國立臺灣大學電機資訊學院資訊工程學系

碩士論文

Institute of Computer Science and Information Engineering

College of Electrical Engineering and Computer Science

National Taiwan University

Master Thesis

# 樹苗偵測演算法

# Sapling Detection

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中華民國 107 年 8 月

August 2018

doi:10.6342/NTU201803034

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## 摘要

偵測一個圖中是否含有某種特殊結構作為導出子圖是一個重要且被廣泛研究的問題，目前已經有許多結構已經被證明出其偵測屬於 NP-complete 類別。而有一些結構存在多項式時間演算法，我們研究其中關於樹苗的特例，並給出一個$O(n^3)$時間的演算法，改進了先前最佳的$O(n^4)$時間演算法。



doi:10.6342/NTU201803034

# Sapling Detection

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## Abstract

The problem of determining whether a graph contains certain structure as induced subgraph has been extensively studied. Many of them have been shown belong to the complexity NP-complete. We study a special case regarding saplings and show an algorithm that solves the problem in $O(n^3)$ time, which is an improvement over the current best $O(n^4)$ time algorithm.



doi:10.6342/NTU201803034

# 致謝

在這幾年的碩士生涯中，最感謝的當然是呂學一教授。在論文的修改上，不厭其煩地提供大大小小的指導，讓這篇論文從完全不可閱讀成為現在的這個樣子，也讓我從中學習到了很多寫證明的方式。除此之外，每次的討論也都得到了老師很多的鼓勵與肯定，這條道路上充滿荊棘與挫折，如果沒有這些鼓勵是很難支撐下去的。另外在做研究之外，教授的教學、生活態度也給了我很大的啟發。

同時也要感激我的父母，給我一個不愁吃穿的生活，父親與母親總是包辦了八成以上的家事與開銷，這樣完美的研究環境，我想是許多人夢寐以求的。最後要感謝我的朋友們(特別是鄭澈，常常跟我討論)、妹妹、尊貴的 Yui 殿下以及家裡的兩位小夥伴冰冰跟小六，總是在需要時陪伴在身邊。僅管研究之路是孤獨的，但在這之外我並不孤獨。



doi:10.6342/NTU201803034

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# 1 Introduction

Let $|S|$ denote the cardinality of set $S$. To *detect* a family of graphs in $G$ is to determine whether $G$ contains an induced subgraph that is isomorphic to a graph in the family. Let $G$ be an undirected $n$-node graph. Let $N_G(v)$ consist of the neighbors of node $v$ in $G$. The *degree* of node $v$ in $G$ is $\deg_G(v) = |N_G(v)|$. A *leaf* of $G$ is a node with degree one in $G$. A *sapling* of $G$ is an induced tree of $G$ containing all leaves of $G$. The NP-complete [26] *k-in-a-tree* problem is to detect saplings in a $k$-leaf graph.[1] Chudnovsky and Seymour [16] gave an $O(n^4)$-time algorithm for THREE-IN-A-TREE (i.e., the case with $k = 3$), which is at the kernel of several state-of-the-art graph detection algorithms. As stated in the following theorem, we reduce the time complexity of THREE-IN-A-TREE to $O(n^3)$.

**Theorem 1.1.** *It takes $O(mn)$ time to detect saplings in an $n$-node $m$-edge 3-leaf graph.*

Below are implications of our result:

- Chudnovsky and Seymour [15, 16] gave the previously only known polynomial-time algorithm, running in $O(n^{11})$ time, of detecting thetas (i.e., induced subdivisions of $K_{2,3}$ [4]) via solving THREE-IN-A-TREE on $O(n^7)$ graphs of $O(n)$ nodes. Theorem 1.1 reduces the time of detecting thetas to $O(n^{10})$.
- A *hole* is an induced simple cycle with at least four nodes. A hole is *odd* (respectively, *even*) if it consists of an odd (respectively, even) number of nodes. A graph is *Berge* if both the graph and its complement are odd-hole-free. The celebrated Strong Perfect Graph Theorem, conjectured by Berge [5] and proved by Chudnovsky, Robertson, Seymour, and Thomas [14], states that a graph is Berge if and only if it is pefect. Although the complexity of detecting odd holes remains open for a long time, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [10] showed that Berge graphs can be recognized in polynomial time. One of the two $O(n^9)$-time bottlenecks in their algorithm is an involved subroutine of detecting pyramids [10, §2]. Chunodvsky and Seymour [16] showed an $O(n^{10})$-time algorithm for detecting pyramids via solving THREE-IN-A-TREE on $O(m^3)$ graphs of $m$ edges. Theorem 1.1 implies that the time of detecting pyramids is $O(m^4 n)$.
- Even-hole-free graphs have been extensively studied in the literature (see, e.g., [2, 19, 20, 21, 24, 25, 40, 46]). Vušković [49] gave an extensive survey. Conforti, Cornuéjols, Kapoor, and Vušković [18, 22] gave the first polynomial-time algorithm for detecting even holes, running in $O(n^{40})$ time [12]. Chudnovsky, Kawarabayashi, and Seymour [12] reduced the running time to $O(n^{31})$. Chudnovsky et al. [12] also observed that the running time can be further reduced to $O(n^{15})$ as long as prisms can be detected efficiently, but Maffray and Trotignon [44] showed that detecting prisms is NP-hard. da Silva and Vušković [25] significantly improved the complexity of recognizing even-hole-free graphs to $O(n^{19})$. The best previously known algorithm, due to Chang and Lu [9], runs in $O(n^{11})$ time. Theorem 1.1 reduces the time of one of the two bottleneck subroutines [9, Lemma 2.3] to $O(n^{10})$.

**Related work** The complexity of $k$-in-a-tree problem for any fixed $k \geq 4$ is open [36]. The analogous $k$-in-a-cycle (respectively, $k$-in-a-path) problem is NP-complete for $k = 2$ (respectively, $k = 3$) [6, 32]. Derhy, Picouleau, and Trotignon [27] studied the four-in-a-tree problem

[1] The original version of the $k$-in-a-tree problem seeks an induced tree containing arbitrary $k$ given nodes, but one can verify that the version requiring that the given $k$ nodes are the leaves of the input graph is equivalent.



doi:10.6342/NTU201803034

problem on graphs having no triangle. Liu and Trotignon [43] studied the $k$-in-a-tree problem on graphs with girth at least $k$. dos Santos, da Silva, and Szwarcfiter [31] studied the $k$-in-a-tree problem on chordal graphs. Golovach, Paulusma, and van Leeuwen [36] studied the $k$-in-a-tree, $k$-in-a-cycle, and $k$-in-a-path problems on AT-free graphs [41]. Bruhn and Saito [8], Fiala, Kaminski, Lidický, and Paulusma [33], and Golovach, Paulusma, and van Leeuwen [37] studied the $k$-in-a-tree problem and $k$-in-a-path problems on claw-free graphs. Lévêque, Lin, Maffray, and Trotignon [42], van 't Hof, Kaminski, and Paulusma [48], and Chudnovsky, Seymour, and Trotignon [17] showed more applications of THREE-IN-A-TREE.

Gitler, Reyes, and Vega [35] XXXX. (They did not use the TREE-IN-A-TREE algorithm or prism/pyramid detection directly.)

Bang-Jensen, Havet, and Maia [3] XXXX.

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Detecing wheels [30] is NP-complete. The above paper also gave a survey for the complexity of various subgraph detection problems.

Trotignon and Vušković [47] even emphasized that their result is the first example that does not fall under the scope of THREE-IN-A-TREE.

Chudnovsky and Kapadia [11] gave an $O(n^{35})$-time algorithm to determine whether an $n$-node theta-free graph has a prism, although Maffray and Trotignon [44] showed that the problem of detecting prisms is NP-complete.

Fomin, Todinca, and Villanger [34] studied induced subgraphs with properties expressible in counting monadic second order logic formula.

Aboulker, Radovanovic, Trotignon, and Vušković [1] studied propeller-free graphs.

Chudnovsky and Lo [13] studied (diamond, odd-hole)-free graph.

Conforti, Cornuéjols, Liu, Vušković, and Zambelli [23] gave the only known polynomial-time algorithm to recognize odd holes in graphs with bounded clique size.

**Technical overview** Hopcroft and Tarjan [39] gave the first linear-time algorithm for computing the triconnected components of a graph. (Gutwenger and Mutzel [38] showed a minor adjustment of the algorithm.) The tree structure of the triconnected components of a biconnected graph can be represented by the linear-time obtainable data structure *SPQR-tree* of Di Battista and Tamassia [28] supporting efficient updates [29].

The rest of the paper is organized as follows: Section 2 gives the topmost level of our proofs for via reducing Theorem 1.1 via Theorem 2.1 to Lemmas 2.2 and 2.3. Sections 3 and 4 prove Lemma 2.2. Section 5 proves Lemma 2.3.



doi:10.6342/NTU201803034

# 2 Proving Theorem 1.1

This section reduces Theorem 1.1 via Theorem 2.1 to Lemmas 2.2 and 2.3.

## 2.1 Preliminaries

Let $R \setminus S$ for sets $R$ and $S$ consist of the elements of $R$ not in $S$. Let $V(G)$ (respectively, $E(G)$) consist of the nodes (respectively, edges) of graph $G$. If $e$ is an edge and $u$ and $v$ are nodes, then let $e = uv$ denote that $u$ and $v$ are the end-nodes of $e$. For subgraph $H$ of graph $G$, let $G[H]$ be the subgraph of $G$ induced by $V(H)$. For any node subset $U$ of $G$, let $G - U = G[V(G) \setminus U]$. Disjoint subgraphs $H_1$ and $H_2$ are *adjacent* in graph $G$ if there is an edge $uv$ of $G$ with $u \in V(H_1)$ and $v \in V(H_2)$. Let $U$ and $V$ be node subsets of $G$. A *$UV$-path* is either a node in $U \cap V$ or a path having one end-node in $U$ and the other end-node in $V$. A *$UV$-rung* [16] is a minimal induced $UV$-path. If $U = \{u\}$, then a $UV$-path is also called a *$uV$-path* and a *$Vu$-path*. If $U = \{u\}$ and $V = \{v\}$, then a $UV$-path is also called a *$uv$-path*. Let *$Uv$-rung*, *$uV$-rung*, and *$uv$-rung* be defined similarly.

The rest of the paper lets $G$ be the input $n$-node $m$-edge graph, which is assumed without loss of generality to be connected with exactly three leaf nodes. The subscripts in notation $N_G$ and $\deg_G$ are omitted.

## 2.2 The characterization of Chudnovsky and Seymour

Let $X \subseteq V(G)$. An *$X$-net* for $G$ is a connected multiple graph $\mathbb{H}$, each of whose *vertices* (i.e., members of $V(\mathbb{H})$) and *arcs* (i.e., members of $E(\mathbb{H})$) is a nonempty subset of $X$, that satisfies the following *Conditions N*:

*N1:* The graph obtained by adding an arc between every two leaf vertices of $\mathbb{H}$ is biconnected.
*N2:* The arcs of $\mathbb{H}$ form a nonempty disjoint partition of the nodes in $X$.
*N3:* Each leaf vertex of $\mathbb{H}$ consists of a distinct leaf node of $G$.
*N4:* For any arc $E = V_1V_2$ of $\mathbb{H}$, each node of $X$ in $E$ is contained by a $V_1V_2$-rung of $G[E]$.
*N5:* For any arc $E$ and vertex $V$ of $\mathbb{H}$, $E \cap V \neq \varnothing$ if and only if $V$ is an end-vertex of $E$ in $\mathbb{H}$.
*N6:* For any nodes $x_1$ and $x_2$ in $X$ contained by distinct arcs $E_1$ and $E_2$ of $\mathbb{H}$, $x_1x_2$ is an edge of $G$ if and only if arcs $E_1$ and $E_2$ share a common end-vertex $V$ in $\mathbb{H}$ with $\{x_1, x_2\} \subseteq V$.

A *triad* of $\mathbb{H}$ is $\Delta(V_1, V_2, V_3) = (V_1 \cap V_2) \cup (V_2 \cap V_3) \cup (V_3 \cap V_1)$ for three vertices $V_1$, $V_2$, and $V_3$ that are pairwise adjacent in $\mathbb{H}$. A nonempty $S \subseteq X$ is *$\mathbb{H}$-local* if $S$ is contained by a vertex, arc, or triad of $\mathbb{H}$. For any subsets $Y$ and $Z$ of $V(G)$, let

$$N(Y) = \bigcup_{y \in Y} N(y) \setminus Y \quad \text{and} \quad N(Y, Z) = N(Y) \cap Z.$$

A nonempty $Y \subseteq V(G) \setminus X$ is *$\mathbb{H}$-local* if $N(Y, X)$ is $\mathbb{H}$-local.

**Theorem 2.1** (Chudnovsky and Seymour [16, Theorem 3.2])**.** *$G$ is sapling-free if and only if there is an $X$-net $\mathbb{H}$ for $G$ such that any nonempty $Y \subseteq V(G) \setminus X$ with connected $G[Y]$ is $\mathbb{H}$-local.*

Our proof of Theorem 1.1 directly uses the if-direction of Theorem 2.1, for which we give a much shorter proof below to make our paper more self-contained. Chudnovsky and Seymour remarkably proved the only-if-direction of Theorem 2.1 in thirty-some pages, among which we directly adopt without proof one lemma [16, (4.1) in pages 395–402] as our Lemma 4.1. Two



doi:10.6342/NTU201803034

of their lemmas [16, (5.3) and (5.4)] are extended to be our Lemma 2.2, for which we give a complete proof in §3 and §4.

*Proof of the if-direction of Theorem 2.1.* Assume a sapling $T$ of $G$ for contradiction. We start with the first claim that if $G[Y]$ is a connected component of $T - X$, then $N(Y, X)$ is contained by an arc of $\mathbb{H}$: Since $Y$ is $\mathbb{H}$-local, any nodes $u \in N(Y, E)$ and $v \in N(Y, F)$ for distinct arcs $E$ and $F$ of $\mathbb{H}$ are contained by a common vertex or triad of $\mathbb{H}$. By Condition N6, $u$ and $v$ are adjacent in $G$, implying a cycle of $T$ in $G[Y \cup \{u, v\}]$, contradiction. The first claim is proved. Condition N6 implies the second claim that if $uv$ is an edge of $T$ with $u \in E$ and $v \in F$ for distinct arcs $E$ and $F$ of $\mathbb{H}$, then $\{u, v\}$ is contained by a common end-vertex of $E$ and $F$ in graph $\mathbb{H}$. By both claims and Conditions N2 and N3, the vertices and arcs of $\mathbb{H}$ intersecting $T$ form a subtree $\mathbb{T}$ of $\mathbb{H}$ with three leaf vertices. Thus, $T$ intersects a vertex of $\mathbb{T}$ and three of its incident arcs in $\mathbb{T}$. Condition N6 implies a triangle in $T$, contradiction. □

### 2.3 Sprouts, abodes, and steady nets

For any nonempty node subsets $S$, $U$, and $V$ of graph $H$, an $(S, U, V)$-*sprout* of $H$ is an induced subgraph of $H$ that is in one of the following four *types S*:

*S1:* A tree intersecting each of sets $S$, $U$, and $V$ at exactly one node.
*S2:* An $SU$-rung plus a node-disjoint $SV$-rung.
*S3:* A $UV$-rung plus a node-disjoint $SU$-rung not intersecting $V$.
*S4:* A $UV$-rung plus a node-disjoint $SV$-rung not intersecting $U$.

Let $\mathbb{H}$ be an $X$-net of $G$. For any node $y$ and node subset $Z$ of $G$, let $N(y, Z) = N(\{y\}, Z)$. Let $S$ be a non-empty node set. $S$ is *secure* in $\mathbb{H}$ if $S$ is contained by an arc $E = UV$ of $\mathbb{H}$ such that $G[E]$ contains no $(S, U, V)$-sprout. $S$ is $\mathbb{H}$-*secure* if $S$ is a vertex of $\mathbb{H}$ or $S$ is secure in $\mathbb{H}$. A set $Y$ inducing a $y_1y_2$-path of $G - X$ is $\mathbb{H}$-*expandable* if one of the following conditions holds with $S_i = N(y_i, X)$:

- $|Y| = 1$ and $N(Y, X)$ is the union of two distinct $\mathbb{H}$-secure sets.
- $|Y| \geq 2$ and

  *X1:* both $S_1$ and $S_2$ are $\mathbb{H}$-secure,
  *X2:* if both $S_1$ and $S_2$ are vertices of $\mathbb{H}$, then vertices $S_1$ and $S_2$ are not adjacent in $\mathbb{H}$, and
  *X3:* if $S_i$ with $i \in \{1, 2\}$ is secure in $\mathbb{H}$, then $S_{3-i}$ is not an end-vertex of the arc of $\mathbb{H}$ containing $S_i$.

We comment that Conditions X2 and X3 are not needed in the proof of Lemma 2.4 but needed in our proof of Lemma 2.2 in §3, which generalizes that of Chudnovsky and Seymour [16, (5.4)]. See the paragraphs containing Equations (6) and (7).

For vertices $U$ and $V$ of $\mathbb{H}$, we call $(U, V)$ a *split pair* for $\mathbb{H}$ if $UV$ is an arc of $\mathbb{H}$ or $\{U, V\}$ is a vertex cut-set of graph $\mathbb{H}$. For any split pair $(U, V)$ for $\mathbb{H}$,

- if $U$ and $V$ are adjacent in $\mathbb{H}$, then a *split component* of $(U, V)$ consists of an arc $UV$;
- otherwise, a *split component* of $(U, V)$ consists of the arcs of $\mathbb{H}[\{U, V\} \cup \mathbb{B}]$ for some connected component $\mathbb{B}$ of $\mathbb{H} - \{U, V\}$ not containing any leaf vertex of $\mathbb{H}$.

For any split pair $(U, V)$ for $\mathbb{H}$,

- a *split arc set* of $(U, V)$ is the union of one or more split components of $(U, V)$ and



doi:10.6342/NTU201803034

- a $UV$*-block* of $\mathbb{H}$ is a subset of $X$ that is the union $C$ of the arcs in a split arc set $\mathbb{C}$ of $(U, V)$.

We call $\mathbb{C}$ the *split arc set of* $\mathbb{H}$ *corresponding to* $C$ and call $C$ *the block of* $\mathbb{H}$ *corresponding to* $\mathbb{C}$. An *abode* in $\mathbb{H}$ of a set $Y$ inducing a path in $G - X$ is a $UV$-block $C$ of $\mathbb{H}$ satisfying the following Conditions A:

*A1:* $N(Y, X) \subseteq C \cup U \cup V$.
*A2:* $N(Y, U) \subseteq C$ or $U \subseteq C \cup N(y)$ holds for an end-node $y$ of path $G[Y]$.
*A3:* $N(Y, V) \subseteq C$ or $V \subseteq C \cup N(y)$ holds for an end-node $y$ of path $G[Y]$.

A set $Y$ is $\mathbb{H}$*-bad* if $G[Y]$ is a minimally non-$\mathbb{H}$-local path of $G - X$. An $X$-net $\mathbb{H}$ is *steady* if there is no parallel arc in $\mathbb{H}$ and any split component of any split pair for $\mathbb{H}$ consists of an arc of $\mathbb{H}$. Thus, if $\{U, V\}$ is a cut-set of a steady $\mathbb{H}$, then each connected component of $\mathbb{H} - \{U, V\}$ contains a leaf vertex of $\mathbb{H}$. The degree of each vertex in a steady $\mathbb{H}$ cannot be two.

**Lemma 2.2.** *If $Y$ is an $\mathbb{H}$-bad non-$\mathbb{H}$-expandable set admitting no abode in a steady $X$-net $\mathbb{H}$ for $G$, then $G[X \cup Y]$ has a sapling.*

### 2.4 Webs

An arc $E = UV$ of an $X$-net $\mathbb{H}$ is

- *trivial* if $|E| = 1$,
- *slim* if $|E| \geq 2$ and $G[E]$ is a $UV$-rung, and
- *risky* if $G[E]$ contains an $(S, U, V)$-sprout for each nonempty subset $S \subseteq E$.

An $X$-net $\mathbb{H}$ is an $X$*-web* if the following *Conditions* W hold:

*W1:* Each arc of $\mathbb{H}$ is trivial, slim, or risky.
*W2:* $\mathbb{H}$ has no parallel risky arc nor degree-2 vertex whose incident arcs are both slim or trivial.

Note that Condition W2 does not rule out, e.g., a degree-3 vertex incident to 3 slim arcs. We comment that Condition W2 is needed in the proofs of Lemmas 2.3 and 6.2 but not in that of Lemma 2.4.
A node set $S$ is $\mathbb{H}$*-safe* for an $X$-net $\mathbb{H}$ if $S$ is a vertex of $\mathbb{H}$ or $S$ consists of two adjacent nodes of $G$ contained by a slim arc of $\mathbb{H}$. A set $Y$ inducing a $y_1y_2$-path in $G - X$ is $\mathbb{H}$*-extendable* if

- $|Y| = 1$ and $N(Y, X)$ is the union of two distinct $\mathbb{H}$-safe sets or
- $|Y| \geq 2$ and each $N(y_i, X)$ with $i \in \{1, 2\}$ is $\mathbb{H}$-safe.

An $X$-net $\mathbb{H}'$ *aids* an $X$-web $\mathbb{H}$ if $\mathbb{H}'$ is steady, each vertex of $\mathbb{H}'$ is a vertex of $\mathbb{H}$, and each arc $UV$ of $\mathbb{H}'$ is a $UV$-block of $\mathbb{H}$.

**Lemma 2.3.** *It takes $O(mn)$ time to ensure the existence of one of the following three items:*

1. *A sapling of $G$.*
2. *An $X$-net $\mathbb{H}'$ for $G$ such that all nonempty $Y \subseteq V(G) \setminus X$ with connected $G[Y]$ are $\mathbb{H}'$-local.*
3. *An $X$-net $\mathbb{H}'$ aiding an $X$-web $\mathbb{H}$ for $G$ and an $\mathbb{H}'$-bad non-$\mathbb{H}$-extendable $Y$ having no abode in $\mathbb{H}$.*



doi:10.6342/NTU201803034

## 2.5 The safe lemma

This subsection proves Lemma 2.4, which is needed to prove Theorem 1.1.

**Lemma 2.4.** *Let $\mathbb{H}$ be an $X$-net of $G$ satisfying Condition W1. If $S$ is a nonempty subset of $X$ contained by a $V_1V_2$-block $C$ of $\mathbb{H}$ such that $G[C]$ does not contain any $(S, V_1, V_2)$-sprout, then $S$ is $\mathbb{H}$-safe.*

**Lemma 2.5** (Menger [45])**.** *Let $G$ be a $k$-connected graph. If $R$ and $S$ are subsets of $V(G)$ with $|R| = |S| = k$, then there are $k$ vertex-disjoint $RS$-paths in $G$.*

*Proof of Lemma 2.4.* A *block* $C$ of $\mathbb{H}$ is a $UV$-block of $\mathbb{H}$ for some split pair $(U, V)$ of $\mathbb{H}$. We call $(U, V)$ the *split pair* of $\mathbb{H}$ for block $C$ if $C$ is a $UV$-block. All sprouts throughout the proof are $(S, V_1, V_2)$-sprouts unless clearly specified otherwise.

Assume for contradiction that $C$ is a minimal block containing a non-$\mathbb{H}$-safe set $S$ such that $G[C]$ does not contain any sprout. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. If $|\mathbb{C}| = 1$, then $C$ is an arc $V_1V_2$ of $\mathbb{H}$. $C$ is not trivial or else $G[C]$, consisting of the single node in $S$, would be a sprout of Type S1. Since $G[C]$ does not contain any sprout, $C$ is not risky. By Condition W1, $G[C]$ is a $V_1V_2$-rung with $|C| \geq 2$. We have $|S| \geq 2$ or else $G[C]$ would be a sprout of Type S1. Let each $P_i$ with $i \in \{1, 2\}$ be the $SV_i$-rung of $G[C]$. $G[P_1 \cup P_2] = G[C]$ or else $G[P_1 \cup P_2]$ would be a sprout of Type S2. Thus, $S$, consisting of two adjacent nodes contained by the slim arc $C$, is $\mathbb{H}$-safe, contradiction.

The rest of the proof argues that $|\mathbb{C}| \geq 2$ also implies that $S$ is $\mathbb{H}$-safe via showing that $S$ is a vertex. We start with proving two claims.

*Claim 1: If $\mathbb{C}$ is a split component of a split pair $(U, V)$, then $\mathbb{C} \cup \{UV\}$ is biconnected.*

Assume a cut-vertex $W$ of $\mathbb{C} \cup \{UV\}$ for contradiction. There is a connected component $\mathbb{B}$ of $(\mathbb{C} \cup \{UV\}) - \{W\}$ not intersecting $\{U, V\}$, implying that $\mathbb{B}$ does not contain any leaf vertex of $\mathbb{H}$. Thus, $W$ is a cut-vertex of the graph obtained from $\mathbb{H}$ by adding an arc between each pair of leaf vertices, contradicting Condition N1. The claim is proved.

*Claim 2: Any node of a $UV$-block $B$ for $\mathbb{H}$ is contained by a $UV$-rung of $G[B]$.*

Let $x$ be an arbitrary node in $B$. Let $E = W_1W_2$ be the arc containing $x$ by Condition N2. Let $\mathbb{B}$ be the split component of $(U, V)$ containing $E$. Lemma 2.5 and Claim 1 imply vertex-disjoint paths $\mathbb{P}_1$ and $\mathbb{P}_2$ of $\mathbb{B} \cup \{UV\}$ between $\{U, V\}$ and $\{W_1, W_2\}$ such that $U_i$ and $W_i$ with $\in \{1, 2\}$ are the end-vertices of $\mathbb{P}_i$. Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are vertex-disjoint, $\mathbb{P}_1 \cup \mathbb{P}_2$ does not intersect arc $UV$. Let each $P_i$ with $i \in \{1, 2\}$ be a $U_iW_i$-rung of $G[B]$ induced by vertex $W_i$ and the arcs of $\mathbb{P}_i$ by Condition N3. Let $Q$ be a $W_1W_2$-rung of $G[E]$ containing $x$ by Condition N3. $G[P_1 \cup Q \cup P_2]$ is a $UV$-rung containing $x$. The claim is proved.

To show that $|\mathbb{C}| \geq 2$ implies that $S$ is a vertex, observe that each arc $E$ of $\mathbb{C}$ is a block with $E \subsetneq C$. Thus, there is a maximal block $B \subsetneq C$ intersecting $S$. Let $\mathbb{B}$ be the split arc set of $\mathbb{H}$ corresponding to $B$. Let $(U_2, W_2)$ be the split pair for $\mathbb{B}$. Let $\mathbb{R} = \{V_1, V_2\}$ and $\mathbb{R}_2 = \{U_2, W_2\}$. By Claim 1, there are vertex-disjoint $\mathbb{R}\mathbb{R}_2$-rungs $\mathbb{P}_i$ with $i \in \{1, 2\}$ such that $R_i \in \mathbb{R}_2$ and $V_i$ are the end-vertices of $\mathbb{P}_i$. Since $(U_2, W_2)$ is a split pair, $\mathbb{P}_1 \cup \mathbb{P}_2$ does not intersect $\mathbb{B}$. We first show $S \not\subseteq B$. Assume $S \subseteq B$ for contradiction. For each $i \in \{1, 2\}$, if $R_i = V_i$, then let $P_i$ be empty; otherwise, let $P_i$ be an $R_iV_i$-rung of $G[C]$ induced by the arcs of $\mathbb{P}_i$. By $S \subseteq B$, $P_1 \cup P_2$ does not intersect $S$. $G[B]$ does not contain any $(S, U_2, W_2)$-sprout $T$ or else $G[P_1 \cup P_2 \cup T]$ would be a sprout. Thus, $B$ contradicts the minimality of $C$. We have $S \not\subseteq B$.

Let $a$ be an arbitrary node in $C \setminus B$. Let $b$ be an arbitrary node in $B$. Let $A = U_1W_1$ be the arc of $\mathbb{C} \setminus \mathbb{B}$ containing $a$. Let $\mathbb{R}_1 = \{U_1, W_1\}$. Claim 1 implies vertex-disjoint $\mathbb{R}\mathbb{R}_i$-rungs $\mathbb{Q}_i$ of $\mathbb{C}$ with $i \in \{1, 2\}$. By Claim 2, Condition N4, and $A \cap B = \varnothing$, there is an $i \in \{1, 2\}$ admitting node-disjoint



doi:10.6342/NTU201803034

- $aV_i$-rung $Q_1(a)$ in the subgraph of $G[C]$ induced by $A$ and the arcs of $\mathbb{Q}_1$ and
- $bV_{3-i}$-rung $Q_2(b)$ in the subgraph of $G[C]$ induced by $B$ and the arcs of $\mathbb{Q}_2$

such that $Q_1(a) - a$ and $Q_2(b) - b$ are not adjacent in $G[C]$. Any nodes $s_1 \in S \setminus B$ and $s_2 \in S \cap B$ are adjacent in $G[C]$ or else $G[Q_1(s_1) \cup Q_2(s_2)]$ would be a sprout of Type S2. By $S \setminus B \neq \varnothing$, $S \cap B \neq \varnothing$, and Condition N6, $S$ is contained by a vertex $U \in \mathbb{R}_1 \cap \mathbb{R}_2$. If there were a node $u \in U \setminus S$, then (1) $u \in B$ would imply that $G[Q_1(s_1) \cup Q_2(u)]$ for any $s_1 \in S \setminus B$ contains a sprout of Type S1 or S2 and (2) $u \notin B$ would imply that $G[Q_1(u) \cup Q_2(s_2)]$ for any $s_2 \in S \cap B$ contains a sprout of Type S1 or S2. Thus, $S = U$. □

### 2.6 Proving Theorem 1.1

We are ready to reduce Theorem 1.1 via Theorem 2.1 and Lemma 2.4 to Lemmas 2.2 and 2.3.

*Proof of Theorem 1.1.* We apply Lemma 2.3. If Item 2 exists, then $G$ is sapling-free by the if-direction of Theorem 2.1. It remains to show that Item 3 implies a sapling in $G[X \cup Y]$. Assume for contradiction that $G[X \cup Y]$ is sapling-free. Since $\mathbb{H}'$ aids $\mathbb{H}$, each arc $UV$ of $\mathbb{H}'$ is a $UV$-block of $\mathbb{H}$. Since $\mathbb{H}'$ is steady and $Y$ is $\mathbb{H}'$-bad, Lemma 2.2 implies that $Y$ either admits an abode in $\mathbb{H}'$ or is $\mathbb{H}'$-expandable.

If $Y$ admits an abode $C$ in $\mathbb{H}'$, then $C$ is a $UV$-block of $\mathbb{H}'$ satisfying Condition A. Since $\mathbb{H}'$ is steady, each split component of split pair $(U, V)$ for $\mathbb{H}'$ is an arc $UV$ of $\mathbb{H}'$, implying that the split arc set of $\mathbb{H}'$ corresponding to $C$ consists of one or more arcs $UV$ of $\mathbb{H}'$. Since each arc $UV$ of $\mathbb{H}'$ is a $UV$-block of $\mathbb{H}$, $C$ is also a $UV$-block of $\mathbb{H}$, implying that $C$ is an abode of $Y$ in $\mathbb{H}$, contradiction.

It remains the case that $Y$ is $\mathbb{H}'$-expandable. We first show that each $\mathbb{H}'$-secure set $S$ is $\mathbb{H}$-safe. If $S$ is a vertex of $\mathbb{H}'$, then $S$ is a vertex of $\mathbb{H}$ and, thus, $\mathbb{H}$-safe. If $S$ is secure in $\mathbb{H}'$, then $S$ is contained by an arc $E' = UV$ of $\mathbb{H}'$, which has to be a $UV$-block of $\mathbb{H}$, such that $G[E']$ does not contain any $(S, U, V)$-sprout. By Condition W1 of $\mathbb{H}$ and Lemma 2.4, $S$ is $\mathbb{H}$-safe.

If $|Y| = 1$, then $N(Y, X) = S_1 \cup S_2$ holds for two distinct $\mathbb{H}'$-secure sets $S_1$ and $S_2$. Since $S_1$ and $S_2$ are both $\mathbb{H}$-safe, $Y$ is $\mathbb{H}$-extendable, contradiction. If $|Y| \geq 2$, then let $y_1$ and $y_2$ be the end-nodes of path $G[Y]$. By Condition X1 of $Y$, both $N(y_1, X)$ and $N(y_2, X)$ are $\mathbb{H}'$-secure and, thus, $\mathbb{H}$-safe, implying that $Y$ is $\mathbb{H}$-extendable, contradiction. □

The rest of the paper proves Lemmas 2.2 and 2.3. Section 3 proves Lemma 2.2 for the case with $|Y| \geq 2$. Section 4 proves Lemma 2.2 for the case with $|Y| = 1$. Section 5 proves Lemma 2.3.

## 3 Proving Lemma 2.2: Part 1

Let $|Y| \geq 2$ throughout this section.

**Lemma 3.1.** *Let $\mathbb{H}$ be a steady $X$-net. Let $\mathbb{L}$ consist of the leaf vertices of $\mathbb{H}$. Let $U$ and $U_3$ be distinct vertices of $\mathbb{H}$ with $U \notin \mathbb{L}$. If there are node sets $N$ and $E$ with $\varnothing \neq N \subseteq U$, $U \setminus E \nsubseteq N$, and $N \nsubseteq E$, then there are vertex-disjoint $U_i\mathbb{L}$-rungs with $i \in \{1, 2, 3\}$ in the graph $\mathbb{H} - U$ such that $UU_1$ is an arc of $\mathbb{H}$ intersecting $N$ and $UU_2$ is an arc of $\mathbb{H}$ intersecting $U \setminus N$.*

*Proof.* Since vertex $U$ has at least three neighbors in $\mathbb{H}$, $U \setminus E \nsubseteq N$ and $N \nsubseteq E$ imply that the vertex set consisting of the neighbors of $U$ other than $U_3$ in $\mathbb{H}$ admits a non-empty disjoint partition $\mathbb{R}_1$ and $\mathbb{R}_2$ such that each arc incident to $U$ and a vertex in $\mathbb{R}_1$ intersects $N$ and



doi:10.6342/NTU201803034

each arc incident to $U$ and a vertex in $\mathbb{R}_2$ intersects $U \setminus N$. Let $\mathbb{R}_3 = \{U_3\}$. Let $\mathbb{H}^*$ be the triconnected graph obtained from the steady $\mathbb{H}$ by (1) replacing vertex $U$ and its incident arcs with a triangle on vertices $V_1$, $V_2$, and $V_3$ and (2) adding an arc between $V_i$ and each vertex in $\mathbb{R}_i$ for all $i \in \{1, 2, 3\}$. Lemma 2.5 implies vertex-disjoint $V_i\mathbb{L}$-rungs $\mathbb{P}_i$ of $\mathbb{H}^*$ with $i \in \{1, 2, 3\}$. The paths $\mathbb{P}_i - V_i$ with $i \in \{1, 2, 3\}$ prove the lemma. □

*Proof of Lemma 2.2 for* $|Y| \geq 2$. Assume for contradiction that $G[X \cup Y]$ has no sapling. Let $\mathbb{L}$ consist of the leaf vertices of $\mathbb{H}$. Since $\mathbb{H}$ is steady, $\mathbb{H}$ has no parallel arcs and degree-2 vertices. By Conditions N3 and N5 of $\mathbb{H}$, any vertex of $\mathbb{H}$ intersecting $N(Y)$ has degree at least 3. Let $N = N(Y, X)$. Let $y_1$ and $y_2$ be the end-nodes of path $G[Y]$. Since $Y$ is $\mathbb{H}$-bad, each

$$N_i = N(Y \setminus \{y_{3-i}\}, X)$$

with $i \in \{1, 2\}$ is $\mathbb{H}$-local. We have $N = N_1 \cup N_2$. Let $Z = Y \setminus \{y_1, y_2\}$. We have

$$N(Z, X) \subseteq N_1 \cap N_2. \tag{1}$$

Since $Y$ is $\mathbb{H}$-bad, both of $N_1$ and $N_2$ are $\mathbb{H}$-local. At least one of the following four cases holds.

*Case 1*: $N$ is contained by the union of two vertices.
*Case 2*: $N$ is contained by the union of a vertex and an arc.
*Case 3*: $N$ is contained by the union of two arcs and Cases 1 and 2 do not hold.
*Case 4*: $N_1$ or $N_2$ is contained by a triad and Case 1 does not hold.

Case 1: $N \subseteq V_1 \cup V_2$ holds for vertices $V_1$ and $V_2$. Let subset $E \subseteq X$ be empty (respectively, consist of the nodes contained by the arc $V_1V_2$) if vertices $V_1$ and $V_2$ are not (respectively, are) adjacent in $\mathbb{H}$. We first show an index $i \in \{1, 2\}$ with

$$V_i \setminus E \nsubseteq N_i \quad \text{and} \quad N_i \nsubseteq E. \tag{2}$$

For each $i \in \{1, 2\}$, Equation (1) and $N_i \neq \varnothing$ together imply $N(V_i \setminus E) \cap Y = \{y_i\}$. Thus, at lease one $i \in \{1, 2\}$ satisfies $V_i \setminus E \nsubseteq N_i$ or else the condition $E = \varnothing$ would imply that $Y$ is $\mathbb{H}$-expandable and the condition $E \neq \varnothing$ would imply that $E$ is an abode of $Y$ in $\mathbb{H}$. Also, at least one index $i \in \{1, 2\}$ satisfies $N_i \nsubseteq E$ or else $N_1 \cup N_2$ would be $\mathbb{H}$-local. If $V_i \setminus E \subseteq N_i$ and $N_{3-i} \subseteq E$ hold for an $i \in \{1, 2\}$, then $N(V_i \setminus E) \cap Y = \{y_i\}$ would imply that $E \neq \varnothing$ is an abode of $Y$. Thus, Equation (2) holds for an $i \in \{1, 2\}$.
We then claim that Equation (2) implies a vertex $V_3 \notin \{V_1, V_2\}$ of $\mathbb{H}$ with

$$N_{3-i} \subseteq V_3 \cup V_i$$

such that $V_3V_1$ (respectively, $V_3V_2$) is an arc of $\mathbb{H}$ intersecting $N_1$ (respectively, $N_2$). Since arc $V_3V_{3-i}$ intersects $N_{3-i}$, we have

$$N_{3-i} \nsubseteq E.$$

Since $V_{3-i}$ has at least three neighbors in $\mathbb{H}$, $N_{3-i} \subseteq V_3 \cup V_i$ implies

$$V_{3-i} \setminus E \nsubseteq N_{3-i}.$$

Hence, Equation (2) holds for both $i \in \{1, 2\}$. The claim implies a vertex $V_4 \notin \{V_1, V_2\}$ with

$$N_j \subseteq V_4 \cup V_{3-i} \tag{3}$$



doi:10.6342/NTU201803034

such that $V_4V_1$ (respectively, $V_4V_2$) is an arc of $\mathbb{H}$ intersecting $N_1$ (respectively, $N_2$). Since arc $V_3V_i$ contains a node $x \in N_i \subseteq V_i$, Equation (3) and $V_3 \notin \{V_1, V_2\}$ together imply

$$x \in N_i \setminus E \subseteq V_4.$$

Thus, arc $V_iV_4$ contains $x$, implying $V_3 = V_4$ by Condition N2 of $\mathbb{H}$. By $N_1 \subseteq V_3 \cup V_2$ and $N_2 \subseteq V_3 \cup V_1$, $N$ is contained by $\Delta(V_1, V_2, V_3)$ of $\mathbb{H}$, contradicting that $N$ is non-$\mathbb{H}$-local.

The rest of the proof ensures the claim. Since vertex $V_i$ has at least three neighbors in $\mathbb{H}$, Equation (2) and Lemma 3.1 imply vertex-disjoint $\mathbb{R}_j\mathbb{L}$-rungs $\mathbb{P}_j$ of the graph $\mathbb{H} - \{V_i\}$ with $j \in \{1, 2, 3\}$ such that if $U_j \in \mathbb{R}_j$ and $L_j \in \mathbb{L}$ are the end-vertices of $\mathbb{P}_j$, then arc $V_iU_1$ of $\mathbb{H}$ intersects $N_i$, arc $V_iU_2$ of $\mathbb{H}$ intersects $V_i \setminus N_i$, and $U_3 = V_{3-i}$.

We prove the claim by showing that $U_1$ is a vertex $V_3$ required by the claim. By $U_1 \in R_1$, we have $U_1 \notin \{V_1, V_2\}$ and that $E_i = U_1V_i$ is an arc of $\mathbb{H}$ intersecting $N_i$. One can verify that it remains to prove

$$N_{3-i} \subseteq U_1 \cup V_i : \tag{4}$$

Since $N$ is non-$\mathbb{H}$-local, we have $N_{3-i} \cap U_1 \neq \varnothing$ by Equation (4), which implies an arc $U_1V_{3-i}$ intersecting $N_{3-i}$ by Condition N5 of $\mathbb{H}$. To prove Equation (4), assume a node $v_{3-i} \in N_{3-i} \setminus (U_1 \cup V_i)$ for contradiction. Let $P_3$ be a $v_{3-i}L_3$-rung in the subgraph of $G$ induced by vertex $V_{3-i}$ and the arcs of $\mathbb{P}_3$. Since the arc $E_i = U_1V_i$ of $\mathbb{H}$ intersects $N_i$, Condition N3 of $\mathbb{H}$ implies a $U_1V_i$-rung $Q_1$ of $G[E_i]$ that intersects $N_i$. Since the arc $F_i = U_2V_i$ of $\mathbb{H}$ intersects $V_i \setminus N_i$, Condition N3 of $\mathbb{H}$ implies a $U_2V_i$-rung $Q_2$ of $G[F_i]$ that intersects $V_i \setminus N_i$. For each $j \in \{1, 2\}$, let $P_j$ be a $U_jL_j$-rung in the subgraph of $G$ induced by vertex $L_j$ and the arcs of $\mathbb{P}_j$. Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are vertex-disjoint,

$$P = G[P_1 \cup Q_1 \cup Q_2 \cup P_2]$$

is an $L_1L_2$-rung that intersects $N$ at exactly one node $v_i$. Since $U_1V_i$ is the arc of $\mathbb{H}$ containing $v_i$ and $U_1 \neq V_{3-i}$, we have $v_i \notin V_{3-i}$. We have $N(v_{3-i}) \cap V(P) \neq \varnothing$ or else $G[P \cup Y \cup P_3]$ would contain a sapling of $G[X \cup Y]$. Since $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are vertex-disjoint and $v_{3-i} \notin U_1 \cup V_i$, Condition N5 of $\mathbb{H}$ implies that $v_{3-i}$ is contained by exactly one vertex of $\mathbb{P}_1 \cup \mathbb{P}_2$ other than $U_1$. With $M = N(v_{3-i}) \cap V(P)$, let each $R_j$ with $j \in \{1, 2\}$ be the $ML_j$-rung of $P$. Either $G[P_1 \cup Q_1 \cup R_2 \cup Y \cup P_3]$ or $G[R_1 \cup Q_2 \cup P_2 \cup Y \cup P_3 \cup \{v_i\}]$ contains a sapling of $G[X \cup Y]$, contradiction.

Case 2: An index $i \in \{1, 2\}$ satisfies

$$N_i \subseteq V \quad \text{and} \quad N_{3-i} \subseteq E \tag{5}$$

for a vertex $V$ and an arc $E = V_1V_2$. We first show $V \setminus E \nsubseteq N_i$. If $V \setminus E \subseteq N_i$ were true, Equation (1) would imply

$$V \subseteq E \cup N(y_i). \tag{6}$$

We have $V \notin \{V_1, V_2\}$ or else $E$ would be an abode of $Y$ in $\mathbb{H}$ by Equation (6). By $V \notin \{V_1, V_2\}$ and Condition N5 of $\mathbb{H}$, we have $V \cap E = \varnothing$. By $V \setminus E \subseteq N_i$ and $N_i \subseteq V$, we have $N_i = V$, implying that $N_i$ is $\mathbb{H}$-secure. Let $R = \{V, V_1, V_2\}$. Let $L$ consist of the leaf vertices of $\mathbb{H}$. Since $\mathbb{H}$ is steady, there are vertex-disjoint $RL$-rungs $\mathbb{P}_j$ of $\mathbb{H}$ with $j \in \{1, 2, 3\}$. For each $j \in \{1, 2, 3\}$, let $R_j \in R$ and $L_j \in L$ be the end-vertices of $\mathbb{P}_j$ and let $P_j$ be an $R_jL_j$-rung in the subgraph of $G$ induced by vertex $L_j$ and the arcs of $\mathbb{P}_j$. $G[E]$ does not contain any $(N_{3-i}, V_1, V_2)$-sprout $T$ or else $G[P_1 \cup P_2 \cup P_3 \cup Y \cup T]$ would contain a sapling of $G[X \cup Y]$. Hence, $N_{3-i}$ is $\mathbb{H}$-secure. Condition X1 holds for $Y$. By $N_{3-i} \subseteq E$, $N_{3-i}$ is not a vertex of $\mathbb{H}$, implying Condition X2 for $Y$. By $V \notin \{V_1, V_2\}$, Condition X3 holds for $Y$, contradicting that $Y$ is non-$\mathbb{H}$-expandable. Thus, $V \setminus E \nsubseteq N_i$.



doi:10.6342/NTU201803034

Let $j$ be an index in $\{1,2\}$ with $V \neq V_j$. Since $N$ is not $\mathbb{H}$-local, we have $N_i \nsubseteq E$. By $V \setminus E \nsubseteq N_i$ and $N_i \nsubseteq E$, Lemma 3.1 implies vertex-disjoint $\mathbb{R}_k\mathbb{L}$-rungs $\mathbb{P}_k$ of graph $\mathbb{H} - \{V\}$ with $k \in \{1,2,3\}$ such that if $U_k \in \mathbb{R}_k$ and $L_k \in \mathbb{L}$ are the end-vertices of $\mathbb{P}_k$, then the arc $VU_1$ of $\mathbb{H}$ intersects $N_i$, the arc $VU_2$ of $\mathbb{H}$ intersects $V \setminus N_i$, and $U_3 = V_j$.

For each $k \in \{1,2,3\}$, let $P_k$ be a $U_kL_k$-rung in the subgraph of $G$ induced by vertex $L_k$ and the arcs of $\mathbb{P}_k$. Since the arc $E_1 = VU_1$ intersects $N_i$, Condition N3 of $\mathbb{H}$ implies a $VU_1$-rung $Q_1$ of $G[E_1]$ that intersects $N_i$. Since the arc $E_2 = VU_2$ intersects $V \setminus N_i$, Condition N3 of $\mathbb{H}$ implies a $VU_2$-rung $Q_2$ of $G[E_2]$ that intersects $V \setminus N_i$. Since $N$ is not $\mathbb{H}$-local, we have $N_{3-i} \nsubseteq V$. Condition N3 of $\mathbb{H}$ implies an $N_{3-i}V_j$-rung $Q_3$ in $G[E]$ that does not intersect $V$. Since $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are vertex-disjoint,

$$P = G[P_1 \cup Q_1 \cup Q_2 \cup P_2]$$

is an $L_1L_2$-rung that intersects $N$ at exactly one node $v_i$ and

$$Q = G[Q_3 \cup P_3]$$

is an $N_{3-i}L_3$-rung that is not adjacent to $P$ in $G$. Therefore, $G[P \cup Q \cup Y]$ contains a sapling of $G[X \cup Y]$, contradiction.

Case 3: Both indices $i \in \{1,2\}$ satisfy

$$N_i \subseteq E_i \quad \text{and} \quad N_i \nsubseteq U_i \tag{7}$$

for distinct arcs $E_1 = U_1V_1$ and $E_2 = U_2V_2$ of $\mathbb{H}$ with $V_1 \neq V_2$. Let $\mathbb{R}_1 = \{U_1, V_1, V_2\}$ and $\mathbb{R}_2 = \{U_2, V_1, V_2\}$. For each $i \in \{1,2\}$, the fact that $\mathbb{H}$ is steady implies vertex-disjoint $\mathbb{R}_i\mathbb{L}$-rungs $\mathbb{P}_{i,j}$ in $\mathbb{H}$ with $j \in \{1,2,3\}$. For any indices $i \in \{1,2\}$ and $j \in \{1,2,3\}$, if $U \in \mathbb{R}_i$ and $V \in \mathbb{L}$ are the end-vertices of path $\mathbb{P}_{i,j}$ in $\mathbb{H}$, then let $P_{i,j}$ be a $UV$-rung in the subgraph of $G$ induced by vertex $V$ and the arcs of $\mathbb{P}_{i,j}$. For each $i \in \{1,2\}$, Equation (7) implies an $E_i$-rung $P_i$ that intersects $N_i \setminus U_i$. Let each $Q_i$ with $i \in \{1,2\}$ be the $N_iV_i$-rung of $P_i$. There cannot be any $(N_i, U_i, V_i)$-sprout $T_i$ in $G[E_i]$ with $i \in \{1,2\}$ or else $G[P_{i,1} \cup P_{i,2} \cup P_{i,3} \cup Y \cup Q_{3-i} \cup T_i]$ would contain a sapling of $G[X \cup Y]$. Hence, $N_1$ and $N_2$ are both $\mathbb{H}$-secure. By Equation (1), we have $N_1 = N(y_1, X)$ and $N_2 = N(y_2, X)$, implying Condition X1. By Equation (7), $N_1$ and $N_2$ are not vertices of $\mathbb{H}$. Thus, Conditions X2 and X3 holds, contradicting that $Y$ with $|Y| \geq 2$ is non-$\mathbb{H}$-expandable.

Case 4: An index $i \in \{1,2\}$ and a triangle of $\mathbb{H}$ on a vertex set $\mathbb{R} = \{U_1, U_2, U_3\}$ satisfy

$$N_i \subseteq \Delta(U_1, U_2, U_3) \quad \text{and} \quad N_i \cap U_j \cap U_k \neq \varnothing \tag{8}$$

for any distinct indices $j$ and $k$ with $\{j,k\} \subseteq \{1,2,3\}$. Let $L$ consist of the leaf nodes of $G$. We show that $G[Q_1 \cup Q_2 \cup Y]$ contains a sapling of $G[X \cup Y]$ by identifying an $LL$-rung $Q_1$ and an $NL$-rung $Q_2$ such that (i) $Q_1$ and $Q_2$ are adjacent in $G$ and (ii) $Q_1$ intersects $N$ at exactly one node. Since $\mathbb{H}$ is steady, Lemma 2.5 implies vertex-disjoint $\mathbb{R}\mathbb{L}$-rungs $\mathbb{P}_j$ with $j \in \{1,2,3\}$ such that $U_j \in \mathbb{R}$ and $L_j \in \mathbb{L}$ are the end-vertices of $\mathbb{P}_j$. Let each $P_j$ with $j \in \{1,2,3\}$ be a $U_jL_j$-rung in the subgraph of $G$ induced by the arcs of $\mathbb{P}_j$. By Condition N5 of $\mathbb{H}$, the three paths $P_j$ with $j \in \{1,2,3\}$ are pairwise nonadjacent in $G$. Let arcs $E_1 = U_2U_3$, $E_2 = U_3U_1$, and $E_3 = U_1U_2$. Let each $v_t$ with $t \in \{1,2,3\}$ be a node in $N_i \cap U_j \cap U_k$ for the indices $j$ and $k$ with $\{t,j,k\} = \{1,2,3\}$ as ensured by Equation (8).

Case 4(a): $N_{3-i}$ intersects an arc $E = V_1V_2$ such that $\{V_1, V_2\}$ intersects at most one of paths $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$. Let $\mathbb{Q}$ be a shortest path of $\mathbb{H}$ between $\{V_1, V_2\}$ and the vertices $\mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3$. Let $V_j$ with $j \in \{1,2\}$ and $V \in V(\mathbb{P}_k)$ with $k \in \{1,2,3\}$ be the end-vertices of $\mathbb{Q}$. Let $\mathbb{Q}_2$ be the $V_{3-j}L_k$-rung in $\mathbb{H}[\mathbb{P}_k \cup \mathbb{Q} \cup \{V_{3-j}\}]$. Let $Q_1 = G[P_s \cup \{v_k\} \cup P_t]$ for indices $s$ and $t$



doi:10.6342/NTU201803034

with $\{k, s, t\} = \{1, 2, 3\}$. Since $N$ is not $\mathbb{H}$-local, Equation (8) implies an $N_{3-j}L_k$-rung $Q_2$ in the subgraph of $G$ induced by the arcs of $\mathbb{Q}_2$ that does not intersect $N_i$. Since $\mathbb{H}[\mathbb{P}_s \cup \mathbb{P}_t]$ and $\mathbb{H}[\mathbb{P}_k \cup \mathbb{Q} \cup \{V_{3-j}\}]$ are vertex-disjoint, $Q_1$ and $Q_2$ are not adjacent in $G$. Since $\mathbb{H}[\mathbb{P}_s \cup \mathbb{P}_t]$ and $\mathbb{Q}_2$ are vertex-disjoint and $N_{3-i}$ is $\mathbb{H}$-local, $Q_1$ intersects $N$ only at $v_k$.

Case 4(b): Case 4(a) does not hold. Since $N$ is non-$\mathbb{H}$-local, $N_{3-i}$ is contained by an arc $E = V_jV_k$ for distinct indices $j$ and $k$ with $\{j, k\} \subseteq \{1, 2, 3\}$ such that $V_j$ is a vertex of $\mathbb{P}_j$ and $V_k$ is a vertex of $\mathbb{P}_k$. Let $t$ be the index in $\{1, 2, 3\} \setminus \{j, k\}$.

We first handle the case with $\{V_j, V_k\} \subseteq \{U_1, U_2, U_3\}$. By $N_{3-i} \not\subseteq V_j \cap V_k$, we assume $N_{3-i} \not\subseteq V_j$ without loss of generality. Condition N4 of $\mathbb{H}$ implies an $N_{3-j}V_k$-rung $Q$ of $G[E]$ that does not intersect $V_j$. Let $Q_1 = G[P_j \cup \{v_k\} \cup P_t]$ and $Q_2 = G[Q \cup P_k]$. By Condition N5 of $\mathbb{H}$, $Q$ does not intersect $V_j$, implying that $Q_1$ and $Q_2$ are non-adjacent in $G$. By Equation (8) and $N_{3-i} \subseteq E$, $Q_1$ intersects $N$ only at $v_k$.

It remains to handle the case with $\{V_j, V_k\} \not\subseteq \{U_1, U_2, U_3\}$. Assume $V_j \notin \{U_1, U_2, U_3\}$ without loss of generality. $N_{3-i}$ does not intersect $P_1 \cup P_2 \cup P_3$ at any node $v$ or else Case (a) would hold for the arc containing $v$. Let $P'_j$ be the $V_jL_j$-rung of $P_j$. Let $P'_k$ be the $V_kL_k$-rung of $P_k$.

- If $N_{3-i} \subseteq V_j$, then Condition N4 of $\mathbb{H}$ implies a $V_jV_k$-rung $Q$ that intersects $N_{3-i}$ only at the end-node of $Q$ in $V_j$. Let $Q_1 = G[P'_j \cup Q \cup P'_k]$ and $Q_2 = G[P_t \cup \{v_k\}]$. Since $\mathbb{P}_j \cup \mathbb{P}_k$ and $\mathbb{P}_t \cup E_k$ are vertex-disjoint, $Q_1$ and $Q_2$ are not adjacent in $G$. By $N_i \subseteq \Delta(U_1, U_2, U_3)$, $E \notin \{E_1, E_2, E_3\}$, and $N_{3-i} \subseteq V_j$, we know that $Q_1$ intersects $N$ only at the end-node of $Q$ in $N_{3-i}$.
- If $N_{3-i} \not\subseteq V_j$, then Condition N4 of $\mathbb{H}$ implies an $N_{3-i}V_k$-rung $Q$ of $G[E]$ that does not intersect $V_j$. Let $Q_1 = G[P_j \cup \{v_k\} \cup P_t]$ and $Q_2 = G[Q \cup P'_k]$. Since $\mathbb{P}_k$ and $\mathbb{P}_j \cup \mathbb{P}_t$ are vertex-disjoint and $Q$ does not intersect $V_j$, $Q_1$ and $Q_2$ are not adjacent in $G$. By $N_i \subseteq \Delta(U_1, U_2, U_3)$ and $E \notin \{E_1, E_2, E_3\}$, we know that $Q_1$ intersects $N$ only at $v_k$.

□

## 4 Proving Lemma 2.2: Part 2

Let $|Y| = 1$ throughout this section. Let $\mathbb{H}$ be an $X$-net. An *$E$-rung* for an arc $E = UV$ of $\mathbb{H}$ is an $UV$-rung. An *$\mathbb{H}$-twig* is a set $B \subseteq X$ such that $G[B \cap E]$ is an $E$-rung for each arc $E$ of $\mathbb{H}$. By Condition N5 of $\mathbb{H}$, if a vertex $V$ of $\mathbb{H}$ is (respectively, is not) incident to an arc $E$ of $\mathbb{H}$, then $|B \cap E \cap V|$ is 1 (respectively, 0). Consider the following *Conditions T* of a nonempty $Y \subseteq V(G) \setminus X$ for an $\mathbb{H}$-twig $B$:

*T1:* An arc $E$ is incident to a vertex $U$ with $N(Y, B) \setminus E = (U \cap B) \setminus E$.
*T2:* An arc $E$ is incident to vertices $U$ and $V$ with $N(Y, B) \setminus E = ((U \cup V) \cap B) \setminus E$.
*T3:* $N(Y, B) = A_1 \cup A_2$, where each $A_i$ with $i \in \{1, 2\}$ either (1) equals $B \cap U_i$ for a vertex $U_i$ or (2) consists of two adjacent nodes of $G$ in $B \cap E_i$ for an arc $E_i$.

**Lemma 4.1** (Chudnovsky and Seymour [16, §4]). *Let $\mathbb{H}$ be a steady $X$-net. If $G[X \cup Y]$ with $|Y| = 1$ has no sapling, then one of Conditions T holds for any $\mathbb{H}$-twig $B$ with non-$\mathbb{H}$-local $N(Y, B)$.*

**Lemma 4.2.** *Let $\mathbb{H}$ be a steady $X$-net. If Condition T3 of a set $Y$ holds for an $\mathbb{H}$-twig $B$, then the following statements hold:*

1. *If $|N(Y, B) \cap V| \geq 3$ holds for a vertex $V$, then $A_i = B \cap V$ holds for an $i \in \{1, 2\}$.*
2. *If $|N(Y, B) \cap V| = 2$ holds for a vertex $V$, then each $A_i$ with $i \in \{1, 2\}$ either equals $B \cap U_i$ for a vertex $U_i$ adjacent to $V$ or consists of two adjacent nodes in $B \cap E_i$ for an incident arc $E_i$ of $V$.*



doi:10.6342/NTU201803034

3. *If $|N(Y,B) \cap E| \geq 2$ holds for an arc $E = V_1V_2$, then (1) $A_i = B \cap U_i$ holds for both $i \in \{1,2\}$ with $\{U_1, U_2\} = \{V_1, V_2\}$ or (2) $A_i \subseteq B \cap E$ holds for an $i \in \{1,2\}$.*

*Proof.* Since $\mathbb{H}$ is steady, $\mathbb{H}$ has no parallel arcs and degree-2 vertices.

Statement 1: $N(Y,B)$ intersects at least three incident arcs of $V$ in $\mathbb{H}$ by Condition N5 of $\mathbb{H}$, implying an $A_i = B \cap U_i$ with $i \in \{1,2\}$ intersecting at least two incident arcs of $V$. Since $\mathbb{H}$ has no parallel arcs, $U_i = V$.

Statement 2: $N(Y,B)$ intersects exactly two incident arcs of $V$ in $\mathbb{H}$ by Condition N5 of $\mathbb{H}$. For the case $A_i = B \cap U_i$ with $i \in \{1,2\}$, we have $U_i \neq V$ or else $A_i$ would intersect at least three incident arcs of $V$ in $\mathbb{H}$. If $V$ is (respectively, is not) adjacent to $U_i$ in $\mathbb{H}$, then $A_i$ intersects one (respectively, zero) incident arc of $V$ in $\mathbb{H}$. For the case $A_i \subseteq B \cap E_i$ with $i \in \{1,2\}$, if $V$ is (respectively, is not) an end-vertex of arc $E_i$ in $\mathbb{H}$, then $A_i$ intersects one (respectively, zero) incident arc of $V$ in $\mathbb{H}$. The statement follows.

Statement 3: For the case $A_i = B \cap U_i$ with $i \in \{1,2\}$, if $U_i$ is (respectively, is not) an end-vertex of $E$, then $|A_i \cap E|$ is 1 (respectively, 0). For the case $A_i \subseteq B \cap E_i$ with $i \in \{1,2\}$, if $E_i$ is (respectively, is not) $E$, then $|A_i \cap E|$ is 2 (respectively, 0). Suppose $A_i \nsubseteq B \cap E$ for both $i \in \{1,2\}$. By $|A_1 \cap E| + |A_2 \cap E| \geq |(A_1 \cup A_2) \cap E| = |N(Y,B) \cap E| \geq 2$, $A_i = B \cap U_i$ holds for both $i \in \{1,2\}$ with $\{U_1, U_2\} = \{V_1, V_2\}$. □

**Lemma 4.3.** *Let $Y$ be an $\mathbb{H}$-bad subset of $V(G) \setminus X$ for a steady $X$-net $\mathbb{H}$. If $N(Y,E)$ for an arc $E = UV$ of $\mathbb{H}$ with $N(Y,X) \nsubseteq E \cup U \cup V$ is not secure in $\mathbb{H}$, then $G[X \cup Y]$ has a sapling.*

*Proof of Lemma 4.3.* Since $S = N(Y,E)$ is not secure in $\mathbb{H}$, $G[E]$ contains an $(S,U,V)$-sprout $T$ in Type S1 or S2 by Conditions N2 and N6 of $\mathbb{H}$. By $N(Y,X) \nsubseteq E \cup U \cup V$, there is an arc $F$ of $\mathbb{H}$ intersecting $N(Y,X) \setminus (E \cup U \cup V)$. Let $W$ be an end-vertex of $F$ with $W \notin \{U,V\}$. Let $Q$ be a $ZW$-rung of $G[F]$ with $Z = N(Y,X) \setminus (U \cup V)$. Let $R = \{U,V,W\}$. Let $L$ consist of the leaf vertices $L_1, L_2, L_3$ of $\mathbb{H}$. Since $\mathbb{H}$ is steady, there are vertex-disjoint $RL$-paths $\mathbb{P}_i$ of $\mathbb{H}$ with $i \in \{1,2,3\}$. Let each $\mathbb{P}_i$ be an $R_iL_i$-path. If $R_i = L_i$, then let $P_i$ be empty. Otherwise, let $P_i$ be an $R_iL_i$-rung of $G$ induced by vertex $L_i$ and the arcs of $\mathbb{P}_i$. Since $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ are vertex-disjoint, $P_1, P_2, P_3$ are pairwise non-adjacent. $G[P_1 \cup P_2 \cup P_3 \cup Q \cup T \cup Y]$ is a sapling. □

**Lemma 4.4.** *Let $\mathbb{H}$ be a steady $X$-net. Let $Y$ be an $\mathbb{H}$-bad non-$\mathbb{H}$-expandable set with $|Y| = 1$. If $G[X \cup Y]$ has no sapling, then $N(Y,X) \subseteq E \cup V_1 \cup V_2$ holds for an arc $E = V_1V_2$ of $\mathbb{H}$.*

*Proof of Lemma 4.4.* Assume for contradiction two nodes $v_1$ and $v_2$ of $N(Y,X)$ with

$$\{v_1, v_2\} \nsubseteq E \cup V_1 \cup V_2 \tag{1}$$

for any arc $E = V_1V_2$ of $\mathbb{H}$. For any $\mathbb{H}$-twig $B$ and any node $v \in X$, the rest of the proof lets $B(v)$ denote an $\mathbb{H}$-twig $(B \setminus E_v) \cup P_v$, where $E_v$ is the arc of $\mathbb{H}$ containing $v$ and $P_v$ is an arbitrary $E_v$-rung containing $v$ as ensured by Condition N4 of $\mathbb{H}$. By Condition N4 of $\mathbb{H}$, there is an $\mathbb{H}$-twig $B \supseteq \{v_1, v_2\}$. By Equation (1), Conditions T1 and T2 of $Y$ do not hold for $B$. Since $\{v_1, v_2\}$ is non-$\mathbb{H}$-local, so is $N(Y,B)$. By Lemma 4.1, Condition T3 of $Y$ holds for $B$. That is,

$$N(Y,B) = A_1 \cup A_2, \tag{2}$$

where each $A_i$ with $i \in \{1,2\}$ either (1) equals $B \cap V_i$ for a vertex $V_i$ of $\mathbb{H}$ or (2) consists of two adjacent nodes of $G$ in $B \cap E_i$ for an arc $E_i$ of $\mathbb{H}$.



doi:10.6342/NTU201803034

Case 1: $A_1 = B \cap V_1$ and $A_2 = B \cap V_2$. Distinct vertices $V_1$ and $V_2$ are non-adjacent in $\mathbb{H}$ by Equation (1), so $V_1 \cap V_2 = \varnothing$. We have $N(Y, X) \neq V_1 \cup V_2$ or else $Y$ would be $\mathbb{H}$-expandable. Let $B_v = B(v)$ for a node

$$v \in ((V_1 \cup V_2) \setminus N(Y, X)) \cup (N(Y, X) \setminus (V_1 \cup V_2)). \tag{3}$$

Since both $V_1$ and $V_2$ intersect $N(Y)$, Equation (2) implies $|N(Y, B) \cap V_i| \geq 3$. By Equation (3),

$$|N(Y, B_v) \cap V_j| \geq 3 \tag{4}$$
$$|N(Y, B_v) \cap V_{3-j}| \geq 2 \tag{5}$$

hold for an index $j \in \{1, 2\}$ with $v \notin V_j$. Since vertices $V_1$ and $V_2$ are not adjacent in $\mathbb{H}$, Equations (4) and (5) imply that $N(Y, B_v)$ is not $\mathbb{H}$-local and Conditions T1 and T2 of $Y$ do not hold for $B_v$. By Lemma 4.1, Condition T3 of $Y$ holds for $B_v$. We have $|N(Y, B_v) \cap V_{3-j}| \neq 2$ or else Equation (4) and Lemmas 4.2(1) and 4.2(2) would imply that vertices $V_1$ and $V_2$ are adjacent in $\mathbb{H}$. By Equation (5), $|N(Y, B_v) \cap V_{3-j}| \geq 3$. Combining with Equation (4) and Lemma 4.2(1), we have $\{A_1, A_2\} = \{B_v \cap V_1, B_v \cap V_2\}$, implying $v \in N(Y, B_v) = B_v \cap (V_1 \cup V_2)$, contradicting $v \in N(Y, X) \setminus (V_1 \cup V_2)$ by Equation (3).

Case 2: Each $G[A_i]$ with $i \in \{1, 2\}$ is an edge of $G[B \cap E_i]$ for an arc $E_i$ of $\mathbb{H}$. We first show

$$N(Y, X) = N(Y, E_1) \cup N(Y, E_2). \tag{6}$$

Assume a node

$$v \in N(Y, X) \setminus (E_1 \cup E_2) \tag{7}$$

for contradiction. Let $B_v = B(v)$. Let $E_v$ be the arc of $\mathbb{H}$ containing $v$. By $N(Y, B) \subseteq E_1 \cup E_2$,

$$N(Y, B_v) \subseteq E_v \cup E_1 \cup E_2. \tag{8}$$

By $\{v_1, v_2\} \subseteq E_1 \cup E_2$ and Equation (7), we have $\{v_1, v_2\} \subseteq N(Y, B_v)$, implying that $N(Y, B_v)$ is not $\mathbb{H}$-local and Conditions T1 and T2 of $Y$ do not hold for $B_v$. By Lemma 4.1, Condition T3 of $Y$ holds for $B_v$ with $N(Y, B_v) = A'_1 \cup A'_2$. By Lemma 4.2(3) on $E_1$ and $E_2$, either

- $A'_1 = B_v \cap V_i$ and $A'_2 = B_v \cap V_{3-i}$ hold for an arc $F = V_1V_2$ and an index $i \in \{1, 2\}$ or
- $A'_1 \subseteq B_v \cap E_i$ and $A'_2 \subseteq B_v \cap E_{3-i}$ hold for an index $i \in \{1, 2\}$.

By the first statement, $V_1$ and $V_2$ are non-leaf vertices of $\mathbb{H}$, implying that $N(Y, B_v)$ intersects at least five arcs of $\mathbb{H}$, contradicting Equation (8). By the second statement, $v \in N(Y, B_v) \subseteq E_1 \cup E_2$, contradicting Equation (7). Thus, Equation (6) holds. Since $Y$ with $|Y| = 1$ is not $\mathbb{H}$-expandable, Equation (6) implies an $N(Y, E_i)$ with $i \in \{1, 2\}$ not secure in $\mathbb{H}$. By Lemma 4.3 and Equation (1), $G[X \cup Y]$ has a sapling, contradiction.

Case 3: $G[A_i]$ with $i \in \{1, 2\}$ is an edge of $G[B \cap E]$ for an arc $E = UV$ of $\mathbb{H}$ and $A_{3-i} = B \cap W$ holds for a vertex $W$. By Equation (1), $W \notin \{U, V\}$. We first show

$$N(Y, X) = N(Y, E) \cup W. \tag{9}$$

Assume a node

$$v \in (N(Y, X) \setminus (E \cup W)) \cup (W \setminus N(Y, X)) \tag{10}$$

for contradiction. Let $B_v = B(v)$. We have

$$|N(Y, B_v) \cap W| \geq 2. \tag{11}$$



doi:10.6342/NTU201803034

By $W \notin \{U, V\}$, we have $v \notin E$, implying

$$|N(Y, B_v) \cap E| = 2 \tag{12}$$

By Equation (11), $N(Y, B_v)$ intersects at least two incident arcs of $W$ in $\mathbb{H}$. Combining with Equation (12), $N(Y, B_v)$ is not $\mathbb{H}$-local. By Lemma 4.1 and $W \notin \{U, V\}$, Condition T3 of $Y$ holds for $B_v$ with $N(Y, B_v) = A'_1 \cup A'_2$. If $|N(Y, B_v) \cap W| \geq 3$, then Lemma 4.2(1) implies an index $j \in \{1, 2\}$ with $B_v \cap W = A'_j \subseteq N(Y, B_v)$, contradicting $v \in B_v$ with Equation (10). The equality of Equation (11) holds. By Equation (12) and Lemma 4.2(3), either

*Statement Z1:* $A'_j = B_v \cap U$ and $A'_{3-j} = B_v \cap V$ hold for an index $j \in \{1, 2\}$ or
*Statement Z2:* $A'_j \subseteq B_v \cap E$ holds for an index $j \in \{1, 2\}$.

By $W \notin \{U, V\}$, each $A'_j$ with $j \in \{1, 2\}$ does not consist of two adjacent nodes in $B_v \cap E_j$ for any incident arc $E_j$ of $W$. By Lemma 4.2(2), $A'_j = B_v \cap W_j$ holds with a vertex $W_j$ adjacent to $W$ in $\mathbb{H}$ for each index $j \in \{1, 2\}$. Since each set $W_j$ with $j \in \{1, 2\}$ intersects $N(Y)$, vertex $W_j$ has at least three incident arcs in $\mathbb{H}$, violating Statement Z2. By Statement Z1, $\{U, V\} = \{W_1, W_2\}$, implying $N(Y, B_v) = B_v \cap (U \cup V)$. Since sets $U$ and $V$ intersect $N(Y)$, there are nodes $u_1 \in N(Y, B_v \cap U) \setminus (E \cup W)$ and $u_2 \in N(Y, B_v \cap V) \setminus (E \cup W)$. By Equation (2), $B \cap \{u_1, u_2\} = \varnothing$. By $B_v = B(v)$ and Condition N6 of $\mathbb{H}$, the arc of $\mathbb{H}$ containing $v$ is the arc $E = UV$ containing $\{u_1, u_2\}$, contradicting $v \notin E$. Hence, Equation (9) holds. Since $Y$ with $|Y| = 1$ is not $\mathbb{H}$-expandable, $N(Y, F)$ is not secure in $\mathbb{H}$ by Equation (9). By Lemma 4.3 and Equation (1), $G[X \cup Y]$ has a sapling, contradiction. □

*Proof of Lemma 2.2 for* $|Y| = 1$. Assume no sapling in $G[X \cup Y]$ for contradiction. Since $\mathbb{H}$ is steady, $\mathbb{H}$ has no parallel arc and degree-2 vertex. By Conditions N3 and N5 of $\mathbb{H}$, any vertex of $\mathbb{H}$ intersecting $N(Y)$ has degree at least 3. Lemma 4.4 implies an arc $E = V_1V_2$ of $\mathbb{H}$ with

$$N(Y, X) \subseteq E \cup V_1 \cup V_2. \tag{13}$$

We first show the following condition for any $i \in \{1, 2\}$ and non-$\mathbb{H}$-local $\mathbb{H}$-twig $B$:

$$((B \cap V_i) \setminus E) \cap N(Y) = \varnothing \quad \text{or} \quad (B \cap V_i) \setminus E \subseteq N(Y). \tag{14}$$

By Lemma 4.1, one of Conditions T of $Y$ holds for $B$.

(a) Condition T1 states $N(Y, B) \setminus F = (B \cap U) \setminus F$ for a vertex $U$ and an incident arc $F$ of $U$. Since $N(Y, B)$ is non-$\mathbb{H}$-local, there are nodes $u \in N(Y, B \cap U) \setminus F$ and $v \in N(Y, B \cap F) \setminus U$. If $E \neq F$, then $v \in N(Y, B \cap F) \setminus (U \cup E)$. By Equation (13), $v \in V_i$ holds for an $i \in \{1, 2\}$, implying $F = UV_i$ by Condition N6 of $\mathbb{H}$. By Equation (13), $u \in V_{3-i}$, implying $U = V_{3-i}$, contradicting that $\mathbb{H}$ has no parallel arcs. Equation (14) follows from $E = F$.

(b) Condition T2 states $N(Y, B) \setminus F = (B \cap (U_1 \cup U_2)) \setminus F$ for an arc $F$ with end-vertices $U_1$ and $U_2$. Since $N(Y, B)$ is non-$\mathbb{H}$-local, there are nodes $u_i \in N(Y, B \cap (F \cup U_i)) \setminus U_{3-i}$ for both $i \in \{1, 2\}$. If $E \neq F$, then $u_i \in N(Y, B \cap (F \cup U_i)) \setminus (U_{3-i} \cup E)$ holds for an $i \in \{1, 2\}$. By Equation (13), $u_i \in V_j$ holds for a $j \in \{1, 2\}$. By Condition N6 of $\mathbb{H}$, we have $U_i = V_j$. By Equation (13), $u_{3-i} \in V_{3-j}$, implying $U_{3-i} = V_{3-j}$, contradicting that $\mathbb{H}$ has no parallel arcs. Equation (14) follows from $E = F$.

(c) Condition T3 states $N(Y, B) = A_1 \cup A_2$ such that (i) $A_i = B \cap U_i$ for a vertex $U_i$ or (ii) $A_i$ consists of two adjacent nodes of $G$ in $B \cap E_i$ for an arc $E_i$. For Case (i), since $N(Y)$ intersects each node set $U_i$, the degree of each vertex $U_i$ is at least three in $\mathbb{H}$. By Equation (13), $U_i \in$



doi:10.6342/NTU201803034

$\{V_1, V_2\}$. For Case (ii), if $E_i \neq E$, then Equation (13) would imply $|N(Y, B) \cap E_i| \leq 1$. Thus, $E_i = E$. Since $A_1 \cup A_2$ equals $B \cap (V_1 \cup V_2)$, $B \cap (V_1 \cup E)$, or $B \cap (V_2 \cup E)$, Equation (14) holds.

Since Equation (13) implies Condition A1 with $C = E$, there is an index $i \in \{1, 2\}$ with nodes

$$u \in N(Y, V_i) \setminus E \quad (15)$$
$$v \in V_i \setminus (N(Y) \cup E) \quad (16)$$

or else $E$ would be an abode of $Y$ in $\mathbb{H}$. Since $Y$ is $\mathbb{H}$-bad, $N(Y, X)$ is non-$\mathbb{H}$-local, implying a non-$\mathbb{H}$-local $\mathbb{H}$-twig $B$. Let $B_u = B(u)$. Since $B$ is non-$\mathbb{H}$-local, so is $B_u$. By $u \in V(P_u) \subseteq B_u$ and Equation (15), we have $u \in ((B_u \cap V_i) \setminus E) \cap N(Y)$. By Equation (14),

$$(B_u \cap V_i) \setminus E \subseteq N(Y). \quad (17)$$

Let $B_v = B_u(v)$. By Equation (15), the degree of $V_i$ in $\mathbb{H}$ is at least three. By Equation (17),

$$((B_v \cap V_i) \setminus E) \cap N(Y) \neq \varnothing. \quad (18)$$

Thus, $B_v$ is not $\mathbb{H}$-local. By Equations (14) and (18), $(B_v \cap V_i) \setminus E \subseteq N(Y)$, contradicting Equation (16). □



doi:10.6342/NTU201803034

## 5 Proving Lemma 2.3

**Lemma 5.1.** *Given an $X$-net $\mathbb{H}$, it takes $O(m)$ time to compute an $X$-net aiding $\mathbb{H}$.*

**Lemma 5.2.** *Let $Y$ be an $\mathbb{H}'$-bad set for an $X$-net $\mathbb{H}'$ aiding an $X$-web $\mathbb{H}$. It takes $O(m)$ time to either (1) obtain a minimal abode of $Y$ in $\mathbb{H}$ or (2) ensure that $Y$ admits no abode in $\mathbb{H}$.*

**Lemma 5.3.** *Let $Y$ be an $\mathbb{H}'$-bad set for an $X$-net $\mathbb{H}'$ aiding an $X$-web $\mathbb{H}$.*

- *It takes $O(m)$ time to determine whether $Y$ is $\mathbb{H}$-expandable.*
- *If $Y$ is $\mathbb{H}$-expandable or a minimal abode in $\mathbb{H}$ of $Y$ is given, then it takes $O(m)$ time to update $\mathbb{H}$ into an $X \cup Y$-web.*

*Proving Lemma 2.3.* We first show that it takes $O(m)$ time to either ensure a sapling of $G$ or obtain an $X$-web $\mathbb{H}$. Let $s_1$, $s_2$, and $s_3$ be the leaves of $G$. It takes $O(m)$ time to obtain a node set $S$ such that $G[S]$ is a shortest $s_2s_3$-path of $G$ and a node set $R$ such that $G[R]$ is a shortest $s_1S$-path of $G$. Let $x_1$ be the node in $R \setminus S$ that is closest to $S$ in path $G[R]$. Let $x_2$ (respectively, $x_3$) be the node in $N(x_1) \cap S$ that is closest to $s_2$ (respectively, $s_3$) in path $G[S]$. Since $s_2$ and $s_3$ are leaves of $G$, $x_2$ and $x_3$ are internal nodes of path $G[S]$. If $x_2 = x_3$, then $G[R \cup S]$ is a sapling of $G$. If $x_2$ and $x_3$ are distinct and non-adjacent, then $G[R \cup S] - I$ is a sapling of $G$, where $I$ consists of the internal nodes of the $x_2x_3$-path in $G[S]$. If $x_2$ and $x_3$ are adjacent in $G$, then there is an $O(m)$-time obtainable $X$-web $\mathbb{H}$ with $X = R \cup S$: Let vertex $V_0 = \{x_1, x_2, x_3\}$ and vertex $V_i = \{s_i\}$ and each arc $E_i = V_0V_i$ with $i \in \{1, 2, 3\}$ consists of the nodes of the $s_ix_i$-rung of $G[X]$. Conditions N and W hold for $\mathbb{H}$.

The lemma follows from repeating the following steps in $O(n)$ iterations:

1. Apply Lemma 5.1 to obtain an $X$-net $\mathbb{H}'$ aiding $\mathbb{H}$ in $O(m)$ time.
2. Spend $O(m)$ time to either ensure that $\mathbb{H}'$ completes Task 2 or obtain an $\mathbb{H}'$-bad set $Y$.
3. Apply Lemmas 5.2 and 5.3 to either complete Task 3 or update $\mathbb{H}$ into an $X \cup Y$-web in $O(m)$ time.

□



doi:10.6342/NTU201803034

### 5.1 Proving Lemma 5.1

An *SPQR-tree* for a biconnected multiple graph $B$ having no self-loops is a linear-time obtainable [38] unique tree $\mathbb{T}$ on graphs that are homeomorphic to subgraphs of $B$ (see, e.g., [28, Lemma 3]) to represent the triconnected components of $B$. Specifically, there is a supergraph $C$ of $B$ with $V(C) = V(B)$ satisfying the following statements, where the edges in $B$ (respectively, $C \setminus B$) are called *actual* (respectively, *virtual*) edges:

- Each vertex of $\mathbb{T}$ is a subgraph of $C$ in one of the following types:
  - *S-vertex*: a simple cycle on three or more nodes. S stands for series.
  - *P-vertex*: three or more parallel edges. P stands for parallel.
  - *Q-vertex*: two parallel edges. Q-vertex simplifies the definitions of other vertices.
  - *R-vertex*: a triconnected simple graph that is not a cycle. R stands for rigid.
- No two S-vertices are adjacent in $\mathbb{T}$ and no two P-vertices are adjacent in $\mathbb{T}$.
- The vertices of $\mathbb{T}$ induce a disjoint partition of the actual edges.
- The end-nodes of each virtual edge form a two-node cutset of $B$.
- Each virtual edge is contained by exactly two vertices that are adjacent in $\mathbb{T}$.

**Lemma 5.4** (Di Battista and Tamassia [28]). *Let $B$ be an $n$-node biconnected multiple graph.*

1. *If two distinct nodes admitting three internally disjoint paths between them in $B$, then the two nodes are contained by either a P-vertex or an R-vertex of the SPQR-tree of $B$.*
2. *It takes $O(m)$ time to compute an SPQR-tree of $B$.*

Throughout the section, each $X$-web $\mathbb{H}$ for $G$ is equipped with the SPQR-tree $\mathbb{T}$ for the biconnected graph $\mathbb{H}^*$ obtained from $\mathbb{H}$ by adding three arcs on the three leaf vertices of $\mathbb{H}$ to form a triangle as ensured by Condition N1 of $\mathbb{H}$. Since $\mathbb{H}$ is connected, there are three internally disjoint paths in $\mathbb{H}^*$ between each pair of leaf vertices of $\mathbb{H}$. Lemma 5.4(1) implies a unique R-vertex of $\mathbb{T}$ that contains the leaf vertices of $\mathbb{H}$. Let $\mathbb{T}$ be rooted at this R-vertex. When we obtain an $X$-web $\mathbb{H}$ or update an $X$-web $\mathbb{H}$ to an $X \cup Y$-web, we always obtain or update the corresponding $\mathbb{T}$ of $\mathbb{H}$ unless explicitly specified otherwise.

Let $\mathbb{H}$ be an $X$-web. For each vertex $t$ of $\mathbb{T}$, let $\mu(t)$ be the graph that $t$ represents. Let $\phi(t)$ be the actual arcs contained by a Q-vertex in the subtree of $\mathbb{T}$ rooted at $t$. Let $C(t)$ be the union of actual arcs contained by $\mu(t')$, for each vertex $t'$ in the subtree rooted at $t$ (note that $\mu(t')$ contains an actual arc if and only if $t'$ is a Q-vertex).

The SPQR-tree characterizes the structure of split-components in the following ways:

**Lemma 5.5.** *Let $B$ be a bi-connected graph. Let $T$ be a rooted SPQR-tree of $B$. For each non-root vertex $t$ of $T$, if the virtual arc between $t$ and the parent of $t$ is between $u$ and $v$, then $B[\{u, v\} \cup \Phi(t)]$ is the union of one or more split-components of $B$ of the split-pair $(u, v)$, where $\Phi(t)$ consists of the actual arcs contained by a Q-vertex in the subtree of $T$ rooted at $t$.*

*Proof.* Classic SPQR-tree property. □

Let *Merge*$(C)$ for a $V_1V_2$-block $C$ of an $X$-web $\mathbb{H}$ be the operation that

- first replaces all arcs of $\mathbb{H}$ intersecting $C$ by an arc $C = V_1V_2$ and
- then deletes the vertices whose incident arcs are all deleted.



doi:10.6342/NTU201803034

One can verify that the resulting $\mathbb{H}'$ remains an $X$-net: Since the arc $C = V_1V_2$ of $\mathbb{H}'$ replaces one or more split components for the split pair $(V_1, V_2)$ of $\mathbb{H}$, any cut-set of $\mathbb{H}'$ is also a cut-set of $\mathbb{H}$. Thus, Condition N1 holds for $\mathbb{H}'$. Condition N2 holds for $\mathbb{H}'$ trivially. Since the leaf vertices remain the same in $\mathbb{H}$ and $\mathbb{H}'$, Condition N3 holds for $\mathbb{H}'$. For each arc $E$ of $\mathbb{H}$ that intersects $C$, Lemma **??** implies a $V_1V_2$-rung $\mathbb{P}$ of $\mathbb{H}$ such that $E$ is an arc of $\mathbb{P}$ and each arc of $\mathbb{P}$ intersects $C$. Thus, if $x$ is a node of $E$, then Condition N4 of $\mathbb{H}$ implies that $x$ is contained by a $V_1V_2$-rung $P$ in the subgraph of $G$ induced by the arcs of $\mathbb{P}$. By Condition N6 of $\mathbb{H}$, $P$ is a $V_1V_2$-rung of $G[C]$. Condition N4 holds for $\mathbb{H}'$. Conditions N5 and N6 of $\mathbb{H}'$ follow from Conditions N5 and N6 of $\mathbb{H}$.

*Proof of Lemma 5.1.* Let $\mathbb{H}'$ be the $X$-net obtained by applying Merge$(C(t))$ to $\mathbb{H}$ for each child $t$ of the root of $\mathbb{T}$. By Lemma 5.5, each $C(t)$ is a block and $\mathbb{H}'$ is well-defined. By the observation, each arc $UV$ of $\mathbb{H}'$ is a $UV$-block of $\mathbb{H}$. Since the root of $\mathbb{T}$ is an R-vertex, $\mathbb{H}'$ is steady. Therefore $\mathbb{H}'$ aids $\mathbb{H}$. The running time is $O(m)$ since $\mathbb{T}$ can be computed in $O(m)$ time and the Merge operations take overall $O(n)$ time. □



doi:10.6342/NTU201803034

### 5.2 Proving Lemma 5.2

For each non-root vertex $t$ of $\mathbb{T}$ with parent $t'$, let $V_1$ and $V_2$ be the *poles* of $t$ if $V_1$ and $V_2$ are the end-vertices of the unique virtual arc contained by both $\mu(t)$ and $\mu(t')$. Let $\mathbb{C}(t) = \mathbb{H}[\{V_1, V_2\} \cup \phi(t)]$. By Lemma 5.5, each $\mathbb{C}(t)$ is a split arc set of $\mathbb{H}$:

> Let $\mathbb{H}^*$ be the biconnected graph obtained from $\mathbb{H}$ by adding three arcs on the three leaf vertices of $\mathbb{H}$ to form a triangle. With $B = \mathbb{H}^*$ and $T = \mathbb{T}$, $\mathbb{C}(t) = \mathbb{H}^*[\{V_1, V_2\} \cup \phi(t)]$. By the choice of the root of $\mathbb{T}$, $\mathbb{C}(t)$ does not contain any arc in $\mathbb{H}^* \setminus \mathbb{H}$.

$\mathbb{T}$ characterize the local minimality (maximality) of $\mathbb{C}(t)$ very well:

**Lemma 5.6.** *Let $\mathbb{H}$ be an $X$-web. Let $t$ be a non-root vertex of $\mathbb{T}$ with poles $V_1$, $V_2$ and children $t_1, \ldots, t_k$. The following holds:*

1. *If $t$ is a P-vertex, then each $\mathbb{C}(t_i)$ with $1 \leq i \leq k$ is a split-component for $(V_1, V_2)$.*
2. *If $t$ is an R-vertex, then $\mathbb{C}$ is a maximal proper split arc set of $\mathbb{C}(t)$ in $\mathbb{H}$ if and only if $\mathbb{C} = \mathbb{C}(t_i)$ for an $1 \leq i \leq k$.*
3. *Let $t$ be an S-vertex such that $\mu(t)$ is a cycle $U_1U_2 \ldots U_{k+1}$ with*
   - *$U_1 = V_1$, $U_{k+1} = V_2$, and*
   - *the virtual arc $U_iU_{i+1}$ is contained by $\mu(t_i)$ for each $1 \leq i \leq k$.*

   *Let $\mathbb{C}_i = \mathbb{C}(t_1) \cup \ldots \cup \mathbb{C}(t_i)$. For each $1 < i \leq k$, $\mathbb{C}_i$ is a split-component of $(U_1, U_{i+1})$. Further, if $\mathbb{C}$ is a minimal split arc set satisfying $\mathbb{C}_i \subsetneq \mathbb{C} \subseteq \mathbb{C}(t)$, then $\mathbb{C} = \mathbb{C}_{i+1}$.*

*Proof.* Suppose that $t$ is a P-vertex. Since $\mu(t)$ consists of only parallel virtual arcs between $V_1$ and $V_2$, the poles of $t_i$ are $V_1$, $V_2$ for each $1 \leq i \leq k$. Hence each $\mathbb{C}(t_i)$ is a split arc set of $(V_1, V_2)$. Since no two P-vertices are adjacent in $\mathbb{T}$, each $\mathbb{C}(t_i)$ is a split-component of $(V_1, V_2)$.

> If $t_i$ is a Q-vertex, then $\mathbb{C}(t_i)$ is an arc of $\mathbb{H}$ between $V_1$ and $V_2$.Hence $t_i$ is either an S-vertex or R-vertex. By Lemma 5.5, $\mathbb{C}(t_i) \setminus \{V_1, V_2\}$ is connected. Therefore $\mathbb{C}(t_i)$ cannot be the union of two or more split-components of $(V_1, V_2)$.

Suppose that $t$ is an R-vertex. By Lemma 5.5, each $\mathbb{C}(t_i)$ with $1 \leq i \leq k$ is a proper split arc set of $\mathbb{C}(t)$. Since $\mu(t)$ is 3-connected, $\mathbb{C}(t_i)$ is maximal:

> Assume for contradiction a split arc set $\mathbb{C}$ of $(U_1, U_2)$ with $\mathbb{C}(t_i) \subsetneq \mathbb{C} \subsetneq \mathbb{C}(t)$, implying an index $j \neq i$ with $\mathbb{C}(t_j) \cap \mathbb{C} \neq \varnothing$. Let $E$ be an arc in $\mathbb{C}(t_j)$. $(U_1, U_2)$ is a split-pair of $\mathbb{C}(t)$ that does not separate $\mathbb{C}(t_i)$ from $E$. Since $\mu(t)$ is 3-connected, such $(U_1, U_2)$ cannot exist.

Conversely, let $\mathbb{C}$ be a maximal proper split arc set of $\mathbb{C}(t)$. There is an index $i$ such that $\mathbb{C}$ intersects $\mathbb{C}(t_i)$. By a similar argument, $\mathbb{C}$ does not intersect $\mathbb{C}(t_j)$ for all $i \neq j$. Hence $\mathbb{C} \subseteq \mathbb{C}(t_i)$ and $\mathbb{C} = \mathbb{C}(t_i)$.

Suppose that $t$ is an S-vertex. By Lemma 5.5, each $\mathbb{C}(t_i)$ with $1 \leq i \leq k$ is a split arc set of $(U_i, U_{i+1})$. Each $\mathbb{C}_i$ with $i > 1$ is a split-component of $(U_1, U_{i+1})$:

> Assume for contradiction that $\mathbb{C}_i$ is not a split-component, implying at least two disjoint split-components $\mathbb{C}'_1$, $\mathbb{C}'_2$ of $(U_1, U_{i+1})$ such that $\mathbb{C}'_1 \cup \mathbb{C}'_2 \subseteq \mathbb{C}_i$. By Lemma 5.5, $U_1$ and $U_{i+1}$ are non-adjacent in $\mathbb{C}_i$. Hence $\{U_1, U_{i+1}\}$ is a split-pair of $\mathbb{C}_i$ that separates $\mathbb{C}'_1$ and $\mathbb{C}'_2$. But by Lemma 5.5, $\mathbb{C}_i \setminus \{U_1, U_{i+1}\}$ is connected, a contradiction.



doi:10.6342/NTU201803034

Let $\mathbb{C}$ be a minimal split arc set with $\mathbb{C}_i \nsubseteq \mathbb{C} \subseteq \mathbb{C}(t)$. Since no two S-vertices can be adjacent in $\mathbb{T}$, $\mathbb{C} = \mathbb{C}_{i+1}$:

> Assume for contradiction $\mathbb{C} \neq \mathbb{C}_{i+1}$. By Lemma 5.5, $\mathbb{C}_{i+1}$ is a split arc set with $\mathbb{C}_i \nsubseteq \mathbb{C}_{i+1} \subseteq \mathbb{C}(t)$. Hence $\mathbb{C}_i \nsubseteq \mathbb{C} \nsubseteq \mathbb{C}_{i+1}$. Let $(U, U_1)$ be the split-pair of $\mathbb{C}$. Since $\mathbb{C}_i$ is a split-component, $\mathbb{C} \setminus \mathbb{C}_i \subsetneq \mathbb{C}(t_{i+1})$ is a split arc set of $(U, U_i)$. Hence $U$ is a cut-vertex of $\mathbb{C}(t_{i+1})$. By Lemma 5.5, $t_{i+1}$ is an S-vertex, a contradiction.

□

By Lemma 5.6, we have

**Lemma 5.7.** *Let $\mathbb{H}$ be an $X$-web. Let $t$ be a non-root vertex of $\mathbb{T}$ with poles $V_1$, $V_2$ and children $t_1, \ldots, t_k$. $C(t)$ is a $V_1V_2$-block and the following holds:*

1. *If $t$ is a P-vertex, then each $C(t_i)$ with $1 \le i \le k$ is a minimal $V_1V_2$-subblock of $C(t)$.*
2. *If $t$ is an R-vertex, then $C$ is a maximal proper subblock of $C(t)$ in $\mathbb{H}$ if and only if $C = C(t_i)$ for an $1 \le i \le k$.*
3. *Let $t$ be an S-vertex such that $\mu(t)$ is a cycle $U_1U_2 \ldots U_{k+1}$ with*
   - *$U_1 = V_1$, $U_{k+1} = V_2$, and*
   - *the virtual arc $U_iU_{i+1}$ is contained by $\mu(t_i)$ for each $1 \le i \le k$.*

   *Let $C_i = C(t_1) \cup \ldots \cup C(t_i)$. For each $1 < i \le k$, $C_i$ is a $(U_1, U_{i+1})$-block. Further, if $C$ is a minimal block satisfying $C_i \subsetneq C \subseteq C(t)$, then $C = C_{i+1}$.*

**Lemma 5.8.** *Let $Y$ be an $\mathbb{H}'$-bad set for an $X$-net $\mathbb{H}'$ aiding an $X$-web $\mathbb{H}$. $Y$ admits an abode in $\mathbb{H}$ if and only if $Y$ admits an abode in $\mathbb{H}'$.*

*Proof.* For the if-part, suppose that $Y$ admits an abode $C$ in $\mathbb{H}'$. Since $\mathbb{H}'$ is steady, $C$ is an arc $UV$ of $\mathbb{H}'$. Since $\mathbb{H}'$ aids $\mathbb{H}$, $C$ is a $UV$-block of $\mathbb{H}$. Conditions A1, A2, and A3 for $Y$ and $C$ in $\mathbb{H}'$ implies Conditions A1, A2, and A3 for $Y$ and $C$ in $\mathbb{H}$. Therefore $C$ is an abode of $Y$ in $\mathbb{H}$.

For the only-if part, suppose that $Y$ admits an abode $C$ in $\mathbb{H}$. By Lemma 5.5, $C \subseteq E$ for an arc $E$ of $\mathbb{H}'$. Let $E = UV$. $E$ is a $UV$-block of $\mathbb{H}'$ that satisfies Conditions A1, A2, and A3 for $Y$. Hence $E$ is an abode of $Y$ in $\mathbb{H}'$. □

Recall that an *abode* of $Y$ in $\mathbb{H}$ is a $V_1V_2$-block $C$ of $\mathbb{H}$ satisfying the following properties:

- $N(Y, X) \subseteq C \cup V_1 \cup V_2$.
- $N(Y, V_1) \subseteq C$ or $V_1 \subseteq N(y) \cup C$ holds for an end-node $y$ of path $G[Y]$.
- $N(Y, V_2) \subseteq C$ or $V_2 \subseteq N(y) \cup C$ holds for an end-node $y$ of path $G[Y]$.

We say that $V_i$ is *full* if $N(Y, V_i) \nsubseteq C$. If $V_i$ is full and $V_1 \subseteq N(y) \cup C$, then we say *$y$ occupies $V_i$*.

**Lemma 5.9.** *Let $Y$ be an $\mathbb{H}'$-bad set for an $X$-net $\mathbb{H}'$ aiding an $X$-web $\mathbb{H}$. Let $t$ be a vertex in $\mathbb{T}$ with poles $V_1$, $V_2$. If $C(t)$ is an abode of $Y$ in $\mathbb{H}$ and $V_1$, $V_2$ are both full, then it takes $O(m)$ time to find a minimal abode of $Y$ in $\mathbb{H}$.*

*Proof.* $C(t)$ is a $V_1V_2$-block. Let $M \subseteq C(t)$ be a minimal abode of $Y$ in $\mathbb{H}$. Since both $V_1$ and $V_2$ are full, $M$ is a $V_1V_2$-block. By Lemma 5.7, if $t$ is not a P-vertex, then $M = C(t)$. Suppose that $t$ is a P-vertex with children $t_1, \ldots, t_k$. Let $j \in \{1, 2\}$ be the index such that $y_j$ occupies $V_1$ and



doi:10.6342/NTU201803034

$y_{3-j}$ occupies $V_2$. By Conditions A1, A2, A3, and Lemma 5.7, for each $1 \le i \le k$, $C(t_i) \subseteq M$ unless

$$\begin{aligned} N(y_j) \cap C(t_i) &= V_1 \cap C(t_i) \\ N(y_{3-j}) \cap C(t_i) &= V_2 \cap C(t_i) \quad \text{and} \\ N(Y \setminus \{y_1, y_2\}) \cap C(t_i) &= \varnothing \end{aligned} \tag{1}$$

holds. Hence $M$ is the union of each $C(t_i)$ that does not satisfy Equation (1) and it takes $O(m)$ to compute $M$. □

*Proof of Lemma 5.2.* If $Y$ admits an abode $C$ in $\mathbb{H}'$, then since $\mathbb{H}'$ is steady and $Y$ is non-$\mathbb{H}$-local, $C$ equals an arc $E = V_1V_2$ of $\mathbb{H}'$, and at least one of $V_1$, $V_2$ is full. It is easy to find $E$ or ensure that no such $E$ exists in $O(m)$ time. By Lemma 5.8 and Lemma 5.1, Task (2) can be completed in $O(m)$ time. The remaining proof deal with Task (1).

Let $N = N(Y, X)$. Let $y_1$ and $y_2$ be the end-nodes of path $G[Y]$. Suppose that $Y$ admits an abode in $\mathbb{H}$. Let $E = V_1V_2$ be an arc of $\mathbb{H}'$ such that $E$ is the abode of $Y$ in $\mathbb{H}'$. Let $M$ be a minimal abode of $Y$ in $\mathbb{H}$. By Lemma 5.7, $E = C(t)$ for a child $t$ of $r$. If $V_1$ and $V_2$ are both full, then it takes $O(m)$ time to compute $M$ by Lemma 5.9. Hence we can assume an index $i \in \{1, 2\}$ such that $V_i$ is full and $V_{3-i}$ is not full. Let $j \in \{1, 2\}$ be the index such that $y_j$ occupies $V_i$. Let $N' = N(Y \setminus \{y_j\}) \cup (N(y_j) \setminus V_i) \cup (V_i \setminus N(y_j))$. Let $t_0$ be the vertex in $\mathbb{T}$ with

- $N' \subseteq C(t_0)$,
- $V_i$ is a pole of $t_0$, and
- $C(t_0)$ is minimal.

It takes $O(m)$ time to find $t_0$ since $|\mathbb{T}| = O(n)$. If $t_0$ is a Q-vertex, then $C(t_0) = M$ and we are done. Suppose that $t_0$ is not a Q-vertex, implying $t_0$ non-leaf. Let $t_1, \ldots, t_k$ be the children of $t_0$.

Case 1: $t_0$ is an R-vertex.

By Lemma 5.7, either

- $M \subseteq C(t_a)$ holds for an index $1 \le a \le k$, or
- $M = C(t)$

holds. It takes $O(m)$ time to to either compute $a$ or ensure $M = C(t)$ by brute force since $k \le n$. If $M = C(t)$ then we are done. Hence we can assume $M \subseteq C(t_a)$. Since $y_j$ occupies $V_i$, $V_i$ is a pole of $t_a$. Let $V$ be the other pole of $t_a$. If $V$ is full then it takes $O(m)$ to compute $M$ by Lemma 5.9. If $V$ is not full, then $t_a$ is a vertex in $\mathbb{T}$ with

- $N' \subseteq C(t_a)$,
- $V_i$ is a pole of $t_a$, and
- $C(t_a) \subsetneq C(t_0)$

, a contradiction to the choice of $t_0$.

Case 2: $t_0$ is an S-vertex.

We can assume that $\mu(t)$ is a cycle $U_1U_2 \ldots U_{k+1}$ such that $U_1 = V_i$ and the virtual arc $U_aU_{a+1}$ is contained by $\mu(t_a)$ for each $1 \le a \le k$. Let $m$ be the largest index with $C(t_m) \cap N' \ne \varnothing$. $m > 1$



doi:10.6342/NTU201803034

since otherwise $t_1$ will be a vertex of $\mathbb{T}$ that contradicts the choice of $t_0$. By Lemma 5.7, one can verify that if

$$N(y_{3-j}) \cap C(t_m) = C(t_m) \cap U_m \quad \text{and} \quad N(Y \setminus \{y_{3-j}\}) \cap C(t_m) = \varnothing$$

, then $M = C(t_1) \cup \ldots \cup C(t_{m-1})$. Otherwise $M = C(t_1) \cup \ldots \cup C(t_m)$. Hence it takes $O(m)$ time to determine $M$.

Case 3: $t_0$ is a P-vertex.
Let $I$ consists of the indices $1 \leq a \leq k$ with $C(t_a) \cap N' \neq \varnothing$. $|I| > 1$ by the choice of $t_0$. By Lemma 5.7, one can verify that $M$ is the union of $C(t_a)$ for each $a \in I$. Hence it takes $O(m)$ time to compute $I$ and hence $M$.

□



doi:10.6342/NTU201803034

## 6 Proving Lemma 5.3

Define the following operations on an $X$-net $\mathbb{H}$:

- *Subdivide*$(V, x_1, x_2)$ for a new vertex $V$ of $\mathbb{H}$ and an edge $x_1x_2$ of $G[E]$ in a slim arc $E$ of $\mathbb{H}$: Let $V = \{x_1, x_2\}$. Suppose that $E = V_1V_2$. Replace the arc $E$ of $\mathbb{H}$ by new arcs $E_i = VV_i$ with $i \in \{1, 2\}$ consisting of the nodes of the minimal $VV_i$-path of $G[E]$.
- *JoinAdd*$(Y, E)$ for an arc $E = V_1V_2$ of $\mathbb{H}$ and an $\mathbb{H}$-bad set $Y$: For each $i \in \{1, 2\}$ and each endpoint $y$ of $G[Y]$, if $y$ has a neighbour in $V_i \setminus E$ then put $y$ into $V_i$. Put $Y$ into $E$.
- *JoinNew*$(Y, V_1, V_2)$ for distinct vertices $V_1$, $V_2$ of $\mathbb{H}$ and an $\mathbb{H}$-bad set $Y$: Add a new arc $E = V_1V_2$. For each $i \in \{1, 2\}$ and each endpoint $y$ of $G[Y]$, if $y$ has a neighbour in $V_i \setminus V_{3-i}$, then put $y$ into $V_i$. Put $Y$ into $E$.

Let $\mathbb{H}$ be an $X$-web. Recall that an induced path $G[Y] = y_1 \ldots y_2$ of $G - X$ is $\mathbb{H}$-*expandable* if the following holds:

1. If $|Y| = 1$ then $N(Y) \cap X$ is the union of two $\mathbb{H}$-secure sets. Otherwise each $N(y_i) \cap X$ for $i \in \{1, 2\}$ is an $\mathbb{H}$-secure set.
2. $N(y, X) = \varnothing$ for each internal node $y$ of $G[Y]$

Let $\mathbb{H}'$ be an $X$-net aiding $X$-web $\mathbb{H}$. Let $Y = y_1 \ldots y_2$ be an $\mathbb{H}'$-bad $\mathbb{H}$-expandable set. If $|Y| > 1$ then let each $S_i = N(y_i) \cap X$ for $i \in \{1, 2\}$. Otherwise let each $S_i$ for $i \in \{1, 2\}$ be an $\mathbb{H}$-secure set such that $N(Y) \cap X = S_1 \cup S_2$. Define $\mathbb{H} + Y$ according to the types of $S_1$ and $S_2$ as follows:

*Case 1:* $S_1 = V_1$ and $S_2 = V_2$ for vertices $V_1$ and $V_2$ of $\mathbb{H}$. Apply JoinNew$(Y, V_1, V_2)$. If $|Y| = 1$ and there is a trivial or risky arc $F$ between $V_1$ and $V_2$, then apply Merge$(E, Y \cup F)$.

*Case 2:* $S_1 = \{x_1, x_2\}$ and $S_2 = V$ for a vertex $V$ of $\mathbb{H}$ and adjacent nodes $x_1, x_2$ contained by a slim arc of $\mathbb{H}$. Apply Subdivide$(V', x_1, x_2)$ and JoinNew$(Y, V, V')$ If $|Y| = 1$ and there is a trivial or risky $F$ between $V$ and $V'$, then apply Merge$(E, Y \cup F)$.

*Case 3:* for each $i \in \{1, 2\}$, $S_i = \{x_i, x'_i\}$ for adjacent nodes $x_i$, $x'_i$ contained by a slim arc of $\mathbb{H}$. Apply Subdivide$(V_1, x_1, x'_1)$, Subdivide$(V_2, x_2, x'_2)$, and JoinNew$(Y, V_1, V_2)$.

Note that $\mathbb{H} + Y$ is unique up to $\mathbb{H}$ (since $S_1$ and $S_2$ satisfy exactly one of the above conditions).

**Lemma 6.1.** *$\mathbb{H} + Y$ is an $X \cup Y$-web. If the $X \cup Y$-net $(\mathbb{H} + Y)'$ aiding $\mathbb{H} + Y$ is not isomorphic to the $X$-net $\mathbb{H}'$ aiding $\mathbb{H}$, then $Y$ is an arc of $(\mathbb{H} + Y)'$.*

*Proof.* The Subdivide operation preserves Conditions N1-N6. Hence $\mathbb{H} + Y$ is an $X \cup Y$-net after the Subdivide operations in Case 2 and Case 3. Since $Y$ is $\mathbb{H}$-expandable, $\mathbb{H} + Y$ is an $X \cup Y$-net after the JoinNew operation. Since Subdivide preserves Condition W1 and $\mathbb{H}$ satisfies Condition W, all the arcs of $\mathbb{H} + Y$ are either slim or risky (one of them is the new arc $Y$). Condition W can be violated if and only if $Y$ is trivial and there is a parallel trivial or risky arc $F$ between the endpoints of $Y$. In this case $F$ is unique by Condition W1 of $\mathbb{H}$ and the Merge$(E, Y \cup F)$ operation justifies (as Lemma **??** guaranteed).

$(\mathbb{H} + Y)'$ and $\mathbb{H}'$ are unique by Lemma 5.1. Suppose that $(\mathbb{H} + Y)'$ is not isomorphic to $\mathbb{H}'$. If $Y$ admits an abode in $\mathbb{H}$, then $(\mathbb{H} + Y)'$ is isomorphic to $\mathbb{H}'$:

> Let $C$ be an abode of $Y$ in $\mathbb{H}$. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. By Lemmas **??** and 5.1, there is an arc $E = U_1U_2$ of $\mathbb{H}'$ that contains $C$. $E$ is a



doi:10.6342/NTU201803034

$U_1U_2$-block of $\mathbb{H}$ with $C \subseteq E'$. Notice that if an arc $E'$ of $\mathbb{H}$ is subdivided, then $E' \in E(\mathbb{C})$. Hence $Y$ is contained by an arc of $\mathbb{H} + Y$ between two vertices of $\mathbb{C}'$, where $\mathbb{C}'$ is the subgraph of $\mathbb{H} + Y$ corresponding to $\mathbb{C}$ with possibly one or two arcs subdivided. This shows that $Y$ is contained by an arc $E \cup Y$ of $(\mathbb{H} + Y)'$ that is a $U_1U_2$-block of $\mathbb{H} + Y$. Since the other part $(\mathbb{H} \setminus \mathbb{C})$ of $\mathbb{H}$ is not modified, $\mathbb{H}'$ is isomorphic to $(\mathbb{H} + Y)'$

Hence we can assume that $Y$ does not admit an abode in $\mathbb{H}$. Assume for contradiction that $Y$ is not an arc of $(\mathbb{H} + Y)'$. By definition of $\mathbb{H} + Y$, there is an arc $E$ of $(\mathbb{H} + Y)'$ that contains $Y$. $E \setminus Y$ is an abode for $Y$ in $\mathbb{H}$

Let $E = U_1U_2$. By Lemma 5.1, $E \setminus Y$ is an $U_1U_2$-block of $\mathbb{H}$. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $E \setminus Y$. Let $E_Y$ be the arc of $\mathbb{H} + Y$ that contains $Y$. By definition of $\mathbb{H} + Y$, both end-vertices of $E_Y$ are either a vertex of $\mathbb{C}$ or a vertex from subdividing an arc of $\mathbb{C}$. Hence $E \setminus Y$ is an abode for $Y$ in $\mathbb{H}$.

, a contradiction. □

Suppose that $Y$ is a non-$\mathbb{H}$-expandable $\mathbb{H}'$-bad set that admits an abode. Let $G[Y] = y_1 \ldots y_2$. Let $C$ be an $V_1V_2$-block that is a minimal abode of $Y$. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. By Lemma 5.2, we can assume that $y_1$ occupies $V_1$ and $V_1 \in V(\mathbb{H}')$. Define $\mathbb{H} +_{\mathbb{C}} Y$ as follows:

*Step 1:* if $V_2$ is a leaf vertex of $\mathbb{C}$ such that the incident arc $E = V_2V_2'$ of $V_2$ in $\mathbb{C}$ is slim and contains $N(Y) \cap V_2$, then perform the following steps:
Let $x_1$ be the node in $N(Y) \cap E$ closest to $V_2$. Let $x_2$ be the neighbor of $x_1$ in $G[E]$ that is not on the $x_1V_2$-path of $G[E]$. Subdivide$(U_2, x_1, x_2)$. Replace $E$ in $\mathbb{C}$ by the new edge $U_2V_2'$. Replace $V_2$ by $U_2$.
*Step 2:* if there is a risky arc $E \notin E(\mathbb{C})$ between (the new) $V_1$ and $V_2$, then apply Merge$(E', C \cup E)$.
*Step 3:* Apply JoinAdd$(Y, E')$.

**Lemma 6.2.** *$\mathbb{H} +_{\mathbb{C}} Y$ is an $X \cup Y$-web. Further, $(\mathbb{H} +_{\mathbb{C}} Y)'$ is isomorphic to $\mathbb{H}'$.*

*proving Lemma 5.3.* By Lemma 5.2, it takes $O(n + \deg(Y))$ time to compute a minimal abode $C$ if $Y$ is non-$\mathbb{H}$-expandable. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. We can update $\mathbb{H}$ to either $\mathbb{H} + Y$ or $\mathbb{H} +_{\mathbb{C}} Y$ in $O(n + \deg(Y))$ time as in Lemmas 6.1 and 6.2. It takes $O(n)$ time to update $\mathbb{T}(\mathbb{H})$ correspondingly: by Lemma 5.4, each Subdivide operation and JoinNew operation takes $O(n)$ time. The Merge operation also takes $O(n)$ time, since by Lemma **??** we are replacing subtrees whose roots share a parent $t$ of $\mathbb{T}(\mathbb{H})$ into a Q-vertex with parent $t$ and the fact that $|\mathbb{T}| = O(n)$. □

# 7 Proving Lemma 6.2

During this section we fix an $X$-web $\mathbb{H}$ of $G$ and a non-$\mathbb{H}$-expandable $\mathbb{H}'$-bad set $Y = y_1 \ldots y_2$ with a minimal abode $C$ that is a $V_1V_2$-block. By Lemma **??**, $C$ is unique. Let $\mathbb{C}$ be the split arc set of $\mathbb{H}$ corresponding to $C$. We can assume that $y_1$ occupies $V_1$ and $V_1 \in V(\mathbb{H}')$.

Recall that $\mathbb{H} +_{\mathbb{C}} Y$ is obtained from $\mathbb{H}$ as follows:



doi:10.6342/NTU201803034

*Step 1:* if $V_2$ is a leaf vertex of $\mathbb{C}$ such that the incident arc $E' = V_2V_2'$ of $V_2$ in $\mathbb{C}$ is slim and contains $N(Y) \cap V_2$, then perform the following steps:
Let $x_1$ be the node in $N(Y) \cap E'$ closest to $V_2$. Let $x_2$ be the neighbor of $x_1$ in $G[E']$ that is not on the $x_1V_2$-path of $G[E']$. Subdivide$(U_2, x_1, x_2)$. Replace $E'$ in $\mathbb{C}$ by the new arc $U_2V_2'$. Replace $V_2$ by $U_2$.

*Step 2:* if there is a risky arc $E' \notin E(\mathbb{C})$ between (the new) $V_1$ and $V_2$, then apply Merge$(E, C \cup E')$.

*Step 3:* Apply JoinAdd$(Y, E)$.

Let $E = U_2V_1$ be the arc of $\mathbb{H} +_{\mathbb{C}} Y$ that contains $Y$. We say that $U_2$ is *subdivided* if Step *(H1)* is executed.

**Lemma 7.1.** *$\mathbb{H} +_{\mathbb{C}} Y$ is an $X \cup Y$-net. $(\mathbb{H} +_{\mathbb{C}} Y)'$ is isomorphic to $\mathbb{H}'$. If $E$ is risky then $\mathbb{H} +_{\mathbb{C}} Y$ is an $X \cup Y$-web. There is an $E$-rung in $\mathbb{H} +_{\mathbb{C}} Y$ that contains $Y$.*

*Proof.* If Step *(H1)* is executed then $U_2$ is a new vertex of degree-two and Step *(H2)* will not be executed. Hence $\mathbb{H} +_{\mathbb{C}} Y$ is well-defined. It is easy to verify that:

- The Subdivide operation preserves N1-N6.
- The Merge operation preserves N1-N6 by Lemma **??**.
- The JoinAdd operation preserves N1, N2, N3, and N5.

We show that Step *(H3)* preserves N4 and N6, implying $\mathbb{H} +_{\mathbb{C}} Y$ an $X \cup Y$-net.

For N4, since $\mathbb{H}$ remains an $X$-net upon the end of Step *(H2)*, it suffices to show that there is an $E$-rung that contains $Y$. This also completes the last statement. If Step *(H1)* is executed or $y_2$ occupies $V_2$, then by definition of operation JoinAdd, $Y$ is an $E$-rung. Hence we can assume $N(y_2) \cap X \subseteq E$. Since $Y$ is non-$\mathbb{H}'$-local, $N(y_2) \nsubseteq V_1$. Since $\mathbb{H}$ remains an $X$-net upon the end of Step *(H2)*, by N4 there is an $E$-rung $P$ with $P \cap Y = \varnothing$ that contains a node in $N(y_2) \setminus V_1$. Let $P'$ a $U_2N(y_2)$-rung in $P$. $P' \cup Y$ is an $E$-rung that contains $Y$.

For N6, let $x_i$ be a node contained by a distinct arc $E_i$ in $\mathbb{H} +_{\mathbb{C}} Y$ for each $i \in \{1, 2\}$. If $\{x_1, x_2\} \subseteq X$ then N6 holds since $\mathbb{H}$ remains an $X$-net upon the end of Step *(H2)*. We can assume without loss of generality $x_1 \in Y$ and $E_1 = E$. Suppose $x_1 \in \text{in}(Y)$. By definition of operation JoinAdd, $x_1 \notin U_2 \cup V_1$. Since $Y$ is $\mathbb{H}'$-bad, $N(x_1) \cap X \subseteq E$. Hence N6 holds. Suppose $x_1 = y_1$. By definition of operation JoinAdd, $x_1 \in V_1$. Since $y_1$ occupies $V_1$, $V_1 \setminus E \subseteq N(y_1)$. If $y_1 \neq y_2$ then $N(y_1) \cap X \subseteq V_1$ and N6 holds. Hence we can assume $x_1 = y_1 = y_2$. If $y_2$ occupies $U_2$ then $N(y_1) \cap X \subseteq (V_1 \cup U_2 \cup E)$ and N6 holds. Otherwise $N(y_2) \cap X \subseteq E$ and N6 also holds. Suppose $x_1 = y_2$. We can assume $y_2 \neq y_1$ by the above cases. If $y_2$ occupies $U_2$ then $N(y_2) \cap X \subseteq U_2 \cup E$ and N6 holds. Otherwise $N(y_2) \cap X \subseteq E$ and N6 also holds.

By Lemma **??** and the fact that the Merge and Subdivide operations preserves $\mathbb{T}(\mathbb{H})$, $\mathbb{T}(\mathbb{H} +_{\mathbb{C}} Y) = \mathbb{T}(\mathbb{H})$. Hence $(\mathbb{H} +_{\mathbb{C}} Y)'$ is isomorphic to $\mathbb{H}'$.

It remains to show that $\mathbb{H} +_{\mathbb{C}} Y$ satisfies W1 and W2 when $E$ is risky. $E$ is non-trivial since $1 < |C \cup Y| \leq |E|$. Suppose that $E$ is risky. W1 holds since each arc of $\mathbb{H} +_{\mathbb{C}} Y$, except $E$ and possibly $U_2V_2$ when Step 1 is executed, is an arc of $\mathbb{H}$. For W2, let $V$ be a degree-two vertex of $\mathbb{H} +_{\mathbb{C}} Y$ with incident arcs $E_1$ and $E_2$. By the definition of $\mathbb{H} +_{\mathbb{C}} Y$, either $E_i = E$ holds for an $i \in \{1, 2\}$ or both $E_1$ and $E_2$ are arcs of $\mathbb{H}$. By W1 of $\mathbb{H}$ and that $E$ is risky, $E_1$ and $E_2$ are not both slim or trivial. Now suppose that $E_1$ and $E_2$ are parallel arcs of $\mathbb{H} +_{\mathbb{C}} Y$. If $E \notin \{E_1, E_2\}$, then by W2 of $\mathbb{H}$, $E_1$ and $E_2$ are not both risky. Hence we can assume without loss of generality $E_1 = E$. If Step *(H1)* is executed then $E$ has no parallel arc, a contradiction. Hence Step *(H1)* is not executed and $U_2 = V_2$. By W2 of $\mathbb{H}$, there is at most one risky arc, say



doi:10.6342/NTU201803034

$F$ of $\mathbb{H}$ between $U_2$ and $V_1$. By Step *(H2)*, $F \subseteq E$ in $\mathbb{H} +_{\mathbb{C}} Y$, implying $E_2$ non-risky. Hence $\mathbb{H} +_{\mathbb{C}} Y$ contains no parallel risky arcs. This completes W2. □

*Proving Lemma 6.2.* By Lemma 7.1, it suffices to show that $E$ is risky. Let non-empty $S \subseteq E$ be an arbitrary set. We show that $G[E]$ contains an $(S, U_2, V_1)$-sprout.

(0) If $y_2 \in U_2$ then $y_2$ occupies $U_2$ and one of the following holds:

0.1 $y_1 = y_2$,
0.2 $N(\text{in}(Y)) \cap E \neq \varnothing$,
0.3 $y_1$ has a non-neighbour in $E \cap V_1$,
0.4 $y_2$ has a non-neighbour in $E \cap U_2$, or
0.5 $y_2$ has a neighbour in $E \setminus U_2$

If all five conditions fail, then $Y$ is $\mathbb{H}$-expandable.

(1) if $S \subseteq Y$, then $G[E]$ contains a $(S, U_2, V_1)$-sprout.

Let $P$ be an $E$-rung that contains $Y$ as Lemma 7.1 guaranteed. Let $s_i$ be the node of $S$ closest to $y_i$ in $G[Y]$ for each $i \in \{1, 2\}$. Let $Y_i$ be the $s_i y_i$-rung in $G[Y]$. $s_1 \neq s_2$ since otherwise $P$ is an $(S, U_2, V_1)$-sprout. If $s_1$ is non-adjacent to $s_2$, then $Y_1 \cup Y_2 \cup (P \setminus Y)$ is an $(S, U_2, V_1)$-sprout. Hence we can assume $s_1 s_2 \in E(G)$, implying $S = \{s_1, s_2\}$ and $|Y| > 1$.

Suppose first $N(\text{in}(Y)) \cap E \neq \varnothing$. Let $y$ be a node of $\text{in}(Y)$ with $N(y) \cap E \neq \varnothing$ that is closest to $S$ in $G[Y]$. Let $P$ be an $E$-rung in $G[E \setminus Y]$ that intersects $N(y)$ (as N4 on $\mathbb{H}$ after Step 2 guaranteed). By $Y$ is $\mathbb{H}$-bad and $N(y_1) \cap X \subseteq V_1$, $N(y) \cap V(P) \subseteq V_1$. $P \cup Y'$ is a $(S, U_2, V_1)$-sprout, where $Y'$ is the $Sy$-rung in $G[Y]$. Therefore,

$$N(\text{in}(Y)) \cap E = \varnothing \tag{1}$$

Suppose that $y_1$ has a non-neighbour $v$ in $E \cap V_1$. Let $P$ be an $E$-rung in $G[E \setminus Y]$ that contains $v$. By Equation (1), $P \cup Y_1$ is a $(S, U_2, V_1)$-sprout. Hence we can assume

$$E \cap V_1 \subseteq N(y_1) \textit{ and } N(y_1) \cap X = V_1. \tag{2}$$

Suppose $y_2 \in U_2$. By (0) $y_2$ has either a non-neighbour $v$ in $E \cap U_2$, or a neighbour $u$ in $E \setminus U_2$. Let $P$ be an $E$-rung in $G[E \setminus Y]$ that contains $v$. Let $Q$ be an $E$-rung in $G[E \setminus Y]$ that contains $u$. Let $Q'$ be the $uV_1$-rung in $Q$. By Equation (1), either $Y_2 \cup P$ or $Y_2 \cup Q'$ is an $(S, U_2, V_1)$-sprout. Hence we can assume

$$y_2 \notin U_2 \tag{3}$$

Let $S' = N(y_2) \cap X$. By Equation (3), Step 1 is non-executed. By definition of $\mathbb{H} +_{\mathbb{C}} Y$,

$$E \setminus Y \textit{ is a } U_2V_1\textit{-block of } \mathbb{H} \textit{ that contains } S' \tag{4}$$

If $G[E \setminus Y]$ contains a $(S', U_2, V_1)$-sprout $T$, then $G[T \cup Y_2]$ is an $(S, U_2, V_1)$-sprout in $G[E]$. Hence

$$G[E \setminus Y] \textit{ contains no } (S', U_2, V_1)\textit{-sprout} \tag{5}$$

By Equations (4), (5) and Lemma 2.4, $S'$ is $\mathbb{H}$-Safe. By Equations (1) and (2), $Y$ is $\mathbb{H}$-expandable, a contradiction. This completes (1).

By (1) we can assume $S \nsubseteq Y$. Let $\mathbb{H}_i$ be the modified $\mathbb{H}$ after Step i for each $i \in \{1, 2\}$. We have:



doi:10.6342/NTU201803034

- $\mathbb{H}_2$ is an $X$-net that satisfies W1
- $E \setminus Y$ is a $U_2V_1$-block of $\mathbb{H}_2$ that contains $S \setminus Y$
- $G[E \setminus Y]$ contains no $(S', U_2, V_1)$-sprout, since otherwise $G[E]$ has an $(S, U_2, V_1)$-sprout

By Lemma 2.4, $S \setminus Y$ is $\mathbb{H}_2$-Safe. By $S \setminus Y \subseteq E$ and the definition of $\mathbb{H}_2$, $S \setminus Y$ is $\mathbb{H}_1$-Safe. Let $\mathbb{H}(S)$ and $G(S)$ be obtained as follows:

*step 1* Let $\mathbb{H}(S)$ be the subgraph of $\mathbb{H}_1$ that corresponds to the $U_2V_1$-block $E \setminus Y$. Let $G(S) = G[E]$.
*step 2* If $S' = \{x_1, x_2\}$ is contained by a slim arc of $\mathbb{H}(S)$, then apply Subdivide$(V(S), x_1, x_2)$; otherwise let $V(S)$ be the vertex of $\mathbb{H}(S)$ that equals $S'$.
*step 3* Add two new nodes $v$ and $v'$ into $G(S)$ with $N_{G(S)}(v) = S'$ and $N_{G(S)}(v') = v$.
*step 4* Add a leaf $V'(S)$ adjacent to $V(S)$ into $\mathbb{H}(S)$. Let the arc $V'(S)V(S)$ be $\{v, v'\}$. Add $v$ into $V(S)$ and $v'$ into $V'(S)$.
*step 5* Add two new nodes $u_2$ and $u_2'$ into $G(S)$ with $N_{G(S)}(u_2) = U_2 \cap E$ and $N_{G(S)}(u_2') = u_2$.
*step 6* Add a leaf $U_2'$ adjacent to $U_2$ into $\mathbb{H}(S)$. Let the arc $U_2U_2'$ be $\{u_2, u_2'\}$. Add $u_2$ into $U_2$ and $u_2'$ into $U_2'$.
*step 7* Add two new nodes $v_1$ and $v_1'$ into $G(S)$ with $N_{G(S)}(v_1) = V_1 \cap E$ and $N_{G(S)}(v_1') = v_1$.
*step 8* Add a leaf $V_1'$ adjacent to $V_1$ into $\mathbb{H}(S)$. Let the arc $V_1V_1'$ be $\{v_1, v_1'\}$. Add $v_1$ into $V_1$ and $v_1'$ into $V_1'$.
*step 9* If $U_2$ (respectively, $V_1$) is a degree-two vertex that incident to two non-risky arcs $E_1$, $E_2$, then apply Merge$(E', E_1 \cup E_2)$.

We say that $V(S)$ is *subdivided* if the Subdivide operation is executed in step *(h2)*. We say that $U_2$ (respectively, $V_1$) is *merged* if the Merge operation is executed in step *(h9)* for $U_2$ (respectively, $V_1$). Let $A = \{v, v', u_2, u_2', v_1, v_1'\}$ be the new nodes.

(2) $\mathbb{H}(S)$ is an $X(S)$-web of $G(S)$, where $X(S) = E \cup A \setminus Y$. $G(S)$ has exactly three leaves. If $G(S)$ has a sapling then $G[E]$ contains a $(S, U_2, V_1)$-sprout.

Since $S \setminus Y$ is $\mathbb{H}_1$-Safe, $\mathbb{H}(S)$ and $G(S)$ are well-defined. All non-leaf vertices (respectively, arcs between non-leaf vertices of $\mathbb{H}(S)$), besides possibly $V(S)$ (respectively, possibly the two subdivided arcs incident to $V(S)$) are vertices (respectively, arcs) of $\mathbb{H}_1$. Hence N1-N6 holds for these vertices and arcs. By definition N1-N6 holds for $\mathbb{H}(S)[\{V, V(S), U_2, U_2', V_1, V_2'\}]$.Therefore $\mathbb{H}(S)$ is an $X(S)$-net. W1 holds since any new arc of $\mathbb{H}(S)$ that is not an arc of $\mathbb{H}_1$ is slim. There is no new pair of parallel arcs in $\mathbb{H}(S)$ and the only new possible degree-two vertices are $U_2$ and $V_1$ since $V(S)$ has at least three neighbours. W2 holds by the last step. Hence $\mathbb{H}(S)$ is an $X(S)$-web of $G(S)$. Since $E \setminus Y$ is a $U_2V_1$-block of $\mathbb{H}_1$, the only possible leaves of $G[E \setminus Y]$ belong to $U_2 \cup V_1$. Hence $G(S)$ has exactly three leaves $v'$, $u_2'$, and $v_1'$. Suppose that $G(S)$ has a sapling $T$. $A \subseteq V(T)$. Let $t$ be the degree-three node in $T$. $T \setminus A$ is an induced subgraph of $G[E]$. By $N_{G(S)}(u_2) = U_2 \cap E$, $N_T(u_2) = V(T) \cap U_2$. Similarly, $N_T(v_1) = V(T) \cap V_1$ and $N_T(v) = V(T) \cap S$. Hence $G[T \setminus A]$ is an $(S, U_2, V_1)$-sprout.

The remaining proof shows that *a subset of $Y$ is $\mathbb{H}'(S)$-bad, non-$\mathbb{H}'(S)$-expandable and admits no abode in* $(\mathbb{H}(S))' = \mathbb{H}^*$. By Lemma 2.2 and (2), $G[E]$ contains a $(S, U_2, V_1)$-sprout as required.

Since $y_1$ occupies $V_1$ in $\mathbb{H}$,

$$v_1 \in N_{G(S)}(y_1) \tag{6}$$

By definition of $\mathbb{H}(S)$,

$$V(S) \notin V(\mathbb{H}) \textit{ if and only if } V(S) \textit{ is subdivided} \tag{7}$$
$$U_2 \notin V(\mathbb{H}) \textit{ if and only if } U_2 \textit{ is subdivided}$$
$$\textit{all the other non-leaf vertex of } \mathbb{H}(S) \textit{ belongs to } V(\mathbb{H})$$



doi:10.6342/NTU201803034

By $\{u_2, v_1\} \cap S = \varnothing$,

$$V(S), U_2 \text{ and } V_1 \text{ are all distinct} \tag{8}$$

By definition of $\mathbb{H}(S)$, an arc $F$ of $\mathbb{H}(S)$ is not an arc of $\mathbb{H}$ if and only if one of the following holds:

$$F \text{ is incident to a leaf of } \mathbb{H}(S) \tag{9}$$
$$F \text{ is incident to } U_2 \text{ and } U_2 \text{ is subdivided}$$
$$F \text{ is incident to } V(S) \text{ and } V(S) \text{ is subdivided}$$
$$\tag{10}$$

By Step *(H1)*, if $U_2$ is subdivided then in $\mathbb{H}_1$,

$$U_2 \text{ is a leaf of } \mathbb{C} \text{ that incident to } V_2' \text{ and } U_2V_2' \text{ is slim} \tag{11}$$

(3) The following statements hold:

*Fact 1* If a set $B \subseteq E$ is $\mathbb{H}^*$-Safe, then $B$ is $\mathbb{H}$-Safe.
*Fact 2* $Y$ admits no abode in $\mathbb{H}^*$.

For Fact 1, suppose first that $B$ is a vertex of $\mathbb{H}^*$ and hence a vertex of $\mathbb{H}(S)$. By $B \subseteq E$, $B \notin \{V_1, U_2, V(S), V_1', U_2', V'(S)\}$ By Equation (7), $B$ is a vertex of $\mathbb{H}$ and hence $\mathbb{H}$-Safe. Suppose that $B$ consists of two adjacent nodes contained by a slim arc $F_1$ of $\mathbb{H}^*$. Since $F_1$ is slim, $F_1$ is an arc of $\mathbb{H}(S)$. If $F_1$ is an arc of $\mathbb{H}$, then $B$ is $\mathbb{H}$-Safe. Therefore we can assume $F_1 \notin E(\mathbb{H})$. Assume first

$$F \text{ is incident to } V(S) \text{ and } V(S) \text{ is subdivided from a slim arc } F = W_1W_2 \tag{12}$$

By $B \subseteq E$, $F_1 = V(S)W_i$ for an $i \in \{1, 2\}$. If $F$ is an arc of $\mathbb{H}$, then $B \subseteq F$ is $\mathbb{H}$-Safe. Hence $F \notin E(\mathbb{H})$. By Equation (9) we can assume $W_i \in \{U_2, V_1', U_2'\}$. If $F = V_1'U_2'$ then $V(S)V_1'$ and $V(S)U_2'$ are both slim arcs of $\mathbb{H}^*$. By definition of $\mathbb{H}(S)$, both $V_1$ and $U_2$ are merged. Since $V(S)$ is subdivided, $V_1U_2$ is a slim arc $E_1$ of $\mathbb{H}_1$. By $B \subseteq E \cap F_1$, $B \subseteq E_1$. If $U_2$ is not subdivided, then $E_1$ is an arc of $\mathbb{H}$ and $B$ is $\mathbb{H}$-Safe. Hence $U_2$ is subdivided. By definition of $\mathbb{H} +_{\mathbb{C}} Y$, $U_2$ is a leaf of $\mathbb{C}$ in $\mathbb{H}$. Since $V_1U_2$ is an arc of $\mathbb{H}_1$, $U_2$ is subdivided from a slim arc $V_1V_2 = E_2$ of $\mathbb{H}$. $B \subseteq E_2$ is $\mathbb{H}$-Safe.

If $W_i = V_1'$ then by Equation (8) $V_1$ is merged, implying $V_1V(S) \in E(\mathbb{H}(S))$.

There are two cases:

Case 1 $F$ is incident to $U'$ for an $U' \in \{U_2', V_1'\}$.
By $B \subseteq E$, Step 9 is executed on $U$, where $U \in \{U_2, V_1\}$ is the neighbour of $U'$ in $\mathbb{H}(S)$. Let $F = U'V$. $V$ is a vertex of $\mathbb{H}^*$ and $\mathbb{H}(S)$. If $V$ is a vertex of $\mathbb{H}$, then $UV$ is a slim arc of $\mathbb{H}$ that contains $B$ and $B$ is $\mathbb{H}$-Safe. Hence $V = V(S)$ is subdivided. By Step *(H2)*, we can assume that $S'$ is two adjacent nodes contained by a slim arc $UW$ of $\mathbb{H}_1$. By $B \subseteq E$, $B \subseteq UW$. If $UW$ is an arc of $\mathbb{H}$, then $B$ is $\mathbb{H}$-Safe. Hence we can assume $W = U_2$ and $U_2 \notin V(\mathbb{H})$. That is, Step *(H2)* is executed. But then $UV_2$ is a slim arc of $\mathbb{H}$ that contains $B$ and $B$ is $\mathbb{H}$-Safe.

Case 2 $F$ is non-incident to $U_2'$ and $V_1'$.
Let $F = UV$. If $F$ is an arc of $\mathbb{H}$, then $B$ is $\mathbb{H}$-Safe. Hence we can assume $F \notin E(\mathbb{H})$. By the definition of $\mathbb{H}(S)$ and that $F$ is non-incident to $U_2$, $F$ is incident to $V(S)$ and $V(S)$ is subdivided. Let $V(S) = V$ be subdivided from a slim arc $UV'$ of $\mathbb{H}_1$. If $UV'$ is an arc of $\mathbb{H}$, then $A$ is $\mathbb{H}$-Safe. Hence we can assume $V' = U_2$ and $U_2 \notin V(\mathbb{H})$. That is, Step *(H2)* is executed. But then $UV_2$ is a slim arc of $\mathbb{H}$ that contains $B$ and $B$ is $\mathbb{H}$-Safe.



doi:10.6342/NTU201803034

For Fact 2, assume for contradiction that $Y$ admits an abode $F$ in $\mathbb{H}^*$. Since $\mathbb{H}^*$ is steady, $F$ is an arc of $\mathbb{H}'(S)$. Let $N = N_{G(S)}(Y)$. Let $N_i = N_{G(S)}(y_i)$ for each $i \in \{1, 2\}$. By $v_1 \in N$, $F$ is incident to either $V_1$ or $V_1'$. Assume first

$$F = VV_1' \tag{13}$$

Case 1 $F$ is incident to $V_1'$.

Let $F = VV_1'$. If $V = V_1$ then by Condition A1, $N \subseteq F \cup V_1 \cup V_1'$. But then $N(Y) \cap X \subseteq V_1$, a contradiction to $Y$ is non-$\mathbb{H}$-local. Hence we can assume that $V \neq V_1$. Since $F$ is a $VV_1'$-block of $\mathbb{H}(S)$ and $V_1'$ is a leaf, $F \setminus V_1V_1' = F \setminus \{v_1, v_1'\}$ is a $VV_1$-block of $\mathbb{H}(S)$. We have $V \neq U_2$, since otherwise (by $\mathbb{H}'(S)$ is steady) $\mathbb{H}'(S)$ contains exactly one degree-three vertex $V = U_2 = V(S)$, a contradiction to Fact 1. Suppose first that $V$ is a vertex of $\mathbb{H}_1$. Since $V \neq U_2$, $V$ is a vertex of $\mathbb{H}$. $C' = F \setminus \{v_1, v_1'\}$ is a $VV_1$-block of $\mathbb{H}$ that is strictly smaller than $C$. We show that $F \setminus \{v_1, v_1'\}$ satisfies Condition A1-A3 for $Y$ in $\mathbb{H}$, a contradiction to the minimality of $C$. By $N \subseteq V \cup F \cup V_1'$ and $V \neq U_2$, $u_2 \notin N$ and $N(y_2) \cap X \subseteq E$.

- By $N \subseteq V \cup F \cup V_1'$ in $\mathbb{H}'(S)$, $N(Y) \cap X \subseteq V \cup C' \cup V_1'$ in $\mathbb{H}$ and A1 holds. Note that if $V = V(S)$ then these two $V$'s may differ on one node $v$, but it does not matter.
- Since $y_1$ occupies $V_1$ on $C$ in $\mathbb{H}$, A2 holds for $V_1$ on $C'$ in $\mathbb{H}$.
- By A3 applied on $F$, either $N_2 \subseteq C'$ or $y_2$ occupies $V$ in $\mathbb{H}'(S)$. If $N_2 \subseteq C'$ then $N(y_2) \cap X \subseteq C'$. Otherwise $y_2$ occupies $V$ in $\mathbb{H}$. Either way, A3 holds.

Hence we can assume $V \notin V(\mathbb{H}_1)$. By definition of $\mathbb{H}(S)$, $V = V(S)$ is subdivided.

□

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**doi:10.6342/NTU201803034**