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有關歐氏空間中凸體的 Brunn-Minkowski 不等式

The generalized Brunn-Minkowski inequality

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有關歐式空間中凸體的 Brunn-Minkowski 不等式
The generalized Brunn-Minkowski inequality

本論文係羅楷綸君（R07221012）在國立臺灣大學數學系完成之碩士學位論文，於民國 109 年 6 月 1 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

凸幾何是一個研究凸函數和凸集的領域。”一集合是個凸集”是個很好的特性，足夠強大使我們可以推導出豐富的結果，又不會太難達成，所以其結果也可以適用在廣闊的情形。在凸幾何中，有一個不等式叫 Brunn-Minkowski 不等式，它給出兩個凸體的體積和它們的 Minkowski 和的體積的關係。

原始的 Brunn-Minkowski 不等式對所有的凸體都適用，而 Böröczky，Lutwak，Yang 和 Zhang 給出了一個針對原點對稱凸體的更強猜想，並且證明了在二維歐式空間的情況。

在這篇論文中我們首先介紹凸幾何中一些基本的概念，然後是 Böröczky，Lutwak，Yang 和 Zhang 的論文中的工作，其中他們證明了 log-Brunn-Minkowski 不等式。

關鍵字: 凸體, Minkowski 和, 混合體積, Brunn-Minkowski 不等式, 椎體積測度.

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Abstract

Convex geometry is a branch of geometry that studies convex functions and convex sets. Because of the strong property, convexity, this research area has many successful theories and applications. In convex geometry, there is an inequality concerning the relationship between the volumes of two convex bodies, and the volume of their Minkowski sum. This important inequality is called Brunn-Minkowski inequality.

The classical Brunn-Minkowski inequality is valid for every two convex bodies. For origin-symmetric convex bodies, stronger inequalities are studied and conjectured by Böröczky, Lutwak, Yang and Zhang. In their work, these inequalities were proved for origin-symmetric convex bodies in two dimensional Euclidean space.

In this thesis we will introduce some basic notions in convex geometry and then the work of Böröczky, Lutwak, Yang and Zhang, in which they proved the log-Brunn-Minkowski inequality.

key words: convex bodies, Minkowski sum, mixed volume, Brunn-Minkowski inequality, cone-volume measure.

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Chapter 1

Introduction

Convex geometry is a branch of geometry that studies convex functions and convex sets. Because of the strong property, convexity, this research area has many successful theories and applications.

Among those important properties, in particular the supporting functions of convex bodies play an important role since one of them uniquely determines the other. In this dissertation, we will mainly study several important properties about the volumes of convex bodies.

First, the Minkowski's theorem on mixed volume says that the volume of a linear combination of convex bodies is a polynomial in the coefficients of the linear combination and the coefficients of the polynomial are the mixed volumes.

Moreover Blaschke's selection theorem also says that any bounded sequence of nonempty compact convex sets in $\mathbb{R}^d$ contains a convergent subsequence with respect to some particular metric. It can be used to show the existence of the solution of some problems.

In convex geometry, there is an extremely important inequality concerning the relationship between the volumes of two convex bodies, and the volume of their Minkowski sum. This important inequality is called the Brunn-Minkowski inequality.

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There are many applications of the Brunn-Minkowski inequality, for example the isoperimetric inequality and the isodiametric inequality which will be introduced in the end of chapter 2.

The classical Brunn-Minkowski inequality is valid for every two convex bodies. For origin-symmetric convex bodies, stronger inequalities are studied and conjectured in [3]. In their work, these inequalities were proved for origin symmetric convex bodies in two dimensional Euclidean space.

In chapter 3, we introduce the methods in [3], which proved the log-Minkowski inequality and hence the log-Brunn-Minkowski inequality. Although we have attempted to prove some similar results in three dimensional Euclidean spaces but failed, we hope to continue to pursue them in the near future.

Below is the organization of this dissertation. In the first section, some definitions and some lemmas are prepared, in particular the definition of the cone-volume measure is very important.

The main inequalities such as log-Brunn-Minkowski inequality were listed in the second section. Then two strong-weak relationships of these inequalities are proved.

Next, the log/L_p-Brunn-Minkowski inequalities and the log/L_p-Minkowski inequalities are showed to be equivalent in the sense that one can easily imply the other.

The Blaschke's extension of the Bonnesen inequality says that $V(C) - 2tV(C, D) + t^2V(D) \leq 0$ for $r(C, D) \leq t \leq R(C, D)$. For the definitions, see section 3.1.

Uniqueness of planar cone-volume measure, which tells us what could happen if two convex bodies have the same cone-volume measure, is very important in the proof of the log-Minkowski inequality.

The finial tool, Lemma 3.6.2, guarantees the existence of the minimum of $\int_{S^1} \log h_Q dV_C$ with some constraints. With these tools, the log-Minkowski inequality can finally be proved.

NOTE: This dissertation is based on Gruber's book [1] and a paper of Böröczky,

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Lutwak, Yang and Zhang [3]. The basic notions in chapter 2 come from [1] and chapter 3 is mainly from [3].

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Chapter 2

Preliminaries

2.1 Convex Bodies and Support Functions

Let $B^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$ be the unit closed ball with center o in $\mathbb{R}^d$. Let int$C$ stands for the interior of C. For $x, y \in \mathbb{R}^d$, let $[x, y] = \{(1-\lambda)x + \lambda y : 0 \leq \lambda \leq 1\}$ be the line segment with endpoints x, y. A set C in $\mathbb{R}^d$ is convex if $[x, y] \subset C \forall x, y \in C$. Let $f : C \to \mathbb{R}$ be a real function on C. The function f is convex if C is convex and

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \text{ for } x, y \in C, 0 \leq \lambda \leq 1.$$

f is strictly convex if C is convex and

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y) \text{ for } x, y \in C, x \neq y, 0 < \lambda < 1.$$

f is called concave, respectively, strictly concave if $-f$ is convex, respectively, strictly convex. $V(C)$ denotes the volume of C. That is, the Lebesgue measure of C in $\mathbb{R}^d$. A convex body is a compact convex set with nonempty interior. Let $\mathcal{C} = \mathcal{C}(\mathbb{R}^d)$ be the space of convex bodies in $\mathbb{R}^d$.

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Let $h_C : \mathbb{R}^d \to \mathbb{R}$ be the support function of C, defined by

$$h_C(u) = \max\{u \cdot x : x \in C\} \text{ for } u \in \mathbb{R}^d.$$

For $u \in S^{d-1}$, let $H_C(u) = \{x : u \cdot x = h_C(u)\}$ be the support hyperplane of C with exterior normal vector u, and $H_C^-(u) = \{x : u \cdot x \leq h_C(u)\}$ be the support halfspace of C with exterior normal vector u.

For a convex body C, define a mapping $p_C : \mathbb{R}^d \to C$ such that

$$\|x - p_C(x)\| = \min_{y \in C} \|x - y\|.$$

That is, for each $x \in \mathbb{R}^d$, the point $p_C(x) \in C$ is the one closest to x. Since C is closed, it is clear that $p_C(x)$ exists for each $x \in \mathbb{R}^d$. To see that $p_C(x)$ is unique for each $x \in \mathbb{R}^d$, assume that there are points $y, z \in C, y \neq z$, both having minimum distance from x. Then $\|y - x\| = \|z - x\|$ and therefore by the parallelogram law,

$$\|(y+z) - 2x\|^2 + \|y - z\|^2 = 2\|y - x\|^2 + 2\|z - x\|^2 = 4\|y - x\|^2.$$

Since $y \neq z$,

$$\|\frac{1}{2}(y+z) - x\|^2 = \frac{1}{4}\|(y+z) - 2x\|^2 < \|y - x\|^2.$$

That is,

$$\|\frac{1}{2}(y+z) - x\| < \|y - x\| = \|z - x\|.$$

Since $\frac{1}{2}(y+z) \in [y, z] \subset C$, there is a point in C which is closer to x, this contradicts our choice of y, z. The uniqueness follows, and so p_C is well-defined.

Theorem 2.1.1. *Let $C \in \mathcal{C}(\mathbb{R}^d)$. Then*

$$C = \cap_{u \in S^{d-1}} H_C^-(u).$$

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Proof. Clearly, $C \subset H_C^-(u)\ \forall u \in S^{d-1}$. So,

$$C \subset \cap_{u \in S^{d-1}} H_C^-(u).$$

Now let $z \in \mathbb{R}^d \setminus C$, then $z \neq p_C(z)$ since $p_C(z) \in C$. Let

$$u = \frac{z - p_C(z)}{\|z - p_C(z)\|} \in S^{d-1}.$$

Suppose $h_C(u) \neq u \cdot p_C(z)$, then by definition of the support functions, $h_C(u) > u \cdot p_C(z)$. So there exists $y \in C$ such that $u \cdot y > u \cdot p_C(z)$. Therefore we can see that $u \cdot (y - p_C(z)) > 0$. Let $D = \|z - p_C(z)\|, v = y - p_C(z)$, and $x = p_C(z) + \lambda v \in [p_C(z), y] \subset C$, where $\lambda \in [0, 1]$. Then

$$\|z - x\|^2 = \|Du - \lambda v\|^2 = D^2 \|u\|^2 - 2D\lambda(u \cdot v) + \lambda^2 \|v\|^2$$

$$= D^2 - \lambda[2D(u \cdot v) - \|v\|^2 \lambda] < D^2 = \|z - p_C(z)\|^2 \text{ as } 0 < \lambda < \frac{2D(u \cdot v)}{\|v\|^2}.$$

Let $\lambda = \min\{1, \frac{D(u \cdot v)}{\|v\|^2}\}$, we conclude that

$$\|x - z\| < \|p_C(z) - z\|$$

This contradicts the definition of $p_C(z)$. So,

$$h_C(u) = u \cdot p_C(z).$$

Observe that

$$u \cdot z = u \cdot (p_C(z) + \|z - p_C(u)\| u) > u \cdot p_C(z) = h_C(u).$$

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So, $z \notin H_C^-(u)$. Therefore,

$$z \notin \cap_{u \in S^{d-1}} H_C^-(u).$$

Hence,

$$C \supset \cap_{u \in S^{d-1}} H_C^-(u).$$

The result follows. □

Corollary 2.1.2. *Let $C, D \in \mathcal{C}$, then*

$$C \subset D \text{ if and only if } h_C \leq h_D.$$

Proof. Suppose that $C \subset D$, then $\forall u \in \mathbb{R}^n$,

$$h_C(u) = \max\{u \cdot x : x \in C\} \leq \max\{u \cdot x : x \in D\} = h_D(u).$$

Suppose that $h_C \leq h_D$, then $\forall u \in S^{d-1}$,

$$H_C^-(u) = \{x : u \cdot x \leq h_C(u)\} \subset \{x : u \cdot x \leq h_D(u)\} = h_D^-(u).$$

By Theorem 2.1.1,

$$C = \cap_{u \in S^{d-1}} H_C^-(u) \subset \cap_{u \in S^{d-1}} H_D^-(u) = D.$$

So, the result follows. □

This corollary also tell us that a convex body is uniquely determined by its support function.

Now, let's define the Minkowski sum and the scalar multiplication in $\mathcal{C}$:

$C + D = \{x + y : x \in C, y \in D\}$ for $C, D \in \mathcal{C}$ is the Minkowski sum of C and D.

$\lambda C = \{\lambda x : x \in C\}$ for $C \in \mathcal{C}, \lambda \in \mathbb{R}$ is the multiplication of C with λ.

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Proposition 2.1.3. *Let $C, D \in \mathcal{C}$ and $\lambda > 0$. Then $C + D, \lambda C \in \mathcal{C}$.*

Proof. Let $u + x, v + y \in C + D$ where $u, v \in C, x, y \in D$, and let $0 \leq \lambda \leq 1$. Then

$$(1 - \lambda)(u + x) + \lambda(v + y) = ((1 - \lambda)u + \lambda v) + ((1 - \lambda)x + \lambda y) \in C + D.$$

So $C + D$ is convex. To show that $C + D$ is compact, observe that $C \times D$ is compact in $\mathbb{R}^{2d}$. $C + D$ is the image of $C \times D$ under the continuous mapping $(x, y) \to x + y$. So $C + D$ is compact. Finally, choose $x \in \text{int } C, y \in \text{int } D$. Then clearly $x + y \in \text{int } (C + D)$. So $C + D$ has nonempty interior and hence is a convex body. The reason that λC is a convex body is similar and simpler, so we omit its proof. □

Proposition 2.1.4. *Let $C, D \in \mathcal{C}$ and $\lambda \geq 0$. Then*

$$h_{C+D} = h_C + h_D, h_{\lambda C} = \lambda h_C.$$

Proof.

$$h_{C+D}(u) = \max\{u \cdot (x + y) : x \in C, y \in D\}$$

$$= \max\{u \cdot x : x \in C\} + \max\{u \cdot y : y \in D\}$$

$$= h_C(u) + h_D(u) \text{ for } u \in \mathbb{R}^d.$$

And

$$h_{\lambda C}(u) = \max\{u \cdot (\lambda x) : x \in C\}$$

$$= \lambda \max\{u \cdot x : x \in C\} = \lambda h_C(u) \text{ for } u \in \mathbb{R}^d$$

□

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2.2 Minkowski's Theorem on Mixed Volumes

Theorem 2.2.1 (Minkowski's theorem on mixed volumes)**.** *Let* $C_1, \ldots, C_m \in \mathcal{C}$*. Then there are coefficients* $V(C_{i_1}, \ldots, C_{i_d}), 1 \le i_1, \ldots, i_d \le m$*, called mixed volumes, which are symmetric in the indices and such that*

$$V(\lambda_1 C_1 + \cdots + \lambda_m C_m) = \sum_{i_1,\ldots,i_d=1}^{m} V(C_{i_1}, \ldots, C_{i_d})\lambda_{i_1} \cdots \lambda_{i_d} for \lambda_1, \ldots, \lambda_m \ge 0. \quad (2.1)$$

For the proof, see e.g. p.89 in [1].

Proposition 2.2.2. *Let* $C, D, D_2, \ldots, D_d \in \mathcal{C}$*. Then*

$$V(\lambda C + \mu D, D_2, \ldots, D_d) = \lambda V(C, D_2, \ldots, D_d) + \mu V(D, D_2, \ldots, D_d)$$

for $\lambda, \mu \ge 0$.

Proof. Let $\lambda, \mu \ge 0$. The quantities

$$V(\lambda_1(\lambda C + \mu D) + \lambda_2 D_2 + \cdots + \lambda_d D_d),$$

$$V((\lambda_1\lambda)C + (\lambda_1\mu)D + \lambda_2 D_2 + \cdots + \lambda_d D_d)$$

have identical polynomial representations in $\lambda_1, \ldots, \lambda_d$. The coefficient of $\lambda_1 \cdots \lambda_d$ in the first polynomial is

$$d!V(\lambda C + \mu D, D_2, \ldots, D_d).$$

The coefficient of $\lambda_1 \cdots \lambda_d$ in the second polynomial can be obtained by representing the second quantity as a polynomial in $\lambda_1\lambda, \lambda_1\mu, \lambda_2, \ldots, \lambda_d$ and then collecting $\lambda_1 \cdots \lambda_d$. Thus it is

$$d!\lambda V(C, D_2, \ldots, D_d) + d!\mu V(D, D_2, \ldots, D_d)$$

Since the coefficients coincide, the proof is complete. □

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Proposition 2.2.3. *Let $C \in \mathcal{C}$. Then*

$$V(C, \dots, C) = V(C)$$

Proof. It is trivial if we put only one convex body C in (2.1) and let $\lambda_1 = 1$. That is,

$$V(C) = V(C, \dots, C).$$

□

2.3 The Blaschke's Selection Theorem

The Hausdorff metric δ^H on the space of nonempty compact convex sets is defined as follows:

$$\delta^H(C, D) = \max\{\max_{x \in C} \min_{y \in D} \|x - y\|, \max_{y \in D} \min_{x \in C} \|x - y\|\} \text{ for } C, D \in \mathcal{C}.$$

If we consider a topology on the space of nonempty compact convex sets or on $\mathcal{C}$, it is always assumed that it is the topology induced by δ^H.

The following is the Blaschke's selection theorem:

Theorem 2.3.1 (Blaschke's selection theorem)**.** *Any bounded sequence of nonempty compact convex sets in $\mathbb{R}^d$ contains a convergent subsequence.*

Proof. Let $C_1, C_2, \cdots \in \mathcal{C}$ be contained in a closed ball B. For the proof that the sequence $C_1, C_2, \cdots$, contains a convergent subsequence, the following will be shown first:

$$\begin{aligned} &\text{The sequence } C_1, C_2, \dots, \text{ contains a subsequence } D_1, D_2, \dots, \text{ such that} \\ &\qquad \delta^H(D_m, D_n) \leq \frac{1}{2^{\min\{m,n\}}} \text{ for } m, n = 1, 2, \dots \end{aligned} \tag{2.2}$$

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For the proof of (2.2) the main step is to prove that

There are sequences $C_{11}, C_{12}, \ldots; C_{21}, C_{22}, \ldots; \ldots,$
where $C_{11}, C_{12}, \ldots$ is a subsequence of $C_1, C_2, \ldots,$ and each subsequent
sequence is a subsequence of the sequence preceding it, such that
$\delta^H(D_{mi}, D_{mj}) \leq \dfrac{1}{2^m}$ for $m = 1, 2, \ldots,$ and $i, j = 1, 2, \ldots$ (2.3)

The first step of the induction is similar to the step from m to $m+1$, thus only the latter will be given. Let $m \geq 1$ and assume that the first m sequences have been constructed already and satisfy the inequality for $1, \ldots, m$. Since the ball B is compact, it can be covered by a finite family of closed balls, each of radius $1/2^{m+2}$ with centre in B. To each convex body in B we associate all ball of this family which intersect it. Clear, these balls cover the convex body. Since there are only finitely many subfamilies of this family of balls, there must be one which corresponds to each convex body from an infinite subsequence of $C_{m1}, C_{m2}, \ldots$, say $C_{m+1\ 1}, C_{m+1\ 2}, \ldots$ Now, given $i, j = 1, 2, \ldots$, for any $x \in C_{m+1\ i}$ there is a ball in our subfamily which contains x. Hence $\|x - c\| \leq 1/2^{m+2}$, where c is the centre of this ball. This ball also intersects $C_{m+1\ j}$. Thus we may choose $y \in C_{m+1\ j}$ with $\|y - c\| \leq 1/2^{m+2}$. This shows that, for each $x \in C_{m+1\ i}$, there is $y \in C_{m+1\ j}$ with $\|x - y\| \leq 1/2^{m+1}$. Similarly, for each $y \in C_{m+1\ j}$ there is $x \in C_{m+1\ i}$ with $\|x - y\| \leq 1/2^{m+1}$. Thus

$$\delta^H(C_{m+1\ i}, C_{m+1\ j}) \leq \frac{1}{2^{m+1}} \text{ for } i, j = 1, 2, \ldots$$

The induction is thus complete, concluding the proof of (2.3). By considering the diagonal sequence $D_1 = C_{11}, D_2 = C_{22}, \ldots$, we see that (2.2) is an immediate consequence of (2.3).

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For the proof of the theorem it is sufficient to show that

$$D_1, D_2, \cdots, \to D, \text{ where } D = \cap_{n=1}^{\infty}(D_n + \frac{1}{2^{n-1}}B^d) \in \mathcal{C}. \tag{2.4}$$

(2.2) implies that

$$D_1 + \frac{1}{2}B^d \supset D_2, D_2 + \frac{1}{2^2} \supset D_3, \dots$$

and thus,

$$D_1 + B^d \supset D_2 + \frac{1}{2}B^d \supset \dots \tag{2.5}$$

Being the intersection of a decreasing sequence of non-empty compact convex sets (see (2.4) and (2.5)), the set D is also non-empty, compact and convex. In order to prove that $D_1, D_2, \dots \to D$, let $\epsilon > 0$. Then

$$D \subset D_n + \frac{1}{2^{n-1}}B^d \subset D_n + \epsilon B^d \text{ for } n \geq 1 + \log_2 \frac{1}{\epsilon}. \tag{2.6}$$

Let $G = \text{int}(D + \epsilon B^d)$. The intersection of the following decreasing sequence of compact sets

$$(D_1 + B^d) \setminus G \supset (D_2 + \frac{1}{2}B^d) \setminus G \supset \dots$$

is contained both in D (see (2.4)) and in $\mathbb{R}^d \setminus G$ and thus is empty. This implies that, from a certain index on, the sets in this sequence are empty. That is,

$$D_n \subset D_n + \frac{1}{2^{n-1}}B^d \subset G \subset D + \epsilon B^d \text{ for all sufficiently large } n. \tag{2.7}$$

(2.6) and (2.7) show that $\delta^H(D_n, D) \leq \epsilon$ for all sufficiently large n. Since $\epsilon > 0$ was arbitrary, $D_1, D_2, \dots \to D$, concluding the proof of (2.4) and thus of the theorem. □

Theorem 2.3.2. *Let $C_1, C_2, \dots \in \mathcal{C}$ and C be a nonempty compact convex set such that $C_1, C_2, \dots \to C$. Then $V(C_1), V(C_2), \dots \to V(C)$.*

Proof. Since volume and Hausdorff metric are translation invariant, we may assume

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that $o \in \text{int}C$. Choose $\rho > 0$ such that $\rho B^d \subset C$. Since $C_1, C_2, \cdots \to C$, we have the following: Let $\epsilon \in (0, \rho)$. Then the inclusions

$$C_n \subset C + \epsilon B^d, C \subset C_n + \epsilon B^d$$

hold for all sufficiently large n. Hence

$$C_n \subset C + \frac{\epsilon}{\rho}C = (1 + \frac{\epsilon}{\rho})C,$$

$$(1 - \frac{\epsilon}{\rho})C + \frac{\epsilon}{\rho}C = C \subset C_n + \frac{\epsilon}{\rho}C.$$

By Proposition 2.1.4,

$$h_{(1-\frac{\epsilon}{\rho})C} + h_{\frac{\epsilon}{\rho}C} = h_{(1-\frac{\epsilon}{\rho})C+\frac{\epsilon}{\rho}C} \leq h_{C_n+\frac{\epsilon}{\rho}C} = h_{C_n} + h_{\frac{\epsilon}{\rho}C}$$

Hence,

$$h_{(1-\frac{\epsilon}{\rho})C} \leq h_{C_n}$$

And we obtain the inclusions

$$(1 - \frac{\epsilon}{\rho})C \subset C_n \subset (1 + \frac{\epsilon}{\rho})C$$

for all sufficiently large n. Therefore,

$$(1 - \frac{\epsilon}{\rho})^d V(C) \leq V(C_n) \leq (1 + \frac{\epsilon}{\rho})^d V(C).$$

for all sufficiently large n. Since $\epsilon \in (0, \rho)$ is arbitrary. So,

$$V(C_n) \to V(C).$$

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$\square$

2.4 The Brunn-Minkowski Inequality

Lemma 2.4.1. *Let $d \geq 2$. Then*

$$(v^{\frac{1}{d-1}} + w^{\frac{1}{d-1}})^{d-1}(\frac{V}{v} + \frac{W}{w}) \geq (V^{\frac{1}{d}} + W^{\frac{1}{d}})^d$$

for $v, w, V, W > 0$, where equality holds if and only if

$$\frac{v}{V^{\frac{d-1}{d}}} = \frac{w}{W^{\frac{d-1}{d}}}.$$

Proof. By Holder's inequality,

$$(v^{\frac{1}{d-1}} + w^{\frac{1}{d-1}})^{\frac{d-1}{d}}(\frac{V}{v} + \frac{W}{w})^{\frac{1}{d}}$$

$$= ((v^{\frac{1}{d}})^{\frac{d}{d-1}} + (w^{\frac{1}{d}})^{\frac{d}{d-1}})^{\frac{d-1}{d}}(((\frac{V}{v})^{\frac{1}{d}})^d + ((\frac{W}{w})^{\frac{1}{d}})^d)^{\frac{1}{d}}$$

$$\geq V^{\frac{1}{d}} + W^{\frac{1}{d}},$$

where equality holds if and only if

$$\frac{v^{\frac{1}{d-1}}}{\frac{V}{v}} = \frac{w^{\frac{1}{d-1}}}{\frac{W}{w}} \Leftrightarrow \frac{v}{V^{\frac{d-1}{d}}} = \frac{w}{W^{\frac{d-1}{d}}}.$$

$\square$

Theorem 2.4.2 (classical Brunn-Minkowski inequality)**.** *Let $C, D \in \mathcal{C}$. Then:*

$$V(C+D)^{\frac{1}{d}} \geq V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}}, \tag{2.8}$$

where equality holds if and only if C and D are homothetic. (A homothety is a compo-

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sition of a dilation and translation.)

Proof. For $d = 1$, C and D are intervals. Thus, $V(C+D) = V(C) + V(D)$ always holds. And C and D are homothetic.

Let $d > 1$ and assume that it holds for $d-1$. If we prove it for d, then by induction we complete the proof.

Let $u \in S^{d-1}$ and, for real t, let $H(t) = \{x : u \cdot x = t\}$ and $H^-(t) = \{x : u \cdot x \leq t\}$. Choose $\alpha_C < \beta_C$ such that $H(\alpha_C)$ and $H(\beta_C)$ are support hyperplanes of C and similarly for D. Then:

$$H(\alpha_C + \alpha_D) \text{ and } H(\beta_C + \beta_D) \text{ are support hyperplanes of } C + D. \tag{2.9}$$

Let $v(\cdot)$ denote $(d-1)$-dimensional volume and put:

$$v_C(t) = v(C \cap H(t)), V_C(t) = V(C \cap H^-(t)) \text{ for } \alpha_C \leq t \leq \beta_C, \text{ and similarly for } D. \tag{2.10}$$

The function $t \to V_C(t)/V(C)$, for $\alpha_C \leq t \leq \beta_C$, assumes the values $0, 1$ for $t = \alpha_C, \beta_C$, is continuous and strictly increasing for $\alpha_C \leq t \leq \beta_C$ and continuously differentiable with derivative $V_C'(t)/V(C) = v_C(t)/V(C) > 0$ for $\alpha_C < t < \beta_C$.

Consider its inverse function $s \to t_C(s)$. Then:

$$\begin{aligned} & t_C(\cdot) \text{ is defined for } 0 \leq s \leq 1, \\ & t_C(0) = \alpha_C, t_C(1) = \beta_C, \\ & t_C(\cdot) \text{ is continuous for } 0 \leq s \leq 1, \\ & t_C(\cdot) \text{ is continuously differentiable with} \\ & t_C'(s) = \frac{V(C)}{v_C(t_C(s))} > 0 \text{ for } 0 < s < 1. \\ & \text{Analogous statements hold for } D. \end{aligned} \tag{2.11}$$

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Thus the function:

$$\begin{aligned}
&t_{C+D}(\cdot) = t_C(\cdot) + t_D(\cdot) \text{ is defined for } 0 \leq s \leq 1,\\
&t_{C+D}(0) = \alpha_C + \alpha_D, t_{C+D}(1) = \beta_C + \beta_D\\
&t_{C+D}(\cdot) \text{ is continuously differentiable with}\\
&t'_{C+D}(s) = \frac{V(C)}{v_C(t_C(s))} + \frac{V(D)}{v_D(t_D(s))} > 0 \text{ for } 0 < s < 1.
\end{aligned} \tag{2.12}$$

Since $H(t_{C+D}(s)) = H(t_C(s)) + H(t_D(s))$ for $0 \leq s \leq 1$, we have:

$$(C + D) \cap H(t_{C+D}(s)) \supset C \cap H(t_C(s)) + D \cap H(t_D(s)) \text{ for } 0 \leq s \leq 1. \tag{2.13}$$

Now (2.9), Fubini's theorem, (2.12) and integration by substitution, (2.13), the induction hypothesis, (2.10), (2.12) and Lemma 2.4.1 together yield (2.8) as follows:

$$\begin{aligned}
&V(C + D) = \int_{\alpha_C+\alpha_D}^{\beta_C+\beta_D} v((C + D) \cap H(t))dt\\
&= \int_0^1 v((C + D) \cap H(t_{C+D}(s)))t'_{C+D}(s)ds\\
&\geq \int_0^1 v(C \cap H(t_C(s)) + D \cap H(t_D(s)))t'_{C+D}(s)ds\\
&\geq \int_0^1 (v_C(t_C(s))^{\frac{1}{d-1}} + v_D(t_D(s))^{\frac{1}{d-1}})^{d-1}(\frac{V(C)}{v_C(t_C(s))} + \frac{V(D)}{v_D(t_D(s))})ds\\
&\geq \int_0^1 (V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}})^d ds = (V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}})^d.
\end{aligned} \tag{2.14}$$

So we have proved the inequality (2.8). Then, assume that equality holds in (2.8). By translating C and D, we may suppose that o is the centroid of both C and D. Let $u \in S^{d-1}$. Since, by assumption, there is equality in (2.8), we have equality throughout (2.14). Thus, in particular,

$$\frac{v_C(t_C(s))}{V(C)^{\frac{d-1}{d}}} = \frac{v_D(t_D(s))}{V(D)^{\frac{d-1}{d}}} \text{ for } 0 < s < 1,$$

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by Lemma 2.4.1. An application of (2.11) then shows that

$$\frac{t'_C(s)}{V(C)^{\frac{1}{d}}} = \frac{t'_D(s))}{V(D)^{\frac{1}{d}}} \text{ for } 0 < s < 1.$$

By fundamental theorem of calculus,

$$\frac{t_C(s)}{V(C)^{\frac{1}{d}}} = \frac{t_D(s))}{V(D)^{\frac{1}{d}}} + \text{const for } 0 < s < 1. \tag{2.15}$$

Since o is the centroid of C, Fubini's theorem, (2.10) and (2.11) show that

$$0 = \int_C u \cdot x dx = \int_{\alpha_C}^{\beta_C} tv(C \cap H(t))dt = \int_{\alpha_C}^{\beta_C} tv_C(t)dt$$

$$= \int_0^1 t_C(s)v_C(t_C(s))t'_C(s)ds = V(C)\int_0^1 t_C(s)ds,$$

and similarly for D.

So, the constant in (2.15) is thus 0. And so

$$h_D(u) = \beta_D = t_D(1) = (\frac{V(D)}{V(C)})^{\frac{1}{d}} t_C(1) = (\frac{V(D)}{V(C)})^{\frac{1}{d}} \beta_C = (\frac{V(D)}{V(C)})^{\frac{1}{d}} h_C(u).$$

Since $u \in S^{d-1}$ was arbitrary,

$$D = (\frac{V(D)}{V(C)})^{\frac{1}{d}} C.$$

That is, C and D are homothetic.

Assume that C and D are homothetic, then the equality obviously holds in (2.8).

Hence we have proved the equality condition. □

Corollary 2.4.3. *Let $C, D \in \mathcal{C}$. Then*

$$V((1-\lambda)C + \lambda D)^{\frac{1}{d}} \geq (1-\lambda)V(C)^{\frac{1}{d}} + \lambda V(D)^{\frac{1}{d}} \textit{ for } 0 \leq \lambda \leq 1,$$

where equality holds for $0 < \lambda < 1$ if and only if C and D are homothetic.

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Theorem 2.4.4. *Let $C, D \in \mathcal{C}$. Then the function*

$$f(\lambda) = V((1-\lambda)C + \lambda D)^{\frac{1}{d}} \text{ is strictly concave for } 0 \leq \lambda \leq 1$$

if C and D are not homothetic. And f is linear if C and D are homothetic.

Proof. Suppose that C and D are not homothetic. Let $0 \leq \lambda_1 < \lambda_2 \leq 1, 0 < \lambda < 1$. Then $(1-\lambda_1)C + \lambda_1 D$ and $(1-\lambda_2)C + \lambda_2 D$ are not homothetic (by Proposition 2.1.4). Therefore, by Corollary 2.4.3,

$$f((1-\lambda)\lambda_1 + \lambda\lambda_2) = V(((1-\lambda)(1-\lambda_1) + \lambda(1-\lambda_2))C + ((1-\lambda)\lambda_1 + \lambda\lambda_2)D)^{\frac{1}{d}}$$

$$= V((1-\lambda)((1-\lambda_1)C + \lambda_1 D) + \lambda((1-\lambda_2)C + \lambda_2 D))^{\frac{1}{d}}$$

$$> (1-\lambda)V((1-\lambda_1)C + \lambda_1 D)^{\frac{1}{d}} + \lambda V((1-\lambda_2)C + \lambda_2 D)^{\frac{1}{d}} = (1-\lambda)f(\lambda_1) + \lambda f(\lambda_2).$$

Suppose that C and D are homothetic. Then by Corollary 2.4.3, $f(\lambda) = (1-\lambda)f(0) + \lambda f(1)$. That is, f is linear. □

The following is a multiplicative version of the Brunn-Mindowski inequality.

Theorem 2.4.5. *Let $C, D \in \mathcal{C}$. Then:*

$$V((1-\lambda)C + \lambda D) \geq V(C)^{1-\lambda}V(D)^{\lambda} \text{ for } 0 \leq \lambda \leq 1,$$

where equality holds for $0 < \lambda < 1$ if and only if C is a translation of D.

Proposition 2.4.6. *The ordinary Brunn-Minkowski inequality and its multiplicative version are equivalent in the sense that each easily implies the other.*

Proof. We first assume that the ordinary Brunn-Minkowski inequality holds. Then

$$V((1-\lambda)C + \lambda D)^{\frac{1}{d}} \geq V((1-\lambda)C)^{\frac{1}{d}} + V(\lambda D)^{\frac{1}{d}}$$

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$$= (1-\lambda)V(C)^{\frac{1}{d}} + \lambda V(D)^{\frac{1}{d}} \geq V(C)^{\frac{1-\lambda}{d}} V(D)^{\frac{\lambda}{d}}.$$

So,

$$V((1-\lambda)C + \lambda D) \geq V(C)^{1-\lambda} V(D)^{\lambda}.$$

Now assume that its multiplicative version holds. Define:

$$\bar{C} = \frac{C}{V(C)^{\frac{1}{d}}}, \bar{D} = \frac{D}{V(D)^{\frac{1}{d}}}, \lambda = \frac{V(D)^{\frac{1}{d}}}{V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}}}.$$

Then

$$(1-\lambda)\bar{C} + \lambda\bar{D} = \frac{C+D}{V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}}}.$$

By multiplicative version of Brunn-Minkowski inequality,

$$V((1-\lambda)\bar{C} + \lambda\bar{D}) \geq V(\bar{C})^{1-\lambda} V(\bar{D})^{\lambda} = 1.$$

So,

$$V(C+D) \geq (V(C)^{\frac{1}{d}} + V(D)^{\frac{1}{d}})^{d}.$$

That is, the ordinary Brunn-Minkowski inequality holds. □

Proof. of Theorem 2.4.5. By Proposition 2.4.6, the inequality holds.

To obtain the equality condition, observe the proof of Proposition 2.4.6. We know that the ordinary Brunn-Minkowski inequality must hold for $(1-\lambda)C$ and λD.

So they are homothetic. That implies C and D are homothetic.

And from the next inequality (the inequality of arithmetic and geometric means), we know $V(C) = V(D)$. So C is a translation of D.

Conversely, if C is a translation of D, then the equality trivially holds. □

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2.5 Minkowski's First and Second Inequalities

Theorem 2.5.1 (Minkowski's first inequality). *Let $C, D \in \mathcal{C}$. Then*

$$V(C, D, \dots, D)^d \geq V(C)V(D)^{d-1},$$

where equality holds if and only if C and D are homothetic.

Proof. By Theorem 2.4.4, the function $f(\lambda) = V((1-\lambda)C + \lambda D)^{\frac{1}{d}}, 0 \leq \lambda \leq 1$, is concave. And f is linear if and only if C and D are homothetic. By Theorem 2.2.1,

$$V((1-\lambda)C + \lambda D) = \sum_{i=0}^{d} \binom{d}{i} (1-\lambda)^i \lambda^{d-i} V(C, \dots, C, D \dots, D), \tag{2.16}$$

with i C's and $(d-i)$ D's in $V(C, \dots, C, D \dots, D)$.

(2.16) shows that f is differentiable. Since f is concave, we have

$$V(D)^{\frac{1}{d}} - V(C)^{\frac{1}{d}} = f(1) - f(0) \geq f'(1) = \frac{1}{d} V(D)^{\frac{1}{d}-1}[dV(D) + d(-1)V(C, D, \dots, D)].$$

Therefore,

$$V(C, D, \dots, D) \geq V(C)^{\frac{1}{d}} V(D)^{1-\frac{1}{d}}.$$

Thus,

$$V(C, D, \dots, D)^d \geq V(C)V(D)^{d-1}.$$

The equality condition was obtained since $f(1) - f(0) = f'(1)$ if and only if f is linear if and only if C and D are homothetic. □

Theorem 2.5.2 (Minkowski's second inequality). *Let $C, D \in \mathcal{C}$. Then*

$$V(C, D, \dots, D)^2 \geq V(C, C, D, \dots, D)V(D).$$

Proof. From the proof in the last theorem, f is concave. And (2.16) shows that f is

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twice differentiable. Thus $f''(1) \leq 0$.

Now let's compute it.

$$[f(\lambda)^d]' = df(\lambda)^{d-1}f'(\lambda).$$

$$[f(\lambda)^d]'' = [df(\lambda)^{d-1}f'(\lambda)]' = d[(d-1)f(\lambda)^{d-2}(f'(\lambda))^2 + f(\lambda)^{d-1}f''(\lambda)].$$

So,

$$[f(\lambda)^d]''|_{\lambda=1} = d[(d-1)V(D)^{\frac{d-2}{d}}[V(D)^{\frac{1}{d}-1}[V(D) - V(C, D, \ldots, D)]]^2 + V(D)^{\frac{d-1}{d}}f''(1)].$$

By (2.16),

$$d(d-1)V(D)+2(-1)d(d-1)V(C, D, \ldots, D)+2\frac{d(d-1)}{2}V(C, C, D, \ldots, D) = [f(\lambda)^d]''|_{\lambda=1}$$

$$= d[(d-1)V(D)^{\frac{d-2}{d}}[V(D)^{\frac{1}{d}-1}[V(D) - V(C, D, \ldots, D)]]^2 + V(D)^{\frac{d-1}{d}}f''(1)]$$

$$\leq d(d-1)V(D)^{-1}[V(D) - V(C, D, \ldots, D)]^2$$

Hence,

$$V(D) - 2V(C, D, \ldots, D) + V(C, C, D, \ldots, D)$$

$$\leq V(D) - 2V(C, D, \ldots, D) + V(D)^{-1}V(C, D, \ldots, D)^2.$$

So,

$$V(C, D, \ldots, D)^2 \geq V(C, C, D, \ldots, D)V(D).$$

□

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2.6 The Isoperimetric and the Isodiametric Inequality

Theorem 2.6.1 (isoperimetric inequality)**.** *Let* $C \in \mathcal{C}(\mathbb{R}^d)$*. Then*

$$\frac{S(C)^d}{V(C)^{d-1}} \geq \frac{S(B^d)^d}{V(B^d)^{d-1}},$$

where equality holds if and only if C *is a solid Euclidean ball. (*$S(\cdot)$ *is the surface area.)*

Proof. First we note that

$$S(C) = \lim_{\lambda \to 0^+} \frac{V(C + \lambda B^d) - V(C)}{\lambda} = dV(B^d, C, \dots, C).$$

by Theorem 2.2.1.

By Minkowski's first inequality,

$$V(B^d, C, \dots, C)^d \geq V(B^d)V(C)^{d-1},$$

where equality holds if and only if C and B^d is homothetic (and so C is a solid Euclidean ball). Note that $S(B^d) = dV(B^d)$, we thus obtain that

$$\frac{S(C)^d}{V(C)^{d-1}} = \frac{d^d V(B^d, C, \dots, C)^d}{V(C)^{d-1}} \geq d^d V(B^d) = \frac{d^d V(B^d)^d}{V(B^d)^{d-1}} = \frac{S(B^d)^d}{V(B^d)^{d-1}}.$$

where equality holds if and only if C is a ball. □

Theorem 2.6.2 (isodiametric inequality)**.** *Let* $C \in \mathcal{C}$*. Then*

$$V(C) \leq (\frac{1}{2} diam C)^d V(B^d),$$

where equality holds if and only if C *is a solid Euclidean ball.*

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Proof. Suppose that C is centrally symmetric. Without loss of generality, we may assume that o is the centre of C. Then $C \subset (\frac{1}{2}\text{diam}C)B^d$ and thus

$$V(C) \leq (\frac{1}{2}\text{diam}C)^d V(B^d),$$

where equality holds if and only if $C = (\frac{1}{2}\text{diam}C)B^d$.

Now, suppose that C is not centrally symmetric. Then Corollary 2.4.3 shows that

$$V(C)^{\frac{1}{d}} = \frac{1}{2}V(C)^{\frac{1}{d}} + \frac{1}{2}V(-C)^{\frac{1}{d}} < V(\frac{1}{2}(C-C))^{\frac{1}{d}} = \frac{1}{2}V(C-C)^{\frac{1}{d}},$$

and thus:

$$V(C) < \frac{1}{2^d}V(C-C).$$

Since $C - C$ is origin-symmetric:

$$V(C-C) \leq (\frac{1}{2}\text{diam}(C-C))^d V(B^d)$$

by the first case. Next note that

$$\text{diam}(C-C) = \max\{\|(u-v)-(x-y)\| : u,v,x,y \in C\}$$

$$\leq \max\{\|u-v\| : u,v \in C\} + \max\{\|x-y\| : x,y \in C\} = 2\text{diam}C.$$

So,

$$V(C) < \frac{1}{2^d}(\text{diam}C)^d V(B^d) = (\frac{1}{2}\text{diam}C)^d V(B^d).$$

□

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Chapter 3

The work of Böröczky, Lutwak, Yang and Zhang

3.1 Some Preparations

A boundary point $x \in \partial C$ of the convex body C is said to have $u \in S^{d-1}$ as one of its outer unit normals provided $x \cdot u = h_C(u)$. A boundary point is said to be singular if it has more than one unit normal vector. It is well known (see, e.g., [2]) that the set of singular boundary points of a convex body has $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ equal to 0.

Let C be a convex body in $\mathbb{R}^d$ and $\nu_C : \partial C \to S^{d-1}$ the generalized Gauss map. For arbitrary convex bodies, the generalized Gauss map is properly defined as a map into subsets of S^{d-1}. However, $\mathcal{H}^{d-1}$-almost everywhere on ∂C it can be defined as a map into S^{d-1}. For each Borel set $\omega \subset S^{d-1}$, the inverse spherical image $\nu_C^{-1}(\omega)$ of ω is the set of all boundary points of C which have an outer unit normal belonging to the set ω. Associated with each convex body C in $\mathbb{R}^d$ is a Borel measure S_C on S^{d-1} called the Aleksandrov-Fenchel-Jessen surface area measure of C, defined by

$$S_C(\omega) = \mathcal{H}^{d-1}(\nu_C^{-1}(\omega))$$

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for each Borel set $\omega \subset S^{d-1}$; i.e., $S_C(\omega)$ is the $(d-1)$-dimensional Hausdorff measure of the set of all points on ∂C that have a unit normal that lies in ω. Let C be a convex body in $\mathbb{R}^d$ that contains the origin in its interior. The cone-volume measure V_C of C is a Borel measure on the unit sphere S^{d-1} defined for a Borel $\omega \subset S^{d-1}$ by

$$V_C(\omega) = \frac{1}{d}\int_{x\in\nu_C^{-1}(\omega)} x\cdot \nu_C(x) d\mathcal{H}^{d-1}(x),$$

and thus

$$dV_C = \frac{1}{d}h_C dS_C. \tag{3.1}$$

Since,

$$V(C) = \frac{1}{d}\int_{u\in S^{d-1}} h_C(u)dS_C(u), \tag{3.2}$$

we can turn the cone-volume measure into a probability measure on the unit sphere by normalizing it by the volume of the body. The cone-volume probability measure $\bar{V}_C$ of C is defined

$$\bar{V}_C = \frac{1}{V(C)}V_C.$$

Suppose C, D are convex bodies in $\mathbb{R}^d$ that contain the origin in their interiors. For $p > 0$, the L_p-mixed volume $V_p(C, D)$ can be defined as

$$V_p(C,D) = \int_{S^{d-1}} (\frac{h_D}{h_C})^p dV_C. \tag{3.3}$$

Suppose that the function $k_t(u) = k(t,u) : I \times S_{d-1} \to (0,\infty)$ is continuous, where $I \subset \mathbb{R}$ is an interval. For fixed $t \in I$, let

$$K_t = \cap_{u\in S^{d-1}}\{x \in \mathbb{R}^d : x\cdot u \le k(t,u)\}$$

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be the Wulff shape associated with the function k_t. We shall make use of the well-known fact that

$$h_{K_t} \leq k_t \text{ and } h_{K_t} = k_t \text{ a.e. w.r.t. } S_{K_t},$$

for each $t \in I$. If k_t happens to be the support function of a convex body then $h_{K_t} = k_t$, everywhere.

The following lemma (proved in e.g. [5]) will be needed.

Lemma 3.1.1. *Suppose $k(t,u) : I \times S^{d-1} \to (0,\infty)$ is continuous, where $I \subset \mathbb{R}$ is an open interval. Suppose also that the convergence in*

$$\frac{\partial k(t,u)}{\partial t} = \lim_{s \to 0} \frac{k(t+s,u) - k(t,u)}{s}$$

is uniform on S^{d-1}. If $\{K_t\}_{t \in I}$ is the family of Wulff shapes associated with k_t, then

$$\frac{dV(K_t)}{dt} = \int_{S^{d-1}} \frac{\partial k(t,u)}{\partial t} dS_{K_t}(u).$$

Suppose C, D are convex bodies in $\mathbb{R}^d$. The inradius $r(C,D)$ and outradius $R(C,D)$ of C with respect to D are defined by

$$r(C,D) = \sup\{t > 0 : x + tD \subset C \text{ and } x \in \mathbb{R}^d\},$$

$$R(C,D) = \inf\{t > 0 : x + tD \supset C \text{ and } x \in \mathbb{R}^d\}.$$

Obviously from the definition, it follows that

$$r(C,D) = 1/R(D,C).$$

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If C, D happen to be origin-symmetric convex bodies, then obviously

$$r(C,D) = \min_{u \in S^{d-1}} \frac{h_C(u)}{h_D(u)} \text{ and } R(C,D) = \max_{u \in S^{d-1}} \frac{h_C(u)}{h_D(u)}. \tag{3.4}$$

It will be convenient to always translate C and D so that for $0 \leq t < r = r(C,D)$, the function $k_t = h_C - th_D$ is strictly positive. Let C_t denote the Wulff shape associated with the function k_t; i.e., let C_t be the convex body given by

$$C_t = \{x \in \mathbb{R}^d : x \cdot u \leq h_C(u) - th_D(u) \text{ for all } u \in S^{d-1}\}. \tag{3.5}$$

Note that $C_0 = C$, and that obviously

$$\lim_{t \to 0} C_t = C_0 = C.$$

From definition (3.5) and Corollary 2.1.2 we immediately have

$$C_t = \{x \in \mathbb{R}^d : x + tD \subset C\}. \tag{3.6}$$

Using (3.6) we can extend the definition of C_t for the case where $t = r = r(C,D)$:

$$C_r = \{x \in \mathbb{R}^d : x + rD \subset C\}.$$

It is not hard to show (see e.g. [2]) that C_r is a degenerate convex set (i.e. has empty interior) and that

$$\lim_{t \to r} V(C_t) = V(C_r) = 0. \tag{3.7}$$

From Lemma 3.1.1 and (3.3), we obtain the well-known fact that for $0 < t < r =$

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$r(C, D)$,

$$\frac{d}{dt}V(C_t) = \int_{S^{d-1}} \frac{\partial k(t,u)}{\partial t} dS_{C_t}(u) = \int_{S^{d-1}} (-h_D) \frac{d}{h_{C_t}} dV_{C_t}(u) = -dV_1(C_t, D).$$

Integrating both sides, and using (3.7), gives

Lemma 3.1.2. *Suppose C and D are convex bodies, and for $0 \le t < r = r(C, D)$, the body C_t is the Wulff shape associated with the positive function $k_t = h_C - th_D$. Then, for $0 \le t \le r = r(C, D)$,*

$$V(C) - V(C_t) = d \int_0^t V_1(C_s, D) ds,$$

where $C_r = \{x \in \mathbb{R}^d : x + rD \subset C\}$.

3.2 The log-Brunn-Minkowski Inequality

Assume $C, D \in \mathcal{C}, o \in \text{int } C \cap \text{ int } D, \lambda \in [0, 1]$. Then the geometric Minkowski combination, $(1 - \lambda) \cdot C +_o \lambda \cdot D$, is defined by

$$(1 - \lambda) \cdot C +_o \lambda \cdot D = \cap_{u \in S^{d-1}} \{x \in \mathbb{R}^d : x \cdot u \le h_C(u)^{1-\lambda} h_D(u)^{\lambda}\}.$$

For $p > 0$, the Minkowski-Firey L_p-combination (or simply L_p-combination), $(1 - \lambda) \cdot C +_p \lambda \cdot D$, is defined by

$$(1 - \lambda) \cdot C +_p \lambda \cdot D = \cap_{u \in S^{d-1}} \{x \in \mathbb{R}^d : x \cdot u \le ((1 - \lambda) h_C(u)^p + \lambda h_D(u)^p)^{1/p}\}.$$

We list the main problems as follows.

The log-Brunn-Minkowski inequality:

Problem 3.2.1. *Show that if C and D are origin-symmetric convex bodies in $\mathbb{R}^d$, then*

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for all $\lambda \in [0,1]$,

$$V((1-\lambda)\cdot C +_o \lambda \cdot D) \geq V(C)^{1-\lambda} V(D)^{\lambda}. \tag{3.8}$$

The L_p-Brunn-Minkowski inequality:

Problem 3.2.2. *Suppose* $p > 0$. *Show that if* C *and* D *are origin-symmetric convex bodies in* $\mathbb{R}^d$, *then for all* $\lambda \in [0,1]$,

$$V((1-\lambda)\cdot C +_p \lambda \cdot D) \geq V(C)^{1-\lambda} V(D)^{\lambda}. \tag{3.9}$$

The log-Minkowski inequality:

Problem 3.2.3. *Show that if* C *and* D *are origin-symmetric convex bodies in* $\mathbb{R}^d$, *then*

$$\int_{S^{d-1}} \log \frac{h_D}{h_C} d\bar{V}_C \geq \frac{1}{d} \log \frac{V(D)}{V(C)}. \tag{3.10}$$

The L_p-Minkowski inequality:

Problem 3.2.4. *Suppose* $p > 0$. *Show that if* C *and* D *are origin-symmetric convex bodies in* $\mathbb{R}^d$, *then*

$$(\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C)^{\frac{1}{p}} \geq (\frac{V(D)}{V(C)})^{\frac{1}{d}}. \tag{3.11}$$

For the rest of this section, we suppose that each convex bodies contain the origin in its interior.

From the definition we can see that

$$(1-\lambda)\cdot C +_1 \lambda \cdot D = (1-\lambda)C + \lambda D.$$

Actually, for fixed C, D, λ, the L_p-combination $(1-\lambda)\cdot C +_p \lambda \cdot D$ is increasing with respect to set inclusion, as p increases.

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That is, if $0 < p < q$, then

$$(1-\lambda) \cdot C +_p \lambda \cdot D \subset (1-\lambda) \cdot C +_q \lambda \cdot D.$$

To see this, it suffices to show that

$$((1-\lambda)h_C(u)^p + \lambda h_D(u)^p)^{1/p} \leq ((1-\lambda)h_C(u)^q + \lambda h_D(u)^q)^{1/q} \ \forall u \in S^{d-1}.$$

Suppose $0 < p < q$. Since $f(x) = x^{q/p}$ is convex, by Jensen's inequality,

$$((1-\lambda)h_C^p + \lambda h_D^p)^{q/p} \leq (1-\lambda)h_C^{p(q/p)} + \lambda h_D^{p(q/p)} = (1-\lambda)h_C^q + \lambda h_D^q.$$

So we have proved it.

Next, we claim that

$$(1-\lambda) \cdot C +_o \lambda \cdot D = \lim_{p \to 0^+}((1-\lambda) \cdot C +_p \lambda \cdot D).$$

Since the monotonicity of $(1-\lambda) \cdot C +_p \lambda \cdot D$, it suffices to show that

$$h_C(u)^{1-\lambda} h_D(u)^{\lambda} = \lim_{p \to 0^+}((1-\lambda)h_C^p + \lambda h_D^p)^{1/p}.$$

To see this, we apply L'Hospital's rule as follows:

$$\lim_{p \to 0^+} \frac{1}{p} \log((1-\lambda)h_C^p + \lambda h_D^p) = \lim_{p \to 0^+} \frac{(1-\lambda)h_C^p \log h_C + \lambda h_D^p \log h_D}{(1-\lambda)h_C^p + \lambda h_D^p}$$

$$= (1-\lambda)\log h_C + \lambda \log h_D = \log h_C^{1-\lambda} h_D^{\lambda}.$$

So,

$$h_C^{1-\lambda} h_D^{\lambda} = \exp(\log(h_C^{1-\lambda} h_D^{\lambda})) = \lim_{p \to 0^+}((1-\lambda)h_C^p + \lambda h_D^p)^{1/p}.$$

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So far, we have proved that if $0 < p \le 1 \le q$,

$$(1-\lambda)\cdot C +_o \lambda \cdot D \subset (1-\lambda)\cdot C +_p \lambda \cdot D \subset (1-\lambda)C + \lambda D \subset (1-\lambda)\cdot C +_q \lambda \cdot D$$

This fact tell us that:

"The log-Brunn-Minkowski inequality"⇒"The L_p-Brunn-Minkowski inequality"⇒"The classical Brunn-Minkowski inequality"⇒"The L_q-Brunn-Minkowski inequality", where $0 < p \le 1 \le q$.

For the other inequalities, there is similar relationship as follows:

"The log-Minkowski inequality"⇒"The L_p-Minkowski inequality"⇒"The L_q-Minkowski inequality", where $0 < p < q$.

To see this, like what we have done, by Jensen's inequality, if $0 < p < q$,

$$(\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C)^{q/p} \le \int_{S^{d-1}} ((\frac{h_D}{h_C})^p)^{q/p} d\bar{V}_C = \int_{S^{d-1}} (\frac{h_D}{h_C})^q d\bar{V}_C.$$

So,

$$(\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C)^{1/p} \le (\int_{S^{d-1}} (\frac{h_D}{h_C})^q d\bar{V}_C)^{1/q}.$$

And by L'Hospital's rule,

$$\lim_{p\to 0^+} \log(\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C)^{1/p} = \lim_{p\to 0^+} \frac{1}{p} \log(\int_{S^{d-1}} (\frac{h_D}{h_C})^p) d\bar{V}_C$$

$$= \lim_{p\to 0^+} \frac{\int_{S^{d-1}} (\frac{h_D}{h_C})^p \log \frac{h_D}{h_C} d\bar{V}_C}{\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C} = \int_{S^{d-1}} \log \frac{h_D}{h_C} d\bar{V}_C$$

And

$$(\int_{S^{d-1}} (\frac{h_D}{h_C})^p d\bar{V}_C)^{1/p} \ge (\frac{V(D)}{V(C)})^{\frac{1}{d}}$$

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if and only if

$$\log(\int_{S^{d-1}}(\frac{h_D}{h_C})^p d\bar{V}_C)^{1/p} \geq \log(\frac{V(D)}{V(C)})^{\frac{1}{d}} = \frac{1}{d}\log\frac{V(D)}{V(C)}.$$

Hence the relationship was proved.

3.3 Equivalence of the L_p-Brunn-Minkowski and the L_p-Minkowski Inequalities

In this section, we show that for each fixed $p \geq 0$ the L_p-Brunn-Minkowski inequality and the L_p-Minkowski inequality are equivalent in that one is an easy consequence of the other. In particular, the log-Brunn-Minkowski inequality are equivalent.

Suppose $p > 0$. If C and D are convex bodies that contain the origin in their interior and $s, t \geq 0$ (not both zero) the L_p-Minkowski combination $s \cdot C +_p t \cdot D$, is defined by

$$s \cdot C +_p t \cdot D = \{x \in \mathbb{R}^d : x \cdot u \leq (sh_C(u)^p + th_D(u)^p)^{1/p} \text{ for all } u \in S^{d-1}\}.$$

We see that for a convex body C and real $s \geq 0$ the relationship between the L_p-scalar multiplication, $s \cdot C$, and Minkowski scalar multiplication sC is given by:

$$s \cdot C = s^{\frac{1}{p}}C.$$

Suppose $p > 0$ is fixed and suppose the following "weak" L_p-Brunn-Minkowski inequality holds for all origin-symmetric convex bodies C and D in $\mathbb{R}^d$ such that $V(C) = 1 = V(D)$:

$$V((1-\lambda) \cdot C +_p \lambda \cdot D) \geq 1, \tag{3.12}$$

for all $\lambda \in (0, 1)$. We claim that from this it follows that the following seemingly "stronger" L_p-Brunn-Minkowski inequality holds: if C and D are origin-symmetric

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convex bodies in $\mathbb{R}^d$, then

$$V(s \cdot C +_p t \cdot D)^{\frac{p}{d}} \geq sV(C)^{\frac{p}{d}} + tV(D)^{\frac{p}{d}}, \tag{3.13}$$

for all $s, t \geq 0$. To see this assume that the "weak" L_p-Brunn-Minkowski inequality (3.12) holds and that C and D are arbitrary origin-symmetric convex bodies. Define tha volume-normalized bodies $\bar{C} = V(C)^{-\frac{1}{d}}C$ and $\bar{D} = V(D)^{-\frac{1}{d}}D$. Then (3.12) gives

$$V((1-\lambda) \cdot \bar{C} +_p \lambda \cdot \bar{D}) \geq 1. \tag{3.14}$$

Let $\lambda = V(D)^{\frac{p}{d}}(V(C)^{\frac{p}{d}} + V(D)^{\frac{p}{d}})^{-1}$. Then

$$(1-\lambda) \cdot \bar{C} +_p \lambda \cdot \bar{D} = \frac{1}{(V(C)^{\frac{p}{d}} + V(D)^{\frac{p}{d}})^{\frac{1}{p}}}(C +_p D).$$

Therefore, from (3.14), we get

$$V(C +_p D)^{\frac{p}{d}} \geq V(C)^{\frac{p}{d}} + V(D)^{\frac{p}{d}}.$$

If we now replace C with $s \cdot C$ and D with $t \cdot D$ and note that $V(s \cdot C)^{\frac{p}{d}} = sV(C)^{\frac{p}{d}}$, we obtain the desired "stronger" L_p-Brunn-Minkowski inequality (3.13).

Lemma 3.3.1. *Suppose $p > 0$. When restricted to origin-symmetric convex bodies in $\mathbb{R}^d$, the L_p-Brunn-Minkowski inequality* (3.9) *and the L_p-Minkowski inequality* (3.11) *are equivalent.*

Proof. Suppose C and D are fixed origin-symmetric convex bodies in $\mathbb{R}^d$. For $0 \leq \lambda \leq 1$, let

$$Q_\lambda = (1-\lambda) \cdot C +_p \lambda \cdot D;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda = ((1-\lambda)h_C^p + \lambda h_D^p)^{\frac{1}{p}}$. It will

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be convenient to consider q_λ as being defined for λ in the open interval $(-\epsilon_0, 1+\epsilon_0)$, where $\epsilon_0 > 0$ is chosen so that for $\lambda \in (-\epsilon_0, 1+\epsilon_0)$, the function q_λ is strictly positive.

We first assume that the L_p-Minkowski inequality (3.11) holds. From (3.2), the fact that $h_{Q_\lambda} = ((1-\lambda)h_C^p + \lambda h_D^p)^{\frac{1}{p}}$ a.e. with respect to the surface area measure S_{Q_λ}, (3.1) and (3.3), and finally the L_p-Minkowski inequality (3.11), we have

$$V(Q_\lambda) = \frac{1}{d}\int_{S^{d-1}} h_{Q_\lambda} dS_{Q_\lambda}$$

$$= \frac{1}{d}\int_{S^{d-1}} ((1-\lambda)h_C^p + \lambda h_D^p)h_{Q_\lambda}^{1-p} dS_{Q_\lambda}$$

$$= (1-\lambda)V(Q_\lambda)\int_{S^{d-1}} (\frac{h_C}{h_{Q_\lambda}})^p d\bar{V}_{Q_\lambda} + \lambda V(Q_\lambda)\int_{S^{d-1}} (\frac{h_D}{h_{Q_\lambda}})^p d\bar{V}_{Q_\lambda}$$

$$\geq V(Q_\lambda)^{1-\frac{p}{d}}((1-\lambda)V(C)^{\frac{p}{d}} + \lambda V(D)^{\frac{p}{d}}). \tag{3.15}$$

This and the inequality of arithmetic and geometric means gives

$$V(Q_\lambda) \geq ((1-\lambda)V(C)^{\frac{p}{d}} + \lambda V(D)^{\frac{p}{d}})^{\frac{d}{p}} \geq V(C)^{1-\lambda}V(D)^{\lambda}, \tag{3.16}$$

which is the L_p-Brunn-Minkowski inequality (3.9).

Now assume that the L_p-Brunn-Minkowski inequality (3.9) holds. As was seen at the beginning of this section, this inequality implies the seemingly stronger L_p-Brunn-Minkowski inequality (3.13). But this inequality tell us that the function $f : [0,1] \to (0,\infty)$, given by $f(\lambda) = V(Q_\lambda)^{\frac{p}{d}}$ for $\lambda \in [0,1]$, satisfies that

$$f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1). \tag{3.17}$$

Now, the convergence as $\lambda \to 0$ in

$$\frac{q_\lambda - q_0}{\lambda} \to \frac{h_C^{1-p}}{p}(h_D^p - h_C^p) = \frac{h_C^{1-p}h_D^p - h_C}{p},$$

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is uniform on S^{d-1}. By Lemma 3.1.1, (3.3) and (3.1), and (3.2),

$$\frac{dV(Q_\lambda)}{d\lambda}|_{\lambda=0} = \int_{S^{d-1}} \frac{h_C^{1-p} h_D^p - h_C}{p} dS_C = \frac{d}{p}(V_p(C,D) - V(C)).$$

Therefore, (3.17) yields

$$V(C)^{\frac{p}{d}-1}(V_p(C,D) - V(C)) = f'(0) = \lim_{\lambda \to 0^+} \frac{f(\lambda) - f(0)}{\lambda}$$

$$\geq \lim_{\lambda \to 0^+} \frac{\lambda(f(1) - f(0))}{\lambda} = f(1) - f(0) = V(D)^{\frac{p}{d}} - V(C)^{\frac{p}{d}}.$$

Then

$$\frac{V_p(C,D)}{V(C)} \geq (\frac{V(D)}{V(C)})^{\frac{p}{d}},$$

which gives the L_p-Minkowski inequality (3.11). □

Lemma 3.3.2. *For origin-symmetric convex bodies in* $\mathbb{R}^d$, *the log-Brunn-Minkowski inequality* (3.8) *and the log-Minkowski inequality* (3.10) *are equivalent.*

Proof. Suppose C and D are fixed origin-symmetric convex bodies in $\mathbb{R}^d$. For $0 \leq \lambda \leq 1$, let

$$Q_\lambda = (1-\lambda) \cdot C +_o \lambda \cdot D;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda = h_C^{1-\lambda} h_D^\lambda$. It will be convenient to consider q_λ as being defined for λ in the open interval $(-\epsilon_0, 1+\epsilon_0)$, where $\epsilon_0 > 0$.

We first assume that the log-Minkowski inequality (3.10) holds. From the fact that $h_{Q_\lambda} = h_C^{1-\lambda} h_D^\lambda$ a.e. with respect to the surface area measure S_{Q_λ}, and the log-Minkowski inequality (3.10), we have

$$0 = \frac{1}{dV(Q_\lambda)} \int_{S^{d-1}} h_{Q_\lambda} \log \frac{h_C^{1-\lambda} h_D^\lambda}{h_{Q_\lambda}} dS_{Q_\lambda}$$

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$$= (1-\lambda)\frac{1}{dV(Q_\lambda)}\int_{S^{d-1}} h_{Q_\lambda}\log\frac{h_C}{h_{Q_\lambda}}dS_{Q_\lambda} + \lambda\frac{1}{dV(Q_\lambda)}\int_{S^{d-1}} h_{Q_\lambda}\log\frac{h_D}{h_{Q_\lambda}}dS_{Q_\lambda}$$

$$\geq (1-\lambda)\frac{1}{d}\log\frac{V(C)}{V(Q_\lambda)} + \lambda\frac{1}{d}\log\frac{V(D)}{V(Q_\lambda)} \tag{3.18}$$

$$= \frac{1}{d}\log\frac{V(C)^{1-\lambda}V(D)^\lambda}{V(Q_\lambda)}.$$

This gives the log-Brunn-Minkowski inequality (3.8).

Now assume that the log-Brunn-Minkowski inequality (3.8) holds. It tell us that the function $f : [0,1] \to (0,\infty)$, given by $f(\lambda) = \log V(Q_\lambda)$ for $\lambda \in [0,1]$, satisfies that

$$f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1). \tag{3.19}$$

Now, the convergence as $\lambda \to 0$ in

$$\frac{q_\lambda - q_0}{\lambda} \to h_C \log\frac{h_D}{h_C},$$

is uniform on S^{d-1}. By Lemma 3.1.1,

$$\frac{dV(Q_\lambda)}{d\lambda}|_{\lambda=0} = \int_{S^{d-1}} h_C\log\frac{h_D}{h_C}dS_C = d\int_{S^{d-1}}\log\frac{h_D}{h_C}dV_C.$$

Therefore, (3.19) yields

$$d\int_{S^{d-1}}\log\frac{h_D}{h_C}d\bar{V}_C = \frac{1}{V(Q_0)}\frac{dV(Q_\lambda)}{d\lambda}|_{\lambda=0} = f'(0) = \lim_{\lambda\to 0^+}\frac{f(\lambda)-f(0)}{\lambda}$$

$$\geq \lim_{\lambda\to 0^+}\frac{\lambda(f(1)-f(0))}{\lambda} = f(1) - f(0) = \log V(D) - \log V(C),$$

which gives the log-Minkowski inequality (3.10). □

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3.4 Blaschke's Extension of the Bonnesen Inequality

In [6], E. Lutwak proved that $V(C,D) = V_1(C,D)$. That is,

$$V(C,D) = \frac{1}{d}\int_{S^{d-1}} h_D(u) dS_C(u) \tag{3.20}$$

From (3.20) we see that for convex bodies C, D, D', we have

$$\begin{aligned} &D \subset D' \Rightarrow V(C,D) \le V(C,D'), \\ &\text{with equality if and only if } h_D = h_{D'} \text{ a.e. w.r.t. } S_C. \end{aligned} \tag{3.21}$$

Theorem 3.4.1. *If C, D are plane convex bodies, then for $r(C,D) \le t \le R(C,D)$,*

$$V(C) - 2tV(C,D) + t^2 V(D) \le 0.$$

The inequality is strict whenever $r(C,D) < t < R(C,D)$. When $t = r(C,D)$ equality will occur if and only if C is the Minkowski sum of a dilation of D and a line segment. When $t = R(C,D)$ equality will occur if and only if D is the Minkowski sum of a dilation of C and a line segment.

Proof. Let $r = r(C,D)$, and suppose $t \in [0,r]$. Recall from (3.5) that

$$C_t = \{x \in \mathbb{R}^d : x \cdot u \le h_C(u) - t h_D(u) \text{ for all } u \in S^{d-1}\},$$

and that from (3.6), we have

$$C_t + tD \subset C. \tag{3.22}$$

But (3.22), together with the monotonicity (3.21), Proposition 2.2.2 (linearity), together

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with Proposition 2.2.3 gives

$$V(C,D) \geq V(C_t + tD, D) = V(C_t, D) + tV(D). \tag{3.23}$$

Now Lemma 3.1.2 and (3.23) gives,

$$\begin{aligned} V(C) - V(C_t) &= 2\int_0^t V(C_s, D)ds \\ &\leq 2\int_0^t (V(C,D) - sV(D))ds \qquad (3.24) \\ &= 2tV(C,D) - t^2V(D). \end{aligned}$$

Thus,

$$V(C) - 2tV(C,D) + t^2V(D) \leq V(C_t). \tag{3.25}$$

From (3.23) and (3.24), we see that equality (3.25) holds if and only if,

$$V(C,D) = V(C_s + sD, D) \ \forall s \in [0,t], \tag{3.26}$$

which, from (3.22) and (3.21), gives

$$h_C = h_{C_s} + sh_D \text{ a.e. w.r.t } S_D \ \forall s \in [0,t].$$

By (3.7) we know $V(C_r) = 0$ and thus C_r is a line segment, possibly a single point. Therefore from (3.25) we have

$$V(C) - 2rV(C,D) + r^2V(D) \leq 0. \tag{3.27}$$

Suppose now the equality holds in (3.27), that is,

$$V(C) - 2rV(C,D) + r^2V(D) = 0. \tag{3.28}$$

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Then, by (3.26) we have,

$$V(C, D) = V(C_r + rD, D).$$

But this in (3.28) gives:

$$V(C) - 2rV(C_r + rD, D) + r^2V(D) = 0,$$

which, using linearity , can be rewritten as

$$V(C) - 2rV(C_r, D) - r^2V(D) = 0,$$

and since $V(C_r) = 0$, using linearity, as

$$V(C) = V(C_r + rD).$$

This and that $C_r + rD \subset C$ forces that $C_r + rD = C$. That is, C is the Minkowski sum of a dilation of D and the line segment C_r. Conversely, suppose that C is the Minkowski sum of a dilation of D and a line segment L. That is, $C = sD + L$. Then $r(C, D) = s, C_r = L$. So, $C = C_r + rD$. Therefore,

$$V(C) - 2rV(C, D) + r^2V(D) = V(C_r + rD) - 2rV(C_r + rD, D) + r^2V(D)$$

$$= (V(C_r) + 2rV(C_r, D) + r^2V(D)) - (2rV(C_r, D) + 2r^2V(D)) + r^2V(D) = V(C_r) = 0.$$

Hence, $V(C) - 2rV(C, D) + r^2V(D) = 0$ if and only if C is the Minkowski sum of a dilation of D and a line segment.

Let $r' = r(D, C) = 1/R(C, D)$, the inequality (3.27) tell us that

$$V(D) - 2r'V(D, C) + r'^2V(C) \leq 0,$$

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with equality if and only if D is the Minkowski sum of a dilation of C and a line segment. That is

$$V(C) - 2RV(C, D) + R^2V(D) \leq 0, \text{ where } R = R(C, D).$$

Finally, let $p(t) = V(C) - 2tV(C, D) + t^2V(D)$. since p is a monic quadratic function , it is a strictly convex function. So,

$$p((1-\lambda)r + \lambda R) < (1-\lambda)p(r) + \lambda p(R) \leq 0 \ \forall 0 < \lambda < 1.$$

Thus,

$$V(C) - 2tV(C, D) + t^2V(D) < 0 \text{ for } r(C, D) < t < R(C, D).$$

□

3.5 Uniqueness of Planar Cone-Volume Measure

Lemma 3.5.1. *If C, D are origin-symmetric plane convex bodies, then*

$$V(D) \int_{S^1} \frac{h_C}{h_D} dV_C \leq V(C) \int_{S^1} \frac{h_D}{h_C} dV_C, \tag{3.29}$$

with equality if and only if C and D are dilates, or C and D are parallelograms with parallel sides.

Proof. Since C and D are origin symmetric, from (3.4) we have

$$r(C, D) \leq \frac{h_C(u)}{h_D(u)} \leq R(C, D).$$

for all $u \in S^1$. Thus, from Theorem 3.4.1, we get

$$V(C) - 2\frac{h_C(u)}{h_D(u)}V(C, D) + (\frac{h_C(u)}{h_D(u)})^2V(D) \leq 0.$$

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Integrating both sides of this, with respect to the measure $h_D dS_C$, and using (3.20) and (3.2), gives

$$0 \geq \int_{S^1} (V(C) - 2\frac{h_C(u)}{h_D(u)}V(C,D) + (\frac{h_C(u)}{h_D(u)})^2 V(D))h_D(u)dS_C(u)$$

$$= -2V(C)V(C,D) + V(D)\int_{S^1} \frac{h_C(u)^2}{h_D(u)}dS_C(u).$$

So,

$$V(D)\int_{S^1} \frac{h_C}{h_D}dV_C = \frac{1}{2}V(D)\int_{S^1} \frac{h_C(u)^2}{h_D(u)}dS_C(u) \leq V(C)V(C,D) = V(C)\int_{S^1} \frac{h_D}{h_C}dV_C.$$

This yields the desired inequality (3.29). Suppose there is equality in (3.29). Thus,

$$V(C) - 2\frac{h_C(u)}{h_D(u)}V(C,D) + (\frac{h_C(u)}{h_D(u)})^2 V(D) = 0, \text{ for all } u \in \text{supp}S_C. \tag{3.30}$$

where $\text{supp}S_C = \{\nu(x) : x \in \partial C, x \text{ is not singular}\} = \{u \in S^{d-1} : S_C(B_r(u) \cap S^{d-1}) > 0$ for all $r > 0\}$.

If C and D are dilates, we're done. So assume that C and D are not dilates. But then $r(C,D) < R(C,D)$. From Theorem 3.4.1, we know that when

$$r(C,D) < \frac{h_C(u)}{h_D(u)} < R(C,D),$$

it follows that

$$V(C) - 2\frac{h_C(u)}{h_D(u)}V(C,D) + (\frac{h_C(u)}{h_D(u)})^2 V(D) < 0,$$

and thus we conclude that

$$h_C(u)/h_D(u) \in \{r(C,D), R(C,D)\} \text{ for all } u \in \text{supp}S_C. \tag{3.31}$$

Note that since C is origin symmetric $\text{supp}S_C$ is origin symmetric as well. Either there

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exists $u_0 \in \text{supp}S_C$ so that $h_C(u_0)/h_D(u_0) = r(C, D)$ or $h_C(u_0)/h_D(u_0) = R(C, D)$. Suppose that $h_C(u_0)/h_D(u_0) = r(C, D)$. Then from (3.30) and the equality conditions of Theorem 3.4.1 we know that C must be a dilation of the Minkowski sum of D and a line segment. But C and D are not dilates, so there exists an $x_0 \neq 0$ so that

$$h_C(u) = |x_0 \cdot u| + r(C, D)h_D(u),$$

for all unit vectors u. This together with $h_C(u_0)/h_D(u_0) = r(C, D)$ shows that x_0 is orthogonal to u_0 and that the only unit vectors at which $h_C/h_D = r(C, D)$ are u_0 and $-u_0$. But $\text{supp}S_C$ must contain at least one unit vector $u_1 \in \text{supp}S_C$ other than $\pm u_0$. From (3.31), and the fact that the only unit vectors at which $h_C/h_D = r(C, D)$ are u_0 and $-u_0$, we conclude $h_C(u_1)/h_D(u_1) = R(C, D)$ and by the same argument we conclude that the only unit vectors at which $h_C/h_D = R(C, D)$ are u_1 and $-u_1$. Now (3.31) allow us to conclude that

$$\text{supp}S_C = \{\pm u_0, \pm u_1\}.$$

This implies that C is a parallelogram. Since C is the Minkowski sum of a dilate of D and a line segment, D must be a parallelogram with sides parallel to those of C. If we had assumed that $h_C(u_0)/h_D(u_0) = R(C, D)$, rather than $r(C, D)$, the same argument would lead to the same conclusion.

It is easily seen that the equality holds in (3.29) if C and D are dilates. Now suppose that C and D are parallelograms with parallel sides. Then

$$h_C(u) = a_1|v_1 \cdot u| + a_2|v_2 \cdot u|, h_D(u) = b_1|v_1 \cdot u| + b_2|v_2 \cdot u|,$$

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where $v_1, v_2 \in S^1$ and $a_1, a_2, b_1, b_2 > 0$. Then $\mathrm{supp}S_C = \{\pm v_1^\perp, \pm v_2^\perp\}$, while

$$\bar{V}_C(\{v_1^\perp\}) = \bar{V}_C(\{-v_1^\perp\}) = \bar{V}_C(\{v_2^\perp\}) = \bar{V}_C(\{-v_2^\perp\}) = 1/4,$$

and $|v_1 \cdot v_2^\perp| = |v_2 \cdot v_1^\perp|$.

Therefore,

$$V(D)\int_{S^1} \frac{h_C}{h_D} d\bar{V}_C = b_1 b_2 |v_1 \cdot v_2^\perp| \frac{1}{2}(\frac{a_1}{b_1} + \frac{a_2}{b_2}) = \frac{1}{2}(a_1 b_2 + a_2 b_1).$$

and

$$V(C)\int_{S^1} \frac{h_D}{h_C} d\bar{V}_C = a_1 a_2 |v_1 \cdot v_2^\perp| \frac{1}{2}(\frac{b_1}{a_1} + \frac{b_2}{a_2}) = \frac{1}{2}(a_1 b_2 + a_2 b_1).$$

Hence,

$$V(D)\int_{S^1} \frac{h_C}{h_D} dV_C = V(C)\int_{S^1} \frac{h_D}{h_C} dV_C.$$

That is, the equality holds in (3.29). □

Theorem 3.5.2. *If C, D are origin-symmetric plane convex bodies that have the same cone-volume measure, then $C = D$ or else C and D are parallelograms with parallel sides.*

Proof. Assume that $V_C = V_D$ and $C \neq D$, then

$$V(C) = \int_{S^1} dV_C = \int_{S^1} dV_D = V(D).$$

So C and D are not dilates (otherwise $C = D$). Thus inequality (3.29) becomes

$$\int_{S^1} \frac{h_C}{h_D} dV_C \leq \int_{S^1} \frac{h_D}{h_C} dV_C.$$

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And by exchanging the roles of C and D,

$$\int_{S^1} \frac{h_D}{h_C} dV_D \leq \int_{S^1} \frac{h_C}{h_D} dV_D.$$

Both inequality have equality condition: C and D are parallelograms with parallel sides. Observe that

$$\int_{S^1} \frac{h_C}{h_D} dV_C \leq \int_{S^1} \frac{h_D}{h_C} dV_C = \int_{S^1} \frac{h_D}{h_C} dV_D \leq \int_{S^1} \frac{h_C}{h_D} dV_D = \int_{S^1} \frac{h_C}{h_D} dV_C.$$

So the equalities hold. That is, C and D are parallelograms with parallel sides. □

3.6 Minimizing the Logarithmic Mixed Volume

Lemma 3.6.1. *Suppose C is a plane origin-symmetric convex body, with $V(C) = 1$, that is not a parallelogram. Suppose also that P_k is an unbounded sequence of origin-symmetric parallelograms all of which have orthogonal diagonals, and such that $V(P_k) \geq 2$. Then, the sequence*

$$\int_{S^1} \log h_{P_k}(u) dV_C(u)$$

is not bounded from above.

Proof. Let $u_{1,k}, u_{2,k}$ be orthogonal unit vectors along the diagonals of P_k. Denote the vertices of P_k by $\pm h_{1,k}u_{1,k}, \pm h_{2,k}u_{2,k}$. Without loss of generality, assume that $0 < h_{1,k} \leq h_{2,k}$. The condition $V(P_k) \geq 2$ is equivalent to $h_{1,k}h_{2,k} \geq 1$. The support function of P_k is given by

$$h_{P_k}(u) = \max\{h_{1,k}|u \cdot u_{1,k}|, h_{2,k}|u \cdot u_{2,k}|\}, \tag{3.32}$$

for $u \in S^1$. Since S^1 is compact, the sequences $u_{1,k}$ and $u_{2,k}$ have convergent subsequences. Again, without loss of generality, we may assume that the sequences $u_{1,k}$ and

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$u_{2,k}$ are themselves convergent with

$$\lim_{k\to\infty} u_{1,k} = u_1 \text{ and } \lim_{k\to\infty} u_{2,k} = u_2,$$

where u_1 and u_2 are orthogonal (since $u_1 \cdot u_2 = \lim_{k\to\infty} u_{1,k} \cdot u_{2,k} = 0$).

It is easy to see that if the cone-volume measure, $V_C(\{\pm u_1\})$, of the two-point set $\{\pm u_1\}$ is positive, then C contains a parallelogram whose area is $2V_C(\{\pm u_1\})$. Since C itself is not a parallelogram and $V(C) = 1$, it must be the case that

$$V_C(\{\pm u_1\}) < \frac{1}{2}. \tag{3.33}$$

For $\delta \in (0, \frac{1}{3})$, consider the neighborhood, U_δ, of $\{\pm u_1\}$, on S^1,

$$U_\delta = \{u \in S^1 : |u \cdot u_1| > 1 - \delta\}.$$

Since $V_C(S^1) = V(C) = 1$, we see that for all $\delta \in (0, \frac{1}{3})$,

$$V_C(U_\delta) + V_C(U_\delta^c) = 1, \tag{3.34}$$

where U_δ^c is the complement of U_δ.

Since the U_δ are decreasing (with respect to set inclusion) in δ and have a limit of $\{\pm u_1\}$,

$$\lim_{\delta\to 0^+} V_C(U_\delta) = V_C(\{\pm u_1\}).$$

This together with (3.33), shows the existence of a $\delta_0 \in (0, \frac{1}{3})$ such that

$$V_C(U_{\delta_0}) < \frac{1}{2}.$$

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But this implies that there is a small $\epsilon_0 \in (0, \frac{1}{2})$ so that

$$\tau_0 = V_C(U_{\delta_0}) - \frac{1}{2} + \epsilon_0 < 0. \tag{3.35}$$

This together with (3.34) gives

$$V_C(U_{\delta_0}) = \frac{1}{2} - \epsilon_0 + \tau_0 \text{ and } V_C(U_{\delta_0}^c) = \frac{1}{2} + \epsilon_0 - \tau_0. \tag{3.36}$$

Since $u_{i,k}$ converge to u_i, we have, $|u_{i,k} - u_i| \leq \delta_0$ whenever k is sufficiently large (for both $i = 1$ and $i = 2$). Then for $u \in U_{\delta_0}$ and k sufficiently large, we have

$$|u \cdot u_{1,k}| \geq |u \cdot u_1| - |u \cdot (u_{1,k} - u_1)| \geq |u \cdot u_1| - |u_{1,k} - u_1| \geq 1 - \delta_0 - \delta_0 \geq \delta_0,$$

where the last inequality follows from the fact that $\delta_0 < \frac{1}{3}$. For all $u \in S^1$, we know that $|u \cdot u_1|^2 + |u \cdot u_2|^2 = 1$. Thus, for $u \in U_{\delta_0}^c$, we have $|u \cdot u_2| \geq (1 - (1 - \delta_0)^2)^{\frac{1}{2}} = (\delta_0(2 - \delta_0))^{\frac{1}{2}} > (5\delta_0^2)^{\frac{1}{2}} > 2\delta_0$, which shows that when k is sufficiently large,

$$|u \cdot u_{2,k}| \geq |u \cdot u_2| - |u \cdot (u_{2,k} - u_2)| \geq |u \cdot u_2| - |u_{2,k} - u_2| \geq 2\delta_0 - \delta_0 = \delta_0.$$

From the last paragraph and (3.32) is follows that when k is sufficiently large,

$$h_{P_k}(u) \geq \begin{cases} \delta_0 h_{1,k} & \text{if } u \in U_{\delta_0}, \\ \delta_0 h_{2,k} & \text{if } u \in U_{\delta_0}^c. \end{cases} \tag{3.37}$$

By (3.37) and (3.34), (3.36), the fact that $0 < h_{1,k} \leq h_{2,k}$ together with (3.35), and finally the fact that $h_{1,k}h_{2,k} \geq 1$ together with $\epsilon_0 \in (0, \frac{1}{2})$, we see that for sufficiently large k,

$$\begin{aligned}\int_{S^1} \log h_{P_k} dV_C &= \int_{U_{\delta_0}} \log h_{P_k} dV_C + \int_{U_{\delta_0}^c} \log h_{P_k} dV_C \\ &\geq \log \delta_0 + V_C(U_{\delta_0}) \log h_{1,k} + V_C(U_{\delta_0}^c) \log h_{2,k}\end{aligned}$$

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$$= \log \delta_0 + (\frac{1}{2} + \tau_0 - \epsilon_0) \log h_{1,k} + (\frac{1}{2} - \tau_0 + \epsilon_0) \log h_{2,k}$$

$$= \log \delta_0 + 2\epsilon_0 \log h_{2,k} + (\frac{1}{2} - \epsilon_0) \log(h_{1,k} h_{2,k}) + \tau_0(\log h_{1,k} - \log h_{2,k})$$

$$\geq \log \delta_0 + 2\epsilon_0 \log h_{2,k}.$$

Since P_k is not bounded, the sequence $h_{2,k}$ is not bounded from above. Thus, the sequence

$$\int_{S^1} \log h_{P_k} dV_C$$

is not bounded from above. □

Lemma 3.6.2. *Suppose C is a plane origin-symmetric convex body that is not a parallelogram, then there exists a plane origin-symmetric convex body C_0 so that $V(C_0) = 1$ and*

$$\int_{S^1} \log h_Q dV_C \geq \int_{S^1} \log h_{C_0} dV_C$$

for every plane origin-symmetric convex body Q with $V(Q) = 1$.

Proof. By letting $\bar{C} = V(C)^{-\frac{1}{2}} C$, we may assume that $V(C) = 1$. Consider the minimization problem,

$$\inf \int_{S^1} \log h_Q dV_C,$$

where the infimum is taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Suppose that Q_k is a minimizing sequence; i.e., Q_k is a sequence of origin-symmetric convex bodies with $V(Q_k) = 1$ and such that $\int_{S^1} \log h_{Q_k} dV_C$ tends to the infimum (which may be $-\infty$). We shall show that the sequence Q_k is bounded and the infimum is finite.

By The John Ellipsoid Theorem (see e.g. [4]), there exist ellipses E_k centered at the origin so that

$$E_k \subset Q_k \subset \sqrt{2} E_k. \tag{3.38}$$

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Let $u_{1,k}, u_{2,k}$, be the principal directions of E_k so that

$$h_{1,k} \leq h_{2,k}, \text{ where } h_{1,k} = h_{E_k}(u_{1,k}) \text{ and } h_{2,k} = h_{E_k}(u_{2,k}).$$

Let P_k be the origin-centered parallelogram that has vertices $\{\pm h_{1,k}u_{1,k}, \pm h_{2,k}u_{2,k}\}$. (Observe that by the Principal Axis Theorem the diagonals of P_k are perpendicular.) Since $E_k \subset \sqrt{2}P_k$, it follows from (3.38) that

$$Q_k \subset \sqrt{2}E_k \subset \sqrt{2}(\sqrt{2}P_k) = 2P_k.$$

So,

$$P_k \subset Q_k \subset 2P_k. \tag{3.39}$$

From this and $V(Q_k) = 1$, we see that $V(P_k) \geq \frac{1}{4}$.

Assume that Q_k is not bounded. Then P_k is not bounded (since $P_k \supset Q_k/2$). Applying Lemma 3.6.1 to $\sqrt{8}P_k$ shows that the sequence $\int_{S^1} \log h_{P_k} dV_C = \int_{S^1} \log h_{\sqrt{8}P_k} dV_C - \log\sqrt{8}$ is not bounded from above. Therefore, from (3.39) we see that the sequence $\int_{S^1} \log h_{Q_k} dV_C$ cannot be bounded from above. However, this is impossible because Q_k was chosen to be a minimizing sequence.

We conclude that Q_k is bounded. By Blaschke's Selection Theorem (Theorem 2.3.1), Q_k has a convergent subsequence that converges to an origin-symmetric convex body C_0, with $V(C_0) = 1$ (by Theorem 2.3.2). (int$C_0 \neq \phi$ since $V(C_0) = 1$.) It follows that $\int_{S^1} \log h_{C_0} dV_C$ is the desired infimum. □

3.7 The log-Minkowski Inequality

In [3], they proved Problem 3.10, Problem 3.8, Problem 3.11 and Problem 3.9 for C, D are in the plane, with their equality conditions. First, the log-Minkowski inequality:

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Theorem 3.7.1. *If C and D are plane origin-symmetric convex bodies, then*

$$\int_{S^1} \log \frac{h_D}{h_C} d\bar{V}_C \geq \frac{1}{2} \log \frac{V(D)}{V(C)}, \tag{3.40}$$

with equality if and only if either C and D are dilates or when C and D are parallelograms with parallel sides.

Proof. Without loss of generality, we can assume that $V(C) = V(D) = 1$. We shall establish the theorem by proving

$$\int_{S^1} \log h_D dV_C \geq \int_{S^1} \log h_C dV_C,$$

with equality if and only if either C and D are dilates or when C and D are parallelograms with parallel sides.

First, assume that C is not a parallelogram. Consider the minimization problem

$$\min \int_{S^1} \log h_Q dV_C,$$

taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Let C_0 denote a solution, whose existence is guaranteed by Lemma 3.6.2. (Our aim is to prove that $C_0 = C$ and thereby demonstrate that C itself can be the only solution to this minimization problem.)

Suppose f is an arbitrary but fixed even continuous function on S^1. Consider the deformation of h_{C_0}, defined on $\mathbb{R} \times S^1$, by

$$q_t(u) = q(t, u) = h_{C_0}(u) e^{tf(u)}.$$

Let Q_t be the Wulff shape associated with q_t. Observe that Q_t is an origin symmetric convex body and that since q_0 is the support function of the convex body C_0, we have

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$Q_0 = C_0$.

Since C_0 is an assumed solution of the minimization problem, the function g_1 defined on $\mathbb{R}$ by

$$g_1(t) = V(Q_t)^{-\frac{1}{2}} \exp\{\int_{S^1} \log h_{Q_t} dV_C\} = \exp\{\int_{S^1} \log h_{Q_t/V(Q_t)^{-1/2}} dV_C\},$$

attains a minimal value at $t = 0$ (since $V(Q_t/V(Q_t)^{-1/2}) = 1$). Since $h_{Q_t} \leq q_t$, this function is dominated by the differentiable function g_2 defined on $\mathbb{R}$ by

$$g_2(t) = V(Q_t)^{-\frac{1}{2}} \exp\{\int_{S^1} \log q_t dV_C\}.$$

Since $q_0 = h_{C_0} = h_{Q_0}$,

$$g_2(0) = g_1(0) \leq g_1(t) \leq g_2(t) \ \forall t \in \mathbb{R}.$$

Thus, $g_2'(0) = 0$. Note that $V(Q_0) = V(C_0) = 1$. By Lemma 3.1.1,

$$0 = g_2'(0) = -\frac{1}{2} \int_{S^1} h_{C_0}(u) f(u) dS_{C_0}(u) \exp\{\int_{S^1} \log q_0 dV_C\}$$

$$+ \exp\{\int_{S^1} \log q_0 dV_C\} \int_{S^1} \frac{h_{C_0}(u) f(u)}{h_{C_0}(u)} dV_C(u).$$

So,

$$\int_{S^1} f(u) dV_{C_0}(u) = \int_{S^1} f(u) dV_C(u).$$

Since f was an arbitrary even continuous function,

$$V_{C_0} = V_C.$$

By Theorem 3.5.2, and the assumption that C is not a parallelogram, we conclude that $C_0 = C$.

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Thus, for each D such that $V(D) = 1$,

$$\int_{S^1} \log h_D dV_C \geq \int_{S^1} \log h_C dV_C,$$

with equality if and only if $C = D$. This is the desired result when C is not a parallelogram.

Now assume that C is a parallelogram whose support function is given by

$$h_C(u) = a_1|v_1 \cdot u| + a_2|v_2 \cdot u|,$$

where $v_1, v_2 \in S^1$ and $a_1, a_2 > 0$. Then $\text{supp} S_C = \{\pm v_1^\perp, \pm v_2^\perp\}$, while $|v_1 \cdot v_2^\perp| = |v_2 \cdot v_1^\perp|$ and $V_C(\{\pm v_i^\perp\}) = 2a_1a_2|v_1 \cdot v_2^\perp|$. So that $4a_1a_2|v_1 \cdot v_2^\perp| = V(C) = 1$. And we can see that

$$\exp \int_{S^1} \log h_D dV_C = \exp(\frac{1}{2}(\log h_D(v_1^\perp) + \log h_D(v_2^\perp))) = \sqrt{h_D(v_1^\perp)h_D(v_2^\perp)}. \tag{3.41}$$

The parallelogram circumscribed about D with sides parallel to those of C has volume

$$4\frac{h_D(v_1^\perp)}{|v_1 \cdot v_2^\perp|} h_D(v_2^\perp) = 16a_1a_2h_D(v_1^\perp)h_D(v_2^\perp),$$

and thus, $16a_1a_2h_D(v_1^\perp)h_D(v_2^\perp) \geq V(D) = 1$, or equivalently

$$h_D(v_1^\perp)h_D(v_2^\perp) \geq \frac{1}{16a_1a_2},$$

with equality if and only if D itself is a parallelogram with sides parallel to those of C. Thus, by (3.41),

$$\int_{S^1} \log h_D dV_C = \log \sqrt{h_D(v_1^\perp)h_D(v_2^\perp)} \geq \log \sqrt{\frac{1}{16a_1a_2}}.$$

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the quality holds if and only if

$$h_D(v_1^{\perp})h_D(v_2^{\perp}) = \frac{1}{16a_1a_2};$$

i.e., if and only if D is a parallelogram with sides parallel to those of C. □

The log-Brunn-Minkowski inequality:

Theorem 3.7.2. *If C and D are origin-symmetric convex bodies in the plane, then for all $\lambda \in [0,1]$,*

$$V((1-\lambda)\cdot C +_o \lambda \cdot D) \geq V(C)^{1-\lambda}V(D)^{\lambda}. \tag{3.42}$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if C and D are dilates or C and D are parallelograms with parallel sides.

Proof. Lemma 3.3.2 shows that the log-Minkowski inequality of Theorem 3.7.1 yields the log-Brunn-Minkowski inequality (3.42) of Theorem 3.7.2. To obtain the equality conditions of (3.42), look the proof of Lemma 3.3.2. Suppose the equality in (3.42) holds. The equality in (3.18) must hold. By the equality conditions in Theorem 3.7.1, we know that either C and Q_λ are dilates or when C and Q_λ are parallelograms with parallel sides. And either D and Q_λ are dilates or when D and Q_λ are parallelograms with parallel sides. All the four possible cases satisfies the equality condition in Theorem 3.7.2. i.e., C and D are dilates or C and D are parallelograms with parallel sides.

Now suppose that C and D are dilates or C and D are parallelograms with parallel sides. If C and D are dilates, then C, D and Q_λ are dilates. So the equality in (3.18) holds. Therefore the equality in (3.42) holds.

If C and D are parallelograms with parallel sides such that

$$h_C(u) = a_1|v_1 \cdot u| + a_2|v_2 \cdot u|, h_D(u) = b_1|v_1 \cdot u| + b_2|v_2 \cdot u|.$$

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where $v_1, v_2 \in S^1$ and $a_1, a_2, b_1, b_2 > 0$. Let $A = |v_1 \cdot v_2^\perp| = |v_2 \cdot v_1^\perp|$. Then

$$h_C(v_1^\perp) = a_2 A, h_C(v_2^\perp) = a_1 A \text{ and } h_D(v_1^\perp) = b_2 A, h_D(v_2^\perp) = b_1 A.$$

So,

$$Q_\lambda \subset \{x : |x \cdot v_1^\perp| \leq a_2^{1-\lambda} b_2^\lambda A, |x \cdot v_2^\perp| \leq a_1^{1-\lambda} b_1^\lambda A\}.$$

Then

$$V(C)^{1-\lambda} V(D)^\lambda \leq V(Q_\lambda) \leq 4 \frac{a_2^{1-\lambda} b_2^\lambda A}{A} a_1^{1-\lambda} b_1^\lambda A$$

$$= (4 \frac{a_2 A}{A} a_1 A)^{1-\lambda} (4 \frac{b_2 A}{A} b_1 A)^\lambda = V(C)^{1-\lambda} V(D)^\lambda.$$

Therefore,

$$V(Q_\lambda) = V(C)^{1-\lambda} V(D)^\lambda.$$

□

The L_p-Minkowski inequality:

Theorem 3.7.3. *Suppose $p > 0$. If C and D are origin-symmetric convex bodies in the plane, then,*

$$(\int_{S^1} (\frac{h_D}{h_C})^p d\bar{V}_C)^{\frac{1}{p}} \geq (\frac{V(D)}{V(C)})^{\frac{1}{2}}, \tag{3.43}$$

with equality if and only if C and D are dilates.

Proof. By (3.40) and Jensen's inequality,

$$(\int_{S^1} (\frac{h_D}{h_C})^p d\bar{V}_C)^{\frac{1}{p}} = \exp\{\frac{1}{p} \log(\int_{S^1} (\frac{h_D}{h_C})^p d\bar{V}_C)\} \geq \exp\{\frac{1}{p} \int_{S^1} \log(\frac{h_D}{h_C})^p d\bar{V}_C\}$$

$$= \exp\{\int_{S^1} \log(\frac{h_D}{h_C}) d\bar{V}_C\} \geq \exp\{\frac{1}{2} \log \frac{V(D)}{V(C)}\} = (\frac{V(D)}{V(C)})^{\frac{1}{2}}.$$

Suppose the equality in (3.43) holds. The equality condition of (3.40) shows that either C and D are dilates or when C and D are parallelograms with parallel sides.

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If C and D are dilates, we are done. If not, then C and D are parallelograms with parallel sides.

The equality condition of Jensen's inequality shows that

$$(\frac{h_D}{h_C})^p = \text{constant on } \mathrm{supp}V_C.$$

Since C and D are parallelograms with parallel sides, C and D are dilates. Conversely, suppose that C and D are dilates, says, $D = tC$ for $t > 0$. Then

$$(\int_{S^1} (\frac{h_D}{h_C})^p d\bar{V}_C)^{\frac{1}{p}} = (t^p)^{\frac{1}{p}} = t = (t^2)^{\frac{1}{2}} = (\frac{V(D)}{V(C)})^{\frac{1}{2}}.$$

□

The L_p-Brunn-Minkowski inequality:

Theorem 3.7.4. *Suppose $p > 0$. If C and D are origin-symmetric convex bodies in the plane, then for all $\lambda \in [0, 1]$,*

$$V((1-\lambda)\cdot C +_p \lambda \cdot D) \geq V(C)^{1-\lambda} V(D)^{\lambda}. \tag{3.44}$$

When $\lambda \in (0, 1)$, equality in the inequality holds if and only if $C = D$.

Proof. Lemma 3.3.1 shows that the L_p-Minkowski inequality of Theorem 3.7.3 yields the L_p-Brunn-Minkowski inequality of Theorem 3.7.4.

To obtain the equality condition, suppose the equality holds for some $\lambda \in (0, 1)$. Look the proof of Lemma 3.3.1. The inequality in (3.15) and (3.16) must be equality.

From the equality conditions of Theorem 3.7.3, we know that equality in inequality (3.15) implies that C and D are dilates. But the inequality of arithmetic and geometric means in (3.16) has equality condition $V(C) = V(D)$. Thus we conclude that equality in (3.44) implies that $C = D$.

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Suppose $C = D$, Then the equality in (3.44) trivially holds (Since $(1-\lambda)\cdot C +_p \lambda \cdot C = C$). $\square$

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