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碩士論文

Department of Mathematics
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Master Thesis

空間形式中均曲率流的幾何性質

The Non-collapsing Property for Mean Curvature flow in S^{n+1}

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感謝同學們互相陪伴的這些日子。

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中文摘要

跟著［1］中的計算，我們利用特定的次橢圓算子以及最大值原理來證明球面上餘維度 1 曲面的某些性質會在均曲率流中會保持。利用同樣的方法，我們證明雙曲空間中餘維度 1 的均曲率流也會保持某種凸性。

關鍵詞：均曲率流、尺度不變量、最大值原理

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1 Introduction

Scale invariant quantities are very usefully in studying the asymptotic behaviour of mean curvature because selfsimilars appear as we do parabolic rescaling near the singularity. In [3] and [4] Huisken and Hamilton both used scale invariant quantity to rule out grim reaper as the limit of curve shortening flow with compact embedding initial condition. In [1], Andrews defined δ non-collapsing property for mean convex hypersurface in $\mathbb{R}^{n+1}$ which is also scale invariant. He used maximal principle to prove this property is preserved under the mean curvature flow and reproved Huisken's celebrated result: every convex hypersurface in $\mathbb{R}^{n+1}$ would converge to a round point under the mean curvature flow. In the first part of this paper, we follow the calculation in [1] to prove that δ non-collapsing is preserved for mean curvature flows in S^{n+1}. In the second part we use a similar method to prove the preservation of certain convexity condition for mean curvature flows in the hyperboloc space.

Definition 1 *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a closed mean-convex hypersurface bounding an open region Ω. For $x \in M$, let ν_x be the unit outer normal of M at x, $H(x)$ be the mean curvature and $\delta > 0$ be a constant. Denote by $B(x,\delta)$ the ball with center $x - \delta/H(x)\nu_x$ and radius $\delta/H(x)$. M is called interior δ non-collapsing if for all $x \in M$ $B(x,\delta)$ is contained in Ω. Suppose that M is convex, then M is called exterior δ non-collapsing if Ω is contained in $B(x,\delta)$ for all $x \in M$.*

Proposition 1 is proved in [1].

Proposition 1 *Let $F : [0,T) \times \bar{M}^n \to \mathbb{R}^{n+1}$ be a family of smooth embeddings evolved by the mean curvature flow.*

$$\frac{\partial F}{\partial t} = -H\nu$$

Suppose that $M_0 = F(0,\bar{M})$ is interior(exterior) δ non-collapsing, then $M_t = F(t,\bar{M})$ is also interior(exterior) δ non-collapsing for all $t \in [0,T)$.

Consider δ non-collapsing for mean-convex hypersurfaces in S^{n+1}. Note that the mean curvature of a geodesic sphere in S^{n+1} with radius $r > 0$ is $n\cot(r)$. Here we define δ non-collapsing in S^{n+1}:

Definition 2 *Let $M^n \hookrightarrow S^{n+1}$ be a closed mean-convex hypersurface and Ω is the domain bounded by M. For $x \in M$, denote by $H(x)$ the mean curvature of M at x, $B(x,\delta)$ the geodesic ball with center $\exp_x(-\cot^{-1}(H(x)/\delta)\nu_x)$ and radius $\cot^{-1}(H(x)/\delta)$. M is interior δ non-collapsing if for any $x \in M$, $B(x,\delta)$ is contained in Ω. Suppose that M is convex, then M is exterior δ non-collapsing if Ω is contained in $B(x,\delta)$ for all $x \in M$.*

In the (n+1)-dimensional hyperbolic space, a horosphere is a hypersurface which is the limit of geodesic spheres, passing through a same point and whose center goes to infinity along a geodesic. In other words, horospheres are the umbilic hypersurfaces with mean curvature equal to n.

Definition 3 *Let $M^n \hookrightarrow \mathbb{H}^{n+1}$ be a closed convex hypersurface, Ω be the domain bounded by M and $\delta > 0$ be a constant. For any $x \in M$, denote by N_x the complete umbilic hypersurface with mean curvature n/δ, tangent to M at x and has the same normal vector with M at x. M is called δ-convex if Ω is enclosed by N_x for any $x \in M$. In particular, a 1-convex hypersurface is supported by horospheres.*

The main result of this paper is the following:

Proposition 2 *Let $F : [0,T) \times \bar{M}^n \to \mathbb{S}^{n+1}$ be a family of smooth embeddings evolved by the mean curvature flow. Suppose that $M_0 = F(0,\bar{M})$ is interior(exterior) δ non-collapsing, then $M_t = F(t,\bar{M})$ is also interior(exterior) δ non-collapsing for all $t \in [0,T)$.*

Proposition 3 *Let $F : [0,T) \times \bar{M}^n \to \mathbb{H}^{n+1}$ be a family of smooth embeddings evolved by the mean curvature flow. Assume that $M_0 = F(0,\bar{M})$ is δ-convex for some $0 < \delta \leq 1$. Then then $M_t = F(t,\bar{M})$ is also δ-convex for all $t \in [0,T)$.*

Remark: It is easy to see that δ-convexity implies $h_{ij} \geq \delta g_{ij}$. In [2], it is proved that a closed hypersurface is 1-convex(supported by horoshperes) if and only if $h_{ij} \geq g_{ij}$.

2 Evolution Equations of MCF in $S^{n+1} \hookrightarrow \mathbb{R}^{n+2}$

$\bar{F} : [0,T) \times \bar{M}^n \to S^{n+1}$ is a family of smooth embeddings evolved by the mean curvature flow in S^{n+1}. $i : S^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ is the standard embedding. $F := i \circ \bar{F}$. Let g_{ij} be the induced metric, $A = \{h_{ij}\}$ be the second fundamental form, H be the mean curvature and ν be the unit outer normal of $F(t,\bar{M})$ in S^{n+1}. In the following, F and ν are considered to be vectors in $\mathbb{R}^{n+2}$. Δ and ∇ are the Laplace-Beltrami operator and covariant derivative with respect to the induced metric. $\langle , \rangle$ and $\|.\|$ are the standard inner product and the standard norm in the Euclidean space.

Lemma 1 *The evolution equations are:*
(a) $\frac{\partial F}{\partial t} = \Delta F + nF$
(b) $\frac{\partial \nu}{\partial t} = \Delta \nu + |A|^2 \nu + 2HF$
(c) $\frac{\partial H}{\partial t} = \Delta H + (|A|^2 + n)H$

proof:

We use normal coordinates in the calculation. Since F is evolved by MCF on sphere,

$$\frac{\partial F}{\partial t} = \Delta F - \langle \Delta F, F \rangle F$$

and

$$0 = \Delta \langle F, F \rangle = 2 \langle \Delta F, F \rangle + 2n$$

Then we have (a). For (b), to find the component of $\frac{\partial \nu}{\partial t}$ in each direction, we compute

$$\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x^i} \right\rangle = -\left\langle \nu, \frac{\partial}{\partial x^i}(-H\nu) \right\rangle = \frac{\partial H}{\partial x^i}$$

$$\left\langle \frac{\partial \nu}{\partial t}, \nu \right\rangle = 0 \text{ and } \left\langle \frac{\partial \nu}{\partial t}, F \right\rangle = -\langle \nu, -H\nu \rangle = H$$

Thus

$$\frac{\partial \nu}{\partial t} = \nabla H + HF$$

To calculate $\nabla_i \nu$

$$\langle \nabla_i \nu, \nu \rangle = 0, \quad \langle \nabla_i \nu, F \rangle = -\left\langle \nu, \frac{\partial F}{\partial x^i} \right\rangle = 0$$

$$\left\langle \nabla_i \nu, \frac{\partial F}{\partial x^j} \right\rangle = -\left\langle \nu, \nabla_i \frac{\partial F}{\partial x^j} \right\rangle = h_{ij}$$

So we get $\nabla_i \nu = g^{km} h_{mi} \frac{\partial F}{\partial x^k}$.

$$\Delta \nu = g^{ij} \nabla_j \nabla_i \nu = g^{ij} \nabla_j g^{km} h_{mi} \frac{\partial F}{\partial x^k}$$

$$g^{ij}(g^{km} \nabla_j h_{mi} \frac{\partial F}{\partial x^k} + g^{km} h_{mi}(-h_{jk}\nu - g_{jk}F))$$

by Codazzi equation

$$= \nabla H - |A|^2 \nu - HF$$

Then we obtain (b)

$$\frac{\partial \nu}{\partial t} = \Delta \nu + |A|^2 \nu + 2HF$$

For (c)

$$\frac{\partial h_{ij}}{\partial t} = \frac{\partial}{\partial t} \left\langle \frac{\partial F}{\partial x^i}, \nabla_j \nu \right\rangle = \left\langle \nabla_i(-H\nu), h_{jk} g^{kl} \frac{\partial F}{\partial x^l} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \nabla_j (\nabla H + HF) \right\rangle$$

$$= -H h_{jk} h_{il} g^{kl} + H g_{ij} + \nabla_j \nabla_i H$$

$$\frac{\partial H}{\partial t} = -g^{ik} g^{lj}(-2H g_{kl}) h_{ij} + g^{ij}(-H h_{jk} h_{il} g^{kl} + H g_{ij} + \nabla_j \nabla_i H) = \Delta H + (|A|^2 + n)H$$

□

3 δ Non-collapsing in S^{n+1}

Lemma 2 *$F : M^n \hookrightarrow S^{n+1}$ is interior(exterior) δ non-collapsing if and only if*

$$Z(x,y) := \frac{H(x)}{2} \|F(y) - F(x)\|^2 + \delta \langle F(y) - F(x), \nu_x \rangle \geq 0 (\leq 0), \quad \forall \quad x, y \in M$$

proof:

It is clear that $F(x) - \delta/H(x)\nu_x$ and $\exp_{F(x)}(-\cot^{-1}(H(x)/\delta)\nu_x)$, considered as vectors in $\mathbb{R}^{n+2}$, are parallel because $\tan\left(\cot^{-1}(H(x)/\delta)\right) = \delta/H(x)$. So the intrinsic distance on $\mathbb{S}^{n+1}$ from a point p to $\exp_{F(x)}(-\cot^{-1}(H(x)/\delta)\nu_x)$ is monotone increasing with respect to the extrinsic distance in $\mathbb{R}^{n+2}$ from p to $F(x) - \delta/H(x)\nu_x$. Together with

$$Z(x,y) = \frac{H(x)}{2} [\|F(y) - F(x) + \frac{\delta}{H(x)} \nu_x\|^2 - \frac{\delta^2}{H(x)^2}]$$

$Z \geq 0$ if and only in $F(y) \in B(x, \delta)$. The assertion follows. □

Let

$$Z(t,x,y) := \frac{H(x,t)}{2} \|F(y,t) - F(x,t)\|^2 + \delta \langle F(y,t) - F(x,t), \nu_x(t) \rangle$$

In the following, H and h_{ij} are the mean curvature and second fundamental form at $F(x,t)$. To use maximal principle we need the evolution equation of $Z(t,x,y)$.

Lemma 3

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta_x-\Delta_y)Z \\
&=(|A|^2+n)Z-2(n-\delta)H+2g_x^{ij}\frac{\partial H}{\partial x^j}\left\langle F(y)-F(x),\frac{\partial F}{\partial x^i}\right\rangle+2(n-\delta)H(1-\langle F(y),F(x)\rangle) \qquad (1)
\end{aligned}
$$

proof:

By direct computation,

$$
\begin{aligned}
\Delta_x Z=&\frac{\Delta_x H}{2}\|F(y)-F(x)\|^2+H\langle F(y)-F(x),-\Delta_x F(x)\rangle+nH+\langle\nabla_x H,\nabla_x\|F(y)-F(x)\|^2\rangle \\
&+\delta\langle-\Delta_x F(x),\nu_x\rangle+\delta\langle F(y)-F(x),\Delta_x\nu_x\rangle-2\langle\nabla_x F(x),\nabla_x\nu_x\rangle \qquad (2)
\end{aligned}
$$

$$
\Delta_y Z=H\langle F(y)-F(x),\Delta_y F(y)\rangle+nH+\delta\langle\Delta_y F(y),\nu_x\rangle \qquad (3)
$$

and

$$
\begin{aligned}
\frac{\partial Z}{\partial t}=&\frac{\Delta_x H+(|A|^2+n)}{2}\|F(y)-F(x)\|^2+H\langle F(y)-F(x),\Delta_y F(y)-\Delta_x F(x)\rangle+nH\|F(y)-F(x)\|^2 \\
&+\delta\langle\Delta_y F(y)-\Delta_x F(x),\nu_x\rangle+n\delta\langle F(y)-F(x),\nu_x\rangle+\delta\langle F(y)-F(x),\Delta_x\nu_x+|A|^2\nu_x+2HF(x)\rangle \qquad (4)
\end{aligned}
$$

So we get

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\Delta_x-\Delta_y)Z=&(|A|^2+n)Z+nH\|F(y)-F(x)\|^2+2\delta H\langle F(y)-F(x),F(x)\rangle \\
&-2nH+2g_x^{ij}\frac{\partial H}{\partial x^j}\left\langle F(y)-F(x),\frac{\partial F}{\partial x^i}\right\rangle+2\delta H \\
=&(|A|^2+n)Z-2(n-\delta)H+2g_x^{ij}\frac{\partial H}{\partial x^j}\left\langle F(y)-F(x),\frac{\partial F}{\partial x^i}\right\rangle+2(n-\delta)H(1-\langle F(y),F(x)\rangle)
\end{aligned}
$$

□

Lemma 4 *At a critical point of Z we can choose a normal coordinate(with respect to the product metric) $\{\partial_j^x,\ \partial_j^y\}_{j=1}^n$ such that $\partial_j^x=\partial_j^y,\ j=1,2,\ldots n-1$ as vectors in $\mathbb{R}^{n+2}$ at this critical point.*

In the remaining of this section, we always use this normal coordinate to calculate at critical points.

proof:

It is sufficient to show that $T_xM\cap T_yM$ is a subspace with dimension at least $n-1$. Let $N=(F(x)-\delta/H\nu_x)$. Since $\nabla_y Z=0$,

$$
F(y)-F(x)+\frac{\delta}{H}\nu_x\perp T_yM\Rightarrow F(y)-F(x)+\frac{\delta}{H}\nu_x\in\text{span}\{F(y),\nu_y\}
$$

$$
\Rightarrow N\in\text{span}\{F(y),\nu_y\}\Rightarrow T_xM,\ T_yM\subset N^\perp
$$

Then the assertion follows. □

At a critical point of Z let $\tilde{\nu_y}$ be a unit vector orthogonal to T_yM and N. Also let $\tilde{\nu_x}$ be a unit vector orthogonal to T_xM, and N. We have Span$\{F(y),\nu_y\}$ =Span$\{N,\tilde{\nu_y}\}$ and Span$\{F(x),\nu_x\}$ =Span$\{N,\tilde{\nu_x}\}$. Thus

$$
F(y)-F(x)+\frac{\delta}{H}\nu_x=\rho_1\tilde{\nu_y}+\rho_2 N \qquad (5)
$$

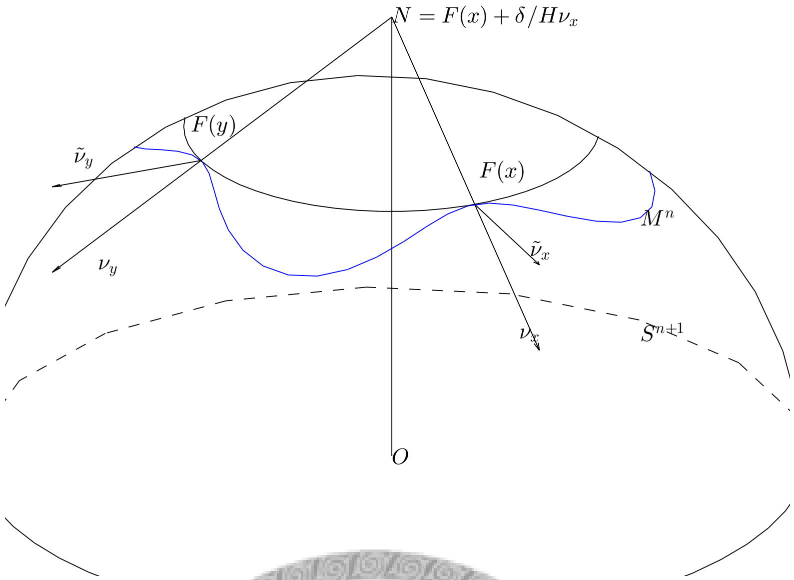

Figure 1: Z=0

for some real number ρ_1 and ρ_2. Note that $\rho_1 = 0$ if and only if $F(y)$ is parallel to N. In this case

$$Z = H(1 - \sqrt{1 + \frac{H^2}{\delta^2}})$$

So $\rho_1 \neq 0$ as Z is closed to 0. The figure above shows the case $Z(x, y) = 0$ and $Z(x, .) \geq 0$

Define the differential operator L by

$$L := \frac{\partial}{\partial t} - \Delta_x - \Delta_y - 2g_x^{ik} g_y^{jl} \left\langle \partial_i^x, \partial_j^y \right\rangle \frac{\partial^2}{\partial x^k \partial y^l}$$

Lemma 5 *The spacial part of L is subelliptic at critical points of Z.*

proof:

First note that the differential operator is independent of coordinate choice. If $\{\tilde{\partial}_1^x, \tilde{\partial}_2^x, \ldots, \tilde{\partial}_n^x, \tilde{\partial}_1^y, \tilde{\partial}_2^y, \ldots, \tilde{\partial}_n^y\}$ is another coordinate. Let $\tilde{\partial}_\mu^x = A_\mu^i \partial_i^x$ and $\tilde{\partial}_\nu^y = B_\nu^i \partial_i^y$. Then

$$\tilde{g}_x^{\mu\rho} \tilde{g}_y^{\nu\sigma} \left\langle \tilde{\partial}_\mu^x, \tilde{\partial}_\nu^y \right\rangle \frac{\partial^2}{\partial \tilde{x}^\rho \partial \tilde{y}^\sigma}$$

$$= (A^{-1})_p^\mu (B^{-1})_q^\nu (A^{-1})_r^\rho (B^{-1})_s^\sigma A_\mu^i B_\nu^j A_\rho^k B_\sigma^l g_x^{pr} g_y^{qs} \left\langle \partial_i^x, \partial_j^y \right\rangle \frac{\partial^2}{\partial x^k \partial y^l}$$

$$= g_x^{ik} g_y^{jl} \left\langle \partial_i^x, \partial_j^y \right\rangle \frac{\partial^2}{\partial x^k \partial y^l}$$

At a critical point the spacial part of L corresponds to the matrix

$$
\left(\begin{array}{cccc|cccc}
1 & & & & 1 & & & \\
& 1 & & & & 1 & & \\
& & \ddots & & & & \ddots & \\
& & & 1 & & & & \langle \partial_n^x, \partial_n^y \rangle \\
\hline
1 & & & & 1 & & & \\
& 1 & & & & 1 & & \\
& & \ddots & & & & \ddots & \\
& & & \langle \partial_n^x, \partial_n^y \rangle & & & & 1
\end{array}\right)
$$

If $\langle \partial_n^x, \partial_n^y \rangle \in (-1, 1)$, then the eigenspaces with eigenvalue 0 or 2 are n-1 dimensional. And the remaining two eigenvalues are $1 + \langle \partial_n^x, \partial_n^y \rangle$ and $1 - \langle \partial_n^x, \partial_n^y \rangle$. If $\langle \partial_n^x, \partial_n^y \rangle = \pm 1$, then the eigenspaces with eigenvalue 0 or 2 are n dimensional. $\square$

Lemma 6 *At a critical point of Z*

$$
LZ = \left[|A|^2 + n + 4\frac{H - \delta h_{nn}}{H}\left(\frac{H^2}{\delta^2} + 1\right) \left\langle \partial_n^y, \frac{F(y) - F(x)}{\|F(y) - F(x)\|} \right\rangle^2 (1 + \rho_2) \right] Z
$$
$$
+2(n - \delta)H(1 - \langle F(y), F(x) \rangle) \tag{6}
$$

proof:

The general form of $\frac{\partial^2 Z}{\partial x^i \partial y^j}$ is

$$
\frac{\partial Z}{\partial x^i} = \frac{1}{2}\frac{\partial H}{\partial x^i}\|F(y) - F(x)\|^2 - H\left\langle F(y) - F(x), \partial_j^x \right\rangle + \delta \left\langle F(y) - F(x), g_x^{km} h_{mi} \partial_k^x \right\rangle
$$
$$
\frac{\partial^2 Z}{\partial x^i \partial y^j} = \frac{\partial H}{\partial x^i} \left\langle F(y) - F(x), \partial_j^y \right\rangle - H \left\langle \partial_j^y, \partial_i^x \right\rangle + \delta g^{km} h_{mi} \left\langle \partial_j^y, \partial_k^x \right\rangle
$$

at a critical point of Z

$$
g_x^{ik} g_y^{jl} \left\langle \partial_i^x, \partial_j^y \right\rangle \frac{\partial^2 Z}{\partial x^k \partial y^l} = \sum_{j=1}^{n-1} \left(\frac{\partial H}{\partial x^j} \left\langle F(y) - F(x), \partial_j^y \right\rangle - H + \delta h_{jj} \right)
$$
$$
+\frac{\partial H}{\partial x^n} \left\langle F(y) - F(x), \left\langle \partial_n^y, \partial_n^x \right\rangle \partial_n^y \right\rangle - \left\langle \partial_n^y, \partial_n^x \right\rangle^2 H - \delta \left\langle \partial_n^y, \partial_n^x \right\rangle^2 h_{nn} \tag{7}
$$

Combining (1) and (7), we have

$$
LZ = (|A|^2 + n)Z - 2(H - \delta h_{nn}) \left\langle \partial_n^x, \tilde{\nu}_y \right\rangle^2
$$
$$
+2\frac{\partial H}{\partial x^n} \left\langle F(y) - F(x), \tilde{\nu}_y \right\rangle \left\langle \partial_n^x, \tilde{\nu}_y \right\rangle + 2(n - \delta)H(1 - \langle F(y), F(x) \rangle) \tag{8}
$$

From $0 = \frac{\partial Z}{\partial x^n}$ we get

$$
\frac{\partial H}{\partial x^n} = 2\|F(y) - F(x)\|^{-2}(H - \delta h_{nn}) \left\langle F(y) - F(x), \partial_n^x \right\rangle
$$

From (5)

$$
\left\langle F(y) - F(x), \tilde{\nu}_y \right\rangle = \left\langle F(y) - F(x), \frac{1}{\rho_1}(F(y) - F(x)) + \frac{\delta}{\rho_1 H}\nu_x - \frac{\rho_2}{\rho_1}N \right\rangle
$$

$$= \frac{Z}{\rho_1 H} + \frac{1}{2\rho_1}\|F(y)-F(x)\|^2 - \frac{\rho_2}{\rho_1}\langle F(y)-F(x), N\rangle$$

Put them together

$$2\frac{\partial H}{\partial x^n}\langle F(y)-F(x),\tilde{\nu}_y\rangle\langle \partial_n^x,\tilde{\nu}_y\rangle$$

$$= 4\rho_1\|F(y)-F(x)\|^{-2}\left(\frac{Z}{\rho_1 H} + \frac{1}{2\rho_1}\|F(y)-F(x)\|^2 - \frac{\rho_2}{\rho_1}\langle F(y)-F(x), N\rangle\right)(H-\delta h_{nn})\langle \partial_n^x,\tilde{\nu}_y\rangle^2$$

$$= 4\|F(y)-F(x)\|^{-2}\frac{H-\delta h_{nn}}{H}\langle \partial_n^x,\tilde{\nu}_y\rangle^2 Z + 2(H-\delta h_{nn})\langle \partial_n^x,\tilde{\nu}_y\rangle^2$$

$$-4\rho_2\|F(y)-F(x)\|^{-2}(H-\delta h_{nn})\langle \partial_n^x,\tilde{\nu}_y\rangle^2\langle F(y)-F(x), N\rangle \tag{9}$$

Insert (9) into (8)

$$LZ = \left(|A|^2 + n + 4\|F(y)-F(x)\|^{-2}\frac{H-\delta h_{nn}}{H}\langle \partial_n^x,\tilde{\nu}_y\rangle^2\right)Z$$

$$-4\rho_2\|F(y)-F(x)\|^{-2}(H-\delta h_{nn})\langle \partial_n^x,\tilde{\nu}_y\rangle^2\langle F(y)-F(x), N\rangle + 2(n-\delta)H(1-\langle F(y),F(x)\rangle) \tag{10}$$

Together with

$$\|F(y)-F(x)\|^{-2}\langle \partial_n^x,\tilde{\nu}_y\rangle^2 = \|F(y)-F(x)\|^{-2}\langle \partial_n^y,\tilde{\nu}_x\rangle^2$$

$$= (1+\frac{\delta^2}{H^2})\|F(y)-F(x)\|^{-2}\langle \partial_n^y,\nu_x\rangle^2 = (\frac{H^2}{\delta^2}+1)\left\langle \partial_n^y, \frac{F(y)-F(x)}{\|F(y)-F(x)\|}\right\rangle^2$$

we have

$$LZ = \left(|A|^2 + n + 4\frac{H-\delta h_{nn}}{H}(\frac{H^2}{\delta^2}+1)\left\langle \partial_n^y, \frac{F(y)-F(x)}{\|F(y)-F(x)\|}\right\rangle^2\right)Z$$

$$-4\rho_2(\frac{H^2}{\delta^2}+1)(H-\delta h_{nn})\left\langle \partial_n^y, \frac{F(y)-F(x)}{\|F(y)-F(x)\|}\right\rangle^2\langle F(y)-F(x), N\rangle + 2(n-\delta)H(1-\langle F(y),F(x)\rangle) \tag{11}$$

Let $l = \|F(y)-F(x)+\delta/H\nu_x\|$, then Z can be expressed by $Z = (H/2)(l^2-\delta^2/H^2)$. By the cosine law

$$\langle F(x),N\rangle = 1, \quad \langle F(y),N\rangle = \frac{1}{2}\left(1^2 + (1+\delta^2/H^2) - l^2\right)$$

$$\langle F(y)-F(x),N\rangle = \frac{1}{2}(\frac{\delta^2}{H^2} - l^2) = \frac{-1}{H}Z \tag{12}$$

Plug (12) into (11)

$$LZ = \left[|A|^2 + n + 4\frac{H-\delta h_{nn}}{H}(\frac{H^2}{\delta^2}+1)\left\langle \partial_n^y, \frac{F(y)-F(x)}{\|F(y)-F(x)\|}\right\rangle^2(1+\rho_2)\right]Z$$

$$+2(n-\delta)H(1-\langle F(y),F(x)\rangle)$$

is obtained. □

proof of proposition 2:

Because

$$|\rho_2| = \left|\left\langle F(y)-F(x)+\frac{\delta}{H}\nu_x, N\right\rangle\right|/(1+\delta^2/H^2) \le (2+\frac{\delta}{H})/(1+\delta^2/H^2)$$

for any $0<\tau<T$, there is a constant $C>0$ such that

$$|c(x,y,t)| := \left||A|^2 + n + 4\frac{H-\delta h_{nn}}{H}(\frac{H^2}{\delta^2}+1)\left\langle \partial_n^y, \frac{F(y)-F(x)}{\|F(y)-F(x)\|}\right\rangle^2(1+\rho_2)\right| < C$$

for all $(x,y,t) \in \bar{M} \times \bar{M} \times [0,\tau]$. Let $\bar{Z} = e^{-Ct}Z$. Suppose that $Z \geq 0$ as $t = 0$ and that $\bar{Z}$ attains $-\epsilon < 0$ first time at (x_0, y_0, t_0), $t_0 \leq \tau$. By the above lemma, we have

$$0 \geq L\bar{Z} = (c-C)\bar{Z} + e^{-Ct_0} 2(n-\delta)H(1 - \langle F(y_0), F(x_0)\rangle)$$

$$\geq (c-C)(-\epsilon) > 0$$

It is a contradiction. In the second inequality we use the fact that $Z \geq 0$ implies $\delta \leq n$. So $Z \geq 0$ for all $t \in [0,\tau]$. Since τ is arbitrary we get $Z \geq 0$ for all $t \in [0,T)$. Hence we know that interior δ non-collapsing is preserved. Similarly, suppose that $Z \leq 0$ as $t = 0$ and that $\bar{Z}$ attains $\epsilon > 0$ first time at (x_0, y_0, t_0), $t_0 \leq \tau$. By the above lemma, we have

$$0 \leq L\bar{Z} = (c-C)\bar{Z} + e^{-Ct_0} 2(n-\delta)H(1 - \langle F(y_0), F(x_0)\rangle)$$

$$\leq (c-C)\epsilon < 0$$

by the same argument, proposition 2 follows. □

4. δ Convexity in $\mathbb{H}^{n+1}$

In this section, $\langle , \rangle$ is the standard inner product in Minkowski space $\mathbb{R}^{n+1,1}$ and $\|p\|^2 = \langle p, p\rangle$ for all $p \in \mathbb{R}^{n+1,1}$. Just like above, we can embed the space form $\mathbb{H}^n$ into $\mathbb{R}^{n+1,1}$. $\mathbb{H}^n = \{p \in \mathbb{R}^{n+1,1} : \|p\|^2 = -1\}$. Let $F : [0,T) \times \bar{M}^n \to \mathbb{H}^{n+1} \hookrightarrow \mathbb{R}^{n+1,1}$ be a family of embeddings evolved by the mean curvature flow in $\mathbb{H}^{n+1}$. Let g_{ij} be the induced metric , $A = \{h_{ij}\}$ be the second fundamental form, H be the mean curvature and ν be the unit outer normal vector of $F(t,M)$. In the following of this section, $F(x)$ and ν are considered to be vectors in $\mathbb{R}^{n+1,1}$.

Lemma 7 *Then the evolution equations are:*
(a) $\frac{\partial F}{\partial t} = \Delta F - nF$
(b) $\frac{\partial \nu}{\partial t} = \Delta \nu + |A|^2 \nu - 2HF$
(c) $\frac{\partial H}{\partial t} = \Delta H + (|A|^2 - n)H$

proof:
Since F is evolved by MCF in $\mathbb{H}^{n+1}$,

$$\frac{\partial F}{\partial t} = \Delta F + \langle \Delta F, F\rangle F$$

and

$$0 = \Delta \langle F, F\rangle = 2\langle \Delta F, F\rangle + 2n$$

Then we have (a). For (b)

$$\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x^i}\right\rangle = -\left\langle \nu, \frac{\partial}{\partial x^i}(-H\nu)\right\rangle = \frac{\partial H}{\partial x^i}$$

$$\left\langle \frac{\partial \nu}{\partial t}, \nu\right\rangle = 0, \quad \left\langle \frac{\partial \nu}{\partial t}, F\right\rangle = -\langle \nu, -H\nu\rangle = H$$

Thus

$$\frac{\partial \nu}{\partial t} = \nabla H - HF$$

$$\langle \nabla_i \nu, \nu \rangle = 0, \quad \langle \nabla_i \nu, F \rangle = -\left\langle \nu, \frac{\partial F}{\partial x^i} \right\rangle = 0$$

By

$$\left\langle \nabla_i \nu, \frac{\partial F}{\partial x^j} \right\rangle = -\left\langle \nu, \nabla_i \frac{\partial F}{\partial x^j} \right\rangle = h_{ij}$$

we have $\nabla_i \nu = g^{km} h_{mi} \frac{\partial F}{\partial x^k}$.

$$\Delta \nu = g^{ij} \nabla_j \nabla_i \nu = g^{ij} \nabla_j g^{km} h_{mi} \frac{\partial F}{\partial x^k}$$

$$g^{ij}(g^{km} \nabla_j h_{mi} \frac{\partial F}{\partial x^k} + g^{km} h_{mi}(-h_{jk}\nu + g_{jk}F))$$

by Codazzi equation

$$= \nabla H - |A|^2 \nu + HF$$

then we obtain (b)

$$\frac{\partial \nu}{\partial t} = \Delta \nu + |A|^2 \nu - 2HF$$

For (c)

$$\frac{\partial h_{ij}}{\partial t} = \frac{\partial}{\partial t} \left\langle \frac{\partial F}{\partial x^i}, \nabla_j \nu \right\rangle = \left\langle \nabla_i(-H\nu), h_{jk} g^{kl} \frac{\partial F}{\partial x^l} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \nabla_j(\nabla H - HF) \right\rangle$$

$$= -H h_{jk} h_{il} g^{kl} - H g_{ij} + \nabla_j \nabla_i H$$

$$\frac{\partial H}{\partial t} = -g^{ik} g^{lj}(-2H g_{kl}) h_{ij} + g^{ij}(-H h_{jk} h_{il} g^{kl} - H g_{ij} + \nabla_j \nabla_i H) = \Delta H + (|A|^2 - n)H$$

□

Note that δ-convexity of $F : M^n \to \mathbb{H}^{n+1} \hookrightarrow \mathbb{R}^{n+1,1}$ is equivalent to

$$Z_0 := \frac{1}{2}\|F(y) - F(x)\|^2 + \delta \langle F(y) - F(x), \nu_x \rangle \leq 0$$

Lemma 8 *At a critical point of Z_0, we can choose a normal coordinate $\{\partial_j^x, \partial_j^y\}_{j=1}^n$(with respect to the product metric) such that $\partial_j^x = \partial_j^y$, $j = 1, 2, \ldots, n-1$ as vectors in $\mathbb{R}^{n+1,1}$ at this point. And we also have $\langle \partial_n^x, \partial_n^y \rangle = 1$ if $\delta = 1$ and $0 \leq \langle \partial_n^x, \partial_n^y \rangle \leq 1$ if $0 < \delta < 1$.*

proof:

Following the same argument in section 3, we have $T_x M \cap T_y M$ is at least n-1 dimensional. Let $N = F(x) - \delta \nu_x$. ∂_n^x and ∂_n^y lie in the 2-dimensional subspace $\pi = \{\partial_1^x, \partial_2^x, \ldots, \partial_{n-1}^x, N\}^\perp$. If $\delta = 1$, then N is a null vector and the restrict inner product on π degenerates. So we can choose $\langle \partial_n^x, \partial_n^y \rangle = 1$. For $\delta < 1$, N is a timelike vector and π is isometric to $\mathbb{R}^2$. Thus $\langle \partial_n^x, \partial_n^y \rangle \leq 1$ □

By the above lemma, the spacial part of operator L defined in section 3 is still subellptic. Note that if N is spacelike then it is not necessarily subelliptic since $\langle \partial_n^x, \partial_n^y \rangle$ might be grater than 1.

Lemma 9 *Assume $Z_0 > -2(1 - \sqrt{1 - \delta^2})$ at a critical point, then*

$$LZ_0 = (|A|^2 - n)Z_0 - \frac{1}{2n}(n|A|^2 - 2\delta n H + n^2)\|F(y) - F(x)\|^2$$

proof:

$$\Delta_x Z_0 = \langle F(y) - F(x), -\Delta_x F(x)\rangle + n + \delta \langle -\Delta_x F(x), \nu_x\rangle + \langle F(y) - F(x), \Delta_x \nu_x\rangle - 2\delta H$$

$$\Delta_y Z_0 = \langle F(y) - F(x), \Delta_y F(y)\rangle + n + \delta \langle \Delta_y F(y), \nu_x\rangle$$

$$\frac{\partial Z_0}{\partial t} = \langle F(y) - F(x), \Delta_y F(y) - \Delta_x F(x)\rangle - n\|F(y) - F(x)\|^2 + \delta \langle \Delta_y F(y) - \Delta_x F(x), \nu_x\rangle$$
$$-n\delta \langle F(y) - F(x), \nu_x\rangle + \delta \left\langle F(y) - F(x), \Delta_x \nu_x + |A|^2 \nu_x - 2HF\right\rangle$$

So

$$(\frac{\partial}{\partial t} - \Delta_x - \Delta_y) Z_0 = (|A|^2 - n) Z_0$$
$$-\frac{|A|^2}{2}\|F(y) - F(x)\|^2 - \frac{n}{2}\|F(y) - F(x)\|^2 + \delta H\|F(y) - F(x)\|^2 - 2n + 2\delta H$$

And at a critical point

$$\frac{\partial Z_0}{\partial x^j} = \left\langle F(y) - F(x), -\partial_j^x\right\rangle + \delta h_j^k \langle F(y) - F(x), \partial_k^x\rangle$$

$$\frac{\partial^2 Z_0}{\partial x^j \partial y^j} = \langle \partial_j^y, -\partial_j^x\rangle + \delta h_j^k \langle \partial_j^y, \partial_k^x\rangle = \begin{cases} -1 + \delta h_{jj} & \text{if } j \neq n \\ \langle \partial_n^y, \partial_n^x\rangle (-1 + \delta h_{nn}) & \text{if } j = n \end{cases}$$

Then we have

$$LZ_0 = \left(\frac{\partial}{\partial t} - \Delta_x - \Delta_y - 2 g_x^{ik} g_y^{jl} \langle \partial_k^x, \partial_l^y\rangle \frac{\partial^2}{\partial_i^x \partial_j^y}\right) Z_0$$
$$= (|A|^2 - n) Z_0 - \frac{1}{2n}(n|A|^2 - 2\delta n H + n^2)\|F(y) - F(x)\|^2 + (1 - \langle \partial_n^x, \partial_n^y\rangle^2)(-1 + \delta h_{nn})$$

If $(-1 + \delta h_{nn}) = 0$ then we have done. By

$$0 = \frac{\partial Z_0}{\partial x^n} = (-1 + \delta h_{nn}) \langle F(y) - F(x), \partial_n^x\rangle$$

If $(-1 + \delta h_{nn}) \neq 0$, then $\partial_n^x \perp F(y)$. So $T_x M \subset \{F(y), N\}^{\perp}$. Because $F(y)$ is parallel to N if and only if $Z_0 = -2(1 - \sqrt{1 - \delta^2})$, $F(y)$ and N are linearly independent. Then $T_x M = \text{span}\{F(y), N\}^{\perp} = T_y M$ and $\langle \partial_n^x, \partial_n^y\rangle = 1$. In both cases we have

$$LZ_0 = (|A|^2 - n) Z_0 - \frac{1}{2n}(n|A|^2 - 2\delta n H + n^2)\|F(y) - F(x)\|^2$$

□

proof of proposition 3:

For the case $\delta = 1$, fix $0 < \tau < T$. There is a constant $C > 0$ such that

$$|(|A|^2 - n)| \leq C$$

in $M \times [0, \tau]$. Let $\bar{Z}_0 = e^{-Ct} Z_0$. By M_0 is 1-convex we know $\bar{Z}_0 \leq 0$ as $t = 0$. If $\bar{Z}_0 = \epsilon > 0$ first time at $(x_0, y_0, t_0) \in M \times M \times [0, \tau]$, by lemma 9 we have at (x_0, y_0, t_0)

$$0 \leq L\bar{Z}_0 \leq (|A|^2 - n - C)\bar{Z}_0 - e^{-Ct}\frac{1}{2n}(H - n)^2\|F(y) - F(x)\|^2 \leq (|A|^2 - n - C)\epsilon < 0$$

It is a contradiction. So $\bar{Z}_0$ and Z_0 remain nonpositive in $[0,\tau]$. Thus Z_0 remains nonpositive as long as the solution exists.

For the case $\delta < 1$,

$$0 \le L\bar{Z}_0 \le (|A|^2-n-C)\bar{Z}_0 - e^{-Ct}\frac{1}{2n}(H-n)^2\|F(y)-F(x)\|^2 - 2ne^{-Ct}(1-\delta)H\|F(y)-F(x)\|^2 \le (|A|^2-n-C)\epsilon < 0$$

we use that 1-convexity implies $H > n > 0$ in the second inequality. Then the result follows. □

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