u p, \partial_v n \rangle \\ -\langle \partial_v p, \partial_u n \rangle & -\langle \partial_v p, \partial_v n \rangle \end{pmatrix},$$

which is also a symmetric tensor.

Fact 1.0.1. The partial derivatives $\partial_u n$ and $\partial_v n$ of n, which is perpendicular to n, can be represented by $\{\partial_u p, \partial_v p\}$:

$$\begin{aligned} \partial_u n &= \frac{FM - GL}{EG - F^2} \partial_u p + \frac{FL - EM}{EG - F^2} \partial_v p, \\ \partial_v n &= \frac{FN - GM}{EG - F^2} \partial_u p + \frac{FM - EN}{EG - F^2} \partial_v p. \end{aligned} \tag{1.0.1}$$

We define the Weingarten map $S = \nabla n : T_pM \to T_pM$. By the symmetry of II, S is a symmetric operator in the sense that it satisfies $\langle SV, W \rangle = \langle V, SW \rangle$ for any $V, W \in T_pM$. Half of the trace of S is called the *mean curvature* $H(p)$ and the determinant of S is called the *Gauss curvature* $K(p)$, respectively. Since the representation matrix of S with respect to $\{\partial_u p, \partial_v p\}$ is $\mathrm{I}^{-1}\mathrm{II}$, we have

Fact 1.0.2. The mean curvature $H(p)$ and the Gauss curvature $K(p)$ are defined by

$$\begin{aligned} H(p) &= \frac{1}{2} \operatorname{tr}(\mathrm{I}^{-1}\mathrm{II}) = \frac{EN + GL - 2FM}{2(EG - F^2)}, \\ K(p) &= \det(\mathrm{I}^{-1}\mathrm{II}) = \frac{LN - M^2}{EG - F^2}. \end{aligned} \tag{1.0.2}$$

It is easy to see

$$S^2 - 2H(p)S + K(p)\mathrm{Id} = 0. \tag{1.0.3}$$

We also define the *third fundamental form* $\mathrm{III} = \mathrm{III}(u, v)$ as

$$\mathrm{III} = dn \cdot dn = \begin{pmatrix} \langle \partial_u n, \partial_u n \rangle & \langle \partial_u n, \partial_v n \rangle \\ \langle \partial_v n, \partial_u n \rangle & \langle \partial_v n, \partial_v n \rangle \end{pmatrix}.$$

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Because of the symmetry of S, $\langle \partial_u n, \partial_u n\rangle = \langle S\partial_u p, S\partial_u p\rangle = \langle S^2\partial_u p, \partial_u p\rangle$ and so on, from (1.0.3) we infer

$$K(p)\mathrm{I} - 2H(p)\mathrm{II} + \mathrm{III} = 0. \tag{1.0.4}$$

We are ready to present several different meanings of the Gauss curvature. To do so let us consider the Gauss map $n : M \to \mathbb{S}^2$ from M to the unit sphere $\mathbb{S}^2$. Then the Gauss curvature appears in its area element.

Fact 1.0.3. The Gauss curvature is written as the ratio of the infinitesimal area elements:

$$|K(p(u_0, v_0))| = \lim_{\varepsilon\to 0} \frac{A_{\Omega_\varepsilon}(n)}{A_{\Omega_\varepsilon}(p)}, \tag{1.0.5}$$

where $\Omega_\varepsilon \subset \Omega$ is an ε-neighborhood of $(u_0, v_0) \in \Omega$.

Proof. It is easy by using (1.0.1) to have

$$\partial_u n \times \partial_v n = \frac{LN - M^2}{EG - F^2}(\partial_u p \times \partial_v p) = K(p)(\partial_u p \times \partial_v p). \tag{1.0.6}$$

If we take an ε-neighborhood $\Omega_\varepsilon \subseteq \Omega$ of $(u_0, v_0) \in \Omega$ for any $\varepsilon > 0$, then since

$$\begin{aligned} A_{\Omega_\varepsilon}(p) &= \int_{\Omega_\varepsilon} |\partial_u p \times \partial_v p|\, dudv, \\ A_{\Omega_\varepsilon}(n) &= \int_{\Omega_\varepsilon} |\partial_u n \times \partial_v n|\, dudv = \int_{\Omega_\varepsilon} |K||\partial_u p \times \partial_v p|\, dudv \end{aligned}$$

are the areas of the image $p(\Omega_\varepsilon) \subseteq M$ and $n(\Omega_\varepsilon) \subseteq \mathbb{S}^2$, respectively. $\square$

A variational approach is also available for the formulation of the curvatures as follows. Let $p : \Omega \to \mathbb{R}^3$ be a regular surface of class C^2. The functional $\mathcal{A}(p)$ defined as

$$\mathcal{A}(p) := \int_\Omega |\partial_u p \times \partial_v p|\, dudv = \int_\Omega dA$$

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is called the *area functional*, whose first and second variation formulas are those we want. Let $q_t = q(u, v, t) = p + tV : \Omega \times (-\varepsilon, \varepsilon)$ be a variation of p with the variation vector field, say,

$$V(u, v) = \varphi^1(u, v)\partial_u p(u, v) + \varphi^2(u, v)\partial_v p(u, v) + \psi(u, v)n(u, v)$$

where $\varphi^i, \psi \in C^1(\Omega)$ $(i = 1, 2)$.

Fact 1.0.4. The first variation of $\mathcal{A}$ at p is then given as

$$d\mathcal{A}(p, V) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(q_t) = -2\int_\Omega \psi \cdot H(p)|\partial_u p \times \partial_v p|\, dudv, \tag{1.0.7}$$

independently of variations in the tangential direction.
While the second variation of $\mathcal{A}$ at a general regular surface p with respect to the normal variation $V = \psi n$ (that is, $\varphi^1 = \varphi^2 = 0$) is given as

$$d^2\mathcal{A}(p, \psi n) = \int_\Omega \left(|\nabla_M \psi|^2 + 2\psi^2 K(p)\right)\, dA, \tag{1.0.8}$$

where the norm $|\nabla_M \psi|^2$ is taken with respect to I, sometimes called the first Beltrami differentiator.

A surface $M \subseteq \mathbb{R}^3$ satisfying $H(p) = 0$ for any point $p \in M$ is said to be *minimal*.

At the end of this chapter, we state a characterization of minimal surfaces as follows:

Fact 1.0.5. Let $p = p(u, v) : \Omega \to \mathbb{R}^3$ be a regular surface of class C^2 and $n : \Omega \to \mathbb{R}^3$ be its Gauss map. Then

$$\partial_v n \times \partial_u p - \partial_u n \times \partial_v p = 2H(p)|\partial_u p \times \partial_v p|n, \tag{1.0.9}$$

or equivalently,

$$d(n \times dp) = -2H(p)ndA$$

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where $n \times dp = (n \times \partial_u p)du + (n \times \partial_v p)dv$ is a differential 1-form on Ω along p. That is to say, $p : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if $n \times dp$ is closed.

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Chapter 2

A discrete surface theory for graphs in $\mathbb{R}^3$

In this chapter, we introduce the definition of discrete normal vector, discrete covariant derivative, discrete mean curvature and discrete Gauss curvature on an embedded trivalent graph. Most of the materials in this chapter come from the paper ([3]) by M. Kotani, H. Naito and T. Omori.

2.1 Definition of curvatures

Let $X = (V, E)$ be a locally finite graph, where V denotes the set of vertices, and E the set of the oriented edges. The oriented edge e is identified with a 1-dimensional cell complex. Thus we can assume that every edge e is identified with the interval $[0, 1]$. The reverse edge is denoted by $\bar{e}$, and E_x is the set of edges which emerge from a vertex $x \in V$.

First, we identify X with the 1-dimensional CW-complex $V \cup (E \times [0, 1])/ \sim$, where the equivalence relation $\sim$ is defined by $o(e) \sim (e, 0)$, $t(e) \sim (e, 1)$ and $(e, a) \sim (\bar{e}, 1 - a)$, where $o(e)$ and $t(e)$ is the origin and terminus of e, respectively. We define an embedding $\Phi : X \to \mathbb{R}^3$ as follows: For $x \in V$, $\Phi(x) \in \mathbb{R}^3$, which satisfies $\Phi(x) \neq \Phi(y)$ if $x \neq y$, for $e(a) \in (E \times [0, 1])/ \sim$, set $\Phi(e(a)) = a\Phi(o(e)) + (1 - a)\Phi(t(e))$. In the followings, we abbreviate $\Phi(e(a))$ to $\Phi(e)$.

Definition 2.1.1. An embedding $\Phi : X \to \mathbb{R}^3$ of a *discrete surface* if

(i) $X = (V, E)$ is a 3-valent graph, that is a graph of degree 3,

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(ii) for each $x \in V$, at least two vectors in $\{\Phi(e) \mid e \in E_x\}$ are linearly independent as vectors in $\mathbb{R}^3$,

(iii) locally oriented, that is, the order of the three edges is assumed to be assigned to each vertex of X.

As said in the introduction, since our targets have necessarily no natural faces, we should take a different approach to develop a surface theory from the existing ones such as Bobenko and Pinkall(1996) [1] or Pinkall and Polthier(1993) [4].

Let $\Phi : X = (V, E) \to M \subseteq \mathbb{R}^3$ be a discrete surface. For each vertex $x \in V$, we assume it is of 3-valent, namely the set $E_x = \{e_1, e_2, e_3\}$ of edges with origin x consists of three oriented edges. In the sequel, we sometimes use the notation $\Phi(x) = \underline{x} \in M$ to denote the vertex in M which corresponds to $x \in V$ and $\Phi(e) = \underline{e} \in M$ to denote the edge in M which corresponds to $e \in E$. The *tangent plane* T_x at $\Phi(x)$ is then the plane with $\underline{n}(x)$ as its *oriented unit normal vector*, $\underline{n}(x)$ at $\Phi(x)$ is defined as

$$
\begin{aligned}
\underline{n}(x) &:= \frac{(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)}{|(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)|} \\
&= \frac{\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1}{|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|}.
\end{aligned}
\tag{2.1.1}
$$

Note that we use the condition of graphs to be 3-valent to define its tangent plane.

Now let $x \in V$ be a vertex, $E_x = \{e_1, e_2, e_3\}$, $x_i := t(e_i)$ $(i = 1, 2, 3)$, and consider the triangle $\triangle(x) = \triangle(\underline{x}_1, \underline{x}_2, \underline{x}_3) \subseteq \mathbb{R}^3$ with ordered vertices $\underline{x}_1, \underline{x}_2, \underline{x}_3$, to each of which the unit normal vectors $\underline{n}_1 := \underline{n}(x_1)$, $\underline{n}_2 := \underline{n}(x_2)$, $\underline{n}_3 := \underline{n}(x_3)$ are assigned respectively. We set $\underline{v}_1 := \underline{e}_1 - \underline{e}_3 = \underline{x}_1 - \underline{x}_3$ and $\underline{v}_2 := \underline{e}_2 - \underline{e}_3 = \underline{x}_2 - \underline{x}_3$. The *first fundamental form* $\mathrm{I}(x)$ at x is now defined as

$$
\mathrm{I}(x) := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \underline{v}_1, \underline{v}_1 \rangle & \langle \underline{v}_1, \underline{v}_2 \rangle \\ \langle \underline{v}_2, \underline{v}_1 \rangle & \langle \underline{v}_2, \underline{v}_2 \rangle \end{pmatrix}, \tag{2.1.2}
$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product of $\mathbb{R}^3$. We also define the directional

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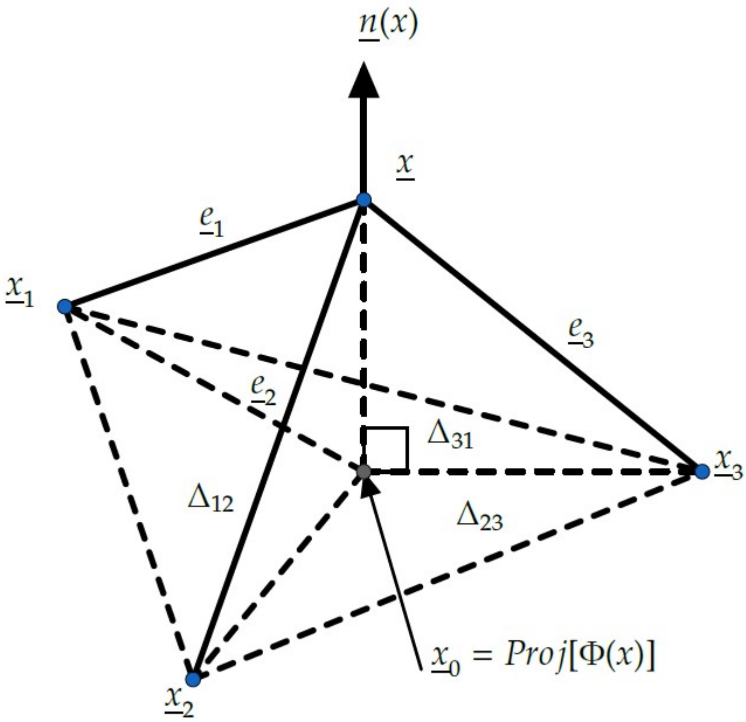

Figure 2.1: The oriented unit normal vector $\underline{n}(x)$ at $\Phi(x)$.

derivative $\nabla_i \underline{n}(x)$ of $\underline{n}$ along $\underline{v}_i$ as

$$\nabla_i \underline{n} = \nabla_i \underline{n}(x) := \text{Proj}[\underline{n}_i - \underline{n}_3] := (\underline{n}_i - \underline{n}_3) - \langle \underline{n}_i - \underline{n}_3, \underline{n}(x) \rangle \, \underline{n}(x)$$

for $i = 1, 2$, where $\underline{n}(x)$ is the unit normal vector of $\triangle(x)$. That is, Proj is the orthogonal projection onto the tangent plane T_x. As is straightforward to check, $\nabla_1 \underline{n}$ and $\nabla_2 \underline{n}$ are in fact written, respectively, as

$$\begin{aligned} \nabla_1 \underline{n} &= \frac{FM_1 - GL}{EG - F^2} \underline{v}_1 + \frac{FL - EM_1}{EG - F^2} \underline{v}_2, \\ \nabla_2 \underline{n} &= \frac{FN - GM_2}{EG - F^2} \underline{v}_1 + \frac{FM_2 - EN}{EG - F^2} \underline{v}_2, \end{aligned} \tag{2.1.3}$$

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where E, F and G are given by (2.1.2) and L, M_1, M_2 and N are defined as

$$\mathrm{II}(x) := \begin{pmatrix} L & M_2 \\ M_1 & N \end{pmatrix} = \begin{pmatrix} -\langle \underline{v}_1, \nabla_1 \underline{n} \rangle & -\langle \underline{v}_1, \nabla_2 \underline{n} \rangle \\ -\langle \underline{v}_2, \nabla_1 \underline{n} \rangle & -\langle \underline{v}_2, \nabla_2 \underline{n} \rangle \end{pmatrix} \tag{2.1.4}$$

in the *second fundamental form* at x.

Remark 2.1.2. The second fundamental form (2.1.4) can be written as

$$\mathrm{II}(x) = \begin{pmatrix} -\langle \underline{v}_1, \underline{n}_1 - \underline{n}_3 \rangle & -\langle \underline{v}_1, \underline{n}_2 - \underline{n}_3 \rangle \\ -\langle \underline{v}_2, \underline{n}_1 - \underline{n}_3 \rangle & -\langle \underline{v}_2, \underline{n}_2 - \underline{n}_3 \rangle \end{pmatrix} \tag{2.1.5}$$

because $\underline{v}_i = \underline{x}_i - \underline{x}_3$ $(i = 1, 2)$ lies on T_x whereas $\nabla_i \underline{n} = \mathrm{Proj}[\underline{n}_i - \underline{n}_3]$ is the orthogonal projection onto T_x.

Note here that $M_1 \neq M_2$ is possible in our case although the classical theory depends on the symmetry of the second fundamental form. But, there exist some graphs with symmetric second fundamental form.

Remark 2.1.3. If the graph X is just K_4 or in other words, the discrete surface $\Phi : X \to \mathbb{R}^3$ is just tetrahedron then the second fundamental form is symmetric.

Proof. Given $X = K_4$ with vertices $V = \{p_1, p_2, p_3, p_4\}$ and let $\underline{p}_a = \Phi(p_a) \in \mathbb{R}^3$ be the vertices in $\mathbb{R}^3$, $a = 1, 2, 3, 4$. W.L.O.G., by renumbering, we set that the right hand rule on $\triangle = \triangle(\underline{p}_2, \underline{p}_3, \underline{p}_4)$ is same direction as the vector from $\triangle$ to $\underline{p}_1$.(see Fig. 2.2) First, we focus on p_1, the set $E_{p_1} = \{e_1 = \{p_1, p_2\}, e_2 = \{p_1, p_3\}, e_3 = \{p_1, p_4\}\}$ of edges with origin p_1 consists of three oriented edges. Then, the unit normal vector on p_1 is defined as

$$\underline{n}(p_1) := \frac{(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)}{|(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)|} = \frac{(\underline{p}_2 - \underline{p}_4) \times (\underline{p}_3 - \underline{p}_4)}{|(\underline{p}_2 - \underline{p}_4) \times (\underline{p}_3 - \underline{p}_4)|},$$

Thus, $\underline{n}(p_1)$ is orthogonal to $\underline{p}_2 - \underline{p}_4$ and $\underline{p}_3 - \underline{p}_4$. Similarly for all unit normal vectors $\underline{n}(p_a) := \underline{n}_a$ on every points.

Let $(i, j, k, l) = (1, 2, 3, 4), (2, 3, 1, 4), (3, 4, 1, 2)$ or $(4, 1, 3, 2)$. Then, easy to see that

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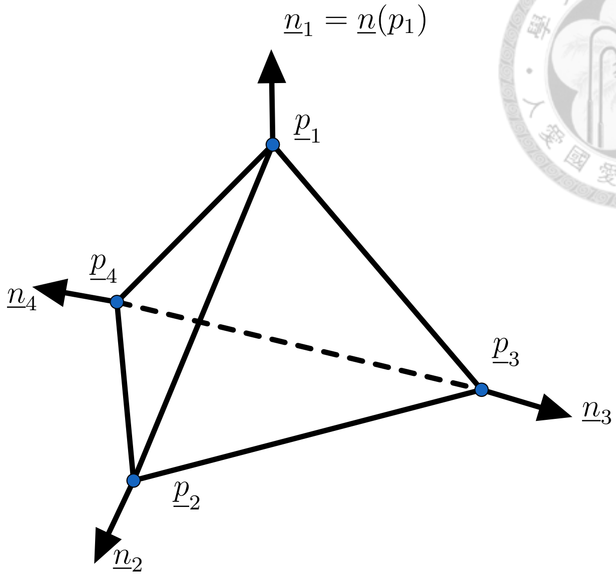

Figure 2.2: A tetrahedron with normal.

the right hand rule on $\triangle = \triangle(\underline{p}_j, \underline{p}_k, \underline{p}_l)$ is same direction as the vector from $\triangle$ to $\underline{p}_i$. By above argument, if we focus on p_j, we have

$$\underline{n}_j \perp (\underline{p}_k - \underline{p}_l).$$

But we back to p_i, from the definition,

$$\begin{aligned} -M_1(p_i) &=< \underline{v}_2, \nabla_1 \underline{n} >=< \underline{p}_k - \underline{p}_l, \underline{n}_j - \underline{n}_l > \\ &=< \underline{p}_k - \underline{p}_l, -\underline{n}_l > . \end{aligned}$$

Similarly,

$$\begin{aligned} -M_2(p_i) &=< \underline{v}_1, \nabla_2 \underline{n} >=< \underline{p}_j - \underline{p}_l, \underline{n}_k - \underline{n}_l > \\ &=< \underline{p}_j - \underline{p}_l, -\underline{n}_l > . \end{aligned}$$

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Thus, we have

$$
\begin{aligned}
M_2(p_i) - M_1(p_i) &= < \underline{p}_k - \underline{p}_l, -\underline{n}_l > - < \underline{p}_j - \underline{p}_l, -\underline{n}_l > \\
&= < \underline{p}_k - \underline{p}_j, -\underline{n}_l > \\
&= 0.
\end{aligned}
$$

This shows that second fundamental form is symmetric at p_i for all i, as desired. □

The rest of the discussion of the symmetry of a discrete surface is in Chapter 3. We now focus on the definition of discrete curvatures.

Definition 2.1.4. For a discrete surface $\Phi : X = (V, E) \to \mathbb{R}^3$, the *mean curvature* $H(x)$ and the *Gauss curvature* $K(x)$ at $x \in V$ are defined, respectively, as

$$H(x) := \frac{1}{2} \operatorname{tr} S_x, \tag{2.1.6}$$

$$K(x) := \det S_x, \tag{2.1.7}$$

where $S_x : T_x \to T_x$, the Weingarten-type map, is defined as $S_x = -\nabla \underline{n}(x)$.
A discrete surface is said to be *minimal* if its mean curvature vanishes at every vertex.

The following result comes from definition of S_x.

Proposition 2.1.5. *The mean curvature $H(x)$ and the Gauss curvature $K(x)$ have, respectively, the following representations:*

$$
\begin{aligned}
H(x) &= \frac{1}{2} \operatorname{tr}(\mathrm{I}(x)^{-1}\mathrm{II}(x)) = \frac{EN + GL - F(M_1 + M_2)}{2(EG - F^2)}, \\
K(x) &= \det(\mathrm{I}(x)^{-1}\mathrm{II}(x)) = \frac{LN - M_1 M_2}{EG - F^2}.
\end{aligned}
\tag{2.1.8}
$$

Proof. Let $\beta_i = \underline{v}_i$ for $i = 1, 2$. Then from (2.1.3),

$$S_x(\beta_1) = -\nabla_{\underline{v}_1} \underline{n}(x) = \frac{GL - FM_1}{EG - F^2} \beta_1 + \frac{EM_1 - FL}{EG - F^2} \beta_2,$$

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$$S_x(\beta_2) = -\nabla_{\underline{v}_2}\underline{n}(x) = \frac{GM_2 - FN}{EG - F^2}\beta_1 + \frac{EN - FM_2}{EG - F^2}\beta_2,$$

Then, the representation of S_x relative to this base is

$$\begin{aligned}
[S_x]_\beta &= \frac{1}{EG - F^2}\begin{pmatrix} GL - FM_1 & GM_2 - FN \\ EM_1 - FL & EN - FM_2 \end{pmatrix} \\
&= \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M_2 \\ M_1 & N \end{pmatrix} = \mathrm{I}(x)^{-1}\mathrm{II}(x).
\end{aligned}$$

And then we get the result. □

The *third fundamental form* $\mathrm{III}(x)$ at $x \in V$ is now defined as

$$\mathrm{III}(x) := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \langle \nabla_1\underline{n}(x), \nabla_1\underline{n}(x)\rangle & \langle \nabla_1\underline{n}(x), \nabla_2\underline{n}(x)\rangle \\ \langle \nabla_2\underline{n}(x), \nabla_1\underline{n}(x)\rangle & \langle \nabla_2\underline{n}(x), \nabla_2\underline{n}(x)\rangle \end{pmatrix}. \tag{2.1.9}$$

Proposition 2.1.6. *With the definition, we have*

$$K(x)\mathrm{I}(x) - 2H(x)\mathrm{II}(x) + \mathrm{III}(x) = \frac{M_1 - M_2}{EG - F^2}\begin{pmatrix} EM_1 - FL & EN - FM_2 \\ FM_1 - GL & FN - GM_2 \end{pmatrix}. \tag{2.1.10}$$

In particular, the second fundamental form $\mathrm{II}(x)$ *is symmetric if and only if*

$$K(x)\mathrm{I}(x) - 2H(x)\mathrm{II}(x) + \mathrm{III}(x) = 0.$$

Proof. A straightforward computation using (2.1.3) gives

$$\begin{aligned}
c_{11} &= \frac{EM_1^2 - 2FLM_1 + GL^2}{EG - F^2}, \\
c_{12} = c_{21} &= \frac{EM_1N - FLN - FM_1M_2 + GLM_2}{EG - F^2}, \\
c_{22} &= \frac{EN^2 - 2FM_2N + GM_2^2}{EG - F^2}.
\end{aligned}$$

These equalities combined with (2.1.2), (2.1.4) and (2.1.8) yield the required equality.

□

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On the other hand, we can obtain the following.

Proposition 2.1.7. *The Gauss curvature $K(x)$ satisfies*

$$\nabla_1 \underline{n}(x) \times \nabla_2 \underline{n}(x) = K(x)(\underline{v}_1 \times \underline{v}_2). \tag{2.1.11}$$

Thus, in particular, the absolute value of the Gauss curvature $K(x)$ is given by

$$|K(x)| = \frac{|\nabla_1 \underline{n}(x) \times \nabla_2 \underline{n}(x)|}{|\underline{v}_1 \times \underline{v}_2|}.$$

Proof. The proof again follows from a direct computation using (2.1.3) as follows:

$$\begin{aligned}
\nabla_1 \underline{n}(x) \times \nabla_2 \underline{n}(x) &= \left(\frac{FM_1 - GL}{EG - F^2}\underline{v}_1 + \frac{FL - EM_1}{EG - F^2}\underline{v}_2\right) \\
&\quad \times \left(\frac{FN - GM_2}{EG - F^2}\underline{v}_1 + \frac{FM_2 - EN}{EG - F^2}\underline{v}_2\right) \\
&= \frac{\underline{v}_1 \times \underline{v}_2}{(EG - F^2)^2}\{(FM_1 - GL)(FM_2 - EN) \\
&\quad - (FL - EM_1)(FN - GM_2)\} \\
&= \frac{\underline{v}_1 \times \underline{v}_2}{(EG - F^2)^2}\{(FM_1FM_2 + GLEN) \\
&\quad - (FLFN + EM_1GM_2)\} \\
&= \frac{(LN - M_1M_2)(EG - F^2)}{(EG - F^2)^2}(\underline{v}_1 \times \underline{v}_2) \\
&= \frac{LN - M_1M_2}{EG - F^2}(\underline{v}_1 \times \underline{v}_2) \\
&= K(x)(\underline{v}_1 \times \underline{v}_2),
\end{aligned}$$

as required. □

Remark 2.1.8. In fact, $H(x)$ and $K(x)$ defined above can be also written by the area-weighted average of the three curvatures around the vertex x.

To this end, we prepare several notations. Let $x \in V$ be a vertex, $E_x = \{e_1, e_2, e_3\}$

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and $(\alpha,\beta)=(1,2),(2,3)$ or $(3,1)$. If we choose the triangle $\triangle_{\alpha\beta}=\triangle(\underline{x}_0,\underline{x}_\alpha,\underline{x}_\beta)$ as

$$\underline{x}_0=\text{Proj}[\Phi(x)],\quad \underline{x}_\alpha=\Phi(t(e_\alpha))\text{ and }\underline{x}_\beta=\Phi(t(e_\beta)),$$

(see Fig. 2.1) then the first, second and third fundamental form of $\triangle_{\alpha\beta}$, are defined as

$$\begin{aligned}
\mathrm{I}_{\alpha\beta}(x)&=\begin{pmatrix}\langle\nabla_{e_\alpha}\Phi,\nabla_{e_\alpha}\Phi\rangle & \langle\nabla_{e_\alpha}\Phi,\nabla_{e_\beta}\Phi\rangle\\ \langle\nabla_{e_\beta}\Phi,\nabla_{e_\alpha}\Phi\rangle & \langle\nabla_{e_\beta}\Phi,\nabla_{e_\beta}\Phi\rangle\end{pmatrix},\\
\mathrm{II}_{\alpha\beta}(x)&=\begin{pmatrix}-\langle\nabla_{e_\alpha}\Phi,\nabla_{e_\alpha}\underline{n}\rangle & -\langle\nabla_{e_\alpha}\Phi,\nabla_{e_\beta}\underline{n}\rangle\\ -\langle\nabla_{e_\beta}\Phi,\nabla_{e_\alpha}\underline{n}\rangle & -\langle\nabla_{e_\beta}\Phi,\nabla_{e_\beta}\underline{n}\rangle\end{pmatrix},\\
\mathrm{III}_{\alpha\beta}(x)&=\begin{pmatrix}\langle\nabla_{e_\alpha}\underline{n},\nabla_{e_\alpha}\underline{n}\rangle & \langle\nabla_{e_\alpha}\underline{n},\nabla_{e_\beta}\underline{n}\rangle\\ \langle\nabla_{e_\beta}\underline{n},\nabla_{e_\alpha}\underline{n}\rangle & \langle\nabla_{e_\beta}\underline{n},\nabla_{e_\beta}\underline{n}\rangle\end{pmatrix},
\end{aligned}$$

respectively, where for $e\in E_x$, and the derivatives are defined as

$$\nabla_e\Phi:=\text{Proj}[\Phi(e)]=\underline{e}-\langle\underline{e},\underline{n}(x)\rangle\,\underline{n}(x),\quad \nabla_e\underline{n}:=\text{Proj}[\underline{n}(t(e))-\underline{n}(o(e))],$$

so that $\nabla_e\Phi,\nabla_e\underline{n}\in T_x$. Under this settings, we can define the mean curvature $H_{\alpha\beta}(x)$ and the Gauss curvature $K_{\alpha\beta}(x)$ for $\triangle_{\alpha\beta}$ similarly as we defined $H(x)$ and $K(x)$ for the triangle $\triangle(x)=\triangle(\underline{x}_1,\underline{x}_2,\underline{x}_3)$,

$$\begin{aligned}
H_{\alpha\beta}&:=\frac{1}{2}\text{tr}\,S_{\alpha\beta},\\
K_{\alpha\beta}&:=\det S_{\alpha\beta},
\end{aligned}$$

where $S_{\alpha\beta}:T_x\to T_x$, the Weingarten-type map, is defined as

$$\begin{aligned}
S_{\alpha\beta}(\nabla_{e_\alpha}\Phi)&=-\nabla_{e_\alpha}\underline{n},\\
S_{\alpha\beta}(\nabla_{e_\beta}\Phi)&=-\nabla_{e_\beta}\underline{n}.
\end{aligned}$$

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Then

$$H(x) = \sum_{\alpha,\beta} \operatorname{sign}_{\alpha\beta}(x) \frac{A_{\alpha\beta}(x)}{A(x)} H_{\alpha\beta}(x), \tag{2.1.12}$$

$$K(x) = \sum_{\alpha,\beta} \operatorname{sign}_{\alpha\beta}(x) \frac{A_{\alpha\beta}(x)}{A(x)} K_{\alpha\beta}(x), \tag{2.1.13}$$

where the summations are taken over any $(\alpha, \beta) \in \{(1, 2), (2, 3), (3, 1)\}$, also, $A(x)$ is half of the denominator of (2.1.1):

$$A(x) := \frac{1}{2}|(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)| = \frac{1}{2}|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|$$

is the area of the triangle $\triangle(x) = \triangle(\underline{x}_1, \underline{x}_2, \underline{x}_3)$, and $A_{\alpha\beta}(x)$:

$$A_{\alpha\beta}(x) := \frac{1}{2}|\nabla_{e_\alpha}\Phi \times \nabla_{e_\beta}\Phi| = \frac{1}{2}|(\underline{x}_\alpha - \underline{x}_0) \times (\underline{x}_\beta - \underline{x}_0)| = \frac{1}{2}|\underline{x}_\alpha \times \underline{x}_\beta + \underline{x}_\beta \times \underline{x}_0 + \underline{x}_0 \times \underline{x}_\alpha|$$

is the area of the triangle $\triangle_{\alpha\beta} = \triangle(\underline{x}_0, \underline{x}_\alpha, \underline{x}_\beta)$. And $\operatorname{sign}_{\alpha\beta}(x)$ is the difference of orientations between $\triangle(x)$ and $\triangle_{\alpha\beta}(x)$:

$$\operatorname{sign}_{\alpha\beta}(x) = \begin{cases} 1, & \text{if } \triangle\,(x) \text{ and } \triangle_{\alpha\beta}\,(x) \text{ are same direction of orientation.} \\ -1, & \text{if } \triangle\,(x) \text{ and } \triangle_{\alpha\beta}\,(x) \text{ are opposite direction of orientation.} \end{cases}$$

2.2 Harmonic and minimal surface

Definition 2.2.1. Let $X = (V, E, m)$ be a weighted graph with weight $m : E \to (0, \infty)$ satisfying $m(e) = m(\bar{e})$. A discrete surface $\Phi : X = (V, E, m) \to \mathbb{R}^3$ is said to be *harmonic with weight* m if it is a harmonic realization with weight m, that is, if it satisfies

$$m(e_{x,1})\Phi(e_{x,1}) + m(e_{x,2})\Phi(e_{x,2}) + m(e_{x,3})\Phi(e_{x,3}) = \underline{0} \tag{2.2.1}$$

for every vertex $x \in V$, where $E_x = \{e_{x,1}, e_{x,2}, e_{x,3}\}$.

Exact representation of H and K in the case of discrete harmonic surfaces is given

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as follows.

Proposition 2.2.2. *Let $X = (V, E, m)$ be a weighted graph with weight $m : E \to (0, \infty)$ satisfying $m(e) = m(\bar{e})$, and $\Phi : X = (V, E, m) \to \mathbb{R}^3$ be a 3-valent discrete harmonic surface, $x \in V$ be fixed and $E_x = \{e_1, e_2, e_3\}$. Then the mean curvature $H(x)$ and the Gauss curvature $K(x)$ are, respectively, written as*

$$H(x) = \frac{m_1 + m_2 + m_3}{8A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\langle \underline{e}_\alpha, \underline{e}_\beta \rangle (\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{n}_\alpha, \underline{e}_\beta \rangle)}{m_\gamma}, \tag{2.2.2}$$

$$K(x) = -\frac{m_1 + m_2 + m_3}{4A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\langle \underline{e}_\alpha, \underline{n}_\beta \rangle \langle \underline{e}_\beta, \underline{n}_\alpha \rangle}{m_\gamma}, \tag{2.2.3}$$

where $m_i = m(e_i)$, $A(x) = |\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|/2$, $\underline{e}_i = \nabla_{e_i}\Phi = \Phi(e_i) \in T_x$ is a tangent vector at $\Phi(x)$, $\underline{n}_i = \underline{n}(t(e_i))$ is the oriented unit normal vector at each adjacent vertex of $\Phi(x)$, for $i = 1, 2, 3$, and the summations are taken over any $(\alpha, \beta, \gamma) = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

Proof. We derive $H(x)$ and $K(x)$ by using (2.1.12) and (2.1.13). We first make the following observations which are easily proved from (2.2.1):

(i) Every $\Phi(e_i)$ lies on the tangent plane T_x at $\Phi(x)$, so that $\underline{e}_i = \nabla_{e_i}\Phi = \Phi(e_i) \in T_x$ for $i = 1, 2, 3$.

(ii) $m_3^{-1}(\underline{e}_1 \times \underline{e}_2) = m_1^{-1}(\underline{e}_2 \times \underline{e}_3) = m_2^{-1}(\underline{e}_3 \times \underline{e}_1)$ and is parallel to $\underline{n}(x)$. This, means that the normal vector of any point in the surface is perpendicular to those edges joining the point.

Let $(\alpha, \beta) = (1, 2)$, $(2, 3)$ or $(3, 1)$ be fixed. The first fundamental form $\mathrm{I}_{\alpha\beta}$ and the second fundamental form $\mathrm{II}_{\alpha\beta}$ of the triangle $\triangle_{\alpha\beta} = \triangle(\Phi(x), t(\underline{e}_\alpha), t(\underline{e}_\beta))$ (see Remark 2.1.8) are, respectively, written as

$$\mathrm{I}_{\alpha\beta} = \begin{pmatrix} \langle \underline{e}_\alpha, \underline{e}_\alpha \rangle & \langle \underline{e}_\alpha, \underline{e}_\beta \rangle \\ \langle \underline{e}_\beta, \underline{e}_\alpha \rangle & \langle \underline{e}_\beta, \underline{e}_\beta \rangle \end{pmatrix}, \quad \mathrm{II}_{\alpha\beta} = \begin{pmatrix} 0 & -\langle \underline{e}_\alpha, \underline{n}_\beta \rangle \\ -\langle \underline{e}_\beta, \underline{n}_\alpha \rangle & 0 \end{pmatrix}$$

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because $\langle \underline{e}_\alpha, \underline{n}_\alpha \rangle = 0 = \langle \underline{e}_\beta, \underline{n}_\beta \rangle$ by (ii). Then we have

$$H_{\triangle_{\alpha\beta}} = \frac{\langle \underline{e}_\alpha, \underline{e}_\beta \rangle \left(\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{e}_\beta, \underline{n}_\alpha \rangle \right)}{2(|\underline{e}_\alpha|^2 |\underline{e}_\beta|^2 - \langle \underline{e}_\alpha, \underline{e}_\beta \rangle^2)}, \tag{2.2.4}$$

$$K_{\triangle_{\alpha\beta}} = -\frac{\langle \underline{e}_\alpha, \underline{n}_\beta \rangle \langle \underline{e}_\beta, \underline{n}_\alpha \rangle}{|\underline{e}_\alpha|^2 |\underline{e}_\beta|^2 - \langle \underline{e}_\alpha, \underline{e}_\beta \rangle^2}. \tag{2.2.5}$$

Here we note that

$$|\underline{e}_\alpha|^2 |\underline{e}_\beta|^2 - \langle \underline{e}_\alpha, \underline{e}_\beta \rangle^2 = \det \mathbf{I}_{\alpha\beta} = |\underline{e}_\alpha \times \underline{e}_\beta|^2 = \frac{4A(x)^2 m_\gamma^2}{(m_1 + m_2 + m_3)^2},$$

where $\gamma \neq \alpha, \beta$. The desired expressions are now immediately obtained from

$$\begin{aligned} \frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} H_{\triangle_{\alpha\beta}} &= \frac{1}{4A(x)\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}} \langle \underline{e}_\alpha, \underline{e}_\beta \rangle \left(\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{e}_\beta, \underline{n}_\alpha \rangle \right) \\ &= \frac{m_1 + m_2 + m_3}{8A(x)^2} \frac{\langle \underline{e}_\alpha, \underline{e}_\beta \rangle \left(\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{e}_\beta, \underline{n}_\alpha \rangle \right)}{m_\gamma} \\ \frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} K_{\triangle_{\alpha\beta}} &= \frac{m_1 + m_2 + m_3}{4A(x)^2} \frac{-\langle \underline{e}_\alpha, \underline{n}_\beta \rangle \langle \underline{e}_\beta, \underline{n}_\alpha \rangle}{m_\gamma} \end{aligned}$$

□

A discrete harmonic surface needs not be minimal in the sense of Definition 2.1.4, but we can provide a sufficient condition for a harmonic surface to be minimal, which is corresponding to the conformality of graphs.

Theorem 2.2.3. *Let $X = (V, E, m)$ be a weighted graph with $m : E \to (0, \infty)$ satisfying $m(e) = m(\overline{e})$. A 3-valent harmonic discrete surface $\Phi : X = (V, E, m) \to \mathbb{R}^3$ is minimal if*

$$\langle \Phi(e_1), \Phi(e_2) \rangle = \langle \Phi(e_2), \Phi(e_3) \rangle = \langle \Phi(e_3), \Phi(e_1) \rangle \tag{2.2.6}$$

holds at every $x \in V$, where $E_x = \{e_1, e_2, e_3\}$. Moreover, if $m : E \to (0, \infty)$ is

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constant, then the condition (2.2.6) *is equivalent to*

$$|\Phi(e_1)| = |\Phi(e_2)| = |\Phi(e_3)| \tag{2.2.7}$$

Proof. We use the same notation as in Proposition 2.2.2. We then sort (2.2.2) by terms involving the common $\underline{n}_\alpha$ to compute

$$\begin{aligned} H(x) &= \frac{m_1+m_2+m_3}{8A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\langle \underline{e}_\alpha, \underline{e}_\beta\rangle(\langle \underline{e}_\alpha, \underline{n}_\beta\rangle + \langle \underline{n}_\alpha, \underline{e}_\beta\rangle)}{m_\gamma} \\ &= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \sum_{(\alpha,\beta,\gamma)} m_\alpha m_\beta \langle \underline{e}_\alpha, \underline{e}_\beta\rangle(\langle \underline{e}_\alpha, \underline{n}_\beta\rangle + \langle \underline{n}_\alpha, \underline{e}_\beta\rangle) \\ &= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \sum_{(\alpha,\beta,\gamma)} \{m_\alpha m_\beta \langle \underline{e}_\alpha, \underline{e}_\beta\rangle\langle \underline{e}_\beta, \underline{n}_\alpha\rangle + m_\gamma m_\alpha \langle \underline{e}_\gamma, \underline{e}_\alpha\rangle\langle \underline{n}_\gamma, \underline{e}_\alpha\rangle\} \\ &= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \sum_{(\alpha,\beta,\gamma)} m_\alpha \langle \langle \underline{e}_\alpha, \underline{e}_\beta\rangle m_\beta \underline{e}_\beta + \langle \underline{e}_\gamma, \underline{e}_\alpha\rangle m_\gamma \underline{n}_\gamma, \underline{e}_\alpha\rangle, \end{aligned}$$

which equals zero provided (2.2.6); $\langle \underline{e}_1, \underline{e}_2\rangle = \langle \underline{e}_2, \underline{e}_3\rangle = \langle \underline{e}_3, \underline{e}_1\rangle$ holds because $m_\beta \underline{e}_\beta + m_\gamma \underline{e}_\gamma = -m_\alpha \underline{e}_\alpha$ is perpendicular to $\underline{n}_\alpha$.

Moreover, if the weight $m : E \to (0, \infty)$ is constant, then the equation (2.2.1) becomes $\underline{e}_1 + \underline{e}_2 + \underline{e}_3 = 0$, which gives

$$|\underline{e}_\alpha|^2 = -\langle \underline{e}_\alpha, \underline{e}_\beta\rangle - \langle \underline{e}_\gamma, \underline{e}_\alpha\rangle$$

$$|\underline{e}_\beta|^2 = -\langle \underline{e}_\alpha, \underline{e}_\beta\rangle - \langle \underline{e}_\gamma, \underline{e}_\beta\rangle$$

after taking the inner product with $\underline{e}_\alpha$ and $\underline{e}_\beta$. This shows $|\underline{e}_\alpha| = |\underline{e}_\beta|$ if and only if $\langle \underline{e}_\gamma, \underline{e}_\alpha\rangle = \langle \underline{e}_\gamma, \underline{e}_\beta\rangle$. □

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Chapter 3

Discrere surface structure on a sphere

In this chapter, we mainly compute the defined discrete curvature of 3-valent graphs on a sphere. We find a criterion for discrete curvatures that corresponds well in the cases above: when normal vectors of the 3-valent graph equal those of the surface at each vertex, the discrete curvature corresponds to the curvature of the continuous surface.

3.1 Plane graphs

A 3-valent discrete surface $\Phi : X = (V, E) \to \mathbb{R}^3$ is said to be a *plane* if its image $\Phi(X)$ lies on a plane in $\mathbb{R}^3$. Since the second fundamental form of a plane vanishes identically, independently of the choice of its side at each point, so do both its mean curvature and Gauss curvature. Since its third fundamental form again vanishes, the second variation of the area functional also vanishes.

3.2 Sphere-shaped graphs

Proposition 3.2.1. *Let $X = (V, E)$ be a finite graph, $\mathbb{S}^2(r) \subseteq \mathbb{R}^3$ be the round sphere with radius $r > 0$ and with center at the origin, and $\Phi : X = (V, E) \to \mathbb{S}^2(r)$ be a 3-valent discrete surface with the property that*

$$\Phi(x) = r\underline{n}(x) \tag{3.2.1}$$

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for every vertex $\underline{x} \in V$, where $\underline{n}(x)$ is the oriented unit normal vector at $x \in V$. Then the mean curvature H and the Gauss curvature K of Φ are given, respectively, as

$$H(x) = -\frac{1}{r},\ K(x) = \frac{1}{r^2} \tag{3.2.2}$$

regardless of $x \in V$.

Proposition 3.2.1 is obtained by direct calculations using (3.2.1). And we have some necessary and sufficient condition for (3.2.1).

Proposition 3.2.2. *Let $X = (V, E)$ be a finite graph, $\mathbb{S}^2(r) \subseteq \mathbb{R}^3$ be the round sphere with radius $r > 0$ and with center at the origin, and $\Phi : X = (V, E) \to \mathbb{S}^2(r)$ be a 3-valent discrete surface. Let $x_0 \in V$, $E_{x_0} = \{e_1, e_2, e_3\}$ and $x_i := t(e_i)$ ($i = 1, 2, 3$), then $\Phi(x_0) = r\underline{n}(x_0)$ if and only if*

$$|\underline{e}_1| = |\underline{e}_2| = |\underline{e}_3|. \tag{3.2.3}$$

Moreover, we have $\Phi(x) = r\underline{n}(x)$ for all $x \in V$ if and only if all of edges in $\Phi(X)$ have equal length, i.e. it is equilateral.

This proposition is just obtained from Φ has range in sphere and the definition of normal $\underline{n}$.

Corollary 3.2.3. *(1) a regular hexahedron, (2) a regular dodecahedron and (3) a regular truncated icosahedron (fullerene C_{60}) are all 3-valent discrete surfaces with constant curvatures:*

$$H(x) = -\frac{1}{r},\ K(x) = \frac{1}{r^2}$$

where $r > 0$ is the radius of the round sphere on which these surfaces lie.

Proof. It is easily from Proposition 3.2.1 and Proposition 3.2.2 because all of them are equilateral. □

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But, how about the surface on sphere with unequal length edges? Here we give some examples. Note that we use spherical coordinate system (r, θ, ϕ) with $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. Then, we will defined two types of graph on the sphere at below. Note that r will be fixed and we only need to consider the graph as $(\theta, \phi)^T$ on $\mathbb{R}^2$ and map it to $\mathbb{R}^3$ with ρ_r, where

$$\rho_r : \begin{pmatrix} \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r \sin\phi \cos\theta \\ r \sin\phi \sin\theta \\ r \cos\phi \end{pmatrix}.$$

In the following, we will calculate the mean curvature and the Gauss curvature of a spherical brick graph $\text{Sph}_k(r, h, v)$. The brick graph $\text{Br}_k(h, v)$ which has two direction, $k = 1$ means the bricks stacked vertically, and $k = 2$ means them stacked horizontally. And the spherical brick graph $\text{Sph}_k(h, v)$ is the image of $\text{Br}_k(h, v)$ under ρ.

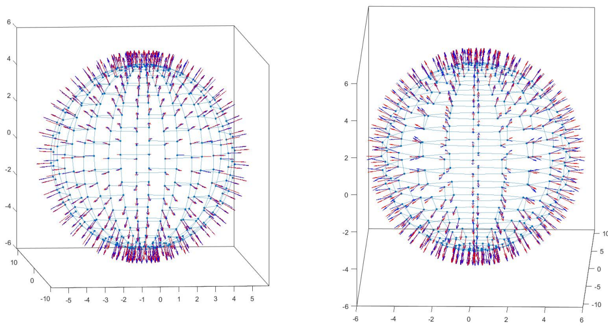

Figure 3.1: The spherical brick graph and the normal on it. The left hand side is type $k = 1$ (or type x) and the right hand side is type $k = 2$ (or type y).

First, for the type $k = 1$, when given h and v, we divide $[0, 2\pi)$ in to $2h$ parts, give $2h + 1$ equal points, $x_i = \frac{\pi i}{h}$ for $i = 0, ..., 2h$, and every part of length $x_{i+1} - x_i = \frac{\pi}{h}$. And divide $[0, \pi]$ in to $v+1$ parts, give $v+2$ equal points, $y_j = \frac{\pi j}{v+1}$ for $j = 0, 1, ..., v+1$, and every part of length $y_{j+1} - y_j = \frac{\pi}{v+1}$.

Secondly, we connect the points. In this process, we ignore line $y = y_0$ and line $y = y_{v+1}$ since them all map to the two poles on the sphere. And, we connect all vertical line, that is all $(x_i, y_j)^T$ to $(x_i, y_{j+1})^T$ with $i = 0, ..., 2h,\ j = 1, ..., v - 1$. For the horizontal line, we connect the whole first and final line, $(x_i, y_1)^T$ to $(x_{i+1}, y_1)^T$

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and $(x_i, y_v)^T$ to $(x_{i+1}, y_v)^T$ with $i = 0, ..., 2h-1$. And now, since we want to make a brick, we might not connect all the horizontal line, but skip one between every two of them. More precisely, we connect $(x_i, y_j)^T$ to $(x_{i+1}, y_j)^T$ when i, j are both even or both odd for $i = 0, ..., 2h-1,\ j = 2, ..., v-1$.

When given h, v we set

$$\underline{a}_1 := \begin{pmatrix} \frac{\pi}{h} \\ 0 \end{pmatrix}, \quad \underline{a}_2 := \begin{pmatrix} 0 \\ \frac{\pi}{v+1} \end{pmatrix}$$

and then the set of vertices $V(\mathrm{Br}_1(h,v))$ and the set of edges $E(\mathrm{Br}_1(h,v))$ of the brick graph of type one can be represented as

$$V(\mathrm{Br}_1(h,v)) = \left\{ \underline{\xi} = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 \;\middle|\; \alpha_1 \in \{0, ..., 2h\}, \alpha_2 \in \{1, ..., v\} \right\},$$

$$E(\mathrm{Br}_1(h,v)) = \left\{ (\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2, \beta_1 \underline{a}_1 + \beta_2 \underline{a}_2) \;\middle|\; \begin{array}{c} \alpha_1 = \beta_1, \alpha_2 - \beta_2 = \pm 1 \\ \alpha_2 = \beta_2 = 1 \text{ or } v, \alpha_1 - \beta_1 = \pm 1 \\ \alpha_2 = \beta_2 = 2, ..., v-1 \text{ with} \\ \alpha_1 - \beta_1 = -1, \text{ if } \alpha_1, \alpha_2 \text{ has same parity or} \\ \alpha_1 - \beta_1 = +1, \text{ if } \alpha_1, \alpha_2 \text{ has different parity} \end{array} \right\}.$$

Definition 3.2.4. For any pair of integers $(h, v) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $h > 0$ and $v > 2$ and $r > 0$, a *spherical brick of type one* (sometimes we call *type x*) $\mathrm{Sph}_1(r, h, v)$ is a 3-valent discrete surface $\Phi_{1,r,h,v} : \mathrm{Br}_1(h,v) = (V(h,v), E(h,v)) \to \mathbb{S}(r) \subset \mathbb{R}^3$ defined by ρ_r, which is defined as above. See Fig. 3.1 for an example. More precisely, $\mathrm{Sph}_1(r, h, v)$ is the embedded graph in $\mathbb{R}^3$ with

$$V(\mathrm{Sph}_1(r,h,v)) = \Phi_{1,r,h,v}(V(h,v))$$
$$E(\mathrm{Sph}_1(r,h,v)) = \{(\Phi_{r,h,v}(\underline{x}), \Phi_{1,r,h,v}(\underline{y})) \mid (\underline{x}, \underline{y}) \in E(h,v)\}.$$

Next, for the type $k = 2$, when given h and v, we again divide $[0, 2\pi)$ in to $2h$ parts, divide $[0, \pi]$ in to $v+1$ parts, get the equal points, $x_i = \frac{\pi i}{h}$ for $i = 0, ..., 2h$, and $y_j = \frac{\pi j}{v+1}$ for $j = 0, 1, ..., v+1$. However, now, we are not going to use these y_js, but

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to use $y'_j = \frac{\pi(j+\frac{1}{2})}{v+1}$ for $j = 0, 1, ..., v$.

And, we connect the points. In the type two, we connect all horizontal line, that is all $(x_i, y'_j)^T$ to $(x_{i+1}, y'_j)^T$ with $i = 0, ..., 2h-1,\ j = 0, ..., v$. For the vertical line, since we again skip one between every two of them. More precisely, we connect $(x_i, y'_j)^T$ to $(x_i, y'_{j+1})^T$ when i, j are both even or both odd for $i = 0, ..., 2h,\ j = 0, ..., v-1$.

When given h, v we set

$$\underline{a}_1 := \begin{pmatrix} \frac{\pi}{h} \\ 0 \end{pmatrix}, \quad \underline{a}_2 := \begin{pmatrix} 0 \\ \frac{\pi}{v+1} \end{pmatrix}$$

and then the set of vertices $V(\mathrm{Br}_2(h, v))$ and the set of edges $E(\mathrm{Br}_2(h, v))$ of the brick graph of type two can be represented as

$$V(\mathrm{Br}_2(h, v)) = \left\{ \underline{\xi} = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 \;\middle|\; \alpha_1 \in \{0, ..., 2h\}, \alpha_2 \in \{0, ..., v\} + \frac{1}{2} \right\},$$

$$E(\mathrm{Br}_2(h, v)) = \{(\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2, \beta_1 \underline{a}_1 + \beta_2 \underline{a}_2) \mid \alpha_2 = \beta_2, \alpha_1 - \beta_1 = \pm 1$$

$$\left. \begin{array}{c} \alpha_1 = \beta_1 = 0, ..., 2h-1 \text{ with} \\ \alpha_2 - \beta_2 = -1, \text{ if } \alpha_1, \alpha_2 - \frac{1}{2} \text{ has same parity or} \\ \alpha_2 - \beta_2 = +1, \text{ if } \alpha_1, \alpha_2 - \frac{1}{2} \text{ has different parity} \end{array} \right\}.$$

Definition 3.2.5. For any pair of integers $(h, v) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $h > 0$ and $v > 2$ and $r > 0$, a *spherical brick of type two* (sometimes we call *type y*) $\mathrm{Sph}_2(r, h, v)$ is a 3-valent discrete surface $\Phi_{2,r,h,v} : \mathrm{Br}_2(h, v) = (V(h, v), E(h, v)) \to \mathbb{S}(r) \subset \mathbb{R}^3$ defined by ρ_r, which is defined as above. See Fig. 3.1 for an example. More precisely, $\mathrm{Sph}_2(r, h, v)$ is the embedded graph in $\mathbb{R}^3$ with

$$V(\mathrm{Sph}_2(r, h, v)) = \Phi_{2,r,h,v}(V(h, v))$$

$$E(\mathrm{Sph}_2(r, h, v)) = \{(\Phi_{r,h,v}(\underline{x}), \Phi_{2,r,h,v}(\underline{y})) \mid (\underline{x}, \underline{y}) \in E(h, v)\}.$$

Now we come to the calculation of the discrete curvatures of $\mathrm{Sph}_k(r, h, v)$. In the following, we fix $r \in \mathbb{R}^+, h \in \mathbb{Z}^+$ and $v > 2$.

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Proposition 3.2.6. *Note that for type one, we pick* $(\alpha_1, \alpha_2) \in \{0, ..., 2h\} \times \{1, ..., v\}$ *($:= \Omega_1$) and for type two, we pick* $(\alpha_1, \alpha_2) \in \{0, ..., 2h\} \times (\{0, ..., v\} + \frac{1}{2})$ *($:= \Omega_2$). A vertex* $\underline{x}(\alpha_1, \alpha_2) = \Phi_{k,r,h,v}(\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2)$ *of* $\mathrm{Sph}_k(r, h, v)$ *is represented as*

$$\underline{x}(\alpha_1, \alpha_2) = r \begin{pmatrix} \sin C_2\alpha_2 \cos C_1\alpha_1 \\ \sin C_2\alpha_2 \sin C_1\alpha_1 \\ \cos C_2\alpha_2 \end{pmatrix}, \tag{3.2.4}$$

where

$$(C_1, C_2) := \left(\frac{\pi}{h}, \frac{\pi}{v+1}\right). \tag{3.2.5}$$

Although the vertices in these two types are different, we still can classify any vertex $\underline{x}(\alpha_1, \alpha_2)$ of $\mathrm{Sph}_k(r, h, v)$ in the following eight cases: The first four cases are of type one, $\mathrm{Sph}_1(r, h, v)$, we have

1. north polar circle case: $\underline{x}_0 = \underline{x}(\alpha_1, 1)$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, 2), \quad \underline{x}_2 := \underline{x}(\alpha_1 + 1, 1), \quad \underline{x}_3 := \underline{x}(\alpha_1 - 1, 1)$$

2. $\vdash$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{x}_1 := \underline{x}(\alpha_1 + 1, \alpha_2), \quad \underline{x}_2 := \underline{x}(\alpha_1, \alpha_2 - 1), \quad \underline{x}_3 := \underline{x}(\alpha_1, \alpha_2 + 1)$$

3. $\dashv$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{x}_1 := \underline{x}(\alpha_1 - 1, \alpha_2), \quad \underline{x}_2 := \underline{x}(\alpha_1, \alpha_2 + 1), \quad \underline{x}_3 := \underline{x}(\alpha_1, \alpha_2 - 1)$$

4. south polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, v)$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, v - 1), \quad \underline{x}_2 := \underline{x}(\alpha_1 - 1, v), \quad \underline{x}_3 := \underline{x}(\alpha_1 + 1, v)$$

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The last four cases are of type two, $\mathrm{Sph}_2(r,h,v)$, we have

5. north polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, \frac{1}{2})$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, 1+\frac{1}{2}), \quad \underline{x}_2 := \underline{x}(\alpha_1+2, \frac{1}{2}), \quad \underline{x}_3 := \underline{x}(\alpha_1-2, \frac{1}{2})$$

6. $\perp$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, \alpha_2 - 1), \quad \underline{x}_2 := \underline{x}(\alpha_1 - 1, \alpha_2), \quad \underline{x}_3 := \underline{x}(\alpha_1 + 1, \alpha_2)$$

7. $\top$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, \alpha_2 + 1), \quad \underline{x}_2 := \underline{x}(\alpha_1 + 1, \alpha_2), \quad \underline{x}_3 := \underline{x}(\alpha_1 - 1, \alpha_2)$$

8. south polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, v+\frac{1}{2})$ and

$$\underline{x}_1 := \underline{x}(\alpha_1, v-\frac{1}{2}), \quad \underline{x}_2 := \underline{x}(\alpha_1-2, v+\frac{1}{2}), \quad \underline{x}_3 := \underline{x}(\alpha_1+2, v+\frac{1}{2})$$

Observe and find out that except for case 2 and 3, every case else seem like $\perp$-case or $\top$-case.

And then, a normal vector of $\mathrm{Sph}_k(r,h,v)$ is computed as follows.

Proposition 3.2.7. *On* $\mathrm{Sph}_k(r,h,v)$*, for any* $(\alpha_1, \alpha_2) \in \Omega_k$*, the outer unit normal vector* $\underline{n}_0 = \underline{n}(\alpha_1, \alpha_2)$ *at* $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ *is based on different classes, and defined as* $\underline{n}_0 = \underline{m}_0 / |\underline{m}_0|$ *where* $\underline{m}_0 = \underline{x}_1 \times \underline{x}_2 + \underline{x}_2 \times \underline{x}_3 + \underline{x}_3 \times \underline{x}_1$*. We have the following*

On $\mathrm{Sph}_1(r,h,v)$*, the normal vector*

1. *north polar circle case:* $\underline{x}_0 = \underline{x}(\alpha_1, 1)$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2 \sin C_1 \begin{pmatrix} -(\cos 2C_2 - \cos C_2) * (\cos C_1 \alpha_1) \\ -(\cos 2C_2 - \cos C_2) * (\sin C_1 \alpha_1) \\ \sin 2C_2 - \sin C_2 \cos C_1 \end{pmatrix}$$

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2. $\vdash$-*case:* $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ *with* $1 < \alpha_2 < v$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2 \begin{pmatrix} +\sin^2 C_2\alpha_2 \sin C_1(\alpha_1+1) + \cos^2 C_2\alpha_2 \sin C_1\alpha_1 - \cos C_2 \sin C_1\alpha_1 \\ -\sin^2 C_2\alpha_2 \cos C_1(\alpha_1+1) - \cos^2 C_2\alpha_2 \cos C_1\alpha_1 + \cos C_2 \cos C_1\alpha_1 \\ \sin C_2\alpha_2 \cos C_2\alpha_2 \sin C_1 \end{pmatrix}$$

3. $\dashv$-*case:* $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ *with* $1 < \alpha_2 < v$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2 \begin{pmatrix} -\sin^2 C_2\alpha_2 \sin C_1(\alpha_1-1) - \cos^2 C_2\alpha_2 \sin C_1\alpha_1 + \cos C_2 \sin C_1\alpha_1 \\ +\sin^2 C_2\alpha_2 \cos C_1(\alpha_1-1) + \cos^2 C_2\alpha_2 \cos C_1\alpha_1 - \cos C_2 \cos C_1\alpha_1 \\ \sin C_2\alpha_2 \cos C_2\alpha_2 \sin C_1 \end{pmatrix}$$

4. *south polar circle:* $\underline{x}_0 = \underline{x}(\alpha_1, v)$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2 v \sin C_1 \begin{pmatrix} (\cos C_2(v-1) - \cos C_2 v) * (\cos C_1\alpha_1) \\ (\cos C_2(v-1) - \cos C_2 v) * (\sin C_1\alpha_1) \\ -\sin C_2(v-1) + \sin C_2 v \cos C_1 \end{pmatrix}$$

On $\mathrm{Sph}_2(r, h, v)$*, the normal vector*

5. *north polar circle:* $\underline{x}_0 = \underline{x}(\alpha_1, \frac{1}{2})$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2 \frac{1}{2} \sin 2C_1 \begin{pmatrix} -(\cos C_2(\frac{1}{2}+1) - \cos C_2\frac{1}{2}) * (\cos C_1\alpha_1) \\ -(\cos C_2(\frac{1}{2}+1) - \cos C_2\frac{1}{2}) * (\sin C_1\alpha_1) \\ \sin C_2(\frac{1}{2}+1) - \sin C_2\frac{1}{2} \cos 2C_1 \end{pmatrix}$$

6. $\perp$-*case:* $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ *with* $1 < \alpha_2 < v$ *and*

$$\underline{m}_0 = 2r^2 \sin C_2\alpha_2 \sin C_1 \begin{pmatrix} (\cos C_2(\alpha_2-1) - \cos C_2\alpha_2) * (\cos C_1\alpha_1) \\ (\cos C_2(\alpha_2-1) - \cos C_2\alpha_2) * (\sin C_1\alpha_1) \\ -\sin C_2(\alpha_2-1) + \sin C_2\alpha_2 \cos C_1 \end{pmatrix}$$

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7. *$\top$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and*

$$\underline{m}_0 = 2r^2 \sin C_2\alpha_2 \sin C_1 \begin{pmatrix} -(\cos C_2(\alpha_2+1) - \cos C_2\alpha_2) * (\cos C_1\alpha_1) \\ -(\cos C_2(\alpha_2+1) - \cos C_2\alpha_2) * (\sin C_1\alpha_1) \\ \sin C_2(\alpha_2+1) - \sin C_2\alpha_2 \cos C_1 \end{pmatrix}$$

8. *south polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, v+\frac{1}{2})$ and*

$$\underline{m}_0 = 2r^2 \sin C_2(v+\frac{1}{2}) \sin 2C_1 \begin{pmatrix} (\cos C_2(v-\frac{1}{2}) - \cos C_2(v+\frac{1}{2})) * (\cos C_1\alpha_1) \\ (\cos C_2(v-\frac{1}{2}) - \cos C_2(v+\frac{1}{2})) * (\sin C_1\alpha_1) \\ -\sin C_2(v-\frac{1}{2}) + \sin C_2(v+\frac{1}{2}) \cos 2C_1 \end{pmatrix}$$

Remark 3.2.8. Since the unit normal vector needs to be divided by the length, we have the following more accurate results (right double arrow means divided by a constant):

On $\mathrm{Sph}_1(r, h, v)$, the normal vector

1. north polar circle case: $\underline{x}_0 = \underline{x}(\alpha_1, 1)$ and

$$\begin{aligned} \underline{m}_0 &\Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{3C_2}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{3C_2}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{3C_2}{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\sin C_2(\cos C_1 - 1) \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} \sin(\frac{3C_2}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{3C_2}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{3C_2}{2}) \end{pmatrix} + \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ -\sin C_2 \end{pmatrix} \end{aligned}$$

2. $\vdash$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\begin{aligned} \underline{m}_0 &\Rightarrow 2\sin C_2\alpha_2 \sin(\frac{C_1}{2}) \begin{pmatrix} \sin C_2\alpha_2(\cos(\frac{C_1(2\alpha_1+1)}{2})) \\ \sin C_2\alpha_2(\sin(\frac{C_1(2\alpha_1+1)}{2})) \\ \cos C_2\alpha_2 \cos(\frac{C_1}{2}) \end{pmatrix} \\ &+ \begin{pmatrix} +\sin C_1\alpha_1(1-\cos C_2) \\ -\cos C_1\alpha_1(1-\cos C_2) \\ 0 \end{pmatrix} \end{aligned}$$

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$$\Rightarrow \begin{pmatrix} \sin C_2\alpha_2(\cos(\frac{C_1(2\alpha_1+1)}{2})) \\ \sin C_2\alpha_2(\sin(\frac{C_1(2\alpha_1+1)}{2})) \\ \cos C_2\alpha_2 \cos(\frac{C_1}{2}) \end{pmatrix} + \frac{(1-\cos C_2)}{2\sin(\frac{C_1}{2})\sin C_2\alpha_2} \begin{pmatrix} +\sin C_1\alpha_1 \\ -\cos C_1\alpha_1 \\ 0 \end{pmatrix}$$

3. $\dashv$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{m}_0 \Rightarrow 2\sin C_2\alpha_2 \sin(\frac{C_1}{2}) \begin{pmatrix} \sin C_2\alpha_2(\cos(\frac{C_1(2\alpha_1-1)}{2})) \\ \sin C_2\alpha_2(\sin(\frac{C_1(2\alpha_1-1)}{2})) \\ \cos C_2\alpha_2 \cos(\frac{C_1}{2}) \end{pmatrix}$$
$$+ \begin{pmatrix} -\sin C_1\alpha_1(1-\cos C_2) \\ +\cos C_1\alpha_1(1-\cos C_2) \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \sin C_2\alpha_2(\cos(\frac{C_1(2\alpha_1-1)}{2})) \\ \sin C_2\alpha_2(\sin(\frac{C_1(2\alpha_1-1)}{2})) \\ \cos C_2\alpha_2 \cos(\frac{C_1}{2}) \end{pmatrix} + \frac{(1-\cos C_2)}{2\sin(\frac{C_1}{2})\sin C_2\alpha_2} \begin{pmatrix} -\sin C_1\alpha_1 \\ +\cos C_1\alpha_1 \\ 0 \end{pmatrix}$$

4. south polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, v)$ and

$$\underline{m}_0 \Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{C_2(2v-1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2v-1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2v-1)}{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ +\sin C_2 v(\cos C_1 - 1) \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \sin(\frac{C_2(2v-1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2v-1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2v-1)}{2}) \end{pmatrix} + \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ +\sin C_2 v \end{pmatrix}$$

On $\mathrm{Sph}_2(r, h, v)$, the normal vector

5. north polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, \frac{1}{2})$ and

$$\underline{m}_0 \Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{C_2(2\frac{1}{2}+1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\frac{1}{2}+1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\frac{1}{2}+1)}{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\sin C_2\frac{1}{2}(\cos 2C_1 - 1) \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} \sin(\frac{C_2(2\frac{1}{2}+1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\frac{1}{2}+1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\frac{1}{2}+1)}{2}) \end{pmatrix} + \frac{(\cos 2C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ -\sin C_2\frac{1}{2} \end{pmatrix}$$

6. $\perp$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{m}_0 \Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{C_2(2\alpha_2-1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\alpha_2-1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\alpha_2-1)}{2}) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ +\sin C_2\alpha_2(\cos C_1 - 1) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sin(\frac{C_2(2\alpha_2-1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\alpha_2-1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\alpha_2-1)}{2}) \end{pmatrix} + \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ +\sin C_2\alpha_2 \end{pmatrix}$$

7. $\top$-case: $\underline{x}_0 = \underline{x}(\alpha_1, \alpha_2)$ with $1 < \alpha_2 < v$ and

$$\underline{m}_0 \Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{C_2(2\alpha_2+1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\alpha_2+1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\alpha_2+1)}{2}) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ -\sin C_2\alpha_2(\cos C_1 - 1) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sin(\frac{C_2(2\alpha_2+1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2\alpha_2+1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2\alpha_2+1)}{2}) \end{pmatrix} + \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ -\sin C_2\alpha_2 \end{pmatrix}$$

8. south polar circle: $\underline{x}_0 = \underline{x}(\alpha_1, v + \frac{1}{2})$ and

$$\underline{m}_0 \Rightarrow 2\sin(\frac{C_2}{2}) \begin{pmatrix} \sin(\frac{C_2(2(v+\frac{1}{2})-1)}{2}) * (\cos C_1\alpha_1) \\ \sin(\frac{C_2(2(v+\frac{1}{2})-1)}{2}) * (\sin C_1\alpha_1) \\ \cos(\frac{C_2(2(v+\frac{1}{2})-1)}{2}) \end{pmatrix}$$

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$$+\sin C_2(v+\frac{1}{2})(\cos 2C_1 - 1)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\sin(\frac{C_2(2(v+\frac{1}{2})-1)}{2}) * (\cos C_1\alpha_1)\\ \sin(\frac{C_2(2(v+\frac{1}{2})-1)}{2}) * (\sin C_1\alpha_1)\\ \cos(\frac{C_2(2(v+\frac{1}{2})-1)}{2})\end{pmatrix} + \frac{(\cos 2C_1 - 1)\sin C_2(v+\frac{1}{2})}{2\sin(\frac{C_2}{2})}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

The remaining calculations we use computer to complete, and we show in the following picture. Fig. 3.2 and Fig. 3.3 show Gauss curvature and mean curvature of spherical brick graph with $(h, v) = (5, 10)$.

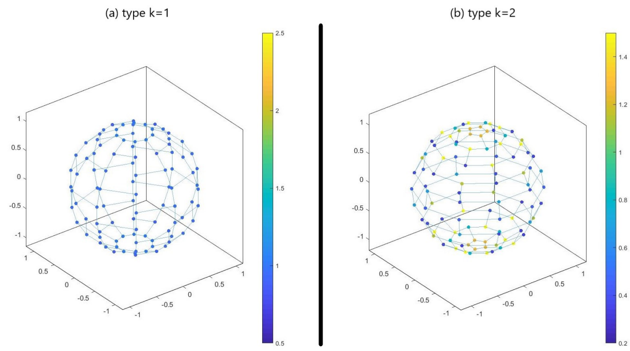

Figure 3.2: Gauss curvature of spherical brick graph.

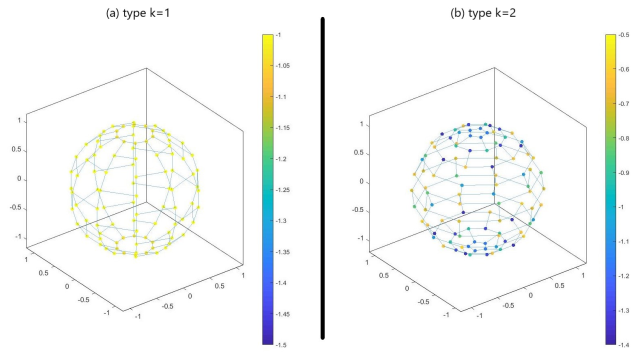

Figure 3.3: Mean curvature of spherical brick graph.

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Chapter 4

Mean curvature flow

The mean curvature flow of a compact, convex surface converges to one point. How about the mean curvature flow of the discrete surface? Given a family of discrete surface $\Phi : X = (V, E) \times [0, \infty) \to \mathbb{R}^3$, consider the mean curvature flow as the following for all $v \in V$,

$$\frac{d\Phi(v,t)}{dt} = H(v,t)\underline{n}(v,t).$$

But, note that the MCF is not trivial for the discrete surface, even if it's just a triangular pyramid. So, in the following, we just consider the MCF of some special tetrahedron.

4.1 M.C.F. of regular tetrahedron

First, for the simplest case, we consider the MCF of a regular tetrahedron. Given a regular tetrahedron, $\triangle_r = (\{p_1, p_2, p_3, p_4\}, E)$, and its coordinate $\Phi : \triangle_r \to \mathbb{R}^3$. Let $\underline{p}_a := \Phi(p_a)$. Since translation and rotation doesn't change the curvature, we can set

$$\underline{p}_1 = r\begin{pmatrix} 0 \\ 0 \\ 3\sqrt{6} \end{pmatrix}, \quad \underline{p}_2 = r\begin{pmatrix} 0 \\ 4\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad \underline{p}_3 = r\begin{pmatrix} -6 \\ -2\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad \underline{p}_4 = r\begin{pmatrix} 6 \\ -2\sqrt{3} \\ -\sqrt{6} \end{pmatrix}.$$

We also get the normal vector of p_1 to be

$$\begin{aligned}
\underline{m}_1 &= (\underline{p}_2 - \underline{p}_4) \times (\underline{p}_3 - \underline{p}_4) \\
&= r\begin{pmatrix} 0-6 \\ 4\sqrt{3} - (-2\sqrt{3}) \\ -\sqrt{6} - (-\sqrt{6}) \end{pmatrix} \times r\begin{pmatrix} -6-6 \\ -2\sqrt{3} - (-2\sqrt{3}) \\ -\sqrt{6} - (-\sqrt{6}) \end{pmatrix} = r^2\begin{pmatrix} -6 \\ 6\sqrt{3} \\ 0 \end{pmatrix} \times \begin{pmatrix} -12 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

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$$= r^2 \begin{pmatrix} 0 \\ 0 \\ 72\sqrt{3} \end{pmatrix} // \underline{p}_1.$$

The unit normal at p_1 is $\underline{n}_1 = \frac{\underline{m}_1}{|\underline{m}_1|} // \underline{p}_1$. And the others are

$$\underline{m}_2 = (\underline{p}_1 - \underline{p}_3) \times (\underline{p}_4 - \underline{p}_3) = r^2 \begin{pmatrix} 0 \\ 48\sqrt{6} \\ -24\sqrt{3} \end{pmatrix} // \underline{p}_2,$$

$$\underline{m}_3 = (\underline{p}_1 - \underline{p}_4) \times (\underline{p}_2 - \underline{p}_4) = r^2 \begin{pmatrix} -72\sqrt{2} \\ -24\sqrt{6} \\ -24\sqrt{3} \end{pmatrix} // \underline{p}_3,$$

$$\underline{m}_4 = (\underline{p}_1 - \underline{p}_2) \times (\underline{p}_3 - \underline{p}_2) = r^2 \begin{pmatrix} 72\sqrt{2} \\ -24\sqrt{6} \\ -24\sqrt{3} \end{pmatrix} // \underline{p}_4.$$

So the unit normal $\underline{n}_2, \underline{n}_3, \underline{n}_4 // \underline{p}_2, \underline{p}_3, \underline{p}_4$, respectively. More precisely, the unit normal vectors are

$$\underline{n}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{n}_2 = \frac{1}{3} \begin{pmatrix} 0 \\ 2\sqrt{2} \\ -1 \end{pmatrix}, \quad \underline{n}_3 = \frac{1}{3} \begin{pmatrix} -\sqrt{6} \\ -\sqrt{2} \\ -1 \end{pmatrix}, \quad \underline{n}_4 = \frac{1}{3} \begin{pmatrix} \sqrt{6} \\ -\sqrt{2} \\ -1 \end{pmatrix}$$

We have the formula of the first fundamental form and the second fundamental form of p_1 (and also p_2, p_3, p_4).

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = 36r^2 \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 72r^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} L & M_2 \\ M_1 & N \end{pmatrix} = -4r \begin{pmatrix} 2\sqrt{6} & \sqrt{6} \\ \sqrt{6} & 2\sqrt{6} \end{pmatrix} = -4\sqrt{6}r \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Now the discrete curvatures are

$$K(p_1) = \frac{LN - M_1 M_2}{EG - F^2} = \frac{4\sqrt{6}r * 4\sqrt{6}r}{72r^2 * 72r^2} = \frac{1}{54r^2},$$

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$$H(p_1) = \frac{EN - F(M_1 + M_2) + GL}{2(EG - F^2)} = \frac{-4\sqrt{6}r * 72r^2(4-1-1+4)}{2 * 72r^2 * 72r^2(4-1)} = -\frac{\sqrt{6}}{18r}.$$

It is easy to verify that the curvatures at all points are the same. Note that the normal vector are a constant multiple of the position, that is

$$\underline{n}_a = \frac{1}{l}\underline{p}_a,$$

where $l = 3\sqrt{6}r$.

Back to the MCF equation

$$\frac{d\Phi(p,t)}{dt} = H(p,t)\underline{n}(p,t).$$

Since the mean curvature is independent of points and the unit normal is also independent of time, we can rewrite the equation of discrete surface $\Phi(p,t) = C(t)\Phi(p,0)$ for some scalar function $C(t)$ which is only dependent on time. And then we have the mean curvature $H(p,t) = \frac{1}{C(t)}H(p,0) = \frac{1}{C(t)}H(0)$. We can reduce the MCF equation to the following

$$\frac{d\Phi(p,t)}{dt} = H(t)\underline{n}(p) \quad \Rightarrow \frac{d}{dt}C(t)\Phi(p,0) = \frac{1}{C(t)}H(0) \cdot \frac{1}{l}\Phi(p,0),$$

We get

$$\frac{dC(t)}{dt} = \frac{H}{l}\frac{1}{C(t)} \quad \Rightarrow C(t) = \sqrt{2(c_1 + \frac{H}{l}t)}.$$

Put $t = 0$ and get the constant $c_1 = \frac{1}{2}$, and then $C(t) = \sqrt{1 - \frac{1}{27r^2}t}$. This means that when t comes to $27r^2$, we have $\Phi(p,t) \to p_0$ with p_0 is the center of the regular tetrahedron $\triangle_r$. This case is similar to the MCF of a compact, convex smooth surface. (see Fig. 4.1)

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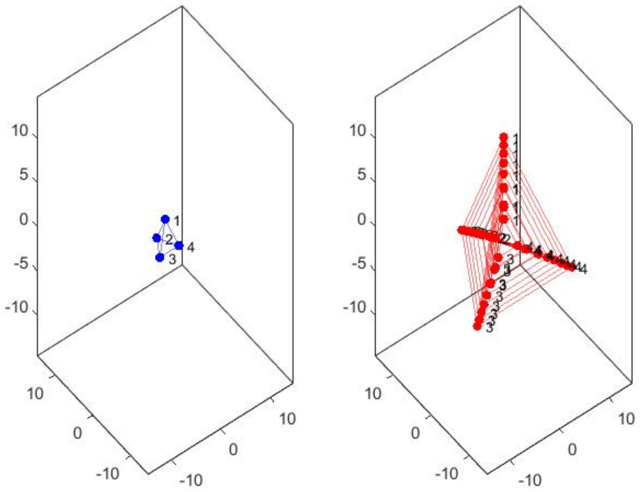

Figure 4.1: A mcf on regular tetrahedron.

4.2 M.C.F. of perpendicular skew line tetrahedron

Now, we consider a little more complicated case.

Given a tetrahedron $\triangle_{psl} = [p_1, p_2, p_3, p_4]$ with $\overrightarrow{p_1p_2} \perp \overrightarrow{p_3p_4}$, $\overrightarrow{p_1p_2} \perp \overrightarrow{AB}$ and $\overrightarrow{p_3p_4} \perp \overrightarrow{AB}$ where A, B are the midpoints of $\overline{p_1p_2}$, $\overline{p_3p_4}$, respectively. Let O be the midpoint of $\overline{AB}$ and W.O.L.G. set O be the origin and $\overline{AB}$ lies on x-axis, and let $\overline{p_1p_2}, \overline{p_3p_4}$ parallel y-axis, z-axis,respectively. Set the length of $\overline{Ap_1}$ (or $\overline{Ap_2}$, $\overline{Bp_3}$, $\overline{Bp_4}$) is b, and $\overline{AO}$ (or $\overline{BO}$) is a for some $a, b > 0$.(see Fig. 4.2)

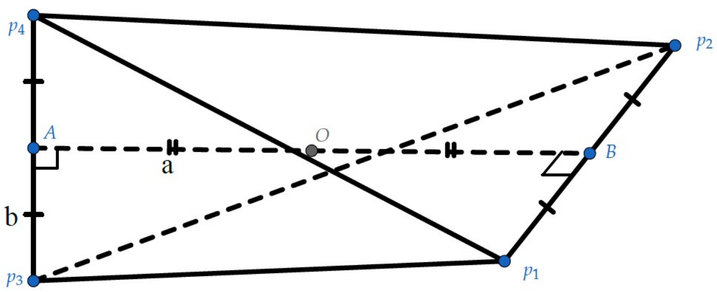

Figure 4.2: perpendicular skew line tetrahedron

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Then, the coordinates of the vertices are

$$p_1 = (a, -b, 0)^t, \quad p_2 = (a, b, 0)^t$$
$$p_3 = (-a, 0, -b)^t, \quad p_4 = (-a, 0, b)^t$$

First we focus on p_1, we have

$$v_1(p_1) = (2a, b, b)^t, \quad v_2(p_1) = (0, 0, 2b)^t$$

and the first fundamental form is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 4a^2 + 2b^2 & 2b^2 \\ 2b^2 & 4b^2 \end{pmatrix}.$$

The unit normal vector of each vertices are

$$n_1 = \frac{1}{l}(b^2, -2ab, 0)^t, \quad n_2 = \frac{1}{l}(b^2, 2ab, 0)^t$$
$$n_3 = \frac{1}{l}(-b^2, 0, -2ab)^t, \quad n_4 = \frac{1}{l}(-b^2, 0, 2ab)^t$$

where $l = \sqrt{b^4 + 4a^2b^2}$ And the second fundamental form is

$$\begin{pmatrix} L & M_2 \\ M_1 & N \end{pmatrix} = \frac{1}{l} \begin{pmatrix} -(2a, b, b) \cdot (2b^2, 2ab, 2ab) & -(2a, b, b) \cdot (0, 0, 4ab) \\ -(0, 0, 2b) \cdot (2b^2, 2ab, 2ab) & -(0, 0, 2b) \cdot (0, 0, 4ab) \end{pmatrix}$$
$$= \frac{1}{l} \begin{pmatrix} -8ab^2 & -4ab^2 \\ -4ab^2 & -8ab^2 \end{pmatrix}.$$

Then, the curvatures are

$$K(p_1) = \frac{1}{b^4 + 4a^2b^2} \frac{48a^2b^4}{16a^2b^2 + 4b^4} = \frac{12a^2}{(4a^2 + b^2)^2}$$
$$H(p_1) = \frac{1}{l} \frac{(4a^2 + 2b^2)(-8ab^2) + (4b^2)(-8ab^2) - (2b^2)(-4ab^2) - (2b^2)(-4ab^2)}{32a^2b^2 + 8b^4}$$
$$= \frac{1}{l} \frac{-32a^3b^2 - 16ab^4 - 32ab^4 + 16ab^4}{32a^2b^2 + 8b^4}$$

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$$= \frac{1}{l}\frac{-4a^3 - 4ab^2}{4a^2+b^2} = \frac{1}{l}\frac{-4a(a^2+b^2)}{4a^2+b^2}.$$

Consider the mean curvature flow which is a realization $\Phi : X \times [0, \infty) \to \mathbb{R}^3$ which various along time t and satisfy the following:

$$\begin{cases} \frac{\partial}{\partial t}\vec{\Phi}(x_i, t) = H(x_i, t) * \vec{n}(x_i, t), \\ \\ \vec{\Phi}(x_i, 0) = p_i. \end{cases}$$

where $x_i \in V$ is vertices of X, $H(x_i, t)$ is mean curvature of $\vec{\Phi}(x_i, t)$ and $\vec{n}(x_i, t)$ is unit normal vector of $\vec{\Phi}(x_i, t)$.

In this case, we have that $\triangle_{psl}$ is constant mean curvature. And notice that the direction of the unit normal vector is related to that of the coordinate of the point, so we have that in any time t, Φ is of form that is similar to $\triangle_{psl}$. That is Φ is just determined from a, b and a, b are just one valued function of t. In this case, we can only focus on point p_1 and rewrite the equations :

$$\begin{cases} \frac{d}{dt}(a, -b, 0) = \frac{1}{l}\frac{-4a(a^2+b^2)}{4a^2+b^2} * \frac{1}{l}(b^2, -2ab, 0), \\ \\ (a(0), -b(0), 0) = (a_0, -b_0, 0). \end{cases}$$

where $a_0, b_0 > 0$ are some fixed constants. But, note that

$$\begin{aligned} \frac{d}{dt}(a(t), -b(t), 0) &= \frac{1}{l}\frac{-4a(a^2+b^2)}{4a^2+b^2} * \frac{1}{l}(b^2, -2ab, 0) \\ &= \frac{-4a(a^2+b^2)}{b^2(4a^2+b^2)^2} * (b^2, -2ab, 0) \end{aligned}$$

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and then we have O.D.E.

$$\begin{cases} a'(t) = \frac{-4a(a^2+b^2)(b)}{b(4a^2+b^2)^2} \\ \\ b'(t) = \frac{-4a(a^2+b^2)(2a)}{b(4a^2+b^2)^2} \end{cases} \tag{4.2.1}$$

We can rewrite (4.2.1) and get

$$\begin{cases} aa'(t) = \frac{-4a^2(a^2+b^2)}{(4a^2+b^2)^2} \\ \\ bb'(t) = \frac{-8a^2(a^2+b^2)}{(4a^2+b^2)^2} \end{cases}$$

This means that

$$\frac{d}{dt}b^2 = 2\frac{d}{dt}a^2$$

and this means $b^2 - 2a^2$ is a constant.

$$b^2(t) - 2a^2(t) = b_0^2 - 2a_0^2 := C$$

or

$$\frac{b^2(t)}{C} - \frac{a^2(t)}{\frac{1}{2}C} = 1$$

Let $A(t) = a^2(t)$ and $B(t) = b^2(t)$. Then we have

$$\begin{cases} A'(t) = 2aa'(t) = \frac{-8a^2(a^2+b^2)}{(4a^2+b^2)^2} = \frac{-8A(A+B)}{(4A+B)^2} \\ \\ B'(t) = 2bb'(t) = \frac{-16a^2(a^2+b^2)}{(4a^2+b^2)^2} = \frac{-16A(A+B)}{(4A+B)^2} \end{cases}$$

and

$$B(t) = C + 2A(t)$$

$$\begin{aligned} A'(t) = 2aa'(t) = \frac{-8a^2(a^2+b^2)}{(4a^2+b^2)^2} = \frac{-8A(A+B)}{(4A+B)^2} \\ = \frac{-8A(C+3A)}{(C+6A)^2} \end{aligned}$$

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This is only one equation, we can solve it:

$$
\begin{aligned}
-8dt &= \frac{(C+6A)^2}{A(C+3A)}dA = (12 + \frac{C^2}{3A^2+CA})dA \\
\Rightarrow -8dt &= (12 + \frac{C}{A} + \frac{-3C}{C+3A})dA \\
\Rightarrow k - 8t &= 12A + C\log(A) - C\log(C+3A) \\
&= \log(\frac{A^C \exp(12A)}{(C+3A)^C}). \qquad (4.2.2)
\end{aligned}
$$

Now, we have three case,

- $C = 0$ that is $b_0^2 = 2a_0^2$, or we can say this tetrahedron is regular, then the equation reduced to

  $$A'(t) = \frac{-8A(0+3A)}{(0+6A)^2} = \frac{-2}{3}.$$

  So,

  $$
  \begin{aligned}
  A(t) &= \frac{-2}{3}t + a_0^2 \quad B(t) = \frac{-4}{3}t + 2a_0^2 = \frac{-4}{3}t + b_0^2, \\
  a(t) &= \sqrt{\frac{-2}{3}t + a_0^2} \quad b(t) = \sqrt{\frac{-4}{3}t + b_0^2}.
  \end{aligned}
  $$

  When $t = \frac{3a_0^2}{2}$, the MCF goes to one point and stop.

- If $C < 0$, this means $b_0^2 < 2a_0^2$ or $B_0 < 2A_0$. Note that $B' = 2A'$, so we have that B goes to 0 earlier than A. It goes to a segment. (see Fig. 4.3)

- If $C > 0$, this means $b_0^2 > 2a_0^2$ or $B_0 > 2A_0$. Note that $B' = 2A'$, so we have that A goes to 0 earlier than B. It goes to a square. (see Fig. 4.4)

From these analysis, we know that the MCF doesn't converge to a point if $C \neq 0$. Since $C \neq 0$, we have the right hand side of equation (4.2.2) goes to negative infinity when A goes to 0. This means that it takes infinity time for the MCF to collapse to either a square or a line segment.

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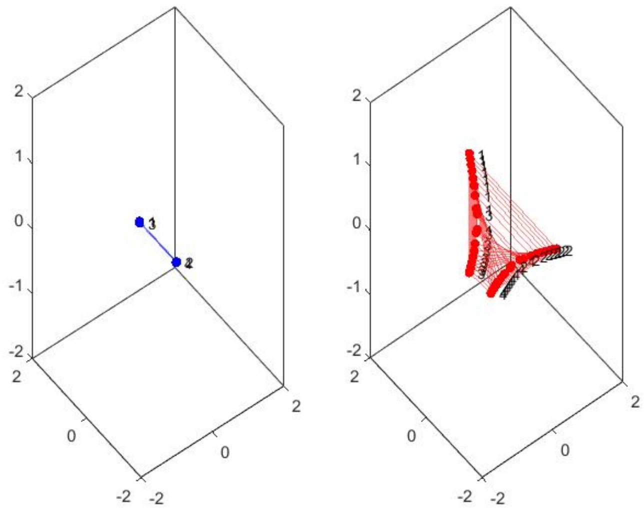

Figure 4.3: A mcf on perpendicular skew line tetrahedron with $b^2 < 2a^2$ and then end likes a segment.

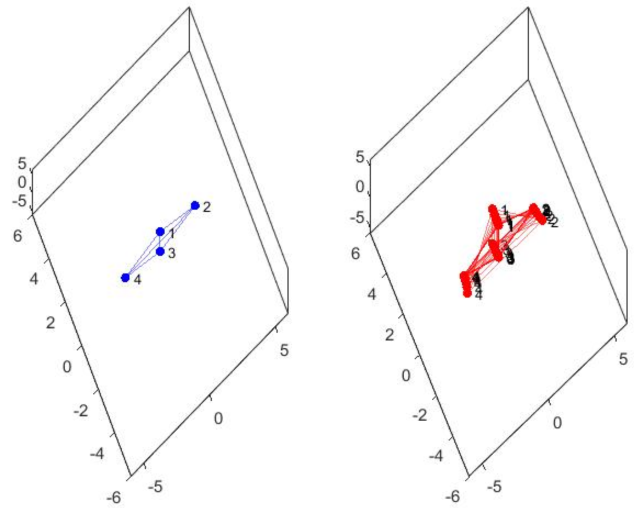

Figure 4.4: A mcf on perpendicular skew line tetrahedron with $b^2 > 2a^2$ and then end likes a square.

Remark 4.2.1. How about if there is a little change of the shape? Notice in the perpendicular skew line tetrahedron, we require $\overrightarrow{p_1p_2} \perp \overrightarrow{p_3p_4}$. Here we give a small change, the angle between $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_3p_4}$ are a little less then $\frac{\pi}{2}$, and we see how the mean curvature flow goes in computer. (see Fig. 4.5) We will see that angle goes smaller then the initial and it will replace the original and become a new type.

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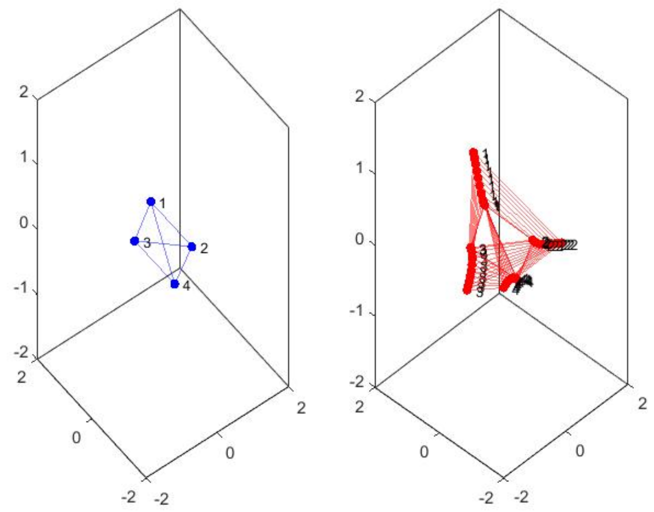

Figure 4.5: A mcf on skew line tetrahedron with angle between $\overrightarrow{p_1p_3}$ and $\overrightarrow{p_2p_4}$ less then $\pi/2$.

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Chapter 5

Convergence of discrete curvatures

We first recall the convergence result from the paper [3]. Then we discuss the convergence of discrete curvatures using the discrete approximation of the sphere constructed in Section 3.2.

5.1 Convergence theorem

The following is the general convergence result from [3].

Proposition 5.1.1. *Let* $\{\Phi_k : X_k = (V_k, E_k) \to \mathbb{R}^3\}_{k=1}^{\infty}$ *be a sequence of 3-valent discrete surfaces with the following properties.*

(i) The sequence of sets of points $\{\Phi_k(V_k)\}_{k=1}^{\infty}$ *converges to a smooth surface* M *in* $\mathbb{R}^3$ *in the Hausdorff topology.*

(ii) For any $p \in M$*, the unit normal vector* $\underline{n}_k(x_k)$ *of* Φ_k *at* $x_k \in V_k$ *converges to the unit normal* $\underline{n}(p)$ *of* M *at* p*, independently of the choice of* $\{x_k\}_{k=1}^{\infty}$ *with* $\Phi_k(x_k) \to p$ *as* $k \to \infty$*.*

(iii) The Weingarten map $S_k : T_{x_k} \to T_{x_k}$ *of* Φ_k *converges to the Weigarten map* $S : T_pM \to T_pM$ *of* M *in the following sense: for* $\{x_k\}_{k=1}^{\infty}$ *with* $\Phi_k(x_k) \to p$ *as* $k \to \infty$ *and for* $\{\underline{v}_k \in T_{x_k}\}_{k=1}^{\infty}$ *converging to some* $\underline{v} \in T_pM$*, it follows*

$$S_k(\underline{v}_k) \to S(\underline{v})$$

in $\mathbb{R}^3$ *as* $k \to \infty$*.*

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Then both the mean curvature $H_k(x_k)$ and the Gauss curvature $K_k(x_k)$ of Φ_k respectively converge to the mean curvature $H(p)$ and the Gauss curvature $K(p)$ of M for $\{x_k\}_{k=1}^{\infty}$ with $\Phi_k(x_k) \to p$ as $k \to \infty$.

Proof. Let $p \in M$ be a point and $\{x_k\}_{k=1}^{\infty}$ be a sequence of points $x_k \in V_k$ such that $\Phi_k(x_k)$ converges to p in $\mathbb{R}^3$. For any tangent vector $\underline{v} \in TpM$, as is easily seen using (ii), it follows that the sequence $\{\underline{v}_k\}_{k=1}^{\infty}$, where $\underline{v}_k$ is the orthogonal projection of $\underline{v}$ onto T_{x_k}, converges to $\underline{v}$. If we take a pair of linearly independent vectors $\{\underline{v}, \underline{w}\} \subseteq T_pM$ so that $\underline{v} \times \underline{w}$ has the same direction as $\underline{n}(p)$, then, the vectors $\{\underline{v}_k, \underline{w}_k\} \subseteq T_{x_k}$ which are respectively obtained from $\{\underline{v}, \underline{w}\} \subseteq T_pM$ as in the above manner are also linearly independent as well as $\underline{v}_k \times \underline{w}_k$ has the same direction as $\underline{n}_k(x_k)$ for sufficiently large $k \in \mathbb{N}$. Then, by (iii),

$$\begin{pmatrix} \langle \underline{v}_k, \underline{v}_k \rangle & \langle \underline{v}_k, \underline{w}_k \rangle \\ \langle \underline{w}_k, \underline{v}_k \rangle & \langle \underline{w}_k, \underline{w}_k \rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle \underline{v}_k, S_k(\underline{v}_k) \rangle & \langle \underline{v}_k, S_k(\underline{w}_k) \rangle \\ \langle \underline{w}_k, S_k(\underline{v}_k) \rangle & \langle \underline{w}_k, S_k(\underline{w}_k) \rangle \end{pmatrix},$$

whose trace is equal to $H_k(x_k)$(resp. determinant is equal to $K_k(x_k)$), converges, as $k \to \infty$, to

$$\begin{pmatrix} \langle \underline{v}, \underline{v} \rangle & \langle \underline{v}, \underline{w} \rangle \\ \langle \underline{w}, \underline{v} \rangle & \langle \underline{w}, \underline{w} \rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle \underline{v}, S(\underline{v}) \rangle & \langle \underline{v}, S(\underline{w}) \rangle \\ \langle \underline{w}, S(\underline{v}) \rangle & \langle \underline{w}, S(\underline{w}) \rangle \end{pmatrix},$$

whose trace is equal to $H(p)$(resp. determinant is equal to $K(p)$). □

The following examples show that the condition of the preceding proposition is optimal in the most general settings.

Example 5.1.2. Let X_k be the regular hexagonal lattice in the plane with the exception at a vertex, say, $(0, 0)$, which is located at $(0, 0, h_k)$, where $h_k > 0$. If the distance of adjacent vertices becomes small with order $1/k$, then

(i) X_k does not converge to the plane in the Hausdorff sense unless h_k converges to 0 as $k \to \infty$.

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(ii) The normal vector does not converge provided kh_k is bounded away from 0 as $k \to \infty$.

(iii) The Weingarten map does not converge provided $k^2 h_k$ is bounded away from 0 as $k \to \infty$.

5.2 Convergence of sphere

In Section 3.2, we define discrete surfaces on sphere, $\mathrm{Sph}_k(r, h, v)$. Here, we fixed $r = 1$, that is we just focus on unit sphere. Given some (h, v), and focus on both types of graphs, $k = 1$ and 2, and compute their curvatures. Moreover, when we put (h, k) to infinity, we will get a finer subdivision and a series of curvatures. Since we usually guess that the discrete curvatures (including Gaussian and mean) approach the smooth one, is it true in our case?

Given a strictly monotone increasing sequence $\{(h^{(i)}, v^{(i)})\}_{i=1}^{\infty}$, and let $X^{(i)}$ be the spherical brick graph $\mathrm{Sph}_k(1, h^{(i)}, v^{(i)})$ with fixed k. By the definition of spherical brick graph (see Definition 3.2.4 and Definition 3.2.5), it is easy to see that the subdivision of spherical graph converges to unit sphere in the Hausdorff topology as i raises. More precisely, given any point $p \in \mathbb{S}^2$, we take the spherical coordinate of p,

$$p = \rho(\theta, \phi)^T = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\rho \\ \cos\phi \end{pmatrix}.$$

For any pair $(h^{(i)}, v^{(i)})$, let $C_1 = C_1^{(i)} = \frac{\pi}{h^{(i)}}$ and $C_2 = C_2^{(i)} = \frac{\pi}{v^{(i)}+1}$. Then for $(\theta, \phi)^T$, we have the following two cases: If p is not a pole, then we have $\phi \neq 0$ or π. Hence, When i is large enough, p will fall on at least one brick, and then take $x^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$

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be any one vertex on this brick. Then the coordinate of $x^{(i)}$ be

$$\Phi_{k,1,h^{(i)},v^{(i)}}(x^{(i)}) = \rho(x^{(i)}) = \underline{x}(\alpha_1^{(i)}, \alpha_2^{(i)}) = \begin{pmatrix} \sin(C_2\alpha_2^{(i)})\cos(C_1\alpha_1^{(i)}) \\ \sin(C_2\alpha_2^{(i)})\sin(C_1\alpha_1^{(i)}) \\ \cos(C_2\alpha_2^{(i)}) \end{pmatrix}.$$

Because any brick in graph $X^{(i)}$ has length and width $(C_1, 2C_2)$ (for $k = 1$) or $(2C_1, C_2)$ (for $k = 2$). We know that the difference between (θ, ϕ) and $(\alpha_1^{(i)}, \alpha_2^{(i)})$ is at most $(2C_1, 2C_2)$. After the realization map ρ, the distance between p and $\Phi(x^{(i)})$ is at most $2\sqrt{C_1^2 + C_2^2}$ and converges to 0 as i increases. And, the other case, p is actually a pole, then we have $\phi = 0$ (or π, resp.). We will take $\alpha_1^{(i)}$ to be any one in $\{0, ..., 2h^{(i)} - 1\}$, and $\alpha_2^{(i)} = 1$ (or $v^{(i)}$, resp.) in type $k = 1$ and $\alpha_2^{(i)} = \frac{1}{2}$ (or $v^{(i)} + \frac{1}{2}$, resp.) in type $k = 2$. Then, the distance between p and $\Phi(x^{(i)})$ is at most C_2 and converges to 0 as i increases.

The above paragraph shows that the convergence of the distance between $x^{(i)}$ and p is independent of position of p, means that $\{X^{(i)}\}_{i=1}^{\infty}$ actually converges to the sphere $\mathbb{S}^2$ in Hausdorff topology. Hence, these subdivision holds the condition (i) in Proposition 5.1.1.

For the condition (ii), recall Remark 3.2.8, we discuss the convergence of normal vector case by case.

1. For the north and south polar circle case in type $k = 1$ and for the $\perp$ and $\top$ case in type $k = 2$, we have the normal vector $\underline{n}^{(i)} = \underline{n}(\alpha_1^{(i)}, \alpha_2^{(i)})$ of $x^{(i)}$ has the following form

$$\begin{pmatrix} \sin(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) * (\cos C_1\alpha_1^{(i)}) \\ \sin(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) * (\sin C_1\alpha_1^{(i)}) \\ \cos(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) \end{pmatrix} \pm \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_2^{(i)} \end{pmatrix}$$

2. For the north and south polar circle case in type $k = 2$, we have the normal vector

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$\underline{n}^{(i)} = \underline{n}(\alpha_1^{(i)}, \alpha_2^{(i)})$ of $x^{(i)}$ has the following form

$$\begin{pmatrix} \sin(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) * (\cos C_1\alpha_1^{(i)}) \\ \sin(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) * (\sin C_1\alpha_1^{(i)}) \\ \cos(\frac{C_2(2\alpha_2^{(i)} \mp 1)}{2}) \end{pmatrix} \pm \frac{(\cos 2C_1 - 1)}{2\sin(\frac{C_2}{2})} \begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_2^{(i)} \end{pmatrix}$$

3. For the $\vdash$ and $\dashv$ case in type $k = 1$, we have the normal vector $\underline{n}^{(i)} = \underline{n}(\alpha_1^{(i)}, \alpha_2^{(i)})$ of $x^{(i)}$ has the following form

$$\begin{pmatrix} (\sin C_2\alpha_2^{(i)})(\cos(\frac{C_1(2\alpha_1^{(i)} \pm 1)}{2})) \\ (\sin C_2\alpha_2^{(i)})(\sin(\frac{C_1(2\alpha_1^{(i)} \pm 1)}{2})) \\ (\cos C_2\alpha_2^{(i)})\cos(\frac{C_1}{2}) \end{pmatrix} + \frac{(1 - \cos C_2)}{2\sin(\frac{C_1}{2})(\sin C_2\alpha_2^{(i)})} \begin{pmatrix} \pm\sin C_1\alpha_1^{(i)} \\ \mp\cos C_1\alpha_1^{(i)} \\ 0 \end{pmatrix}$$

For any choice of $\{x^{(i)}\}_{i=1}^{\infty}$ with $\Phi(x^{(i)}) \to p$ as $i \to \infty$, we have, $C_1, C_2 \to 0$ and $(C_1\alpha_1^{(i)}, C_2\alpha_2^{(i)}) \to (\theta, \phi)$ as $i \to \infty$. Note that $h^{(i)}C_1 = (v^{(i)} + 1)C_2 = \pi$ are fixed. So, the first term in any case of the choice will converge to $\begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}$, which is the normal vector of p. And for the second term (we call it the error term), consider the coefficient of it, we can see the following Taylor expansions (take $C_1 = tC_2$),

$$\begin{aligned} \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})} &= \frac{-t^2}{2}C_2 + \frac{t^2(2t^2-1)}{48}C_2^3 + O(C_2^4), \\ \frac{(\cos 2C_1 - 1)}{2\sin(\frac{C_2}{2})} &= -2t^2C_2 + \frac{t^2(8t^2-1)}{12}C_2^3 + O(C_2^4), \\ \frac{(1 - \cos C_2)}{2\sin(\frac{C_1}{2})(\sin C_2\alpha_2^{(i)})} &= (\frac{1}{2t}C_2 + \frac{t^2-2}{48t}C_2^3 + O(C_2^5))\frac{1}{\sin(y)}. \end{aligned}$$

These means if we fixed the ratio C_1/C_2, which is equal to $(v^{(i)}+1)/h^{(i)}$, then we have the error term converges to 0.

Finally comes to the condition (iii), for $\{x^{(i)}\}_{i=1}^{\infty}$ with $\Phi(x^{(i)}) \to p$ as $i \to \infty$ and for $\{\underline{w}^{(i)} \in T_{x^{(i)}}\}_{i=1}^{\infty}$ converging to some $\underline{w} \in T_p\mathbb{S}^2$, we want to show the Weingarten

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map $S^{(i)} : T_{x^{(i)}} \to T_{x^{(i)}}$ of Φ satisfies

$$S^{(i)}(\underline{w}^{(i)}) \to S(\underline{w})$$

in $\mathbb{R}^3$ as $i \to \infty$, where $S : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ of $\mathbb{S}^2$ is the Weigarten map.

Note that from the definition of the Weigarten map $S^{(i)}$, we have to write $\underline{w}^{(i)}$ as the combination of $\{\underline{x}_a^{(i)} - \underline{x}_b^{(i)}, \underline{x}_c^{(i)} - \underline{x}_b^{(i)}\}$, which $\underline{x}_a^{(i)}, \underline{x}_b^{(i)}, \underline{x}_c^{(i)}$ are the neighbor of $\underline{x}^{(i)}$, like the following

$$\underline{w}^{(i)} = A^{(i)}(\underline{x}_a^{(i)} - \underline{x}_b^{(i)}) + B^{(i)}(\underline{x}_c^{(i)} - \underline{x}_b^{(i)}).$$

Then, we have

$$S^{(i)}(\underline{w}^{(i)}) = -(A^{(i)}(\underline{n}_a^{(i)} - \underline{n}_b^{(i)}) + B^{(i)}(\underline{n}_c^{(i)} - \underline{n}_b^{(i)})).$$

Without loss of generality, we can just think about one direction and $\underline{x}_a^{(i)} - \underline{x}_b^{(i)}$ then

$$\begin{aligned}
S^{(i)}(\underline{w}^{(i)}) &= -(A^{(i)}(\underline{n}_a^{(i)} - \underline{n}_b^{(i)})) \\
&= -\frac{\underline{w}^{(i)}}{\underline{x}_a^{(i)} - \underline{x}_b^{(i)}}(\underline{n}_a^{(i)} - \underline{n}_b^{(i)}) \\
&= -\frac{|\underline{w}^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|}(\underline{n}_a^{(i)} - \underline{n}_b^{(i)}) \\
&= -|\underline{w}^{(i)}|\frac{\underline{n}_a^{(i)} - \underline{n}_b^{(i)}}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|} \\
&= -\underline{w}^{(i)}\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|} + |\underline{w}^{(i)}|\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|}\underline{e}^{(i)} \\
&\to -\underline{w}\lim_{i\to\infty}\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|} + \lim_{i\to\infty}|\underline{w}^{(i)}|\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|}\underline{e}^{(i)} \\
&= S(\underline{w})\lim_{i\to\infty}\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|} + \lim_{i\to\infty}|\underline{w}^{(i)}|\frac{|\underline{n}_a^{(i)} - \underline{n}_b^{(i)}|}{|\underline{x}_a^{(i)} - \underline{x}_b^{(i)}|}\underline{e}^{(i)}
\end{aligned}$$

where $\underline{e}^{(i)}$ is the difference of the direction of $\underline{n}_a^{(i)} - \underline{n}_b^{(i)}$ and $\underline{x}_a^{(i)} - \underline{x}_b^{(i)}$. There are many

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cases about difference between normal vector $\underline{n}_a^{(i)}, \underline{n}_b^{(i)}$ of neighbor of $x^{(i)}$, here we just compute one cases:

For the interior vertex of type $k=2$, that is $\perp$ (or $\top$, resp.) case, then neighbors of $x^{(i)}$ will be $\top$ (or $\perp$, resp.) case, and then the difference of normal vector can be split into two part $\mathrm{P}_1^{(i)} \pm \mathrm{P}_2^{(i)}$,

$$\mathrm{P}_1^{(i)} = \begin{pmatrix} \sin(\frac{C_2(2\alpha_{2a}^{(i)} \mp 1)}{2}) * (\cos C_1\alpha_{1a}^{(i)}) \\ \sin(\frac{C_2(2\alpha_{2a}^{(i)} \mp 1)}{2}) * (\sin C_1\alpha_{1a}^{(i)}) \\ \cos(\frac{C_2(2\alpha_{2a}^{(i)} \mp 1)}{2}) \end{pmatrix} - \begin{pmatrix} \sin(\frac{C_2(2\alpha_{2b}^{(i)} \mp 1)}{2}) * (\cos C_1\alpha_{1b}^{(i)}) \\ \sin(\frac{C_2(2\alpha_{2b}^{(i)} \mp 1)}{2}) * (\sin C_1\alpha_{1b}^{(i)}) \\ \cos(\frac{C_2(2\alpha_{2b}^{(i)} \mp 1)}{2}) \end{pmatrix}$$

and

$$\mathrm{P}_2^{(i)} = \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})}\left(\begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_{2a}^{(i)} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_{2b}^{(i)} \end{pmatrix}\right).$$

But, compare with $\mathrm{P}_3^{(i)} = \underline{x}_a^{(i)} - \underline{x}_b^{(i)}$ where

$$\mathrm{P}_3^{(i)} = \begin{pmatrix} \sin(C_2\alpha_{2a}^{(i)}) * (\cos C_1\alpha_{1a}^{(i)}) \\ \sin(C_2\alpha_{2a}^{(i)}) * (\sin C_1\alpha_{1a}^{(i)}) \\ \cos(C_2\alpha_{2a}^{(i)}) \end{pmatrix} - \begin{pmatrix} \sin(C_2\alpha_{2b}^{(i)}) * (\cos C_1\alpha_{1b}^{(i)}) \\ \sin(C_2\alpha_{2b}^{(i)}) * (\sin C_1\alpha_{1b}^{(i)}) \\ \cos(C_2\alpha_{2b}^{(i)}) \end{pmatrix}.$$

It is easy to verify that $\lim_{i\to\infty} \frac{|\mathrm{P}_1|}{|\mathrm{P}_3^{(i)}|} = 1$ and the difference of direction $\underline{e}^{(i)}$ is in part two $\mathrm{P}_2^{(i)}$. But,

$$\begin{aligned} \mathrm{P}_2^{(i)} &= \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})}\left(\begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_{2a}^{(i)} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sin C_2\alpha_{2b}^{(i)} \end{pmatrix}\right) \\ &= \frac{(\cos C_1 - 1)}{2\sin(\frac{C_2}{2})}\begin{pmatrix} 0 \\ 0 \\ 2\cos\frac{C_2(\alpha_{2a}^{(i)}+\alpha_{2b}^{(i)})}{2}\sin\frac{C_2(\alpha_{2a}^{(i)}-\alpha_{2b}^{(i)})}{2} \end{pmatrix} \end{aligned}$$

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$$= 0 \text{ or } (\cos C_1 - 1)\begin{pmatrix} 0 \\ 0 \\ \cos\frac{C_2(2\alpha^{(i)}_{2a}\pm 1)}{2} \end{pmatrix}$$

The last term holds because we are now in $\perp$ or $\top$ case, these means $\alpha^{(i)}_{2a} - \alpha^{(i)}_{2b} = 0$ or ± 1. So, $\lim_{i\to\infty} \mathrm{P}_2 = 0$ and so is $\lim_{i\to\infty} \underline{e}^{(i)} = 0$.

The rest of the term is $\frac{|\mathrm{P}_2|}{|\mathrm{P}_3|}$, we have $|\mathrm{P}_3| \approx \max\{C_1, C_2\} \sin\phi$. Note we consider in the interior of $X^{(i)}$, so $\sin\phi \neq 0$ and then $\frac{|\mathrm{P}_2|}{|\mathrm{P}_3|} \approx \frac{(\cos C_1 - 1)\cos\phi}{\max\{C_1, C_2\}\sin\phi} \to 0$ as $i \to \infty$.

Combine together we have

$$\begin{aligned}
\frac{|\underline{n}^{(i)}_a - \underline{n}^{(i)}_b|}{|\underline{x}^{(i)}_a - \underline{x}^{(i)}_b|} &= \frac{|\mathrm{P}^{(i)}_1 \pm \mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \\
&\Rightarrow \frac{|\mathrm{P}^{(i)}_1| - |\mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \leq \frac{|\mathrm{P}^{(i)}_1 \pm \mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \leq \frac{|\mathrm{P}^{(i)}_1| + |\mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \\
&\to 1 - 0 \leq \frac{|\mathrm{P}^{(i)}_1 \pm \mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \leq 1 + 0 \\
&\Rightarrow \lim_{i\to\infty} \frac{|\underline{n}^{(i)}_a - \underline{n}^{(i)}_b|}{|\underline{x}^{(i)}_a - \underline{x}^{(i)}_b|} = 1,
\end{aligned}$$

and then

$$\begin{aligned}
\lim_{i\to\infty} S^{(i)}(\underline{w}^{(i)}) &= S(\underline{w}) \lim_{i\to\infty} \frac{|\underline{n}^{(i)}_a - \underline{n}^{(i)}_b|}{|\underline{x}^{(i)}_a - \underline{x}^{(i)}_b|} + \lim_{i\to\infty} |\underline{w}^{(i)}| \frac{|\underline{n}^{(i)}_a - \underline{n}^{(i)}_b|}{|\underline{x}^{(i)}_a - \underline{x}^{(i)}_b|} \underline{e}^{(i)} \\
&= S(\underline{w}) \lim_{i\to\infty} \frac{|\mathrm{P}^{(i)}_1 + \mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} + \lim_{i\to\infty} |\underline{w}^{(i)}| \lim_{i\to\infty} \frac{|\mathrm{P}^{(i)}_1 + \mathrm{P}^{(i)}_2|}{|\mathrm{P}^{(i)}_3|} \lim_{i\to\infty} \underline{e}^{(i)} \\
&= S(\underline{w}) + |\underline{w}| * \underline{0} \\
&= S(\underline{w}),
\end{aligned}$$

as we desired. So, from Proposition 5.1.1, we have that the curvatures of $x^{(i)}$ converge to the curvatures of p. Here, p can pick any point except the pole. Note that this approximation will be fail when we pick p be the pole. Fig. 5.1 and Fig. 5.2 show the error between the discrete curvatures and the smooth one.

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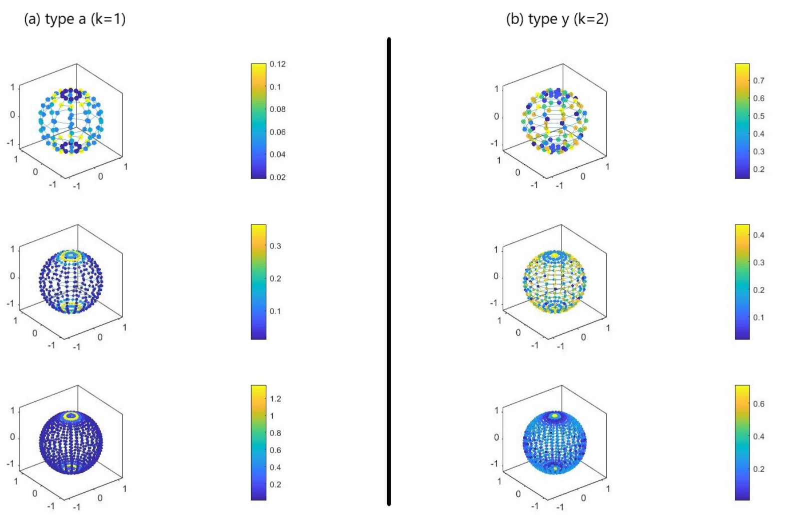

Figure 5.1: Error of Gauss curvature of sphere between discrete sense and smooth sense.

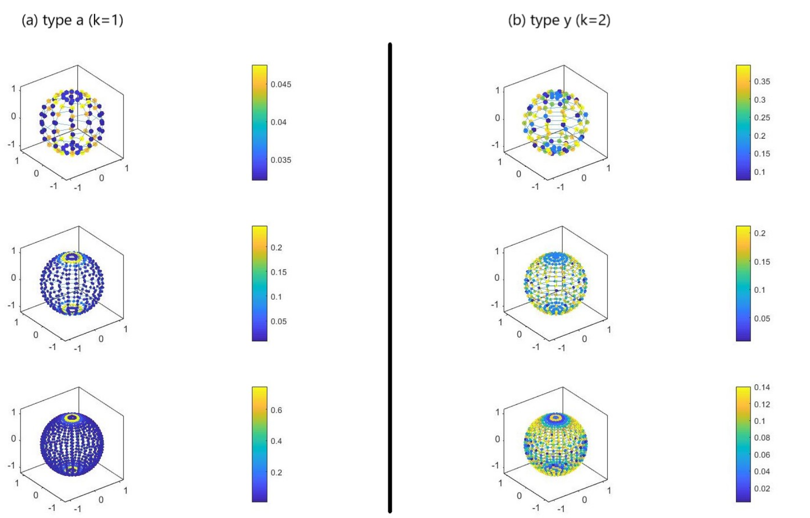

Figure 5.2: Error of mean curvature of sphere between discrete sense and smooth sense.

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Chapter 6

Gauss-Bonnet Theorem

Finally, we comes to the Gauss-Bonnet theorem. In the smooth compact case, we have the following Gauss-Bonnet Theorem.

Theorem 6.0.1 (Gauss-Bonnet Theorem)**.** *Suppose M is a compact two-dimensional Riemannian manifold without boundary. Let K be the Gaussian curvature of M. Then*

$$\int_M K\ dA = 2\pi\chi(M), \tag{6.0.1}$$

where dA is the element of area of the surface. Here, $\chi(M)$is the Euler characteristic of M.

The Gauss-Bonnet theorem connects the geometry of surfaces (in the sense of curvature) to their topology (in the sense of the Euler characteristic). The theorem is true for all Riemannian manifolds satisfying the condition. But, when we consider the discrete case, the theorem fails. We discuss the failure of the discrete Gauss-Bonnet Theorem in the following.

6.1 Discrete Gauss-Bonnet Theorem

For the discrete surface, we may try to establish a similar theorem like the following statement.

Conjecture 6.1.1. Suppose M is a compact two-dimensional Riemannian manifold without boundary and $\Phi : X \to M$ is a 3-valent discrete surface. Let K_X be the

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Gaussian curvature of X in discrete sense. Then,

$$\sum_{v\in X} K_X(v)\, dA_X(v) = 2\pi\chi(X), \tag{6.1.1}$$

where $dA_X(v)$ is the area of the triangle neighboring v. Here, $\chi(X)$is the Euler characteristic of X (usually $=\chi(M)$).

But, the conjecture is wrong in this form. Just consider the regular tetrahedron, both area and curvature are rational number with root of integer, so is the left hand side of (6.1.1). But, the Euler characteristic of a tetrahedron is 2, so the right hand side of the equation is 4π. The two sides can't match.

6.2 Numerical computations for convergence of Gauss-Bonnet Theorem on sphere

Are we going to give up this theorem? No, we believe that when the cut is fine enough, the discrete curvature will approach a smooth curvature. We expect that the discrete sum (6.1.1) will be a Riemann sum of the integral (6.0.1) and will be close to an integral.

More precisely, given a compact two-dimensional Riemannian manifold without boundary M, for example, a sphere, and take a sequence of discrete surface $\Phi : X^{(i)} \to M$. Assume they are finer as i increases, or the distance between adjacent points are monotone decreasing. Then, we believe the following statement is true.

$$\lim_{i\to\infty} \sum_{v\in X^{(i)}} K^{(i)}(v)\, dA^{(i)}(v) = 2\pi\chi(M), \tag{6.2.1}$$

Here, we give a example on sphere. Take M to be a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and take $X^{(i)}$ be the spherical brick graph $\mathrm{Sph}_k(1, h^{(i)}, v^{(i)})$ which is introduced in Section. 3.2. Take $\{h^{(i)}\}$ and $\{v^{(i)}\}$ be strictly monotone increasing sequences. By the definition of

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spherical brick graph, it is easy to see the surfaces are finer as i increases.

By using numerical computations, we obtain distributions of the error of Gauss-Bonnet formula, or

$$\text{error}(i) = \sum_{v \in X^{(i)}} K^{(i)}(v) A^{(i)} - 4\pi.$$

Note that we compute two types of spherical brick graphs (see Section 3.2) and two different choices $\{(h^{(i)}, v^{(i)})\}$ with fixed ratio $h^{(i)} : v^{(i)}$. The detailed information are shown in Fig. 6.1.

As we discuss in Section 5.2, the sequence of the spherical brick graph converges in Hausdorff to the sphere. We expect the error of Gauss-Bonnet formula, error(i), converges to zero. However, error(i) of type x may not converges while error(i) of type y does converge in both parameters (see Fig. 6.1). This shows that the spherical brick graph of type y is a better cut for a sphere. And notice that there are some constant between the limit of error(i) of type x and the zero, this may indicate a slight flaw in type x, we guess that the major error term comes from the points near the pole.

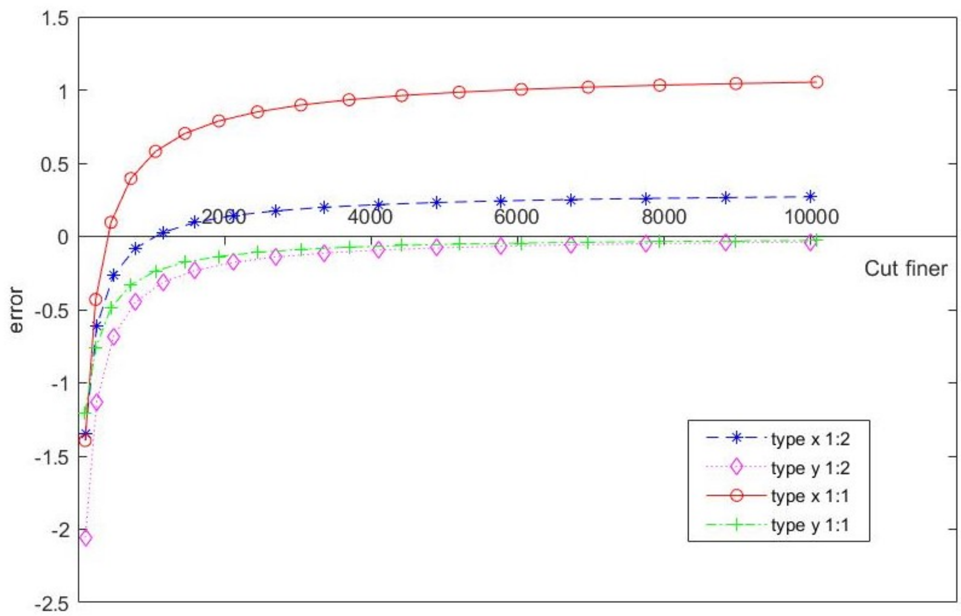

Figure 6.1: Sequence of error of Gauss-Bonnet formula of sphere of two types and two parameters.

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