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動態均衡的無政府狀態價格
On the Price of Anarchy of Dynamic Equilibria

梁峻瑋
Jun-Wei Liang

指導教授：陳和麟 博士
Advisor: Ho-Lin Chen, Ph.D.

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國立臺灣大學碩士學位論文
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本論文係梁峻瑋君（學號 R08921053）在國立臺灣大學電機工程學系完成之碩士學位論文，於民國 110 年 01 月 26 日承下列考試委員審查通過及口試及格，特此證明。

口試委員：

（簽名）

（指導教授）

系主任 （簽名）

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摘　　要

在這篇論文中，我們研究了一個隨時間變化的網路流（network flow）模型，稱之爲「流體排隊模型」。考慮到一個有向圖被注入連續的流量，致使水分子傳播到每一條邊上。對於每一條邊而言，如果流入的流率超過給定的容量，那麼超出的粒子將會形成等待的隊伍，而其他的粒子在給定的時間長度內通過這一條邊。更精確地，我們研究了在賽局觀點上的流體排隊模型，稱之爲動態平衡模型，這被用來描述一些問題。例子包含了網際網路、自駕車中控系統、以及中央處理器挑選任務的過程。先前，有幾位作者研究了流體排隊模型，像是 Ford and Fulkerson [1,2]、Gale [3]、Anderson and Philpott [4]、Fleischer and Tardos [5]。也有一些作者研究了動態平衡模型，像是 Vickrey [6]、Meunier and Wagner [7]、Cominetti, Correa, andLarré [8]、Kaiser and Marcus [9]。我們的研究延續了單調猜想 (monotonicity conjecture) 下具有常數函數流量的網路的 PoA(Price of Anarchy) 上限結果 [10]，以及具有連續函數流量的網路的 PoA 存在性結果 [8,9]。一方面，在串並聯網路（Series-parallel network）研究 PoA 的目的是要測量「分配任務給處理器」這項任務的無效率程度。另一方面，在具備連續函數流量的網路研究 PoA 的目的是要縮小模型和現實中的差距。我們認爲具備動態函數流量的網路相對接近實際狀況，如自駕車中控系統。在現實生活中，交通流量隨著時間而變化。

我們在研究中發現了具備連續函數流量的網路的 PoA 上限。我們證明了平行網路和兩層兩條邊平行網路的 PoA 上限爲 2，串並聯網路有一個 PoA 上限爲「網路的直徑」，所有網路在假設成立下有一個 PoA 上限爲 $2|V|-1$。這是第一篇論文研究具備連續函數流量的網路在流體排隊模型上的 PoA。我們把平行網路和串並聯網路的 PoA 上限從無限大分別壓低到 2 和「網路的半徑」。這兩個被證明出來的上限和具備常數函數流量的網路在流體排隊模型上的情況截然不同。另一方面，類似於在早先抽稅方案上的研究 [11]，我們設計了一個簡單的抽稅方案來改善系統的無效率程度。這會許對於具備連續函數流量的網路會有極大的幫助。

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Abstract

In this thesis, we study a Time-Variant-Model, called the "fluid queuing model". Consider a directed graph injected with a continuous inflow such that water propagates to each edge. For each edge, If the inflow rate exceeds the given capacity, the exceeding particles form a waiting queue, and the other particles pass through this edge under the given delay time. We study the game theory aspect of the fluid queuing model, called the dynamic equilibrium model, which was applied to describe several problems. Examples include the Internet, the self-driving car central control system, and the procedure of CPU-core. Previously, several authors studied on fluid queuing model, likes Ford and Fulkerson [1, 2], Gale [3], Anderson and Philpott [4], Fleischer and Tardos [5]. Some authors studied on dynamic equilibrium model, likes Vickrey [6], Meunier and Wagner [7], Cominetti, Correa, andLarré [8], Kaiser and Marcus [9]. Our study continues the result of PoA bound of networks with constant inflow under monotonicity conjecture [10], and the existence of PoA of networks with dynamic inflow [8,9]. On the one hand, the purpose of studying the PoA in a series-parallel network is to measure the inefficiency of the problem of assigning tasks to processors. On the other hand, the purpose of studying the PoA of networks with dynamic inflow is to narrow the gap between the model and the reality. We think the networks with dynamic inflow will be closer to the actual situation of the central control system of autonomous vehicles. In the real world, traffic will change over time.

We find upper bounds of the PoA of networks with dynamic inflow. We prove the PoA of 2 of parallel-link networks and $(2+2)$-parallel-link networks, the PoA of "network's diameter(called $D(C)$)" of series-parallel networks, the PoA of $2|V|-1$ of general networks with assumption. This thesis is the first study on the PoA for networks with dynamic inflow in the fluid queuing model. That is, we reduce the upper bound of PoA

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of parallel-link networks or series-parallel networks from infinite to 2 and $D(C)$ respectively. The bounds we proved are different from networks with constant inflow in the fluid queuing model. On the other hand, similar to the work of tax scheme [11], we design a simple tax scheme to improve the inefficiency of the system. This may help a lot on networks with dynamic inflow.

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Chapter 1

Introduction

In this thesis, we consider the game theory aspect of the fluid queuing model proposed by Vickrey [6]. In the fluid queuing model, the system consists of a directed graph and a continuous inflow. Each edge of this graph has a capacity and a delay time. When the flow traverses through the edges, queuing occurs if the flow rate exceeds the capacity of the edge. The exceeding particles of the flow would form queuing to wait for the next moment, and the other particles would pass this edge with the delay time of edge as the cost of time. This model views each of the infinite particles as a player. These players selfishly choose the shortest path from source to sink to minimize the travel time, which forms the equilibrium flow. The travel time of each particle is the summation of the waiting time at the source plus the queuing time and delay time of each edge it chooses.

This model can be used to describe several problems. Examples include the Internet, the central control system of self-driving cars, and the task processor of the CPU core. Consider the problem of transferring packets on the Internet. We can model the Internet as a network, packet as particle forming inflow, and network congestion as fluid queuing of each edge. On the example of traffic networks, we can model the traffic network as a network, vehicle as particle forming inflow, the traffic jam on each road as the fluid queuing of each edge. Note that the optimal solution is the self-driving car central control system. On the problem of assigning tasks to processors, we can model each processor execution time as edges of a network, the process as particle forming inflow, and the

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waiting queue of each machine as the fluid queuing of each edge. In this example, the programming statements forming series operations and parallel operations, such as for-loops or if-else. These are the motivation of series-parallel networks.

Previously, Ford and Fulkerson [1,2] provided an algorithm of the fluid queuing model with constant inflow to send the maximal mass of flow with a given time. Gale [3] proved the existence of flow that is the optimal solution at each moment. Anderson and Philpott [4], Fleischer and Tardos [5] improved this result from each moment to continuous time version.

Consider the game theory aspect on the fluid queuing model, which is called dynamic equilibrium model and laterally introduced as "Flow of Model", firstly studied by Vickrey [6]; Meunier and Wagner [7] proved the existence of dynamic equilibria.

For fluid queuing model with dynamic inflow, Cominetti, Correa, and Larré [8] proved the existence of dynamic equilibria of piecewise constant inflow and locally Lebesgue-integrable inflow. Kaiser and Marcus [9] constructively proved the existence of dynamic equilibria of locally Lebesgue-integrable inflow.

The purpose of studying the Price of Anarchy (PoA) is to evaluate and quantify the inefficiency of the system. This analysis of PoA enables us to measure each kind of game and design an improved mechanism for them. The PoA was studied in several games, likes Stackelberg game [12], selfish routing game [13], or network game or called game-theory aspect on fluid queuing model [10]. Focusing on the networks with dynamic inflow, we discuss PoA's bound for parallel-link networks, series-parallel networks, and general networks in this thesis. On the other hand, those two papers in the previous paragraph showed the PoA of the fluid queuing model with locally integrable inflow is finite due to the existence of dynamic equilibria.

In the fluid queuing model, the price of anarchy for networks with constant inflow is tightly bounded by $\frac{e}{e-1}$ under a weak assumption, called the monotonicity conjecture. In particular, the price of anarchy for parallel-link networks tightly bounded by $\frac{4}{3}$ without any assumption [10].

Consider the existence of dynamic equilibria of locally integrable inflow and PoA's

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bound for networks with constant inflow. We want to find PoA's bound of series-parallel networks with locally integrable inflow or called dynamic inflow. The purpose of studying the PoA of series-parallel networks is related to the problem of assigning tasks to processors. That is, we want to study the efficiency of the process-and-CPU system. On the other hand, the purpose of studying the PoA of networks with dynamic inflow is to narrow the gap between the model and the reality. We think the networks with dynamic inflow would be closer to real situations, such as the internet or self-driving car central control system. In the real world, traffic will change over time.

Our Result: We find upper bounds of the PoA of networks with dynamic inflow. We prove that the PoA of 2 is a tight bound for parallel-link networks and $(2+2)$-parallel-link networks; the PoA of series-parallel networks is upper bounded by $D(C)$, where $D(C)$ is closed related to the diameter of the network; the PoA of general networks is upper bounded by $2|V|-1$ with assumption. These results are shown in table 1.1.

This thesis is the first study on the PoA for networks with dynamic inflow in the fluid queuing model. That is, we reduce the upper bound of PoA of parallel-link networks or series-parallel networks from infinite to 2 and $D(C)$ respectively. The bounds we proved are different from networks with constant inflow in the fluid queuing model. On the other hand, similar to the work of tax scheme [11], we design a simple tax scheme, called Delay-time tax scheme, to improve the efficiency of the system. Surprisingly, the delay-time tax scheme did not work on networks with constant inflow but may help a lot on networks with dynamic inflow.

Our main work is to use the total amount of inflow as an upper bound of the maximal throughput of networks to afford the lower bound of the cost of optimal flows (or said optimal time) in the fluid queuing model. This technique helps us finding PoA's tight bound of parallel-link networks and a simple example of series-parallel networks, even a loose bound of series-parallel networks. This technique provides an exactly bound of PoA of series-parallel networks with dynamic inflow. Also, it simplifies the work from finding PoA's bound to calculating the mass of the maximal throughput of networks. On the other hand, we design a tax scheme to improve society's welfare as a possible solution to the system's inefficiency.

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Inflow	Network	PoA
Static	Parallel-link	$\frac{4}{3}$ [10]
Static	General w/ hypo.	$\frac{e}{e-1}$ [10]
Dynamic	General	$< \infty$ [8,9]
Dynamic	Parallel-link	2
Dynamic	General w/ hypo.	$< 2\lvert V\rvert - 1$
Dynamic	Series-parallel	$\leq D(C)$
Dynamic	$(2+2)$-parallel-link	2

Table 1.1: Summary of result. The PoA of 2 is a tight bound of parallel-link networks and $(2+2)$-parallel-link networks; The PoA of $D(C)$ is a loose bound of series-parallel networks; The PoA of $2\lvert V\rvert - 1$ is a loose bound of general networks with assumption.

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Chapter 2

Model

In this chapter, there are three sections. Firstly, we will define the fluid queuing model as the base of this thesis; Secondly, we will introduce the few kinds of flows in the fluid queuing model and imply the term–the Price of Anarchy; Thirdly, we will choose some specific models to study. These contents will be used in the next chapter.

2.1 Fluid Queuing Model

Consider a network C by directed graph $G = (V, E)$ with source $s \in V$ and sink $t \in V$. Each edge e_j has capacity v_j and delay time τ_j, denoted by $e_j = (v_j, \tau_j)$. Sometimes, we use the notation $e_j = (a_j, \sigma_j)$. Moreover, each edge also has an infinite buffer to store the particles.

In this paper, the fluid queuing model is regarded as one of the models of network flow changes over time. The inflow of network, μ_0, is a Lebesgue locally integrable function. Denote the last leaving time by $\hat{\theta} = sup\{\theta | \mu_0(\theta) > 0\}$, and the total amount by $M = \int_0^{\hat{\theta}} \mu_0(\theta) d\theta$. Here, we assume all particles have full information, enable them to imply each moment's situation.

Let's define fluid queuing model here, refer to [10]. For each $e \in E$ at θ, denote the queuing mass by $z_e(\theta)$, and the inflow rate by $f_e^+(\theta)$, where $f_e^+(\theta) : \mathbb{R}^+ \to \mathbb{R}^+$. At this

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moment, the changing speed of $z_e(\theta)$ is:

$$\begin{cases} f_e^+(\theta) - v_e \nearrow, & \text{if } f_e^+(\theta) \geq v_e, \\ v_e - f_e^+(\theta) \searrow, & \text{if } f_e^+(\theta) \leq v_e, z_e(\theta) > 0, \\ 0 \rightarrow, & \text{if } f_e^+(\theta) \leq v_e, z_e(\theta) = 0. \end{cases}$$

So, the particle come-in e at θ will leave e at

$$\theta + \frac{z_e(\theta)}{v_e} + \tau_e,$$

where $\frac{z_e(\theta)}{v_e} + \tau_e$ is named by the travel time of e at θ.

For each $e \in E$ at $\theta + \tau_e$, denote the outflow rate by

$$f_e^-(\theta + \tau_e) = \begin{cases} v_e, & \text{if } z_e(\theta) > 0, \\ \min(v_e, f_e^+(\theta)), & \text{if } z_e(\theta) = 0. \end{cases}$$

Now, for each $v \in V - \{s, t\}$, no particle would stay at v at any θ. That is,

$$\sum_{e=(u,v)\in E} f_e^-(\theta) = \sum_{e=(v,w)\in E} f_e^+(\theta).$$

For source s, particle is allowed to wait at s and leave at any θ. We have:

$$\mu_0(\theta) + \sum_{e=(u,s)\in E} f_e^-(\theta) \geq \sum_{e=(s,w)\in E} f_e^+(\theta),$$

since some particles may stay at s.

Remark 1. *In the previous paper, like [10], they denote that*

$$\mu_0(\theta) + \sum_{e=(u,s)\in E} f_e^-(\theta) = \sum_{e=(s,w)\in E} f_e^+(\theta).$$

But, in this thesis, we use the inequality notation. This is because we do not view the particle waiting at s as part of outflow $\sum_{e=(s,w)\in E} f_e^+(\theta)$. This only has few influence on the continued part of thesis.

In the fluid queuing model, the strategy of each particle is to wait for a moment in s, and then choose a path to leave. For this particle p, the travel time is defined by waiting time at s plus the summation of the travel time of all $e \in$ path at the arriving time of p.

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2.2 Flow of Model

2.2.1 OPT flow

The OPT solution (OPT flow) of the fluid queuing model is the strategy as below: Given network C, inflow μ_0. Each particle from μ_0 is manipulated such that the last particle can arrive at t at the earliest time. We denoted this earliest time by T_{OPT}, called the arriving time of all particles in OPT flow.

Remark 2. *The OPT flow maybe not unique. We default it by the OPT flow without queuing and prefer paths with shorter delay time.*

2.2.2 EQU flow

For each particle as a player, the Nash Equilibrium (EQU flow) of the fluid queuing model is the strategy as below: Given network C, inflow μ_0. For each particle, it has to choose waiting time at s plus the travel time of one of the paths from s to t. And, the cost of each particle is the arriving time. Finally, we denoted the time of the last particle arriving t by T_{EQU}, called the arriving time of all particles in EQU flow.

Now, let's introduce two terms. The PoA, price of anarchy [14], is the ratio between the worst Nash Equilibrium and the optimal solution of social cost. The PoS, price of stability [15], is the ratio between the best Nash Equilibrium and the optimal solution of social cost. However, there is only one unique Nash equilibrium in this network game. We simply denote PoA $:= \frac{T_{EQU}}{T_{OPT}}$ without any ambiguity. Here, PoA is the ratio of the arriving time of all particles in EQU flow to OPT flow.

Remark 3. *The waiting time at s is 0 for all particles in this case.*

Definition 1. *Consider EQU flow of network C, inflow μ_0. For the particles arriving s at θ_0, we denote the earliest time to arrive each $v \in V - s$ by $l_v(\theta_0)$.*

Definition 2. *Consider EQU flow of network C, inflow μ_0. The mass of particle existing in C at θ_0, $Q(C, \mu_0, \theta_0) := \int_{\{\theta|\theta\leq\theta_0, l_t(\theta)>\theta_0\}} \mu_0(\theta)d\theta$. When it is clear from the context, we use $Q(\theta_0)$ to denote $Q(C, \mu_0, \theta_0)$.*

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Definition 3. *Consider EQU flow of network C, inflow μ_0. At the moment θ_0 , we define the shortest travel time $L(C, \mu_0, \theta_0) := l_t(\theta_0) - \theta_0$. And, at the last leaving time $\hat{\theta}$, we short-write $\hat{L} := L(C, \mu_0, \hat{\theta})$. When it is clear from the context, we use $L(\theta_0)$ to denote $L(C, \mu_0, \theta_0)$.*

2.2.3 Throughput flow

The throughput flow of the fluid queuing model is the strategy as below: Given network C, inflow μ_0, period time $\theta = 0 \sim t$. Each particle from μ_0 is manipulated such that sending the maximal mass of particle throughput C at inflow μ_0 during $\theta = 0 \sim t$.

Definition 4. *The maximal mass of particle is denoted by $M_{put}(C, \mu_0, t)$. Sometimes we instead inflow function of infinite inflow rate during $\theta = 0 \sim t$ to afford an upper bound of $M_{put}(C, \mu_0, t)$, denoted by $M_{put}(C, \infty, t)$. When it is clear from the context, we use $M_{put}(t)$ to denote $M_{put}(C, \infty, t)$.*

2.3 Networks of Model

Definition 5 (Parallel-link Network)**.** *Given a network C in fluid queuing model, C is Parallel-link networks if $e = (s, t)$ for all $e \in E$.*

Definition 6 (Series-parallel Network)**.** *Series-parallel networks are defined by Induction: Start from parallel-link networks, each time can do series-linking or parallel-linking operation to another well-defined series-parallel network. All possible results form series-parallel networks.*

Definition 7. *Given the series-parallel network C, the maximal number of each path's nodes among all paths is denoted by $D(C)$.*

Definition 8 (Parallel-group-link Network)**.** *Given $\epsilon > 0$ and a parallel-link network C. If there exists $[L_1, R_1]$, ..., $[L_N, R_N]$, $\frac{R_i}{L_i} \leq 1 + \epsilon$ and $\frac{L_{i+1}}{R_i} \geq \frac{1}{\epsilon}$, and the delay time of each edge of C is in one of the intervals, then C is called parallel-group-link network. Connected from the routing game with groups of similar links, refer to [11].*

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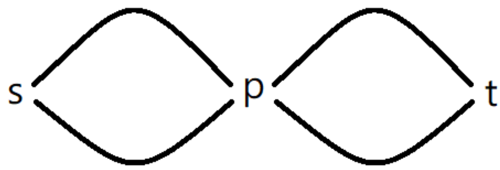

Figure 2.1: The diagram of "$(2 + 2)$-parallel-link Network". In the first stage C_1, the source is s and the sink is p; In the second stage C_2, the source is p and the sink is t.

Definition 9 ($(2 + 2)$-parallel-link Network)**.** *Given parallel-link networks C_1, C_2 and C_1's inflow μ_0. Now, series-link C_1 to C_2, denoted by $C_1 + C_2$. Consider OPT flow and EQU flow on this linking network $C_1 + C_2$, the maximal number of used edges in OPT flow or EQU flow in C_1 and C_2 part are denoted by m_1 and m_2 respectively. If $m_1, m_2 \leq 2$, we called $C_1 + C_2$ is a $(2 + 2)$-parallel-link network.*

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Chapter 3

Main

We will show several PoA's bounds in the fluid queuing model in this chapter. These include some tight upper bound on parallel-link networks or $(2+2)$-parallel-link networks and some finite upper bound on series-parallel networks.

3.1 Parallel-link Networks

First of all, we need Lemma 1 to evaluate the upper bound of the Price of Anarchy for networks.

Lemma 1. *Consider the fluid queuing model with network C, inflow μ_0. For all time θ_0, if*

$$M_{put}(L(\theta_0)) \leq k * Q(\theta_0),$$

then $T_{EQU} \leq (k+1) * T_{OPT}$, where M_{put} is the maximal mass of particle throughput network, Q is the particle existing in network, and T_{OPT}, T_{EQU} are the arriving time of all particle in OPT flow, EQU flow respectively.

Proof. For the shortest travel time $\hat{L} = L(\hat{\theta}) := l_t(\hat{\theta}) - \hat{\theta}$, we have:

$$T_{EQU} - T_{OPT} = l_t(\hat{\theta}) - T_{OPT} = \hat{L} + \hat{\theta} - T_{OPT} \overset{(*)}{\leq} \hat{L},$$

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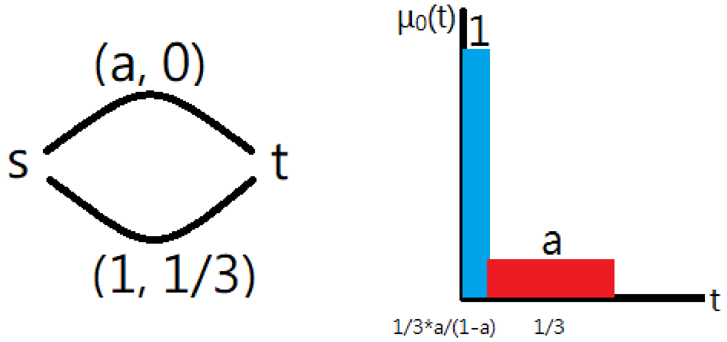

Figure 3.1: The diagram of "PoA of 2 Example". Left part is the parallel-link network with edges $\{e_1 = (a, 0), e_2 = (1, \frac{1}{3})\}$; Right part is the inflow function, which is a step function with range=$\{1, a\}$.

last inequality holds by considering the leaving time and the arriving time of last particle in OPT flow. On the other hand, for the total amount M and the inequality, we have:

$$M \geq Q(\hat{\theta}) \geq \frac{1}{k} M_{put}(L(\hat{\theta})) \overset{(*)}{\geq} \frac{1}{k} * k * M_{put}(\frac{1}{k} L(\hat{\theta})),$$

last inequality holds since we can copy the infinite inflow during $\theta = 0 \sim \frac{1}{k} L(\hat{\theta})$ for k times as an option of infinite inflow during $\theta = 0 \sim L(\hat{\theta})$. On the other hand, since we have to send M mass of particle in OPT flow, so:

$$T_{OPT} \geq \frac{1}{k} L(\hat{\theta}) = \frac{1}{k} \hat{L} \geq \frac{1}{k}(T_{EQU} - T_{OPT}),$$

$$k + 1 \geq \frac{T_{EQU}}{T_{OPT}} = \text{PoA}.$$

□

Theorem 1. *In the fluid queuing model, the Price of Anarchy of* 2 *for parallel-link networks is tight.*

Proof. Gien parallel-link network C, inflow μ_0, denoted $e_j = (v_j, \tau_j)$ for all $e_j \in E$. At any moment θ_0, suppose EQU flow use n edges, then we have:

$$L(\theta_0) \leq \tau_{n+1}.$$

For each edge, the travel time minus the delay time of edge is the queuing time. Hence, we can calculate the mass of particle queuing in C at θ_0 as an lower bound of $Q(\theta_0)$:

$$Q(\theta_0) \geq \sum_{i=1}^{n} v_i[L(\theta_0) - \tau_i],$$

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where $Q(\theta_0)$ is the particle existing in C at θ_0. On the other hand, for each edge, the given time of M_{put} minus the delay time of edge is the total time to pass particle. We have:

$$M_{put}(L(\theta_0)) = \sum_{i=1}^{n} v_i[L(\theta_0) - \tau_i].$$

This implies $Q(\theta_0) \geq M_{put}(L(\theta_0))$. Follow from Lemma 1, we have $PoA \leq 2$. Furthermore, let's provide an example for the Price of Anarchy of 2. Given $1 > a > 0$, consider:

$$E = \{e_1 = (a, 0), e_2 = (1, \frac{1}{3})\}.$$

$$\mu_0(\theta) = \begin{cases} 1, & \theta = 0 \qquad\qquad \sim \frac{1}{3} * \frac{a}{1-a}, \\ a, & \theta = \frac{1}{3} * \frac{a}{1-a} \quad \sim \frac{1}{3} * \frac{a}{1-a} + \frac{1}{3}. \end{cases}$$

In this case, we have:

$$\begin{aligned} T_{OPT} &= \frac{1}{3} * \frac{a}{1-a} + \frac{1}{3}, \\ T_{EQU} &= \frac{1}{3} * \frac{a}{1-a} + \frac{1}{3} + \frac{1}{3}, \\ \text{PoA} &= \frac{T_{EQU}}{T_{OPT}} = \frac{\frac{1}{3} * (2 - 2a + a)}{\frac{1}{3} * (1 - a + a)} = 2 - a \to 2 \quad \text{as} \quad a \to 0^+. \end{aligned}$$

□

Theorem 2. *In the fluid queuing model, if all edges in 1 aggregated network, the Price of Anarchy of* $2 - \frac{1}{1+\epsilon} \approx 1 + \epsilon$ *for parallel-group-link networks is tight, where* ϵ *is given by the network.*

Proof. Given parallel-group-link network C whose edges are in 1 aggregated network, inflow μ_0, denoted $e_j = (v_j, \tau_j)$ for all $e_j \in E$. At the last leaving time $\hat{\theta}$, for the shortest travel time $\hat{L}$, we have the condition:

$$\frac{\hat{L}}{\tau_j} \leq 1 + \epsilon, \qquad \forall e_j \in E.$$

Similar as Lemma 1, we have:

$$T_{EQU} - T_{OPT} = \hat{L} + \hat{\theta} - T_{OPT} \leq (1 - \frac{1}{1+\epsilon})\hat{L} + \tau_1 + \hat{\theta} - T_{OPT} \overset{(*)}{\leq} (1 - \frac{1}{1+\epsilon})\hat{L},$$

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where the last inequality holds by considering the leaving time plus minimal delay time and the arriving time of last particle in OPT flow. Hence, we afford:

$$2 - \frac{1}{1+\epsilon} \geq \frac{T_{EQU}}{T_{OPT}} = \text{PoA}.$$

Finally, similar as Theorem 1, the example for the Price of Anarchy of $2 - \frac{1}{1+\epsilon}$ is shown as below:

$$E = \{e_1 = (a, \frac{1}{3}\frac{1}{1+\epsilon}), e_2 = (1, \frac{1}{3})\},$$

$$\mu_0(\theta) = \begin{cases} 1, & \theta = 0 \qquad\qquad\qquad\qquad \sim(\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon})\frac{a}{1-a}, \\ a, & \theta = (\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon})\frac{a}{1-a} \quad \sim(\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon})\frac{a}{1-a} + \frac{1}{3}. \end{cases}$$

In this case, we have:

$$\begin{aligned} T_{OPT} &= (\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon})\frac{a}{1-a} + \frac{1}{3}, \\ T_{EQU} &= (\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon})\frac{a}{1-a} + (\frac{1}{3} - \frac{1}{3}\frac{1}{1+\epsilon}) + \frac{1}{3}, \\ \text{PoA} &= \frac{T_{EQU}}{T_{OPT}} \to \frac{\frac{2}{3} - \frac{1}{3}\frac{1}{1+\epsilon}}{\frac{1}{3}} = 2 - \frac{1}{1+\epsilon} \approx 2 - (1-\epsilon) = 1 + \epsilon. \end{aligned}$$

holding when $a \to 0$. □

3.2 Series-parallel Networks

Lemma 2. *Consider the fluid queuing model with network C, inflow μ_0, the total capacity of minimal cut face Min-Cut(C), the inequality holds:*

$$M \leq T_{OPT} * \textit{Min-Cut}(C),$$

where M is the total amount and T_{OPT} is the arriving time of all particle in OPT flow.

Proof. In this case, for OPT flow, each particle from s to t has to pass by one of the edges contained in the minimal cut face of C. And, the throughput of the minimal cut face is at most Min-Cut(C) at any moment. As a result, the inequality holds. □

Theorem 3. *In the fluid queuing model, given network C, inflow μ_0. If each of C's edge is used by OPT flow at some moment, the Price of Anarchy is at most $2|V| - 1$.*

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Proof. At the last leaving time $\hat{\theta}$, record $l_v(\hat{\theta})$ for all $v \in E - \{s\}$ and sort them into $\{t_0, t_1, ..., t_n\}$. We discretize the earliest travel time $\hat{L} = l_t(\hat{\theta}) - \hat{\theta}$ into interval (t_{i-1}, t_i). Note that, $n + 1 \leq |V|$.

We know that (t_{i-1}, t_i) contains some edges with some queue mass. For each edge e_j in this interval, we suppose e_j start at $t_{f(j)} \leq t_{i-1}$, denoted $e_j = (v_j, \tau_j)$ and the queuing mass $z_j(t_{f(j)})$, then:

$$\frac{z_j(t_{f(j)})}{v_j} + \tau_j \geq t_i - t_{i-1}.$$

Now, for network C, interval (t_{i-1}, t_i), consider the Min-Cut(C) and an index set of interval $S := \{k | e_k \in [t_{i-1}, t_i]\}$, we have:

$$\frac{\sum_{k \in S} z_k(t_{f(k)})}{\sum_{k \in S} v_k} \leq \frac{M}{\sum_{k \in S} v_k} \leq \frac{M}{\text{Min-Cut}(C)} \overset{(*)}{\leq} T_{OPT},$$

last inequality holds by Lemma 2. This implies $\min_{k \in S} \frac{z_k(t_{f(k)})}{v_k} \leq T_{OPT}$. Now, consider $j = \arg\min \frac{z_k(t_{f(k)})}{v_k}$, we have:

$$t_i - t_{i-1} \leq \frac{z_j(t_{f(j)})}{v_j} + \tau_j \overset{(*)}{\leq} T_{OPT} + T_{OPT},$$

$$\hat{L} = L(C, \mu_0, \hat{\theta}) = l_t(\hat{\theta}) - \hat{\theta} = t_n - t_0 = \sum_{i=1}^{n} t_i - t_{i-1} \leq 2n * T_{OPT},$$

$$\text{PoA} \leq 2n + 1 = 2|V| - 1.$$

Note that, $\tau_j \leq T_{OPT}$ holds since OPT flow uses all edges including e_j. □

Conjecture 1. *In the fluid queuing model, removing each OPT-unused edge in networks only worsens the Price of Anarchy. Hence, the Price of Anarchy is at most* $2|V| - 1$ *for networks.*

Remark 4. *In parallel-link networks, this conjecture is true. In general networks, this conjecture could be false, take Pigou's example for an example. The conjecture is possible to be true in series-parallel networks.*

Theorem 4. *In the fluid queuing model, the Price of Anarchy is at most* $D(C)$ *for series-parallel networks* C*, where* $D(C)$ *denotes the maximal number of each path's node among all* s *to* t *paths.*

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Claim: Given a series-parallel network C, inflow μ_0, any moment θ_0. For the throughput flow M_{put}, the shortest travel time L, and the mass of particle existing in network Q. We have the inequality:

$$M_{put}(C, \infty, L(C, \mu_0, \theta_0)) \leq (D(C) - 1)Q(C, \mu_0, \theta_0),$$

which implies the Theorem from Lemma 1.

Proof. (a)Base Case
In parallel-link network, $D(C) = 2$. The statement is true by Theorem 1.

(b)Parallel-linking
After parallel-linking operation, series-parallel networks C_1 links to C_2. For C's inflow μ_0, define the EQU flow's inflow of C_1 is μ_0^{up}, inflow of C_2 is μ_0^{dn}, and $\mu_0 = \mu_0^{up} + \mu_0^{dn}$; for any moment θ_0. we have:

$$\begin{aligned} L(C_1, \mu_0^{up}, \theta_0) &\geq L(C, \mu_0, \theta_0), \\ L(C_2, \mu_0^{dn}, \theta_0) &\geq L(C, \mu_0, \theta_0), \\ Q(C, \mu_0, \theta_0) &= Q(C_1, \mu_0^{up}, \theta_0) + Q(C_2, \mu_0^{dn}, \theta_0). \end{aligned}$$

Induction hypothesis of Claim:

$$\begin{aligned} \text{given } C_1, \mu_0^{up}, \theta_0: \quad M_{put}(C_1, \infty, L(C_1, \mu_0^{up}, \theta_0)) \leq &\quad (D(C_1) - 1)Q(C_1, \mu_0^{up}, \theta_0), \\ \text{given } C_2, \mu_0^{dn}, \theta_0: \quad M_{put}(C_2, \infty, L(C_2, \mu_0^{dn}, \theta_0)) \leq &\quad (D(C_2) - 1)Q(C_2, \mu_0^{dn}, \theta_0), \end{aligned}$$

This implies:

$$\begin{aligned} M_{put}(C, \infty, L(C, \mu_0, \theta_0)) &\overset{(*)}{\leq} \text{sum of LHS} \\ &\leq \text{sum of RHS} \\ &\leq \max(D(C_1) - 1, D(C_2) - 1)[Q(C_1, \mu_0^{up}, \theta_0) + Q(C_2, \mu_0^{dn}, \theta_0)] \\ &= (D(C) - 1)Q(C, \mu_0, \theta_0), \end{aligned}$$

where $D(C) = \max(D(C_1), D(C_2))$. And, the first inequality holds since the throughput of C is just the sum of throughput of C_1 and C_2 with same period time.

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(c)Series-linking

After series-linking operation, series-parallel networks C_1 links to C_2. For C_1's inflow μ_0, define the EQU flow's outflow of C_1 is μ_1, and μ_1 is C_2's inflow; for any moment θ_0, define $\theta_1 = l_p(\theta_0)$. We have:

$$\begin{aligned} L(C,\mu_0,\theta_0) &= l_t(\theta_0) - \theta_0 \\ &= l_t(\theta_0) - \theta_1 + l_p(\theta_0) - \theta_0 \\ &= L(C_2,\mu_1,\theta_1) + L(C_1,\mu_0,\theta_0), \end{aligned}$$

$$\begin{aligned} Q(C,\mu_0,\theta_0) &\overset{(*)}{\geq} Q(C_1,\mu_0,\theta_0), \\ Q(C,\mu_0,\theta_0) &\overset{(**)}{\geq} Q(C_2,\mu_0,\theta_0). \end{aligned}$$

Inequality (*) holds by consider the mass of particle existing in C_1 at $\theta = \theta_0$ as a lower bound; Inequality (**) holds by consider the mass of particle existing in C_2 at $\theta = \theta_1$ as a lower bound.

Induction hypothesis of Claim:

$$\text{given } C_1,\mu_0,\theta_0: \quad M_{put}(C_1,\infty,L(C_1,\mu_0,\theta_0)) \leq (D(C_1)-1)Q(C_1,\mu_0,\theta_0),$$

$$\text{given } C_2,\mu_1,\theta_1: \quad M_{put}(C_2,\infty,L(C_2,\mu_1,\theta_1)) \leq (D(C_2)-1)Q(C_2,\mu_1,\theta_1),$$

This implies:

$$\begin{aligned} M_{put}(C,\infty,L(C,\mu_0,\theta_0)) &\overset{(*)}{\leq} \text{sum of LHS} \\ &\leq \text{sum of RHS} \\ &\leq [D(C_1)-1+D(C_2)-1]Q(C,\mu_0,\theta_0) \\ &= (D(C)-1)Q(C,\mu_0,\theta_0), \end{aligned}$$

where $D(C) = D(C_1) + D(C_2) - 1$. Moreover, the first inequality holds since we can consider C_2 sending infinite particle without delay time in $\theta = 0 \sim \theta_1$ and C_1 sending infinite particle without delay time in $\theta = \theta_1 \sim l_t(\theta_0)$ as an upper bound of M_{put}. □

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3.3 $(2+2)$-parallel-link Network

Remark 5. *We remove all edges exceed m_1 and m_2, where m_1 and m_2 are the maximal number of used edges in OPT flow or EQU flow in C_1 and C_2 parts respectively. And, this has no influence on our model. On the other hand, we use $e_j = (a_j, \sigma_j)$ in C_1 and $e_j = (v_j, \tau_j)$ in C_2.*

Now, consider the flows on network C_1 and inflow μ_0. Define $T_{EQU}|_{C_1}$,$T_{OPT}|_{C_1}$ as the arriving time of all particle in EQU flow and OPT flow respectively. Let's see a new model named by "Restricted Inflow Model" as below:

Definition 10 (Restricted Inflow Model)**.** *In EQU flow, consider any moment θ_0 such that $\mu_0(\theta_0) \geq a_1 + ... + a_{m_1}$ and $l_t(\theta_0) = \sigma_{m_1}$. We imitate the OPT flow, let $\mu_0(\theta_0) - (a_1 + ... + a_{m_1})$ mass of particles wait at s until $\mu_0 \leq a_1 + ... + a_{m_1}$ and all previous particles have leaven.*

In EQU flow after the operation, the arriving time of each particle is the same as usual. Since the order of any pair of particles is conserved and each edge passing the same mass of particles as usual. Furthermore, the travel time of each particle is at most σ_{m_1} in the Restricted inflow model. Finally, each particle in EQU flow would depart earlier or equal to OPT flow, since there is always no more particle wait at s in EQU flow.

Definition 11 (Different Arriving Time)**.** *Given parallel-link network C_1, inflow μ_0, each particle p from μ_0. Define $d(C_1, \mu_0, p)$ by the different arriving time of EQU flow to OPT flow for particle p. Define $d(C_1, \mu_0) = sup_p d(C_1, \mu_0, p)$.*

Follow from definition, we have:

$$d(C_1, \mu_0, p) \leq \sigma_{m_1}.$$
$$T_{EQU}|_{C_1} - T_{OPT}|_{C_1} \overset{(*)}{\leq} d(C_1, \mu_0) \leq \sigma_{m_1},$$

last inequality holds by considering the last particle as a special case of particle p.

Definition 12 (EO flow)**.** *Given $(2+2)$-parallel-link network $C_1 + C_2$, inflow μ_0. For EQU flow in network C_1 using inflow μ_0, we have the outflow of C_1 as the inflow of C_2,*

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denoted by μ_1. Now, consider the OPT flow of network C_2 using inflow μ_1, this is the EO flow in $(2+2)$-parallel-link network $C_1 + C_2$. And, define the arriving time of all particle in EO flow by the arriving time of all particle in OPT flow of network C_2 using inflow μ_1.

Define n_2 is the maximal number of used edges in EO flow or EQU flow in C_2 part. We apply the above lemma on networks C_2, inflow μ_1, we have:

$$T_{EQU} - T_{EO} \leq \tau_{n_2} \leq \tau_{m_2},$$

where And, m_2 is the maxiaml number in OPT flow or EQU flow in C_2 part. Since we remove all edges exceed m_2, $n_2 \leq m_2$.

Lemma 3. *In the fluid queuing model, given $(2+2)$-parallel-link network C_1 series-linking to C_2 and inflow μ_0, denoted $e_j = (a_j, \sigma_j)$ in C_1 and $e_j = (v_j, \tau_j)$ in C_2. For the OPT flow and EQU flow, we have:*

$$T_{EQU} - T_{OPT} \leq \sigma_{m_1} + \tau_{m_2}.$$

Proof. Follow from above, for the EO flow in this $(2+2)$-parallel-link network, we have:

$$\begin{aligned} T_{EQU} - T_{EO} &\leq \tau_{m_2}, \\ T_{EQU}|_{C_1} - T_{OPT}|_{C_1} &\leq \sigma_{m_1}. \end{aligned}$$

In EO flow, consider a strategy as below: For all particle arriving source of C_2, let them wait at source for $\sigma_{m_1} - d(C_1, \mu_0, p)$ time and leave. In this case, EO flow has the same inflow of C_2 as OPT flow, but delay for exactly σ_{m_1} time. That is:

$$\begin{aligned} \text{OPT's inflow of } C_2 : &\quad \mu_2(\theta), \\ \text{EO's inflow of } C_2 : &\quad \mu_2(\theta + \sigma_{m_1}), \end{aligned}$$

where $\mu_2(\theta)$ is the inflow of C_2 in OPT flow. This relationship implies $T_{EO} - T_{OPT} = \sigma_{m_1}$. However, EO flow may choose some better solutions, implies

$$T_{EO} - T_{OPT} \leq \sigma_{m_1},$$

$$T_{EQU} - T_{OPT} = T_{EQU} - T_{EO} + T_{EO} - T_{OPT} \leq \sigma_{m_1} + \tau_{m_2}.$$

□

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Remark 6. *In the proof of Lemma* 3*, for EO flow, we let the particle wait at the intermediate point* p*. Despite this is not allowed in our model, it only worsens* T_{EQU} *and makes the PoA bigger. The upper bound of PoA is still true if we allowed waiting at the intermediate point* p*. So, we allow it.*

Lemma 4. *Given* $(2+2)$*-parallel-link network* C_1+C_2*,* $T_{OPT} = T_{EO}$ *or* $T_{OPT} \geq \sigma_2+\tau_2$*, where* T_{OPT}*,* T_{EO} *is the arriving time of all particle in OPT flow and EO flow respectively and* (a_1, σ_1)*,* (a_2, σ_2) *denotes the edges of* C_1*,* (v_1, τ_1)*,* (v_2, τ_2) *denotes the edges of* C_2*.*

Proof. Given $(2 + 2)$-parallel-link network $C_1 + C_2$, inflow μ_0, the maximal number of used edges in OPT flow or EQU flow in C_1 part and C_2 part are m_1 and m_2 respectively.

Now, if $m_1 = 1$, then $T_{OPT} = T_{EO}$. Else, consider $m_1 = 2$. Let's divide it into two cases:

(a)$m_1 = 2$ and $a_1 \geq v_1$

During $\theta = 0 \sim \sigma_2$, OPT flow and EO flow have the same inflow and outflow at C_2. Now, if $m_2 = 1$, after $\theta = \sigma + 2$, OPT flow and EO flow still have the same outflow at C_2, implying $T_{OPT} = T_{EO}$. Else, $m_2 = 2$, both OPT and EO active (v_2, τ_2) at $\theta = \sigma_1$.

- If EO flow shutdown $(v_2, \tau_2) first$, then $T_{EO} \leq T_{OPT}$, implies $T_{EQ} = T_{OPT}$.
- If OPT flow shutdown $(v_2, \tau_2) first$ but before $\theta = \sigma_2 + \tau_2$, then OPT flow and EO flow have the same outflow at C_2 later, which will pass the same mass of particle. We let EO flow shutdown (v_2, τ_2) at the same time to afford $T_{EQ} = T_{OPT}$.
- If OPT flow shutdown $(v_2, \tau_2) first$ and after $\theta = \sigma_2 + \tau_2$, then $T_{OPT} \geq \sigma_2 + \tau_2$.

(b)$m_1 = 2$ and $a_1 \leq v_1$

If $m_2 = 1$, consider the EO flow and OPT flow. Maybe EO flow and OPT flow have the total same outflow, then $T_{EO} = T_{OPT}$; Else, EO flow and OPT flow must have the same outflow during $\theta = 0 \sim \sigma_2 + \tau_1$, implies $T_{OPT} \geq \sigma_2 + \tau_1$. Both of the cases are enough to prove PoA ≤ 2.

Else, $m_2 = 2$, which implies $a_1 + a_2 \geq v_2 \geq a_1$ since v_2 is used. Now, if OPT flow

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use (v_2, τ_2), then $T_{OPT} \geq \sigma_2 + \tau_2$. Else, EQU flow use (v_2, τ_2), we can compute:

$$M_{put}(C, \infty, \sigma_2 + \tau_2) = \begin{cases} 0, & \theta = 0 + \tau_1 \sim \quad \sigma_1 + \tau_1, \\ a_1(\sigma_2 - \sigma_1), & \theta = \sigma_1 + \tau_1 \sim \quad \sigma_2 + \tau_1, \\ v_1(\tau_2 - \tau_1), & \theta = \sigma_2 + \tau_1 \sim \quad \sigma_2 + \tau_2. \end{cases}$$

$$M_{put}(C, \mu_0, \sigma_2 + \tau_2) = \begin{cases} 0, & \theta = 0 + \tau_1 \sim \quad \sigma_1 + \tau_1, \\ \int_0^{\sigma_2 - \sigma_1} \min(\mu_0(\theta), a_1) d\theta, & \theta = \sigma_1 + \tau_1 \sim \quad \sigma_2 + \tau_1, \\ v_1(\tau_2 - \tau_1), & \theta = \sigma_2 + \tau_1 \sim \quad \sigma_2 + \tau_2. \end{cases}$$

On the other hand, in EQU flow, let's find a lower bound of the total mass of particle arriving at the source of $C_1 + C_2$ before activating (v_2, τ_2) as an lower bound of M:

$$\begin{cases} 0, & \theta = 0 + \tau_1 \sim \sigma_1 + \tau_1, \\ \text{the mass of particle passing by } C_1 + C_2 = \int_0^{\sigma_2 - \sigma_1} \min(\mu_0(\theta), a_1) d\theta, & \theta = \sigma_1 + \tau_1 \sim \sigma_2 + \tau_1, \\ \text{the mass of particle queuing at } C_2 \text{ to active } (v_2, \tau_2) = v_1(\tau_2 - \tau_1), & \theta = \sigma_2 + \tau_1 \sim \hat{\theta}. \end{cases}$$

As a result, we have:

$$M_{put}(C, \mu_0, \sigma_2 + \tau_2) \leq M.$$

Similar as Lemma 1, this implies $T_{OPT} \geq \sigma_2 + \tau_2$. □

Theorem 5. *In the fluid queuing model, the Price of Anarchy of 2 for $(2+2)$-parallel-link networks is tight.*

Proof. Follow from the Lemma 4, if $T_{OPT} = T_{EO}$, apply Lemma 1 on network C_2 using inflow μ_1, we have:

$$\frac{T_{EQU}}{T_{OPT}} = \frac{T_{EQU}}{T_{EO}} \leq 2.$$

On the other hand, if $T_{OPT} \geq \sigma_2 + \tau_2$, together with Lemma 3, we have:

$$T_{OPT} \geq \sigma_2 + \tau_2 \geq T_{EQU} - T_{OPT},$$

$$\text{PoA} = \frac{T_{EQU}}{T_{OPT}} \leq 2.$$

So, we achieve the goal. □

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Chapter 4

Extension

We will show a self-defined tax scheme and improve the PoA's bound with it in the fluid queuing model. Although it seems helpless for the networks with constant inflow, it is helpful for the parallel-link networks with some extreme inflow cases.

Definition 13 (Delay-time Tax Scheme)**.** *In the fluid queuing model, we increase the delay time of edges by imposing tax in the given networks. The tax is not considered as the cost of society's welfare. That is, the tax scheme will only change the behavior of particles, and the computation of delay time and travel time still uses the original setting. This is called the delay-time tax scheme, refer to [11].*

Remark 7. *Under the definition of the Delay-time Tax Scheme, OPT flow would not change anything after taxing. This is because the computation of delay time and travel time still uses the original setting. As a result, we only discuss EQU flow and the changes on the arriving time of all particle in EQU flow,* T_{EQU}.

Theorem 6. *In the fluid queuing model with constant inflow, the Price of Anarchy is at least* $\frac{4}{3}$ *for parallel-link networks after taxing.*

Proof. Let's show an example of the Price of Anarchy of $\frac{4}{3}$, refer to [10]. Consider the parallel-link network with two edges, the total amount M, and the constant inflow function

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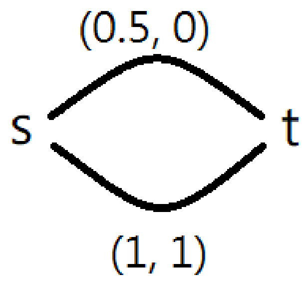

Figure 4.1: The diagram of “PoA of $\frac{4}{3}$ Example” This is the parallel-link network with edges $\{e_1 = (0.5, 0), e_2 = (1, 1)\}$, the total amount $M = 1$, and the constant inflow function $\mu_0 = 1$.

μ_0 as below:

$$E = \{e_1 = (0.5, 0), e_2 = (1, 1)\},$$
$$M = 1,$$
$$\mu_0(\theta) = 1.$$

In this example, we have:

$$T_{OPT} = 1.5, \quad T_{EQU} = 2, \quad \text{PoA} = \frac{4}{3}.$$

Now, for this example, we apply the Delay-time tax scheme into the below 3 cases, and discuss the influence on T_{EQU}:

(a)Taxing on both edges

This can reduce to ”Taxing on single edge” case, since the behavior of particles only be influenced by the difference of two edges’ delay time, but not the exactly value of each edge’s delay time.

(b)Taxing on down edge $e_2 = (1, 1)$

In EQU flow, all particle still choose up edge as usual. Nothing is changed. We have: $T_{EQU} = 2$ and PoA $= \frac{4}{3}$ after taxing.

(c)Taxing on up edge $e_1 = (0.5, 0)$

Denote the delay time of e_1 by x after taxing. Consider EQU flow:

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- if $x > 1$, then all particle choose down edge. We have: $T_{EQU} = 2$ and PoA $= \frac{4}{3}$ after taxing.
- if $x = 1$, then passing two edges spends the same time. That is, there are infinitely many Nash equilibrium, and we consider the worst-case, "the last particle go down edge". We have: $T_{EQU} = 2$ and PoA $= \frac{4}{3}$ after taxing.
- if $x < 1$, then the behavior of particle is as below: During $\theta = 0 \sim 1 - x$, all particles go up edge and queue length is $\frac{1}{2}(1-x)$ finally. From now on, passing two edges spends the same time. That is, there are infinitely many Nash equilibrium, and we consider the worst-case. We have: $T_{EQU} = 2$ and PoA $= \frac{4}{3}$ after taxing.

As a result, the Price of Anarchy for this example is $\frac{4}{3}$ after taxing. □

Theorem 7. *In the fluid queuing model with constant inflow, the Price of Anarchy is at least* $\frac{e}{e-1}$ *after taxing on single edge.*

Proof. Let's show an example of the Price of Anarchy of $\frac{e}{e-1}$, refer to [10]. Consider the series-parallel-link network with $2m$ edges, the total amount M, and the constant inflow function μ_0 as below:

$$\{e_i = (u_i, 0), \quad e^i = (u^i, \alpha\mu^m(\tfrac{1}{u_{OPT}} - \tfrac{1}{u_i}))\}_{i=1\sim m},$$
$$M = \alpha\mu^m,$$
$$\mu_0 = \mu^m,$$

where $m \in \mathbb{N}$, $\alpha > 0$, $\mu^i = \sum_{k=1}^{i} \mu_k$. In this example, we have:

$$T_{OPT} = \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i}), \quad T_{EQU} = \alpha\mu^m, \quad \lim_{m\to\infty} \text{PoA} = \frac{e}{e-1}.$$

Now, for this example, we apply the Delay-time tax scheme on single edge into the below 2 cases, and discuss the influence on T_{EQU}:

(a)Taxing on up edge $e_i = (u^i, \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i}))$:
In EQU flow, all particles go down edge firstly and become part of queuing until $t = \alpha$. At this moment, inflow stop! For non-taxing-up-edges, they are about to be activated; For the only taxing-up-edge e_i, it is not yet to be activated. So, $T_{EQU} > \alpha\mu^m$ and

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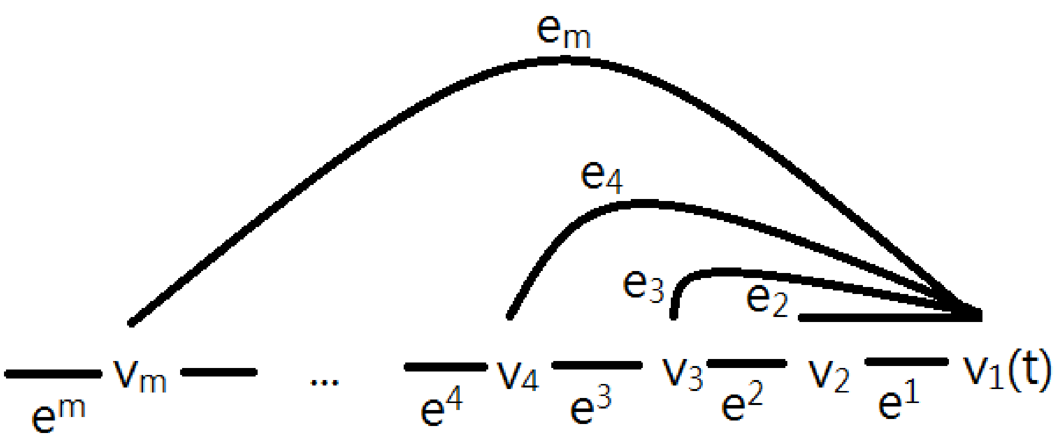

Figure 4.2: The diagram of "PoA of $\frac{e}{e-1}$ Example" This is the series-parallel-link network with edges $\{e_i = (u_i, 0), \quad e^i = (u^i, \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i}))\}_{i=1\sim m}$, the total amount $M = \alpha\mu^m$, and the constant inflow function $\mu_0 = \mu^m$, where $m \in \mathbb{N}$, $\alpha > 0$, $\mu^i = \sum_{k=1}^{i} \mu_k$.

$\lim_{m\to\infty}$ PoA $> \frac{e}{e-1}$.

(b)Taxing on down edge $e^i = (u_i, 0)$:
Denote the delay time of e^i by x after taxing.

- In EQU flow, if $x > \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i})$ or $x = \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i})$, similar as Theorem 6, then all particles coming v_{i+1} go up road e_{i+1}. This makes T_{EQU} the same and $\lim_{m\to\infty}$ PoA $= \frac{e}{e-1}$.

- In EQU flow, if $x < \alpha\mu^m(\frac{1}{u_{OPT}} - \frac{1}{u_i})$, then all particles go down edge firstly. And, the up road e_{i+1} is activated earlier than $t = \alpha$, implies $l_{v_{i+1}}(\theta) = \tau_{i+1}$. Despite the up orad e_{i+1} is activated earlier, but the subsystem from v_{i+1} to v_1 would not finish earlier. This is because the inflow from v_{i+1} coming e_{i+1} and e^i is the same as usual. So, $T_{EQU} = \alpha\mu^m$ and $\lim_{m\to\infty}$ PoA $= \frac{e}{e-1}$.

As a result, the Price of Anarchy for this example is $\frac{e}{e-1}$ after taxing on single edge. $\square$

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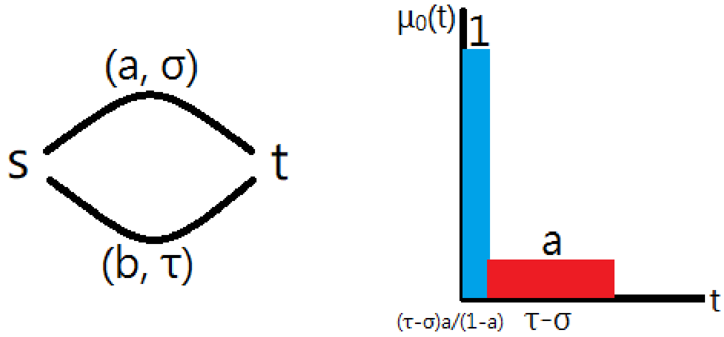

Figure 4.3: The diagram of "PoA of 2 Example". Left part is the parallel-link network with 2 edges $\{e_1 = (a, \sigma), e_2 = (b, \tau)\}$; Right part is the inflow function, which is a step function with range=$\{1, a\}$.

Theorem 8. *In the fluid queuing model with dynamic inflow, the Price of Anarchy can be reduced by the delay-time tax scheme in some cases.*

Proof. Let's show the parallel-link network with 2 edges as example, whose Price of Anarchy is $1 + \epsilon$ after taxing with given ϵ. This example is similar as Chapter 3-Therorem 1. Given a, σ, b, τ such that $a + b \geq 1$, $\sigma < \tau$, consider:

$$E = \{e_1 = (a, \sigma), e_2 = (b, \tau)\}.$$

$$\mu_0(\theta) = \begin{cases} 1, & \theta = 0 \qquad\qquad\qquad\qquad \sim (\tau - \sigma) * \dfrac{a}{1-a}, \\ a, & \theta = (\tau - \sigma) * \dfrac{a}{1-a} \quad \sim (\tau - \sigma) * \dfrac{a}{1-a} + \tau - \sigma. \end{cases}$$

In this case, we have:

$$\begin{aligned} T_{OPT} &= (\tau - \sigma) * \frac{a}{1-a} + \tau, \\ T_{EQU} &= (\tau - \sigma) * \frac{a}{1-a} + \tau + (\tau - \sigma), \\ \text{PoA} &= \frac{T_{EQU}}{T_{OPT}} = \frac{(\tau - \sigma)\frac{1}{1-a} + \sigma}{(\tau - \sigma)\frac{1}{1-a} + (\tau - \sigma)} \to \frac{\tau}{2(\tau - \sigma)} \quad \text{as} \quad a \to 0^+. \end{aligned}$$

Now, we apply the Delay-time tax scheme on this example. Given $\epsilon > 0$, tax on up edge $e_1 = (a, \sigma)$ such that $e_1 = (a, \tau - \epsilon)$. In EQU flow, the behavior of particles are as follow:

$$\begin{cases} \text{All particles go up-road and queue length achieves } a\epsilon, & 0 \sim \epsilon\dfrac{a}{1-a}, \\ \text{Particles go up road at speed } a\text{; down road at speed } 1-a, & \epsilon\dfrac{a}{1-a} \sim (\tau - \sigma)\dfrac{a}{1-a}, \\ \text{Inflow stop! Consume the queuing,} & (\tau - \sigma)\dfrac{a}{1-a} \sim (\tau - \sigma)\dfrac{a}{1-a} + \tau + \epsilon. \end{cases}$$

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As a result, $T_{EQU} = (\tau - \sigma)\frac{a}{1-a} + \tau + \epsilon$ and PoA $= 1 + \frac{(1-a)\epsilon}{\tau - a\sigma} \to 1$ as $\epsilon \to 0^+$. $\square$

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Chapter 5

Conclusions and Future Works

We find upper bounds of the PoA of networks with dynamic inflow. We prove the PoA of 2 of parallel-link networks and $(2+2)$-parallel-link networks, the PoA of $D(C)$ of series-parallel networks, the PoA of $2|V|-1$ of general networks with assumption. We reduce the upper bound of PoA of parallel-link networks or series-parallel networks from infinite to 2 and $D(C)$ respectively. The bounds we proved are different from networks with constant inflow in the fluid queuing model. On the other hand, similar to the work of tax scheme [11], we design the Delay-time tax scheme to improve the system's inefficiency, which may help a lot on networks with dynamic inflow. Our main work is to use the total amount of inflow as an upper bound of the maximal throughput of networks to afford the lower bound of the cost of optimal flows (or said optimal time) in the fluid queuing model. This technique helps us finding PoA's tight bound of parallel-link networks and a simple example of series-parallel networks, even a loose bound of series-parallel networks. This technique provides an exactly bound of PoA of series-parallel networks with dynamic inflow. Also, it simplifies the work from finding PoA's bound to calculating the mass of the maximal throughput of networks. On the other hand, we design a tax scheme to improve society's welfare as a possible solution to the system's inefficiency.

Now, consider we have proved a loose upper bound $D(C)$ of series-parallel networks and the tight bound 2 of parallel-link networks and $(2+2)$-parallel-link networks. The first future work is to find the tight bound of extension-parallel networks or series-parallel

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networks, which probably is 2. In this case, the technique used in the proof of PoA of $(2+2)$-parallel-link networks can be essential. On the other hand, consider the Delay-time tax scheme. We have shown it is helpful on networks with dynamic inflow by an example. Another future work is to prove the PoA of parallel-link networks is $\frac{4}{3}$ after taxing. In networks with dynamic inflow, the equilibrium flows after taxing will prefer no queuing and using the shortest delay time path, similar to the optimal flows. However, in networks with static inflow, the equilibrium flows after taxing will not stop larger delay time edges to queue at shorter delay time edges at the final step, which is different from the optimal flows. As a result, the PoA of networks with dynamic inflow after taxing may be the PoA of networks with constant inflow. Finally, the study of general networks is also a point. For example, we may prove that $T_{OPT} \geq L(\hat{\theta}) = \hat{L} \geq T_{EQU} - T_{OPT}$ in Pigou's example, which implies PoA is 2. However, the technique in Lemma 1 cannot be used directly in this case.

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