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大離差理論在系統風險和投資組合最佳化的應用

Applications of Large Deviation Theory to Systemic Risk and Portfolio Optimization

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摘要

這份論文有兩部分。在第一部分，我們會用重要抽樣法來估計稀有事件的期望值。在常態分配或是布朗運動模型下，可以證明我們提出的方法是有效率的。我們也會說明如何應用重要抽樣法來評估系統風險。在第二部分，我們會應用大離差理論在有限時間的投資最佳化上面。

關鍵詞：重要抽樣法、漸進最佳、大離差、體系風險、投資組合最佳化

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Abstract

There are two parts in this paper. In the first part, we will focus on estimating the expectations under a rare event with the importance sampling method. Under Normal distribution or Brownian motion, we can prove that our proposed method is efficient. We will also show how to apply the importance sampling method to measure the systemic risk. In the second part, we will apply large deviation theory to the finite-horizon investment optimization.

Keywords: importance sampling, asymptotic optimality, large deviation, systemic risk, optimal portfolio

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Chapter 1

Introduction

It is a trend to use Monte Carlo simulation to estimate credit risk. However, it is not efficient when the event become rare. For example, let Z be a random variable that follows the standard normal distribution. We are interested in the expectation $E[Z1(Z < c)]$, where c is a negative constant. It is easy to compute the closed form $-\frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}$. However, when c is very small, we can see that the value will become very small, which makes it inefficient to use crude Monte Carlo to estimate it. That is, we need to sample a lot of times so that we can have a small standard error.

The importance sampling method helps us to solve the "inefficient" problem. More specifically, the importance sampling method helps to reduce the variance [9]. The idea of importance sampling is to find a suitable change of measure so that the rare events we are interested in will become "not rare" under the new measure, which will be called $\tilde{P}$ in this paper. Also, we need to reduce the variance of estimation under this new measure $\tilde{P}$. Since

$$\begin{aligned} Var_{\tilde{P}} &= E_{\tilde{P}}[(f(Z)\frac{dP}{d\tilde{P}})^2] - (E_{\tilde{P}}[f(Z)\frac{dP}{d\tilde{P}}])^2 \\ &= E_{\tilde{P}}[(f(Z)\frac{dP}{d\tilde{P}})^2] - (E[f(Z)])^2, \end{aligned}$$

we can see that the variance equals to 0 if we could minimize the second moment under the measure $\tilde{P}$. However, it is difficult to minimize it directly since the minimization often relates to solving a nonlinear equation. Thus, we seek another way called "asymptotically optimal" property to solve it [9]. That is, we will check if $\lim_{c\to-\infty} \frac{1}{c^2} \ln M_2 =$

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$2\lim_{c\to-\infty}\frac{1}{c^2}\ln M_1$ under the measure $\tilde{P}$, where M_1, M_2 are the first moment and the second moment, respectively. If it does, it implies that the importance sampling we proposed is "efficient".

On the other hand, a lot of researchers use Importance Sampling to apply the theory of large deviation to analyze the asymptotic properties of the tail probability. The famous theories are Cramer Theorem, Schilder's Theorem, Freidlin-Wentzell Theorem, etc. In financial applications, some researchers let the time T go to infinity so that these large deviation theories can be easily applied. [8] provides a large deviations approach to optimal long-term investment. However, some investors would like to earn their money as soon as possible. Thus, we will try to apply the theory of large deviations to the finite-horizon investment.

This paper is composed of two parts. In the first part, we will apply Importance Sampling to estimate the expectation of a random variable under a rare event. In Chapter 2, we are going to apply the method under the standard normal distribution. In Chapter 3 and 4, we will extend it to Brownian motion and geometric Brownian motion. In Chapter 5, we present some applications to estimate systemic risk under different models. In the second part, we will apply the theory of large deviation to the finite-horizon investment, which will be represented in Chapter 6.

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Part 1

Large Deviation Theory applied to Systemic Risk

In this part, we will use importance sampling method to measure the systemic risk under different models. In order to do this, we need to estimate the form $E[1(X < c)]$ and $E[X1(X < c)]$, where X is a random variable and c is a (usually negative) constant. There have been some papers about how to estimate $E[1(X < c)]$ efficiently, such as [1] and [2]. Thus, we will focus on $E[X1(X < c)]$ in this part.

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Chapter 2

Standard Normal Case

Suppose $X_i = \rho X_M + \sqrt{1-\rho^2}Y$, where X_M, Y are independent standard normal distribution, and ρ is correlation between X_i and X_M. We are going to estimate $E[X_i 1(X_M < c)]$. When the value c is very small, it is difficult to use Monte Carlo to estimate. Thus, we will use importance sampling by choosing a new measure $\tilde{P}$ such that the event $\{X_M < c\}$ is no more "rare" under this measure.

2.1 Change measure

Note that

$$
\begin{aligned}
E[X_i 1(X_M < c)] &= E[(\rho X_M + \sqrt{1-\rho^2}Y)1(X_M < c)] \\
&= \rho E[X_M 1(X_M < c)] + \sqrt{1-\rho^2}E[Y1(X_M < c)]
\end{aligned}
$$

We can see that the second term is equal to 0 since X_M and Y are independent. Thus, we just need to consider the first term.

The easiest way to choose a new measure $\tilde{P}$ such that the event $\{X_M < c\}$ is no more "rare" under this measure is to make the mean of X_M be c. That means X_M will be a normal distribution with the mean c and the variance 1 under the new measure $\tilde{P}$. Then

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we can derive that

$$\frac{dP}{d\tilde{P}} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{X_M^2}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{(X_M-c)^2}{2}}}$$
$$= e^{-cX_M+\frac{c^2}{2}}.$$

Due to the above, we can rewrite the first term as:

$$E[X_M 1(X_M < c)] = \tilde{E}[X_M exp(-cX_M + \frac{c^2}{2})1(X_M < c)].$$

2.2 Asymptotic variance analysis

The first moment is

$$\begin{aligned} M_1 &= -\rho E[X_M 1(X_M < c)] \\ &= -\rho \int_{-\infty}^{c} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \frac{\rho}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}. \end{aligned}$$

It implies that

$$\lim_{c\to-\infty} \frac{1}{c^2} \ln M_1 = -\frac{1}{2}.$$

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Suppose $X_M \sim N(-c, 1)$, $Z \sim N(0,1)$ under $\hat{P}$. Then the second moment is

$$\begin{aligned}
M_2 &= \tilde{E}[X_i^2 exp(-2cX_M + c^2)1(X_M < c)] \\
&= e^{c^2}\hat{E}[X_i^2 exp(-2cX_M)1(X_M < c)\frac{d\tilde{P}}{d\hat{P}}] \\
&= e^{c^2}\hat{E}[X_i^2 exp(-2cX_M)1(X_M < c)e^{2cX_M}] \\
&= e^{c^2}\hat{E}[X_i^2 1(X_M < c)] \\
&= \rho^2 e^{c^2}\hat{E}[X_M^2 1(X_M < c)] + (1-\rho^2)e^{c^2}\hat{E}[Y^2 1(X_M < c)] \\
&= \rho^2 e^{c^2}\hat{E}[(Z-c)^2 1(Z < 2c)] + (1-\rho^2)e^{c^2}\hat{E}[1(Z < 2c)] \\
&= \rho^2 e^{c^2}\{\hat{E}[Z^2 1(Z < 2c)] - 2c\hat{E}[Z1(Z < 2c)] + c^2\hat{E}[1(Z < 2c)]\} \\
&\quad + (1-\rho^2)e^{c^2}\Phi(2c) \\
&= \rho^2 e^{c^2}\{\int_{-\infty}^{2c}\frac{z^2}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz - 2c\int_{-\infty}^{2c}\frac{z}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + c^2\Phi(2c)\} \\
&\quad + (1-\rho^2)e^{c^2}\Phi(2c) \\
&= \rho^2 e^{c^2}(1+c^2)\Phi(2c) + (1-\rho^2)e^{c^2}\Phi(2c) \\
&= (\rho^2 c^2 + 1)e^{c^2}\Phi(2c),
\end{aligned}$$

where $\Phi(z)$ is the cdf of a standard normal.

There is an important approximation to $\Phi(z)$:

$$\lim_{z\to-\infty}\Phi(z) = \frac{1}{\sqrt{2\pi}(-z)}e^{-\frac{z^2}{2}}.$$

It implies that

$$\begin{aligned}
&\lim_{c\to-\infty}\frac{1}{c^2}\ln M_2 \\
=& \lim_{c\to-\infty}\frac{1}{c^2}[c^2 + \ln(\rho^2 c^2 + 1) + \ln(\frac{1}{\sqrt{2\pi}(-2c)}) - 2c^2] \\
=& -1.
\end{aligned}$$

Thus, we have the following theorem.

Theorem 1. *Suppose $X_i = \rho X_M + \sqrt{1-\rho^2}Y$, where X_M, Y are independent standard normal distribution, and ρ is correlation between X_i and X_M. Also, let*

$$\begin{aligned}
M_1 &= -E[X_i 1(X_M < c)] \\
&= \tilde{E}[X_i 1(X_M < c)\frac{dP}{d\tilde{P}}]
\end{aligned}$$

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$$M_2 = \tilde{E}[X_i^2 1(X_M < c)(\frac{dP}{d\tilde{P}})^2].$$

Then,

$$\lim_{c \to -\infty} \frac{1}{c^2} \ln M_2 = 2 \lim_{c \to -\infty} \frac{1}{c^2} \ln M_1.$$

2.3 Numerical results

The numerical results are shown below. N is the simulation number. The value simulated by basic Monte Carlo (BMC), importance sampling (IS) and the corresponding standard error (S.E.) are given in the table. We also compare the value with the exact answer (exact).

Table 2.1: Standard normal case

c	exact	N	BMC	S.E.	IS	S.E.
		10000	-0.2418	0.0058	-0.2377	0.0026
-1	-0.2420	40000	-0.2473	0.0029	-0.2416	0.0013
		160000	-0.2403	0.0015	-0.2419	6.3741e-04
		10000	-0.0028	0.0013	-0.0043	7.9976e-05
-3	-0.0044	40000	-0.0041	6.4021e-04	-0.0044	3.8964e-05
		160000	-0.0041	3.0699e-04	-0.0044	1.9402e-05
		10000	0	-	-1.4956e-06	3.3274e-08
-5	-1.4867e-06	40000	0	-	-1.4769e-06	1.7267e-08
		160000	-3.2663e-05	3.2663e-05	-1.5034e-06	8.6994e-09

We can see that the basic Monte Carlo method can hardly sample a rare event when $c = -5$, while the importance sampling method we provide gives an accurate estimation. In addition, the sample error of importance sampling scheme is smaller than that of basic Monte Carlo method.

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Chapter 3

Brownian Motion Case

Let W_{Mt}, Z_t be independent Brownian motion. Define

$$W_{it} = \rho W_{Mt} + \sqrt{1-\rho^2} Z_t,$$

where ρ is a correlation between W_{it} and W_{Mt}. Then W_{it} is also a Brownian motion. In this chapter, we are going to estimate $E[W_{iT} 1(W_{MT} < c)]$.

3.1 Change measure

Note that if c becomes very small, the probability of the event $\{W_{MT} < c\}$ will also be small, which means that it will be inefficient to use crude Monte Carlo method. Thus, we need to find a new measure $\tilde{P}$ such that the event $\{W_{MT} < c\}$ would be no more "rare" under this measure. The following theorem helps us to find a good measure.

Girsanov's Theorem. [6] *Let $W_t, 0 \le t \le T$ be a Brownian motion. Let $\Theta_t, 0 \le t \le T$ be an adapted process. Define*

$$Z_t = exp\{\int_0^t \Theta_u W_u - \frac{1}{2}\int_0^t \Theta_u^2 du\},$$
$$\tilde{W}_t = W_t - \int_0^t \Theta_u du,$$

and assume that

$$E[\int_0^T \Theta_u^2 Z_u^2 du] < \infty.$$

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We also define a new probability measure $\tilde{P}$ by the formula

$$\tilde{P}_A = \int_A Z_\omega dP_\omega \text{ for all } A \in \mathcal{F}.$$

Set $Z = Z_T$. Then $E[Z] = 1$ and under the probability measure $\tilde{P}$, the process $\tilde{W}_t, 0 \leq t \leq T$, is also a Brownian motion.

By the Girsanov's Theorem, we can define

$$\frac{dP}{d\tilde{P}} = e^{-\alpha W_{MT} + \frac{1}{2}\alpha^2 T},$$

$$\tilde{W}_{MT} = W_{MT} - \alpha T,$$

where α is a constant and $\tilde{W}_{MT}$ is a Brownian motion under $\tilde{P}$.

Next, we are going to determine the constant α. If we hope to make the event $\{W_{MT} < c\}$ no more "rare", we can simply let $\tilde{E}[W_{MT}] = c$. That is to say,

$$\tilde{E}[W_{MT}] = \tilde{E}[\tilde{W}_{MT} + \alpha T] = \alpha T = c,$$

which means that $\alpha = \frac{c}{T}$.

3.2 Asymptotic variance analysis

Note that $W_{MT} \sim N(0, T)$ since W_{MT} is a Brownian motion. So we can get the first moment as following:

$$\begin{aligned} M_1 &= -E[W_{iT} 1(W_{MT} < c)] \\ &= -\rho E[W_{MT} 1(W_{MT} < c)] \\ &= -\rho \int_{-\infty}^{c} \frac{x}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx \\ &= \frac{\rho\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{c^2}{2T}}. \end{aligned}$$

It implies that

$$\lim_{c \to -\infty} \frac{1}{c^2} \ln M_1 = -\frac{1}{2T}.$$

Before we consider the second moment, we define a new measure $\hat{P}$ by

$$\frac{dP}{d\hat{P}} = e^{\alpha W_{MT} + \frac{1}{2}\alpha^2 T}$$

$$\hat{W}_{MT} = W_{MT} + \alpha T,$$

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where $\hat{W}_{MT}$ is a Brownian motion under $\hat{P}$. Then we can calculate the second moment as follows:

$$\begin{aligned}
M_2 &= \tilde{E}[W_{iT}^2 1(W_{MT} < c)(\frac{dP}{d\tilde{P}})^2] \\
&= E[W_{iT}^2 1(W_{MT} < c)\frac{dP}{d\tilde{P}}] \\
&= \hat{E}[W_{iT}^2 1(W_{MT} < c)\frac{dP}{d\tilde{P}}\frac{dP}{d\hat{P}}] \\
&= \hat{E}[W_{iT}^2 1(W_{MT} < c)e^{\alpha^2 T}] \\
&= e^{\alpha^2 T}\hat{E}[(\rho W_{MT} + \sqrt{1-\rho^2}Z_T)^2 1(W_{MT} < c)] \\
&= e^{\alpha^2 T}\{\hat{E}[\rho^2 W_{MT}^2 1(W_{MT} < c)] + \hat{E}[(1-\rho^2)Z_T^2 1(W_{MT} < c)]\} \\
&= \rho^2 e^{\alpha^2 T}\hat{E}[W_{MT}^2 1(W_{MT} < c)] + (1-\rho^2)e^{\alpha^2 T}T\hat{E}[1(W_{MT} < c)] \\
&= \rho^2 e^{\alpha^2 T}\hat{E}[(\hat{W}_{MT} - \alpha T)^2 1(\hat{W}_{MT} < 2c)] + (1-\rho^2)e^{\alpha^2 T}T\Phi(\frac{2c}{\sqrt{T}}) \\
&= \rho^2 e^{\alpha^2 T}\{\hat{E}[\hat{W}_{MT}^2 1(\hat{W}_{MT} < 2c)] - 2\alpha T\hat{E}[\hat{W}_{MT}1(\hat{W}_{MT} < 2c)] \\
&\quad + \alpha^2 T^2\hat{E}[1(\hat{W}_{MT} < 2c)]\} + (1-\rho^2)e^{\alpha^2 T}T\Phi(\frac{2c}{\sqrt{T}}) \\
&= \rho^2 e^{\frac{c^2}{T}}\{-2c\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{2c^2}{T}} + T\Phi(\frac{2c}{\sqrt{T}}) + \frac{2c\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{2c^2}{T}} + c^2\Phi(\frac{2c}{\sqrt{T}})\} \\
&\quad + (1-\rho^2)e^{\frac{c^2}{T}}T\Phi(\frac{2c}{\sqrt{T}}) \\
&= e^{\frac{c^2}{T}}\Phi(\frac{2c}{\sqrt{T}})(\rho^2 c^2 + T).
\end{aligned}$$

It implies that

$$\begin{aligned}
&\lim_{c\to-\infty}\frac{1}{c^2}\ln M_2 \\
&= \lim_{c\to-\infty}\frac{1}{c^2}\ln[e^{-\frac{c^2}{T}}\frac{\sqrt{T}}{\sqrt{2\pi}(-2c)}(\rho^2 c^2 + T)] \\
&= -\frac{1}{T}.
\end{aligned}$$

Thus, we have the following theorem.

Theorem 2. *Suppose $W_{it} = \rho W_{Mt} + \sqrt{1-\rho^2}Z_t$, where W_{Mt}, Z_t are Brownian motion, and ρ is correlation between W_{it} and W_{Mt}. Also, let*

$$\begin{aligned}
M_1 &= -E[W_{iT}1(W_{MT} < c)] \\
&= \tilde{E}[W_{iT}1(W_{MT} < c)\frac{dP}{d\tilde{P}}]
\end{aligned}$$

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$$M_2 = \tilde{E}[W_{iT}^2 1(W_{MT} < c)(\frac{dP}{d\tilde{P}})^2].$$

Then,

$$\lim_{c \to -\infty} \frac{1}{c^2} \ln M_2 = 2 \lim_{c \to -\infty} \frac{1}{c^2} \ln M_1.$$

3.3 Numerical results

We perform the sampling again, the numerical results are shown below.

Table 3.1: Brownian motion case

c	exact	N	BMC	S.E.	IS	S.E.
		10000	-0.9652	0.0274	-0.9846	0.0149
-5	-1.0084	40000	-1.0095	0.0140	-1.0089	0.0076
		160000	-1.0157	0.0071	-1.0061	0.0038
		10000	-0.0320	0.0064	-0.0354	6.3683e-04
-15	-0.0360	40000	-0.0426	0.0037	-0.0362	3.2467e-04
		160000	-0.0344	0.0017	-0.0362	1.6229e-04
		10000	0	-	-4.5248e-05	1.0255e-06
-25	-4.5779e-05	40000	0	-	-4.5761e-05	5.2068e-07
		160000	0	-	-4.5571e-05	2.6038e-07

$T = 30, \rho = 0.7.$

Again, we can see that when $c = -25$, the basic Monte Carlo method cannot sample a rare event even we simulate 160000 times, while the importance sampling method we provide gives an accurate estimation. In addition, the sample error of importance sampling scheme is at least an order smaller than that of basic Monte Carlo method when $c < -15$.

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Chapter 4

Geometric Brownian Motion Case

Let two assets be defined by the following:

$$dS_{it} = \mu_i S_{it} dt + \sigma_i S_{it} dW_{it}$$

$$dS_{mt} = \mu_m S_{mt} dt + \sigma_m S_{mt} dW_{mt},$$

where W_{it}, W_{mt} are Brownian motion satisfying $W_{it} = \rho W_{mt} + \sqrt{1-\rho^2} Z_t$. W_{mt}, Z_t are independent Brownian motion.

Let $r_{it} = \ln \frac{S_{it}}{S_{i0}}$, $r_{mt} = \ln \frac{S_{mt}}{S_{m0}}$. In this chapter, we are going to estimate $E[r_{iT} 1(r_{mT} < c)]$.

4.1 Change measure

It can be derived that

$$\ln \frac{S_{iT}}{S_{i0}} = (\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i W_{iT}$$
$$\ln \frac{S_{mT}}{S_{m0}} = (\mu_m - \frac{\sigma_m^2}{2})T + \sigma_m W_{mT}.$$

It means that the event $\{r_{mT} < c\}$ is equivalent to $\{W_{mT} < \frac{c - (\mu_m - \frac{\sigma_m^2}{2})T}{\sigma_m}\}$. Thus, we can again use a similar method as previous chapter. That is, we define

$$\frac{dP}{d\tilde{P}} = e^{-hW_{mT} + \frac{1}{2}h^2 T}$$
$$\tilde{W}_{mT} = W_{mT} - hT,$$

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where h is a constant and $\tilde{W}_{mT}$ is a Brownian motion under $\tilde{P}$. Since we hope that $\tilde{E}[r_{mT}] = c$, we can get

$$\begin{aligned}
\tilde{E}[\ln \frac{S_{mT}}{S_{m0}}] &= (\mu_m - \frac{\sigma_m^2}{2})T + \sigma_m \tilde{E}[W_{mT}] \\
&= (\mu_m - \frac{\sigma_m^2}{2})T + \sigma_m \tilde{E}[\tilde{W}_{mT} + hT] \\
&= (\mu_m - \frac{\sigma_m^2}{2})T + \sigma_m hT \\
&= (\mu_m - \frac{\sigma_m^2}{2} + \sigma_m h) \\
&= c,
\end{aligned}$$

which implies that $h = \frac{c-(\mu_m - \frac{\sigma_m^2}{2})T}{\sigma_m T}$.

4.2 Asymptotic variance analysis

Let $\tilde{c} = \frac{c-(\mu_m - \frac{\sigma_m^2}{2})T}{\sigma_m}$. Then the first moment is

$$\begin{aligned}
M_1 &= E[r_{iT} 1(r_{mT} < c)] \\
&= E[((\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i W_{iT}) 1((\mu_m - \frac{\sigma_m^2}{2})T + \sigma_m W_{mT} < c)] \\
&= (\mu_i - \frac{\sigma_i^2}{2})T \cdot E[1(W_{mT} < \tilde{c})] + \sigma_i \cdot E[W_{iT} 1(W_{mT} < \tilde{c})] \\
&= (\mu_i - \frac{\sigma_i^2}{2})T \cdot \Phi(\frac{\tilde{c}}{\sqrt{T}}) - \frac{\rho \sigma_i \sqrt{T}}{\sqrt{2\pi}} e^{-\frac{\tilde{c}^2}{2T}}.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\lim_{c \to -\infty} \frac{1}{c^2} \ln |M_1| \\
&= \lim_{c \to -\infty} \frac{1}{c^2} \ln |e^{-\frac{\tilde{c}^2}{2T}} (\frac{(\mu_i - \frac{\sigma_i^2}{2})T\sqrt{T}}{\sqrt{2\pi}(-\tilde{c})} - \frac{\rho \sigma_i \sqrt{T}}{\sqrt{2\pi}})| \\
&= \lim_{c \to -\infty} \frac{1}{c^2} (-\frac{[c-(\mu_m - \frac{\sigma_m^2}{2})T]^2}{2T\sigma_m^2}) \\
&= -\frac{1}{2T\sigma_m^2}.
\end{aligned}$$

Before we consider the second moment, we define a new measure $\hat{P}$ by

$$\begin{aligned}
\frac{dP}{d\hat{P}} &= e^{hW_{mT} + \frac{1}{2}h^2 T} \\
\hat{W}_{mT} &= W_{mT} + hT,
\end{aligned}$$

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where $\hat{W}_{mT}$ is a Brownian motion under $\hat{P}$. Then we can calculate the second moment as follows:

$$
\begin{aligned}
M_2 &= \tilde{E}[r_{iT}^2 1(r_{mT} < c)(\frac{dP}{d\tilde{P}})^2] \\
&= \tilde{E}[r_{iT}^2 1(r_{mT} < c)e^{h^2T - 2hW_{mT}}] \\
&= e^{h^2T}\hat{E}[((\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i W_{iT})^2 1(W_{mT} < \tilde{c})] \\
&= e^{h^2T}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2 \hat{E}[1(W_{mT} < \tilde{c})] \\
&\quad + 2(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T\hat{E}[W_{iT}1(W_{mT} < \tilde{c})] \\
&\quad + \sigma_i^2 \hat{E}[W_{iT}^2 1(W_{mT} < \tilde{c})]\} \\
&= e^{h^2T}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2 \hat{E}[1(W_{mT} < \tilde{c})] \\
&\quad + 2(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T\rho\hat{E}[W_{mT}1(W_{mT} < \tilde{c})] \\
&\quad + \sigma_i^2(\rho^2\hat{E}[W_{mT}^2 1(W_{mT} < \tilde{c})] + (1-\rho^2)\hat{E}[Z_T^2 1(W_{mT} < \tilde{c})])\} \\
&= e^{h^2T}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2 \hat{E}[1(\hat{W}_{mT} < \tilde{c} + hT)] \\
&\quad + 2\rho(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T\hat{E}[(\hat{W}_{mT} - hT)1(\hat{W}_{mT} < \tilde{c} + hT)] \\
&\quad + \rho^2\sigma_i^2\hat{E}[(\hat{W}_{mT} - hT)^2 1(\hat{W}_{mT} < \tilde{c} + hT)] + (1-\rho^2)\sigma_i^2 T\Phi(\frac{\tilde{c} + hT}{\sqrt{T}})\} \\
&= e^{h^2T}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2 \Phi(\frac{\tilde{c} + hT}{\sqrt{T}}) \\
&\quad + 2\rho(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T(-\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{(\tilde{c}+hT)^2}{2T}} - hT\Phi(\frac{\tilde{c} + hT}{\sqrt{T}})) \\
&\quad + \rho^2\sigma_i^2\hat{E}[(\hat{W}_{mT}^2 - 2hT\hat{W}_{mT} + h^2T^2)1(\hat{W}_{mT} < \tilde{c} + hT)] \\
&\quad + (1-\rho^2)\sigma_i^2 T\Phi(\frac{\tilde{c} + hT}{\sqrt{T}})\} \\
&= e^{\frac{\tilde{c}^2}{T}}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2\Phi(\frac{2\tilde{c}}{\sqrt{T}}) + 2\rho(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T(-\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{2\tilde{c}^2}{T}} - \tilde{c}\Phi(\frac{2\tilde{c}}{\sqrt{T}})) \\
&\quad + \rho^2\sigma_i^2(-\frac{2\tilde{c}\sqrt{T}}{\sqrt{2\pi}}e^{\frac{2\tilde{c}^2}{T}} + T\Phi(\frac{2\tilde{c}}{\sqrt{T}}) + \frac{2\tilde{c}\sqrt{T}}{\sqrt{2\pi}}e^{\frac{2\tilde{c}^2}{T}} + \tilde{c}^2\Phi(\frac{2\tilde{c}}{\sqrt{T}})) \\
&\quad + (1-\rho^2)\sigma_i^2 T\Phi(\frac{2\tilde{c}}{\sqrt{T}})\} \\
&= e^{\frac{\tilde{c}^2}{T}}\{(\mu_i - \frac{\sigma_i^2}{2})^2 T^2\Phi(\frac{2\tilde{c}}{\sqrt{T}}) + 2\rho(\mu_i - \frac{\sigma_i^2}{2})\sigma_i T(-\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{2\tilde{c}^2}{T}} - \tilde{c}\Phi(\frac{2\tilde{c}}{\sqrt{T}})) \\
&\quad + \sigma_i^2\Phi(\frac{2\tilde{c}}{\sqrt{T}})(\rho^2\tilde{c}^2 + T)\}.
\end{aligned}
$$

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It implies that

$$
\begin{aligned}
&\lim_{c\to-\infty}\frac{1}{c^2}\ln|M_2| \\
&=\lim_{c\to-\infty}\frac{1}{c^2}\ln|e^{-\frac{\tilde{c}^2}{T}}[\frac{(\mu_i-\frac{\sigma_i^2}{2})^2T^2\sqrt{T}}{\sqrt{2\pi}(-2\tilde{c})} \\
&\quad+2\rho(\mu_i-\frac{\sigma_i^2}{2})\sigma_iT(-\frac{\sqrt{T}}{\sqrt{2\pi}}-\frac{\tilde{c}\sqrt{T}}{\sqrt{2\pi}(-2\tilde{c})})+\frac{\sigma_i^2\sqrt{T}}{\sqrt{2\pi}(-2\tilde{c})}(\rho^2\tilde{c}^2+T)]| \\
&=\lim_{c\to-\infty}\frac{1}{c^2}(-\frac{[c-(\mu_m-\frac{\sigma_m^2}{2})T]^2}{T\sigma_m^2}) \\
&=-\frac{1}{T\sigma_m^2}.
\end{aligned}
$$

Thus, we have the following theorem.

Theorem 3. *Suppose $W_{it}=\rho W_{mt}+\sqrt{1-\rho^2}Z_t$, where W_{mt}, Z_t are Brownian motion, and ρ is correlation between W_{it} and W_{mt}. Define two assets*

$$dS_{it}=\mu_iS_{it}dt+\sigma_iS_{it}dW_{it}$$

$$dS_{mt}=\mu_mS_{mt}dt+\sigma_mS_{mt}dW_{mt}.$$

Also, let

$$
\begin{aligned}
M_1&=E[r_{iT}1(r_{MT}<c)] \\
&=\tilde{E}[r_{iT}1(r_{MT}<c)\frac{dP}{d\tilde{P}}] \\
M_2&=\tilde{E}[r_{iT}^21(r_{MT}<c)(\frac{dP}{d\tilde{P}})^2],
\end{aligned}
$$

where

$$r_{it}=\ln\frac{S_{it}}{S_{i0}},r_{mt}=\ln\frac{S_{mt}}{S_{m0}}$$

Then,

$$\lim_{c\to-\infty}\frac{1}{c^2}\ln|M_2|=2\lim_{c\to-\infty}\frac{1}{c^2}\ln|M_1|.$$

In summary, we can get a more general theorem, shown as below.

Theorem 4. *Let X,Y be random variables such that*

$$\begin{pmatrix}X\\Y\end{pmatrix}\sim N\left[\begin{pmatrix}m_1\\m_2\end{pmatrix},\ \begin{pmatrix}\sigma_1^2&\rho\sigma_1\sigma_2\\\rho\sigma_1\sigma_2&\sigma_2^2\end{pmatrix}\right]$$

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Also, let

$$M_1 = E[Y1(X < c)]$$
$$M_2 = \tilde{E}[Y^2 1(X < c)(\frac{dP}{d\tilde{P}})^2],$$

where

$$\frac{dP}{d\tilde{P}} = \frac{e^{-\frac{(x-m_1)^2}{2\sigma_1^2}}}{e^{-\frac{(x-c)^2}{2\sigma_1^2}}}.$$

Then,

$$\lim_{c\to-\infty} \frac{1}{c^2} \ln |M_2| = 2 \lim_{c\to-\infty} \frac{1}{c^2} \ln |M_1|.$$

Proof. Define a random variable Z such that $Y = \rho X + \sqrt{1-\rho^2} Z$. Then

$$Z \sim N(\frac{m_2 - \rho m_1}{\sqrt{1-\rho^2}}, \frac{\sigma_2^2 - \rho^2\sigma_1^2}{1-\rho^2}).$$

Thus,

$$\begin{aligned}
M_1 &= \rho E[X1(X < c)] + \sqrt{1-\rho^2} E[Z1(X < c)] \\
&= \rho(-\frac{\sigma_1}{\sqrt{2\pi}} e^{-\frac{(c-m_1)^2}{2\sigma_1^2}} + m_1 \Phi(\frac{c-m_1}{\sigma_1})) + \sqrt{1-\rho^2} \frac{m_2 - \rho m_1}{\sqrt{1-\rho^2}} \Phi(\frac{c-m_1}{\sigma_1}) \\
&= -\rho \frac{\sigma_1}{\sqrt{2\pi}} e^{-\frac{(c-m_1)^2}{2\sigma_1^2}} + m_2 \Phi(\frac{c-m_1}{\sigma_1}).
\end{aligned}$$

Since

$$\Phi(\frac{c-m_1}{\sigma_1}) \approx \frac{-\sigma_1}{\sqrt{2\pi}(c-m_1)} e^{-\frac{(c-m_1)^2}{2\sigma_1^2}} \text{ as } c \to -\infty,$$

we can derive that

$$\lim_{c\to-\infty} \frac{1}{c^2} \ln |M_1| = -\frac{1}{2\sigma_1^2}.$$

On the other hand,

$$\begin{aligned}
M_2 &= \tilde{E}[(\rho^2 X^2 + 2\rho\sqrt{1-\rho^2} XZ + (1-\rho^2) Z^2) 1(X < c)(\frac{dP}{d\tilde{P}})^2] \\
&= \rho^2 e^{\frac{(c-m_1)^2}{\sigma_1^2}} [-\frac{2\sigma_1 m_1}{\sqrt{2\pi}} e^{-\frac{2(c-m_1)^2}{\sigma_1^2}} + (\sigma_1^2 + (c-2m_1)^2) \Phi(\frac{2c-2m_1}{\sigma_1})] \\
&\quad + 2\rho(m_2 - \rho m_1) e^{\frac{(c-m_1)^2}{\sigma_1^2}} (-\frac{\sigma_1}{\sqrt{2\pi}} e^{-\frac{2(c-m_1)^2}{\sigma_1^2}} - (c-2m_1)\Phi(\frac{2c-2m_1}{\sigma_1})) \\
&\quad + [\sigma_2^2 - \rho^2\sigma_1^2 + (m_2 - \rho m_1)^2] e^{\frac{(c-m_1)^2}{\sigma_1^2}} \Phi(\frac{2c-2m_1}{\sigma_1}).
\end{aligned}$$

Since

$$\Phi(\frac{2c-2m_1}{\sigma_1}) \approx \frac{-\sigma_1}{\sqrt{2\pi} 2(c-m_1)} e^{-\frac{2(c-m_1)^2}{\sigma_1^2}} \text{ as } c \to -\infty,$$

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we can derive that

$$\lim_{c \to -\infty} \frac{1}{c^2} \ln |M_2| = -\frac{1}{\sigma_1^2}.$$

Therefore,

$$\lim_{c \to -\infty} \frac{1}{c^2} \ln |M_2| = 2 \lim_{c \to -\infty} \frac{1}{c^2} \ln |M_1|.$$

□

Remark. Theorem 1, Theorem 2, Theorem 3 *are the special cases of* Theorem 4.

4.3 Numerical results

We show the numerical results again. The results are shown below.

Table 4.1: Geometric Brownian motion case

c	exact	N	BMC	S.E.	IS	S.E.
		10000	-7.8055e-04	2.0346e-04	-1.1732e-03	2.1354e-05
-1	-1.1527e-03	40000	-0.0010	1.1434e-04	-1.1581e-03	1.0481e-05
		160000	-0.0011	5.9253e-05	1.1482e-03	5.2372e-06
		10000	0	-	-8.0165e-06	1.7730e-07
-1.5	-8.2625e-06	40000	-1.4882e-05	1.4882e-05	-8.1879e-06	9.0629e-08
		160000	-8.4565e-06	5.9916e-06	-8.2925e-06	4.5457e-08
		10000	0	-	-7.7696e-09	1.9611e-10
-2	-7.9718e-09	40000	0	-	-7.8575e-09	1.0036e-10
		160000	0	-	-8.0090e-09	5.0809e-11

$T = 0.5, \mu_i = 0.08, \sigma_i = 0.3, S_{i0} = 10, \mu_m = 0.1, \sigma_m = 0.5, S_{m0} = 100, \rho = 0.7.$

Here we can see that when $c = -2$, the basic Monte Carlo method cannot sample a rare event even we simulate 160000 times, while the importance sampling method we provide gives an accurate estimation. In addition, the sample error of importance sampling scheme is at least an order smaller than that of basic Monte Carlo method.

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Chapter 5

Applications to Measuring Systemic Risk

After the 2008 financial crisis, measuring systemic risk has become a crucial issue for the financial stability. Governments are trying to figure out why the regulation failed, how much capital is required and how to address the next financial crisis. Bisias et al. [10] provide a survey on the systemic risk measures and conceptual frameworks that have been developed in the past few years. There are some measures that are widely adopted, such as SRISK and ΔCoVaR. Adrian and Brunnermeier (2011) allow for tail dependence and use a quantile regression approach to estimate the ΔCoVaR. Brownlees and Engle (2012) model time-varying linear dependencies and use a multivariate GARCH-DCC model to compute the SRISK.

In this chapter, we will estimate the systemic risk measurement SRISK in the framework of Stochastic Volatility Model(SV model). SRISK is defined as the expected capital shortfall of a financial entity conditional on a prolonged market decline. SRISK can be considered to be a function of several variables. One of these variables is Long Run Marginal Expected Shortfall(LRMES). Our goal is to estimate LRMES using the Monte Carlo method and compare it with the importance sampling method.

5.1 Stochastic Volatility model

In order to estimate LRMES, we need an appropriate model to compute the expected market return and firm's return. Here we use the Heston model to simulate stochastic volatility

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and assume that stochastic correlation follows Jacobi process.

The overall model is constructed below:

$$\begin{cases} d\ln S_{mt} &= (\mu_m - \frac{V_{mt}}{2})dt + \sqrt{V_{mt}}dW_{mt} \\ dV_{mt} &= \kappa_m(\theta_m - V_{mt})dt + \xi_m\sqrt{V_{mt}}dZ_{mt} \\ d\ln S_{it} &= (\mu_i - \frac{V_{it}}{2})dt + \sqrt{V_{it}}dW_{it} \\ dV_{it} &= \kappa_i(\theta_i - V_{it})dt + \xi_i\sqrt{V_{it}}dZ_{it} \\ d\langle W_m, Z_m\rangle_t &= \rho_m dt \\ d\langle W_i, Z_i\rangle_t &= \rho_i dt \\ d\langle W_m, W_i\rangle_t &= \rho_{it} dt \\ d\rho_{it} &= \alpha_i(m_i - \rho_{it})dt + \beta_i\sqrt{1-\rho_{it}^2}dX_{it} \end{cases}$$

where S_{mt}, S_{it} denote the market index and stock price of the i-th firm, respectively, and V_{mt}, V_{it} are the corresponding stochastic volatility. $W_{mt}, W_{it}, Z_m, Z_i, X_{it}$ are the standard Brownian motions, where $\rho_m, \rho_i, \rho_{it}$ are correlations between each Brownian motion. κ_m, κ_i are the mean reverting speed of corresponding volatility. θ_m, θ_i are the long-run level of the volatility. ξ_m, ξ_i are the volatility of volatility. α_i represents the mean recovery rate. β_i represents the volatility.

Define the i-th firm's LRMES from time t to time $t+h$ by

$$\begin{aligned} \text{LRMES}_{i,t:t+h} &= -E[r_{i,t:t+h}(t+h)|\text{Crisis}_{t:t+h}] \\ &= -\frac{E[r_{i,t:t+h}(t+h)1(r_{m,t:t+h}(t+h) < c)]}{E[1(r_{m,t:t+h}(t+h) < c)]}, \end{aligned} \tag{5.1}$$

where $r_{m,t:t+h}$ and $r_{i,t:t+h}$ are the returns of index of market and the i-th firm over the period $[t, t+h]$.

Sometimes the "Crisis" is a rare event, depending on the value of c. If it is a rare event, then we will introduce the importance sampling method to reduce sample variance. Otherwise, the basic Monte Carlo method is enough.

The numerical results are presented in the following table.

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c	BMC	S.E.	LRMES	IS	S.E.	LRMES
-1	-0.0050	1.389e-04	0.3372	-0.0051	6.462e-05	0.3350
-2	-2.413e-05	8.717e-06	0.4897	-1.133e-05	2.067e-06	0.4627
-3	-	-	-	-5.405e-10	2.876e-10	0.6757

Table 5.1: Results of basic Monte Carlo simulation and importance sampling scheme with $N = 160000, T = 0.5, dt = 0.002, \mu_m = 0.1, \mu_i = 0.08, \kappa_m = 5, \kappa_i = 3, \xi_m = 2, \xi_i = 1, \theta_m = 0.5, \theta_i = 0.3, \alpha_i = 5, \beta_i = 1, m = 0.5, \rho_i = 0.5, \rho_m = 0.5$.

From the numerical results, we can see that the basic Monte Carlo method doesn't work well when c is very small. On the other hand, the standard error was reduced when we use the importance sampling method.

5.2 First Passage Time Case

In this section, we introduce another definition of LRMES. It is called the first passage time problem, or the hitting time problem. We will also use the Heston model, but here we define LRMES as follows:

$$\begin{aligned} \text{LRMES}_{i,0:T} &= -E[r_{i,0:T}(T)|\text{Crisis}_{0:T}] \\ &= -\frac{E[r_{i,0:T}(T)1(\text{inf}_u r_{m,0:T}(u) < c)]}{E[1(\text{inf}_u r_{m,0:T}(u) < c)]}. \end{aligned} \tag{5.2}$$

The only difference between (5.1) and (5.2) is the "Crisis" event. Here we define the "Crisis" to be the minimum market return below c during the time $[0, T]$. Before we show the numerical results, we need to show how to compute the denominator and the numerator of LRMES under the constant volatility model. More specifically, we need to derive the density function.

Let $\hat{W}_{mt} = \alpha t + W_{mt}$, where W_{mt} is a standard Brownian motion. Define $\hat{m}(T) = \min_{0 \le t \le T} \hat{W}_{mt}$, then we can derive the joint distribution of $\hat{m}(T)$ and $\hat{W}_{mt}$. Then derivation is similar to [6]. Let $\hat{Z}(t) = e^{-\alpha W_{mt} - \frac{1}{2}\alpha^2 t} = e^{-\alpha \hat{W}_{mt} + \frac{1}{2}\alpha^2 t}$. By Girsanov's Theorem, $\hat{W}_{mt}$ is a Brownian motion under the measure $\hat{P}$. By using a similar derivation in [6], we can derive the joint density of $(\hat{m}(T), \hat{W}_{mT})$ under $\hat{P}$ is

$$\hat{f}_{\hat{m}(T),\hat{W}_{mT}}(x, y) = -\frac{2(2x - y)}{T\sqrt{2\pi T}} exp(-\frac{(2x - y)^2}{2T}), x < 0, y \ge x.$$

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Hence,

$$\begin{aligned} P(\hat{m}(T) \le m, \hat{W}_{mT} \le w) &= E[1(\hat{m}(T) \le m, \hat{W}_{mT} \le w)] \\ &= \hat{E}[\frac{1}{\hat{Z}(T)} 1(\hat{m}(T) \le m, \hat{W}_{mT} \le w)] \\ &= \hat{E}[e^{\alpha \hat{W}_{mT} - \frac{1}{2}\alpha^2 T} 1(\hat{m}(T) \le m, \hat{W}_{mT} \le w)] \\ &= \int_{-\infty}^{w} \int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}_{\hat{m}(T), \hat{W}_{mT}}(x, y) dx dy. \end{aligned}$$

Therefore, under the measure P,

$$f_{\hat{m}(T), \hat{W}_{mt}}(x, y) = -\frac{2(2x - y)}{T\sqrt{2\pi T}} exp(\alpha y - \frac{1}{2}\alpha^2 T - \frac{(2x-y)^2}{2T}), x < 0, y \ge x.$$

Let $\alpha = \frac{1}{\sigma_m}(\mu_m - \frac{1}{2}\sigma_m^2)$ and $\hat{c} = \frac{c}{\sigma_m}$.

Then we can see that

$$\begin{aligned} S_{mt} &= S_{m0} e^{\sigma_m W_{mt} + (\mu_m - \frac{1}{2}\sigma_m^2)t} \\ &= S_{m0} e^{\sigma_m \hat{W}_{mt}}. \end{aligned}$$

Hence, we can get the denominator

$$\begin{aligned} & E[1(\text{inf}_u r_{m,0:T}(u) < c)] \\ = & E[1(\text{inf}_u \ln \frac{S_{mu}}{S_{m0}} < c)] \\ = & E[1(\text{inf}_u \hat{W}_{mu} < \hat{c})] \\ = & \int_{-\infty}^{\hat{c}} \int_{x}^{\infty} f_{\hat{m}(T), \hat{W}_{mt}}(x, y) dy dx \\ = & \int_{-\infty}^{\hat{c}} \int_{-\infty}^{y} -\frac{2(2x-y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{(2x-y)^2}{2T}} dx dy \\ & + \int_{\hat{c}}^{\infty} \int_{-\infty}^{\hat{c}} -\frac{2(2x-y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{(2x-y)^2}{2T}} dx dy \\ = & e^{2\alpha \hat{c}} \Phi(\frac{\hat{c} + \alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c} - \alpha T}{\sqrt{T}}) \end{aligned}$$

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and the numerator

$$
\begin{aligned}
& E[r_{iT}1(\inf_u r_{m,0:T}(u) < c)] \\
&= E[((\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i W_{iT})1(\inf_u \hat{W}_{mu} < \hat{c})] \\
&= (\mu_i - \frac{\sigma_i^2}{2})T[e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}})] \\
&\quad + \sigma_i\rho E[(\hat{W}_{mT} - \alpha T)1(\inf_u \hat{W}_{mu} < \hat{c})] \\
&= (\mu_i - \frac{\sigma_i^2}{2})T[e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}})] \\
&\quad + \sigma_i\rho[\int_{-\infty}^{\hat{c}}\int_x^{\infty} \frac{2y(2x-y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{(2x-y)^2}{2T}} dy dx \\
&\quad - \alpha T(e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}}))] \\
&= (\mu_i - \frac{\sigma_i^2}{2})T[e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}})] \\
&\quad + \sigma_i\rho[\alpha T\Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}}) + (2\hat{c}+\alpha T)e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) \\
&\quad - \alpha T(e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}}))] \\
&= (\mu_i - \frac{\sigma_i^2}{2})(e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}) + \Phi(\frac{\hat{c}-\alpha T}{\sqrt{T}})) + 2\sigma_i\rho\hat{c}e^{2\alpha\hat{c}}\Phi(\frac{\hat{c}+\alpha T}{\sqrt{T}}).
\end{aligned}
$$

Here the ”Crisis” can also be a rare event, depending on what value c is. Like the last section, we introduce the basic Monte Carlo method to estimate LRMES and compare it with the importance sampling method.

The numerical results are presented in the following table.

c	BMC	S.E.	LRMES	IS	S.E.	LRMES
-1	-2.053e-03	1.519e-04	0.3941	-1.874e-03	2.170e-05	0.4083
-1.5	-1.453e-05	-1.453e-05	0.5812	-1.352e-05	2.557e-07	0.6134
-2	-	-	-	-1.288e-08	3.661e-10	0.8259

Table 5.2: Results of basic Monte Carlo simulation and importance sampling scheme with $N = 40000, T = 0.5, dt = 0.002, \mu_i = 0.08, \mu_m = 0.1, \sigma_i = 0.3, \sigma_m = 0.5, \rho = 0.7$.

As we can see, the basic Monte Carlo method doesn’t work well when c is very small. On the other hand, the standard error was reduced by using the importance sampling method.

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Part 2

Large Deviation Theory applied to Portfolio Optimization

In this part, we are going to apply large deviation theory to the finite-horizon investment optimization.

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Chapter 6

Optimal Finite-Horizon Investment

There have been a lot of research about large deviations approach to optimal long term investment. For example, [8] considered a Bachelies model for the stock price: $S_t = \mu t + \sigma W_t$, where W_t is a Brownian motion. Suppose that an investor trades a number α of shares in stock of price S, and keep it until time T. The wealth at time T is then $X_t^\alpha = \alpha S_T$. The average wealth is $\bar{X}_T^\alpha = \frac{X_T^\alpha}{T}$. The asymptotic version of the outperforming benchmark criterion is then formulated as:

$$\sup_{\alpha \in \mathbb{R}} \lim_{T \to \infty} \frac{1}{T} \ln \mathbb{P}[\bar{X}_T^\alpha \geq x] = - \inf_{\alpha \in \mathbb{R}} I(x, \alpha),$$

where

$$\begin{aligned} I(x, \alpha) &= \sup_{\theta \in \mathbb{R}} [\theta x - \Gamma(\theta, \alpha)] \\ \Gamma(\theta, \alpha) &= \limsup_{T \to \infty} \frac{1}{T} \ln \mathbb{E}[e^{\theta X_T^\alpha}]. \end{aligned}$$

In [8], it is derived that the solution is given by $\alpha^* = x/\mu$, which means that the associated expected wealth $\mathbb{E}[\bar{X}_T^{\alpha^*}]$ is equal to the target x.

In this chapter, we consider a large deviations approach to optimal finite-horizon investment. We first describe some very important large deviations results that will be used in this chapter.

Definition 1. Large-deviations principle. [7] A sequence Y_1, Y_2,... obeys the large-deviations principle (LDP) with rate function $I(\cdot)$ if

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a. For any closed set F,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\frac{1}{n}\sum_{i=1}^{n} Y_i \in F) \le -\inf_{a\in F} I(a);$$

b. For any open set G,

$$\liminf_{n\to\infty} \frac{1}{n} \log P(\frac{1}{n}\sum_{i=1}^{n} Y_i \in G) \ge -\inf_{a\in G} I(a).$$

Freidlin-Wentzell Theorem. [1, 2] *Let W_t be a standard Brownian motion. Then the solution of*

$$dX_t = a(X_t)dt + \sqrt{\epsilon} b(X_t)dW_t$$

satisfies LDP with rate function

$$I(f) = \begin{cases} \frac{1}{2}\int_0^T (\frac{\dot{f}-a}{b})^2 dt, & f \in C_{1,x}[0,T] \text{ and } \int (\dot{f})^2 dt < \infty \\ \infty, & \text{otherwise.} \end{cases}$$

6.1 Constant Investment Strategy

Let the bond price S^0 and the stock price S satisfy

$$\begin{cases} dS_t^0 &= rS_t^0 dt \\ dS_t &= \mu S_t dt + \sigma S_t dW_t. \end{cases}$$

Define the wealth process $V_t^\alpha = \frac{\alpha V_t^\alpha}{S_t} S_t + \frac{(1-\alpha)V_t^\alpha}{S_t^0} S_t^0$, where α is the proportion invested in the stock. Then we can get

$$\begin{aligned} dV_t^\alpha &= \frac{\alpha V_t^\alpha}{S_t} dS_t + \frac{(1-\alpha)V_t^\alpha}{S_t^0} dS_t^0 \\ &= \alpha V_t^\alpha(\mu dt + \sigma dW_t) + (1-\alpha)V_t^\alpha r dt \\ &= V_t^\alpha[((\mu - r)\alpha + r)dt + \sigma\alpha dW_t]. \end{aligned}$$

By Itô's formula, we can derive

$$d\ln V_t^\alpha = (r + (\mu - r)\alpha - \frac{\sigma^2}{2}\alpha^2)dt + \sigma\alpha dW_t.$$

Integrate the above equation from 0 to T, then we can get

$$V_T^\alpha = V_0^\alpha e^{(r+(\mu-r)\alpha-\frac{\sigma^2}{2}\alpha^2)T+\sigma\alpha W_T}.$$

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Define $\tau^\alpha = \inf_{0\le t\le T}\{t : V_t^\alpha \ge x\}$, where $x > 0$. We are interested in the problem $\sup_\alpha P(\tau^\alpha < T)$. Let $X_t^\alpha = \int_0^t a_s ds + \int_0^t b_s dW_s$, where $a_s = r + (\mu - r)\alpha - \frac{\sigma^2}{2}\alpha^2$ and $b_s = \sigma\alpha$. Then

$$\begin{aligned}\{\tau^\alpha < T\} &= \{\sup_{0\le t\le T} V_t^\alpha \ge x\}\\ &= \{\sup_{0\le t\le T} X_t^\alpha \ge \ln\frac{x}{V_0^\alpha}\}\\ &= \{\inf_{0\le t\le T} -X_t^\alpha \le -\ln\frac{x}{V_0^\alpha}\}\\ &= \{\inf_{0\le t\le T} \tilde{X}_t^\alpha \le \tilde{x}\},\end{aligned}$$

where we define $\tilde{X}_t^\alpha = -X_t^\alpha$, $\tilde{x} = -\ln\frac{x}{V_0^\alpha}$, $\tilde{b}_s = b_s$.

To apply the theory of large deviations, we introduce a scaling factor $\sqrt{\epsilon}$ and reformulate $\tilde{X}_t^\alpha$ as $d\tilde{X}_t^\alpha = \tilde{a}_t dt + \sqrt{\epsilon}\tilde{b}_t dW_t$, where $\tilde{a}_t = -(r + (\mu - r)\alpha - \frac{\epsilon\sigma^2}{2}\alpha^2)$.

Define $p^\alpha = P(\tau^\alpha < T)$ and $\hat{a}_t = -(r + (\mu - r)\alpha)$. Then according to Freidlin-Wentzell theorem,

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon\log p^{\alpha,\epsilon}\\ &= -\inf_f I(f)\\ &= -\inf_f \frac{1}{2}\int_0^T (\frac{\dot{f} - \hat{a}_t}{\tilde{b}_t})^2 dt\\ &:= -I(f^*)\end{aligned}$$

Let $L(t, f(t), \dot{f}(t)) = (\frac{\dot{f}(t) - \hat{a}_t}{\tilde{b}_t})^2$. By Euler-Lagrange equation, we have

$$\frac{\partial L}{\partial f} - \frac{d}{dt}\frac{\partial L}{\partial \dot{f}} = 0.$$

After some calculation, we can get

$$\dot{f}(t) = \hat{a}_t + \frac{c_1}{2}\tilde{b}_t^2$$

and so

$$f(t) = c_2 + \int_0^t \hat{a}_s ds + \frac{c_1}{2}\int_0^t \tilde{b}_s^2 ds,$$

where c_1 and c_2 are constant.

Note that $f^*(0) = 0$ and $f^*(T) = \tilde{x}$. Thus, we can derive that $c_1 = \frac{2}{\tilde{b}^2}(\frac{\tilde{x}}{T} - \hat{a})$ and $c_2 = 0$,

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where $\hat{a} = \hat{a}_s$ and $\tilde{b} = \tilde{b}_s$. Therefore,

$$I(f^*) = \frac{T}{2\tilde{b}^2}(\frac{\tilde{x}}{T} - \hat{a})^2. \tag{6.1}$$

So the problem can be reduced to find α such that

$$\begin{aligned}\sup_\alpha \lim_{\epsilon\to 0} \epsilon \log p^{\alpha,\epsilon} &= \sup_\alpha(-I(f^*(\alpha))) \\ &= -\inf_\alpha I(f^*(\alpha))\end{aligned}$$

Note that it is equivalent to find $\alpha \in [0,1]$ such that the function $h(\alpha) = \frac{T}{2\tilde{b}^2}(\frac{\tilde{x}}{T} - \hat{a})^2$ is minimum.

Remark. *If we use the closed-form to consider the above problem, and we let* $\ln \frac{x}{V_0^\alpha} > (r + (\mu - r)\alpha)T$*, then we can get the same answer.*

Proof. By [6], we can derive that

$$p^\alpha = 1 - \Phi(\frac{c - kT}{\sqrt{T}}) + e^{2kc}\Phi(\frac{-c - kT}{\sqrt{T}}),$$

where $c = \frac{1}{\sigma\alpha}\ln\frac{x}{V_0^\alpha}$, $k = \frac{r+(\mu-r)\alpha - \frac{\sigma^2}{2}\alpha^2}{\sigma\alpha}$.
Again, we introduce a scaling factor $\sqrt{\epsilon}$ and rewrite $c = \frac{1}{\sqrt{\epsilon}\sigma\alpha}\ln\frac{x}{V_0^\alpha}$ and $k = \frac{r+(\mu-r)\alpha - \frac{\epsilon\sigma^2}{2}\alpha^2}{\sqrt{\epsilon}\sigma\alpha}$.
Then we will have

$$\begin{aligned}p^{\alpha,\epsilon} &\approx \frac{\sqrt{T}}{\sqrt{2\pi}(c-kT)}e^{-\frac{(c-kT)^2}{2T}} + e^{2kc}\frac{\sqrt{T}}{\sqrt{2\pi}(c+kT)}e^{-\frac{(c+kT)^2}{2T}} \\ &= \frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{(c-kT)^2}{2T}}(\frac{1}{c-kT} + \frac{1}{c+kT})^2.\end{aligned}$$

We use the same notation as above. That is, $\tilde{a} = -(r + (\mu - r)\alpha - \frac{\epsilon\sigma^2}{2}\alpha^2)$ $\hat{a} = -(r + (\mu - r)\alpha)$, $\tilde{b} = \sigma\alpha$, and $\tilde{x} = -\ln\frac{x}{V_0^\alpha}$.
Hence, we can derive that

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon\log p^{\alpha,\epsilon} &= \lim_{\epsilon\to 0}\epsilon[-\frac{(c-kT)^2}{2T} + \log\frac{2\sqrt{T}}{\sqrt{2\pi}} + \log(\frac{c}{c^2 - k^2T^2})] \\ &= \lim_{\epsilon\to 0}\epsilon[\frac{1}{2T}(-\frac{1}{\sqrt{\epsilon}\tilde{b}}\tilde{x} + \frac{\tilde{a}}{\sqrt{\epsilon}\tilde{b}}T)^2 + \log\frac{2\sqrt{T}}{\sqrt{2\pi}} + \log(\frac{\sqrt{\epsilon}\tilde{b}(-\tilde{x})}{\tilde{x}^2 - \tilde{a}^2T^2})] \\ &= -\frac{1}{2T}(-\frac{1}{\tilde{b}}\tilde{x} + \frac{\hat{a}}{\tilde{b}}T)^2 + 0 + 0 \\ &= -\frac{T}{2\tilde{b}^2}(\frac{\tilde{x}}{T} - \hat{a})^2,\end{aligned}$$

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which is the same as (6.1).

Thus,

$$\sup_{\alpha} \lim_{\epsilon \to 0} \epsilon \log p^{\alpha} = -\inf_{\alpha} \frac{T}{2\tilde{b}^2}(\frac{\tilde{x}}{T} - \hat{a})^2.$$

□

6.2 Deterministic Investment Strategy

In the last section, we considered α to be constant. In this section, we will let α be a function of time t, which will be denoted by α_t. In other words, the problem will be $\sup_{\alpha_t} P(\tau^{\alpha_t} < T)$. Similar to the last section, we define $d\tilde{X}_t^{\alpha} = \tilde{a}_t dt + \sqrt{\epsilon}\tilde{b}_t dW_t$, where $\tilde{a}_t = -(r + (\mu - r)\alpha_t - \frac{\epsilon\sigma^2}{2}\alpha_t^2)$, $\tilde{b}_t = \sigma\alpha_t$ and $\hat{a}_t = -(r + (\mu - r)\alpha_t)$.

According to Freidlin-Wentzell theorem, we can rewrite our problem as following:

$$\inf_{\alpha_t} \inf_{f \in B} I(f(t)) = \inf_{\alpha_t} \inf_{f \in B} \frac{1}{2} \int_0^T (\frac{\dot{f}(t) - \hat{a}_t}{\tilde{b}_t})^2 dt,$$

where the set $B = \{f \in C^1[0,T], f(0) = 0, \inf_{0<t\leq T} f(t) \leq \tilde{x} < 0\}$.

Again, by the Euler-Lagrange equation, we can derive that

$$I(f(t)) = \frac{c_1^2\sigma^2}{8} \int_0^T \alpha_t^2 dt,$$

where

$$c_1 = \frac{2(f(t) - \int_0^t \hat{a}_s ds)}{\int_0^t \tilde{b}_s^2 ds}.$$

First, we consider the problem:

$$\inf_{\inf_{0<t\leq T} f(t) \leq \tilde{x}} c_1^2.$$

Since c_1 is constant, we know that $\forall t \in (0,T]$, the value $\frac{2(f(t) - \int_0^t \hat{a}_s ds)}{\int_0^t \tilde{b}_s^2 ds}$ is fixed.

If there exists t_j such that $\int_0^{t_j} \hat{a}_s ds \leq \tilde{x}$, then we choose $f(t_j) = \int_0^{t_j} \hat{a}_s ds$. Note that in this case, $c_1 = 0$.

Thus, we will only consider the case that $\int_0^t \hat{a}_s ds > \tilde{x}$, $\forall t \in (0,T]$.

Let $\hat{x} < \tilde{x}$, then $\forall t \in (0,T]$, $\hat{x} - \int_0^t \hat{a}_s ds < \tilde{x} - \int_0^t \hat{a}_s ds$. Also, $\int_0^t \tilde{b}_s^2 ds > 0$. Hence,

$$\frac{\hat{x} - \int_0^t \hat{a}_s ds}{\int_0^t \tilde{b}_s^2 ds} < \frac{\tilde{x} - \int_0^t \hat{a}_s ds}{\int_0^t \tilde{b}_s^2 ds} < 0.$$

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So we can derive that

$$(\frac{\hat{x}-\int_0^t \hat{a}_s ds}{\int_0^t \tilde{b}_s^2 ds})^2 > (\frac{\tilde{x}-\int_0^t \hat{a}_s ds}{\int_0^t \tilde{b}_s^2 ds})^2.$$

Therefore, we have to choose $f(t_j)=\tilde{x}$, where

$$(\frac{\tilde{x}-\int_0^{t_j} \hat{a}_s ds}{\int_0^{t_j} \tilde{b}_s^2 ds})^2 = \inf_{0<t\leq T}(\frac{\tilde{x}-\int_0^t \hat{a}_s ds}{\int_0^t \tilde{b}_s^2 ds})^2 \tag{6.2}$$

so that we can ensure that c_1^2 is minimized.

That is,

$$c_1 = \frac{2(\tilde{x}-\int_0^{t_j} \hat{a}_s ds)}{\int_0^{t_j} \tilde{b}_s^2 ds}.$$

Then we can derive that

$$f^*(t) = \frac{\tilde{x}-\int_0^{t_j} \hat{a}_s ds}{\int_0^{t_j} \tilde{b}_s^2 ds} \times \int_0^t \tilde{b}_s^2 ds + \int_0^t \hat{a}_s ds.$$

We can easily check that $f^* \in C^1[0,T]$.

Therefore, the problem can be reduced to the following problem:

$$\inf_{\alpha_t} \frac{\sigma^2}{2}(\frac{\tilde{x}-\int_0^{t_j} \hat{a}_s ds}{\int_0^{t_j} \tilde{b}_s^2 ds})^2 \int_0^T \alpha_t^2 dt. \tag{6.3}$$

Due to that fact that it is very hard to solve the problem directly, we can reformulate the problem as following:

Let $0<t_1<t_2<...<t_n=T$, and $g(\alpha_{t_1},\alpha_{t_2},...,\alpha_{t_n}) = c_1^2(\alpha_{t_1},\alpha_{t_2},...,\alpha_{t_n})\sum_{i=1}^n \alpha_{t_i}^2$. It is equivalent to minimize the function g in the space $[0,1]^n$.

Remark. If $\alpha_t=\alpha$ is a constant, we can check that (6.3) is equal to (6.1).

In summary, we can give the theorem.

Theorem 5. *Let the bond price S^0 and the stock price S satisfy*

$$\begin{cases} dS_t^0 &= rS_t^0 dt \\ dS_t &= \mu S_t dt + \sigma S_t dW_t. \end{cases}$$

The wealth process is defined by $V_t^{\alpha_t} = \frac{\alpha_t V_t^{\alpha_t}}{S_t} S_t + \frac{(1-\alpha_t)V_t^{\alpha_t}}{S_t^0} S_t^0$. In addition, we define $\tau^{\alpha_t} = \inf_{0\leq t\leq T}\{t : V_t^{\alpha_t} \geq x\}$, where $x>0$. Let $\int_0^t -(r+(\mu-r)\alpha_s)ds > -\ln\frac{x}{V_0^{\alpha_t}}$,

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$\forall t \in (0, T]$. *Then,*

$$\sup_{\alpha_t} \lim_{\epsilon \to 0} \epsilon \log P(\tau^{\alpha_t, \epsilon} < T)$$
$$= -\inf_{\alpha_t} \frac{\sigma^2}{2} \left(\frac{-\ln \frac{x}{V_0^{\alpha_t}} - \int_0^{t_j} -(r + (\mu - r)\alpha_s) ds}{\int_0^{t_j} (\sigma \alpha_s)^2 ds} \right)^2 \int_0^T \alpha_t^2 dt$$

if we introduce a scaling factor $\sqrt{\epsilon}$ *such that*

$$d \ln \frac{V_t^{\alpha_t}}{V_0^{\alpha_t}} = (r + (\mu - r)\alpha_t - \frac{\epsilon \sigma^2}{2} \alpha_t^2) dt + \sqrt{\epsilon} \sigma \alpha_t dW_t,$$

and if $\exists t_j \in (0, T]$ *such that (6.2) is satisfied.*

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Chapter 7

Conclusion

In this thesis, we applied importance sampling method to estimate the expectation of a random variable with an indicator function under different models. It was shown that our method is efficient under those models. In addition, our importance sampling method can be applied to measure the systemic risk. On the other hand, we can apply the theory of large deviations to the finite-horizon investment optimization.

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