國立臺灣大學理學院數學所

碩士論文

Department or Graduate Institute of Mathematics

College of Science

National Taiwan University

Master Thesis

漸進行為與拉普拉斯轉換的關係

Asymptotic behavior on Laplace Transform

蘇哲寬

Jhe-Kuan Su

指導教授：陳逸昆 教授

Advisor: Professor I-Ken Chen

中華民國 108 年 7 月

July 2019

doi:10.6342/NTU201901693

國立臺灣大學碩士學位論文
口試委員會審定書

漸進行為與拉普拉斯轉換的關係
Asymptotic Behavior on Laplace Transform

本論文係蘇哲寬君（R06221005）在國立臺灣大學數學系完成之碩士學位論文，於民國 108 年 7 月 8 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

陳逸昆 （簽名）

（指導教授）

沈俊嚴

夏俊雄

系主任、所長 ____________ （簽名）

（是否須簽章依各院系所規定）

中文摘要

在這篇論文中，我們主要討論拉普拉斯轉換如何影響一個給定函數的漸進行為。研究發現，一個函數 f(t)在 t 很大的行為會影響 L(f)(s)在 s 很小的漸進行為。更進一步在某些狀況下我們可以藉由 L(f)(s)的漸進行為去推得 f(t)在 t 很大的漸進行為。這篇論文主要使用實分析技巧。我們還將此結果與 Abelian and Tauberian Theorem of Laplace transform 進行比較。

doi:10.6342/NTU201901693

Abstract

In the present thesis, we concern the relation between the function and its Laplace transform. More precisely, the asymptotic behavior when t is large (small) implies asymptotic behavior of its Laplace transform when s is small (large) and vice versa, especially, for critical exponent we provide a simple sufficient and necessary condition. Our approach only involves real analysis. We will also compare the result with Abelian and Tauberian Theorem of Laplace transform.

doi:10.6342/NTU201901693

目　錄

doi:10.6342/NTU201901693

1 Introduction

Throughout this thesis, we assume that f is a measurable function on R and $e^{-st}f(t) \in L_1$ for any $s > 0$.

We define

$$L(f)(s) \equiv \int_0^\infty f * e^{-st} dt \tag{1}$$

and

$$F(t) \equiv \int_0^t f(u) du. \tag{2}$$

It was found that the information about the asymptotic behaviour of F near infinity implies some information about the asymptotic behaviour behaviour of L(f) near zero and vice versa, for example the famous Tauberian Theorem for Laplace transform.

According to Feller in theorem of [1], we know that given $\rho \geq 0$ if we have "scaling behaviour" of f near the infinity, i.e.,

$$\lim_{t\to\infty} \frac{F(xt)}{F(t)} = x^\rho, \tag{3}$$

then L(f) will have the scaling behaviour near origin, i.e.,

$$\lim_{t\to 0} \frac{L(f)(xt)}{L(f)(t)} = x^{-\rho} \tag{4}$$

and vice versa.

In [2], if F has some "logistic" behaviour near infinity, then L(f) has some logistic behaviour near zero and vice versa. In [3], a complex analysis approach is used. With some additional condition, the relation of asymptotic behaviour are characterized.

In the thesis, we will relax the condition and still provide a similar result.

First we found that :

Theorem 1.1.

(a) For integer $n \geq 0$*, when* $\lim_{t\to\infty} \frac{f(t)}{t^n} = 1$*, then*

$$\lim_{s\to 0} \frac{L(f)(s)}{s^{-n-1}} = n!. \tag{5}$$

(b) If $\lim_{t\to\infty} \frac{f(t)}{\frac{1}{t}} = 1$*, then*

$$\lim_{s\to 0} \frac{L(f)(s)}{-\log s} = 1. \tag{6}$$

doi:10.6342/NTU201901693

(c) For integer $n < -1$, if $\lim_{t\to\infty} \frac{f(t)}{t^n} = 1$, then

$$\int_0^\infty e^{-st} f(t)dt - \Sigma_{i=0}^{-n-2} \int_0^\infty \frac{(-1)^i t^i}{i!} f(t)dts^i = \frac{(-1)^{-n} s^{-n-1} \log s}{(-n-1)!} + o(\frac{s^{-n-1} \log s}{(-n-1)!}) \tag{7}$$

as $s \to 0$

Then we use that result to found that

Theorem 1.2.

(a) For integer $n \geq 0$ when $\lim_{t\to\infty} \frac{F(t)}{t^n} = 1$, then

$$\lim_{s\to 0} \frac{L(f)(s)}{s^{-n}} = n! \tag{8}$$

(b) If $\lim_{t\to\infty} \frac{F(t)}{\log t} = 1$ then

$$\lim_{s\to 0} \frac{L(f)(s)}{-\log s} = 1 \tag{9}$$

and vice versa if f is non negative.

(c) For integer $n < -1$ when, $\lim_{t\to\infty} \frac{-F(t)+\int_0^\infty f(u)du}{(n+1)t^{n+1}} = 1$, then

$$L(f)(s) - \Sigma_{i=0}^{-n-2} \int_0^\infty \frac{(-1)^i t^i}{i!} f(t)dts^i = \frac{(-1)^{-n} s^{-n-1} \log s}{(-n-1)!} + o(\frac{s^{-n-1} \log s}{(-n-1)!}) \tag{10}$$

as $s \to 0$, and the converse is not true

The organization of the remaining part of the thesis is as follows:

In Section 2, we will discuss how the behaviour of f affect the behaviour of L(f)(s). In Section 3, we will discuss the relation between the behaviour of F(t) and the behaviour of L(f)(s). In Section 4, we will show that the polynomial terms of L(f) tell nothing about of f. In Section 5, we will compare our result with previous result in [1],[2].

2 Relation between behaviour of the function and its transformed type (1)

The proof of Theorem 1.1 is as below.

doi:10.6342/NTU201901693

Proof.

(a)

If

$f(x) = t^n + h(t)$ where $\frac{h(t)}{t^n} \to 0$ as $t \to \infty$,

then we have

$$\begin{aligned}&\int_0^\infty e^{-st} f(t)dt\\ =&\int_0^\infty e^{-st}(t^n + h(t))dt\\ =&\frac{n!}{s^{n+1}} + \int_0^\infty e^{-st} h(t)dt.\end{aligned} \tag{11}$$

So, it suffice to show that:

$$\int_0^\infty e^{-st} h(t)dt s^{n+1} \to 0 \tag{12}$$

as $s \to 0$.

Since $h(t)t^{-n} \to 0$ as $t \to \infty$,

we have
$\forall \varepsilon > 0, \exists \delta > 0 \; s.t. \; |h(t)| < \varepsilon t^n$ for $t \in [\delta, \infty]$

In doing so,

$$\begin{aligned}&\int_0^\infty e^{-st} h(t)dt\\ =&\int_\delta^\infty e^{-st} h(t)dt + \int_0^\delta e^{-st} h(t)dt,\end{aligned} \tag{13}$$

but

$$\begin{aligned}&|\int_\delta^\infty e^{-st} h(t)dt|\\ \leq&\int_\delta^\infty e^{-st} |h(t)|dt\\ \leq&\int_\delta^\infty e^{-st} \varepsilon t^n dt\\ \leq&\frac{n!\varepsilon}{s^{n+1}}.\end{aligned} \tag{14}$$

doi:10.6342/NTU201901693

Plus,

$$\begin{aligned} &|\frac{\int_0^\delta e^{-st}h(t)dt}{s^{-n-1}}| \\ =&|\int_0^\delta e^{-st}s^{n+1}h(t)dt| \\ =&s^{n+1}\int_0^\delta e^{-st}|h(t)|dt \\ \leq&s^{n+1}\int_0^\delta |h(t)|dt \end{aligned} \tag{15}$$

(Note that h(t) is locally integrable.)

which approach to 0 as $s \to 0$.

So we have $\limsup_{s\to 0} |\int_0^\infty e^{-st}h(t)| \leq \varepsilon$ for any $\varepsilon > 0$.

which implies $\limsup_{s\to 0} |\int_0^\infty e^{-st}h(t)| = 0$

(b) and (c)
For integer $n \geq 1$,
if we have $f(t) = \frac{1}{t^n} + o(\frac{1}{t^n})$ as $t \to \infty$,

define

$$\phi_n(x) := \frac{e^{-x} - \Sigma_{i=0}^{n-2}\frac{(-1)^i x^i}{i!}}{\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}}. \tag{16}$$

For example,

$\phi_1(x) = e^{-x}$,

$\phi_2(x) = \frac{e^{-x}-1}{-x}$,

$\phi_3(x) = \frac{e^{-x}-1+x}{\frac{x^2}{2}}$
...and so on.

It is not hard to know that this series of functions are all non negative.

Claim: for any $n \geq 1$

(1) $\lim_{x\to 0} \phi_n(x) = 1$.

(2) $\lim_{x\to\infty} x\phi_n(x) = n - 1$.

doi:10.6342/NTU201901693

(3) $\phi_n(x)$ is monotone decreasing function.

The first part and second part of the claim is trivial from the definition of $\phi_n(x)$.

So, we only do the third part:

Since the first part and second part of the claim, it suffice to show that all the critical point of this function lie in a decreasing function with initial value 1.

If it has a critical point says x and $\phi_n'(x) = 0$,

then

$$\begin{aligned}
&\phi_n'(x)\\
=&\frac{(-e^{-x} - \Sigma_{i=1}^{n-2}\frac{(-1)^i x^{i-1}}{(i-1)!})(\frac{(-1)^n x^{n-1}}{(n-1)!}) - (-e^{-x} - \Sigma_{i=0}^{n-2}\frac{(-1)^i x^i}{(i)!})(\frac{(-1)^n x^{n-2}}{(n-2)!})}{(\frac{1}{(n-1)!})^2 x^{2n-2}}\\
=&\frac{(-1)^n(n-1)!}{x^n}(-e^{-x}(x+n-1) + \Sigma_{i=0}^{n-2}\frac{(-1)^i x^i(n-1-i)}{i!})
\end{aligned} \tag{17}$$

then we have

$$e^{-x} = \frac{\Sigma_{i=0}^{n-2}\frac{(-1)^i x^i}{i!}(n-1-i)}{x+n-1}. \tag{18}$$

Furthermore,

$$\begin{aligned}
&\phi_n(x)\\
=&\frac{\frac{\Sigma_{i=0}^{n-2}\frac{(-1)^i x^i}{i!}(n-1-i)}{x+n-1} - \Sigma_{i=0}^{n-2}\frac{(-1)^i x^i}{i!}}{\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}}\\
=&\frac{n-1}{x+n-1}.
\end{aligned} \tag{19}$$

That means all critical point lie in a strictly decreasing sequence function $h(x) = \frac{n-1}{x+n-1}$ (note h(0)=1),

but if $\phi_n(x)$ is not monotone decreasing i.e. there exist $x_1 < x_2$ such that $\phi_n(x_1) < \phi_n(x_2)$

Then, if $x_1 \neq 0$, then there is a maximum point y of $\phi_n(x)$ on $(x_1, x_2]$, and a minimum point z of $\phi_n(x)$ on [0,y). Note that y is a critical point,

doi:10.6342/NTU201901693

so $\phi_n(y) = \frac{n-1}{y+n-1} < 1$ which imply $z \neq 0$. z is also a critical point and $\phi_n(z) < \phi_n(y)$, but this contradict the fact that all critical point of $\phi_n(x)$ lie in a strictly decreasing function $f(x) = \frac{n-1}{x+n-1}$.

If $x_1 = 0$, this lead to the contradiction once we notice that $f(x) = \frac{n-1}{x+n-1} < 1$ for any $x > 1$.

By (1) and (3) of the claim, we have $0 \leq \phi_n \leq 1$ for any $x \geq 0$.

By assumption and (2) of claim,we have
$\forall \varepsilon > 0, \exists M(\varepsilon) > 0$
s.t. $\forall t \in [M(\varepsilon), \infty)$, we have $|f(t) - \frac{1}{t^n}| \leq \varepsilon \frac{1}{t^n}$ and $|\phi_n(t)t| \leq (n-1)(1+\varepsilon)$.
Note that

$$\begin{aligned}
&\frac{L(f)(s) - \Sigma_{i=0}^{i=n-2} \frac{(-1)^i s^i}{i!} \int_0^\infty t^i f(t)dt}{\frac{(-1)^{n-1}}{(n-1)!} s^{n-1}} \\
&= \int_0^\infty \phi_n(st) t^{n-1} f(t)dt \\
&= \int_0^{M(\varepsilon)} \phi_n(st) t^{n-1} f(t)dt + \int_{M(\varepsilon)}^{\frac{1}{s}} \phi_n(st) t^{n-1} f(t)dt + \int_{\frac{1}{s}}^\infty \phi_n(st) t^{n-1} f(t)dt \\
&\leq \int_0^{M(\varepsilon)} \phi_n(st) t^{n-1} f(t)dt + \int_{M(\varepsilon)}^{\frac{1}{s}} t^{n-1}|f(t)|dt + (n-1)\int_{\frac{1}{s}}^\infty (1+\varepsilon)^2 \frac{1}{st^2}dt \\
&\leq \int_0^{M(\varepsilon)} \phi_n(st) t^{n-1} f(t)dt + \int_{M(\varepsilon)}^{\frac{1}{s}} (1+\varepsilon)\frac{1}{t}dt + (n-1)\int_{\frac{1}{s}}^\infty (1+\varepsilon)^2 \frac{1}{st^2}dt \\
&\leq \int_0^{M(\varepsilon)} \phi_n(st) t^{n-1} f(t)dt + (1+\varepsilon)(\log\frac{1}{s} - \log M(\varepsilon)) + (n-1)\int_{\frac{1}{s}}^\infty (1+\varepsilon)^2 \frac{1}{st^2}dt \\
&\leq \int_0^{M(\varepsilon)} t^{n-1}|f(t)|dt + (1+\varepsilon)(\log\frac{1}{s} - \log M(\varepsilon)) + (n-1)(1+\varepsilon)^2.
\end{aligned} \tag{20}$$

Hence we have

$$\limsup_{s\to 0} \frac{L(f)(s) - \Sigma_{i=0}^{i=n-2} \frac{(-1)^i s^i}{i!} \int_0^\infty t^i f(t)dt}{(-1)^{n-1} s^{n-1}(-\log s)} \leq 1. \tag{21}$$

On the other hand, for $\frac{\phi_n^{-1}(1-\varepsilon)}{s} > M(\varepsilon)$,

doi:10.6342/NTU201901693

$$
\begin{aligned}
&\int_0^\infty \phi_n(st)t^{n-1}f(t)dt \\
&= \int_0^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} \phi_n(st)t^{n-1}f(t)dt + \int_{\frac{\phi_n^{-1}(1-\varepsilon)}{s}}^\infty \phi_n(st)t^{n-1}f(t)dt \\
&\geq \int_0^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} \phi_n(st)t^{n-1}f(t)dt \\
&= \int_0^{M(\varepsilon)} \phi_n(st)t^{n-1}f(t)dt + (1-\varepsilon)\int_{M(\varepsilon)}^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} t^{n-1}f(t)dt \\
&= \int_0^{M(\varepsilon)} t^{n-1}|f(t)|dt + (1-\varepsilon)\int_{M(\varepsilon)}^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} (1-\varepsilon)\frac{1}{t}dt \\
&= \int_0^{M(\varepsilon)} t^{n-1}|f(t)|dt + (1-\varepsilon)\int_{M(\varepsilon)}^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} (1-\varepsilon)\frac{1}{t}dt \\
&= \int_0^{M(\varepsilon)} t^{n-1}|f(t)|dt + (1-\varepsilon)^2\int_{M(\varepsilon)}^{\frac{\phi_n^{-1}(1-\varepsilon)}{s}} \frac{1}{t}dt] \\
&= \int_0^{M(\varepsilon)} t^{n-1}|f(t)|dt + (1-\varepsilon)^2(\log(\frac{\phi_n^{-1}(1-\varepsilon)}{s}) - \log M(\varepsilon))]
\end{aligned}
\tag{22}
$$

And we conclude that

$$
\liminf_{s\to 0} \frac{L(f)(s) - \Sigma_{i=0}^{i=n-2}\frac{(-1)^i s^i}{i!}\int_0^\infty t^i f(t)dt}{(-1)^{n-1}s^{n-1}(-\log s)} \geq 1. \tag{23}
$$

□

Remark 2.1.

The method to prove the part (a) of Theorem also lead to the following result:

(a) If there is a non-negative integer n such that

$f(x) = t^n + o(t^n)$ *as* $t \to 0$

then

$L(f)(s) = \frac{n!}{s^{n+1}} + o(\frac{1}{s^{n+1}})$ *as* $s \to \infty$.

Remark 2.2. *The converse is not true.*

doi:10.6342/NTU201901693

3 Relation between behaviour of the function and its transformed type (2)

Before we prove the theorem 1.2,we introduce a useful lemma.

Lemma 3.1. $L(\log t)(s) = \frac{\gamma - \log s}{s}$,*where* $\gamma = \int_0^\infty e^{-u} \log u du$.

Proof.

$$\begin{aligned} & L(\log t)(s) \\ =& \int_0^\infty e^{-st} \log t dt \\ =& \int_0^\infty e^{-u} \log \frac{u}{s} \frac{1}{s} du \\ =& \int_0^\infty e^{-u} (log(u) - \log s) \frac{1}{s} du \\ =& \frac{1}{s} (\int_0^\infty e^{-u} \log u du - \int_0^\infty e^{-u} \log s du) \\ =& \frac{1}{s} (\int_0^\infty e^{-u} \log u du - \log s \int_0^\infty e^{-u} du) \\ =& \frac{1}{s} (\gamma - \log s). \end{aligned} \tag{24}$$

□

Then, we will use this to prove Theorem 1

Proof. Here we will break the proof into three steps.

Step 1: The "if" part of (a), (b) and (c)

Step 2: The "only if" part of (b)

Step 3: Offering the counter example of "only if part of (c).

Step 1:

doi:10.6342/NTU201901693

For (a),by Theorem 1.1,

$$\begin{aligned}&\frac{L(f(t))(s)}{s}\\=&L(\int_0^t f(x)dx)(s)\\=&\frac{1}{n+1}(\frac{(n+1)!}{s^{n+2}}+o(\frac{1}{s^{n+2}})).\end{aligned} \tag{25}$$

Therefore

$$L(f)(s)=\frac{n!}{s^{n+1}}+o(\frac{1}{s^{n+1}}). \tag{26}$$

For (b),

$$\begin{aligned}&\frac{L(f(t))(s)}{s}\\=&L(\int_0^t f(x)dx)(s)\\=&L(\log t+g(t))(s)\ (where\ \frac{g(t)}{\log t}\to 0\ as\ t\to\infty)\\=&\frac{\gamma-\log s}{s}+L(g(t))(s)\\=&\frac{\gamma-\log s}{s}+\int_0^\infty e^{-st}g(t)dt.\end{aligned} \tag{27}$$

Note that for any $\varepsilon>0$ there is real number $M(\varepsilon)>0$
s.t. $|g(t)|\leq\varepsilon\log t$ whenever $t>M(\varepsilon)$.
Therefore,

$$\begin{aligned}&\frac{\gamma-\log s}{s}+\int_0^\infty e^{-st}g(t)dt\\=&\frac{\gamma-\log s}{s}+\int_0^{M(\varepsilon)} e^{-st}g(t)dt+\int_{M(\varepsilon)}^\infty e^{-st}g(t)dt\\\leq&\frac{\gamma-\log s}{s}+\int_0^{M(\varepsilon)} e^{-st}g(t)dt+\int_{M(\varepsilon)}^\infty e^{-st}\varepsilon\log tdt\\\leq&(1+\varepsilon)\frac{\gamma-\log s}{s}+\int_0^{M(\varepsilon)} g(t)dt.\end{aligned} \tag{28}$$

That means for any $\varepsilon>0$

$$\limsup_{s\to 0}\frac{L(f(t))(s)}{-\log s}\leq 1+\varepsilon. \tag{29}$$

doi:10.6342/NTU201901693

So we have

$$\limsup_{s\to 0} \frac{L(f(t))(s)}{-\log s} \leq 1. \tag{30}$$

Similarly,

$$\limsup_{s\to 0} \frac{L(f(t))(s)}{-\log s} \geq 1 \tag{31}$$

and the result follows.

For (c), the method is similar to (b)

Step 2:

Note that for any $0 < M < 1$,

$$\begin{aligned}
&L(f)(s)\\
=&\int_0^\infty e^{-st} f(t)dt\\
\geq&\int_0^{-\frac{1}{s}\log M} e^{-st} f(t)dt\\
\geq&\int_0^{-\frac{1}{s}\log M} Mf(t)dt\\
\geq&M\int_0^{-\frac{1}{s}\log M} f(t)dt\\
=&MF(-\frac{1}{s}\log M).
\end{aligned} \tag{32}$$

Since

$$\limsup_{s\to 0} \frac{L(f)(s)}{-\log s} \leq 1, \tag{33}$$

we have

$$\limsup_{s\to 0} \frac{F(-\frac{1}{s}\log M)}{\log\frac{1}{s}} \leq \frac{1}{M}. \tag{34}$$

Hence,

$$\limsup_{s\to 0} \frac{F(\frac{1}{s})}{\log\frac{1}{-s\log M}} \leq \frac{1}{M}. \tag{35}$$

So,

$$\limsup_{s\to 0} \frac{F(\frac{1}{s})}{\log\frac{1}{s}} \leq \frac{1}{M} \tag{36}$$

doi:10.6342/NTU201901693

for any $0 < M < 1$.

Then, we have

$$\limsup_{s\to 0}\frac{F(\frac{1}{s})}{\log\frac{1}{s}} \le 1. \tag{37}$$

On the other hand,

we suppose $\liminf_{t\to\infty}\frac{F(t)}{\log t} < 1$,

then there exist $t_i \nearrow \infty$ such that $F(t_i) \le (1-\varepsilon)\log t_i$.

Let $s_i := t_i^{-\frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}}$.

For convenience, we set $\gamma := \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}$.

Then,

$$\begin{aligned}
&L(f)(s_j)\\
=&\int_0^\infty e^{-s_j t} f(t)dt\\
=&\int_0^\infty e^{-\frac{t}{t_j^\gamma}} f(t)dt\\
=&\int_0^{t_j} e^{-\frac{t}{t_j^\gamma}} f(t)dt + \int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} f(t)dt.
\end{aligned} \tag{38}$$

Set $G(t) \equiv (1+\delta)\log t$ for $t \ge t_j$.
Note that by (39)
$\exists M > 0 s.t. \forall t \ge M, \frac{F(t)}{-\log t} \le 1+\delta$.

For $t_i > M$, by integration by part, we have

$$\begin{aligned}
&\int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} f(t)dt\\
=&-e^{-\frac{t_j}{t_j^\gamma}} F(t_j) + s\int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} F(t)dt\\
\le&-e^{-\frac{t_j}{t_j^\gamma}} F(t_j) + s\int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} G(t)dt\\
\le&-e^{-\frac{t_j}{t_j^\gamma}} F(t_j) + e^{-\frac{t}{t_j^\gamma}} G(t_j) + \int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} dG\\
\le& e^{-\frac{t_j}{t_j^\gamma}} G(t_j) + \int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} dG\\
\le&(1+\delta)e^{-\frac{t_j}{t_j^\gamma}} \log t_j + \int_{t_j}^\infty e^{-\frac{t}{t_j^\gamma}} dG.
\end{aligned} \tag{39}$$

doi:10.6342/NTU201901693

Therefore

$$
\begin{aligned}
&\int_0^{t_j} e^{-\frac{t}{t_j^\gamma}} f(t)dt + \int_{t_j}^{\infty} e^{-\frac{t}{t_j^\gamma}} f(t)dt \\
\le &\int_0^{t_j} f(t)dt + (1+\delta)e^{-\frac{t_j}{t_j^\gamma}} \log t_j + \int_{t_j}^{\infty} e^{-\frac{t}{t_j^\gamma}} dG \\
\le &(1-\varepsilon)\log t_j + (1+\delta)e^{-\frac{t_j}{t_j^\gamma}} \log t_j + \int_{t_j}^{\infty} e^{-\frac{t}{t_j^\gamma}} dG \\
\le &-(1-\frac{\varepsilon}{2})\log s_j - \frac{(1+\delta)}{\gamma} e^{-\frac{t_j}{t_j^\gamma}} \log s_j + \int_{t_j}^{\infty} e^{-\frac{t}{t_j^\gamma}} (1+\delta)\frac{1}{t}dt \\
= &-(1-\frac{\varepsilon}{2})\log s_j - \frac{(1+\delta)}{\gamma} e^{-\frac{t_j}{t_j^\gamma}} \log s_j + (1+\delta)[\log t_j e^{-s_j t_j} + \int_{s_j t_j}^{\infty} \log u e^{-u} du - e^{-s_j t_j}\log s_j] \\
= &-(1-\frac{\varepsilon}{2})\log s_j - \frac{(1+\delta)}{\gamma} e^{-s_j^{\frac{\gamma-1}{\gamma}}} \log s_j + (1+\delta)[-\frac{1}{\gamma}\log s_j e^{-s_j^{\frac{\gamma-1}{\gamma}}} + \int_{s_j^{\frac{\gamma-1}{\gamma}}}^{\infty} \log u e^{-u} du - e^{-s_j^{\frac{\gamma-1}{\gamma}}} \log s_j].
\end{aligned}
\tag{40}
$$

Since $s_j^{\frac{\gamma-1}{\gamma}} \to \infty$, we have

$$
\liminf_{s\to 0} \frac{L(f)(s)}{-\log s} < 1 - \frac{\varepsilon}{2}. \tag{41}
$$

which contradict the assumption.
Hence $\liminf_{t\to\infty} \frac{F(t)}{\log t} \ge 1$.
Combining (36), we finish the step 2.

Step 3:
For $n < -1$,

set $f := \Sigma_{i=0}^{\infty} \delta_{t_i}(t_i^{n+1} - t_{i+1}^{n+1})$,

then

$$
\begin{aligned}
&\int_0^{\infty} \phi_{-n}(st)t^{-n-1} f dt \\
=&\int_0^{\infty} \phi_{-n}(st)t^{-n-1} \Sigma_{i=0}^{\infty} \delta_{t_i}(t_i^{n+1} - t_{i+1}^{n+1}) dt \\
=&\Sigma_{i=0}^{\infty} \phi_{-n}(st_i) t_i^{-n-1}(t_i^{n+1} - t_{i+1}^{n+1}) \\
\le&\Sigma_{i=0}^{\infty} \phi_{-n}(st_i) t_i^{-n-1}(t_i^{n+1}) \\
=&\Sigma_{i=0}^{\infty} \phi_{-n}(st_i).
\end{aligned}
\tag{42}
$$

doi:10.6342/NTU201901693

We assume that $t_1 \geq 2, t_{i+1} \geq (i+1)t_i$ (it is easily follow once we choose the subsequence of t_i).

Set $s_j := \frac{1}{t_{j+1}}$ (here j is large enough such that $\phi_{-n}(x) < (1+\varepsilon)(\frac{1}{x})$ for $x \geq (j+2)$,

then using property of ϕ, we have

$$\begin{aligned}
&\int_0^\infty \phi_{-n}(\frac{t}{t_{j+1}})t^{-n-1}fdt \\
\leq&\Sigma_{i=0}^{\infty}\phi_{-n}(\frac{t_i}{t_{j+1}}) \\
=&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + \Sigma_{i=j+2}^{\infty}\phi_{-n}(\frac{t_i}{t_{j+1}}) \\
\leq&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + \Sigma_{i=j+2}^{\infty}\phi_{-n}(\prod_{k=j+2}^{i} k) \\
\leq&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + \Sigma_{i=j+2}^{\infty}((-n-1)(1+\varepsilon)\frac{1}{\prod_{k=j+2}^{i} k}) \\
=&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + (-n-1)(1+\varepsilon)\Sigma_{i=j+2}^{\infty}(\frac{1}{\prod_{k=j+2}^{i} k}) \\
\leq&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + (-n-1)(1+\varepsilon)\Sigma_{i=j+2}^{\infty}(\frac{1}{(j+2)^{i-j-1}}) \\
\leq&\Sigma_{i=0}^{j+1}\phi_{-n}(\frac{t_i}{t_{j+1}}) + (-n-1)(1+\varepsilon)(\frac{1}{j+1}) \\
\leq&(j+2) + (-n-1)(1+\varepsilon)(\frac{1}{j+1}) \\
<<& \log t_{j+1} \\
=&-\log s_j.
\end{aligned} \tag{43}$$

Given $s \in (s_{j+1}, s_j)$, by monotonicity of ϕ, we know that $\int_0^\infty \phi_{-n}(st)t^{-n-1}fdt$ is monotonically deceasing on s.

Hence,

doi:10.6342/NTU201901693

$$\begin{aligned}
&\int_0^\infty \phi_{-n}(st)t^{-n-1}fdt \\
\leq &\int_0^\infty \phi_{-n}(s_{j+1}t)t^{-n-1}fdt \\
\leq &(j+3)+(-n-1)(1+\varepsilon)(\frac{1}{j+2}) \\
< &< \log t_{j+1} \\
= &-\log s_j \\
\leq &-\log s.
\end{aligned} \tag{44}$$

Let $g(t) := (1+t)^n + \Sigma_{i=0}^{\infty}\delta_{t_i}(t_i^{n+1} - t_{i+1}^{n+1})$.
It is obvious that $\int_0^t g(u)du \neq t^n + o(t^n)$,

but

$L(f)(s) - \Sigma_{i=0}^{-n-2}\int_0^\infty \frac{(-1)^i t^i}{i!} f(t)dts^i = \frac{(-1)^{-n}a_{-n}s^{-n-1}\log s}{(-n-1)!} + o(\frac{s^{-n-1}\log s}{(-n-1)!})$.

Then the result is followed.

□

4 Non-informativeness of polynomial term of the transformed function

Theorem 4.1. *For any $C_i \in R,\ i = 1, 2, ..., m$ there exist f with compact support such that*

$$\int_0^\infty e^{-st}f(t)dt = C_0 + C_1 s + C_2 s^2 ... + C_m s^m + o(s^m). \tag{45}$$

Proof. Note that

$$\begin{aligned}
&\int_0^\infty e^{-st}1_{[a,b]}dt = \frac{e^{-as} - e^{-bs}}{s} \\
=&\Sigma_{i=0}^{\infty}\frac{(-1)^{i-1}(a^{i+1} - b^{i+1})}{(i+1)!}s^i \\
=&\Sigma_{i=0}^{m}\frac{(-1)^{i-1}(a^{i+1} - b^{i+1})}{(i+1)!}s^i + O(s^{m+1}).
\end{aligned} \tag{46}$$

By choosing $f_j := 1_{[0,j+1]}$, for $j = 0, ..., m$, we have

$$\int_0^\infty e^{-st}f_j dt = \Sigma_{i=0}^{m}\frac{(-1)^{i-1}(-(j+1)^{i+1})}{(i+1)!}s^i + O(s^{m+1}). \tag{47}$$

doi:10.6342/NTU201901693

So it suffice to show that the matrix $A \equiv$

$$\begin{bmatrix} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \cdots & \frac{m+1}{1} \\ \frac{-1}{2!} & \frac{-4}{2!} & \frac{-9}{2!} & \cdots & \frac{-(m+1)^2}{2!} \\ \frac{1}{3!} & \frac{8}{3!} & \frac{27}{3!} & \cdots & \frac{(m+1)^3}{3!} \\ \cdots \\ \frac{(-1)^m}{(m+1)!} & \frac{(-1)^m 2^{m+1}}{(m+1)!} & \frac{(-1)^m 3^{m+1}}{(m+1)!} & \cdots & \frac{(-1)^m (m+1)^{m+1}}{(m+1)!} \end{bmatrix}$$

has non zero determinant.

But, the is equal to say the matrix(by elementary matrix operation)

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & m+1 \\ 1^2 & 2^2 & 3^2 & \ldots & (m+1)^2 \\ 1^3 & 2^3 & 3^3 & \ldots & (m+1)^3 \\ \ldots \\ 1^{(m+1)} & 2^{(m+1)} & 3^{(m+1)} & \ldots & (m+1)^{(m+1)} \end{bmatrix}$$

has zero determinant and it is a known fact.

□

5 The difference with the previous result

We will show the difference between asymptotic behaviour and scaling behaviour.

The following theorem is prove in [1]:

Proposition 5.1. *Let u(x) be a measurable function on* $[0, \infty)$ *We define*

$U(t) := \int_0^t u(dx)$

$L(u)(s) := \int_0^\infty e^{-st} dU$ *for* $s > 0$

Then, for any $\rho > 0$*, the following two statements are equivalent:*

(1)

$$\lim_{t \to \infty} \frac{L(u)(\frac{s}{t})}{L(u)(\frac{1}{t})} = \frac{1}{\lambda^\rho} \tag{48}$$

for any $s > 0$

(2)

$$\lim_{t \to 0} \frac{U(ts)}{U(t)} = s^\rho \tag{49}$$

doi:10.6342/NTU201901693

for any $s > 0$

Which is the difference of the condition between this result and the result in chapter 1?

It is obvious that if we have $f(t) = Ct^n + o(t^n)$ as $t \to \infty$ for some $C \in R$, then $\lim_{t\to\infty} \frac{f(xt)}{t}$, but the converse is false even when n=1.

Here we provide example.

Let $g(t) := t(1 + \frac{1}{10}\sin\log\log t)$, it is clear that $g(t) \neq Ct^n + o(t^n)$ for any $C \in R$.

However,

$$
\begin{aligned}
&\frac{g(xt)}{g(t)} \\
=&\frac{xt(1+\frac{1}{10}\sin\log\log xt)}{t(1+\frac{1}{10}\sin\log\log t)} \\
=&x\frac{(1+\frac{1}{10}\sin\log\log xt)}{(1+\frac{1}{10}\sin\log\log t)} \\
=&x(1+\frac{\frac{1}{10}\sin\log\log xt - \frac{1}{10}\sin\log\log t}{(1+\frac{1}{10}\sin\log\log t)}) \\
=&x(1+\frac{1}{10}\frac{\sin\log\log xt - \sin\log\log t}{(1+\frac{1}{10}\sin\log\log t)}) \\
\to& x
\end{aligned}
\tag{50}
$$

as $t \to \infty$.

(note that $|\sin\log\log xt - \sin\log\log t| \leq |\log\log xt - \log\log t|$
$\leq |\log\frac{\log xt}{\log t}| \to 0$)

The following theorem is prove in [2]:

Proposition 5.2. *For any increasing and right continuous function $F : R^+ \to R$ with $F(0^+) = 0$, let $L(dF) \equiv \int_{0^+}^{\infty} e^{-ts}dF(s)$ then the following are equivalent:*

(a) for every positive x and y ($y \neq 1$)

$$\lim_{t\to\infty}\frac{F(tx) - F(t)}{F(ty) - F(t)} = \frac{\log x}{\log y} \tag{51}$$

(b)

$$\lim_{t\to 0}\frac{L(dF)(tx) - L(dF)(t)}{L(dF)(ty) - L(dF)(t)} = \frac{\log x}{\log y}. \tag{52}$$

doi:10.6342/NTU201901693

Here we will show Theorem 5.2 and Theorem 1.2 won't imply each other.

For instance, set $g(t) := \log t(1+\varepsilon \sin\log\log\log t)$ and $h(t) := \log t + \sqrt{\log t}\sin t$.

Clearly, $\lim_{t\to\infty}\frac{g(t)}{\log t}$ doesn't exist.

However,

$$\begin{aligned}
&\frac{g(xt)-g(t)}{g(et)-g(t)}\\
=&\frac{\log xt(1+\varepsilon\sin\log\log\log xt)-\log t(1+\varepsilon\sin\log\log\log t)}{\log et(1+\varepsilon\sin\log\log\log et)-\log t(1+\varepsilon\sin\log\log\log t)}\\
=&\frac{\log x(1+\varepsilon\sin\log\log\log xt)+\log t(1+\varepsilon\sin\log\log\log xt)-\log t(1+\varepsilon\sin\log\log\log t)}{(1+\varepsilon\sin\log\log\log et)+\log t(1+\varepsilon\sin\log\log\log et)-\log t(1+\varepsilon\sin\log\log\log t)}\\
=&\frac{\log x(1+\varepsilon\sin\log\log\log xt)+\log t(\varepsilon(\sin\log\log\log xt-\sin\log\log\log t))}{(1+\varepsilon\sin\log\log\log et)+\log t(\varepsilon(\sin\log\log\log et-\sin\log\log\log t)))}.
\end{aligned} \tag{53}$$

Note that (Here without losing generosity we assume $x>1$.)

$$\begin{aligned}
&|\log t(\varepsilon(\sin\log\log\log xt-\sin\log\log\log t))|\\
\le&|\log t(\varepsilon(\log\log\log xt-\log\log\log t))|\\
\le&|(1-x)t\log t(\varepsilon\frac{1}{\log\log t}\frac{1}{\log t}\frac{1}{t})|\\
\le&|(1-x)(\varepsilon\frac{1}{\log\log t})|\to 0
\end{aligned} \tag{54}$$

as $t\to\infty$.

Hence

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\log x(1+\varepsilon\sin\log\log\log xt)+\log t(\varepsilon(\sin\log\log\log xt-\sin\log\log\log t))}{(1+\varepsilon\sin\log\log\log et)+\log t(\varepsilon(\sin\log\log\log et-\sin\log\log\log t)))}\\
=&\lim_{t\to\infty}\frac{\log x(1+\varepsilon\sin\log\log\log xt)}{(1+\varepsilon\sin\log\log\log et)}\\
=&\log x.
\end{aligned} \tag{55}$$

It is obvious that $\lim_{t\to\infty}\frac{h(t)}{\log t}=1$.

doi:10.6342/NTU201901693

However,

$$\begin{aligned}&\frac{h(2et)-h(t)}{h(et)-h(t)}\\=&\frac{\log 2e+\sqrt{\log 2et}\sin 2et-\sqrt{\log t}\sin t}{1+\sqrt{\log et}\sin et-\sqrt{\log t}\sin t}.\end{aligned} \tag{56}$$

If $t=\frac{\pi n}{e}$,

then it

$$\begin{aligned}=&\frac{\log 2e+\sqrt{\log 2et}\sin 2et-\sqrt{\log t}\sin t}{1+\sqrt{\log et}\sin et-\sqrt{\log t}\sin t}\\=&\frac{\log 2e-\sqrt{\log\frac{\pi n}{e}}\sin\frac{\pi n}{e}}{1-\sqrt{\log\frac{\pi n}{e}}\sin\frac{\pi n}{e}}.\end{aligned} \tag{57}$$

This doesn't approach to $\log x$ when $n\to\infty$.

Also it is shown in P.282 of [3] that:
(Here we simply the result by setting $s_0=0$ and consider s as complex number.)

If $L(f)(s)=\Sigma_{n=0}^{\infty}a_n s^{n-1}+\log s\Sigma_{n=0}^{\infty}b_n s^n$

and the following additional assumption holds.

(1) $L(f)(s)$ is analytic for $Re(s)\geq-\delta,\delta>0$, except at s=0

(2) $L(f)(s)\to 0$ uniformly as $Im(s)\to\infty$ for $-\delta\leq Re(s)\leq\gamma$ for some γ

(3) $\int_{\kappa-i\infty}^{\kappa+i\infty}|L(f)(s)|ds<\infty$ for $-\delta\leq\kappa\leq\gamma$,

Then we have

$$f=a_0-\Sigma_{n=0}^{\infty}(-1)^n b_n n!t^{-n-1}+o(t^{-n-1}) \tag{58}$$

as $t\to\infty$.

doi:10.6342/NTU201901693

References

[1] Feller, William. An introduction to probability theory and its applications. Vol. II. Second edition, 1971,Pages 443

[2] Laurens de Haan. An Abel-Tauber Theorem for Laplace Transforms Journal of the London Mathematical Society, Volume s2-13, Issue 3, 1 August 1976, Pages 537

[3] H.S. Carslaw, J.C. Jaeger. Operation method in applied Mathematics,1947, Pages 282

doi:10.6342/NTU201901693