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多相均曲率流的自相似擴張解

Self-similar Expanding Solutions for a Multiphase Mean Curvature Flow

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國立臺灣大學博士學位論文
口試委員會審定書

多相均曲率流的自相似擴張解
Self-similar Expanding Solutions for a Multiphase Mean Curvature Flow

本論文係廖偉宏君（D01221002）在國立臺灣大學數學學系、所完成之博士學位論文，於民國 108 年 5 月 20 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

（簽名）

（指導教授）

系主任、所長 （簽名）

Acknowledgement

I am very glad to pursue my Ph.D. at National Taiwan University. During this period, I participated in many international conferences and seminars which not only broadened my horizon but also showed me the ideas of other scholars. At the beginning of my research project, I intended to explore the measure theory, but this knowledge is often regarded as a tool and not suitable for research. After I conducted the research for a while, I learned the geometric measure theory and took the course of Leon Simon, a master of geometric measure theory. I talked a lot with the professor and found that he taught us more than in the course and his perspective on the geometric measure theory made me reach a new level in the measure theory.

First of all, I would like to thank the professor Jenn-Fang Hwang in the Academia Sinica. Although he guided me for a short time, he taught me the most important thing "attitude" as a doctoral or a researcher. The correct attitude later helps me a lot in conducting the research. Secondly, I would like to thank Professor Yng-Ing Lee and her postdoctorals for their helps and concerns. At the same time, I would like to thank the professors Chun-Chi Lin and Ting-Jung Kuo in the National Taiwan Normal University for their invitation so that I can give a talk and participate in the conference between Kyushu University and National Taiwan Normal University.

When I was in a elementary school, my father passed away. Since then, my mother had raised us alone. She worked hard and played an important role in our lives so we feel no difference between single-parent family and

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double-parent family. My mother rested in my third year of doctoral life, but I am very grateful to my mother for her support in my life. Without her devotion, I cannot insist on this position. Finally, I also want to thank the professors Shun-Cheng Chang, Fei-Tsen Liang, Ping-Zen Ong, Ki-Seng Tan, Mao-Pei Tsui, and my advisor Ai-Nung Wang for the participations in my oral exam.

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中文摘要

我們考慮由有限多個通過原點的曲面所組成的多相曲面C_0, 其中所有的一維結線是由三曲面兩兩夾角相同所組合形成的。此外各曲面對原點做徑向投影所形成的曲線也是有限長。我們證明了以此C_0做初始條件下存在一多相均曲率流的自相似擴張解, 而這組解是由那些正規三節線和正規四結點所組合出來的曲面。

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Abstract

We consider a multiphase surface C_0 in $\mathbb{R}^3$ consisting of a finite number of surfaces passing through the origin , where all 1-dimensional junctions are regular triple junctions in which three planes meet at the same angle and each surface scales down homothetically to a limit curve of finite length. We prove the existence of self-similar expanding solutions of the mean curvature flow on the multiphase surface initially given by C_0. For this initial C_0, there are multiple solutions that are combinations of the regular triple junctions and regular quadruple points, where four regular triple junctions meet at an angle of approximately $109.5°$.

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Chapter 1

Introduction

Interface problems have long been studied by material scientists and mathematicians, with various scenarios involving a collection of interfaces whose positions and shapes are constrained so as to minimize their total area. In 1873, Plateau [20] observed two singular structures during soap foam experiments, and conjectured that the regular triple junction and the regular quadruple point are the only area-minimizing singular structures in $\mathbb{R}^3$. Around a century later, Taylor [23] gave a mathematical proof of this conjecture.

In the smooth case, there are many useful results for area-minimizing problems in the field of geometric flows. A multiphase mean curvature flow for example is a popular model for the evolution of grain boundaries in polycrystals undergoing heat treatment, which is motivated in [19]. Extending the idea in the smooth case to the singular case, one must formulate a weak evolution equation for the flow near the singularity, as considered by Brakke [3]. Even using Brakke's idea, the short-time existence and long-time behavior and convergence properties are still inevitable. Mantegazza, Novaga, and Tortorelli [15] attacked this problem using a system of equations for curve-shortening flow with the regular initial surface . Under certain hypotheses, they obtained some good results. Recently, Ilmanen, Neves, and Schulze [9] dropped the requirement for a regular initial surface and proved the short-time existence of triple points with non-regular initial surfaces. For $\mathbb{R}^3$ or higher-dimensional spaces, this problem can be studied through the mean curvature flow for the graphical hypersurfaces [6] or the local regularity on the triple junctions without higher-order junctions [22].

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The general flow around the singularity has been studied qualitatively by Brakke, who also considered the resemblance between different structures (see the appendix of [3]). Recently, Kim and Tonegawa [12] proved the global-in-time existence of the mean curvature flow in the sense of Brakke's flow, and their existence theorem does not require any parametrization and imposes no restrictions on the dimension or configuration.

This study is inspired by the work of Mazzeo and Saez [16], who took the first step in proving the short-time existence of triple junctions with non-regular initial conditions. The solutions are far from unique, but those that are regular for $t > 0$ can be clearly described. Additionally, the global behavior of these singular structures can be well understood. The planar network considered in [16, 9] is used to describe the interface problem in $\mathbb{R}^2$, so we consider the similar structure in $\mathbb{R}^3$ to study the problem.

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Chapter 2

Multiphase Surfaces

2.1 Definition

Definition 1. *A multiphase surface* $\Gamma \subset \mathbb{R}^3$ *is a finite union of embedded surfaces or properly embedded "half-planes"* $\{\alpha_i\}_{i=1}^m$*. For* $1 \leq i \neq j \leq m$, $\alpha_i \cap \alpha_j$ *is either empty or a subset of their boundaries i.e., two surfaces can intersect each other only on their boundary not in their interior. In order to well define the boundary of noncompact and embedded surface, we consider the compactification of* $\mathbb{R}^3$ *a union of the sets* $\mathbb{R}^3$ *and* S^∞ *where* $S^\infty := \lim_{r\to\infty} S^r = \lim_{r\to\infty}\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = r^2\}$*. For each surface* α_i*, the boundary of* α_i *is composed of the interior curves or (ideal) boundary curves where the interior curves of* α_i *are a collection of the sets containing a curve for* $1 \leq j \neq i \leq m$ *in* $\mathbb{R}^3$ *and the (ideal) boundary curves of* α_i *are a family of the sets* $\alpha_i \cap S^\infty$*. Moreover, these curves may intersect at certain points.*

The multiphase surface is called regular if all 1-dimensional junctions and 0-dimensional junctions of the multiphase surface in $\mathbb{R}^3$ *are regular triple junctions and regular quadruple points: the former is the intersection of three surfaces meeting at an angle of* $120°$ *and the other is the point junction of four regular triple junctions with an angle of approximately* $109.5°$*. Specifically, there are four grains with six grain boundaries near the quadruple point, and we say that such a point has a tetrahedral structure.*

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Definition 2. *The "half-planes" mentioned in the Definition 1 include the sets homeomorphic to* $H := \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 = 0, x_2 \in \mathbb{R}, x_3 \geq 0\}$ *or homeomorphic to the non-compact but closed subset in* H *up to some rigid transformations. We take the following sets for example*

$$\begin{aligned}
H &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 = 0,\ x_2 \in \mathbb{R},\ x_3 \geq 0\}, \\
U &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 = 0,\ x_2 + x_3 \geq 0,\ x_2 - x_3 \leq 0\}, \\
V &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 = 0,\ 2x_2 + x_3 \geq 1,\ x_2 - x_3 \leq 0\} \\
W &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 = 0,\ 2x_2 + x_3 > 1,\ x_2 - x_3 \leq 0\}
\end{aligned}$$

where $H, U,$ *and* V *are the half-planes in* $\mathbb{R}^3$ *but* W *is not a half-plane.*

Remark 1. The (ideal) boundary curves of α on S^∞ are simply expressed by $\alpha \cap S^\infty$ in the above argument, but it does not mean that $\lim_{r\to\infty}(\alpha \cap S^r)$; instead, we characterize the (ideal) boundary curves by the radial projection i.e., $\lim_{r\to\infty} \frac{1}{r}(\alpha \cap S^r) = \lim_{r\to\infty} \Sigma_r := \Sigma$. Specifically, the embeddedness in the Definition 1 guarantees the existence of limit Σ.

2.2 Examples

To clarify the relation between two surfaces α_i and α_j, $1 \leq i \neq j \leq m$ in the previous section, we give two examples in $\mathbb{R}^3$ to describe the intersection $\alpha_i \cap \alpha_j$.

Example 1. Let

$$\begin{aligned}
H &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in [0, \pi],\ x_2 \in \mathbb{R},\ x_3 = \sin(x_1) - 2\}, \\
U &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 = -2\}, \\
V &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 = -4\},
\end{aligned}$$

we can see that $H \cap U$ has more than one curves and the others $H \cap V$ and $U \cap V$ are empty sets.

Example 2. Given four points $(1, 0, \frac{-1}{\sqrt{2}})$, $(-1, 0, \frac{-1}{\sqrt{2}})$, $(0, 1, \frac{1}{\sqrt{2}})$, and $(0, -1, \frac{1}{\sqrt{2}})$ which are the vertices of the regular tetrahedron, we define the "half-planes" spanned by any two vectors

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of them.

$$R := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ts \\ -(1-t)r \\ -\frac{ts+(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}$$

$$G := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ts-(1-t)r \\ 0 \\ -\frac{ts+(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}$$

$$B := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ts \\ (1-t)r \\ -\frac{ts-(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}$$

$$Y := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ts \\ -(1-t)r \\ -\frac{ts-(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}$$

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ts \\ (1-t)r \\ -\frac{ts-(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}$$

$$P := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -ts+(1-t)r \\ \frac{ts+(1-t)r}{\sqrt{2}} \end{pmatrix},\ t \in [0,1],\ s \in [0,\infty),\ r \in [0,\infty) \right\}.$$

In the Figure 6.1, the intersection of surfaces R and other surfaces except C contains a curve in it. Althrough $R \cap C$ contains no curves, their intersection is a point set $\{0\} \neq \emptyset$.

Remark 2. The Figure 6.1 is the most important and schematical picture of mutiphase surfaces in this paper and we also use the concept of Figure 6.1 to describe an initial condition at the origin.

2.3 Existence

Let C_0 be a finite union of m surfaces meeting at the origin and separating $\mathbb{R}^3$ into k regions i.e., $C_0 = \bigcup_{i=1}^m \sigma_i$ and $\bigcap_{i=1}^m \sigma_i = \{0\}$. We impose the following conditions on C_0:

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(A_1) Each surface in C_0 is simply-connected and embedded in $\mathbb{R}^3$.

(A_2) Each region induced by C_0 is enclosed by at least two surfaces σ_i, $2 \leq i \leq m$, which scales down homothetically to a limit curve Σ_i of finite length on S^2 i.e., $\lim_{r\to\infty} \frac{1}{r}(\sigma_i \cap S^r) = \Sigma_i < \infty$, $2 \leq i \leq m$.

(A_3) The regular triple point is the one and only type of point junction where the curves in (ii) intersect on S^2. In other words, the permitted one dimensional junctions of surfaces are the regular triple junctions. The regular triple point is the intersection of three curves meeting at angle 120° and the regular triple junction is the same aspect for surfaces.

Remark 3. Almgren [1] introduced $(\mathbb{M}, \epsilon, \delta)$ minimal set to model soap films, soap bubble clusters, and combination bubble-films which are the problems of partitioning space into regions of prescribed volumes in such a way as to minimize total interface area. Taylor [23] showed that $(\mathbb{M}, \epsilon, \delta)$ minimal surface in $\mathbb{R}^3$ have precisely the singularities observed in Plateau's problem [20]. In accordance with this minimality around the singularities, we consider no point junctions except the regular triple point on S^2. Furthermore, the embeddedness of surface is for reasons of no interfaces created in the same region.

Remark 4. We give two conventions about the interface problem of immiscible fluids. Based on the topological classification of a network with two triple junctions in [14], we list locally the possible curves with no more than two triple points on S^2 in Table 6.1.

- If a triple point occures, there exist three immiscible fluids concurring at a point. On the other hand, a interface cannot be created in the interior of a fluid.
- When a curve touches the boundary, it means that this curve connects another triple point.

We exclude the cases (i) and (ii) because the former has no singular structures and the other violates the first convention. More specifically, the outside region in the case (ii) has an interface in its interior. Since the cases (iii) and (iv) have the structures like the Brakke spoon, we exclude these cases with the same reason. As the second convention is concerned,

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the case (vii) has no regions enclosed by two surfaces. Hence, the lens-shaped and theta-shaped curves are the possible curves induced by a region endowed with two interfaces.

Define the metric

$$g(\mathbf{x}) = \exp^{\frac{x_1^2+x_2^2+x_3^2}{4}}(dx_1^2 + dx_2^2 + dx_3^2), \tag{2.1}$$

which is complete and negatively curved. In the following argument, we let $\Sigma_1, \cdots, \Sigma_m$ be the prescribed boundary curves of $C_0 \cap S^2$.

Theorem 1 (Main Theorem)**.** *Let C_0 be a finite union of m surfaces meeting at the origin and separating $\mathbb{R}^3$ into k regions. Suppose C_0 satisfy the condition (A_1), (A_2), and (A_3) with its boundary curves $\Sigma_1, \cdots, \Sigma_m$ on S^2, then there exists a regular multiphase surface which is the connected self-expanding solution to mean curvature flow with $\Sigma_1, \cdots, \Sigma_m$ as its boundary and each surface of the solution is a minimal surface for the metric g.*

Remark 5. Because we use the relation in the next chapter to prove Theorem 1, the regular connected self-expanders to mean curvature flow in $\mathbb{R}^3$ have a one-to-one correspondence with the possibly disconnected regular multiphase surfaces in $B_1^3(0)$. The "one-to-one correspondence" in Theorem 1 depends on the following idea: given a deformation from metric g to the standard hyperbolic metric in the class of complete negatively curved metrics, the regular multiphase surface found with respect to g should be continuously deformable to the regular multiphase surface with respect to the standard hyperbolic metric. Similarly, the regular multiphase surface in $\mathbb{H}^3$ should continuously produce a similar structure for g. Because the problem regarding the deformation of the metric is difficult, we do not focus on this here, but simply use the correspondence among similar structures in these metrics.

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Chapter 3

Relation between Poincaré ball model and Euclidean space

Definition 3. *The Poincaré ball model of hyperbolic space is the open submanifold*

$$B_1^3(0) := \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : |\boldsymbol{x}| < 1\}$$

with the Riemannian metric

$$g_B = \frac{4d\boldsymbol{x} \cdot d\boldsymbol{x}}{(1 - |\boldsymbol{x}|^2)^2}.$$

Besides, we introduce a hyperbolic space

$$\mathbb{H}^3 := \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^{3,1} : \langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1,\ x_4 \geq 1\}$$

to define a map between $B_1^3(0)$ and $\mathbb{H}^3$.

Definition 4. *Hyperbolic stereographic projection is the map*

$$\mathcal{S} : \mathbb{H}^3 \to B_1^3(0), \qquad \mathcal{S}((x_1, x_2, x_3, x_4)) = \frac{1}{1 + x_4}(x_1, x_2, x_3) := \boldsymbol{y},$$

Remark 6. We differentiate $\mathcal{S}$ and then obtain

$$d\mathcal{S}_{\mathbf{x}} = \frac{d\mathbf{x}'}{1 + x_4} - \frac{\mathbf{x}'}{(1 + x_4)^2} dx_4.$$

We then obtain

$$\mathcal{S}^* g_B = \frac{4d\mathbf{y} \cdot d\mathbf{y}}{(1 - |\mathbf{y}|^2)^2} = \langle d\mathbf{x}, d\mathbf{x} \rangle$$

Therefore, $\mathcal{S}$ is isometric onto $(B_1^3(0), g_B)$ and conformal to $\mathbb{R}^3$.

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In the following arguments, we will identify each unit tangent sphere of $B_1^3(0)$ with $\partial B_1^3(0)$, and then define a conformal diffeomorphism between $S^2(\infty)$ and $\partial B_1^3(0)$ i.e., $\partial B_1^3(0)$ is the sphere at infinity of the Poincaré ball model.

If $x \in B_1^3(0)$ and $S^2(x) \subset \mathbb{R}^3$ is the unit sphere in the tangent space at x, we define a boundary point from an interior point connected by the geodesic in $B_1^3(0)$ as below

$$\mathbb{B} : S^2(x) \to \partial B_1^3(0)$$
$$\mathbb{B}(u) = \lim_{t\to\infty} \gamma_u(t)$$

where γ_u is the geodesic in $B_1^3(0)$ starting at x in the direction u.

Proposition 1. *Consider a point* $\boldsymbol{n} = \boldsymbol{n}' + n^4 e_4$ *in a subset* $\{\boldsymbol{x} \in \mathbb{R}^{3,1} : \langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0\}$. *Define the map*

$$\mathcal{S}_\infty : S^2(\infty) \to \partial B_1^3(0),\ v = \mathcal{S}_\infty[\boldsymbol{n}] = \frac{1}{n_4}\boldsymbol{n}'.$$

It is a conformal diffeomorphism with its inverse map $\mathcal{S}_\infty^{-1}(v) = [v + e_4]$

Remark 7. These two maps $\mathcal{S}$ and $\mathcal{S}_\infty$ define a bijective map from $\mathbb{H}^3 \cup S^2(\infty)$ to a closed unit ball in $\mathbb{R}^3$. More discussions about Minkowski space and the proof of conformal diffeomorphism between $S^2(\infty)$ and ∂B^3 can refer to the chapter 6 in [10] but we here just use these facts. Furthermore, we can use the Figure 6.2 to know how $\overline{\mathbb{R}^3}$ is stereographically compactified onto $\overline{B_1^3(0)}$. Because these maps are all conformal, the planes passing through the origin are invariant and the angles between two clustering surfaces are fixed under the stereographic compactification. More specifically, given a multiphase surface in $\mathbb{R}^3$, we can find a multiphase surface in Poincaré ball $B_1^3(0)$ whose singular structures are one-to-one correspondence and induce the same (ideal) boundary curves.

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Chapter 4

Self-expanding Solutions to the Multiphase Mean Curvature Flow

4.1 Smooth Case

Definition 5. (Mean Curvature Flow)

A family of smoothly embedded hypersurfaces $(\alpha_t)_{t\in I}$ in $\mathbb{R}^{n+1}$ moves according to the mean curvature if

$$\frac{\partial x}{\partial t} = \vec{H}(x) \tag{4.1}$$

for $x \in \alpha_t$ and $t \in I$, where $I \subset \mathbb{R}^2$ is an open interval. Here, $\vec{H}(x)$ is the mean curvature vector at $x \in \alpha_t$.

Theorem 2. *Let $(\alpha_t)_{t\in I}$ be a family of smoothly embedded hypersurfaces in $\mathbb{R}^{n+1}$. If $(\alpha_t)_{t\in I}$ is a self-similar solution to (4.1), then*

$$\vec{H}(x) = \frac{Cx^{\perp}}{2\lambda^2(t)}, \tag{4.2}$$

where $\lambda(t) = \sqrt{1 + C(t - t_0)}$ for $x \in \alpha_t$ as long as $1 + C(t - t_0) > 0$. This describes expanding self-similar solutions about 0 for $C > 0$ and contracting self-similar solutions about 0 for $C < 0$.

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4.2 Singular Case

We are considering the area-minimizing problem in the multiphase system and the important results in [17, 23] state that the only area-minimizing singular structures in $\mathbb{R}^3$ are the regular triple junction and regular quadruple point, so we hereafter assume that Γ contains the triple junctions or quadruple points.

Definition 6. (Self-expanding Multiphase Solutions) *A family of surfaces* $(\alpha_i)_{i=1}^l$, $l \geq m$, *is said to be a multiphase surface* $\Gamma = \bigcup_{i=1}^l \alpha_i$ *which expands homothetically under mean curvature flow from an initial condition* $C_0 = \bigcup_{i=1}^m \sigma_i$ *if they satisfy the following conditions. Each multiphase surface* Γ_t, $t > 0$, *consists of* l *surfaces*

$$\alpha_i(\cdot, t) : U_i \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad\qquad 1 \leq i \leq l$$

$$\boldsymbol{u} := (u_1, u_2) \longmapsto (x_1(u_1, u_2),\ x_2(u_1, u_2),\ x_3(u_1, u_2)) := \boldsymbol{x}$$

- $\alpha_i(\boldsymbol{u}, t)$, $1 \leq i \leq l$, *is smooth for every time* t *and continuous up to* $t = 0$. *Furthermore, each surface* $\alpha_i(\boldsymbol{u}, t)$ *is regular* $\forall t > 0$ *i.e.,*

  $$\frac{\partial}{\partial u_j}\alpha_i(\boldsymbol{u}, t) \neq 0 \qquad\qquad j = 1, 2$$

  uniformly up to $|\boldsymbol{u}| = 0$

- *the start curves* $\alpha_{i_h}(\gamma, t)$, $1 \leq i_h \leq l$, $h = 1, 2, 3$, *coincide on a curve* $\gamma = \bigcap_{h=1}^3 \partial U_{i_h}$ *for all times* $t > 0$ *and the start points* $\alpha_{j_h}(u_0, t)$, $1 \leq j_h \leq l$, $h = 1, \cdots, 6$, *coincide at a point* $u_0 = \bigcap_{h=1}^6 \partial U_{j_h}$ *for all times* $t > 0$, *but these coincidences may depend on time.*

- $(\alpha_i(\cdot, t))_{i=1}^l$ *are embedded surfaces for all* $t \geq 0$. *If three surfaces* $(\alpha_{i_h})_{h=1}^3$ *meet along a curve, this curve must be a start curve. Similarly, a point clustered with six surfaces* $(\alpha_{j_h})_{h=1}^6$ *must be a start point and moreover, it is the intersection of four start curves. The unit normal vectors* ν_{i_h} *of surfaces at the start curve satisfy the balancing condition i.e.,*

  $$\sum_{h=1}^3 \nu_{i_h} = 0 \qquad\qquad \forall t > 0.$$

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Around a quadruple point, there are four regions $R_1, \cdots, R_4$ separated by six surfaces $\{\alpha_{ij}\}$ for $1 \le i \ne j \le 4$. Let ν_{ij} be an unit normal vector to the surface α_{ij} pointing from R_i to R_j. We impose the skewness on the nonadjacent surfaces which means that the intersection of surfaces is at most a point set. More precisely, we require the orthogonality on the tangent planes of the nonadjacent surfaces i.e.,

$$\langle \nu_{ij}, \nu_{kl} \rangle = 0 \qquad\qquad \text{for } 1 \le i \ne j \ne k \ne l \le 4$$

- *There exist m surfaces $(\alpha_{i_h}(\cdot,t))_{h=1}^m \subset \Gamma_t$ which connect to infinity i.e., for all $t \ge 0$,*

  $$\lim_{|\boldsymbol{u}|\to\infty} |\alpha_{i_h}(\boldsymbol{u},t)| = \infty \qquad\qquad h = 1, 2, \cdots, m.$$

  Each surface $\alpha_{i_h}(\cdot,t)$, $h = 1, 2, \cdots, m$ and $t \ge 0$, is at infinity asymptotically closed to the surfcace $\sigma_h \subset C_0$ i.e.,

  $$d_{\mathcal{H}}(\alpha_{i_h}(U_{i_h},t) \cap (\mathbb{R}^3 \backslash B_r(0)),\ \sigma_h \cap (\mathbb{R}^3 \backslash B_r(0)) \to 0 \qquad\qquad \text{for } r \to \infty$$

  where $d_{\mathcal{H}}$ is the Hausdorff distance.

- *Each surface flows for $|\boldsymbol{u}| > 0$ according to mean curvature flow.*

  $$\left(\frac{d\boldsymbol{x}}{dt}\right)^{\perp} = \vec{H}(\boldsymbol{x})$$

  at every point $\boldsymbol{x} \in \alpha_i \subset \Gamma_t$.

- *For $t = 0$, Γ_0 is the initial configuration C_0.*

- *Γ_t expands homothetically i.e., for $0 < t_1 < t_2$, there exists $\lambda > 1$ such that*

  $$\lambda\Gamma_{t_1} = \{\lambda\alpha_i(\cdot,t_1) : \alpha_i(\cdot,t_1) \subset \Gamma_{t_1}\} = \{\alpha_i(\cdot,t_2) : \alpha_i(\cdot,t_2) \subset \Gamma_{t_2}\} = \Gamma_{t_2}$$

- *α_i is of class $C^0(\mathbb{R}^2 \times [0,\infty)) \cap C^\infty(\mathbb{R}^2 \times (0,\infty))$ for $i = 1, 2, \cdots, l$.*

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4.3 Asymptotic Behaviors

Lemma 1. *If Γ is the regular multiphase surface in $\mathbb{R}^3$, then its (ideal) boundary is the regular triple points connected by curves. More specifically, only the surfaces and the triple junctions can connect to infinity.*

Proof. As Γ is the regular multiphase surface, it contains the regular triple junctions and the regular quadruple points. Away from these singular structures, Γ is a finite union of disjoint surfaces that only induce curves on S^2.
Around the singularities, we first consider the regular triple junction and then the regular quadruple point. The possible structures connecting to infinity are the end of regular triple junction and the 1-dimensional subset of regular triple junction. The end of regular triple junction induces a triple point on S^2, but the 1-dimensional subset of regular triple junction cannot attach to S^2. If this 1-dimensional subset wholly connects to infinity then we get a bi-junction on S^2; in other words, two surfaces intersect along a curve with angle $120°$. Using the relation between the Poincaré ball model and Euclidean space, the angle between two surfaces with the same bounday on S^2 is $0°$ in the Poincaré ball model, so the degeneracy of regular triple junction into bi-junction cannot happen.

We next consider the regular quadruple point. Because it is the intersection of four regular triple junctions, the possible structures connecting to infinity are almost the same in the triple junction case, but with one more structure. This one can be imagined by wholly degenerating one of the regular triple junctions into a point on S^2 until the quadruple point coincides with the triple point on S^2. We introduce the following definition to prove the nonexistence of this case.

Definition 7. *A steradian is defined in $\mathbb{R}^3$ as the solid angle subtended at the center of a unit sphere by a unit area on its surface. For a sphere of radius r, any portion of its surface with area r^2 subtends one steradian.*

At the regular quadruple point x, we use the point x accompanied with four unit vectors $\tau_1(x), \tau_2(x), \tau_3(x)$, and $\tau_4(x)$ to represent the quadruple point with tetrahedral structure i.e., $(x, \tau_1(x), \tau_2(x), \tau_3(x), \tau_4(x))$ where τ_i are the unit tangent vector fields along their regular

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triple junctions. Without loss of generality, we argue the degeneracy in τ_1 direction. Suppose $(x, \tau_1(x), \tau_2(x), \tau_3(x), \tau_4(x))$ degenerates into $(y, 0, \tau_2(y), \tau_3(y), \tau_4(y))$. As the tetrahedral structure is connected, we can find three points on S^2 in the directions of $(\tau_2(y) + \tau_3(y))$, $(\tau_2(y) + \tau_4(y))$, and $(\tau_3(y) + \tau_4(y))$. We connect these three points to y with Poincaré arcs and denote the radii of the Poincaré arcs by r_1, r_2, and r_3. Now, we use $r = \min\{r_1, r_2, r_3\}$ to construct a pseudosphere at y. This pseudosphere prevents a regular quadruple point x from degenerating into y on S^2 because the steradian of this pseudosphere at y is zero. Hence, we conclude that only the end of the triple junction and the surfaces can induce the boundaries on S^2. $\square$

Remark 8. The steradian at the quadruple point with a tetrahedral structure has a lower bound of $\cos^{-1}(\frac{1}{\sqrt{3}})$. Because the tetrahedron has an inscribed spherical cone with angle $\cos^{-1}(\frac{1}{\sqrt{3}})$, this inscribed spherical cone serves as a barrier against degeneracy.

Lemma 2. *Suppose Γ is a area-minimizing multiphase surface in $\mathbb{R}^3$ and all triple junctions in Γ connect to each other by surfaces. Let $\tau : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular triple junction in Γ connecting to infinity. If τ locally induces its (ideal) boundary, a triple point with three curves, on S^2, then τ is at infinity asymptotically closed to a half-line l passing through the origin i.e.,*

$$d_{\mathcal{H}}(\tau(U) \cap (\mathbb{R}^3 \backslash B_r(0)),\ l(x) \cap (\mathbb{R}^3 \backslash B_r(0)) \to 0 \qquad \textit{for } r \to \infty.$$

In other words, a triple point induced by τ is in fact a regular triple point on S^2.

Proof. Let $\alpha \subset \Gamma$ be a surface connecting two regular triple junctions which connect to infinity, it induces a boundary curve Σ on S^2 with triple points as its endpoints. We choose

$$M^1 \to S^2$$

as a smooth immersion of simple closed oriented curve containing Σ. Before continuing the arguments, we need to check Σ is a simple curve so that the choice of M^1 is feasible: Suppose Σ is not a simple curve on S^2, we choose an open neighborhood U in a Poincaré ball such that $\overline{U}$ containing a self-intersection or corner of Σ. Besides, we can find in U a four-junction or bi-junction generating Σ on S^2. After taking the inverse hyperbolic stereographic projection on U, we obtain a four-junction or bi-junction locally existing in $\alpha \subset \mathbb{R}^3$. However, the

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results in [23, 17] show that the 1-dimensional area-minimizing singular structure in $\mathbb{R}^3$ is the regular triple junction. If the surface $\alpha \subset \Gamma$ induces a non-simple curve Σ on S^2, then Γ cannot be the area-minimizer in $\mathbb{R}^3$.

By applying theorem 3 in [2] to this simple closed oriented curve M^1, we obtain a complete area-minimizing locally integral 2-current σ_{M^1} in B^3 with asymptotic boundary M^1. Then using the remark of theorem 3 in [2], we can determine that σ_{M^1} is a smooth and properly embedded complete hypersurface in B^3. Because σ_{M^1} is a complete surface of finite topological type and with well-defined limiting normal planes on its ends, the inverse image σ of the projection on σ_{M^1} is also a complete surface of finite topological type and with well-defined normal planes on each end. Applying theorem 3 in [11] to the inverse image σ, it looks from infinity like a plane passing through the origin. Because Σ and σ induce the same boundary $\alpha \cap M^1$ on S^2, Σ is asymptotic to a plane passing through the origin in $\mathbb{R}^3$. Hence, the asymptote of the regular triple junction is a line emanating from the origin, i.e., the regular triple junction intersects S^2 orthogonally.

$\square$

Remark 9. In Lemma 2, we impose the balancing condition on the triple junction $\tau := \bigcap_{i=1}^{3} \sigma_i$ without further assumptions on the surfaces σ_i away from the singularity τ, so the regular triple junction τ may not have the well-defined limiting normal planes on its whole ends. Nevertheless, the balancing condition on τ locally guarantees the normal plane of each surface σ_i near τ is well-defined; namely, the (ideal) boundary of σ_i behaves well near a triple point induced by τ on S^2. Therefore, we may not have a simple closed oriented curve containing the whole (ideal) boundary of σ_i but a circle on S^2 containing a triple point and a portion of the (ideal) boundary of σ_i. Following the arguments in the proof of Lemma 2, we know that a portion of the surface σ_i near τ approaches to a flat plane through the origin. Hence, the triple junction τ is the intersection of three flat planes through the origin and then we finish the proof. If each surface σ_i behaves well at infinity, for instance,"regular at infinity" which is given as a definition in [21], we can choose a great circle on S^2 as simple closed oriented curve containing the (ideal) boundary of σ_i

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Chapter 5

Proof of Main Theorem

5.1 Flat 2-dimensional Chains $\mathcal{F}_2(\mathbb{R}^3, \mathbb{Z}_{k+1})$

5.1.1 Introduction

Let C_0 be a finite union of m surfaces meeting at the origin and separating $\mathbb{R}^3$ into k regions. Each surface or region induced by C_0 satisfies the conditions (A_1), (A_2), and (A_3) in the section 2.3. The main theorem proves the existence of a regular multiphase surface Γ in $\mathbb{R}^3$ where each surface α is a minimal surface for the metric g with C_0 as an initial condition. Specifically, Γ spans tha same boundary as C_0 on S^2. When m equals to two or three, the existence and uniqueness are clear. However, the existence and uniqueness become more complicated as m is greater than three. For instance, given the four boundary curves Σ_1, Σ_2, Σ_3, and Σ_4, there exist two surfaces α_i connecting Σ_i and Σ_{i+2}, $i = 1, 2$, but there exists no surface σ_i with the triple junction or quadruple point. Hence, the main argument here is to instead prove the existence of at least one connected regular multiphase surface that induces the specified boundary curves.

We consider the minimizing problem in the class $\mathcal{F}_2(\mathbb{R}^3, \mathbb{Z}_{k+1})$ of flat 2-dimensional chains in $\mathbb{R}^3$ with coefficients in $\mathbb{Z}_{k+1}$ and the norm on each nonzero element in $\mathbb{Z}_{k+1}$ equals to one. For a complete discussion of flat chain and multiplicity, refer to [24, 18]. In the following, we briefly describe how the flat chains in $\mathcal{F}_2(\mathbb{R}^3, \mathbb{Z}_{k+1})$ representing the multiphase surface problem in $\mathbb{R}^3$.

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- The space of flat 2-dimensional chains is the completion of the space of polygons with respect to the flat norm.

- The interior of each region in $\mathbb{R}^3$ is assigned to a nonzero element in $\mathbb{Z}_{k+1}$ and the zero element in $\mathbb{Z}_{k+1}$ represents the points not belonging to the interior of any regions in $\mathbb{R}^3$.

- The norm equals to 1 on each nonzero coefficient $a_\alpha \in \mathbb{Z}_{k+1}$, which is part of the definition of the size of a flat chain, i.e.,

$$\mathbb{S}(\Sigma) = \sum_{\alpha} |a_\alpha| \mathbb{M}(\alpha) = \sum_{\alpha} \mathbb{M}(\alpha),$$

  where $\mathbb{M}(\alpha)$ is the mass with respect to g. The mass is equal to the area when α is the surface of class C^1.

5.1.2 Example

Let Σ be a multiphase surface mentioned in Example 2. Using the flat 2-dimensional chains in $\mathcal{F}_2(\mathbb{R}^3, \mathbb{Z}_5)$, we give a representation of Σ as below. Let R, G, B, Y, C, and P be the surfaces defined in Example 2, we call $(\cdot, \cdot, \cdot)$ the region enclosed by three surfaces and define a norm on $\mathbb{Z}_5$ by

$$|[z]| = \begin{cases} 1, & [z] \in \mathbb{Z}_5 \backslash \{[0]\} \\ 0, & [z] = [0]. \end{cases}$$

Hence, we have four regions (R, B, P), (G, B, C), (Y, C, P), and (R, Y, G) separated by surfaces R, G, B, Y, C, and P. We assign a nonzero element in $\mathbb{Z}_5$ to each region i.e.,

$$(R, B, P) = [1] \qquad (G, B, C) = [3] \qquad (Y, C, P) = [2] \qquad (R, Y, G) = [4],$$

and then the coefficient a_α of surface $\alpha \in \{R, G, B, Y, C, P\}$ is defined by the following two steps.
For $1 \leq i \neq j \leq 4$,

- The identities e_{ij} represent the surfaces separating the regions i and j where e_{ij} equals to e_{ji}.

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- We assign an element $[j-i] \in \mathbb{Z}_5$ to the identities e_{ij} and e_{ji}.

According to the above arguments, the correspondences between the surfaces and elements in $\mathbb{Z}_5$ are given by

$$\begin{aligned} R &= [4]-[1]=[3] & G &= [4]-[3]=[1] & B &= [3]-[1]=[2] \\ Y &= [4]-[2]=[2] & C &= [3]-[2]=[1] & P &= [2]-[1]=[1]. \end{aligned}$$

Althrough there may be some surfaces assigned literally to the same element in $\mathbb{Z}_5$, it represents the different things. Nevertheless, it is independent of the assignments that the size of Σ is always defined by

$$\mathbb{S}(\Sigma) = \mathbb{M}(R) + \mathbb{M}(G) + \mathbb{M}(B) + \mathbb{M}(Y) + \mathbb{M}(C) + \mathbb{M}(P).$$

5.2 Existence in a Bounded Domain

Theorem 3. *Let C_0 be a finite union of m surfaces meeting at the origin and separating $\mathbb{R}^3$ into k regions. Each surface or region induced by C_0 satisfies the conditions (A_1), (A_2), and (A_3). Suppose a surface $\sigma \in C_0$ connects to infinity, we define the boundary of σ on a sphere of radius R as*

$$\Sigma^R = \sigma \cap S_R^2(0).$$

Given the boundary curves $\Sigma_1^R, \cdots, \Sigma_m^R$, there exists a connected flat 2-dimensional chain Γ^R with coefficients in $\mathbb{Z}_{k+1}$ such that

$$\mathbb{S}(\Gamma^R) = \inf\{\mathbb{S}(c) : c \in \mathcal{F}_2(B_R^3(0), \mathbb{Z}_{k+1}),\ \partial c = \{\Sigma_1^R, \cdots, \Sigma_m^R\}\}.$$

Moreover, each surface of Γ^R is a minimal surface with respect to g, and all 1-dimensional and 0-dimensional junctions are the regular triple junctions and the regular quadruple points.

Proof. We use the compactness of flat chains to prove the existence of area-minimizer. The argument for applying the compactness theorem is standard, so we check the boundedness on the sizes of flat chain and its boundary. More precise discussions about the compactness theorem can refer to [24, 18, 5]

Let $\{\Gamma_j\}$ be an area-minimizing sequence in $\mathcal{F}_2(B_R^3(0), \mathbb{Z}_{k+1})$ with connected supports and Γ_j spans the boundary curves $\Sigma_1^R, \cdots, \Sigma_m^R$ on $S_R^2(0)$ for all j.

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- $\mathbb{S}(\Gamma_j) \leq c_1$, $0 < c_1 < \infty$:
  Because $\{\Gamma_j\}$ is a area-minimizing sequence in $\mathcal{F}_2(B_R^3(0), \mathbb{Z}_{k+1})$, $\mathbb{S}(\Gamma_j)$ is bounded for all j.

- $\mathbb{S}(\partial\Gamma_j) \leq c_2$, $0 < c_2 < \infty$:
  Let $\{\sigma_i\}_{i=1}^m \subset C_0$ be the surfaces connecting to infinity. The size of $\partial\Gamma_j$ is controlled by the sum of the length of γ_i with a constant $C(R)$ i.e.,

$$\mathbb{S}(\partial\Gamma_j) \leq C(R)\sum_{i=1}^{m} |\gamma_i|,$$

  where $\gamma_i = \lim_{R\to\infty}(\Sigma_i^R/R)$.
  Using the assumption (ii) on an initial condition C_0 and the above inequality, we can conclude that $\mathbb{S}(\partial\Gamma_j)$ is finite for each fixed $R > 0$.

The compactness theorem implies that there is a convergent subsequence $\Gamma_{j_l} \subset \Gamma_j$ with limit Γ^R that spans the boundary curves $\{\Sigma_1^R, \cdots, \Sigma_m^R\}$. Furthermore, the lower semicontinuity of the area functional implies that

$$\mathbb{S}(\Gamma^R) \leq \lim_{l\to\infty} \mathbb{S}(\Gamma_{j_l}).$$

That is, Γ^R is an area-minimizer in $\mathcal{F}_2(B_R^3(0), \mathbb{Z}_{k+1})$.

Regarding the regularity of the minimizer Γ^R, we use regularity theorem 2.6 in [17] for $m = 2$ and $n = 3$. The locally area-minimizing singular structures in a bounded set are Hölder-continuously differentiable curves along which three sections of surfaces meet at equal $120°$ and points at which four such curves and six sections of surfaces meet at $cos^{-1}(-\frac{1}{3}) \approx 109.5°$. For details on area-minimizing singular structures in $\mathbb{R}^3$, refer to [23, 17].

Convergence in the flat norm implies convergence as currents:

$$\int_{\Gamma_{j_l}} f \ d\mathcal{H}^2 \to \int_{\Gamma^R} f \ d\mathcal{H}^2$$

for all $f \in C_0^\infty(B_R^3(0))$.
Suppose that $supp(\Gamma^R)$ is not connected, there exists a surface M in $B_R^3(0)$ such that $B_R^3(0)\backslash M$ has two nontrivial components, and we can take an ϵ-neighborhood $\mathcal{U}_\epsilon$ of M

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such that $\mathcal{U}_\epsilon \cap supp(\Gamma^R) = \emptyset$.

Consider a nonnegative $f \in C_0^\infty(B_R^3(0))$ given by

$$f(x) = \begin{cases} 1, & x \in \mathcal{U}_{\epsilon/2} \\ 0, & x \notin \mathcal{U}_\epsilon \end{cases}$$

As $\{\Gamma_{j_l}\}$ is an area-minimizing sequence in $\mathcal{F}_2(B_R^3(0), \mathbb{Z}_{k+1})$ with connected supports in $B_R^3(0)$, we have

$$0 < \mathcal{H}^2(\mathcal{U}_{\epsilon/2} \cap \Gamma_{j_l}) \leq \int_{\Gamma_{j_l}} f \; d\mathcal{H}^2 \nrightarrow \int_{\Gamma^R} f \; d\mathcal{H}^2 = 0.$$

This inequality contradicts the convergence as currents. Hence, the limit Γ^R must be connected. □

5.3 Existence in an Unbounded Domain

Proof of Main Theorem. Let $\{R_j\}_{j=1}^\infty$ be a sequence of numbers diverging to infinity. We apply Theorem 3 to each bounded set $B_{R_j}^3(0)$ and then obtain a subsequence of flat chains $\{\Gamma_l^{R_j}\}_{l=1}^\infty$ converging to a limit Γ^{R_j} which is the area-minimizer in the class of flat chains spanning the boundary curves $\Sigma_1^{R_j}, \cdots, \Sigma_m^{R_j}$ with connected support in $B_{R_j}^3(0)$. When j goes to infinity, we have a family of convergent subsequences $\{\Gamma_l^{R_j}\}_{l=1}^\infty$, $j \in \mathbb{N}$. Next, we take a diagonal process to derive a convergent subsequence $\{\Gamma_j^{R_j}\}_{j=1}^\infty$ which satisfies the following results

- $\Gamma_j^{R_j}$ converges to Γ in $\mathbb{R}^3$, where $\Gamma := \lim_{j\to\infty} \Gamma_j^{R_j}$.

- Γ is the area-minimizer with connected support in $\mathbb{R}^3$.

Since we mainly care about the structures of regular triple junction and regular quadruple point in the multiphase problem, we need to determine whether there is any structure vanishes in the above process as $j \to \infty$. Suppose that there is a structure vanishing as $j \to \infty$, we use the relation between Poincaré ball model and Euclidean space to have a corresponding structure vanishing in Poincaré ball model. Using the arguments in the proof of Lemma 1, we can construct the barriers to stay them away from infinity. Therefore, it is

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impossible that vanishing in Poincaré ball model. Combining the hypothesis (ii) on C_0 and Lemma 2, Γ is at infinity asymptotically closed to a initial condition C_0. $\square$

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Chapter 6

Appendix

6.1 Equivalent Condition of the Skewness Property

In the next two sections, we give the equivalent condition of skewness property at a quadruple point and also demonstrate the first variation to entropy functional defined by g.

Lemma 3. *At a quadruple point O, the skewness property on the nonadjacent surfaces is equivalent to the balancing condition in [13] i.e., for $1 \leq i \neq j \neq k \neq l \leq 4$,*

$$\langle \nu_{ij}, \nu_{kl} \rangle = 0 \Longleftrightarrow \nu_{ij} + \nu_{jk} + \nu_{kl} + \nu_{li} = 0, \ |\nu_{ij} + \nu_{jk} + \nu_{kl}| \leq 1$$

where ν_{ij} is an unit normal vector pointing from the region R_i to R_j.

Proof. Let O be a quadruple point in $\mathbb{R}^3$ where four triple junctions (curves) and six surfaces clustering. Because we study the local structure around O, without loss of generality, we assume the curves and surfaces to be the lines and planes clustering at the origin in a unit ball.

Given one of the regions near O, there exist three half-planes and triple junctions enclosing this region. Let u, v, and w be the outward-pointing unit tangent vectors along the triple-junctions. See Figure 6.3, for example.

Suppose we have the skewness property on the nonadjacent surfaces. The unit normal vectors of three halfplanes are given by

$$\nu_{uv} := \frac{u \times v}{|u \times v|} \qquad \nu_{vw} := \frac{v \times w}{|v \times w|} \qquad \nu_{wu} := \frac{w \times u}{|w \times u|}.$$

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and their skewed unit normal vectors are defined respectively as follows

$$\tilde{\nu}_{uv} := \frac{(u \times v) \times w}{|(u \times v) \times w|} \qquad \tilde{\nu}_{vw} := \frac{(v \times w) \times u}{|(v \times w) \times u|} \qquad \tilde{\nu}_{wu} := \frac{(w \times u) \times v}{|(w \times u) \times v|}.$$

Consider the sum of unit normal vectors going throught each region once and back to the original one. For example, $\tilde{\nu}_{vw} + \tilde{\nu}_{wu} + \nu_{vw} + \nu_{uw} = \vec{a}$. To determine $\vec{a}$, we take inner product both sides of this equation and use the balancing condition on each triple juntion i.e.,

$$0 = \langle \nu_{vw},\ \tilde{\nu}_{vw} + \tilde{\nu}_{wu} + \nu_{vw} + \nu_{uw} \rangle = \langle \nu_{vw},\ \vec{a} \rangle,$$
$$0 = \langle \nu_{wu},\ \tilde{\nu}_{vw} + \tilde{\nu}_{wu} + \nu_{vw} + \nu_{uw} \rangle = \langle \nu_{wu},\ \vec{a} \rangle,$$
$$0 = \langle \nu_{uv},\ \tilde{\nu}_{vw} + \tilde{\nu}_{wu} + \nu_{vw} + \nu_{uw} \rangle = \langle \nu_{uv},\ \vec{a} \rangle.$$

Owing to the linear independence of u, v, and w, the normal vectors ν_{uv}, ν_{vw}, and ν_{wu} form a basis in $\mathbb{R}^3$. Therefore, the above equations imply that $\vec{a}$ must equal to $\vec{0}$. Using the same argument, it is obvious that the length of sum of three consecutive vectors is less than one. Because the region is arbitrarily chosen from four regions surrounding O, we obtain the balancing condition in [13]. Conversely, if we have the balancing condition in [13] around a quadruple point i.e.,

$$\nu_{ij} + \nu_{jk} + \nu_{kl} + \nu_{li} = 0 \qquad\qquad 1 \leq i \neq j \neq k \neq l \leq 4.$$

Take the inner product both sides of the above equation with ν_{ij} and use the balancing condition of triple junction, we obtain the skewness property $\langle \nu_{ij},\ \nu_{kl} \rangle = 0$. □

Remark 10. Let Λ^{ijk}, Λ^{jlk}, Λ^{kli}, and Λ^{ilj} be the four regular triple junctions clustering at O and for each regular triple junction, say Λ^{ijk}, there are three halfplanes σ_{ij}, σ_{jk}, and σ_{ki} that meet along Λ^{ijk} with the balancing condition

$$\nu_{ij} \cdot \nu_{jk} = \nu_{jk} \cdot \nu_{ki} = \nu_{ki} \cdot \nu_{ij} = -\frac{1}{2}$$

on each point of Λ^{ijk}. In terms of these normal vectors, we can derive the tangent vector field along Λ^{ijk} as

$$\tau^{ijk} = \frac{\nu_{ij} \times \nu_{jk}}{|\nu_{ij} \times \nu_{jk}|} = \frac{2}{\sqrt{3}} \nu_{ij} \times \nu_{jk}.$$

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Next, we compute the inner product of any two tangent vectors at the quadruple point O as bellow

$$\tau^{ijk} \cdot \tau^{jlk} = \frac{4}{3}[(\nu_{ij} \cdot \nu_{jl})(\nu_{jk} \cdot \nu_{lk}) - (\nu_{ij} \cdot \nu_{lk})(\nu_{jk} \cdot \nu_{jl})],$$

and by using the balancing condition on the triple junctions, we obtain

$$\tau^{ijk} \cdot \tau^{jlk} = \frac{4}{3}(-\frac{1}{4} - 0) = -\frac{1}{3}.$$

The above argument and Lemma 3 imply that the geometry near a quadruple point is already determined by the geometry of each triple junction around O.

6.2 First variation around the Singular Structures

Given $\mathbf{x} = (x_1, x_2, x_3) \in \Gamma_t \subset \mathbb{R}^3$, we consider the entropy-type functional with respect to the metric g.

$$F_g(\Gamma_t) = \int_{\Gamma_t} \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} \, d\mathcal{H}^2(\mathbf{x}), \tag{6.1}$$

where

$$g(\mathbf{x}) = \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} \, d\mathbf{x}^2 = \exp^{\frac{C(x_1^2+x_2^2+x_3^2)}{4\lambda^2(t)t}} (dx_1^2 + dx_2^2 + dx_3^2).$$

For the general and related arguments about the entropy-type functional, refer to [4, 7, 8].

Lemma 4. *If a multiphase surface Γ is a regular self-expanding solution to the mean curvature flow, then it is a critical point of the entropy-type functional* (6.1).

Proof. Consider a surface $\alpha \subset \Gamma$ that is smoothly embedded in $\mathbb{R}^3$. We define a smooth family of diffeomorphisms $\{\Phi^s\}_{s\in[0,1]}$ on an open neighborhood $U \subset \alpha$.
Fix $s \in [0, 1]$.

$$\begin{aligned} &\Phi^s : U \to \alpha \\ &(u_1, u_2) \mapsto \Phi^s(u_1, u_2) = (\Phi_1^s(u_1, u_2), \Phi_2^s(u_1, u_2), \Phi_3^s(u_1, u_2)) := \mathbf{x} \end{aligned}$$

with the conditions

$$\begin{aligned} &K \subset\subset U \subset \alpha, \quad \frac{d\mathbf{x}}{ds} = \vec{X}(\mathbf{x}) = \vec{X} \\ &\Phi^0(\mathbf{x}) = \mathbf{x} \quad , \mathbf{x} \in U \\ &\Phi^s(\mathbf{x}) = \mathbf{x} \quad , s \in (0, 1), \quad \mathbf{x} \in U \backslash K. \end{aligned}$$

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The area functional is

$$\mathcal{H}^2(\Phi^s(\alpha \cap K)) = \int_{\alpha\cap K} J\Phi^s_g d\mathcal{H}^2,$$

where

$$J\Phi^s_g = \sqrt{\det\left(\exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} \begin{bmatrix} \frac{\partial\Phi_1^s}{\partial u_1} & \frac{\partial\Phi_2^s}{\partial u_1} & \frac{\partial\Phi_3^s}{\partial u_1} \\ \frac{\partial\Phi_1^s}{\partial u_2} & \frac{\partial\Phi_2^s}{\partial u_2} & \frac{\partial\Phi_3^s}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{\partial\Phi_1^s}{\partial u_1} & \frac{\partial\Phi_1^s}{\partial u_2} \\ \frac{\partial\Phi_2^s}{\partial u_1} & \frac{\partial\Phi_2^s}{\partial u_2} \\ \frac{\partial\Phi_3^s}{\partial u_1} & \frac{\partial\Phi_3^s}{\partial u_2} \end{bmatrix}\right)} = \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} J\Phi^s$$

is the Jacobian of the metric g, $J\Phi^s$ is the Jacobian of the Euclidean metric, and $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure.

On each surface,

$$\begin{aligned} &\frac{d}{ds}\Big|_{s=0} \mathcal{H}^2(\Phi^s(\alpha\cap K)) \\ &= \int_{\alpha\cap K} \langle D(\exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}}) \cdot \vec{X} \rangle d\mathcal{H}^2 + \int_{\alpha\cap K} \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} div_{\alpha}\vec{X}\ d\mathcal{H}^2 \\ &= \int_{\partial(\alpha\cap K)} \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} \langle \vec{X}, \vec{T}\rangle d\mathcal{H}^2 + \int_{\alpha\cap K} \langle \nabla^{\perp}(\exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}}) - \exp^{\frac{C|\mathbf{x}|^2}{4\lambda^2(t)t}} \vec{H}(\mathbf{x}), \vec{X}\rangle d\mathcal{H}^2, \end{aligned} \tag{6.2}$$

where $\vec{T}$ is the unit tangent vector at point $\mathbf{x}$.

If a variational vector $\vec{X}$ has a compact support on each surface, then the first integral in (6.2) obviously equals to zero and the second integral vanishes since α is a self-expanding solutions to the mean curvature flow i.e., $\vec{H} = \frac{C\mathbf{x}^{\perp}}{2\lambda(t)}$ for every point $\mathbf{x} \in \alpha$. If $\vec{X}$ is compactly supported on a neighborhood of each triple junction $(\alpha_i(\mathbf{0}, t))_{i=1}^3$, then the second integral vanishes for the same reason mentioned above and the first one equals to zero which is guaranteed by the balancing condition i.e.,

$$\sum_{i=1}^{3} \nu_i = 0$$

where ν_i is an unit normal vector to the surface α_i.

If $\vec{X}$ is compactly supported around a regular quadruple point $(\alpha_i(\mathbf{0}, t))_{i=1}^6 = O$, then the skewness in Definition 6 or the balancing condition in [13] i.e.,

$$\nu_{ij} + \nu_{jk} + \nu_{kl} + \nu_{li} = 0 \qquad\qquad 1 \le i \ne j \ne k \ne l \le 4$$

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is enough to make the first integral zero and the second one vanishes for the self-similarity of each surface around the point. □

Remark 11. In one dimensional case, the balancing condition at the triple point provides a sufficient relation on any two curves clustering at the triple point. In two dimensional case, we take Figure 6.1 for example. When we consider the surface G, the balancing condition on the triple junction curves enclosing the surface G offers the direct information of the surfaces R, B, Y, and C. After we finish the following argument, the skewness property at the quadruple point provides a direct connection between the surfaces G and P and a interaction between triple junctions and subregions around a quadruple point.

Consider a region near the quadruple point O, there exist three half-planes and triple junctions enclosing this region. See Figure 6.3 for example, we let u, v, and w be the outward-pointing unit tangent vectors along the triple-junctions and denote the angles between any two of them by

$$\theta_1 = \angle(u, v) \qquad \theta_2 = \angle(v, w) \qquad \theta_3 = \angle(w, u).$$

With the same argument in the proof of Lemma 3, we impose the balancing condition on the triple junction which is the intersection of half-planes determined by the unit normal vectors $\tilde{\nu}_{uv}$, $\tilde{\nu}_{vw}$, and $\tilde{\nu}_{wu}$. The balancing condition shows that

$$\tilde{\nu}_{uv} + \tilde{\nu}_{vw} + \tilde{\nu}_{wu} = 0.$$

Applying the triple product expansion of cross product to the above equation, we derive the following equalities

$$\cos^2\theta_1 = \cos^2\theta_2 = \cos^2\theta_3.$$

Using the balancing condition again, it forces that all angles are all equal and they belong to one of the intervals $(0, \frac{\pi}{2})$ or $(\frac{\pi}{2}, \pi)$. Because the same argument is valid for the other regions, we can conclude that the angles between any two unit tangent vectors are all the same i.e.,

$$\angle(u, v) = \angle(v, w) = \angle(w, u) = \angle(z, u) = \angle(z, v) = \angle(z, w) := \theta. \tag{6.3}$$

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Since the region around O is separated into four equal subregions, the angle θ in (6.3) must belong to an interval $(\frac{\pi}{2}, \pi)$. In addition, each subregion has the regular spherical triangle as its boundary on S^2. These results implies that the quadruple point O clustered by the triple junctions determined by u, v, w, and z is a regular quadruple point.

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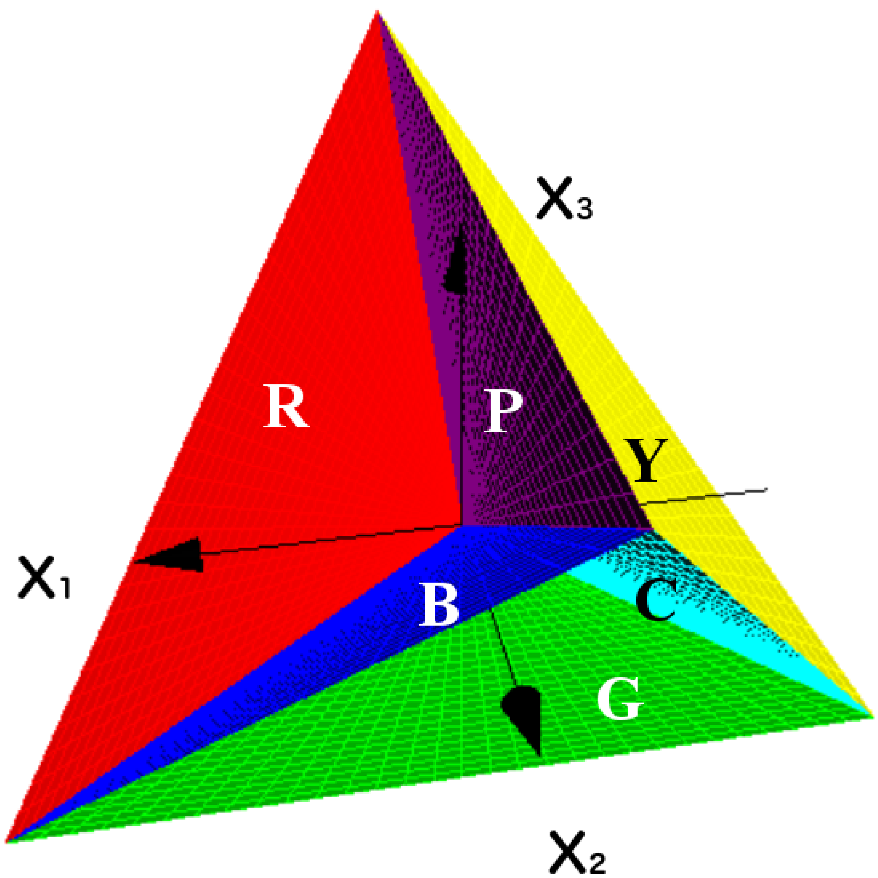

Figure 6.1: Six half-planes

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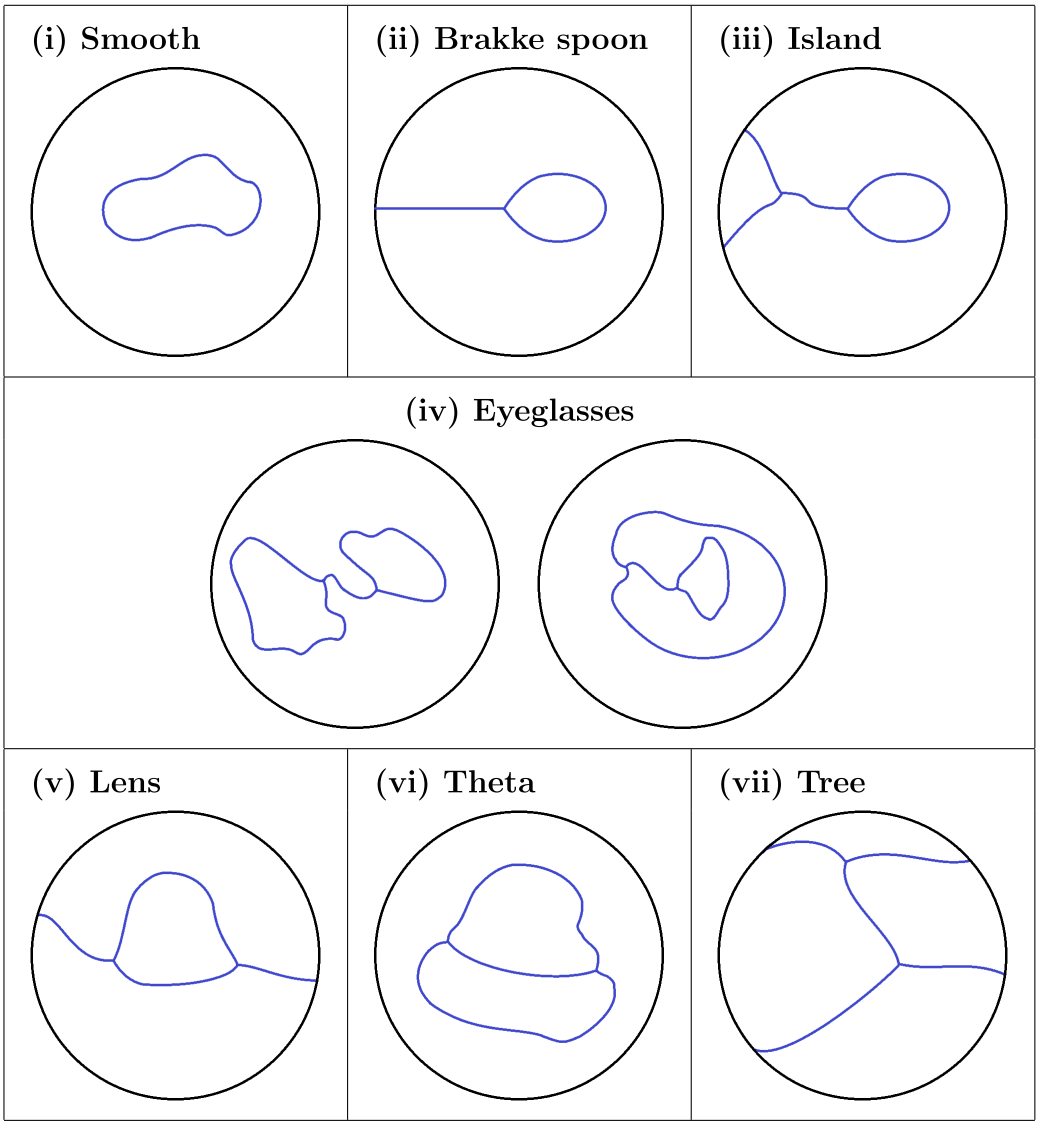

Table 6.1: Classification.

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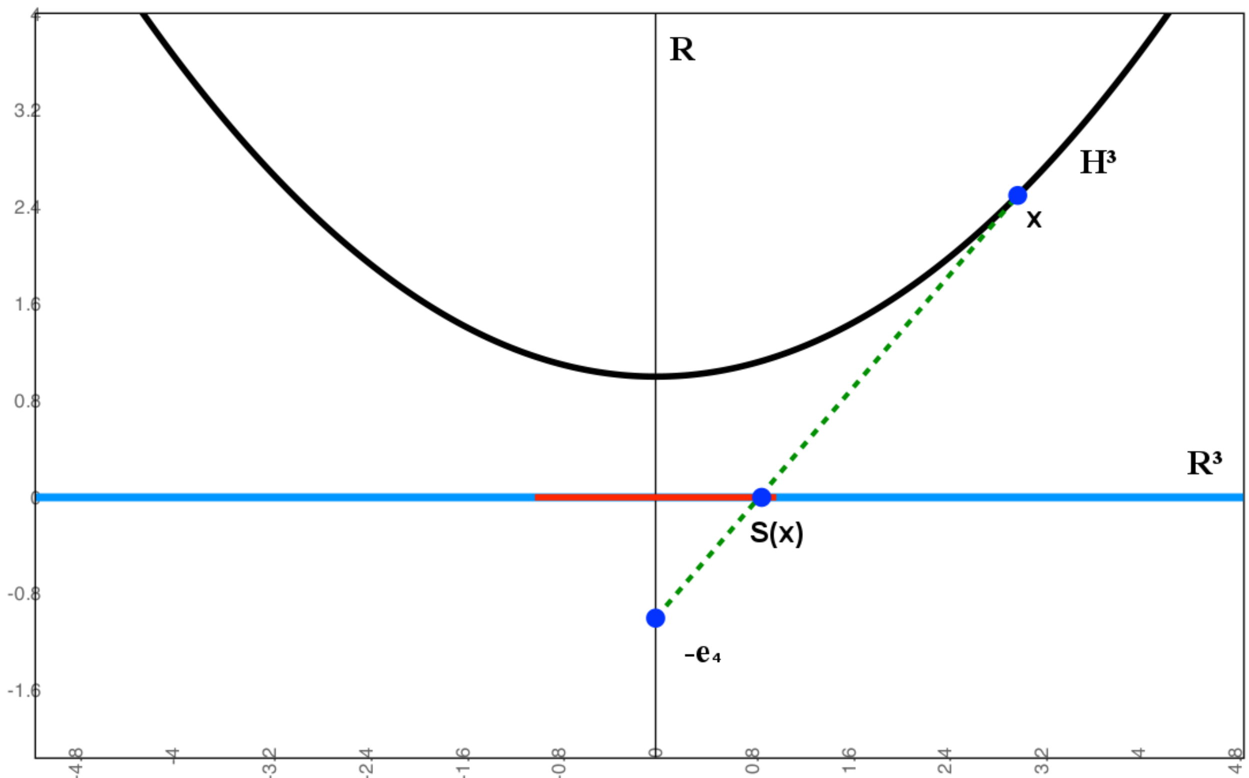

Figure 6.2: The stereographic compactification

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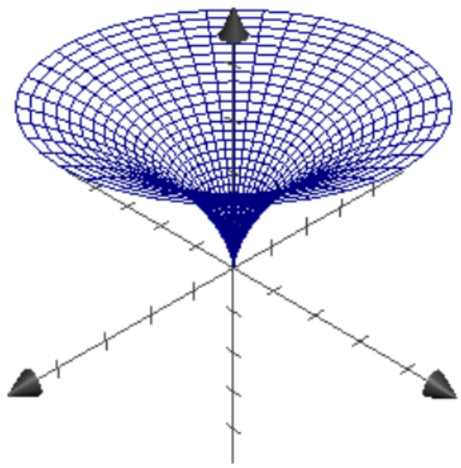
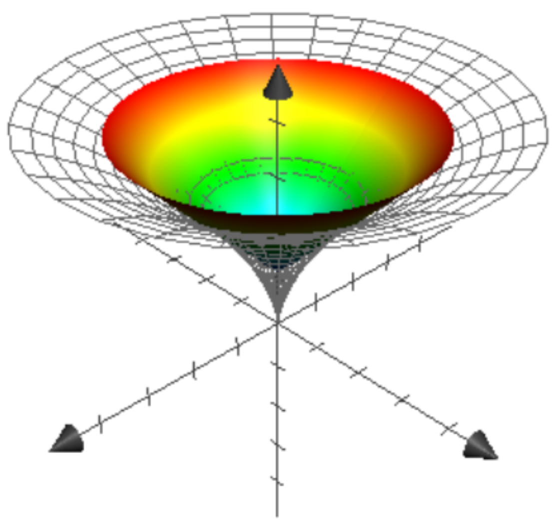
(i) Pseudosphere at the origin. 
(ii) The inscribed tangent cone at some point above the origin. 

Table 6.2: Schematical pictures

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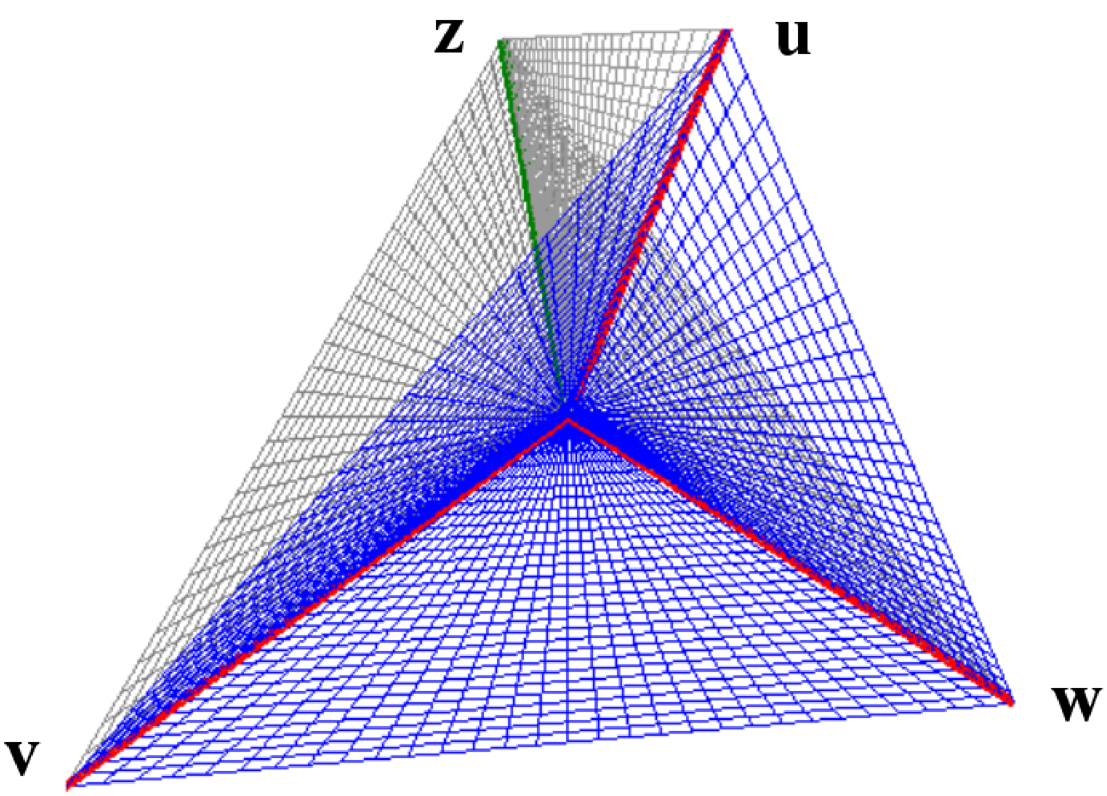

Figure 6.3:

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