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有限特殊線性群的特徵標

On characters of finite special linear groups in non-defining characteristic

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國立臺灣大學碩士學位論文
口試委員會審定書

有限特殊線性群的特徵標
On characters of finite special linear groups in non-defining characteristic

本論文係陳鴻猷君（R05221012）在國立臺灣大學數學系完成之碩士學位論文，於民國 108 年 9 月 9 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

本文討論置換群與有限一般線性群的分解矩陣的各種性質，分析不同性質之間的關係，並證明當特徵 p 不整除 q 時，有限特殊線性群亦擁有 (C, p)-性質。

關鍵詞：有限特殊線性群、模表現論、群論、分解矩陣、有限一般線性群

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Abstract

In this thesis, we consider some properties of decomposition matrices of symmetric groups and finite general linear groups in non-defining characteristic, clarify the relations among these properties, and show that $SL_n(q)$ has an anologue property to $\mathfrak{S}_n$ and $GL_n(q)$ in non-defining characteristic, namely the (C, p)-property.

Keywords: finite special linear group, modular representation, group theory, decomposition matrix, finite general linear group

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Contents

口試委員會審定書 i

致謝 ii

中文摘要 iii

Abstract iv

0 Introduction 1

1 Preliminaries 6

1.1 Partitions 6

1.2 Group Theory 7

1.3 Conjugacy Classes in $GL_n(q)$ 9

1.4 Size of Conjugacy Classes of R_n 12

1.5 Representations and Modules 14

1.6 Characters 17

1.7 The Decomposition Matrix 20

1.8 Harish-Chandra Induction 21

2 Representation Theory of $GL_n(q)$ 22

2.1 Compositions, Tableaux and Permutations 22

2.2 Subgroups 24

2.3 Idempotents 26

2.4 The Module $M_{\mathbf{F}}(\sigma, (1))$ 27

2.5 The Module $M_{\mathbf{F}}(\sigma, (1^k))$ 31

2.6 The Module $M_{\mathbf{F}}(\sigma, \lambda)$, $S_{\mathbf{F}}(\sigma, \lambda)$ and $D_{\mathbf{F}}(\sigma, \lambda)$ 34

2.7 The Module $S_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$ and $D_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$ 39

doi:10.6342/NTU201904116

3 Clifford Theory **42**

3.1 Cyclic Quotient 43

3.2 Direct Product 45

3.3 Classification of Irreducible Representations of SL_n 46

3.4 G-tile and S-tile 48

3.5 Representation Theory of G/S 51

4 Field Theory **53**

4.1 Basic Facts 53

4.2 Elementary Number Theory 54

4.3 Degree Extension Lemma 56

4.4 Lemmas for Kleshchev-Tiep's Theorem 59

5 Kleshchev-Tiep's Theorem **62**

6 Main Theorem **68**

6.1 The Canonical Composition Factor 68

6.2 Main Results 71

6.3 Relations Between Branching Numbers 73

6.4 A Lower Unitriangular Submatrix 74

7 Conclusion **77**

A Appendix **79**

A.1 The Original Problem 79

A.2 The Implication Among Properties 82

A.3 The Decomposition Matrix of GL_2 and SL_2 84

A.4 The Decomposition Matrix of Other Groups 98

References **102**

doi:10.6342/NTU201904116

List of Tables

1 Decomposition matrix of $\mathfrak{S}_6$, $p = 3$ 2
2 The property table for some families of groups 77
3 Decomposition matrix of GL_2, $p > 2$, $p \mid q - 1$ 90
4 Decomposition matrix of SL_2, $p > 2$, $p \mid q - 1$, q odd 91
5 Decomposition matrix of SL_2, $p > 2$, $p \mid q - 1$, q even 92
6 Decomposition matrix of GL_2, $p > 2$, $p \mid q + 1$ 93
7 Decomposition matrix of SL_2, $p > 2$, $p \mid q + 1$, q odd 94
8 Decomposition matrix of SL_2, $p > 2$, $p \mid q + 1$, q even 95
9 Decomposition matrix of GL_2, $p = 2$, q odd 96
10 Decomposition matrix of SL_2, $p = 2$, q odd 97
11 Decomposition matrix of $GL_3(4)$, $p = 3$ 99
12 Decomposition matrix of $SL_3(4)$, $p = 3$ 100
13 Decomposition matrix of $\mathfrak{A}_6$, $p = 3$ 101

doi:10.6342/NTU201904116

0 Introduction

One of the general problems of representation of finite groups is to describe the decomposition matrix of irreducible characters from the ordinary case to the p-modular case. However, complete knowledge of such matrices is known only for few classes of groups, such as the symmetric groups and the general linear groups over finite fields. In this thesis, we are going to study properties of the decomposition matrices for the special linear groups, where most of the ingredients are from Kleshchev-Tiep [K].

Let $\mathbb{F}_q$ be the finite field of q elements. Let $K = \overline{\mathbb{Q}}$ be the algebraic closure of the field of rational numbers, and $F = \overline{\mathbb{F}}_p$ be the algebraic closure of the finite field of p elements, where p is a prime not dividing q.

Let G be a finite group. Denote $\mathrm{irr}_K(G)$ the set of irreducible ordinary characters, and $\mathrm{irr}_F(G)$ the set of irreducible Brauer (p-modular) characters. Let $R_K^+(G)$ be the set of all ordinary characters, and $R_K(G)$ be the free $\mathbb{Z}$-module generated by $\mathrm{irr}_K(G)$, called the set of the virtual ordinary characters. Similarly, let $R_F^+(G)$ be the set of all Brauer characters, and $R_F(G)$ be the free $\mathbb{Z}$-module generated by $\mathrm{irr}_F(G)$, called the set of the virtual Brauer characters.

For any finite generated KG-module V, with its character χ_V, we may take reduction modulo p to get a corresponding FG-module $\overline{V}$. This process is not unique, but different reduction modulo p give the same Brauer character $\phi_{\overline{V}}$ (cf. [S, Theorem 32].) Hence the map between characters $\chi_V \mapsto \phi_{\overline{V}}$ is well-defined, and can be extended to a group homomorphism $d : R_K(G) \to R_F(G)$, with bases $\mathrm{irr}_K(G)$ and $\mathrm{irr}_F(G)$, respectively.

Since for any prime p we have $|\,\mathrm{irr}_F(G)| \leq |\,\mathrm{irr}_K(G)| < \infty$, we may write d into a matrix with respect to the bases and take its transpose, called the decomposition matrix of G. Each row of the matrix describes how an irreducible ordinary character decomposes into irreducible Brauer characters when passing from $R_K(G)$ to $R_F(G)$. For general group G, it is known that the map d is surjective [S, Theorem 33]. We know some finer properties of the map d for specific groups, like the family of symmetric

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groups or finite general linear groups.

For example, Table 1 of [J1] is the decomposition matrix of $G = \mathfrak{S}_6$, the symmetric group of degree 6, for $K = \overline{\mathbb{Q}}$ and $F = \overline{\mathbb{F}}_3$ ($p = 3$). The index of rows and columns are partitions, which serves as labels for the irreducible ordinary characters of $\mathfrak{S}_6$ over K and irreducible Brauer characters of $\mathfrak{S}_6$ over F respectively. The second row of the decomposition matrix means that the irreducible ordinary character $\chi_{(5,1)}$ maps to the Brauer character $\phi_{(6)} + \phi_{(5,1)}$.

	(6)	$(5,1)$	$(4,2)$	(3^2)	$(4,1^2)$	$(3,2,1)$	$(2^2,1^2)$
(6)	**1**						
$(5,1)$	1	**1**					
$(4,2)$			**1**				
(3^2)		1		**1**			
$(4,1^2)$		1			**1**		
$(3,2,1)$	1	1		1	1	**1**	
$(2^2,1^2)$							**1**
(2^3)	1					**1**	
$(3,1^3)$					1	**1**	
$(2,1^4)$				1		**1**	
(1^6)				**1**			

Table 1: Decomposition matrix of $\mathfrak{S}_6$, $p = 3$

Observe that the upper square of this decomposition matrix is a lower unitriangular submatrix. This can be deduced from James' Regularization Theorem for symmetric groups, see [J3, Theorem A]. With appropriate chosen order of labels, the decomposition matrix of the symmetric groups have the following properties:

- In each row, there exists an entry 1.
- In each row, the rightmost nonzero entry is **1**, written in bold.
- In each column, there exists a bold **1**.

These properties mainly come from the fact that the irreducible characters (and conjugacy classes) of the symmetric groups have a good way of labeling, via partitions.

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When considering other related groups, like alternating groups and finite general linear groups, some of these properties remain hold, while some others do not. In this thesis, we concern about which of these nice properties hold for finite special linear groups of non-defining characteristic.

Fix an finite group G, a prime p, $K = \overline{\mathbb{Q}}$, $F = \overline{\mathbb{F}}_p$, and the corresponding decomposition map d. For an ordinary character χ of G, write $\overline{\chi} := d(\chi)$. For a Brauer character ϕ of G, ϕ is said to be liftable if there exists some ordinary character χ of G satisfying $\overline{\chi} = \phi$, and ϕ is almost liftable if there exists some ordinary character χ of G satisfying $\overline{\chi} = a\phi$ for some $a \in \mathbb{N}$.

We say G has (R, p)-property, if property (R) (defined below) holds for G for prime p. We use this terminology throughout the thesis, for properties listed in the introduction.

(R) All irreducible Brauer characters of G are liftable.

(QR) All irreducible Brauer characters of G are almost liftable.

Clearly property (R) implies property (QR). We are interesting about the following problem, which originally comes from an exercise of Serre's (see Appendix, section A.1, for detail story.)

Problem 1. *Find a finite group G and a prime p, such that G has (QR, p)-property, but not (R, p)-property.*

This leads to the definition of the (L, p)-property.

(L) If G has (QR, p)-property, then G has (R, p)-property. That is, if every irreducible Brauer characters are almost liftable, then they are actually all liftable.

Note that if G does not have (QR, p)-property at all, then G automatically has (L, p)-property. Hence G is a solution to Problem 1 for some p, if and only if (L, p)-property fails for G. Therefore, to answer Problem 1, we move to the study of the (L, p)-property, starting from some of the common families of finite groups. Actually, (L, p)-property is a rather weak property, and often proven as a consequence of other stronger property. By considering each irreducible ordinary character, we may strengthen (L) to properties

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below.

(L') For any $\chi \in \mathrm{irr}_K(G)$, $\overline{\chi}$ is either irreducible, or a sum of at least two distinct Brauer characters. In other words, if $\overline{\chi} = a\phi$ for some $\phi \in \mathrm{irr}_F(G)$ and $a \in \mathbb{N}$, then $a = 1$.

(L'') For any $\chi \in \mathrm{irr}_K(G)$, $\overline{\chi}$ contains some $\phi \in \mathrm{irr}_F(G)$ of multiplicity 1.

(C) There exists a partial order $\trianglerighteq$ on $\mathrm{irr}_F(G)$, and a map $\mathrm{irr}_K(G) \to \mathrm{irr}_F(G)$, $\chi \mapsto \phi_\chi$, such that for each $\chi \in \mathrm{irr}_K(G)$, $\overline{\chi}$ contains ϕ_χ of multiplicity 1, and if $\overline{\chi}$ contains $\phi \in \mathrm{irr}_F(G)$, then $\phi \trianglerighteq \phi_\chi$.

It is clear that $(C) \implies (L'') \implies (L') \implies (L)$. For example, James' Regularization Theorem shows that the property (C) holds for symmetric groups for any prime p. Huang [H] proves that the property (L') holds for alternating groups for any prime p , while (L'') and (C) remains unknown.

We also have $(R) \implies (L)$. The Fong-Swan Theorem [S, Theorem 38] shows that for a prime p, property (R) holds for all p-solvable groups, thus these groups have (L,p)-property as well, while any non-abelian p-group is a counterexample of (L',p)-property.

There is a property (U), looks similar to (C), considering each irreducible Brauer character instead. With suitable order of the bases of the decomposition matrix, we may find a lower unitriangular submatrix.

(U) There exists a partial order $\trianglerighteq$ on $\mathrm{irr}_F(G)$, and a map $\mathrm{irr}_F(G) \to \mathrm{irr}_K(G)$, $\phi \mapsto \chi_\phi$, such that for each $\phi \in \mathrm{irr}_F(G)$, $\overline{\chi}_\phi$ contains ϕ of multiplicity 1, and if $\overline{\chi}_\phi$ contains $\phi' \in \mathrm{irr}_F(G)$, then $\phi' \trianglerighteq \phi$.

We have $(R) \implies (U)$, but (U) does not imply (L) (theoretically.) There are no other trivial implication among these properties (See Appendix, section A.2, for details.) Kleshchev-Tiep [K, Proposition 6.3] proves that (U) holds for both finite general and special linear groups in non-defining characteristic, but this is not enough to deduce property (L). Nevertheless, Kleshchev's paper gives strong tools and rich ideas for analyzing properties of decomposition matrix of finite special linear groups, so we may

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achieve our goal easily.

The main result of this thesis is to prove that the property (C) holds for finite special linear groups in non-defining characteristic (Theorem 6.8), and hence implies (L''), (L') and (L). In section 1.3, we start from the conjugacy classes of $GL_n(q)$, labeled as $[(\underline{\sigma}, \underline{\lambda})]$. Then in section 2, we introduce $L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$, modules of $GL_n(q)$ over field $\mathbf{F} = K$ or F, which build up a complete set of non-isomorphic irreducible $\mathbf{F}GL_n(q)$-modules. Next in section 3, we deduce some important lemmas from Clifford's Theorem. Finally we prove the main result of this thesis in section 6. The proof is in fact independent of Kleshchev-Tiep's theorem, the main theorem in [K], which is proved in section 5 for completeness, with the lemmas in part of section 3 and full of section 4. We make a table showing which property holds for which groups in Conclusion, section 7, and some other results in Appendix, section A.

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1 Preliminaries

Let $\mathbb{F}_q$ be the finite field with $q = p_0^f$ elements, and $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. Fixed a prime p not dividing q. In this thesis, K and F will always be a field of characteristic 0 and $p > 0$, respectively. If both K and F work for some result, we will put in the statement $\mathbf{F} = K$ or F.

1.1 Partitions

Given $k \in \mathbb{Z}^{\geq 0}$, a partition $\lambda \vdash k$ is a integer sequence $(\lambda_1, \lambda_2, \cdots)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $k = \sum_i \lambda_i$. For simplicity we may omit zeros and write in a compact form, e.g., $(4, 2^2, 1^3)$ instead of $(4, 2, 2, 1, 1, 1, 0, \cdots)$. Let $r_i \coloneqq \#\{j \in \mathbb{N} \mid \lambda_j = i\}$, then we write λ into $(1^{r_1}, 2^{r_2}, \cdots)$ for expression of r_i.

Let λ, $\mu \vdash n$ and $d \in \mathbb{N}$.

$\lvert\lambda\rvert$	means $\lambda_1 + \lambda_2 + \cdots$, that is, k;
$\lambda + \mu$	is $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \cdots)$.
$d\lambda$	is $(d\lambda_1, d\lambda_2, \cdots)$.
λ'	is the transpose of λ, that is, $\lambda'_i = \#\{j \in \mathbb{N} \mid \lambda_j \geq i\}$.
$\lambda \mathbin{[+]} \mu$	$= (\lambda' + \mu')'$, which combine and rearrange entries of λ and μ.
$[d]\lambda$	$= (d\lambda')'$, which combine and rearrange entries of d copies of λ.
$\lambda \trianglerighteq \mu$	is the dominance order, $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j$ for every i.
$\Delta(\lambda)$	is the greatest common divisor of λ_i.

Let $\underline{k}$ be an a-tuple $(k_1, \cdots, k_a)$, $k_i \in \mathbb{Z}^{\geq 0}$.

$\underline{\lambda} \vdash \underline{k}$	is the multipartition $(\lambda^{(1)}, \cdots, \lambda^{(a)})$, where $\lambda^{(i)} \vdash k_i$;
$\underline{\lambda}'$	is $(\lambda^{(1)\prime}, \cdots, \lambda^{(a)\prime})$;
$\underline{\lambda} \trianglerighteq \underline{\nu}$	if $\lambda^{(i)} \trianglerighteq \nu^{(i)}$ for all i;
$\Delta(\underline{\lambda})$	is the greatest common divisor of $\Delta(\lambda^{(i)})$.

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Lemma 1.1. *Let $\alpha^{(i)}, \beta^{(i)} \vdash n_i$ with $\alpha^{(i)} \trianglerighteq \beta^{(i)}$ for $i = 1, \cdots, m$.*

(1) $\alpha^{(1)} + \cdots + \alpha^{(m)} \trianglerighteq \beta^{(1)} + \cdots + \beta^{(m)}$.

(2) *If $\alpha^{(1)} + \cdots + \alpha^{(m)} = \beta^{(1)} + \cdots + \beta^{(m)}$ then $\alpha^{(i)} = \beta^{(i)}$ for all i.*

(3) $\alpha^{(1)} \, [+] \cdots [+] \, \alpha^{(m)} \trianglerighteq \beta^{(1)} \, [+] \cdots [+] \, \beta^{(m)}$.

(4) *If $\alpha^{(1)} \, [+] \cdots [+] \, \alpha^{(m)} = \beta^{(1)} \, [+] \cdots [+] \, \beta^{(m)}$ then $\alpha^{(i)} = \beta^{(i)}$ for all i.*

Proof. By definition of the dominance order, $\sum_{j=1}^{s} \alpha_j^{(i)} \geq \sum_{j=1}^{s} \beta_j^{(i)}$ $(*)$ for any $s \in \mathbb{N}$ and $i = 1, \cdots, m$. Thus $\sum_{j=1}^{s} \sum_{i=1}^{m} \alpha_j^{(i)} \geq \sum_{j=1}^{s} \sum_{i=1}^{m} \beta_j^{(i)}$ $(**)$ and (1) holds. For (2), since the inequality of $(**)$ is actually equality, then $(*)$ as well and (2) follows. To prove (3), we start from $\alpha^{(i)} \trianglerighteq \beta^{(i)}$ and $(\alpha^{(i)})' \trianglelefteq (\beta^{(i)})'$, then by (1) $\sum_{i=1}^{m} (\alpha^{(i)})' \trianglelefteq \sum_{i=1}^{m} (\beta^{(i)})'$ and hence $[+]_{i=1}^{m} \alpha^{(i)} \trianglerighteq [+]_{i=1}^{m} \beta^{(i)}$. A similar argument to (2) gives (4). □

1.2 Group Theory

Let G be any finite group, $g \in G$ an element.

1_G or e	is the identity element of G;
$\lvert G \rvert$	is the cardinality of G;
$\lvert g \rvert$	is the order of g, i.e. the smallest $m \in \mathbb{N}$ such that $g^m = 1_G$;
$H \leq G$	means H is a subgroup of G.
$H \trianglelefteq G$	means H is a normal subgroup of G.
$O_{p'}(G)$	$= \{g \in G \mid \lvert g \rvert$ is prime to $p\}$, the p'-part of G;
$O_p(G)$	$= \{g \in G \mid \lvert g \rvert$ is a p-power$\}$, the p-part of G;

Conventionally, when $\mathbf{F} = K$, set $O_{p'}(G) = G$ and $O_p(G) = \{1_G\}$.

GL_n	denotes $GL_n(\mathbb{F}_q)$ or $GL_n(q)$, if q is clear;
SL_n	denotes $SL_n(\mathbb{F}_q)$ or $SL_n(q)$;
R_n	satisfies $SL_n \leq R_n \leq GL_n$ and $R_n/SL_n = O_{p'}(GL_n/SL_n)$;
T_n	satisfies $SL_n \leq T_n \leq GL_n$ and $T_n/SL_n = O_p(GL_n/SL_n)$;

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$\mathfrak{S}_n$ denotes the symmetric group of degree n;

$\mathfrak{A}_n$ denotes the alternating group of degree n.

Definition 1.2.

(1) A element $g \in G$ is a p'-element if $g \in O_{p'}(G)$, and is a p-element if $g \in O_p(G)$. If $\mathbf{F} = K$, then every g is a p'-element.

(2) A element $g \in G$ is p-regular exactly if it is a p'-element, and is p-singular if it is not p-regular.

Proposition 1.3. *For any $g \in G$, there is some p'-element $g' \in G$ and p-element $g_p \in G$, such that $g = g'g_p$. This decomposition is unique, both g' and g_p is a power of g, and $g'g_p = g_pg'$.*

Proof. Let $|g| = p^c m$ with $p \not| m$. Then there is some $a, b \in \mathbb{Z}$ such that $ap^c + bm = 1$. Take $g' = g^{ap^c}$ and $g_p = g^{bm}$. Since the only element which is a p-element and a p'-element is 1_G, the uniqueness follows. The other statements are clear. □

Definition 1.4.

(1) Given $g \in G$ finite group and a prime p, the p'-part and p-part of g are g', g_p in the previous proposition, denoted by $(g)_{p'}$ and $(g)_p$, respectively. If $\mathbf{F} = K$, set $(g)_{p'} = g$ and $(g)_p = 1_G$.

(2) Given $r \in \mathbb{N}$ and a prime p, write $r = p^c m$ for non-negative integer c, m, p not dividing m. Then the p'-factor and p-factor of r are m, p^c, denoted by $|r|_{p'}$ and $|r|_p$, respectively. If $\mathbf{F} = K$, set $|r|_{p'} = r$ and $|r|_p = 1$. Do not confuse with p-adic norm, which does not appear in this thesis.

We emphasize that $\mathbb{F}_q^\times$ is the multiplication group of $\mathbb{F}_q$, and we usually apply Definition 1.2, 1.4 and Proposition 1.3 to the elements of $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^d}^\times$.

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1.3 Conjugacy Classes in $GL_n(q)$

Given $\sigma \in \overline{\mathbb{F}}_q^\times$, $\deg(\sigma) = d$, let $B = B(\sigma) \in GL_d(q)$ be the companion matrix of minimal polynomial of σ over $\mathbb{F}_q$. Then the corresponding Jordan block of size r is of the form

$$J_\sigma(r) = \begin{bmatrix} \sigma & 1 & & & \\ & \sigma & 1 & & \\ & & \sigma & & \\ & & & \ddots & 1 \\ & & & & \sigma \end{bmatrix} \quad J_B(r) = \begin{bmatrix} B & I & & & \\ & B & I & & \\ & & B & & \\ & & & \ddots & I \\ & & & & B \end{bmatrix}$$

where $J_\sigma(r) \in GL_r(q^d)$ and $J_B(r) \in GL_{rd}(q)$. With partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ given, we let $J_\sigma(\lambda) = \mathrm{diag}(J_\sigma(\lambda_1), \cdots, J_\sigma(\lambda_m))$ and $J_B(\lambda)$ similarly.

For $g \in GL_n(q)$, write the characteristic polynomial of g as $f_g = f_1^{k_1} \cdots f_a^{k_a}$ for some monic irreducible f_i. Then the Jordan canonical form of g over $\overline{\mathbb{F}}_q^\times$ is

$$J_g = \mathrm{diag}(J_{\sigma_{1,1}}(\lambda^{(1)}), \cdots, J_{\sigma_{1,d_1}}(\lambda^{(1)}), \cdots, J_{\sigma_{a,1}}(\lambda^{(a)}), \cdots, J_{\sigma_{a,d_a}}(\lambda^{(a)}))$$

where $\sigma_{i,j_i} \in \overline{\mathbb{F}}_q^\times$ are roots of f_i for each $j_i = 1, \cdots, d_i$ with $d_i = \deg(f_i)$, and each $\lambda^{(i)} \vdash k_i$. And the rational canonical form of g over $\mathbb{F}_q$ is

$$R_g = \mathrm{diag}(J_{B_1}(\lambda^{(1)}), \cdots, J_{B_a}(\lambda^{(a)}))$$

where B_i the companion matrix of f_i, $\lambda^{(i)} \vdash k_i$.

Definition 1.5. Let $\mathcal{M}_n(\mathbb{F}_q)$ be the set of all $n \times n$ matrices over $\mathbb{F}_q$.

(1) $s \in \mathcal{M}_n(\mathbb{F}_q)$ is semisimple if it has an eigenbasis in $(\overline{\mathbb{F}}_q)^n$. Equivalently, there is an $x \in GL_n(\overline{\mathbb{F}}_q)$ such that xsx^{-1} is a diagonal matrix.

(2) $u \in \mathcal{M}_n(\mathbb{F}_q)$ is unipotent if $(u-1)^m = O$ for some $m \in \mathbb{N}$. Equivalently, there is an $x \in GL_n(\overline{\mathbb{F}}_q)$ such that xux^{-1} is an upper unitrianglar matrix.

By Jordan decomposition [Sp1], every $g \in GL_n(q)$ has a unique decomposition su, where $s \in GL_n(q)$ is semisimple and $u \in GL_n(q)$ is unipotent.

We need to construct an appropriate complete set of representatives for conjugacy classes in $GL_n(q)$.

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Given $\sigma \in \overline{\mathbb{F}}_q^{\times}$, let $[\sigma]$ be the set of all roots of the minimal polynomial of σ. Then σ_1 and σ_2 are (Galois) conjugate if and only if $[\sigma_1] = [\sigma_2]$.

Definition 1.6. Let $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$, $\underline{\tau} = (\tau_1, \cdots, \tau_a) \in (\overline{\mathbb{F}}_q^{\times})^a$. In the following, for all i means for each $i = 1, \cdots, a$.

(1) $\underline{\sigma}$ is p-regular if every σ_i is p-regular as a group element of $\mathbb{F}_{q^{d_i}}^{\times}$.

(2) $\underline{\sigma}$ is non-repeated if for all i, $[\sigma_i]$ are all different.

(3) $\underline{\sigma}$ is p-non-repeated if for all i, $[(\sigma_i)_{p'}]$ are all different.

(4) $\underline{\sigma}$ and $\underline{\tau}$ are (Galois) conjugate if $\deg(\sigma_i) = \deg(\tau_i)$ and $[\sigma_i] = [\tau_i]$ for all i.

(5) $\underline{\sigma}$ and $\underline{\tau}$ are p-conjugate if $\deg(\sigma_i) = \deg(\tau_i)$ and $[(\sigma_i)_{p'}] = [(\tau_i)_{p'}]$ for all i.

Given $\sigma \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = d$ over $\mathbb{F}_q$, then $\{1, \sigma, \cdots, \sigma_{d-1}\}$ is a $\mathbb{F}_q$-basis of $\mathbb{F}_q(\sigma) = \mathbb{F}_{q^d}$, which produces a algebra embedding $\phi^{\sigma} : \mathbb{F}_{q^d} \to \mathcal{M}_d(\mathbb{F}_q)$ by $\phi^{\sigma}(\sigma) = B(\sigma)$, then restricts to a group embedding $\iota^{\sigma} : \mathbb{F}_{q^d}^{\times} \to GL_d(q)$. Similarly, this produces a matrix algebra embedding $\phi_k^{\sigma} : \mathcal{M}_k(\mathbb{F}_{q^d}) \to \mathcal{M}_{kd}(\mathbb{F}_q)$ by $\phi_k^{\sigma}(\sigma E_{ij}) = B(\sigma) \otimes E_{ij}$, where $\otimes$ is the Kronecer product of matrices, and E_{ij} is the $k \times k$ matrix with (i, j)-entry 1 and other entries 0, and then restricts to a group embedding $\iota_k^{\sigma} : GL_k(q^d) \to GL_{kd}(q)$. Note that $\iota^{\sigma}(\sigma) = B(\sigma)$, $\iota_k^{\sigma}(J_{\sigma}(k)) = J_B(k)$ and $\iota_k^{\sigma}(J_1(k)) = J_{I_d}(k)$.

Let $\underline{k} = (k_1, \cdots, k_a) \in \mathbb{N}^a$ and $\underline{\lambda} \vdash \underline{k}$. Then $\pi \in \mathfrak{S}_a$ acts naturally on each a-tuple, such as $\underline{\sigma}$, $\underline{k}$ and $\underline{\lambda}$. Write the action on the right.

Definition 1.7. Let $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a) \in (\overline{\mathbb{F}}_q^{\times})^a$, with $\deg(\sigma_i) = d_i$ for each $i = 1, \cdots, a$. Let $\underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(a)}) \vdash \underline{k} = (k_1, \cdots, k_a) \in \mathbb{N}^a$.

(1) An n-admissible pair with a pairs is of the form $(\underline{\sigma}, \underline{\lambda})$, where both $\underline{\sigma}$ and $\underline{\lambda}$ are a-tuple for some $a \in \mathbb{N}$, $\underline{\sigma}$ is non-repeated, and $n = \sum_{i=1}^{a} k_i d_i$.

(2) We say $(\underline{\sigma}, \underline{\lambda})$ and $(\underline{\tau}, \underline{\nu})$ are equivalent if there exists some $\pi \in \mathfrak{S}_a$ such that $\underline{\sigma}$ and $\underline{\tau}\pi$ are conjugate, and $\underline{\lambda} = \underline{\nu}\pi$. The equivalence class $[(\underline{\sigma}, \underline{\lambda})]$ is called an n-admissible symbol.

doi:10.6342/NTU201904116

(3) We may write pairwisely the n-admissible pair and symbol with $\circ$ product,

$$(\underline{\sigma}, \underline{\lambda}) = (\sigma_1, \lambda^{(1)}) \circ \cdots \circ (\sigma_a, \lambda^{(a)})$$
$$[(\underline{\sigma}, \underline{\lambda})] = \left[([\sigma_1], \lambda^{(1)}) \circ \cdots \circ ([\sigma_a], \lambda^{(a)})\right]$$

(4) The dominance order of (multi)-partition naturally induces the partial order on n-admissible symbol. Denote $[(\underline{\sigma}, \underline{\lambda})] \trianglerighteq [(\underline{\tau}, \underline{\nu})]$ if there exists some $\pi \in \mathfrak{S}_a$ such that $\underline{\sigma}$ and $\underline{\tau}\pi$ are conjugate and $\underline{\lambda} \trianglerighteq \underline{\nu}\pi$.

(5) If $(\underline{\sigma}, \underline{\lambda})$ is an n-admissible pair, then we associate an element $g := su \in GL_n(q)$, where s is semisimple and u is unipotent,

$$\begin{aligned} s = s(\underline{\sigma}, \underline{k}) &:= \operatorname{diag}\left((\iota^{\sigma_1}(\sigma_1))^{(k_1)}, \cdots, (\iota^{\sigma_a}(\sigma_a))^{(k_a)}\right) \\ &= \operatorname{diag}\left(B_1^{(k_1)}, \cdots, B_a^{(k_a)}\right) \\ u = u(\underline{\lambda}, \underline{k}) &:= \operatorname{diag}\left(\iota_{k_1}^{\sigma_1}\left(J_1(\lambda^{(1)})\right), \cdots, \iota_{k_a}^{\sigma_a}\left(J_1(\lambda^{(a)})\right)\right) \\ &= \operatorname{diag}\left(J_{I_{d_1}}(\lambda^{(1)}), \cdots, J_{I_{d_a}}(\lambda^{(a)})\right) \end{aligned}$$

where B_i is the companion matrix of the minimal polynomial of σ_i over $\mathbb{F}_q$.

Note that $\operatorname{diag}(B^{(k)})$ means k copies of B on diagonal, not k power of B.

Then it is not hard to see that

Proposition 1.8.

(1) *$(\underline{\sigma}, \underline{\lambda})$ and $(\underline{\tau}, \underline{\nu})$ are equivalent if and only if their corresponding su are conjugate to each other.*

(2) *The set $\Sigma_K := \{\, [(\underline{\sigma}, \underline{\lambda})] \mid (\underline{\sigma}, \underline{\lambda})$ is an n-admissible pair $\}$ is the complete set of representatives of conjugacy classes of $GL_n(q)$.*

(3) *The set $\Sigma_F := \{\, [(\underline{\sigma}, \underline{\lambda})] \mid (\underline{\sigma}, \underline{\lambda})$ is an n-admissible pair, $\underline{\sigma}$ is p-regular $\}$ is the complete set of representatives of p-regular conjugacy classes of $GL_n(q)$.*

Proof. (1) Note that π means changing the order of blocks of s and u, and if σ_1 is conjugate to τ_1 over $\mathbb{F}_q$, they produce the same companion matrix of their common minimal polynomial.

doi:10.6342/NTU201904116

(2) We show that every $g \in GL_n(q)$ is conjugate to some $s(\underline{\sigma}, \underline{k})u(\underline{\lambda}, \underline{k})$. The rational canonical form of g is

$$R_g = \mathrm{diag}(J_{B_1}(\lambda^{(1)}), \cdots, J_{B_a}(\lambda^{(a)}))$$

so it suffices to prove the case $R_g = J_B(k)$ for some $d \times d$ matirx $B = B(\sigma)$ and $k \in \mathbb{N}$. Let $S = \mathrm{diag}(B^{(k)})$, $U = J_{I_d}(k)$, $N = U - I_d$, then $R_g = S + N$. Now take $D = \mathrm{diag}(I_d, B, \cdots, B^{k-1})$, then we have $D^{-1}SD = S$ and $D^{-1}ND = SN$. Hence $D^{-1}R_gD = S + SN = SU$ and we are done.

(3) Since u is always p-regular (p does not divide q), su is p-regular if and only if s is p-regular, which is equivalent to $\underline{\sigma}$ is p-regular. □

Some other properties of the n-admissible symbols are put in chapter 4.

Given $su = s(\underline{\sigma}, \underline{k})u(\underline{\lambda}, \underline{k})$, if $g \in GL_n(q)$ centralize su, then it centralize both s and u by the uniqueness of the decomposition. That is, the centralizer

$$C_{GL_n(q)}(su) = C_{GL_n(q)}(s) \cap C_{GL_n(q)}(u)$$

The centralizer of s is,

$$C_{GL_n(q)}(s) = (\iota_{k_1}^{\sigma_1} \times \cdots \times \iota_{k_1}^{\sigma_1}) \left(GL_{k_1}(q^{d_1}) \times \cdots \times GL_{k_a}(q^{d_a})\right)$$

The centralizer of u is more complicated. Nevertheless, by [Sp2, I, 2.2], the size of the centralizer of u is,

$$\left|C_{GL_n(q)}(u)\right| = q^N \prod_{i \geq 1} |GL_{r_i}(q)| \tag{1}$$

where $N = N(\underline{\lambda})$ is defined as follows. Write $\lambda = [+]_{i=1}^{a}[d_i]\lambda^{(i)}$ into $(1^{r_1}, 2^{r_2}, \cdots)$. Then $N = \sum_{i \geq 1}(\lambda_i')^2 - r_i^2$. Note that $\lambda_i' = \sum_{j \geq i} r_j$, hence $N \geq 0$.

1.4 Size of Conjugacy Classes of R_n

Recall that R_n is a subgroup of GL_n satisfying $SL_n \leq R_n \leq GL_n$ and $R_n/SL_n = O_{p'}(GL_n/SL_n)$. In the later proof, we need to find the size of conjugacy classes of R_n.

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For $g \in G$, let $\operatorname{Conj}_G(g)$ be the conjugacy class of g in G. The following is the general lemma we use.

Lemma 1.9. *Let $S \trianglelefteq G$, G/S cyclic, and $S \leq R \leq G$. For any $g \in R$, let $c = (G : C_G(g)S)$, $d = (G : R)$. Then $|\operatorname{Conj}_G(g)|/|\operatorname{Conj}_R(g)| = \gcd(c, d)$.*

Proof. Denote $C = C_G(g)$ and $D = C_R(g) = C \cap R$. Then

$$|\operatorname{Conj}_G(g)|/|\operatorname{Conj}_R(g)| = (G : C)/(R : D)$$

The key step is to drag C, D to CS, DS, where we can count their index in G. Now $C \cap S = C \cap R \cap S = D \cap S$, thus $|DS|/|CS| = |D|/|C|$. Consider $\pi : G \to G/S = \langle x \rangle$, then $\pi(CS) = \langle x^c \rangle$, $\pi(R) = \langle x^d \rangle$, and $\pi(CS \cap R) = \langle x^{\operatorname{lcm}(c,d)} \rangle$. Since $CS \cap R = DS$, we have $(G : DS) = \operatorname{lcm}(c, d)$. Therefore

$$\frac{|\operatorname{Conj}_G(g)|}{|\operatorname{Conj}_R(g)|} = \frac{|G|}{|C|}\frac{|D|}{|R|} = \frac{|G|}{|CS|}\frac{|DS|}{|G|}\frac{|G|}{|R|} = \frac{c \cdot d}{\operatorname{lcm}(c, d)} = \gcd(c, d) \qquad \square$$

Lemma 1.10. *Let $u = J_1(\lambda)$ be a Jordan block of $GL_k(q)$, $\lambda \vdash k$. Then* det *maps $C_{GL_k(q)}(u)$ onto $\langle \varepsilon^{\Delta(\lambda)} \rangle \leq \mathbb{F}_q^\times$, where ε is a generator of $\mathbb{F}_q^\times$.*

Proof. Denote $C = C_{GL_n(q)}(u)$ and write λ into $(1^{r_1}, 2^{r_2}, \cdots)$. Then u is similar to $\bigoplus_{i:r_i>0} I_{r_i} \otimes J_1(i)$, hence $D = \bigoplus_{i:r_i>0} GL_{r_i}(q) \otimes I_i$ commutes with u and is a subgroup of C. Consider P_0 a Sylow p_0-subgroup of C. By equation (1), $(C : D) = q^N$, thus we have $C = P_0 D$. Now for any element $a \in P_0$, $a^{q^r} = I_n$ for some $r \in \mathbb{N}$, while $\det^q(a) = \det(a)$ since it's an element of $\mathbb{F}_q^\times$. Hence $\det(P_0) = \{1\}$, $\det(C) = \det(D)$, and $\det(GL_{r_i}(q) \otimes I_i) = \langle \varepsilon^i \rangle$ gives $\det(D) = \langle \varepsilon^{\Delta(\lambda)} \rangle$, as desired. $\square$

Proposition 1.11. *Let $g = su = s(\underline{\sigma}, \underline{k})u(\underline{\lambda}, \underline{k})$ be a representative of a p-regular conjugacy class in $GL_n(q)$ corresponding to $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. Then $g \in R_n$ and*

$$\frac{|\operatorname{Conj}_{GL_n}(g)|}{|\operatorname{Conj}_{R_n}(g)|} = \gcd\{(GL_n : R_n), \Delta(\underline{\lambda})\}$$

Proof. Let $\pi' : GL_n \to GL_n/R_n$. Note that GL_n/R_n is a p-power, thus $\pi'(g)$ must be identity, hence $g \in R_n$.

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Denote $C = C_{GL_n(q)}(g)$. To apply Lemma 1.9, we need to find the index $c = (GL_n : C \cdot SL_n)$, which leads to finding $\det(C \cdot SL_n) = \det(C)$. Then

$$\begin{aligned}
C &= C_{GL_n(q)}(su) = C_{GL_n(q)}(s) \cap C_{GL_n(q)}(u) = C_{C_{GL_n(q)}(s)}(u) \\
&= \prod_{i=1}^{a} C_{\iota_{k_i}^{\sigma_i}(GL_{k_1}(q^{d_1}))}(\iota_{k_i}^{\sigma_i}(J_1(\lambda^{(i)}))) = \prod_{i=1}^{a} \iota_{k_i}^{\sigma_i}\left(C_{GL_{k_1}(q^{d_1})} J_1(\lambda^{(i)})\right)
\end{aligned}$$

Now for k, $d \in \mathbb{N}$, $\sigma \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = d$, consider the following commute diagram

$$\begin{array}{ccc}
GL_k(q^d) & \xrightarrow{\iota_k^{\sigma}} & GL_{kd}(q) \\
\big\downarrow{\scriptstyle \det} & & \big\downarrow{\scriptstyle \det} \\
\mathbb{F}_{q^d}^{\times} & \xrightarrow{N_{\mathbb{F}_{q^d}/\mathbb{F}_q}} & \mathbb{F}_q^{\times}
\end{array}$$

For each d, take a generator $\varepsilon_d \in \mathbb{F}_{q^d}^{\times}$ such that $\varepsilon = N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\varepsilon_d)$. Then

$$\begin{aligned}
\det(C) &= \prod_{i=1}^{a} \det \circ\, \iota_{k_i}^{\sigma_i}\left(C_{GL_{k_1}(q^{d_1})} J_1(\lambda^{(i)})\right) \\
&= \prod_{i=1}^{a} N_{\mathbb{F}_{q^{d_i}}/\mathbb{F}_q} \circ \det\left(C_{GL_{k_1}(q^{d_1})} J_1(\lambda^{(i)})\right) \\
&= \prod_{i=1}^{a} N_{\mathbb{F}_{q^{d_i}}/\mathbb{F}_q}\left(\langle \varepsilon_d^{\Delta(\lambda^{(i)})} \rangle\right) = \prod_{i=1}^{a} \langle \varepsilon^{\Delta(\lambda^{(i)})} \rangle = \langle \varepsilon^{\Delta(\underline{\lambda})} \rangle
\end{aligned}$$

Hence $c = \Delta(\underline{\lambda})$. Applying Lemma 1.9 with $d = (GL_n : R_n)$ yields the result. □

1.5 Representations and Modules

Let G be a finite group, R a commutative ring with 1, and V an R-module. We call (V, ρ) a representation of G over R if $\rho : G \to GL(V)$ is a group homomorphism, where $GL(V)$ is the group of R-module automorphisms of V. In this thesis R will always be a field $\mathbf{F}$, and V is a finite vector space over $\mathbf{F}$.

Given a representation $\rho : G \to GL(V)$, it can be extended to an $\mathbf{F}G$-module, also denoted V. In contrast, given an $\mathbf{F}G$-module V, it defines a representation ρ of G in V over $\mathbf{F}$. We will also call V a representation, although it is the underlying module of ρ.

Definition 1.12. Let V be an $\mathbf{F}G$-module.

doi:10.6342/NTU201904116

(1) An $\mathbf{F}G$-module W is a submodule (subrepresentation) of V, denoted $W \subset V$, if $W \subset V$ as vector space over $\mathbf{F}$, and stable under the action of G.

(2) A submodule W of V is a (direct) summand of V, denoted $W \mid V$, if there is another submodule W' of V such that $V = W \oplus W'$ as vector space.

(3) V is an irreducible representation, or a simple $\mathbf{F}G$-module, if $\{0\}$ and V are the only submodules of V.

(4) Denote $\operatorname{Irr}_{\mathbf{F}}(G)$ to be the set of all isomorphic types of non-isomorphic irreducible representations of G over $\mathbf{F}$.

(5) V is a trivial representation if $V \cong \mathbf{F}$ and ρ acts trivially, denoted id_G.

(6) V is semi-simple or complete reducible, if every submodule of V is a summand of V. Hence $V = \bigoplus W_i$ for some irreducible submodules W_i of V.

(7) The dual of V, denoted V^*, is the $\mathbf{F}$-module $\operatorname{Hom}_{\mathbf{F}}(V, \mathbf{F})$, equipped with the action of G, $(gf)(v) = f(g^{-1}v)$, becoming an $\mathbf{F}G$-module. It is known that $\operatorname{Hom}_{\mathbf{F}}(V_1, V_2) \cong V_1^* \otimes V_2$.

Let $H \leq G$. We may construct some $\mathbf{F}H$-module from an $\mathbf{F}G$-module, or vise versa.

Definition 1.13. Let V be an $\mathbf{F}G$-module, and W be an $\mathbf{F}H$-module.

(1) The restriction of V from G to H, denoted as $\operatorname{Res}^G_H(V)$ or $V{\downarrow}^G_H$, simply restrict the action of $\mathbf{F}G$ to $\mathbf{F}H$. If G is clear, we may also write $V{\downarrow}_H$ for simplicity.

(2) The induction of W from H to G, denoted as $\operatorname{Ind}^G_H(W)$ or $W{\uparrow}^G_H$, is defined to be $\mathbf{F}G \otimes_{\mathbf{F}H} W$. If H is clear, we may also write $W{\uparrow}^G$ for simplicity.

(3) Assume $\mathbf{F} = K$ or $\mathbf{F} = F$ with $p \nmid |G|$. For $\mathbf{F}G$-modules V_1, V_2, define $\langle V_1, V_2 \rangle_G = \dim_{\mathbf{F}} \operatorname{Hom}_{\mathbf{F}G}(V_1, V_2)$. Similarly for $\langle W_1, W_2 \rangle_H$.

(4) For $g \in G$, let ${}^gH \coloneqq gHg^{-1}$. Then ${}^gW \coloneqq g \otimes W$ is naturally an $\mathbf{F}({}^gH)$-module.

(5) The kernel of V, denoted $\ker V$, is the unique maximal subgroup $H \leq G$ such that H acts trivially on the $\mathbf{F}H$-module $V{\downarrow}_H$. It is known that $\ker V \trianglelefteq G$.

The basic properties of restriction and induction is on, for example, [F, II], so we omit them here. We only list some important theorems here.

doi:10.6342/NTU201904116

Theorem 1.14. *Let V be an $\mathbf{F}G$-module, and W be an $\mathbf{F}H$-module.*

(1) *(Frobenius Reciprocity) Assume $\mathbf{F} = K$ or $\mathbf{F} = F$ with $p \nmid |G|$.*

$$\langle V, W{\uparrow}^G \rangle_G = \langle V{\downarrow}_H, W \rangle_H$$

$$\langle W{\uparrow}^G, V \rangle_G = \langle W, V{\downarrow}_H \rangle_H$$

(2) *(Mackey Decomposition) Let $A \leq G$. Then*

$$(W{\uparrow}_H^G){\downarrow}_A^G \cong \bigoplus_x \left({}^x(W{\downarrow}_{H \cap A^x}^G) \right) {\uparrow}_{{}^xH \cap A}^A$$

summing over the complete set of double coset representative $x \in [A \backslash G/H]$.

Let $S \trianglelefteq G$. We may construct some $\mathbf{F}(G/S)$-module from an $\mathbf{F}G$-module, or vise versa.

Definition 1.15. Let V be a $\mathbf{F}G$-module, and W be a $\mathbf{F}(G/S)$-module.

(1) The S-fixed point of V, denoted as V^S, is the abelian group $\{v \in V \mid sv = v \text{ for all } s \in S\}$ equipped with the action of G, which can be viewed as an $\mathbf{F}(G/S)$-module.

(2) The inflation of W, denoted as $\operatorname{infl}_{G/S}^G(W)$, has the same underlying space W, equipped with the action $g \cdot w = \pi(g)w$ for any $g \in G$, $w \in W$ and the canonial homomorphism $\pi : G \to G/S$.

Their basic properties is on, for example, [GR, Chap 4]. It is known that the fixed point construction is adjoint to inflation, so they has an analogue to the Frobenius reciprocity. We list the following relations here for later usage.

Proposition 1.16. *Let $S \trianglelefteq G$, A any other subgroup of G.*

(1) *(Restriction commutes with inflation) Let V be an $\mathbf{F}(G/S)$-module. Then*

$$\operatorname{infl}_{A/A \cap S}^A (V{\downarrow}_{AS/S}^{G/S}) \cong (\operatorname{infl}_{G/S}^G(V)){\downarrow}_A^G$$

Note that we identify $A/A \cap S \cong AS/S$.

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(2) *(Induction commutes with inflation) Assume additionally $S \trianglelefteq A$. Let W be a* $\mathbf{F}(A/S)$*-module. Then*

$$\mathrm{infl}_{G/S}^{G}(W\uparrow_{A/S}^{G/S}) \cong (\mathrm{infl}_{A/S}^{A}(W))\uparrow_{A}^{G}$$

1.6 Characters

Given $K \subset \overline{\mathbb{Q}}$, let $\mathcal{O}_K$ be the ring of integer of K. Pick any prime p and any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ containing p, there is a corresponding p-adic valuation $\nu_{\mathfrak{p}}$. Take the ring $A = \{\alpha \in K \mid \nu_{\mathfrak{p}}(\alpha) \geq 0\}$, which has a unique maximal ideal $\mathfrak{m} = \{\alpha \in K \mid \nu_{\mathfrak{p}}(\alpha) > 0\}$. The residue field $F = A/\mathfrak{m}$ is of characteristic p. If $K = \overline{\mathbb{Q}}$ is algebraically closed, then $F = \overline{\mathbb{F}}_p$ is also algebraically closed.

Definition 1.17. The triple (K, A, F) is called a p-modular system.

Let G be a finite group, $\rho : G \to GL(V)$ an representation over K. Then the (ordinary) character $\chi_V : G \to K$ of G corresponding to V, is defined to be $\chi_V(g) = \mathrm{Tr}(\rho(g))$, with $\rho(g) : v \mapsto \rho(g)v$ written as an invertible matrix with a chosen basis of V. It is clear that the definition of character does not depend on the basis, by the property of the trace.

The properties of characters can be founded in the textbook of Serre [S].

Definition 1.18.

(1) We say χ_V is irreducible if V is.

(2) Write $\mathrm{irr}_K(G)$ the set of all irreducible ordinary characters.

(3) A class function $f : G \to K$, is a function satisfied $f(xgx^{-1}) = f(g)$ for any $g, x \in G$. By the property of the trace, it is clear that characters are class functions.

(4) For two characters χ, ϕ of G, define $\langle \chi, \phi \rangle = |G|^{-1} \sum_{g \in G} \chi(g)\phi(g^{-1})$.

(5) Let $R_K^+(G)$ be the set of all characters of G over K, and $R_K(G)$ be the additive group generated by the characters of G over K. The elements of $R_K(G)$ are called the virtual characters.

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Proposition 1.19. *Let $K = \overline{\mathbb{Q}}$.*

(1) *There are only finite irreducible characters of G, written as $\chi_1, \cdots, \chi_h$.*

(2) *$R_K(G)$ is a $\mathbb{Z}$-module with basis $\{\chi_1, \cdots, \chi_h\}$, and χ_i are mutually orthogonal with the inner product $\langle \cdot, \cdot \rangle$*

(3) *Every class functions are virtual characters, hence the set of all class functions coincides $R_K(G)$. By considering that any class function is constant on a conjugacy class of G, the number of irreducible characters of G is exactly the same as the number of conjugacy classes of G.*

(4) *V and W are isomorphic representations of G if and only if $\chi_V = \chi_W$.*

(5) *If $V = \bigoplus_{i=1}^{h} n_i W_i$ for $W_i \in \mathrm{Irr}_K(G)$, n_i means W_i appears n_i times. Then $\chi_V = \sum_{i=1}^{h} n_i \chi_i$, where χ_i are characters corresponding to W_i.*

In general, if K is a field of characteristic 0 with algebraic closure $\overline{K}$, one can view any KG-module V as a $\overline{K}G$-module $V_{\overline{K}}$ by scalar extension, then define $\chi_V = \chi_{V_{\overline{K}}}$. However, some irreducible $\overline{K}G$-module cannot be realized over K, hence $R_K(G)$ may be a proper subgroup of $R_{\overline{K}}(G)$. To ensure $R_K(G) = R_{\overline{K}}(G)$, a sufficient condition is that K contains the mth root of unity [S, Theorem 24], where $m = \mathrm{lcm}\{|g| \mid g \in G\}$. In this case we say K is sufficiently large for G. In other words, $R_K(G)$ is independent of K as long as K is sufficiently large for G, and we may replace $K = \overline{\mathbb{Q}}$ by any K sufficiently large for G in Proposition 1.19.

Now consider $G_{reg} \coloneqq \{g \in G \mid g \text{ is } p\text{-regular}\}$. Let (K, A, F) be a p-modular system, with K, F sufficiently large for G_{reg}, and $F = A/\mathfrak{m}$. Pick ζ a m'th root of unity of F, where $m' = \mathrm{lcm}\{|g| \mid g \in G_{reg}\}$, and $\tilde{\zeta}$ a m'th root of unity of K which passing from K to F is ζ.

Let V be an FG-module of dimension n. For $g \in G_{reg}$, let $\rho(g) : v \mapsto gv$ for $v \in V$. Then $\rho(g)$ is diagonalizable, and its eigenvalues $t_1, \cdots, t_n$ are all powers of ζ, hence $\tilde{t}_1, \cdots, \tilde{t}_n$ are corresponding powers of $\tilde{\zeta}$. Define $\phi_V(g) = \sum_{i=1}^{n} \tilde{t}_i$. Then $\phi_V : G_{reg} \to K$ is called the Brauer character of V.

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The properties of Brauer characters are also in [S]. It shares many properties with ordinary characters, but without orthogonality.

Definition 1.20.

(1) We say ϕ_V is irreducible if V is.

(2) Write $\mathrm{irr}_F(G)$ the set of all irreducible ordinary characters.

(3) Let $R_F^+(G)$ be the set of all Brauer characters of G over F, and $R_F(G)$ be the additive group generated by the Brauer characters of G over F. The elements of $R_F(G)$ are called the virtual Brauer characters.

Proposition 1.21. *Let $F = \overline{\mathbb{F}}_p$.*

(1) *If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of FG-modules, then $\phi_V = \phi_{V_1} + \phi_{V_2}$.*

(2) *$R_F(G)$ is a $\mathbb{Z}$-module with basis $\{\phi_1, \cdots, \phi_{h'}\}$.*

(3) *The number of irreducible Brauer characters of G is exactly the same as the number of conjugacy classes of G_{reg}. Hence the number of irreducible Brauer characters is equal to or less than the number of irreducible ordinary characters.*

(4) *V and W have the same composition factors if and only if $\phi_V = \phi_W$.*

(5) *If for each $W_i \in \mathrm{Irr}_F(G)$, V has n_i composition factors isomorphic to W_i, then $\phi_V = \sum_{i=1}^{h} n_i \phi_i$, where ϕ_i are Brauer characters corresponding to W_i.*

Note that in the case of characteristic 0, every KG-module is complete reducible. Hence two modules have common composition factors are actually isomorphic.

We may say F is sufficiently large for G with the same definition to K, replacing $F = \overline{\mathbb{F}}_p$ by any F sufficiently large for G in Proposition 1.21.

Given any ring R, let $\mathcal{C}$ be a category of R-modules. Then the Grothendieck group of $\mathcal{C}$ is the abelian group defined by the generators $[V]$ for any $V \in \mathcal{C}$, and relations $[V] = [V_1] + [V_2]$ if $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence.

Then for any field $\mathbf{F} = K$ or F sufficiently large over G, the Grothendieck group of $\mathbf{F}G$-modules has a canonical group isomorphism to $R_{\mathbf{F}}(G)$ by $[V] \mapsto \chi_V$ or ϕ_V.

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Hereafter, we identify $R_{\mathbf{F}}(G)$ with the Grothendieck group of $\mathbf{F}G$-modules, and omit the bracket. That is, when we say $V = V_1 + V_2$ in the Grothendieck group of $\mathbf{F}G$-modules, we actually means $[V] = [V_1] + [V_2]$, hence $\chi_V = \chi_{V_1} + \chi_{V_2}$ or $\phi_V = \phi_{V_1} + \phi_{V_2}$. We use this terminology because the name of the representation is often quite long (e.g. $L_K(\underline{\sigma}, \underline{\lambda})$), which is not suitable to write in character form.

1.7 The Decomposition Matrix

Let $K \subset \overline{\mathbb{Q}}$ be a field of characteristic 0, sufficiently large with respect to G, and $F = A/\mathfrak{m}$ be the field of characteristic p defined in the first paragraph of section 1.6, so (K, A, F) forms a p-modular system.

For a KG-module V, pick a lattice V_1, a finite generated A-submodule of V, generating V as a K-module. Let V_2 be the sum of the image of V_1 under elements of G, hence V_2 is also a lattice of V which is stable under G. Define $\overline{V} = V_2/\mathfrak{m}V_2$. Then $\overline{V}$ is an FG-module, called a reduction modulo p of V, written as $\overline{V} = V \bmod p$ (although it is actually mod $\mathfrak{m}$.)

The following proposition are from [S].

Proposition 1.22. *Let V be an KG-module.*

(1) $\overline{V}$ *is not unique, but they share common composition factors, hence having the same Brauer characters* $\phi_{\overline{V}}$.

(2) *We have* $\phi_{\overline{V}} = \chi_V|_{G_{reg}}$.

Hence the reduction modulo p of an ordinary (virtual) character χ may be defined as $\overline{\chi} \coloneqq \chi|_{G_{reg}}$, and the group homomorphism $d : R_K(G) \to R_F(G)$ defined by reduction modulo p is well-defined, and send characters $R_K^+(G)$ to characters $R_F^+(G)$. The transpose of the matrix form of d is a $|\operatorname{Irr}_K(G)| \times |\operatorname{Irr}_F(G)|$ matrix with non-negative integer entries, called the decompostion matrix. Each row of the decompostion matrix shows how $\overline{\chi}$ decompose into sum of Brauer characters.

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A theorem [S] shows that d is surjective (so the decomposition matrix has full rank), but little else properties are known. Finding out the properties of decomposition matrix is a main objective in the study of representation theory.

We list here some properties of reduction modulo p.

Proposition 1.23. *Reduction modulo p commutes with conjugation, restriction, induction and inflation, in the sense of common composition factor.*

Proof. These are all simple with character and using Proposition 1.22(2). $\square$

1.8 Harish-Chandra Induction

In this thesis, we are concerning about Harish-Chandra induction, a construction of $\mathbf{F}GL_n$-module from $\mathbf{F}GL_r$-module and $\mathbf{F}GL_s$-module with $n = r + s$.

Let $L, U, P \leq GL_n$ be the group of matrices of the form:

$$L = \left[\begin{array}{c|c} g_r & \\ \hline & g_s \end{array}\right] \quad U = \left[\begin{array}{c|c} I_r & * \\ \hline & I_s \end{array}\right] \quad P = \left[\begin{array}{c|c} g_r & * \\ \hline & g_s \end{array}\right]$$

with $g_r \in GL_r$, $g_s \in GL_s$, and I_r, I_s identity. It is easy to check that $L \cong GL_r \times GL_s$, $U \trianglelefteq P$, and $L \cong P/U$ canonically.

Let W_r, W_s be $\mathbf{F}GL_r$-module and $\mathbf{F}GL_s$-module, respectively. Then $W_r \otimes W_s$ gives an $\mathbf{F}L$-module. The Harish-Chandra induction is defined as

$$W_r \circ W_s = \left(\mathrm{infl}_L^P(W_r \otimes W_s)\right) \uparrow_P^{GL_n}$$

Proposition 1.24. *Let W_* be $\mathbf{F}GL_*$-module for $* = r, s, t$.*

(1) $W_r \circ W_s \cong W_s \circ W_r$.

(2) $(W_r \circ W_s) \circ W_t \cong W_s \circ (W_r \circ W_t)$.

(3) $\overline{W_r \circ W_s} \cong \overline{W_s} \circ \overline{W_r}$.

The commutativity and associativity of this Harish-Chandra induction can be easily extended to the case $n = \sum_{i=1}^{a} n_i$ for any $a \in \mathbb{N}$.

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2 Representation Theory of $GL_n(q)$

Here we follow [J] to gives a construction of irreducible representations of $GL_n = GL_n(q)$ over $\mathbf{F} = K$ or F with characteristic not dividing q. $\mathbf{F}$ needs to contain p_0-root of unity, and be sufficiently large for all the groups we have considered in this section.

2.1 Compositions, Tableaux and Permutations

Although we have defined patitions of k in section 1.1, it is natural to consider compositions, an unordered version of partition, when talking about the $\mathbf{F}GL_n$-modules.

A composition λ of a non-negative k, denoted $\lambda \models k$, is a non-negative integer sequence $(\lambda_1, \lambda_2, \cdots)$ where $\sum_i \lambda_i = k$. A composition is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$, written as $\lambda \vdash k$. The transpose $\lambda' \coloneqq (\lambda'_1, \lambda'_2, \cdots)$ is defined as $\lambda'_i = \#\{\lambda_j \mid \lambda_j \geq i\}$. Note that λ' must be a partition, and λ'' is the partition rearranging each terms of λ by order. If $\lambda, \mu \models k$, the dominance order $\lambda \trianglerighteq \mu$ is defined by that of partitions $\lambda' \trianglelefteq \mu'$, that is, $\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \mu'_i$ for all $j \in \mathbb{N}$.

Fixed some $r \in \mathbb{N}$, a partition $\lambda \vdash k$ is r-singular if for some i, $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+r-1} > 0$, otherwise it is r-regular. Hence every nonempty partition is 1-singular, and conventionally every partition is ∞-regular.

For $\lambda \models k$, a λ-tableau is a bijection from $[\lambda] \coloneqq \{(i,j) \mid 1 \leq i,\ 1 \leq j \leq \lambda_i\}$ to $\{1, 2, \cdots, k\}$. For a λ-tableau t, we may draw it by embedding $[\lambda]$ into $\mathbb{N} \times \mathbb{N}$, with x-axis point to south and y-axis point to east, and put the number on its corresponding coordinate. The following are examples of tableaux.

A $(2^2, 3)$-tableau t_1	A $(4,3,2)$-tableau t_2	A $(4,3,2)$-tableau t_3
$\begin{matrix} 2 & 4 & \\ 1 & 7 & \\ 3 & 5 & 6 \end{matrix}$	$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & 9 & & \end{matrix}$	$\begin{matrix} 1 & 4 & 7 & 9 \\ 2 & 5 & 8 & \\ 3 & 6 & & \end{matrix}$

A tableau t is said to be row standard if in each row, the number increase as the y-coordinate increase. The t_1, t_2, t_3 above are all row standard.

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Let t^λ be the unique λ-tableau, put the number in lexicographical order of the coordinate, comparing x-coordinate first. Similarly, let $t^\lambda w_\lambda$ be the unique λ-tableau, put the number in lexicographical order of the coordinate, comparing y-coordinate first. For example, if $\lambda = (4,3,2)$, then $t^\lambda = t_2$ and $t^\lambda w_\lambda = t_3$ above.

For $\lambda \models k$, let t be a λ-tableau, $(i,j) \in [\lambda]$ a node of $[\lambda]$, $m \leq k$.

$$\begin{aligned}
(i,j)t \quad & \text{is the number corresponding to the node.} \\
\mathrm{row}_t(m) \quad & := i, \text{ if } (i,j)t = m. \text{ That is, } m \text{ is on the } i\text{th-row of } t. \\
\mathrm{col}_t(m) \quad & := j, \text{ if } (i,j)t = m. \text{ That is, } m \text{ is on the } j\text{th-column of } t. \\
\mathrm{row}_\lambda(m) \quad & := \mathrm{row}_{t^\lambda}(m), \text{ an abbreviation for tableau } t^\lambda.
\end{aligned}$$

For $w \in \mathfrak{S}_k$, let tw be the λ-tableau such that $(i,j)(tw) = ((i,j)t)w$. $w_\lambda \in \mathfrak{S}_k$ is the permutation consistent with the notation $t^\lambda w_\lambda$ above. For example, if $\lambda = (4,3,2)$, then $w_\lambda = (2\ 4\ 9\ 6\ 5)(3\ 7\ 8)$. Let $R(t) := \{w \in \mathfrak{S}_k \mid \mathrm{row}_t(i) = \mathrm{row}_t(iw) \text{ for all } i\}$, the row stabilizer of t, and $C(t) := \{w \in \mathfrak{S}_k \mid \mathrm{col}_t(i) = \mathrm{col}_t(iw) \text{ for all } i\}$, the column stabilizer of t.

Let d, $k \in \mathbb{N}$ and $n = dk$. If $\lambda \models k$, define $d\lambda \models n$ by $(d\lambda)_i = d(\lambda_i)$.

For $w \in \mathfrak{S}_k$, define $\pi_w \in \mathfrak{S}_n$ by

$$\pi_w : ad - b \mapsto (aw)d - b, \quad 1 \leq a \leq k,\ 0 \leq b \leq d-1$$

That is, if we divide $\{1, 2, \cdots, n\}$ into k packs $v_1 = (1, 2, \cdots, d)$, $v_2 = (d+1, d+2, \cdots, 2d)$, $\cdots$, $v_k = ((k-1)d+1, (k-1)d+2, \cdots, kd)$, then π_w permutes the index of $\{v_i\}$ while not changing its order inside.

Let $W_n \leq GL_n(q)$ be the permutation group. We identify W_n with $\mathfrak{S}_n$ via $\pi \mapsto (\mathbf{e}_i \mapsto \mathbf{e}_{i\pi})$, where $\{\mathbf{e}_i\}$ is the standard basis of $(\mathbb{F}_q)^n$. Then $W_{k\to n} := \{\pi_w \mid w \in \mathfrak{S}_k\}$ is a subgroup of W_n.

For example, if $\lambda = (4,3,2)$, $w = (3\ 5\ 4\ 7\ 8)$, $d = 2$, then we have

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$$t^\lambda = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & 9 & & \end{array} \qquad t^{d\lambda} = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & & \\ 15 & 16 & 17 & 18 & & & & \end{array}$$

$$t^\lambda w = \begin{array}{cccc} 1 & 2 & 5 & 7 \\ 4 & 6 & 8 & \\ 3 & 9 & & \end{array} \qquad t^{d\lambda}\pi_w = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 9 & 10 & 13 & 14 \\ 7 & 8 & 11 & 12 & 15 & 16 & & \\ 5 & 6 & 17 & 18 & & & & \end{array}$$

We can see how π_w moving the packs like w moving the numbers.

2.2 Subgroups of GL_n

Fixed n for GL_n. Let E_{ij} be the $n \times n$ matrix with (i,j)-entry 1 and other entries 0. Let $\Phi \coloneqq \{(i,j) \mid i,j \in \mathbb{N}, i \leq n, j \leq n, i \neq j\}$, with the action of $\mathfrak{S}_n$ on the right, $(i,j)\pi = (i\pi, j\pi)$. For $\alpha \in \mathbb{F}_q$, $(i,j) \in \Phi$, define

$$x_{ij}(\alpha) \coloneqq I_n + \alpha E_{ij}$$

$$h_i(\alpha) \coloneqq I_n + (\alpha - 1)E_{ii}$$

Then the root subgroup of GL_n

$$X_{ij} \coloneqq \{x_{ij}(\alpha) \mid \alpha \in \mathbb{F}_q\}$$

is a multiplicative group isomorphic to the additive group of $\mathbb{F}_q$. Let $H_n \coloneqq \langle h_i(\alpha) \mid 1 \leq i \leq n, \alpha \in \mathbb{F}_q^\times \rangle$ be the set of all invertible diagonal matrices.

A subset $\Gamma \subset \Phi$ is said to be closed if $(i,j), (j,k) \in \Gamma$ implies $(i,k) \in \Gamma$. Then we have the following fundamental theorem:

Theorem 2.1. [J2, Theorem 5.2] *Let Γ be a closed subset of Φ, and $G(\Gamma) \coloneqq \langle X_{ij} \mid (i,j) \in \Gamma \rangle$ be a subgroup of GL_n. Then*

$$G(\Gamma) = \prod_{(i,j)\in\Gamma} X_{ij} = \{I_n + \sum_{(i,j)\in\Gamma} \alpha_{ij}E_{ij} \mid \alpha_{ij} \in \mathbb{F}_q\}$$

where the product can be taken in any order. Once the order is chosen, then each element of $G(\Gamma)$ has a unique expression of $\prod_{(i,j)\in\Gamma} x_{ij}(\alpha_{ij})$.

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For $\mu \models n$, define the subset of Φ,

$$\Phi^+ \coloneqq \{(i,j) \in \Phi \mid j > i\}$$

$$A^+(\mu) \coloneqq \{(i,j) \in \Phi^+ \mid \mathrm{row}_\mu(i) = \mathrm{row}_\mu(j), j = i+1\}$$

$$B^+(\mu) \coloneqq \{(i,j) \in \Phi^+ \mid \mathrm{row}_\mu(i) = \mathrm{row}_\mu(j), j > i+1\}$$

$$C^+(\mu) \coloneqq \{(i,j) \in \Phi^+ \mid \mathrm{row}_\mu(i) = \mathrm{row}_\mu(j)\} = A^+(\mu) \cup B^+(\mu)$$

$$D^+(\mu) \coloneqq \{(i,j) \in \Phi^+ \mid \mathrm{row}_\mu(i) < \mathrm{row}_\mu(j)\}$$

For $\Sigma \subset \Phi$, let $\Sigma^T \coloneqq \{(j,i) \mid (i,j) \in \Sigma\}$. Define $\Phi^- \coloneqq (\Phi^+)^T$ and $S^-(\mu) \coloneqq (S^+(\mu))^T$ for $S = A, B, C, D$. Also let $C(\mu) \coloneqq C^+(\mu) \cup C^-(\mu)$. Below is a example for $\mu = (3,4,1,2)$, where the letters A, B, D means the entry belongs to $A^+(\mu)$, $B^+(\mu)$, $D^+(\mu)$, respectively.

$$\left[\begin{array}{ccc|cccc|c|cc}
1 & A & B & D & D & D & D & D & D & D \\
 & 1 & A & D & D & D & D & D & D & D \\
 & & 1 & D & D & D & D & D & D & D \\
\hline
 & & & 1 & A & B & B & D & D & D \\
 & & & & 1 & A & B & D & D & D \\
 & & & & & 1 & A & D & D & D \\
 & & & & & & 1 & D & D & D \\
\hline
 & & & & & & & 1 & D & D \\
\hline
 & & & & & & & & 1 & A \\
 & & & & & & & & & 1
\end{array}\right]$$

Observe the entries right above the diagonal. They can be A or D, and they are all A only when $\mu = (k)$. Also $C^+(\mu)$, $D^+(\mu)$, $C^-(\mu)$, $D^-(\mu)$ are closed subsets of Φ.

Now for $\mu \models n$, define the subgroups of GL_n,

$$L_\mu \coloneqq \langle H_n, X_{ij} \mid (i,j) \in C(\mu)\rangle$$

$$U_\mu^+ \coloneqq \langle X_{ij} \mid (i,j) \in D^+(\mu)\rangle$$

$$P_\mu^+ \coloneqq \langle H_n, X_{ij} \mid (i,j) \in C(\mu) \cup D^+(\mu)\rangle = U_\mu^+ L_\mu$$

and denote the set of unitriangular matrices $U^+ \coloneqq U^+_{(1^n)}$ and $U^- \coloneqq U^-_{(1^n)}$. Note that $U_\mu^+ \trianglelefteq P_\mu^+$ and $P_\mu^+/U_\mu^+ \cong L_\mu$, thus every $\mathbf{F}L_\mu$-module can be viewed as a $\mathbf{F}P_\mu^+$-module

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with U^+_μ acting trivially (that is, inflation from L_μ to P^+_μ). For example, if $\mu = (3, 1, 2)$, then we have

$$\begin{array}{ccc}
L_\mu & U^+_\mu & P^+_\mu \\
\left[\begin{array}{ccc|c|cc}
* & * & * & & & \\
* & * & * & & & \\
* & * & * & & & \\ \hline
 & & & * & & \\ \hline
 & & & & * & * \\
 & & & & * & *
\end{array}\right]
&
\left[\begin{array}{ccc|c|cc}
1 & & & * & * & * \\
 & 1 & & * & * & * \\
 & & 1 & * & * & * \\ \hline
 & & & 1 & * & * \\ \hline
 & & & & 1 & \\
 & & & & & 1
\end{array}\right]
&
\left[\begin{array}{ccc|c|cc}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\ \hline
 & & & * & * & * \\ \hline
 & & & & * & * \\
 & & & & * & *
\end{array}\right]
\end{array}$$

with the diagonal block invertible.

2.3 Idempotents of GL_n

Let $\mathbf{F} = K$ or F, containing a p_0-root of unity. Pick any non-trivial homomorphism $\chi_{\mathbf{F}} : (\mathbb{F}_q, +) \to \mathbf{F}^\times$. For each $\mu \models n$, there is a linear $\mathbf{F}$-character of U^+,

$$\theta_\mu(u) \coloneqq \chi_{\mathbf{F}}(\sum_{(i,j)\in A^+(\mu)} u_{ij}), \quad u = (u_{ij}) \in U^+$$

Then for each $\mu \models n$ we can define

$$E^+_{\mathbf{F}}(\mu) \coloneqq |U^+|^{-1} \sum_{u\in U^+} \theta_\mu(u^{-1})u \in \mathbf{F}U^+$$

It is clear that $uE^+_{\mathbf{F}}(\mu) = E^+_{\mathbf{F}}(\mu)u = \theta_\mu(u)E^+_{\mathbf{F}}(\mu)$ for any $u \in U^+$, and $(E^+_{\mathbf{F}}(\mu))^2 = E^+_{\mathbf{F}}(\mu)$, hence $E^+_{\mathbf{F}}(\mu)$ is an idempotent of $\mathbf{F}U^+$.

Also, we need another characterization of $E^+_{\mathbf{F}}(\mu)$. For $(i, j) \in \Phi$, let

$$\tilde{X}_{ij} \coloneqq q^{-1} \sum_{\alpha\in\mathbb{F}_q} \chi_{\mathbf{F}}(-\alpha)x_{ij}(\alpha), \qquad \bar{X}_{ij} \coloneqq q^{-1} \sum_{\alpha\in\mathbb{F}_q} x_{ij}(\alpha)$$

Then for $\mu \models n$, define

$$Q^+_{\mathbf{F}}(\mu) \coloneqq |U^+ \cap L_\mu|^{-1} \sum_{u\in U^+\cap L_\mu} \theta_\mu(u^{-1})u, \qquad U^+_{\mathbf{F}}(\mu) \coloneqq |U^+_\mu|^{-1} \sum_{u\in U^+_\mu} u$$

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Proposition 2.2. *For $(i,j) \in \Phi^+$, let $\hat{X}_{ij}(\mu)$ be $\tilde{X}_{ij}$ if $(i,j) \in A^+(\mu)$, and $\bar{X}_{ij}$ otherwise. Then for $\mu \models n$ we have*

$$\begin{aligned}
Q^+_{\mathbf{F}}(\mu) &= \prod_{(i,j)\in C^+(\mu)} \hat{X}_{ij}(\mu) = \prod_{(i,j)\in A^+(\mu)} \tilde{X}_{ij} \prod_{(i,j)\in B^+(\mu)} \bar{X}_{ij} \\
U^+_{\mathbf{F}}(\mu) &= \prod_{(i,j)\in D^+(\mu)} \hat{X}_{ij}(\mu) = \prod_{(i,j)\in D^+(\mu)} \bar{X}_{ij} \\
E^+_{\mathbf{F}}(\mu) &= Q^+_{\mathbf{F}}(\mu) U^+_{\mathbf{F}}(\mu) \\
&= \prod_{(i,j)\in \Phi^+} \hat{X}_{ij}(\mu) = \prod_{(i,j)\in A^+(\mu)} \tilde{X}_{ij} \prod_{(i,j)\in B^+(\mu)\cup D^+(\mu)} \bar{X}_{ij}
\end{aligned}$$

The product can be taken in any order.

Proof. This is just the consequence of Theorem 2.1. Also see [J2, Theorem 9.2] for a proof. □

We emphisize that $C^+(\mu)$ and $D^+(\mu)$ are closed, Hence $U^+ \cap L_\mu$ and U^+_μ are closed under multiplication. This implies that $uQ^+_{\mathbf{F}}(\mu) = \theta_\mu(u) Q^+_{\mathbf{F}}(\mu)$ if $u \in U^+ \cap L_\mu$, and $uU^+_{\mathbf{F}}(\mu) = U^+_{\mathbf{F}}(\mu)$ if $u \in U^+_\mu$.

2.4 The Module $M_{\mathbf{F}}(\sigma, (1))$

In [G, Theorem 13], Green has found all ordinary irreducible characters of GL_n in explicit form. However this form of Green is difficult to analyze, and does not reflect well a natural correspondence between the conjugacy classes of GL_n and its irreducible ordinary characters. The main work of James and Dipper is to create a full list of irreducible representations of GL_n in the module phase, both ordinary and p-modular, with the natural label $(\underline{\sigma}, \underline{\lambda})$ we defined in Definition 1.7.

Let $\langle \varepsilon \rangle = \mathbb{F}^\times_{q^d}$ and $\bar{\varepsilon} = \zeta_{q^d-1} = \exp(2\pi i/(q^d-1)) \in \mathbb{C}$. Then we can send $\mathbb{F}^\times_{q^d}$ to $\mathbb{C}$ injectively by $\varepsilon \mapsto \bar{\varepsilon}$. Define $\langle\ ,\ \rangle_d : \mathbb{F}^\times_{q^d} \times \mathbb{F}^\times_{q^d} \to \mathbb{C}$ by

$$\langle \sigma, \tau \rangle_d := \bar{\varepsilon}^{ij} \quad \text{ if } \sigma = \varepsilon^i, \tau = \varepsilon^j$$

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Note that for $\sigma_1, \sigma_2, \tau \in \mathbb{F}_{q^d}^{\times}$,

$$\langle \sigma_1\sigma_2, \tau\rangle_d = \langle \sigma_1, \tau\rangle_d \langle \sigma_2, \tau\rangle_d$$

If we write $\sigma = \sigma'\sigma_p$, where σ' and σ_p are p'-part and p-part of σ, respectively, then for any p-regular τ we have $\langle \sigma, \tau\rangle_d = \langle \sigma', \tau\rangle_d$.

A conjugacy class $[(\underline{\tau}, \underline{\lambda})]$ is said to be primary if both $\underline{\tau} = (\tau)$ and $\underline{\lambda} = (\lambda)$ are in fact 1-tuples. Hence $(\underline{\tau}, \underline{\lambda}) = (\tau, \lambda)$ in the form of $\circ$ notation.

Theorem 2.3. *(Green) Given $\sigma \in \mathbb{F}_{q^d}^{\times}$ with $\deg(\sigma) = d$, there is an ordinary irreducible character ψ_σ of $GL_d(q)$ which satisfies the following,*

(1) *ψ_σ is zero on all conjugacy classes except for those are primary.*

(2) *For a primary conjugacy class $[(\tau, \lambda)]$, with $\deg(\tau) = a$, $\lambda \vdash b$, $ab = d$, the value of ψ_σ on such conjugacy class is*

$$(-1)^{d+x}(q^a - 1)(q^{2a} - 1)\cdots(q^{(x-1)a} - 1)\{\langle \sigma, \tau\rangle_d + \langle \sigma, \tau\rangle_d^q + \cdots + \langle \sigma, \tau\rangle_d^{q^{a-1}}\}$$

where x is the number of nonzero parts of λ.

(3) *By taking $\tau = 1$, $a = 1$, $b = x = d$, $\lambda = (1^d)$, the degree of ψ_σ is*

$$N := (q - 1)(q^2 - 1)\cdots(q^{d-1} - 1)$$

Moreover, if both σ_1, σ_2 has degree d over $\mathbb{F}_q$, then ψ_{σ_1} and ψ_{σ_2} agree on all p-regular conjugacy classes of GL_d if and only if their p'-part have the same minimal polynomial over $\mathbb{F}_q$, i.e. σ_1 and σ_2 are p-conjugate to each other.

Proof. This ϕ_σ is $(-1)^{d+x} I_d^i[1]$ in [G], where i is any integer satisfied $\sigma = \varepsilon^i$. The virtual character $I_d^i[1]$ is described in the end of chapter 5 of [G], which gives (1) and values of (2). Lemma 7.5 of [G] gives the sign $(-1)^{d+x}$, and lemma 7.6 of [G] show that ϕ_σ is irreducible. Finally, (3) and the last statement is the consequence of (2). $\square$

From now on, write $\mathbf{F} = K$ or F, of characteristic 0 or p, respectively, which is sufficiently large for all the groups hereafter we shall consider.

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We follow Gelfand's procedure to construct the KGL_d-module $M_K(\sigma,(1))$ with corresponding character ψ_σ.

First pick a non-trivial linear complex character $\chi : (\mathbb{F}_q,+) \to \mathbb{C}$. Define a non-trivial linear complex character of U^+,

$$\theta(u) := \chi\left(\sum_{i=1}^{d-1} u_{i,i+1}\right)$$

where $u_{i,i+1}$ is the $(i,i+1)$-entry of $u \in U^+$.

Next let $\mathbb{C}v$ be a one-dimensional $\mathbb{C}U^+$-module with $vu = \theta(u)v$ action on the right. Set G^*_{d-1} be the subgroup of GL_d with $(1,1)$-entry 1 and $(i,1)$-entry 0 for $1 < i \le d$. Observe that $(G^*_{d-1} : U^+) = N$, the degree of ψ_σ in Theorem 2.3(3). Now induce the $\mathbb{C}U^+$-module $\mathbb{C}v$ to $\mathbb{C}G^*_{d-1}$-module, denoted by C.

Now we consider the idempotents acts on the right.

Lemma 2.4. *Let $\mu \models d$, $E^+_{\mathbb{C}}(\mu)$ idempotent defined in section 2.3. Then*

(1) *$vE^+_{\mathbb{C}}((d)) = v$.*

(2) *$vgE^+_{\mathbb{C}}((d)) = 0$ for $g \in G^*_{d-1} \setminus U^+$.*

(3) *$CU^+_{\mathbb{C}}(\mu) = 0$ for $\mu'' \neq (d)$.*

Proof. We make use of $vu = \theta(u)v$ and $uE^+_{\mathbb{C}}((d)) = \theta(u)E^+_{\mathbb{C}}((d))$ for $u \in U^+$.

(1) $vE^+_{\mathbb{C}}((d)) = |U^+|^{-1}\sum_{u\in U^+}\theta(u^{-1})vu = |U^+|^{-1}\sum_{u\in U^+} v = v$.

(2) If we can pick some $u_1, u_2 \in U^+$, $u_1 g = g u_2$ and $\theta(u_1) \neq \theta(u_2)$, then

$$\theta(u_1)vgE^+_{\mathbb{C}}((d)) = vu_1gE^+_{\mathbb{C}}((d)) = vgu_2E^+_{\mathbb{C}}((d)) = \theta(u_2)vgE^+_{\mathbb{C}}((d))$$

hence $vgE^+_{\mathbb{C}}((d)) = 0$.

Write $g = \left[\begin{smallmatrix} 1 & v^t \\ 0 & g_1 \end{smallmatrix}\right]$, where $v \in (\mathbb{F}_q)^{d-1}$ and $g_1 \in GL_{d-1}(q)$. Consider $u_1 = \begin{bmatrix} 1 & w_1^t \\ 0 & I_{d-1} \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 & w_2^t \\ 0 & I_{d-1} \end{bmatrix}$, $w_1, w_2 \in (\mathbb{F}_q)^{d-1}$, then $u_1 g = g u_2$ gives $g_1^t w_1 = w_2$. If there are some w_1, w_2 have different first coordinate, then $\theta(u_1) \neq \theta(u_2)$ and we are done. If for all $w_1 \in (\mathbb{F}_q)^{d-1}$, $w_2 = g_1^t w_1$ have common first coordinate with w_1, then the first row of g_1^t must be $[1, 0, \cdots, 0]$. Now we can replace g by g_1 and continue the process. If

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we never find such pair w_1, w_2, then g would be a upper unitriangular matrix, that is, $g \in U^+$, which contradicts to the assumption.

(3) Write $\mu = (\mu_1, \cdots, \mu_t)$, $0 < \mu_1 < d$. Then $U_{\mathbb{C}}^+(\mu)$ contains $\bar{X}_{i,i+1}$ for $i = \mu_1$. Now for any $v \in C$,

$$v\bar{X}_{i,i+1} = q^{-1} \sum_{\alpha \in \mathbb{F}_q} x_{i,i+1}(\alpha)v = q^{-1} \left(\sum_{\alpha \in \mathbb{F}_q} \chi(\alpha) \right) v = 0$$

by the property of non-trivial linear complex character χ. □

Let $[U^+ \backslash G^*_{d-1}] = \{1 = g_1, g_2, \cdots, g_N\}$ be right coset representatives.

Lemma 2.5. *C is an irreducible $\mathbb{C}G^*_{d-1}$-module.*

Proof. By definition, C is generate by v. If M is an non-trivial $\mathbb{C}G^*_{d-1}$-sub- module of C, then $0 \neq m \in M$ has the form $\sum_{i=1}^N c_i(vg_i)$ for $c_i \in \mathbb{C}$. Pick some $c_r \neq 0$, then $v = c_r^{-1} m g_r^{-1} E_{\mathbb{C}}^+((d)) \in M$ and hence $M = C$. □

Now define $J : GL_d(q) \to \mathbb{C}$ by

$$J(g) = |U^+|^{-1} \sum_{u \in U^+} \theta(u^{-1}) \psi_\sigma(gu)$$

Then with $\psi_\sigma \downarrow_{G^*_{d-1}}$ is irreducible and $\psi_\sigma \downarrow_{U^+}$ contains θ with multiplicity 1, we have

$$(vg_j)g = \sum_{i=1}^N J(g_j g g_i^{-1})(vg_i)$$

for a right coset representative $[U^+ \backslash G^*_{d-1}] = \{1 = g_1, g_2, \cdots, g_N\}$. Now for arbitrary $g \in GL_d(q)$, let

$$(vg_j)g = \sum_{i=1}^N J(g_j g g_i^{-1})(vg_i)$$

Gelfand show that this actions extends C to a $KGL_d(q)$-module $M_K(\sigma, (1))$, and the character of $M_K(\sigma, (1))$ is ψ_σ. Let $M_F(\sigma, (1))$ be the reduction modulo p of $M_K(\sigma, (1))$.

Proposition 2.6. *Let $\mathbf{F} = K$ or F.*

(1) *$M_{\mathbf{F}}(\sigma, (1))$ is an irreducible representation.*

(2) *$M_{\mathbf{F}}(\sigma, (1))$ is a cyclic module generated by v.*

doi:10.6342/NTU201904116

(3) $\dim M_{\mathbf{F}}(\sigma,(1)) = N = (q-1)(q^2-1)\cdots(q^{d-1}-1)$.

(4) *If* $\sigma, \tau \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = \deg(\tau)$, *then* $M_{\mathbf{F}}(\sigma,(1)) \cong M_{\mathbf{F}}(\tau,(1))$ *if and only if* σ, τ *are* p*-conjugate to each other.*

Proof. (2)(3) follows from the definition. A analogue to Lemma 2.5 proves (1). (4) follows from the last statement of Theorem 2.3. □

Lemma 2.7. *Let* $\mu \models d$, $E_{\mathbf{F}}^{+}(\mu)$ *idempotent defined in section 2.3. Then*

(1) $vE_{\mathbf{F}}^{+}((d)) = v$.

(2) $M_{\mathbf{F}}(\sigma,(1))E_{\mathbf{F}}^{+}((d)) = \mathbf{F}v$.

(3) $M_{\mathbf{F}}(\sigma,(1))U_{\mathbf{F}}^{+}(\mu) = 0$ *for* $\mu'' \neq (d)$.

Proof. The space of $M_{\mathbf{F}}(\sigma,(1))$ has no different with C, hence this is just a rewrite of Lemma 2.4. □

2.5 The Module $M_{\mathbf{F}}(\sigma,(1^k))$

Let $d,k \in \mathbb{N}$, $dk = n$, $\sigma \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = d$, and $\mathbf{F} = K$ or F. Define

$$G^{(r)} \coloneqq \{\mathrm{diag}(g_1,\cdots,g_k) \mid g_r \in GL_d(q),\ g_t = I_d \text{ for } t \neq r\}, \quad 1 \leq r \leq k$$

Then clearly $G^{(r)} \cong GL_d(q)$ and $G^{(1)} \times \cdots \times G^{(k)}$ is the Levi subgroup $L_{(d^k)}$. For each r, let $M^{(r)}$ be a $\mathbf{F}G^{(r)}$-module obtain from $M_{\mathbf{F}}(\sigma,(1))$ under the trivial embedding $\iota^{(r)} : GL_d(q) \to G^{(r)} \subset GL_n(q)$ with generator v_r.

Definition 2.8.

(1) Let $M \coloneqq M^{(1)} \otimes \cdots \otimes M^{(r)}$ be a $\mathbf{F}L_{(d^k)}$-module with generator $v_1 \otimes \cdots \otimes v_k$.

(2) Define

$$M_{\mathbf{F}}(\sigma,(1^k)) \coloneqq M^{(1)} \circ \cdots \circ M^{(k)} \cong \mathrm{Ind}_P^G \left(\mathrm{infl}_L^P M\right)$$

be a $\mathbf{F}GL_n(q)$-module, where $G = GL_n(q)$, $P = P_{(d^k)}^{+}$ and $L = L_{(d^k)}$ for temporary abbreviation of Definition 2.8 and Proposition 2.9.

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It is well-known that $GL_n = P^+_{(1^n)} W_n P^+_{(1^n)}$ (the Bruhat decomposition), hence the set $\{\pi u \mid \pi \in W_n, u \in U^+\}$ is a right coset representative of $P^+_{(1^n)} \backslash G$. Since $P^+_{(1^n)} \subset P$, elements of $[P\backslash G]$ is also of the form πu, although they may not represent distinct cosets.

Proposition 2.9.

(1) *$M_{\mathbf{F}}(\sigma, (1^k))$ is a cyclic module generated by $v_1 \otimes \cdots \otimes v_k$.*

(2) *Each element of $M_{\mathbf{F}}(\sigma, (1^k))$ is a linear combination of terms*

$$(v_1 g_1 \otimes \cdots \otimes v_k g_k)\pi u$$

where $g_r \in G^{(r)}$ and πu representative of $[P\backslash G]$.

Proof. Both of them are consequence of the definition of $M_{\mathbf{F}}(\sigma, (1^k))$. □

Lemma 2.10. *Let $\mu \models n$, $E^+_{\mathbf{F}}(\mu)$ be idempotent defined in section 2.3.*

(1) *$(v_1 \otimes \cdots \otimes v_k)E^+_{\mathbf{F}}((d^k)) = v_1 \otimes \cdots \otimes v_k$.*

(2) *For $m \in M$, $\pi \in W_n \setminus W_{k\to n}$ such that $t^{(d^k)}\pi$ is row standard, $u \in U^+$, we have $m\pi u E^+_{\mathbf{F}}(\mu) = 0$ for any $\mu \models n$.*

(3) *$M_{\mathbf{F}}(\sigma, (1^k))U^+_{\mathbf{F}}(\mu) = 0$ if $\mu = (n-r, r)$ for some r not a multiple of d.*

Proof. (1) $E^+_{\mathbf{F}}((d^k)) = Q^+_{\mathbf{F}}((d^k))U^+_{\mathbf{F}}((d^k))$. We have $Q^+_{\mathbf{F}}((d^k)) = E^+_{\mathbf{F}}((d)) \otimes \cdots \otimes E^+_{\mathbf{F}}((d))$ fixes $v_1 \otimes \cdots \otimes v_k$, and $U^+_{\mathbf{F}}((d^k)) \in U^+_{(d^k)}$ acts trivially.

(2) By $uE^+_{\mathbf{F}}(\mu) = \theta_\mu(u)E^+_{\mathbf{F}}(\mu)$, we can ignore it. If π is not in $W_{k\to n}$, then for some a, the ath-row of $t = t^{(d^k)}\pi$ must containing numbers not all consecutive. Pick x such that $\mathrm{row}_t(x) = a$, $\mathrm{row}_t(x+1) \neq a$, and $\mathrm{col}_t(x) \neq d$, that is, x is in the ath-row with next number not $x+1$. Consider $S\pi \coloneqq \{(i,j) \in \Phi \mid \mathrm{row}_t(i) = \mathrm{row}_t(j) = a,\ i \leq x < j\}$ which is non-empty and forms a hollow rectangular. Then $(i,j) \in S\pi$ implies $j > i+1$, $E^+_{\mathbf{F}}(\mu)$ contains $\prod_{(i,j)\in S\pi} \bar{X}_{ij}$ whenever μ is. But then $\pi E^+_{\mathbf{F}}(\mu)$ contains

$$\pi\left(\prod_{(i,j)\in S\pi} \bar{X}_{ij}\right) = \left(\prod_{(i,j)\in S\pi} \bar{X}_{i\pi^{-1},j\pi^{-1}}\right)\pi = \left(\prod_{(i,j)\in S} \bar{X}_{ij}\right)\pi$$

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where $S := \{(i,j) \in \Phi \mid \mathrm{row}_{(d^k)}(i) = \mathrm{row}_{(d^k)}(j) = a,\ i \le x\pi^{-1} < j\}$ (note that π and π^{-1} preserve order in each row,) forming a solid rectangular. Write $x\pi^{-1} = ad - b$, then $\prod_{(i,j)\in S} \bar{X}_{ij} = \iota^{(a)}\left(U^+_{\mathbf{F}}((d-b,b))\right)$, which eliminates $M^{(a)}$ by Lemma 2.7(3), hence M, and m as well.

(3) It suffices to show that for any $m \in M$, $\pi \in W_n$ such that $t = t^{(d^k)}$ is row standard, and $u \in U^+$, we have $m\pi u U^+_{\mathbf{F}}(\mu) = 0$, and the proof is similar to (2), as following. Since u and $U^+_{\mathbf{F}}(\mu)$ commutes, we can ignore it. Because r is not a multiple of d, there is some a such that the ath-row of t contains some element less or equal then r, and some larger then r. Pick x in the ath-row of t such that the next number of x is greater then r. Then $S\pi := \{(i,j) \in \Phi \mid \mathrm{row}_t(i) = \mathrm{row}_t(j) = a,\ i \le x < j\}$ forms a hollow rectangular, and $U^+_{\mathbf{F}}(\mu)$ contains $\prod_{(i,j)\in S\pi} \bar{X}_{ij}$. Now the same argument to (2) shows that $\pi U^+_{\mathbf{F}}(\mu)$ contains $\iota^{(a)}(U^+_{\mathbf{F}}(d-b,b))$ for some b and eliminates m. □

Corollary 2.11. *Let $\mu \models n$. Then $M_{\mathbf{F}}(\sigma,(1^k))E^+_{\mathbf{F}}(\mu) = 0$ if μ is not of a form $d\lambda$ for some $\lambda \vdash k$.*

Proof. Write $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$. Then there is some j such that $r := \sum_{i=1}^{j} \mu_i$ is not a multiple of d. In the view of Proposition 2.2, $E^+_{\mathbf{F}}(\mu)$ contains $U^+_{\mathbf{F}}(\mu)$ and $U^+_{\mathbf{F}}(\mu)$ contains $U^+_{\mathbf{F}}((r, n-r))$, thus the corollary follows from Lemma 2.10(3). □

In order to find out $M_{\mathbf{F}}(\sigma,\lambda)$ for other $\lambda \models k$, we introduce the Hecke algebra arising from $M_{\mathbf{F}}(\sigma,(1^k))$.

Definition 2.12. Let $\mathcal{H}_k = \mathcal{H}_{\mathbf{F},k,\sigma}$ be the algebra of $\mathbf{F}GL_n$-module endomorphisms of $M_{\mathbf{F}}(\sigma,(1^k))$, written on the left.

Theorem 2.13. [J, 4.10] *The algebra $\mathcal{H}_k$ has a basis $\{h_w \mid w \in \mathfrak{S}_k\}$ defined by*

$$h_w : v_1 \otimes \cdots \otimes v_k \mapsto (v_1 \otimes \cdots \otimes v_k)\pi_w U^+_{\mathbf{F}}((d^k))$$

Furthermore, we have

$$M_{\mathbf{F}}(\sigma,(1^k))E^+_{\mathbf{F}}((d^k)) = \mathcal{H}_k(v_1 \otimes \cdots \otimes v_k)$$

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Definition 2.14. For each $w \in \mathfrak{S}_k$, define $T_w \in \mathcal{H}_k$ by

$$T_w : v_1 \otimes \cdots \otimes v_k \mapsto (J(-I_d)q^{(d+d^2)/2})^{l(w)}(v_1 \otimes \cdots \otimes v_k)\pi_w U_{\mathbf{F}}^+((d^k))$$

where $l(w)$ denotes the length of w.

Theorem 2.15. [J, 4.12] *The algebra $\mathcal{H}_k$ is an associative algebra over $\mathbf{F}$ with basis $\{T_w \mid w \in \mathfrak{S}_k\}$, with the multiplication generate by the rule*

$$T_w T_v = \begin{cases} T_{wv} & \textit{if } l(wv) = l(w) + 1 \\ q^d T_{wv} + (q^d - 1)T_w & \textit{otherwise.} \end{cases}$$

where $w, v \in \mathfrak{S}_k$, $v = (i, i+1)$ for some $1 \leq i \leq k-1$.

2.6 The Module $M_{\mathbf{F}}(\sigma, \lambda)$, $S_{\mathbf{F}}(\sigma, \lambda)$ and $D_{\mathbf{F}}(\sigma, \lambda)$

For any $\lambda \models k$, let

$$x_\lambda := \sum_{w \in R(t^\lambda)} T_w \in \mathcal{H}_k$$

where $R(t^\lambda)$ is the row stabilizer of t^λ. Define

$$M_{\mathbf{F}}(\sigma, \lambda) := x_\lambda M_{\mathbf{F}}(\sigma, (1^k))$$

hence $M_{\mathbf{F}}(\sigma, \lambda)$ is an $\mathbf{F}GL_n$-submodule of $M_{\mathbf{F}}(\sigma, (1^k))$. The notation is consistent since $x_{(1^k)} = 1$. Also it is consistent with Harish-Chandra induction, that is, $M_{\mathbf{F}}(\sigma, \lambda) = M_{\mathbf{F}}(\sigma, (\lambda_1)) \circ M_{\mathbf{F}}(\sigma, (\lambda_2)) \circ \cdots$. Hence to investigate $M_{\mathbf{F}}(\sigma, \lambda)$, it suffices to study $M_{\mathbf{F}}(\sigma, (k))$.

Lemma 2.16. [J, 6.3]

(1) *$M_{\mathbf{F}}(\sigma, (k))$ is an irreducible $\mathbf{F}GL_n$-module.*

(2) *$M_{\mathbf{F}}(\sigma, (k))E_{\mathbf{F}}((d^k))$ is the one-dimensional space $\mathbf{F}x_{(k)}(v_1 \otimes \cdots \otimes v_k)$.*

By counting the dimension of $M_{\mathbf{F}}(\sigma, (k))$, we can deduced that

Theorem 2.17. [J, 6.9] *Let $\mu \models n$. Then $M_{\mathbf{F}}(\sigma, (k))E_{\mathbf{F}}^+(\mu)$ is one-dimensional if $\mu = (d^k)$ and is zero otherwise.*

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And its generalization to $M_{\mathbf{F}}(\sigma, \lambda)$,

Theorem 2.18. [J, 7.1] *Let $\lambda, \mu \models k$. Then*

(1) $M_{\mathbf{F}}(\sigma,\lambda)E^+_{\mathbf{F}}(d\mu) = 0$ *unless* $\mu' \trianglerighteq \lambda$.

(2) $M_{\mathbf{F}}(\sigma,\lambda)E^+_{\mathbf{F}}(d\lambda')$ *is a one-dimensional space.*

Now for $\lambda \models k$, we can define

$$S_{\mathbf{F}}(\sigma,\lambda) = M_{\mathbf{F}}(\sigma,\lambda)E^+_{\mathbf{F}}(d\lambda')\mathbf{F}GL_n(q)$$

By the previous theorem, $S_{\mathbf{F}}(\sigma,\lambda)$ is not 0. Since $M_{\mathbf{F}}(\sigma,(k))$ is irreducible, we have $S_{\mathbf{F}}(\sigma,(k)) = M_{\mathbf{F}}(\sigma,(k))$. It is easy to prove that

Proposition 2.19. [J, 7.8] *$S_{\mathbf{F}}(\sigma,\lambda)$ has a unique maximal $\mathbf{F}GL_n(q)$-submodule $S_{\mathbf{F}}(\sigma,\lambda)^{\max}$.*

Corollary 2.20. [J, 7.9] *If $\mathbf{F} = K$, then $S_K(\sigma,\lambda)$ is irreducible.*

Lemma 2.21.

(1) *If $\mu \models n$ is a rearrangement of partition $\lambda \vdash n$, i.e. $\mu'' = \lambda$, then $M_{\mathbf{F}}(\sigma,\mu) \cong M_{\mathbf{F}}(\sigma,\lambda)$ and $S_{\mathbf{F}}(\sigma,\mu) \cong S_{\mathbf{F}}(\sigma,\lambda)$.*

(2) *If $\sigma,\tau \in \overline{\mathbb{F}}_q^\times$, $\deg(\sigma) = \deg(\tau)$, then $M_{\mathbf{F}}(\sigma,\lambda) \cong M_{\mathbf{F}}(\tau,\lambda)$ and $S_{\mathbf{F}}(\sigma,\lambda) \cong S_{\mathbf{F}}(\tau,\lambda)$ if the p'-part of σ, τ have the same minimal polynomial over $\mathbb{F}_q$.*

Proof. (1) Write $M_{\mathbf{F}}(\sigma,\mu) = M_{\mathbf{F}}(\sigma,\mu_1) \circ M_{\mathbf{F}}(\sigma,\mu_2) \circ \dots$ and note that the Harish-Chandra induction is commutative up to isomorphism. The $S_{\mathbf{F}}(\sigma,\lambda)$ case follows from $\mu' = \lambda'$ and the definition.

For (2), from Proposition 2.6 we have $M_{\mathbf{F}}(\sigma,(1^k)) \cong M_{\mathbf{F}}(\tau,(1^k))$ if and only if σ and τ are p-conjugate (defined in Definition 1.6), hence they have isomorphic Hecke algebra, and hence isomorphic $M_{\mathbf{F}}(\sigma,\lambda)$ and $S_{\mathbf{F}}(\sigma,\lambda)$. $\square$

Therefore, when we talk about isomorphic types of $M_{\mathbf{F}}(\sigma,\lambda)$ and $S_{\mathbf{F}}(\sigma,\lambda)$, it suffices to consider partitions $\lambda \vdash n$ rather then $\lambda \models n$.

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By counting the dimension of $S_{\mathbf{F}}(\sigma,\lambda)$, we have two important consequence analogue to the case of symmetric groups. One is the kernel intersection theorem (we omitted here), and the other is Young's rule, described below,

Theorem 2.22 (Young's Rule)**.** [J, 7.19(iii)] *Suppose λ, $\mu \vdash k$. Then the composition factors of $M_{\mathbf{F}}(\sigma,\lambda)$ are of the form $S_{\mathbf{F}}(\sigma,\mu)$, with multiplicity equals the Kostka number $K_{\mu\lambda}$, which means the number of semistandard μ-tableaux of type λ (a definition of this see* [J1, 13]*.)*

Observe that $K_{\mu\lambda}$ is independent of σ.

Corollary 2.23. *In the Grothendieck group of $\mathbf{F}GL_n$-module we may write*

$$M_{\mathbf{F}}(\sigma,\lambda) = S_{\mathbf{F}}(\sigma,\lambda) + \sum_{\mu\trianglerighteq\lambda,\mu\neq\lambda} K_{\mu\lambda} S_{\mathbf{F}}(\sigma,\mu)$$

Proof. By Theorem 2.18 and the definition of $S_{\mathbf{F}}(\sigma,\mu)$, we may deduce that $K_{\mu\lambda} = 0$ unless $\mu \trianglerighteq \lambda$, and $K_{\lambda\lambda} = 1$. One may obtain the result as well by the definition of Kostka numbers. □

Once we have the Young's Rule, the Littlewood-Richardson rule follows,

Theorem 2.24 (Littlewood-Richardson rule)**.** *Let $\sigma \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = d$, $\lambda \vdash k_1$, $\mu \vdash k_2$, and $n = d(k_1 + k_2)$. Then in the Grothendieck group of $\mathbf{F}GL_n$-module,*

$$S_{\mathbf{F}}(\sigma,\lambda) \circ S_{\mathbf{F}}(\sigma,\mu) = \sum_{\nu\vdash k_1+k_2} c^{\nu}_{\lambda\mu} S_{\mathbf{F}}(\sigma,\nu)$$

where $c^{\nu}_{\lambda\mu}$ is the Littlewood-Richardson coefficient, the number of semistandard skew-$\nu\backslash\lambda$-tableau of type μ which the sequence obtained from concatenating reversed rows is a lattice permutation (A definition see [J1, 15, 16]*.)*

Proof. A proof of the case of symmetric groups is in [J1, 16], which proves that the Young's rule implies the Littlewood-Richardson rule. Now an analogue proof follows from Theorem 2.22. □

We need a property of Littlewood-Richardson coefficients later.

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Proposition 2.25. *The Littlewood-Richardson coefficient* $c^{\nu}_{\lambda\mu} = 0$ *unless* $\lambda+\mu \trianglerighteq \nu \trianglerighteq \lambda \,[+]\, \mu$*. In the case* $\nu = \lambda + \mu$ *or* $\nu = \lambda \,[+]\, \mu$*, we have* $c^{\nu}_{\lambda\mu} = 1$*.*

Proof. Assume $c^{\nu}_{\lambda\mu} > 0$, which is the number of the skew-$\nu\backslash\lambda$-tableau of type μ, filling μ_i many number i each, forming a semistandard tableau (nondecreasing on each row, increasing on each column), and the sequence obtained concatenating reversed row is a lattice permutation (each prefix has at least as many positive integers i as integers $i+1$.)

First we claim that for any number i, it can only appear on ith row or below. If i appears on $(i-1)$th row or above, then some $i-1$ must appears before that i in the sequence, that is, on the right of i or on the rows above. But since the tableau is semistandard, $i-1$ cannot put on the right of i, thus it must appears on the rows above, i.e. $i-1$ appears on $(i-2)$th row or above. Continue this we have there is some 1 appears before the first row, which is absurd.

Now the first n rows can only put those number not greater then n. By comparing the total blocks we have $\sum_{i=1}^{n}(\nu_i - \lambda_i) \leq \sum_{i=1}^{n} \mu_i$, which exactly means $\nu \trianglelefteq \lambda + \mu$. If $\nu = \lambda+\mu$, then the ith row can only put number i, so there is only one possible tableau, $c^{\nu}_{\lambda\mu} = 1$.

Next observe that each column can only put strictly increasing numbers, still comparing the total blocks we have $\sum_{i=1}^{n}(\nu'_i - \lambda'_i) \leq \sum_{i=1}^{n} \mu'_i$, which exactly means $\nu' \trianglelefteq \lambda' + \mu'$, thus $\nu \trianglerighteq \lambda \,[+]\, \mu$. Similarly, if $\nu = \lambda \,[+]\, \mu$ then $c^{\nu}_{\lambda\mu} = 1$. □

When $\mathbf{F} = K$, every $S_K(\sigma, \lambda)$ is irreducible, but not for $\mathbf{F} = F$. To find irreducible FGL_n-modules, we define

$$D_{\mathbf{F}}(\sigma, \lambda) \coloneqq S_{\mathbf{F}}(\sigma, \lambda)/S_{\mathbf{F}}(\sigma, \lambda)^{\max}.$$

Clearly $D_{\mathbf{F}}(\sigma, \lambda)$ is irreducible. When $\mathbf{F} = K$, $S_K(\sigma, \lambda) \cong D_K(\sigma, \lambda)$ and we have no interest on it. We immediately have

Lemma 2.26. [J, 7.24] *Let* $\lambda \models k$ *and* $\nu \models n$

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(1) $S_{\mathbf{F}}(\sigma,\lambda)E^{+}_{\mathbf{F}}(\nu)$ *and* $D_{\mathbf{F}}(\sigma,\lambda)E^{+}_{\mathbf{F}}(\nu)$ *are both zero unless* $\nu = d\mu$ *for some* $\mu \models k$ *and* $\mu' \trianglerighteq \lambda$.

(2) $S_{\mathbf{F}}(\sigma,\lambda)E^{+}_{\mathbf{F}}(d\lambda')$ *and* $D_{\mathbf{F}}(\sigma,\lambda)E^{+}_{\mathbf{F}}(d\lambda')$ *are both one-dimensional spaces.*

Lemma 2.27.

(1) *If* $\mu \models n$ *is a rearrangement of partition* $\lambda \vdash n$, *i.e.* $\mu'' = \lambda$, *then* $D_{\mathbf{F}}(\sigma,\mu) \cong D_{\mathbf{F}}(\sigma,\lambda)$.

(2) *If* $\sigma,\tau \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = \deg(\tau)$, *then* $D_{\mathbf{F}}(\sigma,\lambda) \cong D_{\mathbf{F}}(\tau,\lambda)$ *if* σ, τ *are* p*-conjugate to each other.*

Proof. This follows from Lemma 2.21 and the definition of $D_{\mathbf{F}}(\sigma,\lambda)$. □

Theorem 2.28. [J, 7.25] *Let* $\lambda,\mu \vdash k$.

(1) *If* $\lambda \neq \mu$ *then* $D_{\mathbf{F}}(\sigma,\lambda)$ *is not isomorphic to* $D_{\mathbf{F}}(\sigma,\mu)$.

(2) *If some composition factor of* $S_{\mathbf{F}}(\sigma,\lambda)$ *is isomorphic to* $D_{\mathbf{F}}(\sigma,\mu)$, *then* $\mu \trianglerighteq \lambda$.

(3) *Precisely one composition factor of* $S_{\mathbf{F}}(\sigma,\lambda)$ *is isomorphic to* $D_{\mathbf{F}}(\sigma,\lambda)$.

In fact, every composition factor of $S_{\mathbf{F}}(\sigma,\lambda)$ is isomorphic to some $D_{\mathbf{F}}(\sigma,\mu)$. To prove this, we need the submodule theorem (Theorem 2.30) below.

For each $\mathbf{F}GL_n$-module M, define (temporary here) an $\mathbf{F}GL_n$-module ${}^{d}M$ with the same vector space as M, and the action $m * g = m((g^{-1})^T)$. Then if M is irreducible, then ${}^{d}M$ is isomorphic to the dual of M since their Brauer characters are complex conjugate to each other. But from $\psi_{\sigma^{-1}}$ is complex conjugate to ψ_σ, we also have $M_{\mathbf{F}}(\sigma^{-1},(1))$ is isomorphic to the dual of $M_{\mathbf{F}}(\sigma,(1))$, hence

$${}^{d}M_{\mathbf{F}}(\sigma,(1)) \cong M_{\mathbf{F}}(\sigma^{-1},(1))$$

A proof [J, 7.25] show that the map $M \mapsto {}^{d}M$ commutes with Harish-Chandra induction and the action of $\mathcal{H}_k$, hence we have

Lemma 2.29. [J, 7.32, 7.33] *Let* $\lambda \models k$. *Then*

(1) ${}^{d}M_{\mathbf{F}}(\sigma,(\lambda)) \cong M_{\mathbf{F}}(\sigma^{-1},(\lambda))$

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(2) *$M_{\mathbf{F}}(\sigma, (\lambda))$ is isomorphic to the dual of $M_{\mathbf{F}}(\sigma^{-1}, (\lambda))$.*

(3) ${}^{d}S_{\mathbf{F}}(\sigma, (\lambda)) \cong S_{\mathbf{F}}(\sigma^{-1}, (\lambda))$

Then we obtain a non-singular bilinear map $f : M_{\mathbf{F}}(\sigma, \lambda) \times M_{\mathbf{F}}(\sigma^{-1}, \lambda) \to \mathbf{F}$ satisfying $f(m_1 g, m_2) = f(m_1, m_2 g^{-1})$ for $m_1 \in M_{\mathbf{F}}(\sigma, \lambda)$, $m_2 \in M_{\mathbf{F}}(\sigma^{-1}, \lambda)$ and $g \in GL_n$. Define

$$S_{\mathbf{F}}(\sigma^{-1}, \lambda)^{\perp} = \{m_1 \in M_{\mathbf{F}}(\sigma, \lambda) \mid f(m_1, m_2) = 0 \text{ for all } m_2 \in S_{\mathbf{F}}(\sigma^{-1}, \lambda)\}$$

Theorem 2.30 (Submodule Theorem)**.** [J, 7.34] *Let $\lambda \models k$. For every $X \subset M_{\mathbf{F}}(\sigma, \lambda)$, either $X \supset S_{\mathbf{F}}(\sigma, \lambda)$ or $X \subset S_{\mathbf{F}}(\sigma^{-1}, \lambda)^{\perp}$*

Theorem 2.31. [J, 7.35] *Let $\lambda \vdash k$.*

(1) *For every irreducible $\mathbf{F}GL_n$-module D, the composition multiplicity of D in the module $S_{\mathbf{F}}(\sigma, \lambda)^{\max}$ is at most that in $M_{\mathbf{F}}(\sigma, \lambda)/S_{\mathbf{F}}(\sigma, \lambda)$.*

(2) *Every composition factor of $S_{\mathbf{F}}(\sigma, \lambda)^{\max}$ is isomorphic to some $D_{\mathbf{F}}(\sigma, \mu)$ with $\mu \vdash k$, $\mu \trianglerighteq \lambda$, $\mu \neq \lambda$.*

(3) *We have $S_{\mathbf{F}}(\sigma, \lambda)^{\max} = S_{\mathbf{F}}(\sigma, \lambda) \cap S_{\mathbf{F}}(\sigma^{-1}, \lambda)^{\perp}$.*

2.7 The Module $S_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$ and $D_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$

In Definition 1.7 we have defined the n-admissable pair $(\underline{\sigma}, \underline{\lambda})$. Given $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a) \in (\overline{\mathbb{F}}_q^{\times})^a$, $\deg(\sigma_i) = d_i$, and $\lambda^{(i)} \models k_i$, $\underline{k} = (k_1, \cdots, k_a)$, $n = \sum_{i=1}^{a} k_i d_i$, define

$$S_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}) \coloneqq S_{\mathbf{F}}(\sigma_1, \lambda^{(1)}) \circ \cdots \circ S_{\mathbf{F}}(\sigma_a, \lambda^{(a)})$$

as an $\mathbf{F}GL_n$-module, where $\circ$ is the Harish-Chandra induction. Similarly,

$$D_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}) \coloneqq D_{\mathbf{F}}(\sigma_1, \lambda^{(1)}) \circ \cdots \circ D_{\mathbf{F}}(\sigma_a, \lambda^{(a)})$$

For $\mathbf{F} = K$, we are going to find the complete set of non-isomorphic irreducible KGL_n-module $S_K(\underline{\sigma}, \underline{\lambda})$. By Lemma 2.21(1), we may assume $\lambda^{(i)}$ are partitions. Also by Lemma 2.21(2), we may assume $\underline{\sigma}$ is non-repeated, otherwise we may use the

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Littlewood-Richardson rule to decompose $S_K(\sigma, \lambda) \circ S_K(\sigma, \mu)$. Hence $(\underline{\sigma}, \underline{\lambda})$ is an n-admissable pair. Since the Harish-Chandra induction is commutative, $S_K(\underline{\sigma}, \underline{\lambda}) \cong S_K(\underline{\tau}, \underline{\nu})$ if $(\underline{\sigma}, \underline{\lambda})$ and $(\underline{\tau}, \underline{\nu})$ are equivalent. Therefore

$$L_K([(\underline{\sigma}, \underline{\lambda})]) \coloneqq S_K(\underline{\sigma}, \underline{\lambda}), \quad [(\underline{\sigma}, \underline{\lambda})] \in \Sigma_K$$

is well-defined (up to isomorphism), and we shall write $L_K(\underline{\sigma}, \underline{\lambda})$ for simplicity. Then we have

Theorem 2.32. [D1, Theorem 4.7] *The set $\{L_K(\underline{\sigma}, \underline{\lambda}) \mid [(\underline{\sigma}, \underline{\lambda})] \in \Sigma_K\}$ is a complete set of representative of non-isomorphic irreducible KGL_n-module.*

For $\mathbf{F} = F$, we are going to find the complete set of non-isomorphic irreducible FGL_n-module. By Lemma 2.27 and a similar argument to ordinary case above,

$$L_F([(\underline{\sigma}, \underline{\lambda})]) \coloneqq D_F(\underline{\sigma}, \underline{\lambda}), \quad [(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$$

is well-defined (up to isomorphism), and we shall write $L_F(\underline{\sigma}, \underline{\lambda})$ for simplicity. However, $L_F(\underline{\sigma}, \underline{\lambda})$ may not be pairwise non-isomorphic, even not irreducible. A good guess is that the set $\{L_F(\underline{\sigma}, \underline{\lambda}) \mid [(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F\}$ serves our requirement (and this is indeed correct), but it is not easy to prove this fact directly, which is the main goal of [D1].

Theorem 2.33. [D1, Corollary 5.3] *The set $\{L_F(\underline{\sigma}, \underline{\lambda}) \mid [(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F\}$ is a complete set of representative of non-isomorphic irreducible FGL_n-module.*

Finally, we put a lemma here for later usage, whose proof depends on the construction of the modules.

Lemma 2.34. *Let $[(\underline{\sigma}, \underline{\lambda})]$ be an n-admissible symbol. For $\tau \in O_{p'}(\mathbb{F}_q^\times)$, let $\tau\underline{\sigma} = (\tau\sigma_1, \cdots, \tau\sigma_a)$. Then*

$$L_F(\underline{\sigma}, \underline{\lambda}) \otimes L_F(\tau, (n)) \cong L_F(\tau\underline{\sigma}, \underline{\lambda}).$$

Proof. Note that $[(\tau\underline{\sigma}, \underline{\lambda})]$ is an n-admissible symbol by Proposition 4.6 (which is independent from here). We prove it along the construction of $L_F(\underline{\sigma}, \underline{\lambda})$.

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First, the character of the irreducible KGL_d-module $M_K(\sigma,(1))$ are given explicitly in [J, (3.1)], involving a function $\langle x,y\rangle_d$ with the property

$$\langle x_1x_2,y\rangle_d=\langle x_1,y\rangle_d\cdot\langle x_2,y\rangle_d$$

hence $\langle\sigma\tau,y\rangle_d=\langle\sigma,y\rangle_d\cdot\langle\tau,y\rangle_d$ and $\langle\tau,y\rangle_d^{q-1}=1$. It is known that $L_K(\tau,(n))$ is one dimensional and have the character $y\mapsto\langle\tau,y\rangle_d$. Then

$$\begin{array}{rl}
M_K(\sigma,(1))\otimes L_K(\tau,(n))\cong M_K(\sigma\tau,(1)) & \text{ordinary character}\\
M_F(\sigma,(1))\otimes L_F(\tau,(n))\cong M_F(\sigma\tau,(1)) & \text{reduction modulo } p\\
M_F(\sigma,(1^k))\otimes L_F(\tau,(n))\cong M_F(\sigma\tau,(1^k)) & \text{Harish-Chandra induction}\\
M_F(\sigma,\lambda)\otimes L_F(\tau,(n))\cong M_F(\sigma\tau,\lambda) & x_\lambda\in\mathcal{H}_k\\
S_F(\sigma,\lambda)\otimes L_F(\tau,(n))\cong S_F(\sigma\tau,\lambda) & E_F^+(d\lambda')\\
D_F(\sigma,\lambda)\otimes L_F(\tau,(n))\cong D_F(\sigma\tau,\lambda) & \text{subquotient}\\
L_F(\underline{\sigma},\underline{\lambda})\otimes L_F(\tau,(n))\cong L_F(\tau\underline{\sigma},\underline{\lambda}) & \text{Harish-Chandra induction}
\end{array}$$

which proves the lemma. $\square$

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3 Clifford Theory

In section 3 to 5, we are going to prove Theorem 5.2, a theorem of Kleshchev-Tiep. The original proof is in [K].

Clifford theory gives the information about the restriction to a normal subgroup, especially when the quotient is cyclic. Observe that $GL_n/SL_n \cong C_{q-1}$. A good reference to Clifford theory see [F, III].

Let $\mathbf{F} = K$ or F, sufficiently large for any group we have considered in this section.

Theorem 3.1 (Clifford)**.** *Given S a normal subgroup of G, $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. Then*

(1) *$V{\downarrow_S}$ is completely reducible. That is,*

$$V{\downarrow_S} = \bigoplus_{i=1}^{t} W_i$$

where W_i are irreducible $\mathbf{F}S$-modules (but not necessary non-isomorphic.)

(2) *There is an $e \in \mathbb{N}$ such that*

$$V{\downarrow_S} = e\left(\bigoplus_{j=1}^{u} W_j\right)$$

where W_j are non-isomorphic irreducible $\mathbf{F}S$-modules.

(3) *W_j are G-conjugate to each other. In fact, $W_j = {}^{g_j}W_1$, where $\{1 = g_1, \cdots, g_u\}$ is a set of representative of G/I, with $I := \mathrm{Stab}_G(W_1)$ called the inertia group of W_1 in G.*

(4) *There is some $\tilde{W}_1 \in \mathrm{Irr}_{\mathbf{F}}(I)$ such that $\tilde{W}_1{\downarrow_S} = eW_1$ and $V \cong \tilde{W}_1{\uparrow^G}$. This $\tilde{W}_1$ is called an isotypic component of V with respect to W_1.*

Denote $\kappa_S^G(V) := t$ the branching number of V restricted from G to S. If G is clear we also write $\kappa_S(V)$ instead. Since $\dim(W_j) = \dim(W_1)$ for all j, we may also write $\kappa_S^G(V) = \dim(V)/\dim(W_1)$.

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3.1 Cyclic Quotient

If $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$, then $\mathrm{infl}_{G/S}^{G}(L) \in \mathrm{Irr}_{\mathbf{F}}(G)$, which we also denoted L. Hence when we write $V \otimes L$ for $\mathbf{F}G$-module V and $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$, it means $V \otimes \mathrm{infl}_{G/S}^{G}(L)$. Note that both L have dimension 1 over $\mathbf{F}$.

Lemma 3.2. [F, III, 2.14] *Let $S \trianglelefteq G$, G/S be cyclic, and $W \in \mathrm{Irr}_{\mathbf{F}}(G)$. Assume $I = \mathrm{Stab}_G(W) = G$. Then*

(1) *There is some $\mathbf{F}G$-module V satisfied $V{\downarrow_S} \cong W$.*

(2) *For any $U \in \mathrm{Irr}_{\mathbf{F}}(G)$, if W is an irreducible summand of $U{\downarrow_S}$, then $U{\downarrow_S} \cong W$, and $U \cong V \otimes L$ for some $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$.*

Proof. (1) First suppose that $\mathbf{F}$ is algebraically closed. Choose $x \in G$ such that x and S span G, then $x^n \in S$ for $n = (G : S)$. Now consider the representation ρ of S with underlying module W. Since $\mathrm{Stab}_G(W) = G$, ${}^x\rho \cong \rho$, that is, $\rho(xyx^{-1}) = f\rho(y)f^{-1}$ for some $f \in GL(W)$ and all $y \in S$. Hence $f^n\rho(y)f^{-n} = \rho(x^n y x^{-n}) = \rho(x^n)\rho(y)\rho(x^n)^{-1}$, and by Schur's lemma $f^n = c\rho(x^n)$ for some $c \in \mathbf{F}$. Since $\mathbf{F}$ is algebraically closed, there is some $c_1 \in \mathbf{F}$ such that $c_1^n = c$. Define $\rho(x) \coloneqq c_1^{-1}f$, then ρ extends to a representation of G. Let V be the underlying module, then we have $V{\downarrow_S} = W$.

Now assume $\mathbf{F}$ is sufficiently large for G and S. Write $\overline{\mathbf{F}}$ algebraically closure of $\mathbf{F}$. Then every $\overline{\mathbf{F}}G$-module $V_{\overline{\mathbf{F}}}$ is realizable on $\mathbf{F}$, that is, $V_{\overline{\mathbf{F}}} \cong \overline{\mathbf{F}}G \otimes_{\mathbf{F}G} V$ for some $\mathbf{F}G$-module V. By previous statement, there is some $\overline{\mathbf{F}}G$-module $V_{\overline{\mathbf{F}}}$ satisfied $V_{\overline{\mathbf{F}}}{\downarrow_S} = W_{\overline{\mathbf{F}}} \coloneqq \overline{\mathbf{F}}S \otimes_{\mathbf{F}S} W$. Then we have $V{\downarrow_S} \cong W$.

(2) Consider $W{\uparrow^G} \cong W \otimes_{\mathbf{F}S} \mathbf{F}G \cong V \otimes \mathbf{F}(G/S)$. By Frobenius reciprocity U is an irreducible summand of $W{\uparrow^G}$, and thus $U \cong V \otimes L$ for some $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$. Since G/S is cyclic, $\dim_{\mathbf{F}} L = 1$, so $\dim_{\mathbf{F}} U = \dim_{\mathbf{F}} V = \dim_{\mathbf{F}} W$ and $U{\downarrow_S} \cong W$. □

Lemma 3.3. *If G/S is cyclic, then $e = 1$, that is, $V{\downarrow_S}$ is multiplicity free for any $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. Hence $\kappa_S^G(V) = (G : I)$.*

Proof. Assume not, there is some $V \in \mathrm{Irr}_{\mathbf{F}}(G)$ with $e > 1$. Given $W \in \mathrm{Irr}_{\mathbf{F}}(S)$ an

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irreducible summand of $V{\downarrow_S}$, consider $\tilde{W} \in \mathrm{Irr}_{\mathbf{F}}(I)$ where $\tilde{W}{\downarrow_S} = eW$. But by Lemma 3.2 (ii), $\tilde{W}{\downarrow_S} \cong W$, which is a contradiction. The second statement is clear. $\square$

Lemma 3.4. *Let $S \trianglelefteq H \trianglelefteq G$ and $S \trianglelefteq G$, G/S be cyclic, and $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. Let U be an irreducible component of $V{\downarrow_H}$. Then $\kappa_S^G(V) = \kappa_H^G(V) \cdot \kappa_S^H(U)$.*

Proof. Let $\kappa_S^G(V) = t$ and $\kappa_H^G(V) = t_1$. Then

$$V{\downarrow_S} = \bigoplus_{i=1}^{t} W_i \qquad W_i \in \mathrm{Irr}_F(S)$$
$$V{\downarrow_H} = \bigoplus_{j=1}^{t_1} U_j \qquad U_j \in \mathrm{Irr}_F(H)$$

Since $V{\downarrow_S} = (V{\downarrow_H}){\downarrow_S}$, we have $\kappa_S^G(V) = \sum_{j=1}^{t_1} \kappa_S^H(U_j)$, hence it suffices to prove $\kappa_S^H(U_j) = \kappa_S^H(U_1)$ for all j. We may assume W_1 is an irreducible component of $U_1{\downarrow_S}$. Then $\kappa_S^H(U_1) = (H : I_1)$, where $I_1 = \mathrm{Stab}_H(W_1)$. Now for any W_i we have $W_i = {}^{g_i}W_1$ for some $g_i \in G$ and

$$I_i = \mathrm{Stab}_H(W_i) = H \cap g_i \,\mathrm{Stab}_G(W_i) g_i^{-1} = g_i \left(H \cap \mathrm{Stab}_G(W_i)\right) g_i^{-1} = g_i I_1 g_i^{-1}$$

since $H \trianglelefteq G$. Therefore $(H : I_i) = (H : I_1)$ and the lemma follows. $\square$

The following lemma is a crucial part of proof of Kleshchev-Tiep's theorem, which gives a lower bound of $\kappa_S^G(V)$. Note that if $\mathbf{F} = K$, then any cyclic group is a p'-group conventionally.

Lemma 3.5. *Let $S \trianglelefteq G$ with G/S a cyclic p'-group, and $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. Then*

(1) *We have $\kappa_S^G(V) = \#\{L \in \mathrm{Irr}_{\mathbf{F}}(G/S) \mid V \cong V \otimes L\}$.*

(2) *If $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$ and $V \cong V \otimes L$ then $\kappa_S^G(V) \geq (G : \ker L)$.*

Proof. (1) Let $J = \{L \in \mathrm{Irr}_{\mathbf{F}}(G/S) \mid V \cong V \otimes L\}$. Pick $W \in \mathrm{Irr}_{\mathbf{F}}(S)$, $W \mid V{\downarrow_S}$, $I = \mathrm{Stab}_G(W)$, then there is some $\tilde{W} \in \mathrm{Irr}_{\mathbf{F}}(G)$ such that $\tilde{W}{\downarrow_S} = W$ and $\tilde{W}{\uparrow^G} \cong V$. Now If $L \in \mathrm{Irr}_{\mathbf{F}}(G/I)$, then

$$V \otimes L = \tilde{W}{\uparrow^G} \otimes L \cong (\tilde{W} \otimes L{\downarrow_I}){\uparrow^G} \cong \tilde{W}{\uparrow^G} = V$$

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and thus $|J| \geq \kappa_S^G(V)$.

On the other hand, let $L \in \mathrm{Irr}_{\mathbf{F}}(G/S)$ and $V \cong V \otimes L$. Then $\tilde{W}{\uparrow}^G \cong (\tilde{W} \otimes L{\downarrow}_I){\uparrow}^G$. Since there are only one irreducible summand of $V{\downarrow}_I$ containing W when restricts to S, $\tilde{W} \cong \tilde{W} \otimes L\downarrow_I$, and hence $\tilde{W}^* \otimes \tilde{W} \cong \tilde{W}^* \otimes \tilde{W} \otimes L\downarrow_I$. By $\tilde{W}^* \otimes \tilde{W} = \mathrm{Hom}_{\mathbf{F}I}(\tilde{W}, \tilde{W}) \cong \mathrm{id}_I$ we have $\mathrm{id}_I \cong L{\downarrow}_I$. Hence $L \in \mathrm{Irr}_{\mathbf{F}}(G/I)$ and $|J| \leq \kappa_S^G(V)$.

For (2), since $I \leq \ker L$ we have $\kappa_S^G(V) = (G : I) \geq (G : \ker L)$. □

In general, $\kappa_S^G(V)$ splits into two parts.

Lemma 3.6. *Let r be a prime, $S \trianglelefteq G$ with G/S a cyclic group, and $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. Consider $S \leq A, B \leq G$ where $A/S \coloneqq O_r(G/S)$ and $B/S \coloneqq O_{r'}(G/S)$. Take U_A and U_B be an irreducible summand of $V{\downarrow}_A$ and $V{\downarrow}_B$ respectively. Then*

(1) $\kappa_S^G(V) = \kappa_A^G(V) \cdot \kappa_B^G(V)$.

(2) $\kappa_A^G(V) = \kappa_S^B(U_B)$, $\kappa_B^G(V) = \kappa_S^A(U_A)$.

Proof. Pick W an irreducible summand of $V{\downarrow}_S$, and let $I = \mathrm{Stab}_G(W)$, $I_A = \mathrm{Stab}_A(W)$, $I_B = \mathrm{Stab}_B(W)$. If one can prove $I/S = I_A/S \times I_B/S$, then

$$\begin{aligned}\kappa_S^G(V) &= (G : I) = (G/S : I/S) = (A/S : I_A/S) \cdot (B/S : I_B/S) \\ &= (A : I_A) \cdot (B : I_B) = \kappa_S^A(U_A) \cdot \kappa_S^B(U_B)\end{aligned}$$

and thus both (1) and (2) follows from Lemma 3.4.

By definition of stabilizer group $I/S \geq I_A/S \times I_B/S$. Now given $x \in I$, write $xS = yS \cdot zS$, where yS, zS are r-part and r'-part of xS, respectively. Then by Proposition 1.3 both yS and zS are power of xS, hence stabilize W, which gives $I/S \leq I_A/S \times I_B/S$. □

3.2 Direct Product

We put the following lemma here for later usage when applying induction on n for GL_n and related subgroups. Note that $GL_n/SL_n \cong \mathbb{F}_q^\times$ is independent of n.

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Lemma 3.7. *Let $G = G_1 \times G_2$ and $V \in \mathrm{Irr}_{\mathbf{F}}(G)$. For each $i = 1, 2$, $S_i \trianglelefteq G_i$, $G_i/S_i := \langle \bar{x}_i \rangle \cong C_r$ for some $r \in \mathbb{N}$. Consider a subgroup H satisfies $S_1 \times S_2 \leq H \leq G$ and $H/(S_1 \times S_2) := \langle \bar{x}_1 \bar{x}_2 \rangle$. Then $\kappa_H(V) = \gcd(\kappa_1, \kappa_2)$, where $\kappa_1 = \kappa_{S_1 \times G_2}(V)$ and $\kappa_2 = \kappa_{G_1 \times S_2}(V)$.*

Proof. Throughout the proof, $i = 1, 2$.

Write $V = V_1 \otimes V_2$ for $V_i \in \mathrm{Irr}_{\mathbf{F}}(G_i)$, and pick an $W_i \in \mathrm{Irr}_{\mathbf{F}}(S_i)$ such that $W_i \mid (V_i){\downarrow_{S_i}}$. Let $I_i = \mathrm{Stab}_{G_i}(W_i)$, then $\kappa_i = (G_i : I_i)$. Since $W_1 \otimes W_2 \mid V{\downarrow_{S_1 \times S_2}}$, there is some $U \in \mathrm{Irr}_{\mathbf{F}}(H)$, $W_1 \otimes W_2 \mid U{\downarrow_{S_1 \times S_2}}$ and $U \mid V{\downarrow_H}$. Let $J = \mathrm{Stab}_H(W_1 \otimes W_2)$, then $\kappa^H_{S_1 \times S_2}(U) = (H : J)$.

Now we are going to find $m = (H : J)$. Choose $x_i \in G_i$ such that $x_i S_i = \bar{x}_i$, then $H = \langle S_1, S_2, x_1 x_2 \rangle$ and $(x_1 x_2)^m \in J$. From $W_1 \otimes W_2 = (x_1 x_2)^m (W_1 \otimes W_2) = (x_1^m W_1) \otimes (x_2^m W_2)$, we have $x_i^m W_i \cong W_i$ and $x_i^m \in I_i$, hence m is a multiple of both $(G_i : I_i) = \kappa_i$. This proof is clearly invertible, hence $(H : J) = \mathrm{lcm}(\kappa_1, \kappa_2)$.

Finally we consider the dimension as vector space over F,

$$\begin{aligned}\kappa_H(V) &= \frac{\dim(V)}{\dim(U)} = \frac{\dim(V)/\dim(W_1 \otimes W_2)}{\dim(U)/\dim(W_1 \otimes W_2)} \\ &= \frac{(G_1 : I_1) \cdot (G_2 : I_2)}{(H : J)} = \frac{\kappa_1 \cdot \kappa_2}{\mathrm{lcm}(\kappa_1, \kappa_2)} = \gcd(\kappa_1, \kappa_2)\end{aligned}$$ □

3.3 Classification of Irreducible Representations of SL_n

Lemma 3.8. *Let $\mathbf{F} = F$. If G/S is an p-group, and V_1, $V_2 \in \mathrm{Irr}_F(G)$ have common irreducible summand W (up to isomorphism) in their restrictions $(V_1){\downarrow_S}$, $(V_2){\downarrow_S}$, then $V_1 \cong V_2$.*

Proof. First assume $G/S \cong C_p$. Consider $I = \mathrm{Stab}_G W$. If $I = G$, then by Lemma 3.2, $V_1 \cong V_2 \otimes L$ for $L \in \mathrm{Irr}_F(G/S)$, but the only irreducible FC_p-module over F of characteristic p is the trivial module. Now if $I = S$, then by Theorem 3.1(4) we have $\tilde{W} = W$ and hence $V_1 \cong W{\uparrow^G} \cong V_2$.

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For the induction step, Take $S \leq H \leq G$ such that H/S is a maximal subgroup of G/S. Since both $((V_1)\downarrow_H)\downarrow_S$ and $((V_2)\downarrow_H)\downarrow_S$ have common irreducible summand W, by induction hypothesis $(V_1)\downarrow_H \cong (V_2)\downarrow_H$, again by induction hypothesis $V_1 \cong V_2$. □

Lemma 3.9. *Let $S \trianglelefteq G$ with G/S cyclic, $S \leq A \leq G$ with $A/S \coloneqq O_p(G/S)$, and V_1, V_2 irreducible $\mathbf{F}G$-modules. If W is an irreducible summand of both $(V_1)\downarrow_S$ and $(V_2)\downarrow_S$, then there exists $M \in \mathrm{Irr}_{\mathbf{F}}(G/A)$ such that $V_2 = V_1 \otimes M$.*

Proof. First assume $\mathbf{F} = F$, p not dividing $|G|$ or $\mathbf{F} = K$, then $A = S$. Let $I = \mathrm{Stab}_G(W)$ be the inertia group, then there are some $\mathbf{F}I$ module U_i satisfy $V_i \cong (U_i)\uparrow^G$ and $W = (U_i)\downarrow_S$, $i = 1, 2$. Hence by Lemma 3.2, $U_2 = U_1 \otimes N$ for some irreducible $\mathbf{F}(I/S)$-module N. But since G/S is cyclic, N is G/S-invariant and extends to an irreducible $\mathbf{F}(G/S)$-module L. Hence

$$V_2 \cong (U_1 \otimes N)\uparrow^G = (U_1 \otimes (L)\downarrow_I) \cong (U_1)\uparrow^G \otimes L = V_1 \otimes L$$

Now assume p divide $|G|$. For $i = 1, 2$, there is some $W_i \in \mathrm{Irr}_{\mathbf{F}}(A)$ such that $W \mid (W_i)\downarrow_S$ and $W_i \mid (V_i)\downarrow_A$. By Lemma 3.8 we have $W_1 \cong W_2$. Replacing A by S returns to the first case. □

Proposition 3.10. *Let $S \trianglelefteq G$ with G/S cyclic, and $S \leq A \leq G$, $A/S \coloneqq O_p(G/S)$. Define an equivalence relation as following: if $V, U \in \mathrm{Irr}_{\mathbf{F}}(G)$, then $V \sim U$ if $V \cong U \otimes L$ for some $L \in \mathrm{Irr}_{\mathbf{F}}(G/A)$. Pick $V_1, V_2, \cdots, V_m$ as a complete set of representative of equivalence classes of $\mathrm{Irr}_{\mathbf{F}}(G)$, and take restriction to S,*

$$(V_i)\downarrow_S = \bigoplus_{j=1}^{t_i} W_{ij}$$

where $t_i = \kappa_S^G(V_i)$. Then

$$\{W_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq t_i\}$$

is a complete set of representative of the isomorphism classes of the irreducible $\mathbf{F}S$-module.

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Proof. Let $W \in \mathrm{Irr}_{\mathbf{F}}(S)$. Then $W \mid V{\downarrow_S}$ for some $V \in \mathrm{Irr}_{\mathbf{F}}(G)$, and $V \sim V_i$ for some i gives $W \cong W_{ij}$ for some j. Now if $W_{ij} = W_{kl}$, then by Lemma 3.9, $V_i = V_k \otimes L$ for some $L \in \mathrm{Irr}_{\mathbf{F}}(G/A)$, i.e. $i = k$, and thus $j = l$ by Lemma 3.3. □

Therefore, we have the complete list of irreducible representation of SL_n.

Proposition 3.11. *Let $W \in \mathrm{Irr}_{\mathbf{F}}(SL_n)$. Then*

(1) *$W \mid L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}){\downarrow_{SL_n}}$ for some $L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}) \in \mathrm{Irr}_{\mathbf{F}}(GL_n)$.*

(2) *If $W \mid L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}){\downarrow_{SL_n}}$ and $W \mid L_{\mathbf{F}}(\underline{\tau}, \underline{\nu}){\downarrow_{SL_n}}$ for both $L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$ and $L_{\mathbf{F}}(\underline{\tau}, \underline{\nu}) \in \mathrm{Irr}_{\mathbf{F}}(GL_n)$, then $[(\underline{\sigma}, \underline{\lambda})] = \rho \cdot [(\underline{\tau}, \underline{\nu})]$ for some $\rho \in O_{p'}(\mathbb{F}_q^\times)$.*

(3) *Define $[(\underline{\sigma}, \underline{\lambda})] \sim [(\underline{\tau}, \underline{\nu})]$ over $\Sigma_{\mathbf{F}}$, if $[(\underline{\sigma}, \underline{\lambda})] = \rho \cdot [(\underline{\tau}, \underline{\nu})]$ for some $\rho \in O_{p'}(\mathbb{F}_q^\times)$. Let $\Sigma_{\mathbf{F}}/{\sim}$ be the set of equivalence classes, and write*

$$L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}) = \bigoplus_{j=1}^{\kappa_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})} Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; j)$$

where $\kappa_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}) = \kappa_{SL_n}^{GL_n}(L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}))$. Then

$$\{Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; j) \mid 1 \le j \le \kappa_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}), [(\underline{\sigma}, \underline{\lambda})] \in \Sigma/{\sim}\}$$

is a complete set of representative of the isomorphism classes of the irreducible $\mathbf{F}SL_n$-module.

Proof. These follow from Lemma 2.34 and Proposition 3.10. Note that $\rho \cdot [(\underline{\tau}, \underline{\nu})]$ is defined in Proposition 4.6, which is independent from here. □

3.4 G-tile and S-tile

Definition 3.12. Let $S \trianglelefteq G$.

(1) For V_1, $V_2 \in \mathrm{Irr}_{\mathbf{F}}(G)$, define $V_1 \sim_G V_2$ if $(V_2){\downarrow_S} \cong (V_1){\downarrow_S}$.

(2) For W_1, $W_2 \in \mathrm{Irr}_{\mathbf{F}}(S)$, define $W_1 \sim_S W_2$ if $W_2 \cong {}^gW_1$ for some $g \in G$.

(3) If $W \mid V{\downarrow_S}$, let $\eta_S^G(W) := \#[V]_G$ be the branching number of W induced from S to G.

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(4) Let $V \in \mathrm{Irr}_K(G)$ and $D \in \mathrm{Irr}_F(G)$. The submatrix of the decomposition matrix of G with labels $[V]_G \times [D]_G$ is called a G-tile.

(5) Let $W \in \mathrm{Irr}_K(S)$ and $E \in \mathrm{Irr}_F(S)$. The submatrix of the decomposition matrix of S with labels $[W]_S \times [E]_S$ is called a S-tile.

Proposition 3.13. *Write $W_i \mid (V_i){\downarrow}_S^G$, $i = 1, 2$.*

(1) $\kappa_S^G(V_1) = \#[W_1]_S$.

(2) $[V_1]_G = [V_2]_G$ *if and only if* $[W_1]_S = [W_2]_S$.

(3) *There are canonical isomorphic maps $[V]_G \mapsto [W]_S$, $[D]_G \mapsto [E]_S$, and G-tile to S-tile.*

Proof. (1)(2) are trivial from Clifford theorem, and (2) implies (3). $\square$

Therefore, one may discuss the relation of the decomposition map of G and S tilewise.

$$\begin{array}{c||cccc} & D_1 & D_2 & \cdots & D_{\eta^*} \\ \hline\hline V_1 & a_{11} & a_{12} & \cdots & a_{1\eta^*} \\ V_2 & a_{21} & a_{22} & \cdots & a_{2\eta^*} \\ \vdots & & & \vdots & \\ V_\eta & a_{\eta 1} & a_{\eta 2} & \cdots & a_{\eta\eta^*} \end{array} \qquad \begin{array}{c||c} & (D_1){\downarrow}_S \\ \hline\hline (V_1){\downarrow}_S & c \end{array} \qquad \begin{array}{c||ccc} & E_1 & \cdots & E_{\kappa^*} \\ \hline\hline W_1 & b_{11} & \cdots & b_{1\kappa^*} \\ \vdots & & \vdots & \\ W_\kappa & b_{\kappa 1} & \cdots & b_{\kappa\kappa^*} \end{array}$$

G-tile mid-tile S-tile

Consider the case G/S is cyclic. Pick some $[V]_G$ and $[D]_G$, there are corresponding $[W]_S$ and $[E]_S$. Let $\kappa = \kappa_S^G(V)$, $\kappa^* = \kappa_S^G(D)$, $\eta = \eta_S^G(W)$, and $\eta^* = \eta_S^G(E)$. For $V_i \in [V]_G$, $D_{i^*} \in [D]_G$, let a_{i,i^*} be the multiplicity of D_{i^*} in $\overline{V_i}$; for $W_j \in [W]_S$, $E_{j^*} \in [E]_S$, let b_{j,j^*} be the multiplicity of E_{i^*} in $\overline{W_j}$. Let $\mathrm{row}^\Sigma(V_i) = \sum_{i^*=1}^{\eta^*} a_{i,i^*}$, the row sum of the G-tile. Similarly for $\mathrm{col}^\Sigma(D_{i^*})$, $\mathrm{row}^\Sigma(W_j)$ and $\mathrm{col}^\Sigma(E_{j^*})$.

Proposition 3.14. *Let G/S be cyclic.*

(1) *There is some* $c = \mathrm{row}^\Sigma(V_i) = \mathrm{col}^\Sigma(E_{j^*})$

(2) *There is some* $\pi \in \mathfrak{S}_{\eta^*}$, $a_{i,i^*} = a_{1,i^*\pi}$. *Similar for* b_{j,j^*}.

(3) $\kappa \cdot \mathrm{row}^\Sigma(W_j) = \kappa^* \cdot \mathrm{row}^\Sigma(V_i)$.

(4) *If* $c = 1$, *then* $\kappa \mid \kappa^*$, *and* $\mathrm{row}^\Sigma(W_j) = \kappa^*/\kappa$.

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Proof. In the Grothendieck group of KG-modules and KS-modules, consider

$$\overline{V}_i = \sum_{i^*=1}^{\eta^*} a_{i,i^*} D_{i^*}, \qquad \overline{W}_j = \sum_{j^*=1}^{\kappa^*} b_{j,j^*} E_{j^*}$$

Then we have both

$$(\overline{V}_i){\downarrow_S} = \left(\sum_{i^*=1}^{\eta^*} a_{i,i^*} \right) (D_1){\downarrow_S} = \mathrm{row}^{\Sigma}(V_i) \cdot \sum_{j^*=1}^{\kappa^*} E_{j^*}$$

$$(\overline{V}_i){\downarrow_S} = \sum_{j=1}^{\kappa} \overline{W}_j = \sum_{j=1}^{\kappa} \sum_{j^*=1}^{\kappa^*} b_{j,j^*} E_{j^*} = \mathrm{col}^{\Sigma}(E_{j^*}) \cdot \sum_{j^*=1}^{\kappa^*} E_{j^*}$$

Hence (1) follows from $(\overline{V}_i){\downarrow_S} = (\overline{V}_1){\downarrow_S}$ for all i.

For (2), consider $V_1 \cong V_i \otimes L$ for some $L \in \mathrm{Irr}_K(G/S)$. Note that $(G/S)_{reg} \cong G/A$ for $A/S = O_p(G/S)$, and $\bar{L} \in \mathrm{Irr}_F(G/A)$. Hence $D_{i^*} \otimes \bar{L} \in [D_{i^*}]_G$. Construct $\pi \in \mathfrak{S}_{\eta^*}$ by $D_{i^*\pi} = D_{i^*} \otimes \bar{L}$. Then

$$\overline{V}_1 = \overline{V_i \otimes L} = \sum_{i^*=1}^{\eta^*} a_{i,i^*} (D_{i^*} \otimes \bar{L}) = \sum_{i^*=1}^{\eta^*} a_{i,i^*} D_{i^*\pi}$$

gives $a_{i,i^*} = a_{1,i^*\pi}$. Similarly, for $W_1 \cong {}^g W_j$ for some $g \in G$, construct $\pi \in \mathfrak{S}_{\kappa^*}$ by $E_{j^*\pi} = {}^g E_{j^*}$. Then $b_{j,j^*} = b_{1,j^*\pi}$.

For (3), consider two ways to sum up $\sum_{j=1}^{\kappa} \sum_{j^*=1}^{\kappa^*} b_{j,j^*}$ and use (1). (4) follows by (3) and (1). $\square$

There are 3 good cases that one may deduce the S-tile from G-tile directly.

Proposition 3.15. *Let G/S be cyclic.*

(1) *If $\kappa = 1$, then the S-tile is a $1 \times \kappa^*$ matrix with all entries c.*

(2) *If $\kappa^* = 1$, then the S-tile is a $\kappa \times 1$ matrix with all entries c/κ.*

(3) *If $c = 1$, then in a suitable order of labels, the S-tile is a $\kappa \times \kappa^*$ matrix, which just put κ^*/κ identity matrices of size κ in a row.*

Proof. These are all obvious from Proposition 3.14. $\square$

When none of the case above, one cannot obtain full S-tile without other methods. For example, let $G = \mathfrak{S}_{13}$ and $S = \mathfrak{A}_{13}$, $p = 3$, and take the G-tile $[S_K^{(5,3^2,1^2)}]_G \times$

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$[D_F^{(7,3,2,1)}]_G$. The G-tile is a 1×1 matrix $[2]$, and the corresponding S-tile is a 2×2 matrix. Our methods here can only find that the matrix is of the form $\left[\begin{smallmatrix} r & s \\ s & r \end{smallmatrix}\right]$ with $r + s = 2$, while it is in fact $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. One may find this in [web], with the dimension $\dim S_K^{(5,3^2,1^2)} = 16016$ and $\dim D_F^{(7,3,2,1)} = 1428$.

3.5 Representation Theory of G/S

Let $S \trianglelefteq G$. The relation of irreducible $\mathbf{F}(G/S)$-modules and $\mathbf{F}G$-modules is much simpler then that of irreducible $\mathbf{F}S$-modules and $\mathbf{F}G$-modules. We modify the Proposition 2.5, 2.19 of [L] into module phase.

Proposition 3.16. *Let $U_i \in \mathrm{Irr}_{\mathbf{F}}(G/S)$, $i = 1, \cdots, h$, and $V_i = \mathrm{infl}_{G/S}^G(U_i)$. Then*

(1) *V_i is irreducible.*

(2) *The set $\{V_i\}$ is characterized by $\{V \in \mathrm{Irr}_{\mathbf{F}}(G/S) \mid \ker V \geq S\}$.*

(3) *The inflation map is a bijective map from $\mathrm{Irr}_{\mathbf{F}}(G/S)$ to $\{V_i\}$.*

Proof. Consider an inverse of inflation. For an $\mathbf{F}G$-module V with $\ker V \geq S$, let $\mathrm{quot}_{G/S}^G(V)$ be the same $\mathbf{F}$-module V, with the multiplication $(gS)v = gv$, becoming an $\mathbf{F}(G/S)$-module. This is well-defined since $Sv = v$. It is trivial to check that the inflation map and the quotient map are inverse to each other. Hence it remains to check (1). If $0 \neq W \subset V_i$, then $0 \neq \mathrm{quot}_{G/S}^G(W) \subset U_i$, hence $\mathrm{quot}_{G/S}^G(W) = U_i$ and $W = V_i$. $\square$

Proposition 3.17. *Let Δ_G, $\Delta_{G/S}$ be the decomposition matrix of G, G/S (for the same p), respectively. Then Δ_G is of the form*

$$\Delta_G = \left[\begin{array}{cc} \Delta_{G/S} & O \\ \hline * & * \end{array}\right]$$

Proof. Write $\Delta_{G/S} = (d'_{ij})$ with the row labels $U_1, \cdots, U_m \in \mathrm{Irr}_K(G)$ and column labels $C_1, \cdots, C_{m^*} \in \mathrm{Irr}_F(G)$ (they are not cyclic groups here). Similarly, write $\Delta_G = (d_{ij})$ with the row labels $V_1, \cdots, V_h \in \mathrm{Irr}_K(G)$ and column labels $D_1, \cdots, D_{h^*} \in \mathrm{Irr}_F(G)$, and set $V_i = \mathrm{infl}_{G/S}^G(U_i)$, $D_j = \mathrm{infl}_{G/S}^G(C_j)$.

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We prove it in character phase. Let χ_{V_i}, χ_{U_i} be the character of V_i, U_i, and ϕ_{D_j}, ϕ_{C_j} be the character of D_j, C_j, respectively. Then for each $1 \leq i \leq m$ and each $g \in G$, we have

$$\chi_{V_i}(g) = \chi_{U_i}(gS) = \sum_{j=1}^{m^*} d'_{ij}\phi_{C_i}(gS) = \sum_{j=1}^{m^*} d'_{ij}\phi_{D_i}(g)$$

Hence $d_{ij} = d'_{ij}$ for $1 \leq j \leq m^*$ and $d_{ij} = 0$ for $m^* < j \leq h^*$. □

Corollary 3.18. *If G has (C,p), (L'',p), (L',p)-property, then G/S also has (C,p), (L'',p), (L',p)-property, respectively.*

Proof. The cases (L'',p)-property and (L',p)-property are obvious from the definition. For (C,p)-property, define the partial order as following. Let C_1, $C_2 \in \mathrm{Irr}_F(G/S)$, then $C_1 \trianglerighteq C_2$ if $\mathrm{infl}_{G/S}^G(C_1) \trianglerighteq \mathrm{infl}_{G/S}^G(C_2)$. Then we may pass the (C,p)-property from G to G/S. □

The following result is not used in this thesis. We put it here just for completeness.

Corollary 3.19. *Assume that S is a p-group. Then if G/S has (R,p), (U,p)-property, then G also has (R,p), (U,p)-property, respectively.*

Proof. If we can prove that $|\mathrm{Irr}_F(G)| = |\mathrm{Irr}_F(G/S)|$, then the O of Δ_G in Proposition 3.17 does not appear, hence the properties involving surjectivity may be passed from G/S to G.

Now we show that the number of p-regular conjugacy classes of G equals the number of p-regular conjugacy classes of G/S. Given p-regular g_1, $g_2 \in G$, if $g_2 = hg_1h^{-1}$ for some $h \in G$, then clearly $g_2S = (hS)(g_1S)(h^{-1}S)$. On the other hand, if $g_2S = (hS)(g_1S)(h^{-1}S)$, write $g_2s_2 = mg_1s_1m^{-1} = (mg_1m^{-1})(ms_1m^{-1})$ for some s_1, $s_2 \in S$, $m \in hS$. Then by considering the uniqueness of decomposition of p'-part and p-part, we have $g_2 = mg_1m^{-1}$ and $s_2 = ms_1m^{-1}$. This completes the proof. □

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4 Field Theory

In this chapter, we will clarify the relations between σ in a larger field $\mathbb{F}_{q^d}$, and its p'-part $\sigma' = (\sigma)_{p'}$ in a smaller field $\mathbb{F}_q$.

One relation which has been highly used in Dipper-James's paper [D1] states that every irreducible representation of GL_n has a p-regular label. Also when we bring a representation of GL_n from K to F by reduction modulo p, it appears as the label of the canonical composition factor.

4.1 Basic Facts

The following propositions are basic for finite fields. We list here without proof and shall use them freely.

Proposition 4.1. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$. Then $\sigma \in \mathbb{F}_{q^d}^\times$ for some $d \in \mathbb{N}$. Hence we can talk about p'-part $(\sigma)_{p'}$ and p-part $(\sigma)_p$ of σ, viewing $\mathbb{F}_{q^d}^\times$ a multiplicative group.*

Proposition 4.2. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$. The following are equivalent.*

(1) *$\sigma \in \mathbb{F}_{q^d}$ and $\sigma \notin \mathbb{F}_{q^c}$ for any $c \in \mathbb{N}$, $c < d$.*

(2) *There is some monic irreducible $\mu_\sigma(T) \in \mathbb{F}_q[T]$ of degree d such that σ is a root of $\mu_\sigma(T)$, called the minimal polynomial of σ over $\mathbb{F}_q$.*

(3) *$\mathbb{F}_q(\sigma) \cong \mathbb{F}_{q^d}$ as a vector space over $\mathbb{F}_q$ with basis $\{1, \sigma, \cdots, \sigma^{d-1}\}$.*

(4) *$\sigma^{q^d} = \sigma$ and $\sigma, \sigma^q, \cdots, \sigma^{q^{d-1}}$ are all distinct.*

In this case, we say σ is of degree d over $\mathbb{F}_q$, denote as $\deg(\sigma) = d$.

Proposition 4.3. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$, $\deg(\sigma) = d$. The following sets coincide.*

(1) *The set of roots of $\mu_\sigma(T)$*

(2) *$\{\sigma, \sigma^q, \cdots, \sigma^{q^{d-1}}\}$*

(3) *$\{\sigma^{q^j} \mid j \in \mathbb{Z}, j \geq 0\}$*

Denote this set $[\sigma]$, which has d elements, and can be view as equivalence class under

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the relation having common minimal polynomial.

Proposition 4.4. *Let $\sigma_1, \sigma_2 \in \overline{\mathbb{F}}_q^\times$. The following are equivalent.*

(1) $[\sigma_1] = [\sigma_2]$.

(2) σ_1 *and* σ_2 *have common minimal polynomial.*

(3) *There is some non-negative integer j such that $\sigma_2 = \sigma_1^{q^j}$.*

In this case, we say σ_1 and σ_2 are (Galois) conjugate over $\mathbb{F}_q$.

Definition 4.5. Given $\tau \in \mathbb{F}_q^\times$.

(1) For $\sigma \in \overline{\mathbb{F}}_q^\times$, $\deg(\sigma) = d$, define $\tau[\sigma] = \{\tau\sigma, \tau\sigma^q, \cdots, \tau\sigma^{q^{d-1}}\}$.

(2) For $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$, define $\tau\underline{\sigma} = (\tau\sigma_1, \cdots, \tau\sigma_a)$ pairwisely.

(3) For n-admissible pair $(\underline{\sigma}, \underline{\lambda})$, define $\tau \cdot (\underline{\sigma}, \underline{\lambda}) = (\tau\underline{\sigma}, \underline{\lambda})$.

Proposition 4.6. *Let $\tau \in \mathbb{F}_q^\times$, $\sigma, \sigma_1, \sigma_2 \in \overline{\mathbb{F}}_q^\times$, and $(\underline{\sigma}, \underline{\lambda})$ an n-admissible pair.*

(1) $\tau[\sigma] = [\tau\sigma]$.

(2) $\deg(\tau\sigma) = \deg(\sigma)$.

(3) $[\sigma_1] = [\sigma_2]$ *if and only if* $[\tau\sigma_1] = [\tau\sigma_2]$.

(4) $\tau \cdot [(\underline{\sigma}, \underline{\lambda})] \coloneqq [\tau \cdot (\underline{\sigma}, \underline{\lambda})]$ *is well-defined.*

(5) $[\sigma_1] = [\sigma_2]$ *implies* $[(\sigma_1)_{p'}] = [(\sigma_2)_{p'}]$ *and* $[(\sigma_1)_p] = [(\sigma_2)_p]$.

Proof. (1) follows from $\tau^q = \tau$ and Proposition 4.2. (2)(3) follows from (1), and (4) follows from (3). For (5), write $\sigma_2 = \sigma_1^{q^j}$ for some $j \geq 0$. By the uniqueness of decomposition, we have both $(\sigma_2)_{p'} = {(\sigma_1)_{p'}}^{q^j}$ and $(\sigma_2)_p = {(\sigma_1)_p}^{q^j}$. □

4.2 Elementary Number Theory

The following lemmas are from elementary number theory. They focus on the order of p-part of multiplication groups of finite fields.

In the section 4.2, fixed a prime p.

Definition 4.7. Let $r \in \mathbb{N}$.

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(1) Let $N_r(m) = 1 + r + \cdots + r^{(m-1)}$. If $r > 1$, then $N_r(m) = \dfrac{r^m - 1}{r - 1}$. Hence one may prove that $N_r(km) = N_{r^m}(k)N_r(m)$.

(2) Let $e(r)$ be the minimal number m such that $p \mid N_r(m)$. If no such m exists, set $e(r) = \infty$. Observe if $p \mid r - 1$, then $e(d) = p$; if $p \mid r$, then $e(d) = \infty$; otherwise, $e(d) \mid p - 1$.

Lemma 4.8. *Let $r \in \mathbb{N}$, l a prime, and $p \mid r - 1$. Take $N = N_r(l)$.*

(1) *If $p > 2$, then* $|N|_p = \begin{cases} 1, & l \neq p \\ p, & l = p \end{cases}$.

(2) *If $p = 2$, then* $|N|_2 = \begin{cases} 1, & l \neq 2 \\ 2, & l = 2, \quad r \equiv 1 \mod 4 \\ 2^b \text{ for some } b \geq 2, & l = 2, \quad r \equiv 3 \mod 4 \end{cases}$.

We call $p = 2$, $l = 2$, $r \equiv 3 \mod 4$ the exceptional case.

Proof. The case $r = 1$ is clear, so assume $r > 1$.

Since $r \equiv 1 \mod p$, $r^k \equiv 1 \mod p$ for all $k \in N$. Then if $l \neq p$, then $N = 1 + r + \cdots + r^{l-1} \equiv l \mod p$, hence has trivial p-factor.

For $l = p$ case, let $r = 1 + p^c m$ with $p \nmid m$. Then

$$r^p = 1 + p \cdot p^c m + (p(p-1)/2) \cdot p^{2c} m^2 + \cdots$$

If $p > 2$ then $r^p = 1 + p^{c+1}m + p^{2c+1}m'$ for some $m' \in \mathbb{N}$, where $2c + 1 > c + 1$. Or if $c \geq 2$ then $r^p = 1 + p^{c+1}m + p^{2c}m'$ for some $m' \in \mathbb{N}$, also $2c > c + 1$. Hence in both case $|r^p - 1|_p / |r - 1|_p = p$.

For $p = 2$ and $c = 1$ case, which means $r \equiv 3 \mod 4$, write $r = 1 + 2m$, then $r^2 = 1 + 4m + 4m^2 = 1 + 8(m(m-1)/2)$, thus $|r^2 - 1|_2 / |r - 1|_2 \geq 2^2$. □

Lemma 4.9. *Let $r \in \mathbb{N}$ with $p \mid r - 1$. Take $N = N_r(k)$ for any $k \in \mathbb{N}$. Then $|k|_p$ divides $|N|_p$. More precisely,*

(1) *$|N|_p = |k|_p$, except for the case $p = 2$, k is even, $r \equiv 3 \mod 4$.*

(2) *In the exceptional case, $|N|_p$ is a proper multiple of $|k|_p$.*

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Proof. If $k' = lk$ for some prime l, then $N' := N_r(k') = N_{r^k}(l)N_r(k)$, so we can replace r by r^k in Lemma 4.8. For non-exceptional case we have $|N'|_p = |l|_p|N|_p$, and then applying induction on k yields the result.

In the exceptional case, the initial step $|r^2-1|_2/|r|_2$ is a proper multiple of 2. Now for any even k', write $k' = lk$ with k even, then $r^k \equiv 1 \mod 4$ which is non-exceptional, so $|N'|_2 = |l|_2|N|_2$ is still valid. The result follows from induction on even k. $\square$

Lemma 4.10. *Let $r \in \mathbb{N}$ with $p \nmid r$, $p \nmid r-1$. Take $N = N_r(k)$ for any $k \in \mathbb{N}$. Write $e = e(r)$, then*

(1) *$e \nmid k$, then $|N|_p = 1$.*

(2) *$e \mid k$, then $|N|_p$ is a proper multiple of $|k|_p$.*

Proof. Since $r > 1$, Consider $N(r-1) = r^k - 1$. Then e is the smallest number m satisfied $p \mid r^m - 1$, that is, e is the order of r in $\mathbb{Z}_p^\times$. Hence $p \mid r^k - 1$ if and only if $e \mid k$. This gives (1). When $e \mid k$, we have $N = N_{r^e}(k/e)N_r(e)$. Since $p \mid r^e - 1$, by Lemma 4.9 $|N_{r^e}(k/e)|_p$ is a multiple of $|k/e|_p = |k|_p$, and $N_r(e)$ is a multiple of p. This gives (2). $\square$

4.3 Degree Extension Lemma

The following lemma is of importance in Dipper-James's paper [D1], although the proof is scattered in several places of Kleshchev-Tiep's paper [K].

Lemma 4.11. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$ be a p-regular element with $\deg(\sigma) = d$ over $\mathbb{F}_q$, $p \mid q^d - 1$. Then the following are equivalent:*

(1) *$a = dp^c$ for some non-negative integer c.*

(2) *There exists some p-element $\upsilon \in \overline{\mathbb{F}}_q^\times$ such that $\deg(\sigma\upsilon) = a$ over $\mathbb{F}_q$.*

Proof. (1) $\Rightarrow$ (2): Taking $\upsilon = 1$ for $c = 0$, we may assume $c \geq 1$. Let $P_i = O_p(\mathbb{F}_{q^{dp^i}}^\times)$ for $i = 0, 1, \cdots, c$. Then for $i < j$, $P_i < P_j$ via $\mathbb{F}_{q^{dp^i}} \subset \mathbb{F}_{q^{dp^j}}$. The following diagram

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may help, where every groups in the diagram are cyclic.

$$\begin{tikzcd}
\mathbb{F}^{\times}_{q^{dp^c}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \mathbb{F}^{\times}_{q^{dp^{c-1}}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \cdots \arrow[r, dash] & \mathbb{F}^{\times}_{q^{dp}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \mathbb{F}^{\times}_{q^{d}} \arrow[d, dash, "p\text{-part}"] \\
P_c \arrow[r, dash] & P_{c-1} \arrow[r, dash] & \cdots \arrow[r, dash] & P_1 \arrow[r, dash] & P_0
\end{tikzcd}$$

Taking $r = q^{dp^i}$ and $k = p$ in Lemma 4.9, we have p divides $(P_{i+1} : P_i)$ for $i = 0, 1, \cdots, c-1$. Pick any $\upsilon \in P_c \setminus P_{c-1}$, hence υ is of degree p^c over $\mathbb{F}_{q^d}$. Evidently σ and υ are p'-part and p-part of $\sigma\upsilon$, hence by Proposition 1.3, both σ and υ are power of $\sigma\upsilon$. Therefore

$$\mathbb{F}_q(\sigma\upsilon) = \mathbb{F}_q(\sigma)(\upsilon) = \mathbb{F}_{q^d}(\upsilon) = \mathbb{F}_{q^{dp^c}}$$

and $\sigma\upsilon$ is of degree dp^c over $\mathbb{F}_q$.

(2) $\Rightarrow$ (1): By the same argument above, $\mathbb{F}_q(\sigma\upsilon) = \mathbb{F}_{q^d}(\upsilon)$, thus we may assume $\upsilon \in \mathbb{F}_{q^{dk}}$ for some $k = p^b m$, p does not divide m. But then by Lemma 4.9(1), $\mathbb{F}^{\times}_{q^{dk}}$ and $\mathbb{F}^{\times}_{q^{dp^b}}$ have the same p-part. Hence $\upsilon \in \mathbb{F}_{q^{dp^c}}$, υ is of degree p^c over $\mathbb{F}_{q^d}$ for some $c \leq b$, and $a = dp^c$. $\square$

Corollary 4.12. *Follow the notation and assumption in Lemma 4.11. Then the υ in Lemma 4.11 has the following property.*

(1) *υ has degree p^c over $\mathbb{F}_{q^d}$.*

(2) *υ has degree dp^c over $\mathbb{F}_q$.*

(3) *The choice of υ only depends on d and c. In particular, we can choose one υ for all p-regular $\sigma \in \overline{\mathbb{F}}^{\times}_q$ with $\deg(\sigma) = d$.*

(4) *If $p = 2$, $q^d \equiv 3 \mod 4$ and $c = 1$, then υ can be chosen to be $|\upsilon| = 4$.*

Proof. (1)(2)(3) follows from the proof of Lemma 4.11. For (4), $P_0 = \langle -1 \rangle$, pick some $\upsilon \in P_1$ such that $\upsilon^2 = -1$, then $\upsilon \in P_1 \setminus P_0$, and the rest of the proof is the same. $\square$

The following is a generalized version of Lemma 4.11.

Lemma 4.13. *Let $\sigma \in \overline{\mathbb{F}}^{\times}_q$ be a p-regular element with $\deg(\sigma) = d$ over $\mathbb{F}_q$. Write $e = e(q^d)$. Then the following are equivalent:*

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(1) *$a = d$ or $a = dep^c$ for some non-negative integer c.*

(2) *There exists some p-element $\upsilon \in \overline{\mathbb{F}}_q^{\times}$ such that* $\deg(\sigma\upsilon) = a$ *over* $\mathbb{F}_q$.

Proof. When $p \mid q^d - 1$, $e = p$, and this is just Lemma 4.11. Assume $p \nmid q^d - 1$. Then $e \mid p - 1$, and $p \mid q^{de} - 1$.

(1) $\Rightarrow$ (2): If $a = d$, take $\upsilon = 1$. A similar argument to Lemma 4.11 gives the following diagram:

$$\begin{tikzcd}
\mathbb{F}^{\times}_{q^{dep^c}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \mathbb{F}^{\times}_{q^{dep^{c-1}}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \cdots \arrow[r, dash] & \mathbb{F}^{\times}_{q^{dep}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \mathbb{F}^{\times}_{q^{de}} \arrow[r, dash] \arrow[d, dash, "p\text{-part}"] & \mathbb{F}^{\times}_{q^{d}} \arrow[d, dash, "p\text{-part}"] \\
P_c \arrow[r, dash] & P_{c-1} \arrow[r, dash] & \cdots \arrow[r, dash] & P_1 \arrow[r, dash] & P_0 \arrow[r, dash] & P_d
\end{tikzcd}$$

Taking $r = q^{dep^{(c-1)}}$ and $k = p$ in Lemma 4.9, we have p divides $(P_c : P_{c-1})$, so in particular $P_c \setminus P_{c-1}$ is nonempty. Pick any $\upsilon \in P_c \setminus P_{c-1}$, then υ is of degree ep^c over $\mathbb{F}_{q^d}$. Now $\mathbb{F}_q(\sigma\upsilon) = \mathbb{F}_{q^d}(\upsilon) = \mathbb{F}_{q^{dep^c}}$ gives $\sigma\upsilon$ is of degree dep^c over $\mathbb{F}_q$.

(2) $\Rightarrow$ (1): By $\mathbb{F}_q(\sigma\upsilon) = \mathbb{F}_{q^d}(\upsilon)$, we may assume υ is of degree k over q^d. If $e \nmid k$, taking $r = q^d$ in Lemma 4.10 shows that $\mathbb{F}^{\times}_{q^{dk}}$ and $\mathbb{F}^{\times}_{q^d}$ have the same p-part. This forces $k = 1$ and $a = d$. If $e \mid k$, write $k = ep^c m$, $p \nmid m$. Taking $r = q^{dep^c}$ in Lemma 4.9 shows that $\mathbb{F}^{\times}_{q^{dk}}$ and $\mathbb{F}^{\times}_{q^{dep^c}}$ have the same p-part. This forces $m = 1$ and $a = dep^c$. □

The following theorem is the main result of Dipper-James's paper [D1]. It is one of the crucial part of proving the complete set of non-isomorphic irreducible representation of GL_n over F (Theorem 2.33), and plays an important role on Kleshchev's theorem and the p-regularization of the n-admissible symbol.

Theorem 4.14. [D1, Theorem 5.1] *Let $\sigma \in \overline{\mathbb{F}}_q^{\times}$,* $\deg(\sigma) = d$, *$\lambda \vdash k \in \mathbb{N}$, and $n = dk$. Write $e = e(q^d)$. By virtue of division algorithm, there is an unique pair of partition (μ, γ) satisfied*

$$\lambda = ([e]\mu) \,[+]\, \gamma, \quad \gamma \text{ is } e\text{-regular}$$

Then if $\mu \neq 0$ and $\gamma \neq 0$, we have

$$L_F(\sigma, \lambda) \cong L_F(\sigma\upsilon, \mu) \circ L_F(\sigma, \gamma)$$

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And if $\mu \neq 0$ and $\gamma = 0$, we have

$$L_F(\sigma, \lambda) \cong L_F(\sigma\upsilon, \mu)$$

where $\upsilon \in \overline{\mathbb{F}}_q^\times$ is some p-element such that $\deg(\sigma\upsilon) = de$.

Corollary 4.15. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$, $\deg(\sigma) = a$, and $\lambda \vdash k \in \mathbb{N}$. Write $\sigma' = (\sigma)_{p'}$, $\deg(\sigma') = d$, and let $\mu \coloneqq [a/d]\lambda$. Then*

$$L_F(\sigma, \lambda) \cong L_F(\sigma', \mu)$$

Proof. Assume $a > d$. Write $e(q^d) = e$. By Lemma 4.13, $a/d = ep^c$ for some non-negative integer c. Again by the same Lemma, for each $i = 0, 1, \cdots, c-1$, there is some p-element $\upsilon_i \in \overline{\mathbb{F}}_q^\times$ such that $\deg(\sigma'\upsilon_i) = dep^i$, and take $\upsilon_c = (\sigma)_p$. Let $\mu_i \coloneqq [p^{c-i}]\lambda$. Then we have the series

$$(\sigma', \mu) \to (\sigma'\upsilon_0, \mu_0) \to \cdots \to (\sigma'\upsilon_{c-1}, \mu_{c-1}) \to (\sigma'\upsilon_c, \mu_c) = (\sigma, \lambda)$$

Applying Theorem 4.14 to each arrow completes the proof. $\square$

4.4 Lemmas for Kleshchev-Tiep's Theorem

From now on, we are going to establish lemmas used in Kleshchev-Tiep's paper, which talk about the properties of $\sigma \in \overline{\mathbb{F}}_q^\times$, and its p'-part σ', under multiplication of some element $\tau \in \mathbb{F}_q^\times$.

Lemma 4.16. *Let $\sigma \in \overline{\mathbb{F}}_q^\times$ with p'-part σ', and p-part υ, $\deg(\sigma) = kd$ and $\deg(\sigma') = d$. If one of the following criterion holds,*

(1) *$p > 2$ or $p = 2$, $q^d \equiv 1 \mod 4$.*

(2) *$p = 2$, $q^d \equiv 3 \mod 4$, $k > 2$.*

(3) *$p = 2$, $q^d \equiv 3 \mod 4$, $k = 2$, $|\upsilon| = 4$.*

Then there exists some $\alpha \in O_p(\mathbb{F}_q^\times)$, $|\alpha| = |\gcd(k, q-1)|_p$, such that $\sigma\alpha$ is conjugate to σ over $\mathbb{F}_q$.

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Proof. If p does not divide $q-1$, or $k=1$, then just take $\alpha=1$, so assume $p \mid q-1$ and $k>1$. Hence $p \mid q^d-1$, and by Lemma 4.11 we have $k=p^b$ and the p-factor of $\gcd(k,q-1)$ is p^c for some integers $b \geq c \geq 1$.

For case (1), take $\alpha = \sigma^{q^{dp^{b-c}}-1}$. Since $(\sigma')^{q^d-1}=1$, α is a power of υ, thus a p-element. It suffices to prove that $|\alpha|=p^c$. By Lemma 4.9, the p-part of $(q^{dp^b}-1)/(q^{dp^{b-c}}-1)$ is p^c, thus $\sigma^{q^{dp^b}-1}=1$ implies $\alpha^{p^c}=1$. On the other hand, the p-part of $(q^{dp^{b-1}}-1)/(q^{dp^{b-c}}-1)$ is p^{c-1}, so if $\alpha^{p^{c-1}}=1$, then $\sigma^{q^{dp^{b-1}}-1}=1$, contradicts to $\deg(\sigma)=dp^b$.

For case (2) and (3), note that if $4 \mid q-1$ implies $q^d \equiv 1 \mod 4$, hence we only need to find $|\alpha|=2$. Take $\alpha=\sigma^{q^{2^{b-1}d}-1}$. By similar argument in case (1), then α is a 2-element, and $\alpha \neq 1$ or it will contradict to $\deg(\sigma)=2^a d$. If $k>2$, then $b>1$, note that $q^{2^{b-1}d} \equiv 1 \mod 4$ is non-exceptional case of Lemma 4.9, so a similar argument in case (1) holds. If $k=2$, then $b=1$, since $\upsilon^4=1$, $\alpha^2=\sigma^{2(q^d-1)}-\upsilon^{2(q^d-1)}=1$. □

Lemma 4.17. *Given $\sigma \in \overline{\mathbb{F}}_q^\times$ with p'-part σ' and p-part υ, $\deg(\sigma)=kd$, and deg$(\sigma')=d$. Let $I=\{\tau \in O_p(\mathbb{F}_q^\times) \mid [\sigma]=[\sigma\tau]\}$. Then we have $|I|$ divides $|\gcd(k,q-1)|_p$. More precisely,*

(1) $|I|=|\gcd(k,q-1)|_p$*, unless* $p=2$, $q^d \equiv 3 \mod 4$, $k=2$, *and* $|\upsilon| \geq 8$.

(2) *In the exceptional case* $|I|=1$.

Proof. If p does not divide $q-1$, then $|I|=1=|\gcd(k,q-1)|_p$, so assume $p \mid q-1$. For any $\tau \in I$, write $\sigma\tau=\sigma^{q^j}$ for some integer $j \geq 0$, then $(\sigma')^{1-q^j}=\upsilon^{q^j-1}\tau^{-1}=1$. Hence $\tau=\upsilon^{q^j-1}$, and by $\deg(\sigma')=d$, we have $(\sigma')^{q^j-1}=1$ if and only if $d \mid j$. Note that $\tau^q=\tau$,

$$\tau^k=\tau^{1+q^j+\cdots+q^{(k-1)j}}=(\upsilon^{q^j-1})^{1+q^j+\cdots+q^{(k-1)j}}=\upsilon^{q^{kj}-1}=1$$

since $\deg(\upsilon)=kd$ by Corollary 4.12. Hence $|I| \mid k$, as well as $|\gcd(k,q-1)|_p$.

Now (1) follows from an element $\alpha \in I$ of degree exactly $|\gcd(k,q-1)|_p$ found in Lemma 4.16. For (2), consider the case $p=2$, $q^d \equiv 3 \mod 4$, $k=2$, where

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$|\gcd(k, q-1)|_p = 2$ and $|I| \mid 2$. We show that if $\tau = -1 \in I$, then $|\upsilon| = 4$. Write again $\sigma\tau = \sigma^{q^j}$. From $\deg(\sigma) = 2d$, j can be chosen from $0 \leq j < 2j$. Since q is odd, $j = 0$ is excluded. By $(\sigma')^{1-q^j} = \upsilon^{q^j-1}\tau^{-1} = 1$, we have $d \mid j$, thus $d = j$. Now $\tau = \upsilon^{q^d-1} = -1$ and $q^d \equiv 3 \mod 4$ gives $|\upsilon| = 4$. Note that $1, -1 \in \mathbb{F}_{q^d}$ when $q^d \equiv 3 \mod 4$, so $\deg(\upsilon) = 2d$ implies $|\upsilon| \geq 4$. □

Lemma 4.18. *Let $d \in \mathbb{N}$ and $p^c \mid (q-1)$ for some integer $c \geq 0$. Then there exists an p-element $\upsilon \in \overline{\mathbb{F}}_q^{\times}$, depending only on c, d, such that for any p'-element $\sigma \in \overline{\mathbb{F}}_q^{\times}$ of degree d, we have $\deg(\sigma\upsilon) = dp^c$ and $|I| = p^c$, where $I = \{\tau \in O_p(\mathbb{F}_q^{\times}) \mid [\sigma\upsilon] = [\sigma\upsilon\tau]\}$.*

Proof. The existence of such υ follows from Lemma 4.11 and Corollary 4.12. For $p = 2$, $q \equiv 3 \mod 4$, $c = 1$, we can pick $|\upsilon| = 4$ by Corollary 4.12 (4). Therefore, we avoid the exceptional case in Lemma 4.17, taking $k = p^c = |\gcd(k, q-1)|_p$ gives $|I| = p^c$. □

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5 Kleshchev-Tiep's Theorem

Here we are going to derive the main result of Kleshchev-Tiep.

Let $\mathbf{F} = K$ or F be sufficiently large for GL_n. In [K], $\mathbf{F}$ is asked to be albegraically closed, but it seems that sufficiently large is enough. Recall that any irreducible KGL_n-module is of the form $L_K(\underline{\sigma}, \underline{\lambda})$ for some $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_K$, and any irreducible FGL_n-module is of the form $L_F(\underline{\sigma}, \underline{\lambda})$ for some $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$.

Definition 5.1. Let $[(\underline{\sigma}, \underline{\lambda})]$ be an n-admissible symbol, p a prime.

(1) The p'-branching number is defined by

$$\kappa_{p'}(\underline{\sigma}, \underline{\lambda}) = \#\{\tau \in O_{p'}(\mathbb{F}_q^\times) \mid \tau \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\}$$

(2) The p-branching number is defined by

$$\kappa_p(\underline{\sigma}, \underline{\lambda}) = |\gcd(q-1, \Delta(\underline{\lambda}'))|_p$$

(3) The ordinary branching number is defined by

$$\kappa(\underline{\sigma}, \underline{\lambda}) = \#\{\tau \in \mathbb{F}_q^\times \mid \tau \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\}$$

When $V = L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$, we also denote $\kappa_{p'}(V)$, $\kappa_p(V)$ instead of $\kappa_{p'}(\underline{\sigma}, \underline{\lambda})$, $\kappa_p(\underline{\sigma}, \underline{\lambda})$.

The Kleshchev-Tiep's Theorem is stated as following.

Theorem 5.2 (Kleshchev-Tiep (2008))**.**

(1) *Let $V = L_F(\underline{\sigma}, \underline{\lambda})$ be an irreducible FGL_n-module corresponding to $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. Then the branching number of $V{\downarrow}_{SL_n}$ is*

$$\kappa_{SL_n}(V) = \kappa_{p'}(\underline{\sigma}, \underline{\lambda}) \cdot \kappa_p(\underline{\sigma}, \underline{\lambda})$$

Note that $\kappa_{p'}(\underline{\sigma}, \underline{\lambda})$ and $\kappa_p(\underline{\sigma}, \underline{\lambda})$ are the p'-factor and p-factor of $\kappa_{SL_n}(V)$.

(2) *Let $V_K = L_K(\underline{\sigma}, \underline{\lambda})$ be an irreducible KGL_n-module corresponding to $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_K$. Then the branching number of $(V_K){\downarrow}_{SL_n}$ is*

$$\kappa_{SL_n}(V_K) = \kappa(\underline{\sigma}, \underline{\lambda})$$

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The proof of Theorem 5.2 splits into several steps. Recall that T_n and R_n satisfy $SL_n \leq T_n, R_n \leq GL_n$, $T_n/SL_n = O_p(GL_n/SL_n)$, $R_n/SL_n = O_{p'}(GL_n/SL_n)$.

Using Lemma 3.6 to split $\kappa_{SL_n}(V)$ into $\kappa_{T_n}(V) \cdot \kappa_{R_n}(V)$. The $\kappa_{T_n}(V)$ part is rather simple, using Lemma 3.5 and Lemma 2.34. The rest is to find out $\kappa_{R_n}(V)$. The strategy is to find a lower bound of $\kappa_{R_n}(V)$, and use total counting to force $\kappa_{R_n}(V)$ to match its lower bound.

In the rest of the section, let $\sigma \in \overline{\mathbb{F}}_q^{\times}$, $\deg(\sigma) = d$, $n = kd$.

Lemma 5.3. *Let $m \in \mathbb{N}$, R be any group satisfies $SL_n \leq R \leq GL_n$. If $\kappa_R\left(L_K(\sigma, \lambda)\right) \geq m$ for all $\lambda \vdash k$, then $\kappa_R\left(L_F(\sigma, \lambda)\right) \geq m$ for all $\lambda \vdash k$.*

Proof. By Theorem 2.28, the composition factors of $S_F(\sigma, \lambda)$ are of the form $L_F(\sigma, \mu)$ for $\mu \trianglerighteq \lambda$, and exactly one of them is $L_F(\sigma, \lambda)$. Hence applying induction on dominance order on partition makes sense.

If $\lambda = (k)$, then $L_F(\sigma, \lambda) = S_F(\sigma, \lambda)$, so the lemma holds.

Assume for any $\mu \trianglerighteq \lambda$, $\mu \neq \lambda$, we have $L_F(\sigma, \mu)\downarrow_R^{GL_n} = \bigoplus_{i=1}^{m(\mu)} L_i^{\mu}$, where $L_i^{\mu} \in \mathrm{Irr}_F(R)$ and $m(\mu) \geq m$. Since GL_n/R is cyclic, by Lemma 3.3 $m(\mu) = (GL_n : I_i)$ for any $I_i = \mathrm{Stab}_G(L_i^{\mu})$. Now consider $S_F(\sigma, \lambda)$ in the Grothendieck group of FGL_n-module, we have

$$S_F(\sigma, \lambda) = L_F(\sigma, \lambda) + \sum_{\mu \trianglerighteq \lambda, \mu \neq \lambda} L_F(\sigma, \mu)$$

Taking restriction gives

$$S_F(\sigma, \lambda)\downarrow_R^{GL_n} = L_F(\sigma, \lambda)\downarrow_R^{GL_n} + \sum_{\mu \trianglerighteq \lambda, \mu \neq \lambda} \sum_{i=1}^{m(\mu)} L_i^{\mu}$$

On the other hand, $S_F(\sigma, \lambda)$ is a reduction modulo p of $S_K(\sigma, \lambda) = L_K(\sigma, \lambda)$. By assumption of the statement, $L_K(\sigma, \lambda)\downarrow_R^{GL_n} = \bigoplus_{j=1}^{t} V_K(j)$, where $V_K(j) \in \mathrm{Irr}_K(R)$ and $t \geq m$. Since reduction modulo p and restriction commute,

$$S_F(\sigma, \lambda)\downarrow_R^{GL_n} = \overline{L_K(\sigma, \lambda)\downarrow_R^{GL_n}} = \overline{\bigoplus_{j=1}^{t} V_K(j)} = \bigoplus_{j=1}^{t} V_j$$

Where V_j is a reduction modulo p of $V_K(j)$.

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Now assume the contrary, $L_F(\sigma, \lambda)\downarrow_R^{GL_n} = \bigoplus_{i=1}^{m(\lambda)} L_i$, where $L_i \in \mathrm{Irr}_F(R)$ but $m(\lambda) < m$. For convenience let L_1 be a composition factor of V_1. Since for any j, $V_K(j) \cong {}^gV_K(1)$ for some $g \in GL_n$, thus ${}^gL_1$ is a composition factor of V_j. Hence there are at least t conjugates of L_1, which cannot be all summands of $L_F(\sigma, \lambda)\downarrow_R^{GL_n}$, and thus there are some $L_i^\mu \cong {}^gL_1$. But the indices of inertia group of L_i^μ and ${}^gL_1$ are $m(\mu)$, t, respectively, which leads to a contradiction as $m(\mu) \geq m > t$. $\square$

Lemma 5.4. *Let $\lambda \vdash k$ and $1 \neq \alpha \in \mathbb{F}_q^\times$. Assume $[\sigma\alpha] = [\sigma]$. Then $\kappa_R(L_F(\sigma, \lambda)) \geq |\alpha|$ for $R = \ker(L_K(\alpha, (n)))$*

Proof. By virtue of Lemma 2.34, $L_K(\sigma, \lambda) \otimes L_K(\alpha, (n)) \cong L_K(\sigma\alpha, \lambda) \cong L_K(\sigma, \lambda)$. Hence the assumption of Lemma 3.5(2) is fulfilled, and thus $\kappa_R(L_K(\sigma, \lambda)) \geq |\alpha|$ as $|\alpha| = (GL_n : R)$. Since λ is arbitrary, the result follows from Lemma 5.3. $\square$

Lemma 5.5. *Let $r \in \mathbb{N}$ with $r > 1$, and $|r-1|_p = p^c$ with $c \in \mathbb{N}$. Take $N = \dfrac{r^{p^a} - 1}{r - 1}$ for any $a \in \mathbb{N}$. Then p^a divides $|N|_p$. More precisely,*

(1) *$|N|_p = p^a$, except the case $p = 2$, $c = 1$, $r \equiv 3 \mod 4$.*

(2) *In the exceptional case, $|N|_p$ is a proper multiple of p^a.*

Proof. Take $k = p^d$ in Lemma 4.9. $\square$

Lemma 5.6. *Assume that σ is an p'-element, $p^c \mid \gcd(n, q-1)$ with $c \geq 1$. Then for any $\lambda \vdash k$ with $p^c \mid \lambda'$, we have $\kappa_{R_n}(L_F(\sigma, \lambda)) \geq p^c$.*

Proof. Let $d = \deg(\sigma)$. Take $a = dp^c$ in Lemma 4.11, there exists some υ such that $\deg(\sigma\upsilon) = dp^c$. If $p = 2$, $q \equiv 3 \mod 4$, $c = 1$, by Corollary 4.12 we can take $|\upsilon| = 4$. Therefore, by Lemma 4.16 there is some $\alpha \in \mathbb{F}_q^\times$, $|\alpha| = |\gcd(p^c, q-1)|_p = p^c$, such that $[\sigma\upsilon] = [\sigma\upsilon\alpha]$. In particular, $\alpha \neq 1$, which matches the assumption of Lemma 5.4. Thus $\kappa_R(L_F(\sigma\upsilon, \nu)) \geq |\alpha|$ for $R = \ker(L_\mathbb{C}(\alpha, (n)))$ and all $\nu \vdash n/\deg(\sigma\tau) = n/(dp^c) = k/p^c$. By $(GL_n : R) = |\alpha| = p^c \leq (GL_n : R_n)$, we have $R \geq R_n$ and $\kappa_{R_n}(L_F(\sigma\upsilon, \nu)) \geq p^c$ for any $\nu \vdash k/p^c$. Now since $p^c \mid \lambda'$, there exists some $\mu \vdash k/p^c$ such that $\lambda = [p^c]\mu$. Then $L_F(\sigma\tau, \mu) \cong L_F(\sigma, \lambda)$ by Corollary 4.15 and the proof is complete. $\square$

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Lemma 5.7. *Assume $p^c \mid \gcd(n, q-1))$ with $c \geq 1$. Let $V = L_F(\underline{\sigma}, \underline{\lambda})$ be an irreducible FGL_n-module, $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$, and $p^c \mid \Delta(\underline{\lambda}')$. Then $\kappa_{R_n}(V) \geq p^c$.*

Proof. Write $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$, $\underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(a)})$. For $i = 1, \cdots, a$, $\deg(\sigma_i) = d_i$, $\lambda^{(i)} \vdash k_i$, and $n = \sum_{i=1}^{a} k_i d_i$. Apply induction on a. The case $a = 1$ is Lemma 5.6. Assume $a \geq 2$, set

$$\begin{aligned} &r = k_1 d_1, && A = GL_r, && A_1 = R_r, && W_A = L_F(\sigma_1, \lambda^{(1)}), \\ &s = n - r, && B = GL_s, && B_1 = R_s, && W_B = L_F(\sigma_2, \lambda^{(2)}) \circ \cdots \circ L_F(\sigma_a, \lambda^{(a)}). \end{aligned}$$

then $W_A \in \mathrm{Irr}_F(A)$, $W_B \in \mathrm{Irr}_F(B)$ and $V = W_A \circ W_B$. Since $p^c \mid (\lambda^{(i)})'$ for any i, we have $p^c \mid \gcd(r, q-1)$ and $p^c \mid \gcd(s, q-1)$. By induction hypothesis,

$$\kappa_{A_1}(W_A) = \kappa_{A_1 \times B}(W_A \otimes W_B) = p^\alpha \geq p^c$$
$$\kappa_{B_1}(W_B) = \kappa_{A \times B_1}(W_A \otimes W_B) = p^\beta \geq p^c$$

Now choose suitable $x \in A$, $y \in B$ satisfied $\det(x) = \tau$, $\langle \tau \rangle = O_p(\mathbb{F}_q^\times)$, $\det(y) = \tau^{-1}$. Then $\bar{x} = xA_1$ generates A/A_1 and $\bar{y} = yB_1$ generates B/B_1. Since $GL_n/SL_n \cong \mathbb{F}_q^\times$ is independent of n, we have $(A : A_1) = (B : B_1) = (GL_n : R_n)$.

Consider the standard parabolic subgroup $P = QL < GL_n$ with upper unitriangular subgroup Q and Levi subgroup $L = GL_r \times GL_s = A \times B$. Let $H = \langle A_1, B_1, xy \rangle$, then $H/(A_1 \times B_1) = \langle \bar{x}\bar{y} \rangle$ and applying Lemma 3.7,

$$\kappa_H(W_A \otimes W_B) = \gcd(p^\alpha, p^\beta) \geq p^c$$

Note that $H = L \cap R_n$. For this, observe that $\langle A_1, B_1, xy \rangle \leq L \cap R_n$, and check $(L : H) = (L/(A_1 \times B_1) : K/(A_1 \times B_1)) = (GL_n : R_n) = (L : L \cap R_n)$. Multiply Q on the left gives $QH = P \cap R_n$. The following lattice may help.

$$\begin{array}{ccccc} GL_n & \text{———} & P = QL & \text{———} & L \\ | & & | & & | \\ R_n & \text{———} & QH & \text{———} & H \end{array}$$

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From $P \cdot R_n = GL_n$ we have $[P\backslash G/R_n] = \{1\}$. Then

$$\begin{aligned} V{\downarrow}^{GL_n}_{R_n} &= \left((\mathrm{infl}^P_L(W_A \otimes W_B)){\uparrow}^{GL_n}_P\right){\downarrow}^{GL_n}_{R_n} \\ &\cong \left((\mathrm{infl}^P_L(W_A \otimes W_B)){\downarrow}^P_{QH}\right){\uparrow}^{R_n}_{QH} \\ &\cong \left(\mathrm{infl}^{QH}_H\left((W_A \otimes W_B){\downarrow}^L_H\right)\right){\uparrow}^{R_n}_{QH} \end{aligned}$$

Hence $\kappa^{GL_n}_{R_n}(V) \geq \kappa^L_H(W_A \otimes W_B) \geq p^c$. □

Lemma 5.8. *Let $V = L_F(\underline{\sigma}, \underline{\lambda})$ be an irreducible FGL_n-module, $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. Then $\kappa_{R_n}(V) = \gcd\left((GL_n : R_n), \Delta(\underline{\lambda}')\right)$.*

Proof. Write $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$, $\underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(a)})$. For $i = 1, \cdots, a$, $\deg(\sigma_i) = d_i$, $\lambda^{(i)} \vdash k_i$, and $n = \sum_{i=1}^a k_i d_i$.

Note that $(GL_n : R_n)$ is a power of p, thus $\gcd\left((GL_n : R_n), \Delta(\underline{\lambda}')\right) = p^{c(V)}$ for some non-negative integer $c(V)$. Since $p^{c(V)}$ divides $\sum_{i=1}^a |(\lambda^{(i)})'| = n$ and $(GL_n : R_n)$ divides $|\mathbb{F}_q^\times| = q-1$, applying Lemma 5.7 gives $\kappa_{R_n}(V) \geq p^{c(V)}$ if $c(V) \geq 1$. The case $c(V) = 0$ holds trivially.

To force $\kappa_{R_n}(V)$ to match their lower bound, we gives a counting argument. Take $S = R_n$ and $G = A = GL_n$ in Lemma 3.10, the set X containing all irreducible summand of restriction of any non-isomorphic irreducible FGL_n-module, is exactly the set Y containing all non-isomorphic irreducible FR_n-module.

Now $|X|$ equals $\sum \kappa_{R_n}(V)$, sum over $V = L_F(\underline{\sigma}, \underline{\lambda})$ for $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. On the other hand, by Proposition 1.11, $|Y|$ equals to $\sum \gcd\left((GL_n : R_n), \Delta(\underline{\lambda})\right)$, sum over all $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. Since $\underline{\lambda}$ only depends on $\underline{k}$, $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$ means $[(\underline{\sigma}, \underline{\mu})] \in \Sigma_F$ for any $\underline{\mu} \vdash \underline{\lambda}$. In particular, we may write $|Y| = \sum \gcd\left((GL_n : R_n), \Delta(\underline{\lambda}')\right) = \sum p^{c(V)}$, sum over $V = L_F(\underline{\sigma}, \underline{\lambda})$ for $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_F$. This forces the inequality of each $\kappa_{R_n}(V) \geq p^{c(V)}$ is actually equality, hence proves the lemma. □

Now we can prove Theorem 5.2.

Proof. By Lemma 3.6 we have $\kappa_{SL_n}(V) = \kappa_{T_n}(V) \cdot \kappa_{R_n}(V)$. From Lemma 3.5, $\kappa_{T_n}(V) = \#\{L \in \mathrm{Irr}_F(GL_n/T_n) \mid V \cong V \otimes L\}$. The tensor product is describe in Lemma 2.34,

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which proves exactly $\kappa_{T_n}(V) = \kappa_{p'}(\underline{\sigma}, \underline{\lambda})$. Lemma 5.8 gives $\kappa_{R_n}(V) = \kappa_p(\underline{\sigma}, \underline{\lambda})$, which proves (1).

In the case $\mathbf{F} = K$, any finite group G is a p'-group, and $O_{p'}(G)$ is G itself. Hence again by Lemma 3.5 and Lemma 2.34 we have $\kappa_{SL_n}(V_K) = \kappa(\underline{\sigma}, \underline{\lambda})$. □

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6 Main Theorem

6.1 The Canonical Composition Factor

Given $V_K = L_K(\underline{\sigma}, \underline{\lambda}) \in \mathrm{Irr}_K(GL_n)$, where $[(\underline{\sigma}, \underline{\lambda})]$ is an n-admissable symbol, we consider $\overline{V_K}$, its reduction modulo p. Then $(\underline{\sigma}, \underline{\lambda})$ may not be p-regular, even not p-non-repeated. Nevertheless, we can find a corresponding p-regular n-admissable symbol $[(\underline{\sigma}^*, \underline{\lambda}^*)]$, called the p-regularization of $[(\underline{\sigma}, \underline{\lambda})]$, such that $V = L_F(\underline{\sigma}^*, \underline{\lambda}^*) \in \mathrm{Irr}_F(GL_n)$ has some good properties related to V_K.

Let $[(\underline{\sigma}, \underline{\lambda})]$ be an n-admissable symbol, where $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$, $\underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(a)})$ with $\lambda^{(i)} \vdash k_i$. For each σ_i, let $\sigma_i' = (\sigma)_{p'}$, $\upsilon_i = (\sigma)_p$, $d_i = \deg(\sigma_i)$ and $f_i = \deg(\sigma_i)/\deg(\sigma_i')$. Consider the index set $A = \{1, \cdots, a\}$ with the equivalence relation $i_1, i_2 \in A$, $i_1 \sim i_2$ if $[\sigma_{i_1}'] = [\sigma_{i_2}']$. Under this equivalence relation, A is split into parts $A_1, \cdots, A_b$. For each $j = 1, \cdots, b$, pick some $i_h \in A_j$ and take $\sigma_j^* = \sigma_{i_h}'$ as a representative. Let $\lambda^{(j)*} = [+]_{i_h \in A_j} [f_{i_h}] \lambda^{(i_h)}$, and set $\underline{\sigma}^* = (\sigma_1^*, \cdots, \sigma_b^*)$, $\underline{\lambda}^* = (\lambda^{(1)*}, \cdots, \lambda^{(b)*})$. It is clear that $(\underline{\sigma}^*, \underline{\lambda}^*)$ is p-regular and p-non-repeated.

Definition 6.1. The p-regular n-admissable symbol $[(\underline{\sigma}^*, \underline{\lambda}^*)]$ defined above is called the p-regularization of $[(\underline{\sigma}, \underline{\lambda})]$.

By Proposition 4.6, $[\sigma_1] = [\sigma_2]$ implies $[\sigma_1'] = [\sigma_2']$. Hence the p-regularization is well-defined.

Lemma 6.2. *Let $V_K = L_K(\sigma, \alpha^{(1)}) \circ \cdots \circ L_K(\sigma, \alpha^{(m)})$ and $\alpha = [+]_{i=1}^m \alpha^{(i)}$. Then V_K has a composition factor of $L_K(\sigma, \alpha)$ of multiplicity one, and all other factor is of the form $L_K(\sigma, \beta)$ with $\beta \trianglerighteq \alpha$.*

Proof. By Theorem 2.24, in the Grothendieck group of KGL_n-modules we have

$$L_K(\sigma, \alpha) \circ L_K(\sigma, \beta) = \sum_{\gamma \vdash |\alpha| + |\beta|} c_{\alpha\beta}^{\gamma} L_K(\sigma, \gamma)$$

where $c_{\alpha\beta}^{\gamma}$ are Littlewood-Richardson coefficients. By Proposition 2.25 if $\delta = \alpha \ [+] \ \beta$, then $c_{\alpha\beta}^{\delta} = 1$ and $c_{\alpha\beta}^{\gamma} > 0$ implies $\gamma \trianglerighteq \delta$. Now the result follows by induction on m. $\square$

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Lemma 6.3. *Given σ an p'-element, let $V = L_F(\sigma, \lambda^{(1)}) \circ \cdots \circ L_F(\sigma, \lambda^{(m)})$ and $\lambda = [+]_{i=1}^m \lambda^{(i)}$. Then V has a composition factor of $L_F(\sigma, \lambda)$ of multiplicity one, and all other factor is of the form $L_F(\sigma, \mu)$ with $\mu \trianglerighteq \lambda$.*

Proof. By Theorem 2.28, if $\deg(\sigma) = d$, $\beta \vdash k$, then in the Grothendieck group of FGL_{kd}-modules we may write

$$\overline{L_K(\sigma, \beta)} = L_F(\sigma, \beta) + \sum_{\alpha \trianglerighteq \beta, \alpha \neq \beta} w_{\beta\alpha} L_F(\sigma, \alpha)$$

This gives an integer lower unitriangular matrix $(w_{\beta\alpha})$ with index as partition ordered by dominance order. Its inverse $(x_{\beta\alpha})$ is also an integer lower unitriangular matrix. Note that the set $\Phi_{\trianglerighteq} = \{(\alpha, \beta) \mid \alpha \trianglerighteq \beta\}$ is a closed subset (see §2.2) of $\Phi_{\geq} = \{(\alpha, \beta) \mid \alpha \geq \beta\}$. Hence if all (α, β)-entries of $(w_{\beta\alpha})$ with $(\alpha, \beta) \in \Phi_{\geq} \setminus \Phi_{\trianglerighteq}$ are zero, then so do such entries of $(x_{\beta\alpha})$. Therefore we may write

$$L_F(\sigma, \beta) = \overline{L_K(\sigma, \beta)} + \sum_{\alpha \trianglerighteq \beta, \alpha \neq \beta} x_{\beta\alpha} \overline{L_K(\sigma, \alpha)}$$

Now replace β by $\lambda^{(i)}$ and apply Harish-Chandra induction to get

$$V = \overline{V_K} + \sum x_{\alpha^{(1)}, \cdots, \alpha^{(m)}} \overline{L_K(\sigma, \alpha^{(1)}) \circ \cdots \circ L_K(\sigma, \alpha^{(m)})}$$

which sum over $\alpha^{(i)} \trianglerighteq \lambda^{(i)}$ for all i and $\alpha^{(j)} \neq \lambda^{(j)}$ for at least one j, $V_K = L_K(\sigma, \lambda^{(1)}) \circ \cdots \circ L_K(\sigma, \lambda^{(m)})$, and $x_{\alpha^{(1)}, \cdots, \alpha^{(m)}} \in \mathbb{Z}$.

Now if $L_F(\sigma, \mu)$ is a composition factor of V, then either it's a composition factor of $\overline{V_K}$, which $\mu = \lambda$ with multiplicity one or $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$; or it's a composition factor of some $\overline{L_K(\sigma, \alpha^{(1)}) \circ \cdots \circ L_K(\sigma, \alpha^{(m)})}$, which $\mu \trianglerighteq [+]_{i=1}^m \alpha^{(i)} \trianglerighteq [+]_{i=1}^m \lambda^{(i)} = \lambda$, and $\mu \neq \lambda$ by the strict inequality of the second $\trianglerighteq$. This proves the lemma. $\square$

Theorem 6.4. *Let $V_K = L_K(\underline{\sigma}, \underline{\lambda})$ be an irreducible KGL_n-module. Then $\overline{V_K}$ has $L_F(\underline{\sigma}^*, \underline{\lambda}^*)$ as a composition factor with multiplicity one, and all other factor is of the form $L_F(\underline{\sigma}^*, \underline{\nu})$ with $\underline{\nu} \trianglerighteq \underline{\lambda}^*$.*

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Proof. Adopt the notation before Definition 6.1, such as σ_i, σ_i', k_i, d_i, f_i, partition set $A_1, \cdots, A_b$, and the corresponding σ_i^*, λ_i^*.

Now consider some $i \in A_j$ for some j. Then by Corollary 4.15,

$$\begin{aligned}\overline{L_K(\sigma_i, \lambda^{(i)})} &= L_F(\sigma_i, \lambda^{(i)}) + \sum_{\beta \trianglerighteq \lambda^{(i)}, \beta \neq \lambda^{(i)}} x_\beta^{(i)} L_F(\sigma_i, \beta) \\ &= L_F(\sigma_i', [f_i]\lambda^{(i)}) + \sum_{\beta \trianglerighteq \lambda^{(i)}, \beta \neq \lambda^{(i)}} x_\beta^{(i)} L_F(\sigma_i', [f_i]\beta)\end{aligned}$$

in the Grothendieck group of $FGL_{k_i d_i}$-modules, $x_\beta^{(i)} \in \mathbb{Z}$.

For $i_1, \cdots, i_m \in A_j$, let $V_{K,j} = L_K(\sigma_{i_1}, \lambda^{(i_1)}) \circ \cdots \circ L_K(\sigma_{i_m}, \lambda^{(i_m)})$ and let $V_j = L_F(\sigma_j^*, [f_{i_1}]\lambda^{(i_1)}) \circ \cdots \circ L_F(\sigma_j^*, [f_{i_m}]\lambda^{(i_m)})$, then

$$\overline{V_{K,j}} = V_j + \sum x_{\beta^{(i_1)}, \cdots, \beta^{(i_m)}} L_F(\sigma_j^*, f_{i_1}\beta^{(i_1)}) \circ \cdots \circ L_F(\sigma_j^*, [f_{i_m}]\beta^{(i_m)})$$

which sum over $\beta^{(i_h)} \trianglerighteq \lambda^{(i_h)}$ for all $i_h \in A_j$, and $\beta^{(i_h)} \neq \lambda^{(i_h)}$ for at least one $i_h \in A_j$, $x_{\beta^{(i_1)}, \cdots, \beta^{(i_m)}} \in \mathbb{Z}$.

By Lemma 6.3, any composition factor of RHS is of the form $L_F(\sigma_j^*, \mu)$, either it is a composition factor of V_j, which has $\mu = \lambda^{(j)*}$ with multiplicity one or $\mu \trianglerighteq \lambda^{(j)*}$ and $\mu \neq \lambda^{(j)*}$; or it is a composition factor of some $L_F(\sigma_j^*, [f_{i_1}]\beta^{(i_1)}) \circ \cdots \circ L_F(\sigma_j^*, [f_{i_m}]\beta^{(i_m)})$, which has

$$\mu \trianglerighteq \left[\biguplus_{i_h \in A_j}\right] [f_{i_h}]\beta^{(i_h)} \trianglerighteq \left[\biguplus_{i_h \in A_j}\right] [f_{i_h}]\lambda^{(i_h)} = \lambda^{(j)*}$$

with strict inequality from the second $\trianglerighteq$. Hence in particular $L_F(\sigma_j^*, \lambda^{(j)*})$ is a composition factor of $\overline{V_{K,j}}$ with multiplicity one.

Finally, taking Harish-Chandra induction over all $V_{K,j}$ gives V_K. Hence $L_F(\underline{\sigma}^*, \underline{\lambda}^*)$ is a composition factor of $\overline{V_K}$ with multiplicity one. □

Corollary 6.5. *$GL_n(q)$ has (C, p)-property, hence (L'', p)-property, (L', p)-property, and (L, p)-property for p not dividing q.*

Proof. This is the direct result of the previous theorem. □

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6.2 Main Results

In Definition 1.7, we have defined a partial order of n-admissible symbol, hence a partial order of irreducible representations of GL_n. Now we are going to define a partial order for irreducible representations of SL_n.

Definition 6.6. Let $[(\underline{\sigma}, \underline{\lambda})]$, $[(\underline{\tau}, \underline{\nu})]$ be n-admissible symbol.

(1) Denote $[(\underline{\sigma}, \underline{\lambda})] \trianglerighteq_{p'} [(\underline{\tau}, \underline{\nu})]$ if there exists some $\rho \in O_{p'}(\mathbb{F}_q^\times)$ such that $\rho \cdot [(\underline{\sigma}, \underline{\lambda})] \trianglerighteq [(\underline{\tau}, \underline{\nu})]$. Write $[(\underline{\sigma}, \underline{\lambda})] =_{p'} [(\underline{\tau}, \underline{\nu})]$ if $\rho \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\tau}, \underline{\nu})]$.

(2) If $V_1 = L_F(\underline{\sigma}, \underline{\lambda})$, $V_2 = L_F(\underline{\tau}, \underline{\nu})$, then write $V_1 \trianglerighteq_{p'} V_2$ if $[(\underline{\sigma}, \underline{\lambda})] \trianglerighteq_{p'} [(\underline{\tau}, \underline{\nu})]$, and $V_1 =_{p'} V_2$ if $[(\underline{\sigma}, \underline{\lambda})] =_{p'} [(\underline{\tau}, \underline{\nu})]$.

(3) By Proposition 3.11, we have

$$L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})\downarrow_{SL_n} = \bigoplus_{i=1}^{\kappa} Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; i)$$

for $\kappa = \kappa_{SL_n}(L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}))$. Define any total order on these $Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; i)$, for example, $Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; i) \geq Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; j)$ if $i \geq j$.

(4) For $W_i \in \mathrm{Irr}_F(SL_n)$, write $W_i \mid (V_i)\downarrow_S$ for some $V_i \in \mathrm{Irr}_F(GL_n)$, $i = 1, 2$. Denote $W_1 \trianglerighteq W_2$ if either $V_1 \trianglerighteq_{p'} V_2$ but $V_1 \neq_{p'} V_2$, or $V_1 =_{p'} V_2$ and $W_1 \geq W_2$ with total order defined in (3).

Lemma 6.7.

(1) *The order $\trianglerighteq_{p'}$ defined in Definition 6.6(1) is a partial order on $\Sigma_{\mathbf{F}}/O_{p'}(\mathbb{F}_q^\times)$.*

(2) *The order $\trianglerighteq$ defined in Definition 6.6(4) is a partial order on $\mathrm{Irr}_F(SL_n)$.*

Proof. (1) Clear. (2) If $W_1 \cong W_2$, then by Proposition 3.11, $V_1 \cong L_F(\rho, (n)) \otimes V_2$ for some $\rho \in O_{p'}(\mathbb{F}_q^\times)$, which is equivalent to $V_1 =_{p'} V_2$, so the reflexivity follows. The transitivity is clear. For the anti-symmetry, if $W_1 \trianglerighteq W_2$ and $W_2 \trianglerighteq W_1$, then $V_1 =_{p'} V_2$, $W_1 \geq W_2$ and $W_2 \geq W_1$ gives $W_1 = W_2$ (up to isomorphism.) $\square$

Now we can prove our main theorem of this thesis.

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Theorem 6.8. *For any $W_K \in \mathrm{Irr}_K(SL_n)$, there is some $W \in \mathrm{Irr}_F(SL_n)$ such that W is a summand of $\overline{W_K}$ with multiplicity 1, and if $U \in \mathrm{Irr}_F(SL_n)$ is a summand of $\overline{W_K}$, then $U \trianglerighteq W$.*

Proof. Write $W_K = Y_K(\underline{\sigma}, \underline{\lambda}; i)$. By Theorem 6.4, in the Grothedieck group of GL_n we have

$$\overline{L_K(\underline{\sigma}, \underline{\lambda})} = L_F(\underline{\sigma}^*, \underline{\lambda}^*) + \sum_{\underline{\nu} \trianglerighteq \underline{\lambda}^*, \underline{\nu} \neq \underline{\lambda}^*} x_{\underline{\nu}} L_F(\underline{\sigma}^*, \underline{\nu})$$

where $x_{\underline{\nu}} \in \mathbb{Z}$. Note that this implies $[(\underline{\sigma}^*, \underline{\nu})] \trianglerighteq_{p'} [(\underline{\sigma}^*, \underline{\lambda}^*)]$ and $[(\underline{\sigma}^*, \underline{\nu})] \neq_{p'} [(\underline{\sigma}^*, \underline{\lambda}^*)]$, since the multipartition part must be inequality.

Consider the G-tile $[L_K(\underline{\sigma}, \underline{\lambda})]_G \times [L_F(\underline{\sigma}^*, \underline{\lambda}^*)]_G$. We claim that $c = 1$ in the mid-tile, by showing $[L_F(\underline{\sigma}^*, \underline{\lambda}^*)]_G$ contains only one element $L_F(\underline{\sigma}^*, \underline{\lambda}^*)$. Assume $L_F(\underline{\sigma}^*, \underline{\nu}) \cong L_F(\rho\underline{\sigma}^*, \underline{\lambda}^*)$ for some $\rho \in O_{p'}(\mathbb{F}_q^\times)$, $\underline{\nu} \trianglerighteq \underline{\lambda}^*$. Then $\rho\underline{\sigma}^* = \underline{\sigma}^*\pi$ for some $\pi \in \mathfrak{S}_a$, and $\underline{\nu} = \underline{\lambda}^*\pi^{-1} \trianglerighteq \underline{\lambda}^*$. This implies $\underline{\lambda}^*\pi^{-1} = \underline{\lambda}^*$, and $c = 1$ follows.

Now by Proposition 3.14(4), $\overline{Y_K(\underline{\sigma}, \underline{\lambda}; i)}$ contains κ^*/κ composition factors of the form $Y_F(\underline{\sigma}^*, \underline{\lambda}^*; j)$. Pick the minimum j_0 and set $W = Y_F(\underline{\sigma}^*, \underline{\lambda}^*; j_0)$, then W is the required composition factor. □

Corollary 6.9. *$SL_n(q)$ has (C, p)-property, hence (L'', p)-property, (L', p)-property, and (L, p)-property for p not dividing q.*

Proof. This is the direct result of the previous theorem. □

Corollary 6.10. *$PSL_n(q)$ has (C, p)-property, hence (L'', p)-property, (L', p)-property, and (L, p)-property for p not dividing q.*

Proof. By definition, $PSL_n(q) = SL_n(q)/Z(SL_n(q))$, where $Z(SL_n(q))$ is the center of $SL_n(q)$. Then by Corollary 3.18, $PSL_n(q)$ has (C, p)-property, and thus other properties. □

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6.3 Relations Between Branching Numbers

Given $V_K = L_K(\underline{\sigma}, \underline{\lambda})$ irreducible representation of GL_n over K, and its canonical p-regularization $V = L_F(\underline{\sigma}^*, \underline{\lambda}^*)$ over F, we want to find the relation between their branching numbers, $\kappa_{SL_n}(V_K)$ and $\kappa_{SL_n}(V)$. Recall that by Theorem 5.2, we have the branching number in K,

$$\kappa_{SL_n}(V_K) = \#\{\alpha \in \mathbb{F}_q^\times \mid \alpha \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\}$$

and the branching number in F,

$$\begin{aligned}
\kappa_{SL_n}(V) &= \kappa_{p'}(\underline{\sigma}^*, \underline{\lambda}^*) \cdot \kappa_p(\underline{\sigma}^*, \underline{\lambda}^*) \\
\kappa_{p'}(\underline{\sigma}^*, \underline{\lambda}^*) &= \#\{\alpha \in O_{p'}(\mathbb{F}_q^\times) \mid \alpha \cdot [(\underline{\sigma}^*, \underline{\lambda}^*)] = [(\underline{\sigma}^*, \underline{\lambda}^*)]\} \\
\kappa_p(\underline{\sigma}^*, \underline{\lambda}^*) &= |\gcd(n, q-1, \Delta((\underline{\lambda}^*)'))|_p
\end{aligned}$$

Here we also split $\kappa_{SL_n}(V_K)$ into its p'-factor $\kappa_{p'}(V_K)$ and p-factor $\kappa_p(V_K)$.

Lemma 6.11. *Let $[(\underline{\sigma}, \underline{\lambda})] \in \Sigma_K$, $V_K = L_K(\underline{\sigma}, \underline{\lambda})$, and $V = L_F(\underline{\sigma}^*, \underline{\lambda}^*)$. Then*

(1) *$\kappa_{p'}(V_K)$ divides $\kappa_{p'}(\underline{\sigma}^*, \underline{\lambda}^*)$.*

(2) *$\kappa_p(V_K)$ divides $\kappa_p(\underline{\sigma}^*, \underline{\lambda}^*)$.*

Therefore $\kappa_{SL_n}(V_K)$ divides $\kappa_{SL_n}(V)$.

Proof. Adopt the notation before Definition 6.1, such as σ_i, σ_i', k_i, d_i, f_i, partition set $A_1, \cdots, A_b$, and the corresponding σ_i^*, λ_i^*.

(1) Let $I' = \{\beta \in O_{p'}(\mathbb{F}_q^\times) \mid \beta \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\} = \langle \rho \rangle$. We are going to prove that ρ also fixes the p-regularization of $[(\underline{\sigma}, \underline{\lambda})]$.

By Proposition 4.6, $[\rho\sigma_{i_1}] = [\sigma_{i_2}]$ implies $[\rho\sigma_{i_1}'] = [\sigma_{i_2}']$. Then ρ send $[\sigma_{i_1}]$ with $i_1 \in A_{j_1}$ to $[\sigma_{i_2}]$ with $i_2 \in A_{j_2}$, hence actually ρ send every $[\sigma_{i_1}]$ with $i_1 \in A_{j_1}$ to $[\sigma_{i_2}]$ with $i_2 \in A_{j_2}$, and ρ^{-1} do the inverse. Therefore A_{j_1} is bijective to A_{j_2} via ρ, and also the corresponding $f_{i_1} = f_{i_2}$ and $\lambda^{(i_1)} = \lambda^{(i_2)}$. This implies $\lambda^{(j_1)*} = \lambda^{(j_2)*}$, and ρ send $[\sigma_{j_1}^*]$ to $[\sigma_{j_2}^*]$, which means ρ fixes $[(\underline{\sigma}^*, \underline{\lambda}^*)]$ and $|I'|$ divides $\kappa_{p'}(\underline{\sigma}^*, \underline{\lambda}^*)$.

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(2) Let $I = \{\alpha \in O_p(\mathbb{F}_q^\times) \mid \alpha \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\} = \langle \tau \rangle$. By definition, $\kappa_p(V_K) = |\tau| = p^e$ for some e. We are going to prove that $p^c \mid \lambda^{(j)*\prime}$ for each j.

Consider some $i_1 \in A_j$ for some j. Since τ fixed $[(\underline{\sigma}, \underline{\lambda})]$, $[\tau\sigma_{i_1}] = [\sigma_{i_2}]$ for some i_2, with corresponding $\lambda^{(i_2)} = \lambda^{(i_1)}$. Moreover, by Proposition 4.6, $[\tau\sigma_{i_1}] = [\sigma_{i_2}]$ implies $[\sigma'_{i_1}] = [\sigma'_{i_2}]$, we have $i_2 \in A_j$. Therefore, there is an orbit $\mathcal{O} \subset A_j$ such that $[\sigma_{i_1}], [\tau\sigma_{i_1}], \cdots, [\tau^{p^c-1}\sigma_{i_1}]$ are p^c distinct elements, $0 \leq c \leq e$, each $[\tau^{h-1}\sigma_{i_1}] = [\sigma_{i_h}]$ for some $i_h \in \mathcal{O}$, with corresponding $\lambda^{(i_h)} = \lambda^{(i_1)}$. Hence τ^{p^c} fixed $[\sigma_{i_h}]$ for all $i_h \in \mathcal{O}$, so $I_{i_h} = \{\alpha \in O_p(\mathbb{F}_q^\times) \mid [\alpha\sigma_{i_h}] = [\sigma_{i_h}]\} = \langle \tau^{p^c} \rangle$, and applying Lemma 4.17 we have $|I_{i_h}|$ divides $|\gcd(f_{i_h}, q-1)|_p$, in particular $|I_{i_h}| = p^{e-c} \mid f_{i_h}$. Note that f_{i_h} are all the same over $i_h \in \mathcal{O}$, thus $p^e \mid \lambda'_{\mathcal{O}}$, where

$$\lambda'_{\mathcal{O}} = \left(\underset{i_h \in \mathcal{O}}{\left[+\right]} [f_{i_h}] \lambda^{(i_h)} \right)' = \sum_{i_h \in \mathcal{O}} f_{i_h} \lambda^{(i_h)\prime} = p^c f_{i_1} \lambda^{(i_1)\prime}$$

Now consider each orbit of A_j, we have $p^e \mid \lambda^{(j)*\prime} = \sum_{\mathcal{O} \subset A_j} \lambda'_{\mathcal{O}}$. Therefore p^e divides $\Delta((\underline{\lambda}^*)')$, and of course n and $q-1$, hence $\kappa_p(\underline{\sigma}^*, \underline{\lambda}^*)$. $\square$

6.4 A Lower Unitriangular Submatrix

For completeness, we show that SL_n have (U,p)-property for p not dividing q, which is a consequence of Kleshchev's theorem.

Proposition 6.12. *Given an p-regular n-admissible symbol $[(\underline{\tau}, \underline{\nu})]$, there is some n-admissible symbol $[(\underline{\sigma}, \underline{\lambda})]$, such that*

(1) $[(\underline{\sigma}^*, \underline{\lambda}^*)] = [(\underline{\tau}, \underline{\nu})]$.

(2) $\kappa_{SL_n}(L_K(\underline{\sigma}, \underline{\lambda})) = \kappa_{SL_n}(L_F(\underline{\tau}, \underline{\nu}))$.

Proof. (1) Write $\underline{\tau} = (\tau_1, \cdots, \tau_a)$ and $\underline{\nu} = (\nu^{(1)}, \cdots, \nu^{(a)})$, the index set $A = \{1, \cdots, a\}$. Let $\kappa_p(\underline{\tau}, \underline{\nu}) = p^c$ for some integer $c \geq 0$, and $\kappa_{p'}(\underline{\tau}, \underline{\nu}) = |J'|$, where $J' = \{\beta \in O_{p'}(\mathbb{F}_q^\times) \mid \beta \cdot [(\underline{\tau}, \underline{\nu})] = [(\underline{\tau}, \underline{\nu})]\} = \langle \rho \rangle$. Since $p^c \mid (\nu^i)'$ for all i, choose $\lambda^{(i)}$ such that $\nu^{(i)} = [p^c]\lambda^{(i)}$, and let $\underline{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(a)})$.

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Given some τ_{i_1}, consider the orbit $\mathcal{O} \subset A$ such that $[\tau_{i_1}], [\rho\tau_{i_1}], \cdots, [\rho^{b-1}\tau_{i_1}]$ are all distinct, $[\rho^{h-1}\tau_{i_1}] = [\tau_{i_h}]$ for some $i_h \in \mathcal{O}$, and the corresponding $\nu^{(i_h)} = \nu^{(i_1)}$, $1 \le h \le b$. We may repick τ_{i_h} such that $\rho^{h-1}\tau_{i_1} = \tau_{i_h}$ without changing $[(\underline{\tau}, \underline{\nu})]$. By Proposition 4.6, all τ_{i_h} have same degree d, hence by Lemma 4.18, there is some p-element $\upsilon \in \overline{\mathbb{F}}_q^{\times}$, such that $\sigma_{i_h} = \tau_{i_h}\upsilon$ have degree $p^c d$.

Then $\underline{\sigma} = (\sigma_1, \cdots, \sigma_a)$ is constructed by considering all orbits of A. It is not hard to see that $[(\underline{\sigma}^*, \underline{\lambda}^*)] = [(\underline{\tau}, \underline{\nu})]$, so this proves (1).

(2) First we show that $\kappa_{p'}(L_K(\underline{\sigma}, \underline{\lambda})) = \kappa_{p'}(\underline{\tau}, \underline{\nu})$. Let $I' = \{\beta \in O_{p'}(\mathbb{F}_q^{\times}) \mid \beta \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\}$. If $\beta \in I'$ send $[\sigma_{i_1}]$ to $[\sigma_{i_2}]$, then the corresponding $\lambda^{(i_1)} = \lambda^{(i_2)}$, hence $\nu^{(i_1)} = \nu^{(i_2)}$ and β also send $[\tau_{i_1}]$ to $[\tau_{i_2}]$, so $I' \subset J'$.

Conversely, we want to prove $\rho \in I'$. If p not divide $q-1$, then $[(\underline{\sigma}, \underline{\lambda})] = [(\underline{\tau}, \underline{\nu})]$, so assume $p \mid q-1$. From the orbit $\mathcal{O}$ above, we have $\tau_{i_h} = \rho^{h-1}\tau_{i_1}$, thus ρ sends σ_{i_h} to $\sigma_{i_{h+1}}$ for $1 \le h \le b-1$. Thus it suffices to show that $[\alpha\tau_{i_1}] = [\tau_{i_1}]$ implies $[\alpha\sigma_{i_1}] = [\sigma_{i_1}]$ for $\alpha = \rho^b$. There is some integer $j \ge 0$ such that $\alpha\tau_{i_1} = (\tau_{i_1})^{q^j}$, so $\alpha = (\tau_{i_1})^{q^j - 1}$. Take $|\upsilon| = p^e$ for some integer $e \ge 0$. Since $p \mid q-1$, by Lemma 4.9 $p^e \mid q^{p^e} - 1$. Then with $\alpha^q = \alpha$ we have

$$\alpha^{p^e} = \alpha^{1+q^j+\cdots+q^{(p^e-1)j}} = (\tau_{i_1})^{q^{jp^e}-1} = (\tau_{i_1}\upsilon)^{q^{jp^e}-1}$$

hence $[\alpha^{p^e}\sigma_{i_1}] = [\sigma_{i_1}]$. Now α is p-regular, so α also fixed $[\sigma_{i_1}]$.

Next we shall prove that $\kappa_p(L_K(\underline{\sigma}, \underline{\lambda})) = \kappa_p(\underline{\tau}, \underline{\nu})$, that is, $|I| = p^c$, where $I = \{\beta \in O_p(\mathbb{F}_q^{\times}) \mid \beta \cdot [(\underline{\sigma}, \underline{\lambda})] = [(\underline{\sigma}, \underline{\lambda})]\}$. But since $[(\underline{\sigma}, \underline{\lambda})]$ is p-non-repeated by construction, this implies $I = \bigcap_{i=1}^{a} I_i$, $I_i = \{\beta \in O_p(\mathbb{F}_q^{\times}) \mid [\beta\sigma_i] = [\sigma_i]\}$ for each i. Then Lemma 4.18 claim that $|I_i| = p^c$ for each i, thus also for $|I|$. □

Theorem 6.13. *Let W be an irreducible FSL_n-module. Then there is an irreducible KSL_n-module W_K, such that W is a composition factor of $\overline{W_K}$ with multiplicity 1, and if U is a composition factor of $\overline{W_K}$, then $U \trianglerighteq W$.*

Proof. Write $W = Y_F(\underline{\tau}, \underline{\nu}; j_0)$. Then by Proposition 6.12, there is some $[(\underline{\sigma}, \underline{\lambda})] \in$

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Σ_K such that $[(\underline{\sigma}^*, \underline{\lambda}^*)] = [(\underline{\tau}, \underline{\nu})]$ and $\kappa_{SL_n}(L_K(\underline{\sigma}, \underline{\lambda})) = \kappa_{SL_n}(L_F(\underline{\tau}, \underline{\nu}))$. A similar consideration to Theorem 6.8 shows that each $\overline{Y_K(\underline{\sigma}, \underline{\lambda}; i)}$ contains $\kappa^*/\kappa = 1$ composition factors of the form $Y_F(\underline{\sigma}^*, \underline{\lambda}^*; j)$. Relabel i such that $\overline{Y_K(\underline{\sigma}, \underline{\lambda}; i)} \cong Y_F(\underline{\tau}, \underline{\nu}; i)$ for $i = 1, \cdots, \kappa$, then $W_K = Y_K(\underline{\sigma}, \underline{\lambda}; j_0)$ is the required module. □

Corollary 6.14. *$SL_n(\mathbb{F}_q)$ has (U, p)-property for p not dividing q.*

Proof. This is equivalent to the previous theorem. □

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7 Conclusion

The main result is Theorem 6.8, that $SL_n(q)$ has (C,p)-property of non-defining characteristic. In the following we list a table of related known result. Recall that $(C) \Rightarrow (L'') \Rightarrow (L') \Rightarrow (L)$ and $(R) \Rightarrow (L), (U)$.

	(C)	(L'')	(L')	(L)	(R)	(U)
$\mathfrak{S}_n$	◯	◯	◯	◯	×	◯
$\mathfrak{A}_n$	?	?	◯	◯	×	?
$p \nmid q$, $GL_n(q)$	◯	◯	◯	◯	×	◯
$p \nmid q$, $SL_n(q)$	◯	◯	◯	◯	×	◯
abelian group	◯	◯	◯	◯	◯	◯
p-solvable group	×	×	×	◯	◯	◯

Table 2: The property table for some families of groups

- $\mathfrak{S}_n$, symmetric group.

  James' Regularization Theorem [J1] is exactly (C), and (U) is the consequence that the labels of irreducible Brauer characters are p-regular partitions. A counterexample of (R) is $\mathfrak{S}_6$ for $p = 3$, which the decomposition matrix is listed in Table 1.

- $\mathfrak{A}_n$, alternating group.

  Huang shows that (L') holds [H]. The difficulty of alternating groups is that the Mullineax map distorts the dominance order of partitions, while (C) and (U) needs some partial order. It is also not easy to prove that (L'') is true or not, either, again due to the Mullineax map, which combines entries when generating the decomposition matrix of $\mathfrak{A}_n$ from $\mathfrak{S}_n$.

- $GL_n(q)$, $p \nmid q$, finite general linear group of non-defining characteristic.

  An analogue of James' Regularization Theorem for $GL_n(q)$ gives property (C) [K, Theorem 5.4], which combines the result of [J] and the end-part of [D2]. Kleshchev-Tiep also proves (U) for both $GL_n(q)$ and $SL_n(q)$, $p \nmid q$ [K, Theorem 6.3].

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- $SL_n(q)$, $p \nmid q$, finite special linear group of non-defining characteristic.

  Kleshchev-Tiep only proves (U), while he gives a powerful Kleshchev-Tiep's theorem (Theorem 5.2), which can be use to prove (C), the main result of this thesis, implying (L''), (L'), (L).

- abelian group

  Since every characters are of degree 1, they remains irreducible after reduction modulo p, hence (C) holds automatically. Since d is surjective, the decomposition matrix is of full rank, so (R) also holds.

- p-solvable group

  The well-known Fong-Swan Theorem [S, Theorem 38] shows that (R) holds, while any non-abelian p-group is a counterexample of (L').

Counterexamples of (R) for $\mathfrak{A}_n$, $GL_n(q)$ and $SL_n(q)$ are in Appendix A.4.

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A Appendix

A.1 The Original Problem

Problem 1 in the introduction comes from exercise 16.6 in the textbook of Serre [S]. We start from the *cde*-triangle.

Fixed a finite group G and a prime p. Recall that given $K \subset \overline{\mathbb{Q}}$ a field of characteristic 0, we may pick a valuation ring A, and obtain its residue field $F = A/\mathfrak{m}$ of characteristic p (see the first paragraph of section 1.6.) Here we need K and F to be sufficiently large for G.

Let $R_K(G)$ be the Grothendieck group of KG-modules (see the last paragraph of section 1.6), with the basis $\beta_K = \{[V_i]_K \mid V_i \in \mathrm{Irr}_K(G)\}$. Similarly, let $R_F(G)$ be the Grothendieck group of FG-modules, with the basis $\beta_F = \{[E_j]_F \mid E_j \in \mathrm{Irr}_F(G)\}$. Then the decomposition map is defined to be

$$d : R_K(G) \to R_F(G), \quad [V]_K \mapsto [\overline{V}]_F$$

where $\overline{V}$ is a reduction modulo p of V (see section 1.7). The decomposition matrix of G is the transpose of the matrix of d with respect to β_K, β_F.

Now we briefly recall the definitions and properties of projective modules.

Definition A.1. Let R be a ring, P be an left R-module.

(1) We say P is projective if any of the following equivalent condition holds:

  (i) P is a direct summand of some free R-module.

  (ii) Given any surjective R-module homomorphism $f : E \to E'$, and any R-module homomorphism $g' : P \to E'$, there exists a homomorphism $g : P \to E$ such that $g' = f \circ g$.

  (iii) For any exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of left R-modules, the sequence $0 \to \mathcal{F}(E_1) \to \mathcal{F}(E) \to \mathcal{F}(E_2) \to 0$ is also exact, where $\mathcal{F}$ is the functor $E \mapsto \mathrm{Hom}_R(P, E)$.

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(2) Assume R is artinian. We say an R-module homomorphism $f : E \to E'$ is essential, if $f(E) = E'$, and $f(U) \neq E'$ for any proper submodule U of E. Note that if f is essential, then f must be surjective.

(3) We say P is a projective envelope of an R-module E, if P is projective, and there exists an essential homomorphism $f : P \to E$.

In particular, the group ring FG is artinian.

Proposition A.2. *Let E be an FG-module.*

(1) *There exists a projective envelope of E, unique up to isomorphism.*

(2) *Let P_j be the projective envelope of $E_j \in \mathrm{Irr}_F(G)$ for each j. Then P_j is indecomposible, and every projective FG-module is a direct sum of these P_j.*

Proof. For a proof, see [S, Proposition 41] and its corollaries. □

Therefore, we may define $\mathcal{P}_F(G)$ to be the Grothendieck group of projective FG-modules with the basis $\beta_{\mathcal{P}} = \{[P_j]_{\mathcal{P}}\}$, where P_j is the projective envelope of $E_j \in \mathrm{Irr}_F(G)$ in the same order as β_F. Then we naturally have the Cartan homomorphism:

$$c : \mathcal{P}_F(G) \to R_F(G), \quad [P]_{\mathcal{P}} \mapsto [P]_F$$

which roughly means the decomposition of a projective module into its composition factors.

To define the map e, we consider the projective AG-modules, where A is the valuation ring with $F = A/\mathfrak{m}$. Given $\hat{E}$ an AG-module, the quotient $\hat{E}/\mathfrak{m}\hat{E}$ is an FG-module. Denote this map $\pi_{\mathfrak{m}}$.

Proposition A.3.

(1) *Every projective AG-module is a direct sum of indecomposible projective modules Q_j, which is characterized (up to isomorphism) by $\pi_{\mathfrak{m}}(Q_j) = P_j$.*

(2) *Let $\mathcal{P}_A(G)$ be the Grothendieck group of projective AG-modules. Then we may identify $\mathcal{P}_A(G)$ and $\mathcal{P}_F(G)$ via the map $\pi_{\mathfrak{m}}$.*

doi:10.6342/NTU201904116

Proof. For a proof, see [S, Proposition 42] and its corollaries. □

Now the map e is given by

$$e : \mathcal{P}_F(G) \to R_K(G), \quad [P]_{\mathcal{P}} \mapsto [\hat{P}]_K$$

where $\hat{P} = KG \otimes_{AG} \pi_{\mathfrak{m}}^{-1}(P)$. That is, first pass P to the AG-module via the inverse of $\pi_{\mathfrak{m}}$, then tensor with K to obtain the KG-module $\hat{P}$. It roughly means that the there is an inverse of reduction modulo p on projective FG-modules.

Here we list some basic properties of the *cde*-triangle.

Proposition A.4.

(1) $c = d \circ e$. *Hence the following diagram commutes.*

$$\begin{tikzcd} \mathcal{P}_F(G) \arrow[rr, "c"] \arrow[dr, "e"'] & & R_F(G) \\ & R_K(G) \arrow[ur, "d"'] & \end{tikzcd}$$

(2) *With the basis β_K, β_F, $\beta_{\mathcal{P}}$, the matrix of d is transpose to the matrix of e. Hence the matrix of c (the Cartan matrix) is symmetric.*

(3) *d is surjective. Hence e is a split injection.*

Proof. See section 15.4 and 16.1 of [S]. □

Note that in this thesis, the decomposition matrix of G is actually the matrix of e, the way as the decomposition matrix of symmetry groups list by James.

Despite of $R_K(G)$, we are often more interesting on $R_K^+(G) = \{[V]_K\}$, the case that there indeed exists a KG-module V, which can be characterized as $\{\sum_i c_i[V_i]_K \mid c_i \in \mathbb{Z}, c_i \geq 0, V_i \in \mathrm{Irr}_K(G)\}$. Similarly we may define $R_F^+(G)$ and $\mathcal{P}_F^+(G)$. Then in general, d no more sends $R_K^+(G)$ onto $R_F^+(G)$, while it sends $R_K(G)$ onto $R_F(G)$. We then have the condition (R).

(R) $d(R_K^+(G)) = R_F^+(G)$. That is, d sends $R_K^+(G)$ onto $R_F^+(G)$.

For the map e, we may also consider a condition (E).

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(E) $e(\mathcal{P}_F^+(G)) = e(\mathcal{P}_F(G)) \cap R_K^+(G)$. That is, e sends $R_K^+(G)$ onto the "positive part" of its original image.

The exercise 16.6 of Serre [S] asks the reader to prove that

The condition (E) is equivalent to the condition (R).

But Huang had difficulty to prove it. He then wrote a letter to the original author, J. P. Serre, asking for help. Serre sent back the modified exercise,

The condition (E) is equivalent to the condition (QR).

and gave a proof for it, with a slightly weaker condition.

(QR) There is some $N \in \mathbb{N}$ such that $N \cdot R_F^+(G) \subset d(R_K^+(G))$.

In the last part of the letter, Serre left an open problem, that whether there exists any group G (and a prime p) such that (QR) is true but (R) is false, in order to ensure the original exercise is wrong. This is Problem 1, which remains unsolved in this thesis.

A proof for the modified exercise is in the Appendix A, B of [H]. Note that the condition (R), (QR) is equivalent to the property (R), (QR) listed in the introduction, respectively.

A.2 The Implication Among Properties

Here we list again the properties in the introduction.

(R) All irreducible Brauer characters of G are liftable.

(QR) All irreducible Brauer characters of G are almost liftable.

(L) If G has (QR,p)-property, then G has (R,p)-property.

(L') For any $\chi \in \mathrm{irr}_K(G)$, if $\overline{\chi} = a\phi$ for some $\phi \in \mathrm{irr}_F(G)$, then $a = 1$.

(L'') For any $\chi \in \mathrm{irr}_K(G)$, $\overline{\chi}$ contains some $\phi \in \mathrm{irr}_F(G)$ of multiplicity 1.

(C) There exists a partial order $\trianglerighteq$ on $\mathrm{irr}_F(G)$, and a map $\mathrm{irr}_K(G) \to \mathrm{irr}_F(G)$, $\chi \mapsto \phi_\chi$, such that for each $\chi \in \mathrm{irr}_K(G)$, $\overline{\chi}$ contains ϕ_χ of multiplicity 1, and if $\overline{\chi}$ contains $\phi \in \mathrm{irr}_F(G)$, then $\phi \trianglerighteq \phi_\chi$.

(U) There exists a partial order $\trianglerighteq$ on $\mathrm{irr}_F(G)$, and a map $\mathrm{irr}_F(G) \to \mathrm{irr}_K(G)$, $\phi \mapsto \chi_\phi$,

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such that for each $\phi \in \mathrm{irr}_F(G)$, $\overline{\chi}_\phi$ contains ϕ of multiplicity 1, and if $\overline{\chi}_\phi$ contains $\phi' \in \mathrm{irr}_F(G)$, then $\phi' \trianglerighteq \phi$.

We are going to find out all implication among these properties excluding (QR). For any two properties (A), (B), write $(A) \implies (B)$ if for any finite group G and prime p, G has (A,p)-property implies G has (B,p)-property, and $(A) \not\Longrightarrow (B)$ means there exists some group G and some prime p, such that G has (A,p)-property, but not (B,p)-property. In this case we say that a counterexample is found for $(A) \not\Longrightarrow (B)$.

Proposition A.5. *We have $(C) \implies (L'') \implies (L') \implies (L)$, $(R) \implies (L)$, and $(R) \implies (U)$.*

Proof. Clearly $(C) \implies (L'') \implies (L')$ from the text of definition. If (L') and (QR) holds, then for all $\phi \in \mathrm{irr}_F(G)$, there is some $\chi \in \mathrm{irr}_K(G)$ such that $\overline{\chi} = a\phi$, but then $a = 1$ and (R) holds, which proves $(L') \implies (L)$. If (R) holds, then (L) holds logically from the definition, hence $(R) \implies (L)$. Finally if (R) holds, for each $\phi \in \mathrm{irr}_F(G)$, there is some $\chi \in \mathrm{irr}_K(G)$ such that $\overline{\chi} = \phi$, then the map $\phi \mapsto \chi$ and any partial order gives (U). This proves $(R) \implies (U)$. □

In the rest of this section, we will show that there are no other implications among these properties.

Lemma A.6. *Given $(A) \implies (a)$ and $(B) \implies (b)$. If $(A) \not\Longrightarrow (b)$, then none of $(A), (a)$ imply any of $(B), (b)$.*

Proof. Assume some of $(A), (a)$ imply some of $(B), (b)$. Then $(A) \implies (b)$. □

Proposition A.7. *We have*

(1) $(R) \not\Longrightarrow (L')$. *Hence none of $(R), (U), (L)$ imply any of $(C), (L''), (L')$.*

(2) $(C) \not\Longrightarrow (R)$. *Hence none of $(C), (L''), (L'), (L)$ imply (R).*

(3) $(U) \not\Longrightarrow (R)$.

(4) $(L') \not\Longrightarrow (L'')$. *Hence $(L') \not\Longrightarrow (C)$, either.*

(5) $(C) \not\Longrightarrow (U)$. *Hence none of $(C), (L''), (L'), (L)$ imply (U).*

doi:10.6342/NTU201904116

(6) $(L'') \not\Rightarrow (C)$.

Proof. For (1), any non-abelian p-group is a counterexample. Here we take $G = Q_8$ the quaternion group and $p = 2$. There are four irreducible ordinary characters of degree 1, and one of degree 2. Let $\chi_1 \in \operatorname{irr}_K(Q_8)$ be the trivial character, and $\chi_2 \in \operatorname{irr}_K(Q_8)$ be that of degree 2. Since the only $\phi \in \operatorname{irr}_F(Q_8)$ is the trivial character, we have $\overline{\chi_2} = 2\phi$ by considering degree, hence (L') fails, while $\overline{\chi_1} = \phi$ and (R) holds.

For (2)(3), take $G = \mathfrak{S}_6$ and $p = 3$ as a counterexample. By James' Regularization Theorem, both (C) and (U) holds for $\mathfrak{S}_n$ with any n and any prime p. From Table 1 in the introduction, $\phi_{(5,1)}$ is not liftable, hence (R) fails.

For (4), take $G = ON$ and $p = 2$, see [web]. One may check that each row contains single nonzero entry 1, or at least 2 nonzero entries, while the row χ_{16} does not have an entry 1.

For (5), take $G = S_4(4)$ and $p = 2$, see [web]. One may check that (C) holds with the original order on the table of block 1. Observe that $\# \operatorname{Irr}_F(G) = 16$. If (U) holds, then there should be at least 13 columns containing at least 3 entries zero, but there are only 11 such columns.

For (6), take $G = Fi_{23}$ and $p = 3$, see [web]. One may check that each row contains some entry 1. If (C) holds, then there is a (minimal) column containing only entries 1 and 0, but there is no such column. □

The only implication unchecked is that $(U) \not\Rightarrow (L)$, which a real example has not been found. If there exists some example, it is a solution to Problem 1.

A.3 The Decomposition Matrix of GL_2 and SL_2

In this section, we are going to find the decomposition matrix of $GL_2(q)$ and $SL_2(q)$ for p not dividing q. We split the procedure into several parts.

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(Step 1) Elements of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$

Write $q-1=p^{c_1}m_1$ and $q^2-1=p^{c_2}m_2$, where c_1, c_2, m_1, m_2 are nonnegative integers, and p does not divide m_1, m_2. It is clear that $c_1 \le c_2$ and $m_1 \mid m_2$.

By Proposition 1.3, every element $\sigma \in \mathbb{F}_{q^2}^\times$ decomposes into its p'-part σ' and p-part υ. Write $e \in \mathbb{F}_{q^2}^\times$ the identity, $\sigma'_{(d)} \in \mathbb{F}_{q^2}^\times$ if it is p-regular of degree d, and $\upsilon_{(d)} \in \mathbb{F}_{q^2}^\times$ if it is a p-element of degree d and is not identity. Then the following multiplication table gives the number of each kind of elements. Note that only $\{\sigma'_{(1)}\}$ and $\{\sigma'_{(1)}\upsilon_{(1)}\}$ are elements of $\mathbb{F}_q^\times$, the others are those of $\mathbb{F}_{q^2}^\times$.

	1	$s_1 := p^{c_1}-1$	$s_2 := p^{c_2}-p^{c_1}$
$r_1 := m_1$	$\#\{\sigma'_{(1)}\}$	$\#\{\sigma'_{(1)}\upsilon_{(1)}\}$	$\#\{\sigma'_{(1)}\upsilon_{(2)}\}$
$r_2 := m_2 - m_1$	$\#\{\sigma'_{(2)}\}$	$\#\{\sigma'_{(2)}\upsilon_{(1)}\}$	$\#\{\sigma'_{(2)}\upsilon_{(2)}\}$

Let $\tau \in \mathbb{F}_q^\times$ be an element of degree 1, τ' for those p-regular. Similarly, let $\omega \in \mathbb{F}_{q^2}^\times$ be an element of degree 2, ω' for those p-regular. The following table shows their corresponding p'-part. The entry $-$ means the element is already p-regular.

element	form	p'-part	form	#
τ	$\sigma'_{(1)}$	$-$	$-$	r_1
	$\sigma'_{(1)}\upsilon_{(1)}$	τ'	$\sigma'_{(1)}$	$r_1 s_1$
ω	$\sigma'_{(2)}$	$-$	$-$	r_2
	$\sigma'_{(2)}\upsilon_{(1)}$	ω'	$\sigma'_{(2)}$	$r_2 s_1$
	$\sigma'_{(2)}\upsilon_{(2)}$	ω'	$\sigma'_{(2)}$	$r_2 s_2$
	$\sigma'_{(1)}\upsilon_{(2)}$	τ'	$\sigma'_{(1)}$	$r_1 s_2$

From now on, we write τ', ω', υ instead of $\sigma'_{(1)}$, $\sigma'_{(2)}$, $\upsilon_{(1)}$, respectively.

(Step 2) Conjugacy classes of GL_2

For simplicity, we write (σ) instead of $(\sigma,(1))$, and write e the identity of $\mathbb{F}_{q^2}^\times$, in order to make difference from the partition $(1) \vdash 1$. The following table shows the conjugacy classes of GL_2, where $C_2^t := t(t-1)/2$ for $t \in \mathbb{N}$.

doi:10.6342/NTU201904116

symbol	$(\tau,(2))$	$(\tau,(1^2))$	$(\tau_1\circ\tau_2)$	(ω)
#	$q-1$	$q-1$	C_2^{q-1}	C_2^q
Jordan	$\begin{bmatrix}\tau & 1\\ & \tau\end{bmatrix}$	$\begin{bmatrix}\tau & \\ & \tau\end{bmatrix}$	$\begin{bmatrix}\tau_1 & \\ & \tau_2\end{bmatrix}$	$\begin{bmatrix}\omega & \\ & \omega^q\end{bmatrix}$
#Cent.	$q(q-1)$	$\lvert GL_2\rvert$	$(q-1)^2$	q^2-1
#Conj.	q^2-1	1	$q(q+1)$	$q(q-1)$

(Step 3) Counting dimension of irreducible representations of GL_2

We make use of the following dimension formulas.

Theorem A.8.

(1) *Let V_1, V_2 be $\mathbf{F}GL_r$-module, $\mathbf{F}GL_s$-module, respectively, $n = r+s$. Then*

$$\dim_{\mathbf{F}} V_1 \circ V_2 = (G:P)\cdot \dim V_1 \cdot \dim V_2$$

where $G = GL_n$, $P = P^+_{(r,s)}$ defined in §2.2.

(2) *Let $M_{\mathbf{F}}(\sigma,\lambda)$ be the $\mathbf{F}GL_n$-module defined in §2.6. Then*

$$\dim_{\mathbf{F}} M_{\mathbf{F}}(\sigma,\lambda) = \prod_{i=1}^{dk}(q^i-1)\Big/\prod_{j=1}^{\infty}\left(\prod_{i=1}^{\lambda_j}(q^{di}-1)\right)$$

(3) *Let $S_{\mathbf{F}}(\sigma,\lambda)$ be the $\mathbf{F}GL_n$-module defined in §2.6. Then Theorem 2.22 (Young's Rule) gives a way to find $\dim_{\mathbf{F}} S_{\mathbf{F}}(\sigma,\lambda)$ from $\dim_{\mathbf{F}} M_{\mathbf{F}}(\sigma,\lambda)$.*

Proof. (1) follows from the definition of Harish-Chandra induction. (2) comes from [J, 6.5, 6.8], which gives $\dim_{\mathbf{F}} M_{\mathbf{F}}(\sigma,(k))$. One may use (1) to extend the formula to (2). For (3), note that Kostka numbers form a unitriangular matrix, hence we may find $\dim_{\mathbf{F}} S_{\mathbf{F}}(\sigma,(k))$ one by one, or just use inverse matrix. □

Hence we may calculate the dimension of irreducible $\mathbf{F}GL_2$-modules.

irr. rep'n	$L_K(\tau,(2))$	$L_K(\tau,(1^2))$	$L_K(\tau_1\circ\tau_2)$	$L_K(\omega)$
#	$q-1$	$q-1$	C_2^{q-1}	C_2^q
$\dim_K$	1	q	$q+1$	$q-1$

The $\dim_F L_F(\underline{\sigma},\underline{\lambda})$ heavily depends on p, and have no general formula.

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(Step 4) The p-regularization and James-Mathas theorem

The p-regularization is defined right before Definition 6.1. Roughly speaking, it consists the following 3 steps.

(1) Decompose each $\sigma_i = \sigma_i' \upsilon_i$ into its p'-part and p-part.

(2) Replace each $(\sigma_i, \lambda^{(i)})$ by $(\sigma_i', [f_i]\lambda^{(i)})$, where $f_i = \deg \sigma_i / \deg \sigma_i'$

(3) If $[\sigma_i] = [\sigma_j]$, then replace $(\sigma_i, \lambda^{(i)}) \circ (\sigma_j, \lambda^{(j)})$ by $(\sigma_i, \lambda^{(i)} \, [+] \, \lambda^{(j)})$.

Theorem 6.4 tells that each $\overline{L_K(\underline{\sigma}, \underline{\lambda})}$ contains $L_F(\underline{\sigma}^*, \underline{\lambda}^*)$ of multiplicity 1, and every composition factor is of the form $L_F(\underline{\sigma}^*, \underline{\nu})$ with $\underline{\nu} \trianglerighteq \underline{\lambda}^*$.

There is another theorem useful to find the entries of a decomposition matrix, which classifies which $\overline{L_K(\underline{\sigma}, \underline{\lambda})}$ remains irreducible over F.

Theorem A.9 (James-Mathas theorem)**.** *Let $V = L_K(\underline{\sigma}, \underline{\lambda}) \in \mathrm{Irr}_0(GL_n)$, and $\overline{V}$ be a reduction modulo p of V. Write $(\underline{\sigma}, \underline{\lambda}) = (\sigma_1, \lambda^{(1)}) \circ \cdots \circ (\sigma_a, \lambda^{(a)})$. Then $\overline{V}$ remains irreducible if and only if the following two conditions hold:*

(1) *If $i \neq j$, $\deg \sigma_i = \deg \sigma_j$, then σ_i, σ_j are not p-conjugate to each other.*

(2) *For each i and all nodes (r, t), (s, t) in the Young diagram of $\lambda^{(i)}$, write h_{rt}, h_{st} the corresponding hook length, $d_i = \deg \sigma_i$. Then*

$$|N_{q^{d_i}}(h_{rt})|_p = |N_{q^{d_i}}(h_{st})|_p$$

where $N_r(m) = (r^m - 1)/(r - 1)$, defined in §4.2.

Given a label $[(\underline{\sigma}, \underline{\lambda})]$ and its p-regularization $[(\underline{\sigma}^*, \underline{\lambda}^*)]$, we say $[(\underline{\sigma}, \underline{\lambda})]$ is of type I, if in the construction of p-regularization above, all $f_i = |\Delta((\underline{\lambda}^*)')|_p = p^c$ in step (2), and step (3) never happens. Otherwise it is of type II.

The following table describes the number of p-regular and p-singular irreducible ordinary representations, their p-regularizations, and if they satisfy the criterion in James-Mathas theorem or not.

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irr. rep'n	form	#p-reg	#p-sing	p-reg'n	type*	JM?
$L_K(\tau,(2))$	τ'	r_1		$-$	I	yes
	$\tau'\upsilon$		r_1s_1	$(\tau',(2))$	I	yes
$L_K(\tau,(1^2))$	τ'	r_1		$-$	I/II	**
	$\tau'\upsilon$		r_1s_1	$(\tau',(1^2))$	I/II	**
$L_K(\tau_1\circ\tau_2)$	$\tau_1'\circ\tau_2'$	$C_2^{r_1}$		$-$	I	yes
	$\tau_1'\circ\tau_2'\upsilon$		$2C_2^{r_1}s_1$	$(\tau_1'\circ\tau_2')$	I	yes
	$\tau_1'\upsilon_1\circ\tau_2'\upsilon_2$		$C_2^{r_1}s_1^2$	$(\tau_1'\circ\tau_2')$	I	yes
	$\tau'\circ\tau'\upsilon$		r_1s_1	$(\tau',(1^2))$	II	no
	$\tau'\upsilon_1\circ\tau'\upsilon_2$		$r_1C_2^{s_1}$	$(\tau',(1^2))$	II	no
$L_K(\omega)$	ω'	$r_2/2$		$-$	I	yes
	$\omega'\upsilon$		$r_2s_1/2$	(ω')	I	yes
	$\omega'\upsilon_{(2)}$		$r_2s_2/2$	(ω')	I	yes
	$\tau'\upsilon_{(2)}$		$r_1s_2/2$	$(\tau',(1^2))$	II/I	yes

* [type for $p>2$] / [type for $p=2$] ** yes, if $p\nmid q+1$; no, if $p\mid q+1$.

(Step 5) The branching numbers

Given $L_{\mathbf{F}}(\underline{\sigma},\underline{\lambda})\in\mathrm{Irr}_{\mathbf{F}}(GL_n)$, write κ, κ^* the branching number from GL_n to SL_n over K, F, and write η, η^* the branching number from SL_n to GL_n over K, F, respectively. Write $\kappa^*=\kappa_{p'}\kappa_p$ for its p'-factor and p-factor.

Proposition A.10. *Let $G=GL_n$ and $S=SL_n$. Given the label $[(\underline{\sigma},\underline{\lambda})]$.*

(1) $\kappa=\#\{\rho\in\mathbb{F}_q^\times\mid\rho\cdot[(\underline{\sigma},\underline{\lambda})]=[(\underline{\sigma},\underline{\lambda})]\}$

(2) $\kappa_{p'}=\#\{\rho\in O_{p'}(\mathbb{F}_q^\times)\mid\rho\cdot[(\underline{\sigma},\underline{\lambda})]=[(\underline{\sigma},\underline{\lambda})]\}$

(3) $\kappa_p=|\gcd(n,q-1,\Delta(\underline{\lambda}'))|_p$

(4) $\eta\kappa=q-1$ *and* $\eta^*\kappa_{p'}=m_1$.

Proof. (1)(2)(3) is Theorem 5.2. By Proposition 3.11,

$$\eta=\#\{\text{ orbit of }[(\underline{\sigma},\underline{\lambda})]\text{ act by }\rho\in\mathbb{F}_q^\times\}$$

$$\eta^*=\#\{\text{ orbit of }[(\underline{\sigma},\underline{\lambda})]\text{ act by }\rho\in O_{p'}(\mathbb{F}_q^\times)\}$$

hence (4) follows by the orbit-stabilizer theorem. □

Corollary A.11. *Let $G=GL_2$ and $S=SL_2$.*

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(1) $\kappa = 1$, *except for* q *odd, label* $(\tau \circ -\tau)$ *or* (ω) *with* $\omega^q = -\omega$. *In the exceptional case,* $\kappa = 2$.

(2) $\kappa_{p'} = 1$, *except for* q *odd,* $p > 2$, *label* $(\tau \circ -\tau)$ *or* (ω) *with* $\omega^q = -\omega$. *In the exceptional case,* $\kappa_{p'} = 2$.

(3) *In above two exceptional case, there are* $(q-1)/2$ *labels of the form* $(\tau \circ -\tau)$, *and another* $(q-1)/2$ *labels of the form* (ω) *with* $\omega^q = -\omega$.

(4) $\kappa_p = 1$, *except for* $p = 2$, *label* $(\tau, (1^2))$. *In the exceptional case,* $\kappa_p = 2$.

Proof. Assume $\rho, \tau \in \mathbb{F}_q^\times$, $\omega \in \mathbb{F}_{q^2}^\times$, $\deg \omega = 2$. Then $\rho\tau = \tau$ implies $\rho = e$; $\rho\tau_1 = \tau_2$, $\rho\tau_2 = \tau_1$ implies $\rho = e, -e$, and it is clear that there are $(q-1)/2$ labels of the form $(\tau \circ -\tau)$; take ε_2 a generator of $\mathbb{F}_{q^2}^\times$, Then $\rho = \varepsilon_2^{(q+1)j}$, and $\omega = \varepsilon_2^i$, $0 \leq i < q^2 - 1$, $q - 1 \nmid i$. Then $\rho\omega = \omega^q$ implies $(q+1)j = i(q-1)$. Hence either $q - 1 \mid j$ and $\rho = e$, or $\rho \neq e$, $(q-1)/2 \mid j$, thus $i = (2t+1)(q+1)/2$, $0 \leq t < q - 1$. So there are $(q-1)/2$ labels of the form (ω) with $\omega^q = -\omega$. In order that $-e \in \mathbb{F}_q^\times$, we need q being odd. This gives (1) and (3). For (2), in order that $-e \in O_{p'}(\mathbb{F}_q^\times)$, we need further $p > 2$. Finally, (4) follows by checking every labels of GL_2 for the criterion in James-Mathas theorem. □

(Step 6) The decomposition matrix

We split into several cases. Note that $p \nmid q$.

- $p > 2$, $p \nmid q^2 - 1$.

  That is, $p \nmid |GL_2|$ and $p \nmid |SL_2|$. By Proposition 43 of [S], both the decomposition matrix of GL_2 and SL_2 are identity. We notice that this result is consistent to James-Mathas theorem, that is, every label $[(\underline{\sigma}, \underline{\lambda})]$ satisfies the irreducibility criterion.

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- GL_2, $p > 2$, $p \mid q-1$

  In this case, $p \nmid q+1$. Thus $m_2 = (q+1)m_1$ and $r_2 = m_1 q$. Also $c_1 = c_2$ gives $s_2 = 0$.

$\dim_K$	$\mathrm{Irr}_K(GL_2)$	#	p-reg'n	type	JM
1	$L_K(\tau,(2))$	$q-1$	$(\tau',(2))$	I	yes
q	$L_K(\tau,(1^2))$	$q-1$	$(\tau',(1^2))$	I	yes
$q+1$	$L_K(\tau_1 \circ \tau_2)$	$(q-1)(q-2-s_1)/2$	$(\tau_1' \circ \tau_2')$	I	yes
		$(q-1)s_1/2$	$(\tau_1',(1^2))$	II	no
$q-1$	$L_K(\omega)$	$(q-1)q/2$	(ω')	I	yes
		q^2-1			

$\dim_F$	$\mathrm{Irr}_F(GL_2)$	#
1	$L_F(\tau',(2))$	r_1
q	$L_F(\tau',(1^2))$	r_1
$q+1$	$L_F(\tau_1' \circ \tau_2')$	$r_1(r_1-1)/2$
$q-1$	$L_F(\omega')$	$r_1 q/2$
		$r_1(r_1+q+3)/2$

$\dim_K$		$\dim_F$ / type	1 $(\tau',(2))$	q $(\tau',(1^2))$	$q+1$ $(\tau_1' \circ \tau_2')$	$q-1$ (ω')
1	$(\tau,(2))$	I	**1**			
q	$(\tau,(1^2))$	I		**1**		
$q+1$	$(\tau_1 \circ \tau_2)$	I			**1**	
		II	$\underline{1}$	**1**		
$q-1$	(ω)	I				**1**

Table 3: Decomposition matrix of GL_2, $p > 2$, $p \mid q-1$

The bold **1** means we find it by Theorem 6.4. The underlined $\underline{1}$ means we find it by counting the dimension.

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- SL_2, $p > 2$, $p \mid q-1$, q odd

  When q is odd with label $(\tau \circ -\tau)$ or (ω) with $\omega^q = -\omega$, we have $\kappa = \kappa_{p'} = 2$, otherwise 1. We have $\kappa_p = 1$, $\eta = (q-1)/\kappa$, $\eta^* = r_1/\kappa_{p'}$.

  If $L_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})\downarrow_{SL_n}$ is irreducible, write $Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda})$ instead of $Y_{\mathbf{F}}(\underline{\sigma}, \underline{\lambda}; 1)$.

  Pick ε_2 a generator of $\mathbb{F}_{q^2}^{\times}$, let ω_0 be the p'-part of $\varepsilon_2^{(q+1)/2}$. Then $\omega_0 = \varepsilon_2^{ap^{c_1}(q+1)/2}$ for some a, b satisfying $ap^{c_1} + 2bm_1 = 1$. Note that ap^{c_1} is odd. Then $\omega_0^{q-1} = \varepsilon_2^{(q-1)ap^{c_1}(q+1)/2} = -e$, so ω_0 is p-regular with $\omega_0^q = -\omega_0$.

$\dim_K$	$\mathrm{Irr}_K(SL_2)$	#	type
1	$Y_K(e,(2))$	1	I
q	$Y_K(e,(1^2))$	1	I
$(q+1)/2$	$Y_K((e \circ -e); i)$, $i = 1, 2$	1 each	I
$q+1$	$Y_K(e \circ \tau)$, $\tau \neq -e$	$(q-3-s_1)/2$	I
		$s_1/2$	II
$(q-1)/2$	$Y_K((\omega_0); i)$, $i = 1, 2$	1 each	I
$(q-1)$	$Y_K(\omega)$, $\omega^q \neq -\omega$	$(q-1)/2$	I
		$(q-1)+5$	

$\dim_F$	$\mathrm{Irr}_F(SL_2)$	#
1	$Y_F(e,(2))$	1
q	$Y_F(e,(1^2))$	1
$(q+1)/2$	$Y_F((e \circ -e); i)$, $i = 1, 2$	1 each
$q+1$	$Y_F(e \circ \tau')$, $\tau' \neq -e$	$(r_1-2)/2$
$(q-1)/2$	$Y_F((\omega_0); i)$, $i = 1, 2$	1 each
$q-1$	$Y_F(\omega')$, $(\omega')^q \neq -\omega'$	$(q-1)/2$
		$(r_1+q-1)/2+5$

		$\dim_F$	1	q	$(q+1)/2$	$q+1$	$(q-1)/2$	$q-1$
$\dim_K$		type	$(e,(2))$	$(e,(1^2))$	$(e \circ -e); i$	$(e \circ \tau')$	$(\omega_0); i$	(ω')
1	$(e,(2))$	I	**1**					
q	$(e,(1^2))$	I		**1**				
$(q+1)/2$	$(e \circ -e); i$	I			**1**			
$q+1$	$(e \circ \tau)$	I				**1**		
		II	1	**1**				
$(q-1)/2$	$(\omega_0); i$	I					**1**	
$q-1$	(ω)	I						**1**

Table 4: Decomposition matrix of SL_2, $p > 2$, $p \mid q-1$, q odd

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- SL_2, $p > 2$, $p \mid q-1$, q even

  We have $\kappa = \kappa_{p'} = \kappa_p = 1$, $\eta = q-1$, $\eta^* = r_1$ for all labels. Hence the decomposition matrix of SL_2 is similar to that of GL_2.

$\dim_K$	$\mathrm{Irr}_K(SL_2)$	#	type
1	$Y_K(e,(2))$	1	I
q	$Y_K(e,(1^2))$	1	I
$q+1$	$Y_K(e \circ \tau)$	$(q-2-s_1)/2$	I
		$s_1/2$	II
$q-1$	$Y_K(\omega)$	$q/2$	I
		$(q-1)+2$	

$\dim_F$	$\mathrm{Irr}_F(SL_2)$	#
1	$Y_F(e,(2))$	1
q	$Y_F(e,(1^2))$	1
$q+1$	$Y_F(e \circ \tau')$	$(r_1-1)/2$
$q-1$	$Y_F(\omega')$	$q/2$
		$(r_1+q-1)/2+2$

$\dim_K$		$\dim_F$ / type	1 $(e,(2))$	q $(e,(1^2))$	$q+1$ $(e\circ\tau')$	$q-1$ (ω')
1	$(e,(2))$	I	**1**			
q	$(e,(1^2))$	I		**1**		
$q+1$	$(e\circ\tau)$	I			**1**	
		II	1	**1**		
$q-1$	(ω)	I				**1**

Table 5: Decomposition matrix of SL_2, $p > 2$, $p \mid q-1$, q even

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- GL_2, $p > 2$, $p \mid q+1$.

  In this case, $p \nmid q-1$. Thus $c_1 = s_1 = 0$ and $r_1 = m_1 = q-1$, so every element of $\mathbb{F}_q^\times$ is p-regular. Also we have $q-1 \mid m_2$, $q-1 \mid r_2$, hence we may write $r_0 \coloneqq r_2/(q-1)$.

$\dim_K$	$\mathrm{Irr}_K(GL_2)$	#	p-reg'n	type	JM
1	$L_K(\tau,(2))$	$q-1$	$(\tau',(2))$	I	yes
q	$L_K(\tau,(1^2))$	$q-1$	$(\tau',(1^2))$	I	no
$q+1$	$L_K(\tau_1 \circ \tau_2)$	$(q-1)(q-2)/2$	$(\tau_1' \circ \tau_2')$	I	yes
$q-1$	$L_K(\omega)$	$(q-1)(q-s_2)/2$	(ω')	I	yes
		$(q-1)s_2/2$	$(\tau',(1^2))$	II	yes
		q^2-1			

$\dim_F$	$\mathrm{Irr}_F(GL_2)$	#
1	$L_F(\tau',(2))$	$q-1$
$q-1$	$L_F(\tau',(1^2))$	$q-1$
$q+1$	$L_F(\tau_1' \circ \tau_2')$	$(q-1)(q-2)/2$
$q-1$	$L_F(\omega')$	$r_2/2$
		$(r_2+(q-1)(q+2))/2$

		$\dim_F$	1	$q-1$	$q+1$	$q-1$
$\dim_K$		type	$(\tau',(2))$	$(\tau',(1^2))$	$(\tau_1' \circ \tau_2')$	(ω')
1	$(\tau,(2))$	I	$\mathbf{1}$			
q	$(\tau,(1^2))$	I	$\underline{1}$	$\mathbf{1}$		
$q+1$	$(\tau_1 \circ \tau_2)$	I			$\mathbf{1}$	
$q-1$	(ω)	I				$\mathbf{1}$
		II		$\mathbf{1}$		

Table 6: Decomposition matrix of GL_2, $p > 2$, $p \mid q+1$

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- SL_2, $p > 2$, $p \mid q+1$, q odd

  When q is odd with label $(\tau \circ -\tau)$ or (ω) with $\omega^q = -\omega$, we have $\kappa = \kappa_{p'} = 2$, otherwise 1. We have $\kappa_p = 1$, $\eta = (q-1)/\kappa$, $\eta^* = r_1/\kappa_{p'}$.

  Pick ε_2 a generator of $\mathbb{F}_{q^2}^{\times}$. If $\omega^q = -\omega$, then $\omega = \varepsilon_2^{(2t+1)(q+1)/2}$ for some $t \in \mathbb{N}$. Then the order of ω is $2(q-1)$, which is prime to p, hence every such ω is p-regular. Take $\omega_0 = \varepsilon_2^{(q+1)/2}$. In particular, $\omega_0^q \neq \omega_0$, hence $\deg \omega_0 = 2$.

$\dim_K$	$\mathrm{Irr}_K(SL_2)$	#	type
1	$Y_K(e,(2))$	1	I
q	$Y_K(e,(1^2))$	1	I
$(q+1)/2$	$Y_K((e \circ -e);i)$, $i = 1,2$	1 each	I
$q+1$	$Y_K(e \circ \tau)$, $\tau \neq -e$	$(q-3)/2$	I
$(q-1)/2$	$Y_K((\omega_0);i)$, $i = 1,2$	1 each	I
$(q-1)$	$Y_K(\omega)$, $\omega^q \neq -\omega$	$(q-1-s_2)/2$	I
		$s_2/2$	II
		$(q-1)+5$	

$\dim_F$	$\mathrm{Irr}_F(SL_2)$	#
1	$Y_F(e,(2))$	1
$q-1$	$Y_F(e,(1^2))$	1
$(q+1)/2$	$Y_F((e \circ -e);i)$, $i = 1,2$	1 each
$q+1$	$Y_F(e \circ \tau')$, $\tau' \neq -e$	$(q-3)/2$
$(q-1)/2$	$Y_F((\omega_0);i)$, $i = 1,2$	1 each
$q-1$	$Y_F(\omega')$, $(\omega')^q \neq -\omega'$	$(r_0-1)/2$
		$(r_0+q-2)/2+5$

		$\dim_F$	1	$q-1$	$(q+1)/2$	$q+1$	$(q-1)/2$	$q-1$
$\dim_K$		type	$(e,(2))$	$(e,(1^2))$	$(e \circ -e);i$	$(e \circ \tau')$	$(\omega_0);i$	(ω')
1	$(e,(2))$	I	**1**					
q	$(e,(1^2))$	I	1	**1**				
$(q+1)/2$	$(e \circ -e);i$	I			**1**			
$q+1$	$(e \circ \tau)$	I				**1**		
$(q-1)/2$	$(\omega_0);i$	I					**1**	
$q-1$	(ω)	I						**1**
		II		**1**				

Table 7: Decomposition matrix of SL_2, $p > 2$, $p \mid q+1$, q odd

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- SL_2, $p > 2$, $p \mid q+1$, q even

  We have $\kappa = \kappa_{p'} = \kappa_p = 1$, $\eta = \eta^* = (q-1)$ for all labels. Hence again the decomposition matrix of SL_2 is similar to that of GL_2.

$\dim_K$	$\mathrm{Irr}_K(SL_2)$	#	type
1	$Y_K(e,(2))$	1	I
q	$Y_K(e,(1^2))$	1	I
$q+1$	$Y_K(e \circ \tau)$	$(q-2)/2$	I
$q-1$	$Y_K(\omega)$	$(q-s_2)/2$	I
		$s_2/2$	II
		$(q-1)+2$	

$\dim_F$	$\mathrm{Irr}_F(SL_2)$	#
1	$Y_F(e,(2))$	1
$q-1$	$Y_F(e,(1^2))$	1
$q+1$	$Y_F(e \circ \tau')$	$(q-2)/2$
$q-1$	$Y_F(\omega')$	$r_0/2$
		$(r_0+q-2)/2+2$

		$\dim_F$	1	$q-1$	$q+1$	$q-1$
$\dim_K$		type	$(e,(2))$	$(e,(1^2))$	$(e \circ \tau')$	(ω')
1	$(e,(2))$	I	**1**			
q	$(e,(1^2))$	I	1	**1**		
$q+1$	$(e \circ \tau)$	I			**1**	
$q-1$	(ω)	I				**1**
		II		**1**		

Table 8: Decomposition matrix of SL_2, $p > 2$, $p \mid q+1$, q even

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- GL_2, $p = 2$, q odd.

  In this case, $p \mid q-1$ and $p \mid q+1$. Let $r_0 = r_2/r_1$ and $s_0 = s_2/(s_1+1)$. For type I, if all $f_i = p^c$ for some c, write I_c for distinction.

$\dim_K$	$\mathrm{Irr}_K(GL_2)$	#	p-reg'n	type	JM
1	$L_K(\tau,(2))$	$q-1$	$(\tau',(2))$	I	yes
q	$L_K(\tau,(1^2))$	$q-1$	$(\tau',(1^2))$	II	no
$q+1$	$L_K(\tau_1 \circ \tau_2)$	$(q-1)(q-2-s_1)/2$	$(\tau_1' \circ \tau_2')$	I	yes
		$(q-1)s_1/2$	$(\tau_1',(1^2))$	II	no
$q-1$	$L_K(\omega)$	$(q-1)(q-s_0)/2$	(ω')	I_0	yes
		$(q-1)s_0/2 = r_1 s_2/2$	$(\tau',(1^2))$	I_1	yes
		q^2-1			

$\dim_F$	$\mathrm{Irr}_F(GL_2)$	#
1	$L_F(\tau',(2))$	r_1
$q-1$	$L_F(\tau',(1^2))$	r_1
$q+1$	$L_F(\tau_1' \circ \tau_2')$	$r_1(r_1-1)/2$
$q-1$	$L_F(\omega')$	$r_2/2$
		$r_1(r_1+r_0+3)/2$

		$\dim_F$	1	$q-1$	$q+1$	$q-1$
$\dim_K$		type	$(\tau',(2))$	$(\tau',(1^2))$	$(\tau_1' \circ \tau_2')$	(ω')
1	$(\tau,(2))$	I	$\mathbf{1}$			
q	$(\tau,(1^2))$	II	$\underline{1}$	$\mathbf{1}$		
$q+1$	$(\tau_1 \circ \tau_2)$	I			$\mathbf{1}$	
		II	$\underline{2}$	$\mathbf{1}$		
$q-1$	(ω)	I_0				$\mathbf{1}$
		I_1		$\mathbf{1}$		

Table 9: Decomposition matrix of GL_2, $p = 2$, q odd

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- SL_2, $p=2$, q odd

  When q is odd with label $(\tau \circ -\tau)$ or (ω) with $\omega^q = -\omega$, we have $\kappa = 2$, otherwise 1. When the label is $(\tau, (1^2))$, we have $\kappa_p = 2$, otherwise 1. We have $\kappa_{p'} = 1$, $\eta = (q-1)/\kappa$ and $\eta^* = r_1$. Note that either $4 \mid q-1$ and $s_0 = 1$, or $4 \mid q+1$ and $s_1 = 1$. Hence in each case there is a type of $\mathrm{Irr}_K(SL_2)$ missing.

  Pick ε_2 a generator of $\mathbb{F}_{q^2}^\times$. Let $\omega_0 = \varepsilon_2^{(q+1)/2}$. Then $\omega_0^q = -\omega_0$.

$\dim_K$	$\mathrm{Irr}_K(SL_2)$	#	type
1	$Y_K(e,(2))$	1	I
q	$Y_K(e,(1^2))$	1	II
$q+1$	$Y_K(e\circ\tau)$, $\tau \neq -e$	$(q-2-s_1)/2$	I
		$(s_1-1)/2$	II
$(q+1)/2$	$Y_K((e\circ -e);i)$, $i=1,2$	1 each	II
$q-1$	$Y_K(\omega)$, $\omega^q \neq -\omega$	$(q-s_0)/2$	I_0
		$(s_0-1)/2$	I_1
$(q-1)/2$	$Y_K((\omega_0);i)$, $i=1,2$	1 each	I_1
		$(q-1)+5$	

$\dim_F$	$\mathrm{Irr}_F(SL_2)$	#
1	$Y_F(e,(2))$	1
$(q-1)/2$	$Y_F((e,(1^2));i)$, $i=1,2$	1 each
$q+1$	$Y_F(e\circ\tau')$	$(r_1-1)/2$
$q-1$	$Y_F(\omega')$	$r_0/2$
		$(r_0+r_1-1)/2+3$

		$\dim_F$	1	$(q-1)/2$	$(q-1)/2$	$q+1$	$q-1$
$\dim_K$		type	$(e,(2))$	$(e,(1^2));2$	$(e,(1^2));1$	$(e\circ\tau')$	(ω')
1	$(e,(2))$	I	**1**				
q	$(e,(1^2))$	II	1	1	**1**		
$q+1$	$(e\circ\tau)$	I				**1**	
		II	2	1	**1**		
$(q+1)/2$	$(e\circ -e);2$	II	1	**1**			
$(q+1)/2$	$(e\circ -e);1$	II	1		**1**		
$q-1$	(ω)	I_0					**1**
		I_1		1	**1**		
$(q-1)/2$	$(\omega_0);2$	I_1		**1**			
$(q-1)/2$	$(\omega_0);1$	I_1			**1**		

Table 10: Decomposition matrix of SL_2, $p=2$, q odd

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A.4 The Decomposition Matrix of Other Groups

Counterexample for (R,p)-property of GL_n

Pick $G = GL_3(4)$, $p = 3$. Then $q = 4$ and $p \mid q-1$. For $d = 1,2,3$, write $q^d - 1 = p^{c_d} m_d$ and set $c_0 = 0$. Let $r_d := \sum_{t|d} \mu(t) m_{d/t}$ and $s_d := \sum_{t|d} \mu(t) p^{c_{d/t}}$. Let $\sigma'_{(d)}$ be p'-element of degree d, and $\upsilon_{(d)}$ be p-element of degree d. It is not hard to prove that $\deg(\sigma'_{(d_1)}\upsilon_{(d_2)}) = \mathrm{lcm}(d_1, d_2)$.

$\#\{\sigma'_{(1)}\}$	$1 = r_1 = m_1$	$\#\{\upsilon_{(1)}\}$	$3 = s_1 = p^{c_1}$
$\#\{\sigma'_{(2)}\}$	$4 = r_2 = m_2 - m_1$	$\#\{\upsilon_{(2)}\}$	$0 = s_2 = p^{c_2} - p^{c_1}$
$\#\{\sigma'_{(3)}\}$	$6 = r_3 = m_3 - m_1$	$\#\{\upsilon_{(3)}\}$	$6 = s_3 = p^{c_3} - p^{c_1}$

deg	element	form	p'-part	#	
1	τ	$\sigma'_{(1)}\upsilon_{(1)}$	τ'	$r_1 s_1$	$= 3$
2	ω	$\sigma'_{(2)}\upsilon_{(1)}$	ω'	$r_2 s_1$	$= 12$
3	δ	$\sigma'_{(3)}\upsilon_{(1)}$	δ'	$r_3 s_1$	$= 18$
		$\sigma'_{(3)}\upsilon_{(3)}$	δ'	$r_3 s_3$	$= 36$
		$\sigma'_{(1)}\upsilon_{(3)}$	τ'	$r_1 s_3$	$= 6$

$\dim_K$		$\dim_K$		$\dim_K$	
1	$M_K(\tau,(3))$	1	$S_K(\tau,(3))$	1	$L_K(\tau,(3))$
21	$M_K(\tau,(2,1))$	20	$S_K(\tau,(2,1))$	20	$L_K(\tau,(2,1))$
105	$M_K(\tau,(1^3))$	64	$S_K(\tau,(1^3))$	64	$L_K(\tau,(1^3))$
1	$M_K(\tau,(2))$	1	$S_K(\tau,(2))$	21	$L_K((\tau_1,(2))\circ(\tau_2))$
5	$M_K(\tau,(1^2))$	4	$S_K(\tau,(1^2))$	84	$L_K((\tau_1,(1^2))\circ(\tau_2))$
1	$M_K(\tau)$	1	$S_K(\tau)$	105	$L_K((\tau_1)\circ(\tau_2)\circ(\tau_3))$
3	$M_K(\omega)$	3	$S_K(\omega)$	63	$L_K((\omega)\circ(\tau))$
45	$M_K(\delta)$	45	$S_K(\delta)$	45	$L_K(\delta)$

$\dim L_K$	symbol	p-reg'n	#	type	JM	$\#L_F$
1	$(\tau,(3))$	$(\tau',(3))$	3	I	yes	1
20	$(\tau,(2,1))$	$(\tau',(2,1))$	3	I	no	1
64	$(\tau,(1^3))$	$(\tau',(1^3))$	3	II	no	1
21	$(\tau_1,(2))\circ(\tau_2)$	$(\tau',(2,1))$	6	II	no	–
84	$(\tau_1,(1^2))\circ(\tau_2)$	$(\tau',(1^3))$	6	II	no	–
105	$(\tau_1)\circ(\tau_2)\circ(\tau_3)$	$(\tau',(1^3))$	1	II	no	–
63	$(\omega)\circ(\tau)$	$(\omega')\circ(\tau')$	18	I	yes	2
45	(δ)	(δ')	18	I_1	yes	2
		$(\tau',(1^3))$	2	I_0		

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To find the full decomposition matrix, we make use of Theorem 2.31(1). For example, the composition multiplicity of $D_F(e,(3))$ in $S_F(e,(2,1))$ is at most that in $M_F(e,(2,1))/S_F(e,(2,1)) \cong S_F(e,(3))$, hence the multiplicity is 1, and one may calculate $\dim_F D_F(e,(2,1)) = 19$. From the G-tile $[S_K(e,(2,1))]_G \times [D_F(e,(3))]_G$, Proposition 3.14(2) shows that $D_F(e,(3))$ is of multiplicity 1 in each $S_F(\tau,(2,1))$.

We omit the rest of the calculations and give the decomposition matrix here.

		$\dim_F$	1	19	45	63	45
$\dim_K$		type	$(e,(3))$	$(e,(2,1))$	$(e,(1^3))$	$(\omega')\circ(e)$	(δ')
1	$(\tau,(3))$	I	**1**				
20	$(\tau,(2,1))$	I	1	**1**			
64	$(\tau,(1^3))$	II		1	**1**		
21	$(\tau_1,(2))\circ(\tau_2)$	II	2	**1**			
84	$(\tau_1,(1^2))\circ(\tau_2)$	II	1	2	**1**		
105	$(\tau_1)\circ(\tau_2)\circ(\tau_3)$	II	3	3	**1**		
63	$(\omega)\circ(\tau)$	I				**1**	
45	(δ)	I_1					**1**
		I_0			**1**		

Table 11: Decomposition matrix of $GL_3(4)$, $p = 3$

Observe that $L_F(e,(2,1))$ is not liftable, so $GL_3(4)$ does not have (R,p)-property for $p = 3$.

Counterexample for (R,p)-property of SL_n

Now consider $S = SL_3(4)$, $p = 3$. Since $\eta^*\kappa_{p'} = r_1 = 1$, thus they are both 1. We have $\eta = 3/\kappa$, and $\kappa = 1$ unless the labels $(\tau_1 \circ \tau_2 \circ \tau_3)$ and (δ) with $\deg(\delta)_{p'} = 1$. In these cases $\kappa = 3$. Those δ can be characterized as ε_3^{7i}, $i = 1, 2$, where ε_3 is a generator of $\mathbb{F}_{64}^\times$. Finally, $\kappa_p = 1$ unless the labels $(e \circ (1^3))$. In this case $\kappa_p = 3$.

In the following table, $i = 1, 2, 3$.

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$\dim L_K$	#	label	type	$\dim L_F$	#	label
1	1	$(\tau, (3))$	I	1	1	$(e, (3))$
20	1	$(\tau, (2, 1))$	I	19	1	$(e, (2, 1))$
64	1	$(\tau, (1^3))$	II	15	1 each	$(e, (1^3)); i$
21	2	$(\tau_1, (2)) \circ (\tau_2)$	II			
84	2	$(\tau_1, (1^2)) \circ (\tau_2)$	II			
35	1 each	$(\tau_1) \circ (\tau_2) \circ (\tau_3); i$	II			
63	6	$(\omega) \circ (\tau)$	I	63	2	$(\omega') \circ (\tau')$
45	6	$(\delta),\ (\delta)_p \neq e$	I_1	45	2	(δ')
15	2 each	$(\delta); i,\ (\delta)_p = e$	I_0			

Since each G-tile is one of the good case in Proposition 3.15, we have the decomposition matrix of $SL_3(4)$ for $p = 3$.

			$\dim_F$	1	19	15	15	15	63	45
			#	1	1	1	1	1	2	2
$\dim_K$	#		type	$(e, (3))$	$(e, (2, 1))$	$(e, (1^3)); 3$	$(e, (1^3)); 2$	$(e, (1^3)); 1$	$(\omega') \circ (e)$	(δ')
1	1	$(\tau, (3))$	I	**1**						
20	1	$(\tau, (2, 1))$	I	1	**1**					
64	1	$(\tau, (1^3))$	II		1	1	1	**1**		
21	2	$(\tau_1, (2)) \circ (\tau_2)$	II	2	**1**					
84	2	$(\tau_1, (1^2)) \circ (\tau_2)$	II	1	2	1	1	**1**		
35	1	$(\tau_1) \circ (\tau_2) \circ (\tau_3); 3$	II	1	1	**1**				
35	1	$(\tau_1) \circ (\tau_2) \circ (\tau_3); 2$	II	1	1		**1**			
35	1	$(\tau_1) \circ (\tau_2) \circ (\tau_3); 1$	II	1	1			**1**		
63	6	$(\omega) \circ (\tau)$	I						**1**	
45	6	$(\delta),\ (\delta)_p \neq e$	I_1							**1**
15	2	$(\delta); 3,\ (\delta)_p = e$	I_0			**1**				
15	2	$(\delta); 2,\ (\delta)_p = e$	I_0				**1**			
15	2	$(\delta); 1,\ (\delta)_p = e$	I_0					**1**		

Table 12: Decomposition matrix of $SL_3(4)$, $p = 3$

Observe that $Y_F(e, (2, 1))$ is not liftable, hence $SL_3(4)$ does not have (R, p)-property for $p = 3$.

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Counterexample for (R, p)-property of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

Here we take $G = \mathfrak{S}_6$, $S = \mathfrak{A}_6$, $p = 3$. The decomposition matrix of $\mathfrak{A}_6$ comes from that of $\mathfrak{S}_6$, with the method of G-tile and S-tile in Proposition 3.15.

	(6)	(3^2)	$(5,1)$	$(3,2,1)$	$(4,2)$	$(2^2,1^2)$	$(4,1^2)$
(6)	1						
(1^6)		1					
$(5,1)$	1		1				
$(2,1^4)$		1		1			
$(4,2)$					1		
$(2^2,1^2)$						1	
(3^2)		1	1				
(2^3)	1			1			
$(4,1^2)$			1				1
$(3,1^3)$				1			1
$(3,2,1)$	1	1	1	1			**1**

$G = \mathfrak{S}_6$ for $p = 3$

	$(6),(3^2)$	$(5,1),(3,2,1)$	$(4,2),(2^2,1^2)$	$(4,1^2)+$	$(4,1^2)-$
$(6),(1^6)$	1				
$(5,1),(2,1^4)$	1	1			
$(4,2),(2^2,1^2)$			1		
$(3^2),(2^3)$	1	1			
$(4,1^2),(3,1^3)$		1		1	1
$(3,2,1)+$	1	1		**1**	
$(3,2,1)-$	1	1			**1**

$G = \mathfrak{A}_6$ for $p = 3$

Table 13: Decomposition matrix of $\mathfrak{A}_6$, $p = 3$

Then the column $(5,1)$ of $\mathfrak{S}_6$ and the column $(5,1),(3,2,1)$ of $\mathfrak{A}_6$ show that both $\mathfrak{S}_6$ and $\mathfrak{A}_6$ do not have (R,p)-property for $p = 3$, respectively.

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