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點對點檔案傳輸之賽局分析

A Game-Theoretic Analysis of P2P File-Sharing Systems

蔡瑋倫
Wei-Lun Tsai

指導教授：陳健輝博士
Advisor: Gen-Huey Chen, Ph.D.
共同指導教授：陳和麟博士
Co-Advisor: Ho-Lin Chen, Ph.D.

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A Game-Theoretic Analysis of P2P File-Sharing Systems

本論文係蔡瑋倫君（學號 R06922082）在國立臺灣大學資訊工程學系完成之碩士學位論文，於民國 108 年 7 月 22 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

陳健輝
（指導教授）

系 主 任 莊永裕

摘要

點對點式網路架構常用於使用者之間的檔案傳輸與分享，藉以改善傳統主從式架構伺服器負擔過重以及易受攻擊等問題。然而，實驗結果發現點對點網路架構容易造成搭便車問題，於是我們必須借助賽局理論以設計良好的獎勵機制督促使用者貢獻自己的資源，以維持系統運作 [16]。

我們的研究從 Chiranjeeb Buragohain 等人在 2003 年所提出的模型 [2] 延伸而來。原論文根據每位使用者的貢獻來決定他/她是否能從社群獲得資源的機率函數，貢獻與機率成正相關，而效益函數則是所獲得資源去扣除自己開放頻寬給其他使用者下載的成本，在兩個人的環境下恰有兩個不崩潰的均質納許均衡，促使社群高貢獻的均衡點是穩定的。在我們的論文額外考慮了使用者對其他人所擁有資源的需求有所節制以及多重使用者的情況。在此情形下，我們發現當需求幾乎沒有節制的時候不影響原本的納許均衡；當需求有些節制的時候會壓低原本促使社群高貢獻的均衡點的貢獻量，同時該均衡點轉為不穩定，可能收斂到其他均衡點；當使用者的需求極低 (資源同質性高) 的時候整個系統反而會崩潰 (使用者均不貢獻)。此外，我們也觀察了不同條件之下納許均衡的效率隨著模型參數 (單位資源所產生之效益、需求的節制、社群人數) 的變化。

關鍵字：賽局理論、納許均衡、點對點、檔案分享、獎勵機制

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Abstract

A peer-to-peer (P2P) network is commonly used for file-sharing among different users. This kind of structure can solve some common problems of centralized networks. However, experiments show that free-riding is a major problem for the P2P networks, so we have to design a good incentive mechanism with the help of game theory in order to encourage users to contribute to the community and maintain the network [16].

We use the model proposed by Buragohain *et al.* [2] in 2003. In the original paper, the author determines the probability function, from the contribution of each user, which controls the probability that a user can retrieve resources from the community. The probability increases with the contribution. The utility function is determined by the retrieved resources with the contribution cost subtracted. In a two-player file-sharing game, there are two non-collapsing Nash equilibria, one of which with a greater contribution is stable. In our thesis, we further consider a multi-player file-sharing game where the need for resources of each user is limited. In this game, we've discovered that when the limitation is not obvious, the original Nash equilibria are not affected. When the limitation is a little influential, the contribution of the Nash equilibrium with a greater contribution will be lowered and it will become unstable. When the limitation is drastic, the system will collapse. Besides, we've also observed how the efficiency of Nash equilibria changes with system parameters under different conditions. The parameters include the benefit drawn by one unit of resources, the limitation of need for

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resources, and the number of users in the network which will be defined later.

Keywords: Game Theory, Nash Equilibrium, Peer-to-Peer, File-Sharing, Incentive Mechanism

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Chapter 1

Introduction

A peer-to-peer (P2P) network is a distributed system that consists of many users which are often directly connected to each other, and they can be both providers and consumers of resources at the same time in the network. In constrast to P2P networks, a centralized network also consists of many users, but only servers provide all the resources and are connected to clients. The clients can only consume the resources and they are not necessarily connected to each other.

The most significant advantages of P2P networks over centralized networks are scalability and robustness. When a new user joins a P2P network, he/she not only increases the network load but also provides some resources to the system (as a small server), so the network load is usually balanced and the P2P network is scalable. When a node is attacked or fails to work for some reason, the other parts of the network can still work as usual because only a very small part of the system is affected. In a centralized network, an attack against one of the main servers can severely reduce the performance since the resources are completely on the servers. Therefore P2P networks are more robust than centralized networks.

However, a major problem for the P2P networks is "free-riding." Free-riding means that most users only consume the resources but forget to provide enough resources to maintain the network. Since making contribution definitely takes some cost, it is intuitive that free-riding is a dominant strategy. Unfortunately, if everyone chooses this dominant strategy, there will be no resources in the network and therefore the system will collapse.

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Experiments in [9] showed this phenomenon. Hence, some incentive mechanisms are needed to overcome this free-riding problem.

Incentive mechanisms incorporated by the P2P file-sharing networks in the past were mainly based on monetary payment schemes or reciprocity-based schemes [5]. In monetary payment schemes, users must pay money before consuming resources and can get paid when providing resources to others. Mojonation and Karma [12], and some studies such as [6, 10, 14, 17] used this kind of schemes. The implementation is not easy in practice since it requires infrastructure for accounting and micropayments. Contrary to monetary payment schemes, we can also use reciprocity-based schemes. They include direct reciprocity and indirect reciprocity. In direct reciprocity schemes, the quality of resources user A wants to provide to user B is based on the quality of resources A retrieved from B in the past. BitTorrent [3] uses this kind of schemes based on the tit-for-tat strategy. In indirect reciprocity schemes, also called *reputation based* schemes, the quality of resources a user deserves to obtain highly depends on his/her "overall" generosity. The word "overall" here means that as long as user A's reputation is high, it is not necessary for A to provide good quality resources to user B even if A wants to retrieve good quality resources from B. Some studies such as [2, 7] used this kind of schemes. We should note that this is an advantage when a user is not interested in anything the other one can offer. It is the only difference between direct reciprocity and indirect reciprocity. Nowadays, the incentive mechanisms are further enhanced. For example, Hu *et al.* [8] combined monetary payment schemes and indirect reciprocity schemes. Zhang *et al.* [18] used a Blockchain-based mechanism to resolve the difficulty of finding a trusted third party (TTP) in a real P2P system.

[2] is a representative paper about reputation based schemes. In [2] the authors proposed a *differential service-based* incentive scheme to improve the system's performance (i.e., reduce free-riding). First, they considered the case of a "homogeneous" system where the value of resources is independent of users who own them and users who retrieve them. In this case, there exists two non-collapsing Nash equilibria with different contribution levels. Only the one resulting in the better overall performance is stable (i.e., easily real-

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ized). Second, they studied the case of a "heterogenous" system through simulation, since no closed form solution is possible. In this case, the numerical experiments showed that the system also converges to the desirable Nash equilibrium if a good initial condition is given, and that the average contribution is almost independent of the number of users. Finally, they gave some suggestions on how to modify current P2P systems to implement the proposed incentive scheme. We need a function of the contribution level of user A to control the probability that A can retrieve resources from another user B. Also, the probability function should be a part of the system's architecture. It means that the setting should be exactly the same for all users and cannot be modified by them. In order to prevent users from reporting their contribution levels incorrectly, a *neighbour audit scheme* in which users can verify the information of their neighbors is required. In order to encourage new users to join the system, they can be given a default contribution level at the beginning.

Our research is continued from [2]. In the original paper, the resources a player possesses are not limited. To our best knowledge, there are almost no research papers discussing the case of limited resources, so we will consider this environment in our thesis. We only study the case of a homogeneous system of two players, three players, and multiple players, but with a fixed maximum benefit of resources from each player, and the probability function satisfying some "good" assumptions that we will introduce in the next chapter. Our main contribution is to find some important Nash equilibria under different parameter settings, analyze their stability and efficiency including the *price of anarchy (PoA)* and *price of stability (PoS)*, and observe how they vary with related parameters. We define the PoA to be the ratio of the maximum total utility among all possibilities to that of the "worst" Nash equilibrium, and define the PoS to be the ratio of the maximum total utility among all possibilities to that of the "best" Nash equilibrium.

The rest of the thesis is organized as follows. In Chapter 2, we explain the meaning of our newly proposed model and introduce the related parameters. In Chapter 3, we analyze a homogeneous system of two players. In Chapter 4, we analyze a homogeneous system of three players, but without considering the stability of Nash equilibria. In Chapter 5, we analyze a homogeneous system of multiple players, but only considering symmetric Nash

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equilibria. That is, all players have the same strategy. Finally in Chapter 6, we conclude our analysis, describe additional possibly extended models, and discuss some aspects that can be improved in the future.

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Chapter 2

Model

In this chapter, we're going to introduce the system parameters inherited from [2] that will be used in this thesis. Assume that there are N players (users) P_1, P_2, ..., P_N in the system. All parameters, as in the original paper, are dimensionless.

Definition 2.1 (**Contribution**)**.** Let d_i be the contribution of P_i which is a nonnegative number. The meaning of the contribution can be very widespread. For example, [2] says we may think of d_i as the disk space contribution integrated over a fixed period of time, or the number of downloads served by this peer to other peers. In this thesis, we usually see d_i as the amount of downloadable resources owned by P_i. Since this parameter is also a *strategy* one player can decide, the term "strategy" and "contribution" have the same meaning in this thesis.

Definition 2.2 (**Benefit**)**.** The value of resources owned by a player may vary depending mainly on other users who retrieve them. For example, if Alice has lots of music, whereas Bob has lots of Japanese animation, I may prefer Bob's resources to Alice's. Hence we let b denote how much the "unit" contribution made by one player is worth to another player in a homogeneous system. That is, if a player P_i retrieves one unit of contribution from another player P_j, then P_i's utility will increase by b. Details of the utility function will be introduced later.

Definition 2.3 (**Probability as Service Differentiator**)**.** In a differential service, the probability that a player P_i can retrieve resources from other players should increase with

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his/her contribution d_i. This mechanism encourages the players to share their file resources. In this thesis, a player P_i can retrieve resources from other players with probability $p(d_i)$, and is rejected with probability $1-p(d_i)$.

Proposition 2.1. To achieve the goal of a service differentiator, the probability function $p(d)$ must be non-decreasing (i.e., $p'(d) \geq 0$ for $d \geq 0$). To meet the definition of "probability," p should satisfy $p(0) = 0$ and $\lim_{d \to \infty} p(d) = 1$. To ensure each player has only one best strategy in each iteration, we assume $p'(d)$ to be decreasing (i.e., $p''(d) < 0$ when $0 \leq p(d) < 1$). To ensure $bdp'(d) = C$ has at most two solutions for every constant $C > 0$, we also assume $dp'(d)\Big|_{d=0} = \lim_{d \to \infty} dp'(d) = 0$, and there exists a threshold d_0 such that $(dp'(d))' > 0$ for $d < d_0$ and $(dp'(d))' < 0$ for $d > d_0$. We assume all probability functions $p(d)$ satisfy all our assumptions in this proposition unless otherwise specified.

Definition 2.4 (Utility). Let the total utility u_i that P_i will derive in the homogeneous system be $u_i = -d_i + \sum_{j \neq i} \min\{K,\ bd_j p(d_i)\}$. The term $-d_i$ is the cost of P_i to join the system, which is proportional to his/her contribution. The other term $\sum_{j \neq i} \min\{K,\ bd_j p(d_i)\}$ is the total expected benefit of P_i. It is obvious that $\min\{K,\ bd_j p(d_i)\}$ for some j is the expected benefit gained from some player P_j. In this term, d_j is the amount of resources P_j can provide, so multiplying it by $p(d_i)$ gives the expected amount of resources P_i can acquire. Multiplying it by b again obtains the expected "benefit." In that term K denotes the maximum benefit one player can derive from another player. Normally K is greater than 0.

Proposition 2.2. Suppose all d_i's have the same value of d. If $b \leq \dfrac{1}{n-1}$, the utility function u_i is therefore not greater than $(n-1)\, b\, d\, p(d) - d = d\left((n-1)\, b\, p(d) - 1\right) \leq d\left((n-1)\, b - 1\right) \leq d\,(1-1) = 0$. This means that any homogeneous solution is not better than the origin. It may cause the system to collapse. To avoid this problem, we should assume $b > \dfrac{1}{n-1}$ in this thesis.

After the definitions and propositions, here is one important lemma about Proposition 2.1 that will commonly be referred to when the parameter b varies.

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Lemma 2.1. *Assume the two equations $b_1 xp'(x) = C$ and $b_2 xp'(x) = C$, where $0 < b_1 < b_2$, have solutions. Let the solutions to $b_1 xp'(x) = C$ be $d_{1\ell}$ and d_{1h}, where $d_{1\ell} \leq d_{1h}$. Let the solutions to $b_2 xp'(x) = C$ be $d_{2\ell}$ and d_{2h}, where $d_{2\ell} \leq d_{2h}$. Then $d_{2\ell} < d_{1\ell}$ and $d_{2h} > d_{1h}$.*

Proof. Since $xp'(x)\Big|_{x=0} = 0$ and $xp'(x)\Big|_{x=d_{1\ell}} = C/b_1$, by the intermediate value theorem there must exist at least one $x_\ell < d_{1\ell}$ such that $xp'(x)\Big|_{x=x_\ell} = C/b_2$ ($\because b_2 > b_1$). Since $xp'(x)\Big|_{x=d_{1h}} = C/b_1$ and $\lim_{x\to\infty} xp'(x) = 0$, by the intermediate value theorem there must exist at least one $x_h > d_{1h}$ such that $xp'(x)\Big|_{x=x_h} = C/b_2$ ($\because b_2 > b_1$). Therefore $x_\ell < d_{1\ell} \leq d_{1h} < x_h$. Since $b_2 xp'(x) = C$ has at most two solutions, we can simply say $x_\ell = d_{2\ell}$ and $x_h = d_{2h}$. $\therefore d_{2\ell} < d_{1\ell}$ and $d_{2h} > d_{1h}$. □

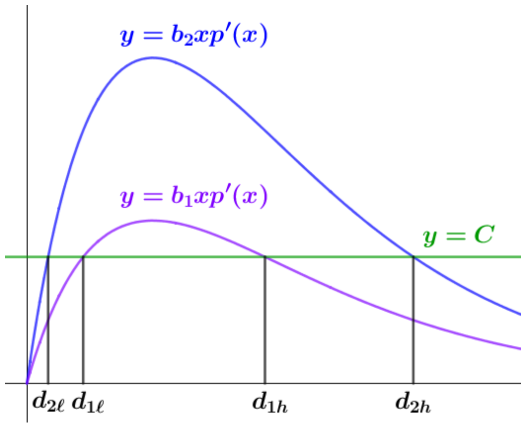

Figure 2.1: A geometric illustration of Lemma 2.1

After introducing the system parameters, we're going to derive some important lemmas related to the probability function that will be heavily used in the later chapters.

2.1 Useful Properties

Before the lemmas, we also define two symbols that will be used in the whole thesis.

Definition 2.5. Let u_{opt} be the maximum total utility in an n-player file-sharing game. That is, $u_{opt} = \max_{\substack{d_i \geq 0 \\ \text{for } 1 \leq i \leq n}} u(d_1, d_2, ..., d_n)$.

Definition 2.6. Let d_o be the unique solution to the equation $bxp(x) = K$. Since $K > 0$, d_o cannot be 0 and we can see it as the intersection of $p(x)$ and $\dfrac{K}{bx}$.

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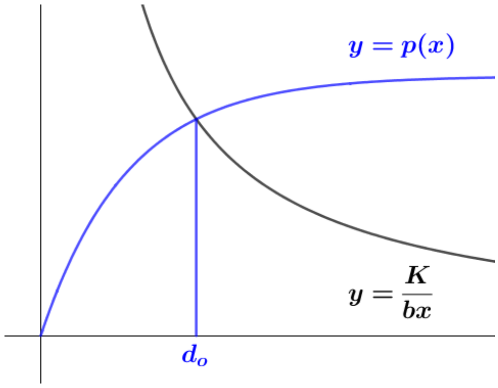

Figure 2.2: A geometric solution of d_o

Lemma 2.2. *If* $0 < d_1 < d_2$*, then* $d_1\, p(d_2) < d_2\, p(d_1)$.

Proof. We're going to prove this lemma with the technique of "change of variables" covered in the calculus course. Write the probability function in an integral form,

$$p(d_2) = \int_0^{d_2} p'(x)\, dx \overset{x=(d_2/d_1)u}{=\!=\!=\!=} \int_{0\cdot\frac{d_1}{d_2}}^{d_2\cdot\frac{d_1}{d_2}} p'\left(\frac{d_2}{d_1}u\right) d\left(\frac{d_2}{d_1}u\right) = \frac{d_2}{d_1}\int_0^{d_1} p'\left(\frac{d_2}{d_1}u\right) du,$$

and it can be rearranged into $\frac{d_1}{d_2}p(d_2) = \int_0^{d_1} p'\left(\frac{d_2}{d_1}u\right) du$. Compare it with

$$p(d_1) = \int_0^{d_1} p'(u)\, du.$$

Since $d_2 > d_1$ (which implies $\frac{d_2}{d_1}u > u$ for $u > 0$) and $p'(x)$ is decreasing if greater than zero, we can always pick some $d_0 \in (0, d_1)$ such that $p'(u) > 0$ $\Big($i.e., $p'(u) > p'(\frac{d_2}{d_1}u)\Big)$ for all $u \in (0, d_0)$ and $p'(u) = 0$ $\Big($i.e., $p'(\frac{d_2}{d_1}u) = 0\Big)$ for all $u \in (d_0, d_1)$. Hence

$$\begin{aligned}
\frac{d_1}{d_2}p(d_2) = \int_0^{d_1} p'\left(\frac{d_2}{d_1}u\right) du &= \int_0^{d_0} p'\left(\frac{d_2}{d_1}u\right) du + \int_{d_0}^{d_1} p'\left(\frac{d_2}{d_1}u\right) du \\
&< \int_0^{d_0} p'(u)\, du + \int_{d_0}^{d_1} p'\left(\frac{d_2}{d_1}u\right) du \\
&= \int_0^{d_0} p'(u)\, du + \int_{d_0}^{d_1} p'(u)\, du \\
&= \int_0^{d_1} p'(u)\, du = p(d_1).
\end{aligned}$$

We can obtain the result by multiplying both sides of the above inequality by d_2. □

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Corollary 2.3. *If* $d_1\, p(d_2) > d_2\, p(d_1) > 0$*, then* $d_1 > d_2 > 0$*. If* $d_1\, p(d_2) = d_2\, p(d_1) > 0$*, then* $d_1 = d_2 > 0$*. If* $0 < d_1\, p(d_2) < d_2\, p(d_1)$*, then* $0 < d_1 < d_2$*.*

Proof. By the law of trichotomy, exactly one of the three conditions $d_1 < d_2$, $d_1 = d_2$, or $d_1 > d_2$ is true. Consider the first statement in our corollary. If $0 < d_1 < d_2$, by Lemma 2.2 we can deduce $d_1\, p(d_2) < d_2\, p(d_1)$. If $0 < d_1 = d_2$, then $d_1\, p(d_2) = d_2\, p(d_1)$. Both of the above assumptions violate the statement, so only $d_1 > d_2 > 0$ can be the conclusion of it. The reader can use the same method to prove the remaining two statements. □

Lemma 2.4. *If* $d_1\, p'(d_2) = d_2\, p'(d_1) > 0$*, then* $d_1 = d_2$*.*

Proof. The structure of this proof is very similar to Corollary 2.3. If $0 < d_1 < d_2$ and $p'(d_2) > 0$, then $p'(d_1) > p'(d_2)$ and $d_2 p'(d_1) > d_1 p'(d_2)$ since $p'(x)$ is decreasing. Similarly if $0 < d_2 < d_1$ and $p'(d_1) > 0$, then $p'(d_2) > p'(d_1)$ and $d_1 p'(d_2) > d_2 p'(d_1)$. The above two cases both violate the lemma assumption. From the above, only $d_1 = d_2 > 0$ can satisfy the assumption, so it is our conclusion. □

Lemma 2.5. *If* $p(d_1) = p(d_2)$ *for some* $d_1 < d_2$*, then* $p(x) = 1$ *and* $p'(x) = 0$ *for all* $x \geq d_1$*.*

Proof. Write the probability function $p(x)$ in an integral form.

$$
\begin{aligned}
p(d_2) - p(d_1) &= \int_0^{d_2} p'(x)\, dx - \int_0^{d_1} p'(x)\, dx \\
&= \int_{d_1}^{d_2} p'(x)\, dx = 0. \qquad (2.1)
\end{aligned}
$$

Suppose for contradiction that $p'(d_1) > 0$. Then we can definitely find a $d_{mid} \in (d_1, d_2)$ such that $p'(x) > 0$ for all $x \in (d_1, d_{mid})$, and

$$
\int_{d_1}^{d_2} p'(x)\, dx = \int_{d_1}^{d_{mid}} p'(x)\, dx + \int_{d_{mid}}^{d_2} p'(x)\, dx > \int_{d_{mid}}^{d_2} p'(x)\, dx \geq 0,
$$

which violates Equation (2.1). Hence $p'(d_1) = 0$ and $p'(x) = 0$ for all $x \geq d_1$ since $p'(x)$ is decreasing. In addition $p'(x) = 0$ implies $p(x) = 1$, so $p(x) = 1$ for all $x \geq d_1$. □

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Corollary 2.6. *If* $p(d_1) = p(d_2)$*,* $p'(d_1) > 0$ *and* $p'(d_2) > 0$*, then* $d_1 = d_2$*.*

Proof. W.L.O.G., assume $d_1 \leq d_2$. If $d_1 < d_2$, then by Lemma 2.5 $p'(d_1) = p'(d_2) = 0$ causes a contradiction. Therefore, $d_1 = d_2$. □

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Chapter 3

Nash Equilibrium Analysis for Two-Player File-Sharing Games

In the previous chapter, we've introduced some basic elements of our model. For simplicity, we consider a homogeneous system of two players first. It's easy to see that the model can be simplified to the following.

$$\begin{cases} u_1(d_1) = -d_1 + \min\{K,\ b\, d_2\, p(d_1)\} \\ u_2(d_2) = -d_2 + \min\{K,\ b\, d_1\, p(d_2)\} \\ u(d_1, d_2) = u_1(d_1) + u_2(d_2). \end{cases}$$

We also use the notation $u(d) = u(d, d)$ if both d_1 and d_2 have the same value of d.

In this chapter, we are going to find all Nash equilibria under different parameter settings, analyze their stability and efficiency (PoA and PoS), and observe how they vary with system parameters b and K. Before calculating the PoA and PoS, we should find the points where the maximum total utility occurs.

3.1 Maximum Total Utility

In this section, we hope to find the maximum total utility in different parameter settings. The method used in this chapter is to calculate the gradient with respect to d_1 or d_2 at each point in the domain of $u(d_1, d_2)$. Since u is bounded above ($u \leq 2K$), we can guarantee

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the existence of a maximum, and it cannot occur at the points where $(\frac{\partial u}{\partial d_1})^+ = (\frac{\partial u}{\partial d_1})^- \neq 0$ or $(\frac{\partial u}{\partial d_2})^+ = (\frac{\partial u}{\partial d_2})^- \neq 0$. Based on this observation, we can exclude these points first (called an elimination procedure), then compare the values of the remaining points, and then finally choose the optimal points from them.

Observing the formula in this model, the reader may guess that u_{opt} occurs when $bd_2p(d_1) = K$ and $bd_1p(d_2) = K$. In fact this is true under some "good" parameter settings. In this section, we will introduce these "good" conditions, and explain why u_{opt} occurs at such places.

By symmetry, it suffices to consider only the upper left part of the domain of $u(d_1, d_2)$ in the following analysis. It can be partitioned into three regions with respect to the two equations $bp(d_2) = 1$ and $bd_1p(d_2) = K$. These regions can be defined formally.

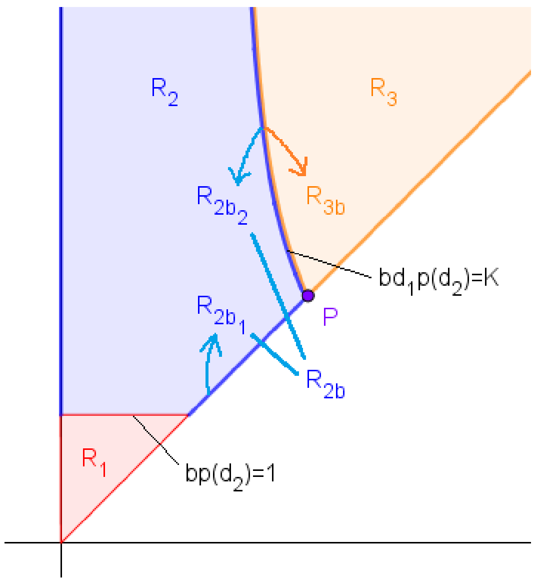

Figure 3.1: A simple diagram of Definition 3.1.
The symbols R_{2b_1} and R_{2b_2} will be defined later.

Definition 3.1. Let region 1 be $R_1 = \{ (d_1, d_2) \mid 0 \leq d_1 \leq d_2 \wedge bp(d_2) \leq 1 \}$. Let region 2 be $R_2 = \{ (d_1, d_2) \mid 0 \leq d_1 \leq d_2 \wedge bp(d_2) > 1 \wedge bd_1p(d_2) \leq K \}$. Let region 3 be $R_3 = \{ (d_1, d_2) \mid 0 \leq d_1 \leq d_2 \wedge bp(d_2) > 1 \wedge bd_1p(d_2) \geq K \}$. Let R_{2b} be the rightmost boundary of R_2. That is, $R_{2b} = R_2 \cap \{ (d_1, d_2) \mid d_1 = d_2 \vee bd_1p(d_2) = K \}$. Let R_{3b} be the leftmost boundary of R_3. That is, $R_{3b} = R_3 \cap \{ (d_1, d_2) \mid bd_1p(d_2) = K \} \subseteq R_{2b}$. Let P be the point (d, d) where $bdp(d) = K$. The letter "b" here means "boundary."

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The reason why we want to partition the domain is explained as follows. Utilities in R_1 are always nonpositive (which will be proven later), which is obviously not better than the origin $(0, 0)$, so this region will be excluded eventually if we hope that the model has a positive u_{opt}. We also note that the term $\min\{K,\ bd_1p(d_2)\}$ causes the gradient of $u(d_1, d_2)$ to be discontinuous at the curve $bd_1p(d_2) = K$, so the values should be calculated in R_2 and R_3 separately.

In the following several pages we're going to perform our elimination procedure. The procedure can be divided into three stages. The first stage is to remove the points where u_{opt} cannot occur in R_2 and R_3. By Lemma 3.1 R_2 can be minimized to R_{2b}, and by Lemma 3.2 R_3 can be minimized to R_{3b}. Since R_{2b} contains R_{3b}, we only consider the points where u_{opt} cannot occur in R_{2b} and remove them in the second stage. By Lemma 3.3 and Lemma 3.4 R_{2b} can be minimized to the point P or completely eliminated. Finally, we'll show that the maximum total utility within R_1 is exactly 0 and find out circumstances in which P will be better than R_1.

Now we perform the first stage of the elimination procedure.

Lemma 3.1. *After we remove these points where u_{opt} cannot occur in R_2, the region R_2 should be minimized to R_{2b}.*

Proof. One property of R_2 is the inequality $bd_1p(d_2) \leq K$. According to this, the utility $u = -d_1 + \min\{K, bd_1p(d_2)\} - d_2 + \min\{K, bd_2p(d_1)\}$ can be simplified to $u = -d_1 + bd_1p(d_2) - d_2 + \min\{K, bd_2p(d_1)\}$. Since the "min" term may cause the partial derivatives to be discontinuous, for simplicity we use the notation of partial derivatives as usual to represent the less of the left derivative and right derivative.

$$\because \quad \frac{\partial}{\partial d_1}K = 0 \quad \text{and} \quad \frac{\partial}{\partial d_1}bd_2p(d_1) = bd_2p'(d_1) \geq 0. \quad \therefore \quad \frac{\partial}{\partial d_1}\min\{K, bd_2p(d_1)\} \geq 0.$$

$$\therefore \quad \frac{\partial u}{\partial d_1} = -1 + bp(d_2) + 0 + \frac{\partial}{\partial d_1}\min\{K, bd_2p(d_1)\} \geq -1 + bp(d_2) > 0.$$

According to this derivative, we can say for each pair of points (ℓ, d_2) and (r, d_2) in R_2, $u(\ell, d_2) < u(r, d_2)$ if $\ell < r$. Hence the result follows. □

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Lemma 3.2. *After we remove these points where u_{opt} cannot occur in R_3, the region R_3 should be minimized to R_{3b}.*

Proof. One property of R_3 is the inequality $bd_1p(d_2) \geq K$. According to this, we can further deduce $bd_2p(d_1) \geq bd_1p(d_2) \geq K$ by Lemma 2.2. The utility $u = -d_1 - d_2 + \min\{K, bd_2p(d_1)\} + \min\{K, bd_1p(d_2)\}$ is then simplified to $u = -d_1 - d_2 + 2K$, and therefore $\frac{\partial u}{\partial d_1} = -1$ in this region. According to this derivative, we say for each pair of points (ℓ, d_2) and (r, d_2) in R_3, $u(\ell, d_2) > u(r, d_2)$ if $\ell < r$. Hence the result follows. □

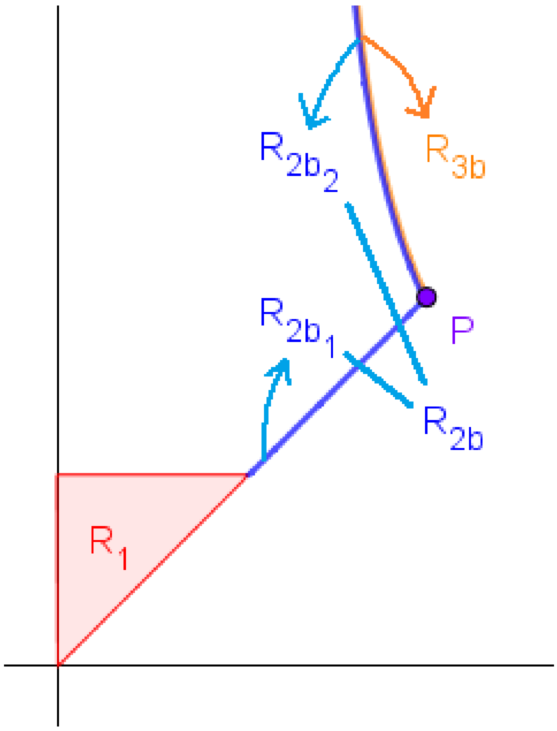

Figure 3.2: A simple diagram of the remaining regions after the first stage of the elimination procedure

Now we perform the second stage of the elimination procedure.

Definition 3.2. Let $R_{2b_1} = R_{2b} \cap \{ (d_1, d_2) \mid d_1 = d_2 \}$, and let $R_{2b_2} = R_{2b} \cap \{ (d_1, d_2) \mid bd_1p(d_2) = K \}$. Then $R_{3b} = R_{2b_2}$ and $R_{2b} = R_{2b_1} \cup R_{2b_2}$.

Lemma 3.3. *If R_{2b_1} exists, it should be minimized to the single point P after we remove these points where u_{opt} cannot occur in R_{2b_1}.*

Proof. In R_{2b_1} there is a condition $d_1 = d_2$, so $bd_2p(d_1) = bd_1p(d_2) \leq K$. If we let $d = d_1 = d_2$, then the utility can be simplified to $u = -d_1 + bd_1p(d_2) - d_2 + bd_2p(d_1) = 2(bdp(d) - d)$, and the derivative is

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$$\frac{\partial u}{\partial d} = 2\frac{\partial}{\partial d}\Big(bdp(d) - d\Big) = 2\Big(bp(d) + bdp'(d) - 1\Big) \geq 2\Big(bp(d) - 1\Big) > 0$$

by the condition $bp(d) > 1$ stated in R_2. Hence the result follows. □

Lemma 3.4. *After we remove these points where u_{opt} cannot occur in R_{2b_2}, the region R_{2b_2} should be minimized to the point P or completely eliminated.*

Proof. According to the constraint $d_1 p(d_2) = K/b$ stated in R_{2b_2}, we first differentiate both sides of the equation with respect to d_2, in order to obtain $\partial d_1/\partial d_2$.

$$\frac{\partial}{\partial d_2}\Big(d_1 p(d_2)\Big) = \frac{\partial}{\partial d_2}\Big(\frac{K}{b}\Big) \quad \Longrightarrow \quad \frac{\partial d_1}{\partial d_2} p(d_2) + d_1 p'(d_2) = 0.$$

According to the property $bd_1p(d_2) = K$ in R_{2b_2}, we can further deduce $bd_2p(d_1) \geq bd_1p(d_2) = K$ by Lemma 2.2, so the utility is then simplified to $u = -d_1 - d_2 + 2K$. Differentiate u with respect to d_2.

$$\begin{aligned}\frac{\partial u}{\partial d_2} &= \frac{\partial}{\partial d_2}\Big(-d_1 - d_2 + 2K\Big) = -\frac{\partial d_1}{\partial d_2} - 1 = \frac{d_1\, p'(d_2)}{p(d_2)} - 1 = \frac{d_1\, p'(d_2) - p(d_2)}{p(d_2)} \\ &= \frac{d_1\, p'(d_2) - \int_0^{d_2} p'(t)dt}{p(d_2)} < \frac{d_1\, p'(d_2) - d_2\, p'(d_2)}{p(d_2)} = \frac{p'(d_2)}{p(d_2)}\big(d_1 - d_2\big).\end{aligned}$$

Therefore $\frac{\partial u}{\partial d_2} < 0$ since $p'(d_2) \geq 0$, $p(d_2) > 0$, and $d_1 \leq d_2$. According to this derivative, we can increase u only by decreasing d_2. As $\frac{\partial d_1}{\partial d_2} \leq 0$, only the condition $d_1 = d_2$ can stop our traversal. If this condition is reached, then we arrive the point P. If this condition can never be reached (i.e., $d_1 = d_2$ can only happen when $b\, p(d_2) \leq 1$), then the whole region R_{2b_2} should be eliminated. In this case $P \in R_1$. Hence the result follows. □

Lemma 3.5. *The maximum total utility achieved in R_1 is 0.*

Proof. We first observe that $bp(d_1) \leq bp(d_2) \leq 1$ because of our assumption $d_1 \leq d_2$. Multiplying d_2 on both sides of $bp(d_1) \leq 1$ gives $bd_2p(d_1) \leq d_2$. Multiplying d_1 on both sides of $bp(d_2) \leq 1$ gives $bd_1p(d_2) \leq d_1$. Then they can be applied to the following.

$$u_1(d_1) = -d_1 + \min\{K,\ b\, d_2\, p(d_1)\} \leq -d_1 + b\, d_2\, p(d_1) \leq -d_1 + d_2.$$
$$u_2(d_2) = -d_2 + \min\{K,\ b\, d_1\, p(d_2)\} \leq -d_2 + b\, d_1\, p(d_2) \leq -d_2 + d_1.$$

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Adding these two inequalities together, we'll discover that

$$u_1(d_1) + u_2(d_2) \leq (-d_1 + d_2) + (-d_2 + d_1) = 0.$$

Hence the result follows. □

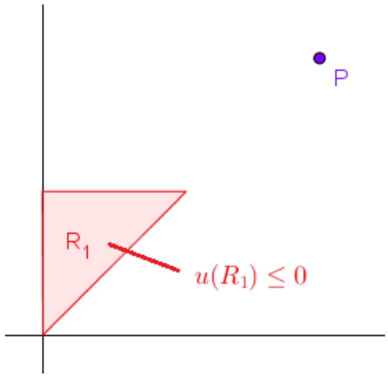

Figure 3.3: A simple diagram of the remaining regions after the second stage of the elimination procedure

The comparison between R_1 and P in the final stage is illustrated in the following theorem.

Theorem 3.6. *If* $d_o \geq K$*, then* $u_{opt} = 0$*. If* $d_o < K$*, then* $u_{opt} = 2(K - d_o) > 0$*.*

Proof. By Lemma 3.5, the maximum achieved in R_1 is 0. If $d_o \geq K$, then the total utility at the point P is $u(d_o, d_o) = -d_o + bd_o p(d_o) - d_o + bd_o p(d_o) = 2(K - d_o) \leq 0$. $\therefore u_{opt} = 0$ in this case. If $d_o < K$, then $u(d_o, d_o) = 2(K - d_o) > 0$, which is better than R_1. $\therefore u_{opt} = 2(K - d_o) > 0$ in this case. □

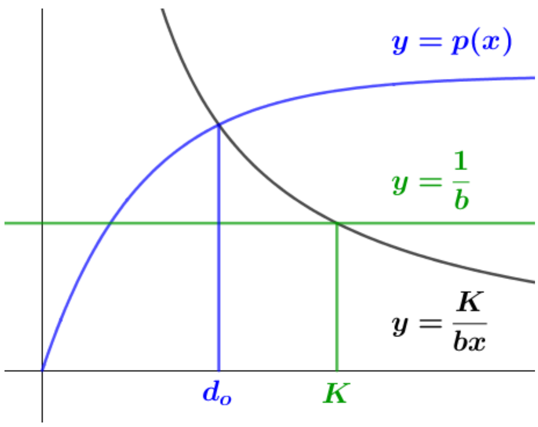

Figure 3.4: A geometric perspective of the condition $d_o < K$ such that $u(d_o) > 0$

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Corollary 3.7. *Let d_ℓ be the less solution to $bxp'(x) = 1$. If $d_o \geq d_\ell$, then $u_{opt} = u(d_o)$.*

Proof. $\because\ bp(d_o) \geq bp(d_\ell) > bd_\ell p'(d_\ell) = 1 \quad \therefore\ d_o < K$ by Definition 2.6. In this case $u_{opt} > 0$ by Theorem 3.6. However the maximum total utility within R_1 is not greater than 0, so u_{opt} can only occur at the point $P = (d_o, d_o)$. □

We close this section with the following conclusive table.

Table 3.1: The maximum total utility of two-player file-sharing games

Condition	**Utility**
$d_o \geq K$	0
$d_o < K$	$2(K - d_o)$

3.2 Nash Equilibria

After analyzing the maximum total utility, we still have to find Nash equilibria in order to analyze the PoA and PoS.

Lemma 3.8. *The player P_i does not want to change his/her strategy d_i if and only if one of the following cases occurs.*

Case I. $\left(\frac{\partial u_i}{\partial d_i}\right)^-$ *does not exist (i.e., $d_i = 0$) and* $\left(\frac{\partial u_i}{\partial d_i}\right)^+ \leq 0$.

Case II. $\left(\frac{\partial u_i}{\partial d_i}\right)^- \geq 0$ *and* $\left(\frac{\partial u_i}{\partial d_i}\right)^+ \leq 0 : \begin{cases} \left(\frac{\partial u_i}{\partial d_i}\right)^- = \left(\frac{\partial u_i}{\partial d_i}\right)^+ = 0 \text{ ....}(A) \\ \left(\frac{\partial u_i}{\partial d_i}\right)^- \geq 0 \textit{ and } \left(\frac{\partial u_i}{\partial d_i}\right)^+ = -1 \text{ ......}(B) \end{cases}$

In case I, $d_i = 0$ and $bd_jp'(0) \leq 1$. In case II-(A), $0 < bd_jp(d_i) < K$ and $bd_jp'(d_i) = 1$. In case II-(B), $bd_jp(d_i) = K$ and $bd_jp'(d_i) \geq 1$.

Proof. Recall the utility function $u_i = -d_i + \min\{K,\ bd_jp(d_i)\}$. Differentiate it with respect to d_i.

$$\frac{\partial u_i}{\partial d_i} = \begin{cases} \frac{\partial}{\partial d_i}\Big(-d_i + bd_jp(d_i)\Big) = bd_jp'(d_i) - 1 \geq -1 & \text{if } bd_jp(d_i) \leq K \\ \frac{\partial}{\partial d_i}\Big(-d_i + K\Big) = -1 & \text{if } bd_jp(d_i) \geq K. \end{cases}$$

Since bd_j is a fixed nonnegative number, and $p'(x)$ is a nonnegative non-increasing function, $\frac{\partial u_i}{\partial d_i}$ is non-increasing for all $d_i \geq 0$. Hence the result follows. □

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Now we are going to discuss the places where these Nash equilibria occur by cases in the following theorems.

Lemma 3.9. *In a Nash equilibrium* (d_1, d_2)*, if a player* P_i*'s strategy* d_i *satisfies case I, then the other player* P_j*'s strategy* d_j *must also satisfy case I. That is,* $d_1 = d_2 = 0$.

Proof. If $d_i = 0$, then $u_j = -d_j + \min\{K,\ b \cdot 0 \cdot p(d_j)\} = -d_j$ for all $d_j \geq 0$, and $\frac{\partial u_j}{\partial d_j} = -1$. According to Lemma 3.8, the player P_j can make an optimal strategy only by letting $d_j = 0$. In this case $\frac{\partial u_i}{\partial d_i}$ is also -1, so the Nash equilibrium can only be $(0, 0)$. □

Definition 3.3. If $bxp'(x) = 1$ has two different solutions, let d_ℓ be the less one, and let d_h be the greater one. If the equation has only one solution, let d_ℓ and d_h both denote it.

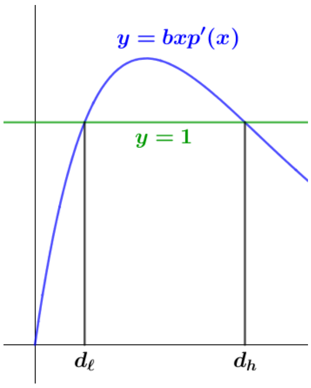

Figure 3.5: A geometric solution of d_ℓ and d_h in Definition 3.3

Theorem 3.10. *There exists a Nash equilibrium* (d_1, d_2) *such that* d_1 *satisfies case II-(A) and* d_2 *satisfies case II-(B), if and only if* d_1 *and* d_2 *both satisfy*

$$\begin{cases} b\, d_2\, p'(d_1) = 1 & \textit{and} \quad b\, d_2\, p(d_1) < K \\ b\, d_1\, p'(d_2) \geq 1 & \textit{and} \quad b\, d_1\, p(d_2) = K. \end{cases}$$

In addition, the Nash equilibrium $N_{side1} = (d_1, d_2)$ *is unique and exists if and only if* d_ℓ *and* d_h *both exist and* $d_\ell < d_o < d_h$.

Proof. First, we prove necessity by contradiction. Assume neither d_ℓ nor d_h exists, or both d_ℓ and d_h exist but $d_\ell = d_h$, or both d_ℓ and d_h exist, $d_\ell < d_h$ but $d_o \notin (d_\ell, d_h)$. Each condition implies $bd_op'(d_o) \leq 1$. Now we're going to explain why $bd_2p'(d_1) < 1$ in this assumption. By Definition 2.6 the point (d_o, d_o) must lie on the curve $bxp(y) = K$, and

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by Corollary 2.3 the constraint $bd_1p(d_2) = K > bd_2p(d_1)$ tells us $d_1 > d_2$. We start from $(x, y) = (d_o, d_o)$ and move along that curve in the correct direction ($x \geq d_o > y$). If $p'(d_o) = 0$, then $p'(x) = 0$ and $byp'(x) = 0$. If $p'(d_o) > 0$, then $byp'(x) \leq byp'(d_o) < bd_op'(d_o) \leq 1$. $\therefore byp'(x) < 1$. If $(x, y) = (d_1, d_2)$, then $bd_2p'(d_1) < 1$ is a contradiction. Nash equilibria cannot exist in this case.

Second, we prove sufficiency. Assume both d_ℓ and d_h exist, and $d_o \in (d_\ell, d_h)$. This condition implies $bd_op'(d_o) > 1$ instead. By Corollary 2.3, we should move from (d_o, d_o) in the same direction ($x \geq d_o > y$) again, so that $byp(x) < K$ and $bxp(y) = K$ always hold. Besides, we know $bxp'(y) \geq bd_op'(d_o)$, and $byp'(x)$ is decreasing, as the point (x, y) goes far away from (d_o, d_o). Since there is a point at infinity $\lim\limits_{\substack{x_o \to \infty \\ y_o \to 0}} (x_o, y_o)$ on the curve such that $\lim\limits_{\substack{x_o \to \infty \\ y_o \to 0}} by_op'(x_o) = 0$, by the intermediate value theorem there must exist one point (x, y) in this direction such that $byp'(x) = 1$. In this case if $(x, y) = (d_1, d_2)$, the Nash equilibrium N_{side1} exists. If $(x, y) \neq (d_1, d_2)$, then either ($x \geq d_1$ and $y < d_2$) or ($x \leq d_1$ and $y > d_2$) happens. If the former happens and $p'(x) > 0$, then $byp'(x) \leq byp'(d_1) < bd_2p'(d_1) = 1$. If the former happens and $p'(x) = 0$, then $byp'(x) = 0$. If the latter happens, then $byp'(x) \geq byp'(d_1) > bd_2p'(d_1) = 1$. The reader may discover that $byp'(x) \neq bd_2p'(d_1) = 1$ in both cases, so the point is unique. □

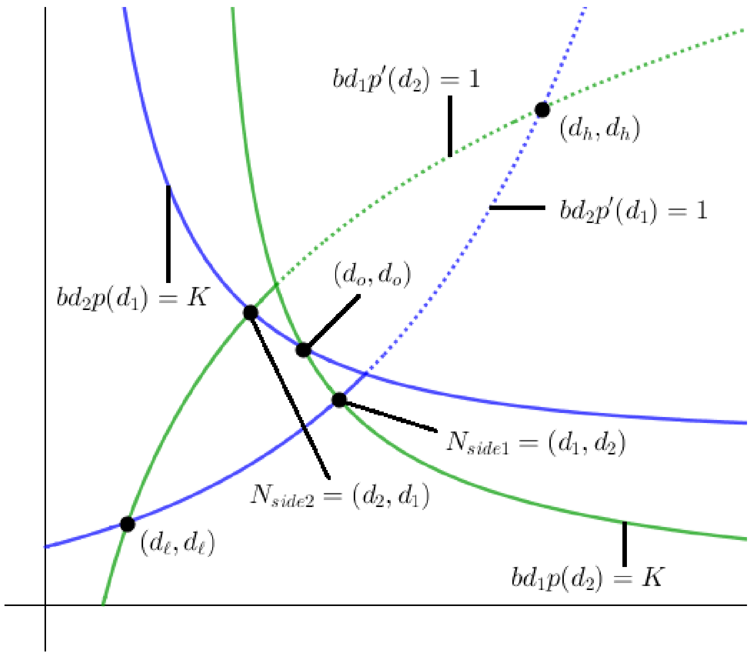

Figure 3.6: A simple diagram of N_{side1} and N_{side2} in Theorem 3.10 and its corollary

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Corollary 3.11. *The Nash equilibrium $N_{side2} = (d_2, d_1)$ exists if and only if $N_{side1} = (d_1, d_2)$ exists.*

Theorem 3.12. *There exists a Nash equilibrium (d_1, d_2) such that the strategies of both players satisfy case II-(A), if and only if d_1 and d_2 both satisfy*

$$\begin{cases} b\, d_2\, p'(d_1) = 1 \quad \textit{and} \quad b\, d_2\, p(d_1) < K \\ b\, d_1\, p'(d_2) = 1 \quad \textit{and} \quad b\, d_1\, p(d_2) < K. \end{cases}$$

In addition, the Nash equilibrium $N_\ell = (d_\ell, d_\ell)$ exists if and only if d_ℓ exists and $d_\ell < d_o$. The Nash equilibrium $N_h = (d_h, d_h)$ exists if and only if d_h exists and $d_h < d_o$.

Proof. According to the constraints $bd_2p'(d_1) = bd_1p'(d_2) > 0$, we can deduce $d_1 = d_2 > 0$ by Lemma 2.4. Thus, these d_i's are in fact the solutions of $bxp'(x) = 1$, by Definition 3.3 one of which is d_ℓ and the other d_h. Since $bd_\ell p(d_\ell) < K = bd_op(d_o) \iff d_\ell < d_o$, and $bd_hp(d_h) < K = bd_op(d_o) \iff d_h < d_o$, the result follows. □

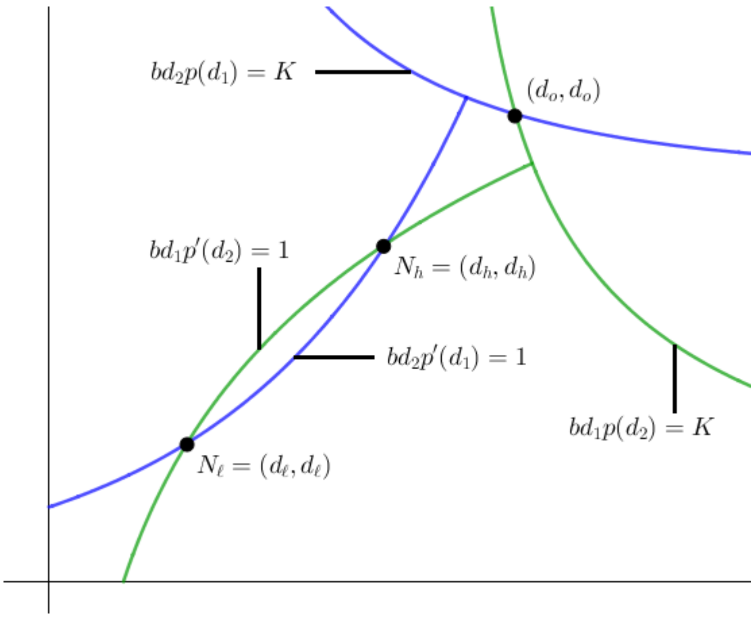

Figure 3.7: A simple diagram of N_ℓ and N_h in Theorem 3.12

Theorem 3.13. *There exists a Nash equilibrium (d_1, d_2) such that the strategies of both players satisfy case II-(B), if and only if d_1 and d_2 both satisfy*

$$\begin{cases} b\, d_2\, p'(d_1) \geq 1 \quad \textit{and} \quad b\, d_2\, p(d_1) = K \\ b\, d_1\, p'(d_2) \geq 1 \quad \textit{and} \quad b\, d_1\, p(d_2) = K. \end{cases}$$

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In addition, the Nash equilibrium $N_o = (d_o, d_o)$ is unique and exists if and only if d_ℓ and d_h both exist and $d_\ell \leq d_o \leq d_h$.

Proof. According to the constraints $bd_2p(d_1) = bd_1p(d_2) > 0$, we can deduce $d_1 = d_2 > 0$ by Corollary 2.3. In this case, these d_i's can only be the solution of $bxp(x) = K$, and by Definition 2.6 it is d_o. Since $bd_op'(d_o) \geq 1 \iff d_\ell \leq d_o \leq d_h$, the result follows. □

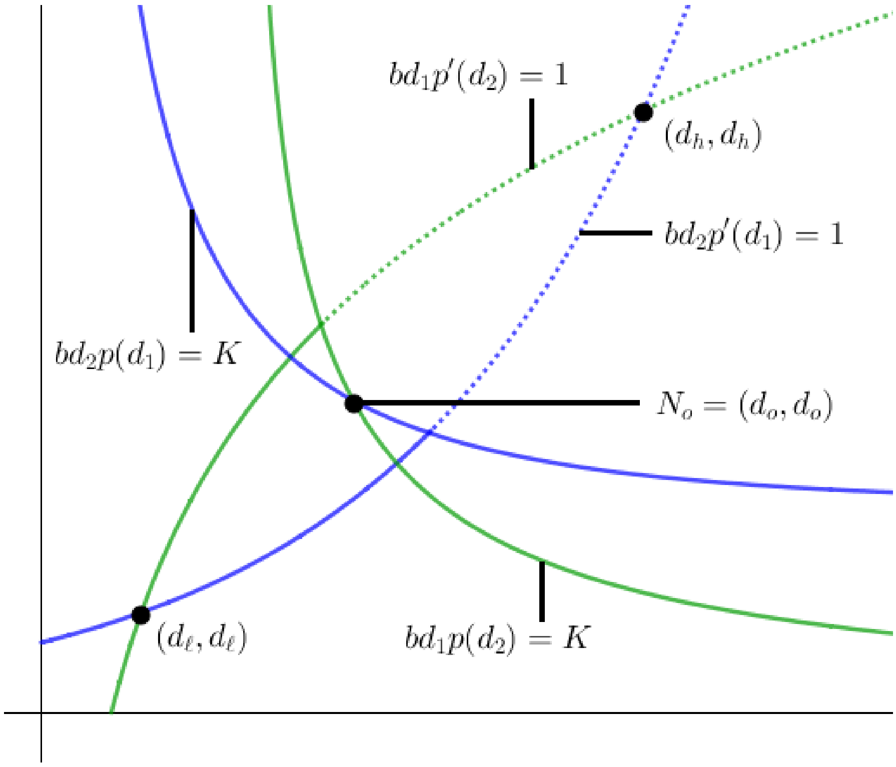

Figure 3.8: A simple diagram of N_o in Theorem 3.13

After discussing conditions for the existence of Nash equilibria, we subsequently want to discuss their stability.

Theorem 3.14. $(0, 0)$ *is always a stable Nash.*

Proof. Consider an extremely small rectangular area whose bottom-left corner is $(0, 0)$. Assume its height is h and its width is w. We want to show that in this area both $\frac{\partial u_1}{\partial d_1}$ and $\frac{\partial u_2}{\partial d_2}$ are negative if h and w are small enough, and neither of these derivatives converges to 0. If this is true, then any point in this area must have a tendency to converge to $(0, 0)$ and we are done. Since this area is extremely small, we assume $bd_1p(d_2) < K$ and $bd_2p(d_1) < K$. Then

$$\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 \quad \text{and} \quad \frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1.$$

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Observing the formula, we discover that the maximum value of $\frac{\partial u_1}{\partial d_1}$ occurs at the top-left corner, and the maximum value of $\frac{\partial u_2}{\partial d_2}$ occurs at the bottom-right corner. To achieve our goal, we can take h such that $bhp'(0) - 1 < 0$ and take w such that $bwp'(0) - 1 < 0$. Therefore both $\frac{\partial u_1}{\partial d_1} \leq bhp'(0) - 1$ and $\frac{\partial u_2}{\partial d_2} \leq bwp'(0) - 1$ in the whole area, and neither of them converges to 0. We can say $(0, 0)$ is a stable Nash. $\square$

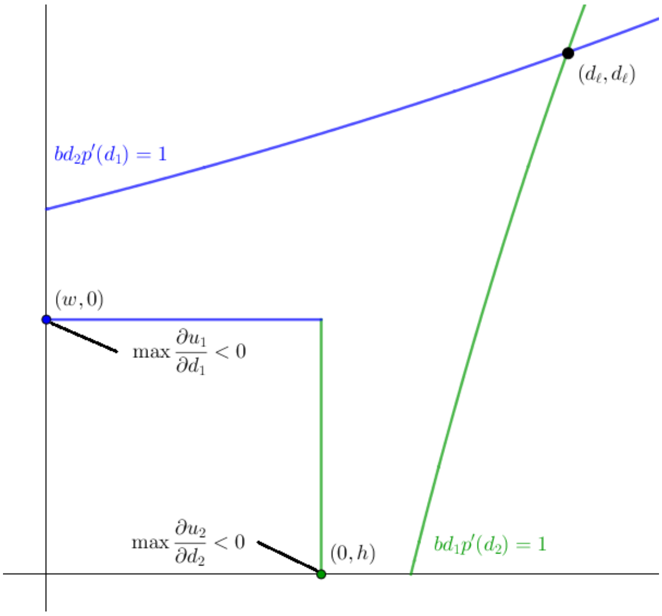

Figure 3.9: A geometric illustration of Theorem 3.14

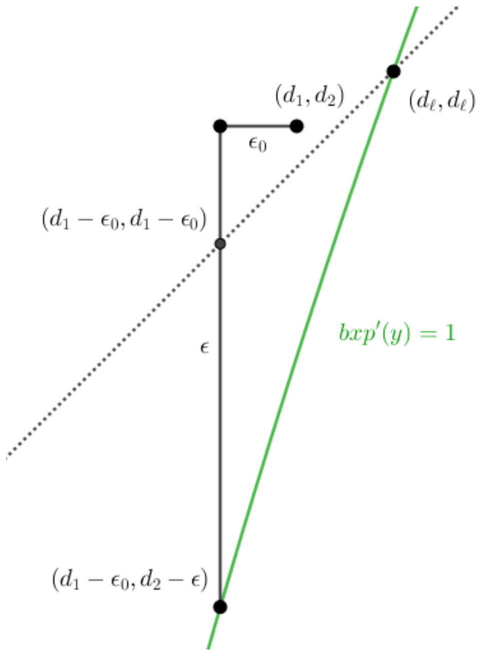

Figure 3.10: A geometric illustration of Lemma 3.15

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Lemma 3.15. *Suppose there are two nonnegative strategies* $d_1 < d_2 \le d_\ell$ *and the initial condition* $bd_1p'(d_2) < 1$. *If we can find two positive numbers* $\epsilon,\ \epsilon_0 > 0$ *such that* $b(d_1 - \epsilon_0)p'(d_2 - \epsilon) = 1$, *then* $\epsilon > \epsilon_0 > 0$.

Proof. Since $d_1 - \epsilon_0 < d_\ell$, we deduce $b(d_1 - \epsilon_0)p'(d_1 - \epsilon_0) < bd_\ell p'(d_\ell) = 1$. Compare it with $b(d_1 - \epsilon_0)p'(d_2 - \epsilon) = 1$, we also deduce $p'(d_1 - \epsilon_0) < p'(d_2 - \epsilon)$ and therefore $d_1 - \epsilon_0 > d_2 - \epsilon$. This inequality implies $\epsilon - \epsilon_0 > d_2 - d_1 > 0$, so $\epsilon > \epsilon_0$. □

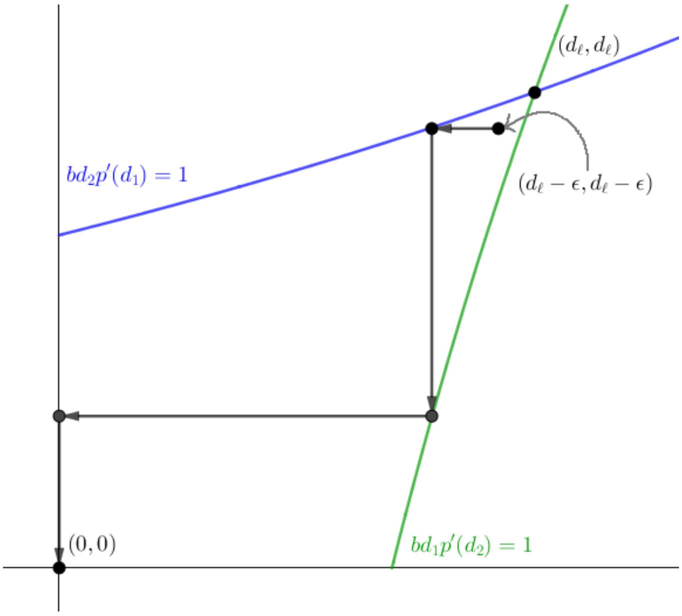

Figure 3.11: A geometric illustration of Theorem 3.16

Theorem 3.16. N_ℓ*, if exists, must be an unstable Nash.*

Proof. Consider the starting point (d_ℓ^-, d_ℓ^-) where $d_\ell^- = d_\ell - \epsilon$ for an arbitrarily small $\epsilon > 0$. If N_ℓ exists, by Theorem 3.12 $bd_\ell^- p(d_\ell^-) < bd_o p(d_o) = K$. Therefore if d_1 and d_2 are non-increasing during the iterative process, then $bd_1p(d_2) < K$, $bd_2p(d_1) < K$, and

$$\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 \quad \text{and} \quad \frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1.$$

At the starting point we have $\frac{\partial u_1}{\partial d_1} = \frac{\partial u_2}{\partial d_2} \le bd_\ell^- p'(d_\ell^-) - 1 < 0$. Without loss of generality, assume d_1 decreases first. It should decrease to 0 or the value such that $\frac{\partial u_1}{\partial d_1} = 0$. If d_1 becomes 0, then $\frac{\partial u_2}{\partial d_2} = b \cdot 0 \cdot p'(d_2) - 1 = -1$ and therefore the system converges to $(0, 0)$. If d_1 is adjusted to achieve $\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 = 0$, then $d_1 < d_2$ and $\frac{\partial u_2}{\partial d_2} =$

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$bd_1p'(d_2) - 1 < 0$. It's d_2's turn to decrease. Since $\frac{\partial u_2}{\partial d_2} < 0$, we should decrease d_2 to 0 or the value such that $\frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1 = 0$. By Lemma 3.15, if d_2 is not decreased to 0, the decrement of d_2 should be greater than that of d_1 in the previous round. Hence the decrement of d_i in each round cannot converge to 0. Based on this fact, the system must finally converge to $(0, 0)$ in finitely many rounds. Since $(0, 0)$ is a Nash equilibrium, it's impossible for the system to go back to N_ℓ again. Therefore N_ℓ is unstable. □

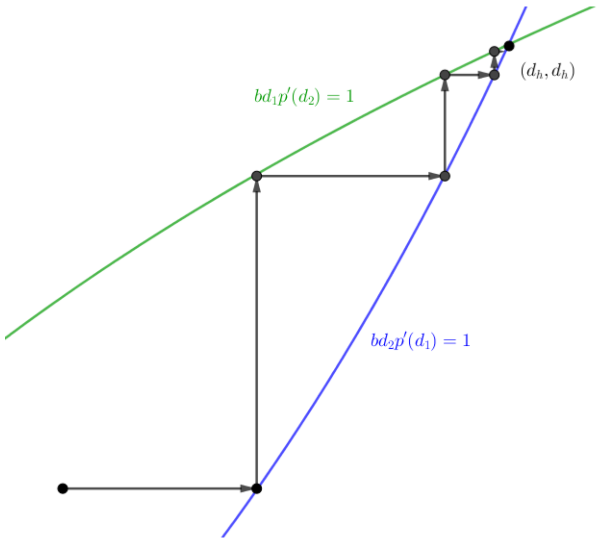

Figure 3.12: A geometric illustration of Lemma 3.17

Lemma 3.17. *If $d_h \le d_o$ and the starting point (x, y) satisfies $byp'(x) \ge 1$, $bxp'(y) \ge 1$, $x > d_\ell$, and $y > d_\ell$, then it must converge to (d_h, d_h).*

Proof. First, we want to show by contradiction that $d_1, d_2 \le d_h$ if $bd_2p'(d_1) \ge 1$ and $bd_1p'(d_2) \ge 1$. If $d_1 > d_h$ and $d_1 \ge d_2$, then $bd_2p'(d_1) \le bd_1p'(d_1) < bd_hp'(d_h) = 1$ is a contradiction. If $d_2 > d_h$ and $d_2 \ge d_1$, then $bd_1p'(d_2) \le bd_2p'(d_2) < bd_hp'(d_h) = 1$ is a contradiction. Hence neither d_1 nor d_2 can be greater than d_h.

Second, since $x \le d_h \le d_o$ and $y \le d_h \le d_o$, we deduce $bxp(y) \le K$ and $byp(x) \le K$, and

$$\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 \quad \text{and} \quad \frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1.$$

Without loss of generality, assume d_1 increases first. This move should let $\frac{\partial u_1}{\partial d_1} = 0$ and $\frac{\partial u_2}{\partial d_2} > 0$. The reader may discover that the move doesn't leave the area $\frac{\partial u_1}{\partial d_1} \ge 0$ and $\frac{\partial u_2}{\partial d_2} \ge 0$ and therefore $d_1, d_2 \le d_h \le d_o$ and the derivatives are not affected by the bound K. It's d_2's turn to increase. The two players will take turn increasing their contributions.

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Finally, since d_1 and d_2 are both monotonic (increasing) and bounded above (not greater than d_h), by the monotone convergence theorem they must converge eventually. The system converges only if $\frac{\partial u_1}{\partial d_1} = \frac{\partial u_2}{\partial d_2} = 0$. By Lemma 2.4 both d_1 and d_2 can only be d_ℓ or d_h at the same time. Since $x > d_\ell$ and $y > d_\ell$, according to monotonicity the system must converge to (d_h, d_h). □

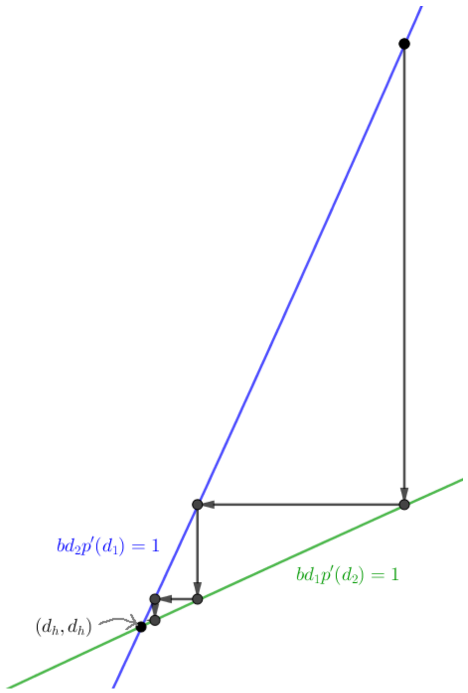

Figure 3.13: A geometric illustration of Lemma 3.18

Lemma 3.18. *If $d_h < d_o$ and the starting point (x, y) which is arbitrarily close to (d_h, d_h) satisfies $byp'(x) \leq 1$, $bxp'(y) \leq 1$, and $x, y \geq d_h$, then it must converge to (d_h, d_h).*

Proof. The proof is very similar to Lemma 3.17. Since (x, y) is arbitrarily close to (d_h, d_h), we deduce $bxp(y) < K$ and $byp(x) < K$, and

$$\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 \quad \text{and} \quad \frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1.$$

Without loss of generality, assume d_2 decreases first. This move should let $\frac{\partial u_2}{\partial d_2} = 0$ and $\frac{\partial u_1}{\partial d_1} < 0$. After that it's d_1's turn to decrease to let $\frac{\partial u_1}{\partial d_1} = 0$ and $\frac{\partial u_2}{\partial d_2} < 0$. The two players will take turn decreasing their contributions (strategies). The reader may discover that none of the moves leaves the area $\frac{\partial u_1}{\partial d_1} \leq 0$ and $\frac{\partial u_2}{\partial d_2} \leq 0$. Based on this fact, we want to show by contradiction that $d_1, d_2 \geq d_h$ during the iterative process.

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If it's d_2's turn to decrease and after that $d_1 \geq d_h > d_2$, then $\frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1 \geq bd_hp'(d_2) - 1 > bd_hp'(d_h) - 1 = 0$ which leaves the area $\frac{\partial u_2}{\partial d_2} \leq 0$. Therefore d_2 should be greater than or equal to d_h. This argument can also be applied to the case when it's d_1's turn to decrease.

Finally, since d_1 and d_2 are both monotonic (decreasing) and bounded below (not less than d_h), by the monotone convergence theorem they must converge eventually. The system converges only if $\frac{\partial u_1}{\partial d_1} = \frac{\partial u_2}{\partial d_2} = 0$. By Lemma 2.4 both d_1 and d_2 can only be d_ℓ or d_h at the same time. Since $d_1 \geq d_h$ and $d_2 \geq d_h$, the system must converge to (d_h, d_h). □

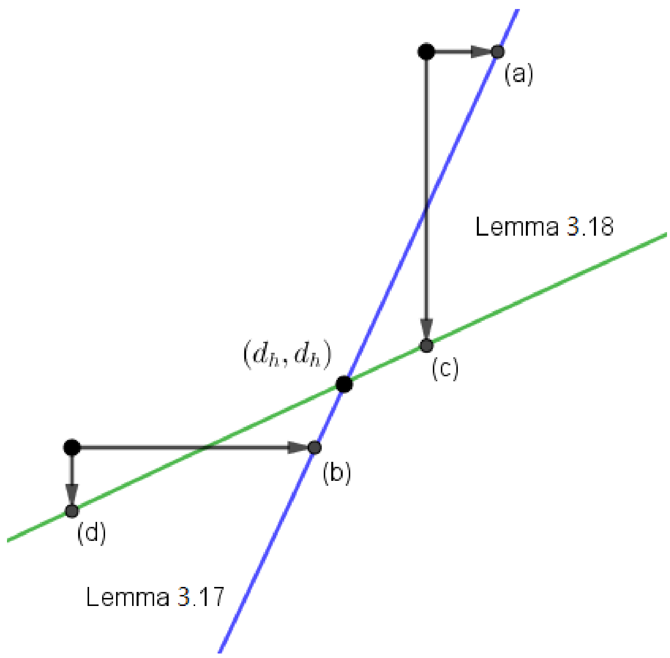

Figure 3.14: A geometric illustration of Theorem 3.19

Theorem 3.19. *N_h, if exists, must be a stable Nash equilibrium if $d_\ell < d_h$, and be unstable if $d_\ell = d_h$.*

Proof. If $d_\ell = d_h$, then N_h and N_ℓ are the same. By Theorem 3.16 N_h is unstable. If $d_\ell < d_h$, we can consider an extremely small region centered at (d_h, d_h). By Theorem 3.12 $d_h < d_o$, we deduce $bd_1p(d_2) < K$ and $bd_2p(d_1) < K$, and

$$\frac{\partial u_1}{\partial d_1} = bd_2p'(d_1) - 1 \quad \text{and} \quad \frac{\partial u_2}{\partial d_2} = bd_1p'(d_2) - 1.$$

Consider an arbitrary starting point (x, y) in the area. If (x, y) satisfies $\frac{\partial u_1}{\partial d_1} \geq 0 \wedge \frac{\partial u_2}{\partial d_2} \leq 0$,

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then (a) $\frac{\partial u_1}{\partial d_1} = 0 \wedge \frac{\partial u_2}{\partial d_2} \leq 0$ or (b) $\frac{\partial u_1}{\partial d_1} = 0 \wedge \frac{\partial u_2}{\partial d_2} \geq 0$ happens after we increase d_1 a little, or (c) $\frac{\partial u_2}{\partial d_2} = 0 \wedge \frac{\partial u_1}{\partial d_1} \leq 0$ or (d) $\frac{\partial u_2}{\partial d_2} = 0 \wedge \frac{\partial u_1}{\partial d_1} \geq 0$ happens after we decrease d_2 a little. If (a) happens, then $bd_2p'(d_1) = 1$ and $bd_1p'(d_2) \leq 1$. The constraint gives $d_2 \geq d_1$. Suppose for contradiction that $d_\ell < d_1 < d_h$. Then $bd_2p'(d_1) \geq bd_1p'(d_1) > bd_hp'(d_h) = 1$ contradicts $bd_2p'(d_1) = 1$. Hence $d_2 \geq d_1 \geq d_h$, and by Lemma 3.18 the system converges to N_h. By symmetry, the same conclusion holds for (c). If (b) happens, we can simply use Lemma 3.17 to show that the system converges to N_h. By symmetry, the same conclusion holds for (d). To sum up, the system converges to N_h if (x, y) satisfies $\frac{\partial u_1}{\partial d_1} \geq 0 \wedge \frac{\partial u_2}{\partial d_2} \leq 0$.

By symmetry, we can also say the system converges to N_h if (x, y) satisfies $\frac{\partial u_1}{\partial d_1} \leq 0 \wedge \frac{\partial u_2}{\partial d_2} \geq 0$. According to Lemma 3.17, the same conclusion also holds if (x, y) satisfies $\frac{\partial u_1}{\partial d_1} \geq 0 \wedge \frac{\partial u_2}{\partial d_2} \geq 0$. Now we consider the last case when $\frac{\partial u_1}{\partial d_1} \leq 0 \wedge \frac{\partial u_2}{\partial d_2} \leq 0$. Suppose for contradiction that $d_\ell < d_1 < d_h$ and $d_1 \leq d_2$. Then $bd_2p'(d_1) \geq bd_1p'(d_1) > bd_hp'(d_h) = 1$ contradicts $bd_2p'(d_1) \leq 1$. Hence $d_1, d_2 \geq d_h$ in this case. By Lemma 3.18 the system also converges to N_h. Therefore all points very close to (d_h, d_h) will converge to N_h. It is stable. □

Theorem 3.20. *N_o, if exists, is unstable when $d_\ell \leq d_o < d_h$ or $d_\ell = d_o = d_h$, and is stable when $d_\ell < d_o = d_h$.*

Proof. If $d_\ell = d_o$, then we can repeat the proof in Theorem 3.16 to say N_o is unstable. If $d_\ell < d_o < d_h$, we want to show that in any arbitrarily small region centered at N_o, there must exist at least one point which will converge to N_{side1} (or N_{side2}). Consider the iterative process in the reverse direction (starting from N_{side1}). We want to construct a path from N_{side1} to N_o with the following procedure. Recall the constraint of N_{side1}: $(b\ d_2\ p(d_1) < K) \wedge (b\ d_1\ p(d_2) = K)$. Let d_2 increase first such that $(b\ d_2\ p(d_1) = K) \wedge (b\ d_1\ p(d_2) \geq K)$ and we say the system arrives at the point P_1. Then d_1 decreases such that $(b\ d_2\ p(d_1) \leq K) \wedge (b\ d_1\ p(d_2) = K)$ and the system arrives at the point P_2. The two players will take turn making an ultimate adjustment of their strategies under the constraint $(b\ d_2\ p(d_1) \leq K) \wedge (b\ d_1\ p(d_2) \geq K)$ and obtain the subsequent points P_3, P_4, P_5, and so on. In the following paragraphs we want to show this procedure will converge to (d_o, d_o) eventually.

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First, we show the boundedness. Consider the constraint $b\, d_1\, p(d_2) \geq K \geq b\, d_2\, p(d_1)$. By Corollary 2.3, we deduce $d_1 \geq d_2$. Then we want to prove $d_1 \geq d_o \geq d_2$ by contradiction. If $d_o > d_1 \geq d_2$, then $bd_1p(d_2) < bd_op(d_o) = K$ contradicts $b\, d_1\, p(d_2) \geq K$. If $d_1 \geq d_2 > d_o$, then $bd_2p(d_1) > bd_op(d_o) = K$ contradicts $b\, d_2\, p(d_1) \leq K$. Hence d_1 is bounded below by d_o, and d_2 is bounded above by d_o.

Second, we show the monotonicity. Recall the inequality $bd_2p'(d_1) \geq 1$ of the constraint of N_{side1}. Since in our iterative process d_1 is non-increasing and d_2 is non-decreasing, $bd_2p'(d_1) \geq 1$ is always true. Since $d_1 \geq d_o \geq d_2$, we have $bd_1p'(d_2) \geq bd_op'(d_o) > 1$. Therefore $p'(d_1) > 0$ and $p'(d_2) > 0$. It means that neither of $p(d_1)$ and $p(d_2)$ reaches 1 during the iterative process, and either $(b\ d_2\ p(d_1) = K) \wedge (b\ d_1\ p(d_2) > K)$ or $(b\ d_2\ p(d_1) < K) \wedge (b\ d_1\ p(d_2) = K)$ happens in each move. In fact it also implies that d_1 is "strictly decreasing" and d_2 is "strictly increasing." The monotonicity is proven.

By the monotone convergence theorem, the procedure must eventually converge to some point. If it converges, then solving the equation $bd_1p(d_2) = K = bd_2p(d_1)$ by Corollary 2.3 gives us $d_1 = d_2 = d_o$ and we can say the procedure finally converges to (d_o, d_o). It means that in any arbitrarily small region centered at N_o, we can always find some point P_N for a sufficiently large N. Since $bd_2p(d_1) \leq K$ and $bd_2p'(d_1) \geq 1$ in this area, we deduce $\frac{\partial u_1}{\partial d_1} \geq 0$. Since $bd_1p(d_2) \geq K$ in this area, we deduce $\frac{\partial u_2}{\partial d_2} = -1$. We can guarantee that the system naturally goes from P_N to P_{N-1}, and goes from P_{N-1} to P_{N-2}, and so on. Finally it reaches N_{side1} and can never go back to N_o. N_o is unstable.

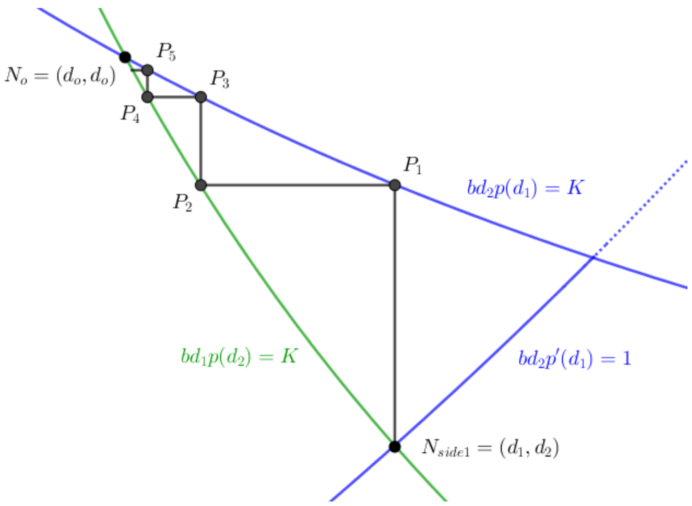

Figure 3.15: A geometric illustration of the case $d_\ell < d_o < d_h$

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If $d_\ell < d_o = d_h$, then there must be some point around N_h that causes $bd_1p(d_2) > K$ or $bd_2p(d_1) > K$. If one of these inequalities holds, the corresponding player should decrease his/her contribution (strategy) such that $bd_1p(d_2) \leq K$ and $bd_2p(d_1) \leq K$. In this case one of $d_1 \leq d_h$ and $d_2 \leq d_h$ must hold. Otherwise, both d_1 and d_2 are larger than d_h which causes $bd_1p(d_2)$ and $bd_2p(d_1)$ to be larger than K. This is a contradiction. Without loss of generality, we only assume $d_1 \leq d_h$ in our proof. Since the points are arbitrarily close to N_h, we should always keep in mind that $d_1, d_2 > d_\ell$. This section can be split into two cases $d_1 \geq d_2$ and $d_1 < d_2$.

If $d_1 \geq d_2$, then we deduce $bd_1p'(d_2) \geq bd_1p'(d_1) \geq 1$. Since $bd_1p'(d_2) \geq bd_2p'(d_1)$ must hold when $d_1 \geq d_2$, either $bd_1p'(d_2) \geq bd_2p'(d_1) \geq 1$ or $bd_1p'(d_2) > 1 \geq bd_2p'(d_1)$ happens. If the former happens, the system will automatically converge to N_h by Lemma 3.17. If the latter happens, we can either decrease d_1 to reach $bd_2p'(d_1) = 1$ or increase d_2 to reach $bd_1p'(d_2) = 1$. If we can decrease d_1 to x, then x must be larger than d_2. Otherwise, $bd_2p'(x) \geq bd_2p'(d_2) > 1$ is a contradiction. Since $x > d_2$, we deduce $bxp'(d_2) > bd_2p'(d_2) \geq 1$. By Lemma 3.17 the system will automatically converge to N_h. If we can increase d_2 to x, then x must be larger than or equal to d_1. Otherwise, $bd_1p'(x) > bd_1p'(d_1) \geq 1$ is a contradiction. Additionally, $x \leq d_h$. Otherwise, $bd_1p'(x) < bd_1p'(d_h) \leq bd_hp'(d_h) = 1$ is a contradiction. Since $d_1 \leq x$ (new d_2) $\leq d_h$, neither $bd_1p(x)$ nor $bxp(d_1)$ exceeds K, $\frac{\partial u_2}{\partial d_2}$ is not affected by K, and therefore increasing d_2 is possible. We reach $bxp'(d_1) \geq bd_1p'(x) = 1$ and by Lemma 3.17 the system will automatically converge to N_h.

If $d_1 < d_2$, then we deduce $bd_2p'(d_1) > bd_1p'(d_1) > 1$. Since $bd_2p'(d_1) > bd_1p'(d_2)$ must hold when $d_1 < d_2$, either $bd_2p'(d_1) > bd_1p'(d_2) > 1$ or $bd_2p'(d_1) > 1 \geq bd_1p'(d_2)$ happens. If $bd_2p'(d_1) > bd_1p'(d_2) \geq 1$ happens, the system will automatically converge to N_h by Lemma 3.17. If $bd_2p'(d_1) > 1 > bd_1p'(d_2)$ happens, we can either decrease d_2 to reach $bd_1p'(d_2) = 1$ or increase d_1 to reach $bd_2p'(d_1) = 1$. If we want to decrease d_2 now, the argument of the case to decrease d_1 in the previous paragraph can also be used to show the convergence. If we want to increase d_1 instead, then either (a) $bd_2p(d_1) = K$ or (b) $bd_2p'(d_1) = 1$ happens first. If (a) happens first, then this step fails to satisfy $bd_2p'(d_1) = 1$.

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It implies $bd_2p'(d_1) > 1$ still remains. In this case if $bd_1p'(d_2) \geq 1$, we simply use Lemma 3.17 to show that the system will finally converge to N_h. If $bd_1p'(d_2) < 1$, we can decrease d_2 again and use the previous argument of the case $bd_2p'(d_1) > 1 > bd_1p'(d_2)$ to show the convergence. If (b) happens first, then we can also use the argument of the case to increase d_2 in the previous paragraph to show the convergence. Hence we are done.

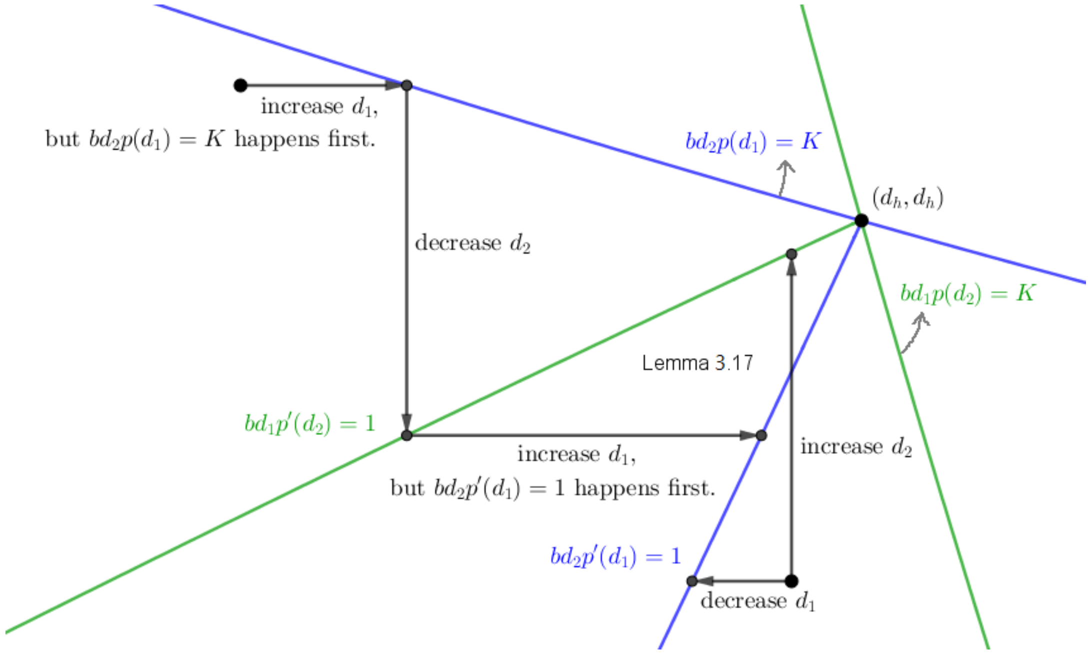

Figure 3.16: A geometric illustration of the case $d_\ell < d_o = d_h$

□

We close this section with the following conclusive table.

Table 3.2: Summary of Nash equilibria of two-player file-sharing games in ascending order of their total utility

Point	**Stability**	**Condition**
$(0,0)$	YES	-
N_ℓ	NO	$d_\ell < d_o$
N_{side1}, N_{side2}	unknown	$d_\ell < d_o < d_h$
N_o	(almost) NO	$d_\ell \leq d_o \leq d_h$
N_h	(almost) YES	$d_h < d_o$

3.3 The PoA and PoS

Finally, we're going to calculate the PoS and PoA and observe their properties. Our analysis is split into three different cases depending on the value of d_o. Before the analysis, we give two definitions to show that we don't care about the origin.

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Definition 3.4. A Nash equilibrium not on the origin is a non-collapsing Nash equilibrium.

Definition 3.5. If we only consider these non-collapsing Nash equilibria when calculating a PoA, the result is a non-collapsing PoA. In this section we only care about the non-collapsing PoA.

The objective of Lemma 3.21, Lemma 3.22, and Theorem 3.23 is to sort all the Nash equilibria by the total utility function.

Lemma 3.21. *Given two Nash equilibria* $N_x = (d_{x_1}, d_{x_2})$ *and* $N_y = (d_{y_1}, d_{y_2})$*, if* $d_{x_1} \geq d_{y_1} \geq d_\ell$ *and* $d_{x_2} \geq d_{y_2} \geq d_\ell$*, then* $u(N_x) \geq u(N_y)$ *.*

Proof. Since $bp(d_\ell) \geq bd_\ell p'(d_\ell) = 1$, we ensure that $bp(d_{x_i}) - 1 \geq 0$ and $bp(d_{y_i}) - 1 \geq 0$ are always true for all parameters not less than d_ℓ. In addition, $bd_{x_i} p(d_{x_j}) \leq K$ and $bd_{y_i} p(d_{y_j}) \leq K$ are always true for all parameters because N_x and N_y are Nash equilibria. We can write

$$u(N_x) = d_{x_1}\Big(bp(d_{x_2}) - 1\Big) + d_{x_2}\Big(bp(d_{x_1}) - 1\Big), \text{ and}$$
$$u(N_y) = d_{y_1}\Big(bp(d_{y_2}) - 1\Big) + d_{y_2}\Big(bp(d_{y_1}) - 1\Big).$$

It is clear to see that $bp(d_{x_2}) - 1 \geq bp(d_{y_2}) - 1 \geq 0$ and $bp(d_{x_1}) - 1 \geq bp(d_{y_1}) - 1 \geq 0$, so $u(N_x) \geq u(N_y)$. □

Lemma 3.22. *If* $N_{side1} = (d_1, d_2)$ *exists, then the order of the parameters should be the following:* $d_\ell < d_2 < d_o < d_1 < d_h$*.*

Proof. Recall the constraint of N_{side1}:

$$\begin{cases} b\, d_2\, p'(d_1) = 1 \quad \text{and} \quad b\, d_2\, p(d_1) < K \\ b\, d_1\, p'(d_2) \geq 1 \quad \text{and} \quad b\, d_1\, p(d_2) = K. \end{cases}$$

Since $b\, d_1\, p(d_2) = b\, d_o\, p(d_o) > b\, d_2\, p(d_1)$, by Theorem 2.3 we deduce $d_2 < d_o < d_1$. Compare $b\, d_2\, p'(d_1) = 1$ with $b\, d_h\, p'(d_h) = 1$. If $d_1 \geq d_h$, then $p'(d_1) \leq p'(d_h)$. The inequality along with $d_2 < d_o < d_h$ together implies $b\, d_2\, p'(d_1) < 1$ which is a contradiction. Therefore $d_1 < d_h$. Compare $b\, d_2\, p'(d_1) = 1$ with $b\, d_\ell\, p'(d_\ell) = 1$. If $d_2 \leq d_\ell$, then $d_\ell < d_o < d_1$ implies $p'(d_1) < p'(d_\ell)$ and therefore $b\, d_2\, p'(d_1) < 1$ which is

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a contradiction. Therefore $d_\ell < d_2$. Combining $d_\ell < d_2$, $d_2 < d_o < d_1$, and $d_1 < d_h$, we obtain our final result. □

Theorem 3.23. *If $d_o = d_\ell$, then the order of existing Nash equilibria should be $u(O) < u(N_o)$. If $d_\ell < d_o < d_h$, then the order of existing Nash equilibria should be $u(O) < u(N_\ell) < u(N_{side1}) = u(N_{side2}) \leq u(N_o)$. If $d_o = d_h$, then $u(O) < u(N_\ell) \leq u(N_o)$. If $d_o > d_h$, then $u(O) < u(N_\ell) \leq u(N_h)$.*

Proof. Since $d_\ell > 0$ and $bp(d_\ell) > bd_\ell p'(d_\ell) = 1$, we deduce $u(0,0) = 0 < 2d_\ell\Big(bp(d_\ell) - 1\Big) = u(N_\ell)$. If $N_{side1} = (d_1, d_2)$ exists, then $d_\ell < d_2 < d_o < d_1 < d_h$ by Lemma 3.22. We can deduce $u(N_{side1}) = 2d_2\big[bp(d_1) - 1\big] > 2d_\ell\Big(bp(d_\ell) - 1\Big) = u(N_\ell)$ by Lemma 3.21. Besides, $u(N_{side1}) = u(N_{side2})$ by symmetry. N_o exists only if $d_\ell \leq d_o$. According to this inequality, we deduce $bp(d_o) \geq bp(d_\ell) > 1$ and therefore $d_o < K$. By Theorem 3.6, N_o has the maximum total utility if it exists. Since N_{side1} exists only if N_o exists, we can say $u(N_{side1}) \leq u(N_o)$. If $d_o \geq d_h$, the result also follows from Lemma 3.21. □

Theorem 3.24 states the conclusion when neither d_ℓ nor d_h exists, or both exist but $d_o < d_\ell$.

Theorem 3.24. *If neither d_ℓ nor d_h exists, or both exist but $d_o < d_\ell$, then the maximum total utility of all existing Nash equilibria must be 0.*

Proof. Since in this case the only existing Nash equilibrium is $(0,0)$ and $u(0,0) = 0$, the result follows. □

Lemma 3.25 is an auxiliary proposition helping us in observing how the PoA, PoS vary with the parameters b.

Lemma 3.25. *If K and $p(x)$ remain fixed, and b is the only varying parameter, then: (a) $\dfrac{\partial d_o}{\partial b} < 0$, (b) $\dfrac{\partial d_\ell\Big(bp(d_\ell) - 1\Big)}{\partial b} < 0$, and (c) $\dfrac{\partial d_\ell}{\partial b} < \dfrac{\partial d_o}{\partial b}$ when $d_\ell = d_o$.*

Proof. Part (a) can be directly deduced from the definition $bd_o p(d_o) = K$. For part (b), recall the definition $bd_\ell p'(d_\ell) = 1$ first. Since d_ℓ is the less solution to $bd_\ell p'(d_\ell) = 1$, by Lemma 2.1 we have $\partial d_\ell / \partial b < 0$. It means that when b increases, d_ℓ decreases, $p'(d_\ell)$

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increases, $\frac{1}{p'(d_\ell)}$ decreases, and therefore $\frac{\partial}{\partial b}\Big(\frac{1}{p'(d_\ell)}\Big) < 0$. Also write $\frac{\partial}{\partial b}\Big(\frac{1}{p'(d_\ell)}\Big) = \frac{\partial(bd_\ell)}{\partial b} = d_\ell + b\frac{\partial d_\ell}{\partial b}$, so $d_\ell + b\frac{\partial d_\ell}{\partial b} < 0$.

$$\begin{aligned}\frac{\partial\Big(bd_\ell p(d_\ell) - d_\ell\Big)}{\partial b} &= d_\ell p(d_\ell) + b\frac{\partial d_\ell}{\partial b}p(d_\ell) + bd_\ell p'(d_\ell)\frac{\partial d_\ell}{\partial b} - \frac{\partial d_\ell}{\partial b}\\ &= d_\ell p(d_\ell) + b\frac{\partial d_\ell}{\partial b}p(d_\ell)\\ &= p(d_\ell)\Big(d_\ell + b\frac{\partial d_\ell}{\partial b}\Big) < 0.\end{aligned}$$

For part (c), we go back to $bd_op(d_o) = K$. According to this equality, $\frac{\partial}{\partial b}\Big(\frac{K}{p(d_o)}\Big) = \frac{\partial(bd_o)}{\partial b} = d_o + b\frac{\partial d_o}{\partial b} > 0$. Comparing with $d_\ell + b\frac{\partial d_\ell}{\partial b} < 0$ from (b), we obtain (c). □

Theorem 3.26 states the relationship between the PoA, PoS and the parameters b, K when $d_\ell \le d_o \le d_h$.

Theorem 3.26. *If both d_ℓ and d_h exist, and $d_\ell \le d_o \le d_h$, then the $PoS = 1$ and the $PoA = \frac{u_{opt}}{u(d_\ell)} = \frac{u(N_o)}{u(d_\ell)} = \frac{d_o\Big(bp(d_o)-1\Big)}{d_\ell\Big(bp(d_\ell)-1\Big)}$. Furthermore, when b and $p(x)$ are fixed, and K is the only varying parameter, the PoA approaches 1 as K decreases such that d_o approaches d_ℓ, and the PoA approaches its maximum $\frac{d_h\Big(bp(d_h)-1\Big)}{d_\ell\Big(bp(d_\ell)-1\Big)}$ as K increases such that d_o approaches d_h. When K and $p(x)$ are fixed, and b is the only varying parameter, the PoA approaches infinity as b keeps increasing, and the PoA approaches its minimum as b decreases such that d_o approaches d_h.*

Proof. By Corollary 3.7 $u_{opt} = u(N_o)$, so the $PoS = 1$. By Theorem 3.23, the worst non-collapsing Nash equilibrium is the point (d_ℓ, d_ℓ). Hence the $PoA = \frac{u(d_o)}{u(d_\ell)}$. If b, $p(x)$ are fixed and only K varies, then only d_o varies with it and the denominator doesn't change. Since $bp(d_o) \ge bp(d_\ell) > 1$, the PoA increases with d_o (and K).

Consider the case when b is the only varying parameter. We should also note that the PoA can be written as $\frac{K - d_o}{d_\ell\Big(bp(d_\ell)-1\Big)}$. By Lemma 3.25 $\frac{\partial d_o}{\partial b} < 0$ and $\frac{\partial d_\ell\Big(bp(d_\ell)-1\Big)}{\partial b} < 0$, so the numerator increases, the denominator decreases, and the PoA increases with b. If K, $p(x)$ are fixed and b is the only increasing parameter, by part (c) of Lemma 3.25 the

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inequality $d_\ell \le d_o \le d_h$ always remains, so the PoA increases unboundedly. If K, $p(x)$ are fixed and b is the only decreasing parameter, by part (c) of Lemma 3.25 the inequality $d_\ell \le d_o$ remains, but d_o may exceed d_h. Therefore the PoA achieves its minimum as d_o achieves its maximum (d_h). □

Theorem 3.27 states how the side Nash equilibria affect the "stable" PoS and PoA.

Theorem 3.27. *If N_{side1} and N_{side2} both exist and are stable Nash equilibria, then the "stable" PoS and PoA are* $\dfrac{u_{opt}}{u(N_{side1})} = \dfrac{u(d_o)}{u(N_{side1})} < 2$.

Proof. The side Nash equilibria exist only if $d_\ell < d_o < d_h$, in this case other existing non-collapsing Nash equilibria are unstable. Hence if these side Nash equilibria are stable, the "stable" PoS and PoA should be $\dfrac{u_{opt}}{u(N_{side1})} = \dfrac{u(d_o)}{u(N_{side1})}$. Let $N_{side1} = (d_1, d_2)$. Recall the constraint of N_{side1}: $\begin{cases} bd_1p(d_2) = K \\ bd_2p'(d_1) = 1. \end{cases}$ Then the ratio can also be written as $\dfrac{d_o\Big(bp(d_o)-1\Big) + d_o\Big(bp(d_o)-1\Big)}{d_1\Big(bp(d_2)-1\Big) + d_2\Big(bp(d_1)-1\Big)} = \dfrac{2\,(K-d_o)}{K - d_1 + \frac{p(d_1)}{p'(d_1)} - d_2}$. As long as we can prove $K - d_1 + \frac{p(d_1)}{p'(d_1)} - d_2 > K - d_o$, then we are done. Since $p(d_1) > d_1p'(d_1)$, we first deduce $K - d_1 + \frac{p(d_1)}{p'(d_1)} - d_2 > K - d_2$. Since $d_1 \ge d_2$, we also deduce $d_1 \ge d_o \ge d_2$ from the definition $bd_op(d_o) = K$, and therefore $K - d_2 \ge K - d_o$. Hence we are done. □

Theorem 3.28 states the relationship between the PoA, PoS and the parameters b, K when $d_h < d_o$.

Theorem 3.28. *If both d_ℓ and d_h exist, and $d_h < d_o$, the* $PoS = \dfrac{u_{opt}}{u(N_h)} = \dfrac{d_o\Big(bp(d_o)-1\Big)}{d_h\Big(bp(d_h)-1\Big)}$ *and the* $PoA = \dfrac{u_{opt}}{u(N_\ell)} = \dfrac{d_o\Big(bp(d_o)-1\Big)}{d_\ell\Big(bp(d_\ell)-1\Big)}$. *If we only consider the non-collapsing stable Nash equilibria, then the "stable" PoA becomes* $\dfrac{u_{opt}}{u(N_h)}$. *Furthermore, when b and $p(x)$ are fixed, and K is the only varying parameter, the PoS approaches 1 and the PoA approaches its greatest lower bound* $\dfrac{d_h\Big(bp(d_h)-1\Big)}{d_\ell\Big(bp(d_\ell)-1\Big)}$ *as K decreases such that d_o approaches d_h, and both the $PoS = \Theta(K)$ and $PoA = \Theta(K)$ approach infinity as K*

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keeps increasing. When K and $p(x)$ are fixed, and b is the only varying parameter, the PoS approaches 1 and the PoA approaches its least upper bound as b increases such that d_o approaches d_h, and the PoS approaches its maximum and the PoA approaches its minimum as b keeps decreasing until d_h does not exist.

Proof. By Corollary 3.7, u_{opt} occurs at $u(d_o)$. By Theorem 3.23, N_h has the maximum total utility, and N_ℓ has the minimum total utility among all existing non-collapsing Nash equilibria. Hence the PoS and PoA in our theorem follow. If we only consider the non-collapsing stable Nash equilibria, then N_h is the only one. Hence the "stable" PoA in our theorem follows.

According to the proof in Theorem 3.26, the PoS and PoA both increase with d_o (and K). We should also note that the numerator can be expressed as $K - d_o$. When K is very large, $p(d_o)$ approaches 1 and therefore $d_o = \dfrac{K}{bp(d_o)} \approx \dfrac{K}{b}$, so $K - d_o \approx K - \dfrac{K}{b} = K(1 - \dfrac{1}{b}) = \Theta(K)$.

According to the proof in Theorem 3.26, the PoA increases with b. We should also note that the PoS can be written as $\dfrac{d_o}{d_h} \cdot \dfrac{p(d_o) - 1/b}{p(d_h) - 1/b}$. If $K, p(x)$ are fixed and b is the only increasing parameter, then d_o and $p(d_o)$ decrease, and d_h and $p(d_h)$ increase. In addition, adding the same quantity to both the numerator and denominator of an improper fraction decreases its value. Therefore we can deduce the PoS decreases with b instead. □

We close this chapter with the following tables concluding Theorem 3.24, Theorem 3.26, and Theorem 3.28.

Table 3.3: Summary of the PoS and PoA with K as the only varying parameter. We assume d_o starts at d_ℓ and keeps increasing.

Condition	$d_\ell \le d_o \le d_h$ **(Phase 1)**	$d_h < d_o$ **(Phase 2)**
PoS	1	$u(d_o)/u(d_h)$ **(stable PoA)**
Monotonicity	-	increasing
Starting at 1	-	Yes
PoA	$u(d_o)/u(d_\ell)$	
Monotonicity	increasing	
Starting at 1	Yes	

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Table 3.4: Summary of the PoS and PoA with b as the only varying parameter. We assume b starts at its valid minimum value (i.e. $bxp'(x) = 1$ has exactly one solution.) and keeps increasing.

Condition	$d_o > d_h$ **(Phase 1)**	$d_h \geq d_o \geq d_\ell$ **(Phase 2)**
PoS	$u(d_o)/u(d_h)$ **(stable PoA)**	1
Monotonicity	decreasing	-
Terminating at 1	YES	-
PoA	$u(d_o)/u(d_\ell)$	
Monotonicity	increasing	
Starting at 1	No	

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Chapter 4

Nash Equilibrium Analysis for Three-Player File-Sharing Games

After the analysis of two-player file-sharing games, we want to consider a three-player file-sharing game. Like Chapter 3, the model can be simplified to the following again.

$$\begin{cases} u_1(d_1) = -d_1 + \min\{K,\ b\, d_2\, p(d_1)\} + \min\{K,\ b\, d_3\, p(d_1)\} \\ u_2(d_2) = -d_2 + \min\{K,\ b\, d_1\, p(d_2)\} + \min\{K,\ b\, d_3\, p(d_2)\} \\ u_3(d_3) = -d_3 + \min\{K,\ b\, d_1\, p(d_3)\} + \min\{K,\ b\, d_2\, p(d_3)\} \\ u = u_1(d_1) + u_2(d_2) + u_3(d_3). \end{cases}$$

We also use the notation $u(d) = u(d, d, d)$ if d_1, d_2, and d_3 have the same value of d.

In this chapter, we do almost the same thing as in Chapter 3, including finding all Nash equilibria under different parameter settings, analyzing their efficiency (PoA and PoS), and observing how they vary with system parameters b and K. The only exception is that we don't care about their stability here. Similarly, we begin with the section which aims to find the maximum total utility.

4.1 Maximum Total Utility

Although for two-player games we detailedly analyzed gradients of almost all points in the domain, the technique is too complicated to apply to the three-player games. In the

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light of this, we use another way in this chapter to prove that the point where u_{opt} occurs can still be (d, d, d) where $b\,d\,p(d) = K$ under some particular parameter settings.

The structure of our proof is illustrated below. Observing the formula of our model, we can split u into two parts.

$$u = \sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} \min\{K,\ b\,d_i\,p(d_j)\} - (d_1 + d_2 + d_3).$$

After that, we want to show for each surface $\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} b\,d_i\,p(d_j) = C$ decided by a constant C, the point where both the (part 1) $\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} \min\{K,\ b\,d_i\,p(d_j)\}$ and (part 2) $d_1 + d_2 + d_3$ attain their "own" maximum and minimum value respectively is (d, d, d) where $6\,b\,d\,p(d) = C$. Hence the maximum u "within that surface" occurs on the diagonal. According to this conclusion, we also partition the whole domain (the positive first octant of $\mathbb{R}^3$) into infinitely many surfaces of the same type (corresponding to different C), apply Lemma 4.1 and Lemma 4.7, and then obtain the same conclusion for each surface. Therefore u_{opt} must occur (at least) at some point on the diagonal.

The following are some lemmas and theorems related to the proof.

Lemma 4.1. *In any surface* $\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} bd_ip(d_j) = C \ge 0$*,* $\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} \min\{K,\ bd_ip(d_j)\}$ *attains its maximum value on (but not limited to) the diagonal.*

Proof. Consider the point (d, d, d) on the diagonal. This point makes all $bd_ip(d_j)$ have the same value of $C/6$. If $C/6 \le K$, then

$$\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} \min\{K,\ b\,d_i\,p(d_j)\} = \sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} C/6 = C.$$

Since $\sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} \min\{K,\ b\,d_i\,p(d_j)\} \le \sum_{\substack{1 \le i,j \le 3,\\ i \ne j}} b\,d_i\,p(d_j) = C$, it attains its maximum. If $C/6 > K$, then $\min\{K,\ b\,d_i\,p(d_j)\} = K$. By definition, it also attains its maximum. Hence the result follows. $\square$

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Lemma 4.2. *In any straight line $d_1 + d_2 = C > 0$, the value of $p(d_1) + p(d_2)$ is (non-strictly) decreasing with d_1 when $d_1 > d_2$, and is (non-strictly) increasing with d_1 when $d_1 < d_2$.*

Proof. Differentiate the value with respect to d_1.

$$\frac{\partial}{\partial d_1}\Big(p(d_1) + p(d_2)\Big) = p'(d_1) + p'(d_2)\frac{\partial d_2}{\partial d_1} = p'(d_1) - p'(d_2).$$

If $d_1 > d_2$, then $p'(d_1) \leq p'(d_2)$. If $d_1 < d_2$, then $p'(d_1) \geq p'(d_2)$. Hence the result follows. □

Lemma 4.3. *In any straight line $d_1 + d_2 = C > 0$, the value of $d_1 p(d_2) + d_2 p(d_1)$ is (non-strictly) decreasing with d_1 when $d_1 > d_2$, and is (non-strictly) increasing with d_1 when $d_1 < d_2$.*

Proof. Differentiate the value with respect to d_1.

$$\begin{aligned}\frac{\partial}{\partial d_1}\Big(d_1 p(d_2) + d_2 p(d_1)\Big) &= p(d_2) + d_1 p'(d_2)\frac{\partial d_2}{\partial d_1} + \frac{\partial d_2}{\partial d_1} p(d_1) + d_2 p'(d_1) \\ &= p(d_2) - p(d_1) + d_2 p'(d_1) - d_1 p'(d_2).\end{aligned}$$

If $d_1 > d_2$, then $p(d_2) \leq p(d_1)$ and $d_2 p'(d_1) \leq d_1 p'(d_2)$. $\therefore p(d_2) - p(d_1) + d_2 p'(d_1) - d_1 p'(d_2) \leq 0$. If $d_1 < d_2$, then $p(d_2) \geq p(d_1)$ and $d_2 p'(d_1) \geq d_1 p'(d_2)$. $\therefore p(d_2) - p(d_1) + d_2 p'(d_1) - d_1 p'(d_2) \geq 0$. Hence the result follows. □

Lemma 4.4. *In any plane $d_1 + d_2 + d_3 = C > 0$, if $d_x \geq d_y$ for some players P_x and P_y at a point, it always has a value of $\sum_{\substack{1 \leq i,j \leq 3,\\ i \neq j}} d_i\, p(d_j)$ greater than or equal to another point where d_x is increased by δ and d_y is decreased by δ, for any $\delta > 0$.*

Proof. W.L.O.G., take $x = 1$ and $y = 2$. We can expand the formula as the following.

$$\sum_{\substack{1 \leq i,j \leq 3,\\ i \neq j}} d_i\, p(d_j) = p(d_3) \cdot (d_1 + d_2) + \Big(d_1 p(d_2) + d_2 p(d_1)\Big) + d_3 \cdot \Big(p(d_1) + p(d_2)\Big).$$

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By Lemma 4.3, (d_1, d_2, d_3) has a value of $d_1p(d_2) + d_2p(d_1)$ greater than or equal to $(d_1 + \delta, d_2 - \delta, d_3)$. By Lemma 4.2, (d_1, d_2, d_3) has a value of $p(d_1) + p(d_2)$ greater than or equal to $(d_1 + \delta, d_2 - \delta, d_3)$. Since the other terms don't change, we are done. $\square$

Lemma 4.5. *In any plane $d_1 + d_2 + d_3 = C > 0$, the maximum value of $\sum_{\substack{1 \le i,j \le 3, \\ i \ne j}} d_i\, p(d_j)$ can occur at the point where $d_1 = d_2 = d_3 = C/3$.*

Proof. Define the function $f(d_1, d_2, d_3) = \sum_{\substack{1 \le i,j \le 3, \\ i \ne j}} b\ d_i\ p(d_j)$ first for simplicity. In this proof, we want to compare $(\frac{C}{3}, \frac{C}{3}, \frac{C}{3})$ with another arbitrary point $(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$ on the same plane, and deduce $f(\frac{C}{3}, \frac{C}{3}, \frac{C}{3}) \ge f(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$. Without loss of generality, we can consider only two cases.

Case 1. $\delta_1 \ge \delta_2 \ge 0$

By Lemma 4.4, $f(\frac{C}{3}, \frac{C}{3} + \delta_2, \frac{C}{3} - \delta_2) \ge f(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$. By applying the same lemma again we deduce $f(\frac{C}{3}, \frac{C}{3}, \frac{C}{3}) \ge f(\frac{C}{3}, \frac{C}{3} + \delta_2, \frac{C}{3} - \delta_2)$. $\therefore f(\frac{C}{3}, \frac{C}{3}, \frac{C}{3}) \ge f(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$.

Case 2. $\delta_1 \ge 0 \ge \delta_2 \ge -\delta_1$

By Lemma 4.4, $f(\frac{C}{3} + \delta_1 + \delta_2, \frac{C}{3}, \frac{C}{3} - (\delta_1 + \delta_2)) \ge f(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$. By applying the same lemma again we deduce $f(\frac{C}{3}, \frac{C}{3}, \frac{C}{3}) \ge f(\frac{C}{3} + \delta_1 + \delta_2, \frac{C}{3}, \frac{C}{3} - (\delta_1 + \delta_2))$. $\therefore f(\frac{C}{3}, \frac{C}{3}, \frac{C}{3}) \ge f(\frac{C}{3} + \delta_1, \frac{C}{3} + \delta_2, \frac{C}{3} - (\delta_1 + \delta_2))$.

Since the inequality holds for both cases, the result follows. $\square$

Lemma 4.6. *If $x > y \ge 0$ and the values of d_1 and d_2 are not both 0, (d_1, d_2, x) always has a value of $\sum_{\substack{1 \le i,j \le 3, \\ i \ne j}} d_i\, p(d_j)$ greater than (d_1, d_2, y).*

Proof. Expand $\sum_{\substack{1 \le i,j \le 3, \\ i \ne j}} d_i\, p(d_j)$ again to observe which terms are affected by d_3.

$$\sum_{\substack{1 \le i,j \le 3, \\ i \ne j}} d_i\, p(d_j) = p(d_3) \cdot (d_1 + d_2) + \Big(d_1 p(d_2) + d_2 p(d_1)\Big) + d_3 \cdot \Big(p(d_1) + p(d_2)\Big).$$

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Focus on the first term. $\because d_1 + d_2 > 0$. $\therefore p(x) \cdot (d_1 + d_2) \geq p(y) \cdot (d_1 + d_2)$.

Focus on the last term. $\because p(d_1)+p(d_2) > 0$. $\therefore x\cdot\Big(p(d_1)+p(d_2)\Big) > y\cdot\Big(p(d_1)+p(d_2)\Big)$.

Hence the result follows. □

Lemma 4.7. *In any surface* $\sum_{\substack{1\leq i,j\leq 3,\\ i\neq j}} b\, d_i\, p(d_j) = C > 0$, $d_1 + d_2 + d_3$ *attains its minimum value on (but not limited to) the diagonal.*

Proof. Define two functions $f(d_1, d_2, d_3) = \sum_{\substack{1\leq i,j\leq 3,\\ i\neq j}} b\, d_i\, p(d_j)$ and $g(d_1, d_2, d_3) = d_1 + d_2 + d_3$. Pick one point (d, d, d) and another arbitrary point $(d + \delta_1, d + \delta_2, d + \delta_3)$ on the same surface. If we can prove $g(d, d, d) \leq g(d + \delta_1, d + \delta_2, d + \delta_3)$, then we are done.

We first introduce an auxiliary point $(d + \delta_1, d + \delta_2, d - (\delta_1 + \delta_2))$ which lies on the same plane as (d, d, d). By Lemma 4.5, $f(d, d, d) \geq f(d + \delta_1, d + \delta_2, d - (\delta_1 + \delta_2))$. $\because$ (d, d, d) and $(d+\delta_1, d+\delta_2, d+\delta_3)$ lie on the same surface. $\therefore f(d, d, d) = f(d+\delta_1, d+\delta_2, d+\delta_3)$. That is, $f(d+\delta_1, d+\delta_2, d+\delta_3) \geq f(d+\delta_1, d+\delta_2, d-(\delta_1+\delta_2))$. According to this inequality, we can deduce $d+\delta_3 \geq d-(\delta_1+\delta_2)$ by Lemma 4.6. Since $d+\delta_3 \geq d-(\delta_1+\delta_2)$, it is obvious that $g(d + \delta_1, d + \delta_2, d + \delta_3) \geq g(d + \delta_1, d + \delta_2, d - (\delta_1 + \delta_2)) = g(d, d, d)$. This inequality is our goal. Hence the result follows. □

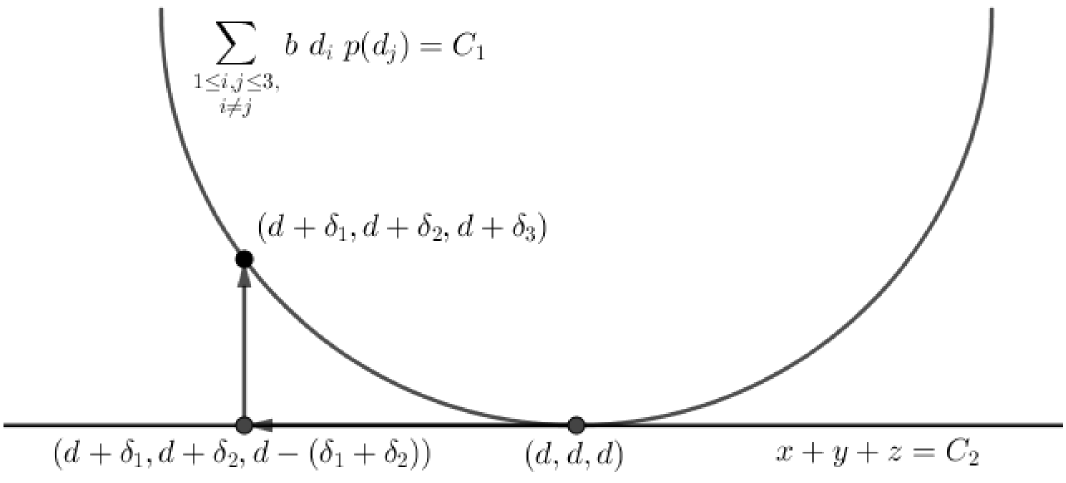

Figure 4.1: A geometric illustration of Lemma 4.7

By Lemma 4.1 and Lemma 4.7, there must be a point on the diagonal where u_{opt} occurs under some parameter settings. The following theorem tells us what the settings are.

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Theorem 4.8. *If $d_o \geq 2K$, then $u_{opt} = 0$. If $d_o < 2K$, then $u_{opt} = 3(2K - d_o) > 0$.*

Proof. Recall the utility formula of the point (d, d, d) on the diagonal. We should note that $d \leq d_o \iff b\, d\, p(d) \leq K$, and $d \geq d_o \iff b\, d\, p(d) \geq K$.

$$u = 3 \cdot \big(-d + 2 \cdot \min\{K,\ b\, d\, p(d)\} \big) = \begin{cases} 3\, d\, \big(2\, b\, p(d) - 1 \big) & \text{if } d \leq d_o \\ 3\, \big(2K - d \big) & \text{if } d \geq d_o. \end{cases}$$

If $d_o \geq 2K$, then $2bp(d_o) \leq 1$. In this case $u(d_o) \leq 0$. When $d < d_o$, $2bp(d) \leq 2bp(d_o) \leq 1$ and therefore $u(d) \leq 0$. When $d > d_o$, $2K - d < 2K - d_o$ and therefore $u(d) < u(d_o) \leq 0$. Hence $u_{opt} = 0$. If $d_o < 2K$, then $2bp(d_o) > 1$. In this case $u(d_o) > 0$. When $d < d_o$, $2bp(d) \leq 2bp(d_o)$ and therefore $u(d) \leq u(d_o)$. When $d > d_o$, $2K - d < 2K - d_o$ and therefore $u(d) < u(d_o)$. Hence $u_{opt} = u(d_o) > 0$. □

Corollary 4.9. *Let $d_{\ell\ell}$ be the less solution to $bxp'(x) = \frac{1}{2}$. If $d_o \geq d_{\ell\ell}$, then $u_{opt} = u(d_o)$.*

Proof. $\because\ bp(d_o) \geq bp(d_{\ell\ell}) > bd_{\ell\ell}p'(d_{\ell\ell}) = \frac{1}{2}$ $\quad \therefore\ d_o < 2K$ by Definition 2.6. In this case $u_{opt} = u(d_o) > 0$ by Theorem 4.8. □

We close this section with the following conclusive table.

Table 4.1: The maximum total utility of three-player games.

Condition	**Utility**
$d_o \geq 2K$	0
$d_o < 2K$	$3(2K - d_o)$

4.2 Nash Equilibria

As in the previous chapter, we are going to find Nash equilibria in order to calculate the PoA and PoS in the next section.

Lemma 4.10. *The player P_i does not want to change his/her strategy d_i if and only if one of the following cases occurs.*

Case I. $\left(\frac{\partial u_i}{\partial d_i}\right)^-$ *does not exist (i.e., $d_i = 0$) and* $\left(\frac{\partial u_i}{\partial d_i}\right)^+ \leq 0$.

Case II. $\left(\frac{\partial u_i}{\partial d_i}\right)^- \geq 0$ *and* $\left(\frac{\partial u_i}{\partial d_i}\right)^+ \leq 0$.

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Proof. Assume the other two players are P_j and P_k whose strategies are d_j and d_k, respectively. Recall the utility function $u_i = -d_i + \min\{K,\ bd_jp(d_i)\} + \min\{K,\ bd_kp(d_i)\}$. Differentiate it with respect to d_i. W.L.O.G., we let $d_j \geq d_k$.

$$\frac{\partial u_i}{\partial d_i} = \begin{cases} \frac{\partial}{\partial d_i}\Big(- d_i + bd_jp(d_i) + bd_kp(d_i)\Big) = b(d_j + d_k)p'(d_i) - 1 & \text{if } bd_jp(d_i) \leq K \\ \frac{\partial}{\partial d_i}\Big(- d_i + K + bd_kp(d_i)\Big) = bd_kp'(d_i) - 1 & \text{if } bd_jp(d_i) \geq K, \text{ and } bd_kp(d_i) \leq K \\ \frac{\partial}{\partial d_i}\Big(- d_i + K + K\Big) = -1 & \text{if } bd_kp(d_i) \geq K. \end{cases}$$

Since bd_j and bd_k are fixed nonnegative numbers, and $p'(x)$ is a nonnegative non-increasing function, $\frac{\partial u_i}{\partial d_i}$ is non-increasing for all $d_i \geq 0$. Hence the result follows. □

Lemma 4.11. *In a Nash equilibrium* (d_1, d_2, d_3)*, if a player* P_i*'s strategy* d_i *satisfies case I, then so do such strategies* d_j *and* d_k *of the other players* P_j *and* P_k*. That is,* $d_1 = d_2 = d_3 = 0$.

Proof. If $d_i = 0$, then the other players P_j and P_k fall into the case discussed in the previous chapter. In this case $bd_jp(d_i) = bd_kp(d_i) = 0$, so $\frac{\partial u_i}{\partial d_i} = b(d_j + d_k)p'(0) - 1$. To ensure P_i is in case I, we must guarantee $\frac{\partial u_i}{\partial d_i} \leq 0$. Now we want to collect all Nash equilibria in the two-player file-sharing game and find those which can make $\frac{\partial u_i}{\partial d_i} \leq 0$. Consider any Nash equilibrium (d_j, d_k) except $(0, 0)$. By the theorems related to Nash equilibria in Chapter 3, the strategies $d_j, d_k \in [d_\ell, d_h]$. It implies $b\ d_j\ p'(d_j) \geq 1$ and $b\ d_k\ p'(d_k) \geq 1$, and therefore $b\ d_j\ p'(0) \geq 1$ and $b\ d_k\ p'(0) \geq 1$. The inequality $b(d_j + d_k)p'(0) \geq 2$ implies $\frac{\partial u_i}{\partial d_i} \geq 2 - 1 = 1 > 0$ which is a contradiction. If $(d_j, d_k) = (0, 0)$, then $\frac{\partial u_i}{\partial d_i} = -1 \leq 0$ which is what we need. Therefore, only $d_i = d_j = d_k = 0$ satisfies our conclusion. □

Corollary 4.12. *In a Nash equilibrium* (d_1, d_2, d_3)*, if a player* P_i*'s strategy* d_i *satisfies case II, then so do such strategies* d_j *and* d_k *of the other players* P_j *and* P_k*. That is,* $d_1, d_2, d_3 > 0$.

In the remaining of this section we are going to discuss the Nash equilibria mentioned in Corollary 4.12. W.L.O.G., assume $d_1 \geq d_2 \geq d_3$. According to the derivative stated

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in Lemma 4.10, all conditions of possible Nash equilibria are listed below. Take P_1 for example.

Case 1. $bd_2p(d_1) < K \quad \Longrightarrow \quad bd_2p'(d_1) + bd_3p'(d_1) = 1.$

Case 2. $bd_2p(d_1) = K$ and $bd_3p(d_1) < K \quad \Longrightarrow \quad \begin{cases} bd_2p'(d_1) + bd_3p'(d_1) \geq 1 \\ bd_3p'(d_1) \leq 1. \end{cases}$

Case 3. $bd_2p(d_1) = K$ and $bd_3p(d_1) = K \quad \Longrightarrow \quad bd_2p'(d_1) + bd_3p'(d_1) \geq 1.$

Case 4. $bd_2p(d_1) > K$ and $bd_3p(d_1) < K \quad \Longrightarrow \quad bd_3p'(d_1) = 1.$

Case 5. $bd_2p(d_1) > K$ and $bd_3p(d_1) = K \quad \Longrightarrow \quad bd_3p'(d_1) \geq 1.$

Case 6. $bd_2p(d_1) > K$ and $bd_3p(d_1) > K \quad \Longrightarrow \quad \dfrac{\partial u_1}{\partial d_1} = -1$ (impossible).

We can write down the conditions for all players and arrange them into the following table. In this table, two adjacent cells are connected together if they do not contradict each other, but the validity of a whole combination (from column A to column C) still remains to be verified.

Table 4.2: Condition matching for each Nash equilibrium.

	First Player (Column A)	Second Player (Column B)	Third Player (Column C)
1	* $bd_2p(d_1) < K$ $\Longrightarrow bd_2p'(d_1) + bd_3p'(d_1) = 1$	* $bd_1p(d_2) < K$ $\Longrightarrow bd_1p'(d_2) + bd_3p'(d_2) = 1$	* $bd_1p(d_3) < K$ $\Longrightarrow bd_1p'(d_3) + bd_2p'(d_3) = 1$
2	* $bd_2p(d_1) = K$ and $bd_3p(d_1) < K$ $\Longrightarrow \begin{cases} bd_2p'(d_1) + bd_3p'(d_1) \geq 1 \\ bd_3p'(d_1) \leq 1 \end{cases}$	* $bd_1p(d_2) = K$ and $bd_3p(d_2) < K$ $\Longrightarrow \begin{cases} bd_1p'(d_2) + bd_3p'(d_2) \geq 1 \\ bd_3p'(d_2) \leq 1 \end{cases}$	* $bd_1p(d_3) = K$ and $bd_2p(d_3) < K$ $\Longrightarrow \begin{cases} bd_1p'(d_3) + bd_2p'(d_3) \geq 1 \\ bd_2p'(d_3) \leq 1 \end{cases}$
3	* $bd_2p(d_1) = K$ and $bd_3p(d_1) = K$ $\Longrightarrow bd_2p'(d_1) + bd_3p'(d_1) \geq 1$	* $bd_1p(d_2) = K$ and $bd_3p(d_2) = K$ $\Longrightarrow bd_1p'(d_2) + bd_3p'(d_2) \geq 1$	* $bd_1p(d_3) = K$ and $bd_2p(d_3) = K$ $\Longrightarrow bd_1p'(d_3) + bd_2p'(d_3) \geq 1$
4	* $bd_2p(d_1) > K$ and $bd_3p(d_1) < K$ $\Longrightarrow bd_3p'(d_1) = 1$	* $bd_1p(d_2) > K$ and $bd_3p(d_2) < K$ $\Longrightarrow bd_3p'(d_2) = 1$	* $bd_1p(d_3) > K$ and $bd_2p(d_3) < K$ $\Longrightarrow bd_2p'(d_3) = 1$
5	* $bd_2p(d_1) > K$ and $bd_3p(d_1) = K$ $\Longrightarrow bd_3p'(d_1) \geq 1$	* $bd_1p(d_2) > K$ and $bd_3p(d_2) = K$ $\Longrightarrow bd_3p'(d_2) \geq 1$	* $bd_1p(d_3) > K$ and $bd_2p(d_3) = K$ $\Longrightarrow bd_2p'(d_3) \geq 1$

Definition 4.1. Let (Ax, By, Cz) denote a combination in Table 4.2 which contains the x-th row of column A, the y-th row of column B, and the z-th row of column C. If some specific entry is dropped, it means that the corresponding column is not important (don't care).

The following lemma shows that the matchings not shown in the table are invalid.

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Lemma 4.13. *(A_1, B_1, C_1), (A_1, B_2, C_2), (A_1, B_4, C_4), (A_3, B_3, C_3), (A_3, B_4, C_4), (A_4, B_4, C_3) are the only valid combinations in Table 4.2.*

Proof. In this proof, we first investigate the valid matchings between column A and column B, then column B and column C, and finally between A and column C or the validity of the whole combinations.

Let's see the first part. A_1 says $bd_2p(d_1) < K$, and B_3 and B_5 both say $bd_3p(d_2) = K$. Connecting these cells together will result in a contradiction $bd_2p(d_1) < bd_3p(d_2)$ because our assumption $d_1 \geq d_2 \geq d_3$ should imply $bd_2p(d_1) \geq bd_3p(d_2)$. $\therefore (A_1, B_3)$ and (A_1, B_5) are invalid. Both A_2 and A_3 say $bd_2p(d_1) = K$, and B_1 says $bd_1p(d_2) < K$. Connecting these cells together will result in a contradiction $bd_1p(d_2) < bd_2p(d_1)$ because our assumption $d_1 \geq d_2$ should imply $bd_1p(d_2) \geq bd_2p(d_1)$ by Lemma 2.2. $\therefore$ (A_2, B_1) and (A_3, B_1) are invalid. Both A_2 and A_4 say $bd_3p(d_1) < K$, and both B_3 and B_5 say $bd_3p(d_2) = K$. Connecting these cells together will result in a contradiction $bd_3p(d_1) < bd_3p(d_2)$ because our assumption $d_1 \geq d_2$ should imply $bd_3p(d_1) \geq bd_3p(d_2)$. $\therefore (A_2, B_3)$, (A_2, B_5), (A_4, B_3), and (A_4, B_5) are invalid. If we connect A_3 to B_2, then the constraints $bd_2p(d_1) = K$ and $bd_1p(d_2) = K$ will result in $d_1 = d_2$, and the constraints $bd_3p(d_1) = K$ and $bd_3p(d_2) < K$ will result in $d_1 > d_2$. They contradict each other. $\therefore (A_3, B_2)$ is invalid. If we connect A_3 to B_5, then the constraints $bd_2p(d_1) = K$ and $bd_1p(d_2) > K$ will result in $d_1 > d_2$ by Corollary 2.3, and the constraints $bd_3p(d_1) = K$ and $bd_3p(d_2) = K$ along with $p'(d_1) > 0$ and $p'(d_2) > 0$ will result in $d_1 = d_2$ by Corollary 2.6. The two constraints contradict each other. $\therefore (A_3, B_5)$ is invalid. Both A_4 and A_5 say $bd_2p(d_1) > K$, and B_1, B_2, and B_3 says $bd_1p(d_2) \leq K$. Connecting these cells together will result in a contradiction $bd_2p(d_1) > bd_1p(d_2)$ because our assumption $d_1 \geq d_2$ should imply $bd_1p(d_2) \geq bd_2p(d_1)$. $\therefore (A_4, B_1)$, (A_4, B_2), (A_4, B_3), (A_5, B_1), (A_5, B_2), and (A_5, B_3) are invalid.

Now we consider the second part. B_1 says $bd_1p(d_2) < K$, and C_2, C_3, C_4, and C_5 say $bd_1p(d_3) \geq K$. Connecting these cells together will result in a contradiction $bd_1p(d_3) > bd_1p(d_2)$ because our assumption $d_2 \geq d_3$ should imply $bd_1p(d_2) \geq bd_1p(d_3)$.

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$\therefore$ (B_1, C_2), (B_1, C_3), (B_1, C_4), and (B_1, C_5) are invalid. If we connect B_2 and B_3 to C_1, then the constraints $bd_1p'(d_2) + bd_3p'(d_2) \geq 1$ and $bd_1p'(d_3) + bd_2p'(d_3) = 1$ will result in $bd_1p'(d_2) = bd_1p'(d_3)$ and $bd_3p'(d_2) = bd_2p'(d_3)$ since $bd_1p'(d_2) \leq bd_1p'(d_3)$ and $bd_3p'(d_2) \leq bd_2p'(d_3)$. According to $bd_1p'(d_2) = bd_1p'(d_3)$, $d_1 > 0$, $p'(d_2) > 0$, and $p'(d_3) > 0$, we can deduce $p'(d_2) = p'(d_3) > 0$ and therefore $d_2 = d_3$. If $bd_1p(d_2) = K$ (in B_2 and B_3), then $bd_1p(d_3) = K$ which contradicts $bd_1p(d_3) < K$ in C_1. $\therefore$ (B_2, C_1) and (B_3, C_1) are invalid. Both B_4 and B_5 say $bd_3p'(d_2) \geq 1$, and C_1 says $0 \leq bd_1p'(d_3) = 1 - bd_2p'(d_3) \leq 1 - bd_3p'(d_2)$. If we connect B_4 and B_5 to C_1, then $bd_3p'(d_2) = 1$ and $bd_1p'(d_3) = 0$. The latter implies $d_1 = 0$ or $p'(d_3) = 0$. However, $d_1 = 0$ contradicts $bd_1p(d_2) > K$, and $p'(d_3) = 0$ contradicts $b(d_1 + d_2)p'(d_3) = 1$. $\therefore$ (B_4, C_1) and (B_5, C_1) are invalid. B_2 says $bd_1p(d_2) = K$ and $p'(d_2) > 0$, and C_3 says $bd_1p(d_3) = K$ and $p'(d_3) > 0$. By Corollary 2.6, we deduce $d_2 = d_3$, but this contradicts $bd_3p(d_2) < K = bd_2p(d_3)$. $\therefore$ (B_2, C_3) is invalid. Both B_3 and B_5 say $bd_3p(d_2) = K$, and both C_2 and C_4 say $bd_2p(d_3) < K$. Connecting these cells together will result in a contradiction $bd_2p(d_3) < bd_3p(d_2)$ because our assumption $d_2 \geq d_3$ should imply $bd_2p(d_3) \geq bd_3p(d_2)$ by Lemma 2.2. $\therefore$ (B_3, C_2), (B_3, C_4), (B_5, C_2) and (B_5, C_4) are invalid. Both B_2 and B_3 say $bd_1p(d_2) = K$, and both C_4 and C_5 say $bd_1p(d_3) > K$. Connecting these cells together will result in a contradiction $bd_1p(d_2) < bd_1p(d_3)$ because our assumption $d_2 \geq d_3$ should imply $bd_1p(d_2) \geq bd_1p(d_3)$ by Lemma 2.2. $\therefore$ (B_2, C_4), (B_2, C_5), (B_3, C_4) and (B_3, C_5) are invalid. If we connect B_4 to C_2, then $1 = bd_3p'(d_2) \leq bd_2p'(d_3) \leq 1$ and it implies $d_2 = d_3$ by Lemma 2.4. However it contradicts $bd_1p(d_2) > K = bd_1p(d_3)$. $\therefore$ (B_4, C_2) is invalid.

Finally we check the validity between column B and column C or the whole combination. A_1 says $bd_2p(d_1) < K$, and C_3 and C_5 say $bd_2p(d_3) = K$. Connecting these cells together will result in a contradiction $bd_2p(d_1) < bd_2p(d_3)$ because our assumption $d_1 \geq d_3$ should imply $bd_2p(d_1) \geq bd_2p(d_3)$ by Lemma 2.2. $\therefore$ (A_1, C_3) and (A_1, C_5) are invalid. If we connect B_2 to C_2, then the argument in the connection (B_2, C_3) mentioned in the previous paragraph can be used to deduce $d_2 = d_3$. This contradicts $bd_2p(d_1) = K > bd_3p(d_1)$ in A_2. $\therefore$ (A_2, B_2, C_2) is invalid. In C_3, the con-

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straint $bd_1p(d_3) = bd_2p(d_3) = K$ implies $d_1 = d_2$. If we connect A_2 and A_3 to B_4, then $bd_1p(d_2) > K = bd_2p(d_1)$ contradicts $d_1 = d_2$. $\therefore (A_2, B_4, C_3)$ and (A_3, B_4, C_3) are invalid. A_2 says $bd_2p(d_1) = K$ and $p'(d_1) > 0$, and C_5 says $bd_2p(d_3) = K$ and $p'(d_3) > 0$. If we connect A_2 to C_5, by Corollary 2.6 we deduce $d_1 = d_3$. However this contradicts $bd_1p(d_3) > K > bd_3p(d_1)$. $\therefore (A_2, C_5)$ is invalid. In A_3, $bd_2p(d_1) = K = bd_3p(d_1)$ implies $d_2 = d_3$. If we connect B_4 and C_5, then $bd_3p(d_2) < K = bd_2p(d_3)$ contradicts $d_2 = d_3$. $\therefore (A_3, B_4, C_5)$ is invalid. If we connect B_4 to C_4, then by Lemma 2.4 $bd_3p'(d_2) = bd_2p'(d_3) = 1$ implies $d_2 = d_3$. However this contradicts $bd_2p(d_1) \geq K > bd_3p(d_1)$ in A_2 and A_4. $\therefore (A_2, B_4, C_4)$ and (A_4, B_4, C_4) are invalid. If we connect A_4 to B_4, then $bd_3p'(d_1) = bd_3p'(d_2) = 1$ implies $d_1 = d_2$. However this contradicts $bd_1p(d_3) > K = bd_2p(d_3)$ in C_5. $\therefore (A_4, B_4, C_5)$ is invalid. If we connect B_5 to C_5, then $bd_3p(d_2) = K = bd_2p(d_3)$ implies $d_2 = d_3$ by Corollary 2.3. However this contradicts $bd_2p(d_1) > K = bd_3p(d_1)$ in A_5. $\therefore (A_5, B_5, C_5)$ is invalid.

After checking all combinations, we can deduce that the remaining valid ones are $(A_1, B_1, C_1), (A_1, B_2, C_2), (A_1, B_4, C_4), (A_3, B_3, C_3), (A_3, B_4, C_4), (A_4, B_4, C_3)$. $\square$

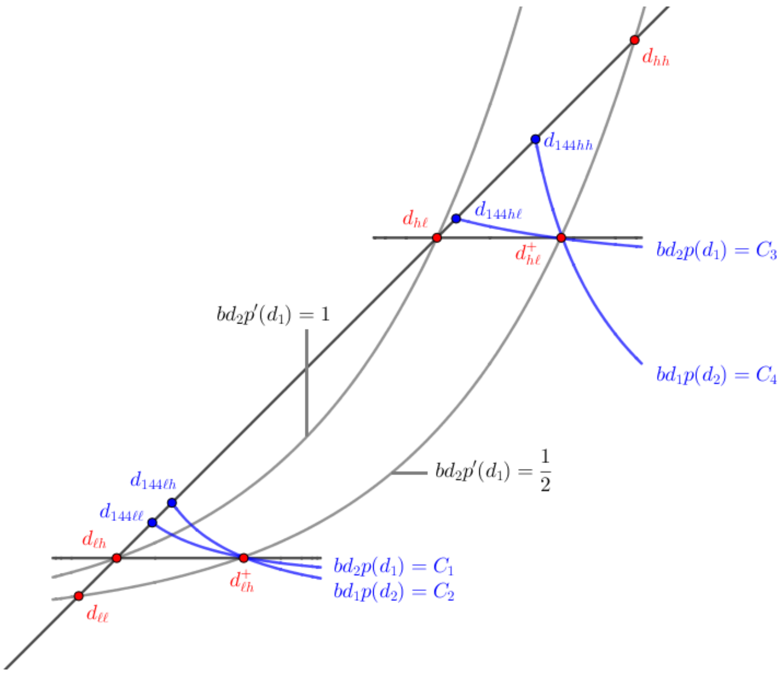

Figure 4.2: A geometric illustration of Definition 4.2

Before finding all Nash equilibria according to these combinations, we must define some variables beforehand.

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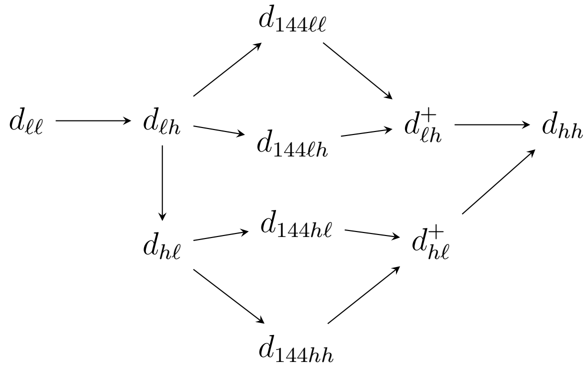
Definition 4.2. If $bxp'(x) = \dfrac{1}{2}$ has two different solutions, let $d_{\ell\ell}$ be the less one, and let d_{hh} be the greater one. If the equation has only one solution, let $d_{\ell\ell}$ and d_{hh} both denote it. If $bxp'(x) = 1$ has two different solutions, let $d_{\ell h}$ be the less one, and let $d_{h\ell}$ be the greater one. If the equation has only one solution, let $d_{\ell h}$ and $d_{h\ell}$ both denote it. Let d_o be the unique solution to the equation $bxp(x) = K$. Let $d^+_{\ell h}$ be the unique solution to $bd_{\ell h}p'(x) = \dfrac{1}{2}$. Let $d^+_{h\ell}$ be the unique solution to $bd_{h\ell}p'(x) = \dfrac{1}{2}$. Let $d_{144\ell\ell}$ be the unique solution to $bxp(x) = bd_{\ell h}p(d^+_{\ell h})$. Let $d_{144\ell h}$ be the unique solution to $bxp(x) = bd^+_{\ell h}p(d_{\ell h})$. Let $d_{144h\ell}$ be the unique solution to $bxp(x) = bd_{h\ell}p(d^+_{h\ell})$. Let d_{144hh} be the unique solution to $bxp(x) = bd^+_{h\ell}p(d_{h\ell})$.

Lemma 4.14. *The parameters in Definition 4.2, if exist, have the partial order shown in the following directed acyclic graph, where $A \to B$ means $A < B$.*



Proof. We first focus on the parameters related to $bdp'(d)$. By the property of $dp'(d)$, $d_{\ell\ell} < d_{\ell h} < d_{h\ell} < d_{hh}$. Besides, $\dfrac{p'(d_{\ell h})}{p'(d^+_{\ell h})} = \dfrac{bd_{\ell h}p'(d_{\ell h})}{bd_{\ell h}p'(d^+_{\ell h})} = \dfrac{1}{1/2} = 2$. $\therefore p'(d_{\ell h}) > p'(d^+_{\ell h})$ and $d_{\ell h} < d^+_{\ell h}$. By a similar argument, $d_{h\ell} < d^+_{h\ell}$. If $d_{144\ell\ell} \leq d_{\ell h}$, then $bd_{144\ell\ell}p(d_{144\ell\ell})$ $\leq bd_{\ell h}p(d_{\ell h}) < bd_{\ell h}p(d^+_{\ell h})$, with the latter inequality coming from $p'(d^+_{\ell h}) > 0$, is a contradiction. If $d_{144\ell\ell} \geq d^+_{\ell h}$, then $bd_{144\ell\ell}p(d_{144\ell\ell}) \geq bd^+_{\ell h}p(d^+_{\ell h}) > bd_{\ell h}p(d^+_{\ell h})$ is also a contradiction. Hence $d_{\ell h} < d_{144\ell\ell} < d^+_{\ell h}$. By a similar argument, we can deduce $d_{\ell h} < d_{144\ell h} < d^+_{\ell h}$, $d_{h\ell} < d_{144h\ell} < d^+_{h\ell}$, and $d_{h\ell} < d_{144hh} < d^+_{h\ell}$. By definition $bd_{\ell h}p'(d^+_{\ell h}) = \dfrac{1}{2} = bd_{hh}p'(d_{hh})$, and $\dfrac{p'(d^+_{\ell h})}{p'(d_{hh})} = \dfrac{d_{hh}}{d_{\ell h}} > 1$ implies $d^+_{\ell h} < d_{hh}$. By a similar argument, $d^+_{h\ell} < d_{hh}$. □

After variable definitions, we can formally define all Nash equilibria and discuss the ranges of these parameters.

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Theorem 4.15. *The Nash equilibria corresponding to the combination* (A_1, B_1, C_1) *can only be* $N_{111\ell} = (d_{\ell\ell}, d_{\ell\ell}, d_{\ell\ell})$ *and* $N_{111h} = (d_{hh}, d_{hh}, d_{hh})$. $N_{111\ell}$ *exists if and only if* $d_{\ell\ell} < d_o$, *and* N_{111h} *exists if and only if* $d_{hh} < d_o$.

Proof. The "derivatives" in (A_1, B_1, C_1) can be organized as

$$b(d_2 + d_3)p'(d_1) = 1,\ b(d_1 + d_3)p'(d_2) = 1,\ \text{and}\ b(d_1 + d_2)p'(d_3) = 1.$$

From the first two equations, since $d_2 + d_3 \leq d_1 + d_3$ and $p'(d_1) \leq p'(d_2)$, then $p'(d_1) = p'(d_2) > 0$ and $d_1 = d_2$. From the last two equations, we also deduce $d_2 = d_3$ by a similar argument. Therefore $d_1 = d_2 = d_3 = d$ and the equality becomes $b(2d)p'(d) = 1$. The utilities in (A_1, B_1, C_1) become $bdp(d) < K$.

The solutions to $bdp'(d) = \frac{1}{2}$ are only $d_{\ell\ell}$ and d_{hh}. It's obvious that $bd_{\ell\ell}p(d_{\ell\ell}) < K = bd_op(d_o) \iff d_{\ell\ell} < d_o$, and $bd_{hh}p(d_{hh}) < K = bd_op(d_o) \iff d_{hh} < d_o$. Hence the result follows. □

Theorem 4.16. *The Nash equilibria corresponding to the combination* (A_1, B_2, C_2) *can only be* $N_{122\ell} = (d_{122\ell x}, d_{122\ell y}, d_{122\ell y})$ *and* $N_{122h} = (d_{122hx}, d_{122hy}, d_{122hy})$. *If* $N_{122\ell}$ *exists, then* $d_{\ell\ell} < d_{122\ell x} \leq d^+_{\ell h}$ *and* $d_{\ell\ell} < d_{122\ell y} \leq d_{\ell h}$. *If* N_{122h} *exists, then* $d^+_{h\ell} \leq d_{122hx} < d_{hh}$ *and* $d_{h\ell} \leq d_{122hy} < d_{hh}$.

Proof. Since $b(d_2 + d_3)p'(d_1) \leq b(d_1 + d_3)p'(d_2) \leq b(d_1 + d_2)p'(d_3)$ and $bd_2p'(d_3) \geq bd_3p'(d_2)$, the derivatives in (A_1, B_2, C_2) can be simplified to $b(d_2 + d_3)p'(d_1) = 1$ and $bd_2p'(d_3) \leq 1$. Observe the utility constraints. $bd_1p(d_2) = K = bd_1p(d_3)$ along with $p'(d_2) > 0$ and $p'(d_3) > 0$ gives $d_2 = d_3$ by Corollary 2.6. Besides, $bd_1p(d_2) = K > bd_2p(d_1)$ implies $d_1 > d_2$ by Corollary 2.3. Therefore $d_1 > d_2 = d_3$ and $b(d_2+d_3)p'(d_1) = 1$ becomes $bd_2p'(d_1) = \frac{1}{2}$. In addition, we deduce $bd_2p'(d_1) < bd_2p'(d_3)$ because $d_1 > d_3$ and $p'(d_1) > 0$, so the constraint $bd_2p'(d_3) \leq 1$ can be extended to $\frac{1}{2} < bd_2p'(d_3) \leq 1$.

Let $d_2 = d_3 = d_y$. Then $\frac{1}{2} < bd_yp'(d_y) \leq 1$ implies $d_{\ell\ell} < d_y \leq d_{\ell h} \vee d_{h\ell} \leq d_y < d_{hh}$ by the property of $dp'(d)$. If the less solution is $d_{122\ell y}$, then $d_{\ell\ell} < d_{122\ell y} \leq d_{\ell h}$ is proven. If the greater solution is d_{122hy}, then $d_{h\ell} \leq d_{122hy} < d_{hh}$ is proven.

The constraint $bd_2p'(d_1) = \frac{1}{2}$, according to our setting, becomes $bd_{122\ell y}p'(d_{122\ell x}) = \frac{1}{2}$ and $bd_{122hy}p'(d_{122hx}) = \frac{1}{2}$. Since $p'(x)$ is decreasing when it is greater than 0, $d_{122\ell y}$ and

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$d_{122\ell x}$ both increase or decrease together, and so do d_{122hy} and d_{122hx}. Observing the ranges of $d_{122\ell y}$ and d_{122hy} and their corresponding solutions, we finally deduce $d_{\ell\ell} < d_{122\ell x} \leq d^+_{\ell h}$ and $d^+_{h\ell} \leq d_{122hx} < d_{hh}$. □

Theorem 4.17. *The Nash equilibria corresponding to the combination* (A_1, B_4, C_4) *can only be* $N_{144\ell} = (d^+_{\ell h}, d_{\ell h}, d_{\ell h})$ *and* $N_{144h} = (d^+_{h\ell}, d_{h\ell}, d_{h\ell})$.

Proof. The equalities $bd_3p'(d_2) = 1$ and $bd_2p'(d_3) = 1$ together implies $d_2 = d_3$ by Lemma 2.4. By Definition 4.2, d_2 $(= d_3)$ can only be $d_{\ell h}$ or $d_{h\ell}$. Substituting it into $bd_2p'(d_1) + bd_3p'(d_1) = 1$ gives $bd_2p'(d_1) = bd_3p'(d_1) = \frac{1}{2}$. If $bd_{\ell h}p'(d_1) = \frac{1}{2}$, then $d_1 = d^+_{\ell h}$. If $bd_{h\ell}p'(d_1) = \frac{1}{2}$, then $d_1 = d^+_{h\ell}$. Therefore $N_{144\ell}$ and N_{144h} are our results. □

Theorem 4.18. *The Nash equilibria corresponding to the combination* (A_3, B_3, C_3) *can only be* $N_{333} = (d_o, d_o, d_o)$. N_{333} *exists if and only if* $d_{\ell\ell} \leq d_o \leq d_{hh}$.

Proof. From the utility equalities, it's clear to conclude that $d_1 = d_2 = d_3$ by Corollary 2.3, and it is also equal to d_o by Definition 2.6. We can deduce $bd_op'(d_o) \geq \frac{1}{2}$ from the derivative inequalities. By the property of $dp'(d)$, the inequality is equivalent to $d_{\ell\ell} \leq d_o \leq d_{hh}$. □

Theorem 4.19. *The Nash equilibria corresponding to the combination* (A_3, B_4, C_4) *can only be* $N_{344\ell} = (d_{344\ell x}, d_{344\ell y}, d_{344\ell y})$ *and* $N_{344h} = (d_{344hx}, d_{344hy}, d_{344hy})$. *If* $N_{344\ell}$ *exists, then* $d_{\ell h} < d_{344\ell x} \leq d^+_{\ell h}$ *and* $d_{344\ell y} = d_{\ell h}$. *If* N_{344h} *exists, then* $d_{h\ell} < d_{344hx} \leq d^+_{h\ell}$ *and* $d_{344hy} = d_{h\ell}$.

Proof. The equalities $bd_3p'(d_2) = 1$ and $bd_2p'(d_3) = 1$ together implies $d_2 = d_3$ by Lemma 2.4, and the utility constraint $bd_1p(d_2) > K = bd_2p(d_1)$ gives $d_1 > d_2$ by Corollary 2.3. Therefore $d_1 > d_2 = d_3$, and $b(d_2+d_3)p'(d_1) \geq 1$ becomes $bd_2p'(d_1) \geq \frac{1}{2}$. Since $d_1 > d_2$ and $p'(d_2) > 0$ together implies $p'(d_1) < p'(d_2)$, $1 = bd_2p'(d_2) > bd_2p'(d_1)$. If we let $d_x = d_1$ and $d_y = d_2 = d_3$, then the constraints become $\frac{1}{2} \leq bd_yp'(d_x) < 1$ and $bd_yp'(d_y) = 1$. By Definition 4.2, d_y can only be $d_{\ell h}$ and $d_{h\ell}$. If $\frac{1}{2} \leq bd_{\ell h}p'(d_x) < 1$, then $d_{\ell h} < d_x \leq d^+_{\ell h}$. If $\frac{1}{2} \leq bd_{h\ell}p'(d_x) < 1$, then $d_{h\ell} < d_x \leq d^+_{h\ell}$. Hence the result follows. □

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Theorem 4.20. *The Nash equilibria corresponding to the combination* (A_4, B_4, C_3) *can only be* $N_{443} = (d_{443x}, d_{443x}, d_{443y})$*. If* N_{443} *exists, then* $d_{\ell h} \leq d_{443x} \leq d_{h\ell}$ *and* $d_{\ell h} \leq d_{443y} \leq d_{h\ell}$*.*

Proof. From C_3's derivative constraint $bd_3p'(d_1) = 1 = bd_3p'(d_2)$, we know $d_1 = d_2$. Let it be d_{443x} and let d_3 be d_{443y}. Then the equality becomes $bd_{443y}p'(d_{443x}) = 1$. Since $p'(x)$ is decreasing when it is greater than 0, d_{443y} and d_{443x} both increase or decrease together. By the property of $dp'(d)$, their minimum is $d_{\ell h}$ and their maximum is $d_{h\ell}$. Hence the result follows. □

4.3 The PoA and PoS

Now we similarly want to calculate the PoA and PoS. The analysis in this section is still split into three different cases depending on the value of d_o, and we still ignore the collapsing Nash equilibrium $(0, 0, 0)$.

The objective of Lemma 4.21 and Theorem 4.22 is to sort all the Nash equilibria by the total utility function.

Lemma 4.21 (Generalized Lemma 3.21). *Given two points* $X = (d_{x_1}, d_{x_2}, d_{x_3})$ *and* $Y = (d_{y_1}, d_{y_2}, d_{y_3})$*, if* $d_{x_i} \geq d_{y_i} \geq d_{\ell\ell}$ *for* $1 \leq i \leq 3$*, all terms in the form of* $bd_{x_i}p(d_{x_j}) \leq K$ *for* $i \neq j$ *in* $u(X)$*, and all terms in the form of* $bd_{y_i}p(d_{y_j}) \leq K$ *for* $i \neq j$ *in* $u(Y)$*, then* $u(X) \geq u(Y)$*.*

Proof. Since $bp(d_{\ell\ell}) \geq bd_{\ell\ell}p'(d_{\ell\ell}) = \frac{1}{2}$, then $bp(d_{x_i}) + bp(d_{x_j}) - 1 \geq 0$ and $bp(d_{y_i}) + bp(d_{y_j}) - 1 \geq 0$ are always true for all parameters not less than $d_{\ell\ell}$. We can write

$$u(X) = d_{x_1}\Big(bp(d_{x_2})+bp(d_{x_3})-1\Big)+d_{x_2}\Big(bp(d_{x_1})+bp(d_{x_3})-1\Big)+d_{x_3}\Big(bp(d_{x_1})+bp(d_{x_2})-1\Big), \text{ and}$$

$$u(Y) = d_{y_1}\Big(bp(d_{y_2})+bp(d_{y_3})-1\Big)+d_{y_2}\Big(bp(d_{y_1})+bp(d_{y_3})-1\Big)+d_{y_3}\Big(bp(d_{y_1})+bp(d_{y_2})-1\Big).$$

It is clear to see that $bp(d_{x_2})+bp(d_{x_3})-1 \geq bp(d_{y_2})+bp(d_{y_3})-1 \geq 0$, $bp(d_{x_1})+bp(d_{x_3})-1 \geq bp(d_{y_1}) + bp(d_{y_3}) - 1 \geq 0$, and $bp(d_{x_1}) + bp(d_{x_2}) - 1 \geq bp(d_{y_1}) + bp(d_{y_2}) - 1 \geq 0$, so $u(X) \geq u(Y)$. □

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Theorem 4.22. *$N_{111\ell}$, if exists, has the minimum total utility, and N_{111h}, if exists, has the maximum total utility among all existing Nash equilibria discussed above.*

Proof. Recall all previous theorems about Nash equilibria from Theorem 4.15 to Theorem 4.20. None of the strategies of these points are less than $d_{\ell\ell}$. If N_{111h} exists, then by Theorem 4.15 $d_{hh} < d_o$ and none of these strategies are greater than d_o. That is, $bd_i p(d_j) \leq K$ for these strategies. Since N_{111h} has the greatest contributions (strategies) of each player among all possible Nash equilibria, by Lemma 4.21 N_{111h} has the maximum total utility.

Now we're going to show $N_{111\ell}$ has the minimum total utility. If some Nash equilibrium other than $(0, 0, 0)$ exists, it implies $bd_i p(d_j) \leq K$ for some $d_i,\ d_j \geq d_{\ell\ell}$. Therefore $bd_{\ell\ell} p(d_{\ell\ell}) \leq K$ and $d_{\ell\ell} \leq d_o$. $N_{111\ell}$ must also exist. For Nash equilibria corresponding to the combinations (A_1, B_1, C_1), (A_1, B_2, C_2), and (A_3, B_3, C_3), they all have the property $bd_i p(d_j) \leq K$ for all parameters d_i and d_j. We can simply use Lemma 4.21 to show that the total utilities of these Nash equilibria are greater than that of $N_{111\ell}$. For other combinations (A_1, B_4, C_4), (A_3, B_4, C_4), and (A_4, B_4, C_3), we discuss them by cases. Consider $N_{144\ell} = (d^+_{\ell h}, d_{\ell h}, d_{\ell h})$ of (A_1, B_4, C_4) first. This combination implies $bd^+_{\ell h} p(d_{\ell h}) \geq K$ and $bd_{\ell h} p(d^+_{\ell h}) \leq K$. Therefore we can say

$$\begin{aligned} u(N_{144\ell}) &= 2K - d^+_{\ell h} + 2bd_{\ell h} p(d^+_{\ell h}) + 2bd_{\ell h} p(d_{\ell h}) - 2d_{\ell h} \\ &\geq 2bd_{\ell h} p(d^+_{\ell h}) - d^+_{\ell h} + 2bd_{\ell h} p(d^+_{\ell h}) + 2bd_{\ell h} p(d_{\ell h}) - 2d_{\ell h} \\ &= 4bd_{\ell h} p(x) - x + 2bd_{\ell h} p(d_{\ell h}) - 2d_{\ell h} \,\Big|_{x = d^+_{\ell h}}. \end{aligned}$$

Consider the auxiliary function $f(x) = 4bd_{\ell h} p(x) - x + 2bd_{\ell h} p(d_{\ell h}) - 2d_{\ell h}$ and its derivative $f'(x) = 4bd_{\ell h} p'(x) - 1$. By definition $f'(d^+_{\ell h}) = 4bd_{\ell h} p'(d^+_{\ell h}) - 1 = 4 \cdot \dfrac{1}{2} - 1 = 1$ and therefore $f'(x) \geq 1$ for all $0 \leq x \leq d^+_{\ell h}$. We deduce $u(N_{144\ell}) \geq u(d_{\ell h}, d_{\ell h}, d_{\ell h})$ from this and deduce $u(d_{\ell h}, d_{\ell h}, d_{\ell h}) \geq u(N_{111\ell})$ from Lemma 4.21. Similarly, we can say $u(N_{144h}) \geq u(N_{111\ell})$ if replacing $d^+_{\ell h}$ with $d^+_{h\ell}$ and replacing $d_{\ell h}$ with $d_{h\ell}$ in the above argument. Also, this argument can be used to explain why $u(N_{344\ell}) \geq u(N_{111\ell})$ and $u(N_{344h}) \geq u(N_{111\ell})$. Finally we consider $N_{443} = (d_{443x}, d_{443x}, d_{443y})$ of (A_4, B_4, C_3). This combination implies $bd_{443x} p(d_{443x}) \geq K$ and $bd_{443x} p(d_{443y}) \leq K$, so we can say

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$$\begin{aligned}u(N_{443}) &= 2(K - d_{443x}) + 2bd_{443x}p(bd_{443y}) + 2bd_{443y}p(bd_{443x}) - d_{443y}\\ &\geq 2(bd_{443x}p(d_{443y}) - d_{443x}) + 2bd_{443x}p(bd_{443y}) + 2bd_{443y}p(bd_{443x}) - d_{443y}\\ &= 2d_{443x}(2bp(d_{443y}) - 1) + d_{443y}(2bp(bd_{443x}) - 1)\\ &\geq 2d_{443y}(2bp(d_{443y}) - 1) + d_{443y}(2bp(bd_{443y}) - 1) \quad (\because bp(d_{443y}) \geq bp(d_{\ell h}) \geq 1)\\ &= u(d_{443y}, d_{443y}, d_{443y}).\end{aligned}$$

Since $bd_{443y}p(d_{443y}) \leq K$, we also deduce $u(d_{443y}, d_{443y}, d_{443y}) \geq u(N_{111\ell})$ by Lemma 4.21. In the end, we can say $N_{111\ell}$, if exists, has the minimum total utility. □

After clearly comparing the total utilities of all possible Nash equilibria, we can discuss the PoS and the non-collapsing PoA in the following three cases.

Lemma 4.23 and Theorem 4.24 together states the conclusion when neither $d_{\ell\ell}$ nor d_{hh} exists, or both exist but $d_o < d_{\ell\ell}$.

Lemma 4.23. *When $d_o < d_{\ell\ell}$, there are no non-collapsing Nash equilibria. When $d_o = d_{\ell\ell}$, the only non-collapsing Nash equilibrium is N_{333}.*

Proof. We go through all Nash equilibria mentioned in several previous theorems here. By Theorem 4.15, $N_{111\ell}$ and N_{111h} cannot exist since $d_o \not> d_{\ell\ell}$ and $d_o \not> d_{hh}$. By Theorem 4.16, all the strategies $d_{122\ell x}$, $d_{122\ell y}$, d_{122hx} and d_{122hy} are greater than $d_{\ell\ell}$ (and d_o). Therefore $bd_{122\ell x}p(d_{122\ell y}) = bd_{122hx}p(d_{122hy}) > bd_{\ell\ell}p(d_{\ell\ell}) \geq bd_op(d_o) = K$ and the constraint $bd_1p(d_2) = K$ cannot be satisfied. $N_{122\ell}$ and N_{122h} cannot exist. We also figure out that the least contribution (strategy) in Theorem 4.17 is $d_{\ell h}$ which is greater than d_o, so $bd_2p(d_1) > bd_op(d_o) = K$ and the constraint $bd_2p(d_1) < K$ cannot be satisfied. $N_{144\ell}$ and N_{144h} cannot exist. Similarly, all strategies in Theorem 4.19 and Theorem 4.20 are not less than $d_{\ell h}$, so $bd_2p(d_1) > bd_op(d_o) = K$ violates the constraint $bd_2p(d_1) = K$ in (A_3, B_4, C_4), and $bd_1p(d_3) > bd_op(d_o) = K$ violates the constraint $bd_1p(d_3) = K$ in (A_4, B_4, C_3). $N_{344\ell}$, N_{344h}, and N_{443} cannot exist. Finally, Theorem 4.18 says N_{333} exists if and only if $d_{\ell\ell} \leq d_o \leq d_{hh}$, so we are done. □

Theorem 4.24. *If neither $d_{\ell\ell}$ nor d_{hh} exists, or both exist but $d_o < d_{\ell\ell}$, then the maximum total utility of all existing Nash equilibria must be* 0.

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Proof. By Lemma 4.11 and Lemma 4.23, the only existing Nash equilibrium in this case is $(0,0,0)$ and $u(0,0,0) = 0$. The result follows. □

Lemma 4.25 is an auxiliary proposition helping us in observing how the PoA, PoS vary with the parameters b.

Lemma 4.25 (Generalized Lemma 3.25). *If K and $p(x)$ remain fixed, and b is the only varying parameter, then (a)* $\frac{\partial d_o}{\partial b} < 0$, *(b)* $\frac{\partial d_{\ell\ell}\left(2bp(d_{\ell\ell}) - 1\right)}{\partial b} < 0$, *and (c)* $\frac{\partial d_{\ell\ell}}{\partial b} < \frac{\partial d_o}{\partial b}$ *when $d_{\ell\ell} = d_o$.*

Proof. Part (a) can be directly deduced from the definition $bd_op(d_o) = K$. For part (b), recall the definition $bd_{\ell\ell}p'(d_{\ell\ell}) = \frac{1}{2}$ first. Since $d_{\ell\ell}$ is the less solution to $bd_{\ell\ell}p'(d_{\ell\ell}) = \frac{1}{2}$, by Lemma 2.1 we have $\partial d_{\ell\ell}/\partial b < 0$. It means that when b increases, $d_{\ell\ell}$ decreases, $p'(d_{\ell\ell})$ increases, $\frac{1/2}{p'(d_{\ell\ell})}$ decreases, and therefore $\frac{\partial}{\partial b}\left(\frac{1/2}{p'(d_{\ell\ell})}\right) < 0$. Also write $\frac{\partial}{\partial b}\left(\frac{1/2}{p'(d_{\ell\ell})}\right) = \frac{\partial(bd_{\ell\ell})}{\partial b} = d_{\ell\ell} + b\frac{\partial d_{\ell\ell}}{\partial b}$, so $d_{\ell\ell} + b\frac{\partial d_{\ell\ell}}{\partial b} < 0$.

$$
\begin{aligned}
\frac{\partial\left(2bd_{\ell\ell}p(d_{\ell\ell}) - d_{\ell\ell}\right)}{\partial b} &= 2d_{\ell\ell}p(d_{\ell\ell}) + 2b\frac{\partial d_{\ell\ell}}{\partial b}p(d_{\ell\ell}) + 2bd_{\ell\ell}p'(d_{\ell\ell})\frac{\partial d_{\ell\ell}}{\partial b} - \frac{\partial d_{\ell\ell}}{\partial b} \\
&= 2d_{\ell\ell}p(d_{\ell\ell}) + 2b\frac{\partial d_{\ell\ell}}{\partial b}p(d_{\ell\ell}) \\
&= 2p(d_{\ell\ell})\left(d_{\ell\ell} + b\frac{\partial d_{\ell\ell}}{\partial b}\right) < 0.
\end{aligned}
$$

For part (c), we go back to $bd_op(d_o) = K$. According to this equality, $\frac{\partial}{\partial b}\left(\frac{K}{p(d_o)}\right) = \frac{\partial(bd_o)}{\partial b} = d_o + b\frac{\partial d_o}{\partial b} > 0$. Comparing with $d_{\ell\ell} + b\frac{\partial d_{\ell\ell}}{\partial b} < 0$ deduced above, we obtain part (c). □

Theorem 4.26 states the relationship between the PoA, PoS and the parameters b, K when $d_{\ell\ell} \le d_o \le d_{hh}$.

Theorem 4.26 (Generalized Theorem 3.26). *If both $d_{\ell\ell}$ and d_{hh} exist, and $d_{\ell\ell} \le d_o \le d_{hh}$, then the $PoS = 1$ and the* $PoA = \frac{u_{opt}}{u(d_{\ell\ell})} = \frac{u(N_{333})}{u(d_{\ell\ell})} = \frac{d_o\left(2bp(d_o) - 1\right)}{d_{\ell\ell}\left(2bp(d_{\ell\ell}) - 1\right)}$. *Furthermore, when b, $p(x)$ are fixed, and K is the only varying parameter, the PoA approaches 1 as K decreases such that d_o approaches $d_{\ell\ell}$, and the PoA approaches its maximum*

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$\dfrac{d_{hh}\Big(bp(d_{hh})-1\Big)}{d_{\ell\ell}\Big(bp(d_{\ell\ell})-1\Big)}$ *as K increases such that d_o approaches d_{hh}. When K, $p(x)$ are fixed, and b is the only varying parameter, the PoA approaches infinity as b keeps increasing, and the PoA approaches its minimum as b decreases such that d_o approaches d_{hh}.*

Proof. By Corollary 4.9 $u_{opt} = u(N_{333})$, so the $PoS = 1$. By Theorem 4.22, the worst non-collapsing Nash equilibrium is the point $(d_{\ell\ell}, d_{\ell\ell}, d_{\ell\ell})$. Hence the $PoA = \dfrac{u(d_o)}{u(d_{\ell\ell})}$. If b, $p(x)$ are fixed and only K varies, then only d_o varies with it and the denominator doesn't change. Since $2bp(d_o) \geq 2bp(d_{\ell\ell}) > 2bd_{\ell\ell}p'(d_{\ell\ell}) = 1$, the PoA increases with d_o (K).

Consider the case when b is the only varying parameter. We should also note that the PoA can be written as $\dfrac{2K-d_o}{d_{\ell\ell}\Big(2bp(d_{\ell\ell})-1\Big)}$. By Lemma 4.25 $\dfrac{\partial d_o}{\partial b} < 0$ and $\dfrac{\partial d_{\ell\ell}\Big(2bp(d_{\ell\ell})-1\Big)}{\partial b} < 0$, so the numerator increases, the denominator decreases, and the PoA increases with b. If K, $p(x)$ are fixed and b is the only increasing parameter, by part (c) of Lemma 4.25 the inequality $d_{\ell\ell} \leq d_o \leq d_{hh}$ always remains, so the PoA increases unboundedly. If K, $p(x)$ are fixed and b is the only decreasing parameter, by part (c) of Lemma 4.25 the inequality $d_{\ell\ell} \leq d_o$ remains, but d_o may exceed d_{hh}. Therefore the PoA achieves its minimum as d_o achieves its maximum (d_{hh}). □

Theorem 4.27 states the relationship between the PoA, PoS and the parameters b, K when $d_{hh} < d_o$.

Theorem 4.27 (Generalized Theorem 3.28). *If both $d_{\ell\ell}$ and d_{hh} exist, and $d_{hh} < d_o$, then the $PoS = \dfrac{u_{opt}}{u(N_{111h})} = \dfrac{d_o\Big(2bp(d_o)-1\Big)}{d_{hh}\Big(2bp(d_{hh})-1\Big)}$ and the $PoA = \dfrac{u_{opt}}{u(N_{111\ell})} = \dfrac{d_o\Big(2bp(d_o)-1\Big)}{d_{\ell\ell}\Big(2bp(d_{\ell\ell})-1\Big)}$. If we only consider the non-collapsing stable Nash equilibria, then the "stable" PoA becomes $\dfrac{u_{opt}}{u(N_{111h})}$. Furthermore, when b and $p(x)$ are fixed, and K is the only varying parameter, the PoS approaches 1 and the PoA approaches its greatest lower bound $\dfrac{d_{hh}\Big(bp(d_{hh})-1\Big)}{d_{\ell\ell}\Big(bp(d_{\ell\ell})-1\Big)}$ as K decreases such that d_o approaches d_{hh}, and both the $PoS = \Theta(K)$ and $PoA = \Theta(K)$ approach infinity as K keeps increasing. When K and $p(x)$ are fixed, and b is the only varying parameter, the PoS approaches 1 and the PoA approaches its least upper bound as b increases such that d_o approaches d_{hh}, and the PoS approaches*

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its maximum and the PoA approaches its minimum as b keeps decreasing until d_{hh} does not exist.

Proof. By Corollary 4.9, u_{opt} occurs at $u(d_o)$. By Theorem 4.22, N_{111h} has the maximum total utility, and $N_{111\ell}$ has the minimum total utility among all existing non-collapsing Nash equilibria. Hence the PoS and PoA in our theorem follow. If we only consider the non-collapsing stable Nash equilibria, then N_{111h} is the only one. Hence the "stable" PoA in our theorem follows.

According to the proof in Theorem 4.26, the PoS and PoA both increase with d_o (and K). We should also note that the numerator can be expressed as $2K - d_o$. When K is very large, $p(d_o)$ approaches 1 and therefore $d_o = \dfrac{K}{bp(d_o)} \approx \dfrac{K}{b}$, so $2K - d_o \approx 2K - \dfrac{K}{b} = K(2 - \dfrac{1}{b}) = \Theta(K)$.

According to the proof in Theorem 4.26, the PoA increases with b. We should also note that the PoS can be written as $\dfrac{d_o}{d_{hh}} \cdot \dfrac{2p(d_o) - 1/b}{2p(d_{hh}) - 1/b}$. If K, $p(x)$ are fixed and b is the only increasing parameter, then d_o and $2p(d_o)$ decrease, and d_{hh} and $2p(d_{hh})$ increase. In addition, adding the same quantity to both the numerator and denominator of an improper fraction decreases its value. Therefore we can deduce the PoS decreases with b instead.

□

We close this chapter with the following tables concluding Theorem 4.24, Theorem 4.26, and Theorem 4.27.

Table 4.3: Summary of the PoS and PoA with K as the only varying parameter. We assume d_o starts at $d_{\ell\ell}$ and keeps increasing.

Condition	$d_{\ell\ell} \leq d_o \leq d_{hh}$ **(Phase 1)**	$d_{hh} < d_o$ **(Phase 2)**
PoS	1	$u(d_o)/u(d_{hh})$ **(stable PoA)**
Monotonicity	-	increasing
Starting at 1	-	Yes
PoA	$u(d_o)/u(d_{\ell\ell})$	
Monotonicity	increasing	
Starting at 1	Yes	

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Table 4.4: Summary of the PoS and PoA with b as the only varying parameter. We assume b starts at its valid minimum value (i.e. $bxp'(x) = \frac{1}{2}$ has exactly one solution.) and keeps increasing.

Condition	$d_o > d_{hh}$ **(Phase 1)**	$d_{hh} \geq d_o \geq d_{\ell\ell}$ **(Phase 2)**
PoS	$u(d_o)/u(d_{hh})$ **(stable PoA)**	1
Monotonicity	decreasing	-
Terminating at 1	YES	-
PoA	$u(d_o)/u(d_{\ell\ell})$	
Monotonicity	increasing	
Starting at 1	No	

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Chapter 5

Nash Equilibrium Analysis for Multi-Player File-Sharing Games

After the analysis of three-player file-sharing games, we eventually want to generalize the result to n-player file-sharing games. In this chapter the model is exactly in the form of what we've described in Chapter 2, and n denotes the number of players.

$$\begin{cases} u_i(d_i) = -d_i + \sum_{k \neq i} \min\{K,\ b\, d_k\, p(d_i)\}, \quad \text{for } 1 \leq i \leq n \\ \\ u(d_1, d_2, ..., d_n) = \sum_{i=1}^{n} u_i(d_i). \end{cases}$$

We also use the notation $u(d) = u(d, d, ..., d)$ if all the d_i's have the same value of d.

In this chapter, we still do almost the same thing as in Chapter 4. The difference is that we only consider the "symmetric" Nash equilibria (i.e., the same contribution for all players) here, and we don't care about their stability either.

5.1 Maximum Total Utility

The structure of the proof is exactly the same as that in Chapter 4. The following are some related lemmas and theorems.

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Lemma 5.1 (Generalized Lemma 4.1). *In any surface* $\sum_{\substack{1\le i,j\le n,\\ i\ne j}} bd_ip(d_j) = C \ge 0$, $\sum_{\substack{1\le i,j\le n,\\ i\ne j}} \min\{K,\ bd_ip(d_j)\}$ *attains its maximum value on (but not limited to) the diagonal.*

Proof. Consider the case when all d_i's are the same. This case makes all $bd_ip(d_j)$ have the same value of $C/(n(n-1))$. If $C/(n(n-1)) \le K$, then

$$\sum_{\substack{1\le i,j\le n,\\ i\ne j}} \min\{K,\ b\ d_i\ p(d_j)\} = \sum_{\substack{1\le i,j\le n,\\ i\ne j}} C/(n(n-1)) = C.$$

Since $\sum_{\substack{1\le i,j\le n,\\ i\ne j}} \min\{K,\ b\ d_i\ p(d_j)\} \le \sum_{\substack{1\le i,j\le n,\\ i\ne j}} b\ d_i\ p(d_j) = C$, it attains its maximum. If $C/(n(n-1)) > K$, then $\min\{K,\ b\ d_i\ p(d_j)\} = K$. By definition, it also attains its maximum. Hence the result follows. □

Lemma 5.2 (Generalized Lemma 4.4). *In any plane* $\sum_{i=1}^{n} d_i = C > 0$*, if* $d_x \ge d_y$ *for some players* P_x *and* P_y *at a point, it always has a value of* $\sum_{\substack{1\le i,j\le n,\\ i\ne j}} d_i\ p(d_j)$ *greater than or equal to another point where* d_x *is increased by* δ *and* d_y *is decreased by* δ*, for any* $\delta > 0$.

Proof. W.L.O.G., take $x = 1$ and $y = 2$. We can expand the formula as the following.

$$\sum_{\substack{1\le i,j\le n,\\ i\ne j}} d_i\ p(d_j) = (d_1 + d_2)\cdot\sum_{i=3}^{n} p(d_i) + \sum_{\substack{3\le i,j\le n,\\ i\ne j}} d_i\ p(d_j) + \Big(d_1p(d_2) + d_2p(d_1)\Big) + \Big(p(d_1) + p(d_2)\Big)\cdot\sum_{i=3}^{n} d_i.$$

By Lemma 4.3, $d_1p(d_2) + d_2p(d_1) \ge (d_1+\delta)p(d_2-\delta) + (d_2-\delta)p(d_1+\delta)$. By Lemma 4.2, $p(d_1) + p(d_2) \ge p(d_1+\delta) + p(d_2-\delta)$. Since the other terms don't change, then we are done. □

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Lemma 5.3 (Generalized Lemma 4.5). *In any plane* $\sum_{i=1}^{n} d_i = C > 0$*, the maximum value of* $\sum_{\substack{1\leq i,j\leq n,\\ i\neq j}} d_i\, p(d_j)$ *can occur at the point where* $d_i = C/n$ *for all* $1 \leq i \leq n$.

Proof. Define the function $f(d_1, d_2, ..., d_n) = \sum_{\substack{1\leq i,j\leq n,\\ i\neq j}} b\, d_i\, p(d_j)$ first for simplicity. In this proof, we want to compare $(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n})$ with another arbitrary point $(\frac{C}{n}+\delta_1, \frac{C}{n}+\delta_2, ..., \frac{C}{n}+\delta_{n-1}, \frac{C}{n}-\sum_{i=1}^{n-1}\delta_i)$ on the same plane, and deduce $f(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n}) \geq f(\frac{C}{n}+\delta_1, \frac{C}{n}+\delta_2, ..., \frac{C}{n}+\delta_{n-1}, \frac{C}{n}-\sum_{i=1}^{n-1}\delta_i)$. Consider the following argument. If a point $P(d_1, d_2, ..., d_n)$ on the same plane is not $(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n})$, there must be some $d_i > \frac{C}{n}$ and some $d_j < \frac{C}{n}$. If $|d_i - \frac{C}{n}| \geq |d_j - \frac{C}{n}|$, then we can adjust the point to a new one Q where d_i becomes $d_i - (\frac{C}{n} - d_j) \geq \frac{C}{n}$ and d_j becomes $\frac{C}{n}$. In this case $f(P) \leq f(Q)$. If $|d_i - \frac{C}{n}| \leq |d_j - \frac{C}{n}|$, then we can adjust the point to another one R where d_i becomes $\frac{C}{n}$ and d_j becomes $d_j + (d_i - \frac{C}{n}) \leq \frac{C}{n}$. In this case $f(P) \leq f(R)$. Then we can repeat the above procedure until the point becomes $(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n})$. The procedure will be executed only at most n times since in each iteration there must exist at least one d_i which becomes $\frac{C}{n}$. Hence $f(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n})$ is the maximum value on the plane. □

Lemma 5.4 (Generalized Lemma 4.6). *If the strategies* d_i*'s of all players* P_i*'s (except for* P_k*'s* d_k*) are not all* 0*, then the value of* $\sum_{\substack{1\leq i,j\leq n,\\ i\neq j}} d_i\, p(d_j)$ *increases with* d_k.

Proof. Expand $\sum_{\substack{1\leq i,j\leq n,\\ i\neq j}} d_i\, p(d_j)$ again to observe which terms are affected by d_k.

$$\sum_{\substack{1\leq i,j\leq n,\\ i\neq j}} d_i\, p(d_j) = p(d_k)\cdot\Big(\sum_{i\neq k} d_i\Big) + \Big(\sum_{\substack{1\leq i,j\leq n,\\ i\neq j,\ i\neq k,\ j\neq k}} d_i\, p(d_j)\Big) + d_k\cdot\Big(\sum_{i\neq k} p(d_i)\Big).$$

Focus on the first term.

$$\because \sum_{i\neq k} d_i > 0. \quad \therefore p(d_k+\delta)\cdot\Big(\sum_{i\neq k} d_i\Big) \geq p(d_k)\cdot\Big(\sum_{i\neq k} d_i\Big) \text{ for } \delta > 0.$$

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Focus on the last term.

$$\because \sum_{i \neq k} p(d_i) > 0. \quad \therefore (d_k + \delta) \cdot \Big(\sum_{i \neq k} p(d_i) \Big) > d_k \cdot \Big(\sum_{i \neq k} p(d_i) \Big) \text{ for } \delta > 0.$$

Hence the result follows. □

Lemma 5.5 (Generalized Lemma 4.7). *In any surface* $\sum_{\substack{1 \le i,j \le n, \\ i \neq j}} b\, d_i\, p(d_j) = C > 0$, $\sum_{i=1}^{n} d_i$ *attains its minimum value on (but not limited to) the diagonal.*

Proof. Define two functions $f(d_1, d_2, ..., d_n) = \sum_{\substack{1 \le i,j \le n, \\ i \neq j}} b\; d_i\; p(d_j)$ and $g(d_1, d_2, ..., d_n) = \sum_{i=1}^{n} d_i$. Pick one point $(d, d, ..., d)$ and another arbitrary point $(d + \delta_1, d + \delta_2, ..., d + \delta_n)$ on the same surface. If we can prove $g(d, d, ..., d) \le g(d + \delta_1, d + \delta_2, ..., d + \delta_n)$, then we are done.

We first introduce an auxiliary point $(d+\delta_1, d+\delta_2, ..., d+\delta_{n-1}, d-\sum_{i=1}^{n-1} \delta_i)$ which lies on the same plane as (d, d, d). By Lemma 5.3, $f(d, d, ..., d) \ge f(d + \delta_1, d + \delta_2, ..., d + \delta_{n-1}, d - \sum_{i=1}^{n-1} \delta_i)$. $\because$ $(d, d, ..., d)$ and $(d + \delta_1, d + \delta_2, ..., d + \delta_n)$ lie on the same surface. $\therefore$ $f(d, d, ..., d) = f(d + \delta_1, d + \delta_2, ..., d + \delta_n)$. That is, $f(d + \delta_1, d + \delta_2, ..., d + \delta_n) \ge f(d + \delta_1, d + \delta_2, ..., d + \delta_{n-1}, d - \sum_{i=1}^{n-1} \delta_i)$. According to this inequality, we can deduce $d + \delta_n \ge d - \sum_{i=1}^{n-1} \delta_i$ by Lemma 5.4. Since $d + \delta_n \ge d - \sum_{i=1}^{n-1} \delta_i$, it is obvious that $g(d + \delta_1, d + \delta_2, ..., d + \delta_n) \ge g(d + \delta_1, ..., d + \delta_{n-1}, d - \sum_{i=1}^{n-1} \delta_i) = g(d, d, ..., d)$. This inequality is our goal. Hence the result follows. □

By Lemma 5.5, there must be a point on the diagonal where u_{opt} occurs under some parameter settings. The following theorem tells us what the settings are.

Theorem 5.6 (Generalized Theorem 4.8). *If* $d_o \ge (n-1)K$, *then* $u_{opt} = 0$. *If* $d_o < (n-1)K$, *then* $u_{opt} = n\,\big(\,(n-1)K - d_o\,\big) > 0$.

Proof. Recall the utility formula of the point $(d, d, ..., d)$ on the diagonal. We should note that $d \le d_o \iff b\, d\, p(d) \le K$, and $d \ge d_o \iff b\, d\, p(d) \ge K$.

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$$u = n\cdot\big(-d+(n-1)\cdot\min\{K,\ b\,d\,p(d)\}\big) = \begin{cases} n\,d\,\big(\,(n-1)\,b\,p(d)-1\,\big) & \text{if} \quad d \le d_o \\ n\,\big(\,(n-1)K-d\,\big) & \text{if} \quad d \ge d_o. \end{cases}$$

If $d_o \ge (n-1)K$, then $(n-1)bp(d_o) \le 1$. In this case $u(d_o) \le 0$. When $d < d_o$, $(n-1)bp(d) \le (n-1)bp(d_o) \le 1$ and therefore $u(d) \le 0$. When $d > d_o$, $(n-1)K - d < (n-1)K - d_o$ and therefore $u(d) < u(d_o) \le 0$. Hence $u_{opt} = 0$. If $d_o < (n-1)K$, then $(n-1)bp(d_o) > 1$. In this case $u(d_o) > 0$. When $d < d_o$, $(n-1)bp(d) \le (n-1)bp(d_o)$ and therefore $u(d) \le u(d_o)$. When $d > d_o$, $(n-1)K - d < (n-1)K - d_o$ and therefore $u(d) < u(d_o)$. Hence $u_{opt} = u(d_o) > 0$. □

Corollary 5.7 (Generalized Corollary 4.9). *Let d_L be the less solution to $bxp'(x) = \dfrac{1}{n-1}$. If $d_o \ge d_L$, then $u_{opt} = u(d_o)$.*

Proof. $\because bp(d_o) \ge bp(d_L) > bd_L p'(d_L) = \dfrac{1}{n-1} \quad \therefore d_o < (n-1)K$ by Definition 2.6. In this case $u_{opt} = u(d_o) > 0$ by Theorem 5.6. □

We close this section with the following conclusive table.

Table 5.1: The maximum total utility of multi-player games.

Condition	**Utility**
$d_o \ge (n-1)K$	0
$d_o < (n-1)K$	$n\,\big(\,(n-1)K - d_o\,\big)$

5.2 Nash Equilibria

Before calculating the PoA and PoS, it is necessary to generalize some theorems in the previous chapter first.

Lemma 5.8 (Generalized Lemma 4.10). *The player P_i does not want to change his/her strategy d_i if and only if one of the following cases occurs.*

Case I. $\left(\dfrac{\partial u_i}{\partial d_i}\right)^-$ *does not exist (i.e., $d_i = 0$) and* $\left(\dfrac{\partial u_i}{\partial d_i}\right)^+ \le 0$.

Case II. $\left(\dfrac{\partial u_i}{\partial d_i}\right)^- \ge 0$ *and* $\left(\dfrac{\partial u_i}{\partial d_i}\right)^+ \le 0$.

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Proof. Assume there are n players P_1 to P_n whose strategies are d_1 to d_n. Recall the utility function $u_i = -d_i + \sum_{k \neq i} \min\{K,\ bd_k p(d_i)\}$. Differentiate it with respect to d_i. W.L.O.G., we assume the strategies d_1 to d_n (except for d_i) are in ascending order. If there exists some $1 \leq j \leq n$ such that $bd_k p(d_i) \leq K$ for all $1 \leq k \leq j$ (but $k \neq i$) and $bd_k p(d_i) \geq K$ for all $j+1 \leq k \leq n$ (but $k \neq i$), then

$$\frac{\partial u_i}{\partial d_i} = \frac{\partial}{\partial d_i}\Big(-d_i + \sum_{\substack{1 \leq k \leq j \\ k \neq i}} bd_k p(d_i) + (n-j)K\Big) = -1 + \sum_{\substack{1 \leq k \leq j \\ k \neq i}} bd_k p'(d_i).$$

If $bd_k p(d_i) \geq K$ for all $k \neq i$, then

$$\frac{\partial u_i}{\partial d_i} = \frac{\partial}{\partial d_i}\Big(-d_i + (n-1)K\Big) = -1.$$

Since bd_k's (for all $k \neq i$) are fixed nonnegative numbers, $p'(x)$ is a nonnegative non-increasing function, and j cannot be incremented as d_i goes up, $\frac{\partial u_i}{\partial d_i}$ is non-increasing for all $d_i \geq 0$. Hence the result follows. □

However, since the Nash equilibria in which at least two players have different strategies are too difficult to analyze, we simply assume strategies of players are all the same in this section. The derivative of utility is shown below.

$$\frac{\partial u_i}{\partial d_i} = \begin{cases} \frac{\partial}{\partial d_i}\Big(-d_i + \sum_{k \neq i} bd_k p(d_i)\Big) = b\Big(\sum_{k \neq i} d_k\Big)p'(d_i) - 1 & \\ \qquad\qquad\qquad\qquad = (n-1)bd_i p'(d_i) - 1 & \text{if } bd_i p(d_i) \leq K \\ \frac{\partial}{\partial d_i}\Big(-d_i + (n-1)K\Big) = -1 & \text{if } bd_i p(d_i) \geq K. \end{cases}$$

According to the above conclusion, all possible Nash equilibria we care about in this section can only be O (the origin), N_L, N_o, and N_H stated in the following theorems.

Definition 5.1. If $bxp'(x) = \dfrac{1}{n-1}$ has two different solutions, let d_L be the less one, and let d_H be the greater one. If it has only one solution, let d_L and d_H both denote it.

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Theorem 5.9. *The Nash equilibrium N_L, where all players have the same strategy d_L, exists if and only if $d_L < d_o$. The Nash equilibrium N_H, where all players have the same strategy d_H, exists if and only if $d_H < d_o$.*

Proof. The two points correspond to the case when $(n-1)bd_ip'(d_i)-1 = 0$ and $bd_ip(d_i) < K$. Since $bd_ip(d_i) < K = bd_op(d_o)$ if and only if $d_i < d_o$, the result follows. □

Theorem 5.10. *N_o exists if and only if $d_L \leq d_o \leq d_H$.*

Proof. The point corresponds to the case when $(n-1)bd_ip'(d_i)-1 \geq 0$ and $bd_ip(d_i) = K$. Since $bd_op'(d_o) \geq \frac{1}{n-1}$ if and only if $d_L \leq d_o \leq d_H$, the result follows. □

5.3 The Symmetric PoA and PoS

We can eventually discuss the PoS and non-collapsing PoA. Since we assume all players have the same strategy, the definitions of the PoS and non-collapsing PoA should be modified a little.

Definition 5.2. We say a Nash equilibrium is "symmetric" if all players have the same contribution on that point.

Definition 5.3. Let the symmetric PoS = $\frac{u_{opt}}{u(\text{the best symmetric Nash equilibrium})}$.

Definition 5.4. Let the non-collapsing symmetric PoA

$$= \frac{u_{opt}}{u(\text{the worst symmetric Nash equilibrium except for the origin})}.$$

Lemma 5.11 is convenient for us to compare the values of total utility of two different symmetric Nash equilibria.

Lemma 5.11 (Generalized Lemma 4.21). *Given two points $X = (d_{x_1}, d_{x_2}, ..., d_{x_n})$ and $Y = (d_{y_1}, d_{y_2}, ..., d_{y_n})$, we can deduce $u(X) \geq u(Y)$ if $d_o \geq d_{x_i} \geq d_{y_i} \geq d_L$ for $1 \leq i \leq n$.*

Proof. Since $bp(d_L) \geq bd_Lp'(d_L) = \frac{1}{n-1}$, then $\sum_{i \neq \text{ some fixed } k} bp(d_{x_i}) - 1 \geq 0$ and $\sum_{i \neq \text{ some fixed } k} bp(d_{y_i}) - 1 \geq 0$ are always true for all parameters not less than d_L. In addition,

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$bd_{x_i}p(d_{x_j}) \leq K$ and $bd_{y_i}p(d_{y_j}) \leq K$ are always true for all parameters not greater than d_o. We can write

$$u(X) = \sum_{k=1}^{n} d_{x_k} \cdot \Big(\sum_{i \neq k} bp(d_{x_i}) - 1\Big), \text{ and}$$
$$u(Y) = \sum_{k=1}^{n} d_{y_k} \cdot \Big(\sum_{i \neq k} bp(d_{y_i}) - 1\Big).$$

It is clear to see that $\sum_{i \neq k} bp(d_{x_i}) - 1 \geq \sum_{i \neq k} bp(d_{y_i}) - 1 \geq 0$ for all $1 \leq k \leq n$, so $u(X) \geq u(Y)$. □

Theorem 5.12 states the conclusion when neither d_L nor d_H exists, or both exist but $d_o < d_L$.

Theorem 5.12. *If neither d_L nor d_H exists, or both exist but $d_o < d_L$, then the maximum total utility of all existing Nash equilibria must be* 0.

Proof. Since in this case the only existing symmetric Nash equilibrium is the origin O and $u(O) = 0$, the result follows. □

Lemma 5.13 is an auxiliary proposition helping us in observing how the PoA, PoS vary with the parameters b.

Lemma 5.13 (Generalized Lemma 4.25). *If b is the only varying parameter and all the other parameters are fixed, then (a)$\dfrac{\partial d_o}{\partial b} < 0$, (b)$\dfrac{\partial d_L\Big((n-1)bp(d_L) - 1\Big)}{\partial b} < 0$, and (c)$\dfrac{\partial d_L}{\partial b} < \dfrac{\partial d_o}{\partial b}$ when $d_L = d_o$.*

Proof. Part (a) can be directly deduced from the definition $bd_op(d_o) = K$. For part (b), recall the definition $bd_Lp'(d_L) = \dfrac{1}{n-1}$ first. Since d_L is the less solution to $bd_Lp'(d_L) = \dfrac{1}{n-1}$, by Lemma 2.1 we have $\partial d_L/\partial b < 0$. It means that when b increases, d_L decreases, $p'(d_L)$ increases, $\dfrac{1/(n-1)}{p'(d_L)}$ decreases, and therefore $\dfrac{\partial}{\partial b}\Big(\dfrac{1/(n-1)}{p'(d_L)}\Big) < 0$. Also write $\dfrac{\partial}{\partial b}\Big(\dfrac{1/(n-1)}{p'(d_L)}\Big) = \dfrac{\partial(bd_L)}{\partial b} = d_L + b\dfrac{\partial d_L}{\partial b}$, so $d_L + b\dfrac{\partial d_L}{\partial b} < 0$.

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$$\begin{aligned}\frac{\partial\Big((n-1)bd_Lp(d_L)-d_L\Big)}{\partial b} &= (n-1)d_Lp(d_L)+(n-1)b\frac{\partial d_L}{\partial b}p(d_L)\\ &\quad+(n-1)bd_Lp'(d_L)\frac{\partial d_L}{\partial b}-\frac{\partial d_L}{\partial b}\\ &= (n-1)d_Lp(d_L)+(n-1)b\frac{\partial d_L}{\partial b}p(d_L)\\ &= (n-1)p(d_L)\Big(d_L+b\frac{\partial d_L}{\partial b}\Big)<0.\end{aligned}$$

For part (c), we go back to $bd_op(d_o) = K$. According to this equality, $\frac{\partial}{\partial b}\Big(\frac{K}{p(d_o)}\Big) = \frac{\partial(bd_o)}{\partial b} = d_o + b\frac{\partial d_o}{\partial b} > 0$. Comparing with $d_L + b\frac{\partial d_L}{\partial b} < 0$ deduced above, we obtain part (c). □

Lemma 5.14 is an auxiliary proposition helping us in observing how the PoA, PoS vary with the parameters n.

Lemma 5.14. *If n is the only varying parameter and all the other parameters are fixed, then* $\frac{\partial d_L\Big((n-1)bp(d_L)-1\Big)}{\partial n} < 0.$

Proof. Recall the definition $(n-1)d_Lp'(d_L) = \frac{1}{b}$ first. Since d_L is the less solution to $(n-1)d_Lp'(d_L) = \frac{1}{b}$, by Lemma 2.1 we have $\partial d_L/\partial n < 0$. Differentiating both sides of the equation with respect to n gives

$$d_Lp'(d_L)+(n-1)\frac{\partial d_L}{\partial n}p'(d_L)+(n-1)d_Lp''(d_L)\frac{\partial d_L}{\partial n}=0.$$
$$p'(d_L)\cdot\Big(d_L+(n-1)\frac{\partial d_L}{\partial n}\Big)=-(n-1)d_Lp''(d_L)\frac{\partial d_L}{\partial n}<0.$$

Since $p'(d_L) > 0$, we deduce $d_L + (n-1)\frac{\partial d_L}{\partial n} < 0$.

$$\begin{aligned}\frac{\partial\Big((n-1)bd_Lp(d_L)-d_L\Big)}{\partial n} &= bd_Lp(d_L)+b(n-1)\frac{\partial d_L}{\partial n}p(d_L)\\ &\quad+(n-1)bd_Lp'(d_L)\frac{\partial d_L}{\partial n}-\frac{\partial d_L}{\partial n}\\ &= bd_Lp(d_L)+b(n-1)\frac{\partial d_L}{\partial n}p(d_L)\\ &= bp(d_L)\Big(d_L+(n-1)\frac{\partial d_L}{\partial n}\Big)<0.\end{aligned}$$

□

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Theorem 5.15 states the relationship between the PoA, PoS and the parameters b, K, n when $d_L \leq d_o \leq d_H$.

Theorem 5.15 (Generalized Theorem 4.26). *If both d_L and d_H exist, and $d_L \leq d_o \leq d_H$, then the $PoS = 1$ and the $PoA = \frac{u_{opt}}{u(d_L)} = \frac{u(N_o)}{u(d_L)} = \frac{d_o\Big((n-1)bp(d_o)-1\Big)}{d_L\Big((n-1)bp(d_L)-1\Big)}$. Furthermore, when K is the only varying parameter and all the other parameters are fixed, the PoA approaches 1 as K decreases such that d_o approaches d_L, and the PoA approaches its maximum $\frac{d_H\Big(bp(d_H)-1\Big)}{d_L\Big(bp(d_L)-1\Big)}$ as K increases such that d_o approaches d_H. When b is the only varying parameter and all the other parameters are fixed, the PoA approaches infinity as b keeps increasing, and the PoA approaches its minimum as b decreases such that d_o approaches d_H. When n is the only varying parameter and all the other parameters are fixed, the PoA approaches infinity as n keeps increasing, and the PoA approaches its minimum as n decreases such that d_H approaches d_o.*

Proof. By Corollary 5.7 $u_{opt} = u(N_o)$, so the $PoS = 1$. By Lemma 5.11, the worst non-collapsing Nash equilibrium is the point $(d_L, d_L, ..., d_L)$. Hence the $PoA = \frac{u(d_o)}{u(d_L)}$. If only K varies and all the other parameters are fixed, then only d_o varies with it and the denominator doesn't change. Since $(n-1)bp(d_o) \geq (n-1)bp(d_L) > (n-1)bd_Lp'(d_L) = 1$, the PoA increases with d_o (and K).

Consider the case when b is the only varying parameter. We should also note that the PoA can be written as $\frac{(n-1)K - d_o}{d_L\Big((n-1)bp(d_L)-1\Big)}$. By Lemma 5.13 $\frac{\partial d_o}{\partial b} < 0$ and $\frac{\partial d_L\Big((n-1)bp(d_L)-1\Big)}{\partial b} < 0$, so the numerator increases, the denominator decreases, and the PoA increases with b. If b is the only increasing parameter and all the other parameters are fixed, by part (c) of Lemma 5.13 the inequality $d_L \leq d_o \leq d_H$ always remains, so the PoA increases unboundedly. If b is the only decreasing parameter and all the other parameters are fixed, by part (c) of Lemma 5.13 the inequality $d_L \leq d_o$ remains, but d_o may exceed d_H. Therefore the PoA achieves its minimum as d_o achieves its maximum (d_H).

Consider the case when n is the only varying parameter. The PoA can be written as

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$\dfrac{(n-1)K - d_o}{d_L\Big((n-1)bp(d_L) - 1\Big)}$. By Lemma 5.14 $\dfrac{\partial d_L\Big((n-1)bp(d_L) - 1\Big)}{\partial n} < 0$, so the numerator increases, the denominator decreases, and the PoA increases with n. If n is the only increasing parameter and all the other parameters are fixed, it is obvious that the inequality $d_L \leq d_o \leq d_H$ always remains, so the PoA increases unboundedly. If n is the only decreasing parameter and all the other parameters are fixed, then d_L may exceed d_o or d_H may fall below d_o. Therefore the PoA achieves its minimum as d_L achieves its maximum (d_o) first or d_H achieves its minimum (d_o) first. □

Theorem 5.16 states the relationship between the PoA, PoS and the parameters b, K, n when $d_H < d_o$.

Theorem 5.16 (Generalized Theorem 4.27). *If both d_L and d_H exist, and $d_H < d_o$, then the $PoS = \dfrac{u_{opt}}{u(N_H)} = \dfrac{d_o\Big((n-1)bp(d_o) - 1\Big)}{d_H\Big((n-1)bp(d_H) - 1\Big)}$ and the $PoA = \dfrac{u_{opt}}{u(N_L)} = \dfrac{d_o\Big((n-1)bp(d_o) - 1\Big)}{d_L\Big((n-1)bp(d_L) - 1\Big)}$. If we only consider the non-collapsing stable Nash equilibria, then the "stable" PoA becomes $\dfrac{u_{opt}}{u(N_H)}$. Furthermore, when K is the only varying parameter and all the other parameters are fixed, the PoS approaches 1 and the PoA approaches its greatest lower bound $\dfrac{d_H\Big(bp(d_H) - 1\Big)}{d_L\Big(bp(d_L) - 1\Big)}$ as K decreases such that d_o approaches d_H, and both the $PoS = \Theta(K)$ and $PoA = \Theta(K)$ approach infinity as K keeps increasing. When b is the only varying parameter and all the other parameters are fixed, the PoS approaches 1 and the PoA approaches its least upper bound as b increases such that d_o approaches d_H, and the PoS approaches its maximum and the PoA approaches its minimum as b keeps decreasing until d_H does not exist. When n is the only varying parameter and all the other parameters are fixed, the PoS approaches 1 and the PoA approaches its least upper bound as n increases such that d_H approaches d_o, and the PoS approaches its maximum and the PoA approaches its minimum as n keeps decreasing until d_H does not exist.*

Proof. By Corollary 5.7, u_{opt} occurs at $u(d_o)$. By Lemma 5.11, N_H has the maximum total utility, and N_L has the minimum total utility among all existing non-collapsing Nash equilibria. Hence the PoS and PoA in our theorem follow. If we only consider the non-collapsing stable Nash equilibria, then N_H is the only one. Hence the "stable" PoA in our

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theorem follows.

According to the proof in Theorem 5.15, the PoS and PoA both increase with d_o (and K). We should also note that the numerator can be expressed as $(n-1)K - d_o$. When K is very large, $p(d_o)$ approaches 1 and therefore $d_o = \dfrac{K}{bp(d_o)} \approx \dfrac{K}{b}$, so $(n-1)K - d_o \approx (n-1)K - \dfrac{K}{b} = K((n-1) - \dfrac{1}{b}) = \Theta(K)$.

According to the proof in Theorem 5.15, the PoA increases with b. We should also note that the PoS can be written as $\dfrac{d_o}{d_H} \cdot \dfrac{(n-1)p(d_o) - 1/b}{(n-1)p(d_H) - 1/b}$. If b is the only increasing parameter and all the other parameters are fixed, d_o and $(n-1)p(d_o)$ decrease, and d_H and $(n-1)p(d_H)$ increase. In addition, adding the same quantity to both the numerator and denominator of an improper fraction decreases its value. Therefore we can deduce the PoS decreases with b instead.

According to the proof in Theorem 5.15, the PoA increases with n. We should also note that the PoS can be written as $\dfrac{d_o}{d_H} \cdot \dfrac{bp(d_o) - 1/(n-1)}{bp(d_H) - 1/(n-1)}$. If n is the only increasing parameter and all the other parameters are fixed, then d_H increases and so does the denominator. In addition, adding the same quantity to both the numerator and denominator of an improper fraction decreases its value. Therefore we can deduce the PoS decreases with n instead. $\square$

We close this chapter with the following tables concluding Theorem 5.12, Theorem 5.15, and Theorem 5.16.

Table 5.2: Summary of the PoS and PoA with K as the only varying parameter. We assume d_o starts at d_L and keeps increasing.

Condition	$d_L \le d_o \le d_H$ **(Phase 1)**	$d_H < d_o$ **(Phase 2)**
PoS	1	$u(d_o)/u(d_H)$ **(stable PoA)**
Monotonicity	-	increasing
Starting at 1	-	Yes
PoA	$u(d_o)/u(d_L)$	
Monotonicity	increasing	
Starting at 1	Yes	

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Table 5.3: Summary of the PoS and PoA with b as the only varying parameter. We assume b starts at its valid minimum value (i.e. $bxp'(x) = \frac{1}{n-1}$ has exactly one solution.) and keeps increasing.

Condition	$d_o > d_H$ **(Phase 1)**	$d_H \geq d_o \geq d_L$ **(Phase 2)**
PoS	$u(d_o)/u(d_H)$ **(stable PoA)**	1
Monotonicity	decreasing	-
Terminating at 1	YES	-
PoA	$u(d_o)/u(d_L)$	
Monotonicity	increasing	
Starting at 1	No	

Table 5.4: Summary of the PoS and PoA with n as the only varying parameter. We assume n starts at its valid minimum value (i.e. $bxp'(x) = \frac{1}{n-1}$ has exactly one solution.) and keeps increasing.

Condition	$d_H < d_o$ **(Phase 1)**	$d_L \leq d_o \leq d_H$ **(Phase 2)**
PoS	$u(d_o)/u(d_H)$ **(stable PoA)**	1
Monotonicity	decreasing	-
Terminating at 1	uncertain	-
PoA	$u(d_o)/u(d_L)$	
Monotonicity	increasing	
Starting at 1	No	

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Chapter 6

Conclusion and Future Work

In the last chapter, we're going to briefly conclude our analysis, describe additional possibly extended models, and discuss some aspects that can be improved in the future.

In a two-player file-sharing game, we detailedly examine all Nash equilibria including their stability. When the need for resources is almost not limited, there are two non-collapsing Nash equilibria, one of which with a greater contribution is stable. When the need is a little limited, the contribution of the Nash equilibrium with a greater contribution will be lowered and it will become unstable. Besides, there exist two additional side Nash equilibria in this case. When the limitation is drastic, the system will collapse. In a three-player file-sharing game, we still examine all Nash equilibria, yet without stability. In a multi-player file-sharing game, we only examine symmetric Nash equilibria without stability. The conclusion of the PoA and PoS remains the same when the number of players increases from two to three. It remains the same for an arbitrary number of players if we only consider the symmetric Nash equilibria.

We give an intuitive explanation of the PoS and PoA here. The PoS and PoA both increase with K since the Nash equilibria (the consequence of selfishness) naturally falls behind the maximum total utility (which increases with the amount of resources). We also discover that the two parameters b and n both represent the flexibility of the model. If we increase b and n, ideally the best Nash equilibrium will be improved and the worst Nash equilibrium will be deteriorated. Hence the PoS decreases with b, n but the PoA increases with b, n. When the need for resources is a little limited, the PoS can remain 1 because

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the maximum total utility is not too far away such that the best Nash equilibrium is able to catch up.

After the analysis in multi-player file-sharing games, the reader may make a guess of the following conjectures. First, $(d_L, d_L, ..., d_L)$ is always unstable, and $(d_H, d_H, ..., d_H)$ is always stable in a multi-player file-sharing game. Second, $u(d_L, d_L, ..., d_L)$ is the least among all non-collapsing Nash equilibria, and $u(d_H, d_H, ..., d_H)$ is the greatest among all non-collapsing Nash equilibria. If the second conjecture is true, the PoA and PoS derived in multi-player file-sharing games are always true even if we take all Nash equilibria into consideration.

In this thesis, we assume each player can provide at most the benefit K of resources to all other players. This is a simple assumption. If we further consider a more realistic situation where they have different limitations K_j, the utility function becomes

$$u_i(d_i) = -d_i + \sum_{j \neq i} \min\{K_j,\ b\, d_j\, p(d_i)\}, \text{ for } 1 \leq i \leq n.$$

If each player has his/her own desired resources of the "total" benefit K distributed on all the other players, the utility function becomes

$$u_i(d_i) = -d_i + \min\{K,\ b\, p(d_i) \sum_{j \neq i} d_j\}, \text{ for } 1 \leq i \leq n.$$

If the benefits of these resources (K_i) are different, the utility function becomes

$$u_i(d_i) = -d_i + \min\{K_i,\ b\, p(d_i) \sum_{j \neq i} d_j\}, \text{ for } 1 \leq i \leq n.$$

If all players have their unique "files" of different benefits (K_j) and each player will try their best to retrieve all files from all the other players, the utility function becomes

$$u_i(d_i) = -d_i + \sum_{j \neq i} \text{if}\ \{b\, d_j\, p(d_i) \geq K_j\} \cdot K_j, \text{ for } 1 \leq i \leq n,$$

where the value of the "if" function is defined to be 1 if the condition is true, and defined

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to be 0 if the condition is false. Since a file is valid only if all portions of it are retrieved, we use the “if” function here. They are also good research problems.

Finally, the reader may discover that in the results of [2] and our thesis, the common problems of P2P such as whitewashing attacks and sybil attacks from malicious users are still not taken into consideration. In fact there are many studies [1, 4, 11, 13, 15] focusing on these problems. Maybe we can study these papers in the future and improve our models to concretely solve the problems.

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