國立臺灣大學理學院數學研究所

博士論文

Department of Mathematics

College of Science

National Taiwan University

Doctoral Dissertation

在 N 維空間中的費滋漢那古默系統之行波解

Travelling wave solutions of the diffusive FitzHugh-Nagumo system in $\mathbf{R}^N$

黃志強

Chih-Chiang Huang

指導教授：陳俊全 博士

Advisor: Chiun-Chuan Chen Ph.D.

中華民國 101 年 6 月

June 2012

Acknowledgements

I wish to express sincere gratitude to the following people.

- My advisor: Professor Chiun-Chuan Chen;
- Members of dissertation committee: Professor I-Liang Chern, Professor Jann-Long Chern, Professor Jong-Shenq Guo, Professor Chun-Hsiung Hsia, Professor Tai-Chia Lin, Professor Chih-Wen Shih;
- my classmates and friends: Chuen-Hsin Chang, Der-Chiun Chen, Li-Chang Hung, Chiun-Chang Lee;
- My family: my parents, my wife, my sister and my brother;

Without them, much of my dissertation could not have been written.

Abstract

In this thesis, we study the existences of travelling waves of the diffusive FitzHugh-Nagumo system (DFHN) in $\mathbf{R}^N$. This system has a skew-gradient structure as defined by Yanagida as well as a non-local gradient structure. In addition, by a suitable transformation, it also has a monotone-system structure on some parameter ranges. For bounded domains, the variational approach is applied to construct steady states of (DFHN) with Dirichlet or/and Neumann condition. For unbounded cylindrical domains, we study the travelling wave solutions via all of the three structures mentioned above when the diffusion coefficients in the equations are equal. By using the nonlocal variational energy, we establish the existence of a travelling front solution for (DFHN). Our existence result also obtains a variational characterization for the wave speed. On the other hand, using the skew-gradient structure, we give a mini-max formulation of the travelling wave and its speed. For whole domains, we employ the method of super- and subsolutions to establish the existence of monostable-type traveling wave solutions in $\mathbf{R}^N$. Moreover, we construct infinitely many standing periodic solutions in $\mathbf{R}^1$ based on the reflection method.

keywords: FitzHugh-Nagumo system; travelling waves; skew-gradient structure; variational method; the method of super- and subsolutions

Contents

Chapter 1

Introduction

In this thesis, we are concerned with the diffusive FitzHugh-Nagumo system (DFHN) in $\mathbf{R}^N$.

$$u_t = u_{\xi\xi} + \Delta_y u + f(u) - v, \tag{1.0.1}$$
$$v_t = d(v_{\xi\xi} + \Delta_y v) + \delta(u - \gamma v), \tag{1.0.2}$$

where $(\xi, y) \in \Omega := \mathbf{R}^1 \times \Omega_y$ with Ω_y being $\mathbf{R}^{N-1}$ or being a bounded C^{2,α_0} domain in $\mathbf{R}^{N-1}$, $d \geq 0$, $\delta, \gamma > 0$, and $f(u) = u(1-u)(u-\beta)$ for $0 < \beta < \frac{1}{2}$. Moreover, if Ω_y is a bounded domain, we impose the Dirichlet or Neumann boundary condition on it.

(DFHN), a typical model for excitable media, arises as a simplification of the Hodgkin-Huxley for nerve-impulse propagation ([8], [12] and [32]). Here u is the membrane potential of the nerve cells and v represents the effects of sodium ions and potassium ions. In the past decades, (DFHN) has become one of the frequently-used reaction-diffusion systems to describe different phenomena in many fields, such as physics, chemistry and biology ([13], [27], [34], [35] and [37]).

Here we interest in the existence of steady states, standing waves and travelling waves in $\mathbf{R}^N$. By setting $x = \xi - ct$, (DFHN) is reduced to a elliptic system with a unknown variable c, called the "wave speed".

$$u_{xx} + \Delta_y u + cu_x + f(u) - v = 0, \tag{1.0.3}$$
$$d(v_{xx} + \Delta_y v) + cv_x + \delta(u - \gamma v) = 0. \tag{1.0.4}$$

Among variant interesting structures related to (DFHN), we list three ones we used in this thesis as follows.

1. The skew-gradient structure

For variational approach, the functions u and v need to be in the same weighted space, i.e., $d = 1$. Moreover, setting $v = \sqrt{\delta}\tilde{v}$ and dropping the tilde we obtain system (1.0.3)-(1.0.4) enjoying the skew-gradient structure defined by Yanagida [45]; namely, the system

$$u_{xx} + \Delta_y u + cu_x + f(u) - \sqrt{\delta}v = 0, \tag{1.0.5}$$
$$v_{xx} + \Delta_y v + cv_x + \sqrt{\delta}u - \delta\gamma v = 0 \tag{1.0.6}$$

satisfying $\frac{\partial}{\partial v}(f(u) - \sqrt{\delta}v) = -\frac{\partial}{\partial u}(\sqrt{\delta}u - \delta\gamma v)$.

The corresponding energy of the above system (1.0.5)-(1.0.6) is

$$E_1[u,v] = \frac{1}{2}\int_\Omega e^{cx}(u_x^2 - v_x^2) + \frac{1}{2}\int_\Omega e^{cx}(|\nabla_y u|^2 - |\nabla_y v|^2) + \int_\Omega e^{cx} H(u,v),$$

where $H(u,v) = F(u) + \sqrt{\delta}uv - \frac{1}{2}\gamma\delta v^2$ and $F(u) = -\int_0^u f(s)ds = \frac{1}{4}u^4 - \frac{\beta+1}{3}u^3 + \frac{\beta}{2}u^2$.

2. The nonlocal-gradient structure
Observing that (1.0.6) is a linear equation, we can formally solve v, expressed in term of u. Denote v by $B_c[u]$. Consequently, system (1.0.5)-(1.0.6) is reduced to a single equation

$$u_{xx} + \Delta_y u + cu_x + f(u) - B_c[u] = 0. \tag{1.0.7}$$

Moreover, equation (1.0.7) is the Euler-Lagrange equation of the nonlocal gradient energy $E_2[u]$, defined by

$$E_2[u] = \frac{1}{2}\int_\Omega e^{cx}(u_x^2 + |\nabla_y u|^2) + \int_\Omega e^{cx} F(u) + \frac{\sqrt{\delta}}{2}\int_\Omega e^{cx} u B_c[u].$$

3. The monotone-system structure
For technical restriction, we assume $d = 1$. By transforming $w = -v + \sigma u$, $\sigma > 0$ determined later, system (1.0.3)-(1.0.4) is rewritten as

$$u_{xx} + \Delta_y u + cu_x + f(u) - \sigma u + w = 0, \tag{1.0.8}$$

$$w_{xx} + \Delta_y w + cw_x + \sigma(f(u) + (\delta\gamma - \sigma - \frac{\delta}{\sigma})u) + (\sigma - \delta\gamma)w = 0. \tag{1.0.9}$$

Usually, we expect the solution u is bounded, for example $0 \le u \le 1$. To ensure system (1.0.8)-(1.0.9) is monotone, we impose $A = f'(u) + (\delta\gamma - \sigma - \frac{\delta}{\sigma}) \ge 0$ on $u \in [0,1]$. The condition that $\gamma \ge \frac{1}{\sigma} + \frac{\sigma}{\delta} + \frac{1-\beta}{\delta}$ is sufficient and necessary for $A \ge 0$ on $u \in [0,1]$. By choosing $\sigma = \sqrt{\delta}$, we obtain a optimal parameter range for γ such that system (1.0.8)-(1.0.9) is a monotone system on $u \in [0,1]$.

The above three structures will be discussed more in chapter 2-5. This thesis is organized as follows. In chapter 2, we survey the existence of waves from literature and focus on the statements of the main theorems and simple descriptions of proofs. Next, in Chapter 3, the steady states of (DFHN) in a bounded domain are established. We employ the direct method to obtain nontrivial minimizers and use the reflection method to construct periodic solutions in $\mathbf{R}^1$. By applying the method of sub- and supersolutions, Chapter 4 is devoted to the existence of monostable-type travelling waves for (DFHN). By using the nonlocal structure of (DFHN), the existence of travelling frons is established in Chapter 5. Moreover, we obtain a variational characterization for the wave speed. From the skew-gradient structure, we set a mini-max formulation of the travelling wave and its speed.

Chapter 2

Literature review

Among variant interesting problems related to (DFHN), the existence of wave solutions is one of the main issues. There has been tremendous work on this problem.

Assume $N = 1$. When $d = 0$, Rinzel and Terman [40] completely analyzed critical values of γ based on the results of Carpenter [2], Casten *et al.* [4] and Keener [18]. If δ is small, they showed that travelling pules, fronts and back waves exist for $\gamma \in (0, \gamma_1)$, $\gamma \in (\gamma_0, \infty)$ and $\gamma \in (\gamma_0, \gamma_2)$ respectively (See Fig. 2.1 for the definition of γ_i, $i = 0, 1, 2$.).

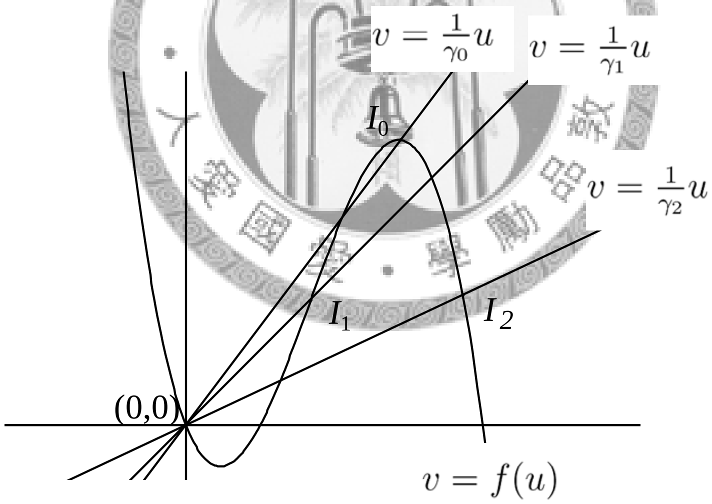

Figure 2.1: There are three critical values γ_0, γ_1 and γ_2 of γ, which are defined as follows: $v = \frac{1}{\gamma_i}u$ is the line passing through the origin and $I_i \in \{v = f(u)\}$ for $i = 0, 1, 2$, where I_0 and I_1 are the local maximum and the inflection point of the curve $v = f(u)$ respectively and I_2 satisfies the condition that the line I_1I_2 is parallel to the u axis.

By a direct calculation, $\gamma_1 = \frac{9}{2\beta^2 - 5\beta + 2}$. As one will see, this γ_1 is also a crucial critical value in our main theorem.

2.1 Steady states on bounded domains

In this section, assume Ω is a bounded smooth domain and the steady states of (DFHN) satisfy the Dirichlet boundary condition. Let $\epsilon = \delta/d$. The system we study is as follows:

$$\Delta u + f(u) - v = 0, \tag{2.1.1}$$

$$\Delta v + \epsilon(u - \gamma v) = 0, \tag{2.1.2}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{2.1.3}$$

The corresponding energy of system (2.1.1)-(2.1.3) is

$$\Phi[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega F(u) + \frac{1}{2}\int_\Omega uB[u], \tag{2.1.4}$$

where $B[u] = \epsilon(-\Delta + \epsilon\gamma)^{-1}[u]$.

2.1.1 Bistable cases

The existence results of the steady states for system (2.1.1)-(2.1.3) were constructed first by Klaasen and Mitidieri [20]. According to variational approaches, they obtained two pairs of solutions for (DFHN), where one is a minimizer of the energy and the other is a mountain pass solution. The main theorems in [20] are stated as follows.

Theorem 2.1.1. ([*20*]) *Assume $\gamma > \frac{9}{2\beta^2-5\beta+2} =: \gamma_1$. There exists $R_0 > 0$ such that if Ω contains a ball B_{R_0} with the radius R_0 then system* (2.1.1)-(2.1.3) *has a nontrivial C^2-solution pair $(u_1, B[u_1])$ which satisfies*

$$\inf_{H_0^1(\Omega)} \Phi[u] = \Phi[u_1] < 0. \tag{2.1.5}$$

Moreover, there exists another nontrivial C^2-solution pair $(u_2, B[u_2])$ which satisfies

$$\inf_{\sigma\in\Sigma}\max_{0\le s\le 1} \Phi[\sigma(s)] = \Phi[u_2] > 0, \tag{2.1.6}$$

where $\Sigma = \{\sigma \in C([0,1]; H_0^1(\Omega)) | \sigma(0) = 0, \sigma(1) = u_1\}$.

On the other hand, nonexistence theorems were also established in [20].

Theorem 2.1.2. ([*20*]) *If Ω is a ball $B_R(0)$ and one of the following assumption is supposed:*
(i) $\epsilon, \gamma > 0$ are fixed and $R > 0$ is sufficiently small;
(ii) $\epsilon\gamma^2 \ge 1$, $\gamma < \frac{4}{(1-\beta)^2}$ and any $R > 0$;
(iii) $\epsilon\gamma^2 < 1$, $2\sqrt{\epsilon} - \epsilon\gamma > \frac{4}{(1-\beta)^2}$ and any $R > 0$.
Then system (2.1.1)-(2.1.3) *has no nontrivial weak solutions.*

Alternatively, Reinecke and Sweers [39] obtained the existence of steady states of system (2.1.1)-(2.1.3) by considering the following eigenvalue problem.

$$\frac{1}{\lambda}\Delta u + f(u) - v = 0, \tag{2.1.7}$$

$$\frac{1}{\lambda}\Delta v + \epsilon(u - \gamma v) = 0, \tag{2.1.8}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{2.1.9}$$

According to their existence results, the "boundary layer solution" of system (2.1.7)-(2.1.9) was established by transforming (2.1.7)-(2.1.8) to a quasimonotone system. Consequently, the new system enjoys the maximum principle. By this structure, a pointwise estimate was obtained. Let

$$\gamma^{(\epsilon)} = \begin{cases} \frac{1-\beta}{\epsilon} + \frac{2}{\sqrt{\epsilon}} & \text{if } 0 < \epsilon < \gamma_1^2; \\ \frac{1-\beta}{\epsilon} + \frac{\gamma_1}{\epsilon} + \frac{1}{\gamma_1} & \text{if } \epsilon \geq \gamma_1^2. \end{cases} \tag{2.1.10}$$

When γ is greater than $\gamma^{(\epsilon)}$, the existence theorem was established as follows.

Theorem 2.1.3. ([*39*]) *Assume $\partial\Omega$ is C^3. For all $\epsilon > 0$ and $\gamma > \gamma^{(\epsilon)}$, there exist $\lambda^* > 0$ and a function $\Lambda \in C^1([\lambda^*, \infty), C^2(\overline{\Omega}) \times C^2(\overline{\Omega}))$ such that $(u_\lambda, v_\lambda) := \Lambda(\lambda)$ is a pair of positive solution of system* (2.1.7)-(2.1.9) *for all $\lambda \geq \lambda^*$. Moreover, $p_1 < \max_\Omega u_\lambda < p_2$, $\frac{p_1}{\gamma} < \max_\Omega v_\lambda < \frac{p_2}{\gamma}$ and $\lim_{\lambda\to\infty} \Lambda(\lambda) = (p_2, \frac{p_2}{\gamma})$ uniformly on all compact subsets of Ω, where $0 < p_1 < p_2$ and p_1, p_2 solve $u^2 - (\beta+1)u + (\beta + \frac{1}{\gamma}) = 0$.*

Remark. By scaling for space variables, that λ is large in Theorem 2.1.3 is equivalent to that Ω contains a large ball in Theorem 2.1.1.

As $\gamma \gg 1$, Reinecke et al. [39] and Klaasen et al. [20] obtained the solutions of system (2.1.1)-(2.1.3) by different approaches. It is natural to ask what relations are between those solutions. Consequently, Matsuzawa [26] proved that the global minimizer in [20] identifies the boundary layer solution in [39] under the following conditions.
$(C1)$ $\gamma > \max\{\frac{1}{\beta}, \frac{2}{\sqrt{\epsilon}} + \frac{\beta}{\epsilon}\}$.
$(C2)$ $\epsilon\gamma - 2\sqrt{\epsilon} > M := \frac{(1-\beta)^2}{2} + \frac{1+\beta}{2}\sqrt{(1-\beta)^2 + \frac{4}{\gamma}} + \frac{3}{\gamma}$.
$(C3)$ $\frac{2\beta^2 - 5\beta + 2}{9} > \frac{\epsilon\gamma - M}{2} - \frac{1}{2}\sqrt{(\epsilon\gamma - M)^2 - 4\epsilon}$.

Theorem 2.1.4. ([*26*]) *If the conditions $(C1)$-$(C3)$ hold, then there exists $\lambda^\flat > 0$ such that u_λ in Theorem 2.1.3 coincides with the global minimizer in Theorem 2.1.1 for all $\lambda > \lambda^\flat$.*

2.1.2 Monostable cases

For monostable case ($0 < \gamma < \frac{4}{(1-\beta)^2}$), by the nonexistence theorem (see Theorem 2.1.2), if Ω is sufficiently small or $\epsilon := \frac{\delta}{d} \geq \frac{1}{\gamma^2}$ then system (2.1.1)-(2.1.3) only has trivial solution. Klaasen [19] obtained a sufficient condition to insure the existence of steady states of (2.1.1)-(2.1.3).

Theorem 2.1.5. ([*19*]) *Let $0 < \gamma < \frac{4}{(1-\beta)^2}$ and $\delta > 0$. If R and d are chosen such that*

$$\left((\frac{R}{R-1})^N - 1 \right) \left(1 + \frac{\delta R^4}{d + \delta\gamma R^2} + \frac{\beta^3}{6}(2-\beta) \right) < \frac{1-2\beta}{6}. \tag{2.1.11}$$

Then for all smooth domain containing B_R system (2.1.1)-(2.1.3) *has two nontrivial classical solutions.*

2.2 Travelling waves in $\mathbf{R}^1$

In this section, we consider the following travelling wave equation.

$$u_{xx} + cu_x + f(u) - v = 0, \tag{2.2.1}$$
$$dv_{xx} + cv_x + \delta(u - \gamma v) = 0. \tag{2.2.2}$$

2.2.1 Bistable cases

For the bistable case, there are also various types of solutions. By the shooting method, Klaasen and Troy [22] obtained the existences of standing pulses and infinitely many periodic solutions. The main theorem is stated as follows.

THEOREM 2.2.1. ([*22*]) *Let* $\gamma > \max\{\gamma_1, \frac{2}{\sqrt{\delta}} + \frac{1-\beta}{\delta}\}$. *Then system* (2.2.1)-(2.2.2) *with* $c = 0$ *has a nonconstant pair* (u, v) *satisfying* $(u, u_x, v, v_x)(\pm\infty) = 0$ *and* $(u_x, v_x)(0) = 0$. *Moreover, system* (2.2.1)-(2.2.2) *with* $c = 0$ *has an infinite number of periodic solution.*

The global bifurcation structure of front and back waves were studied by Ikeda, Mimura and Nishiura [17] when $\gamma > \gamma_0$.

THEOREM 2.2.2. ([*17*]) *Let* $d = \frac{\tau}{\sigma}$, $\delta = \tau\sigma$ *and* $c = s\tau$, *where* $\tau, \sigma > 0$ *and* $s \in \mathbf{R}$ *are parameters. Assume* $\tau = O(\sigma)$ *or* $O(\frac{1}{\sigma})$, *then there exists* $\sigma_0 > 0$ *such that for all* $0 < \sigma \leq \sigma_0$ *the "following bifurcation phenomena"(see Fig. 2.2) holds.*

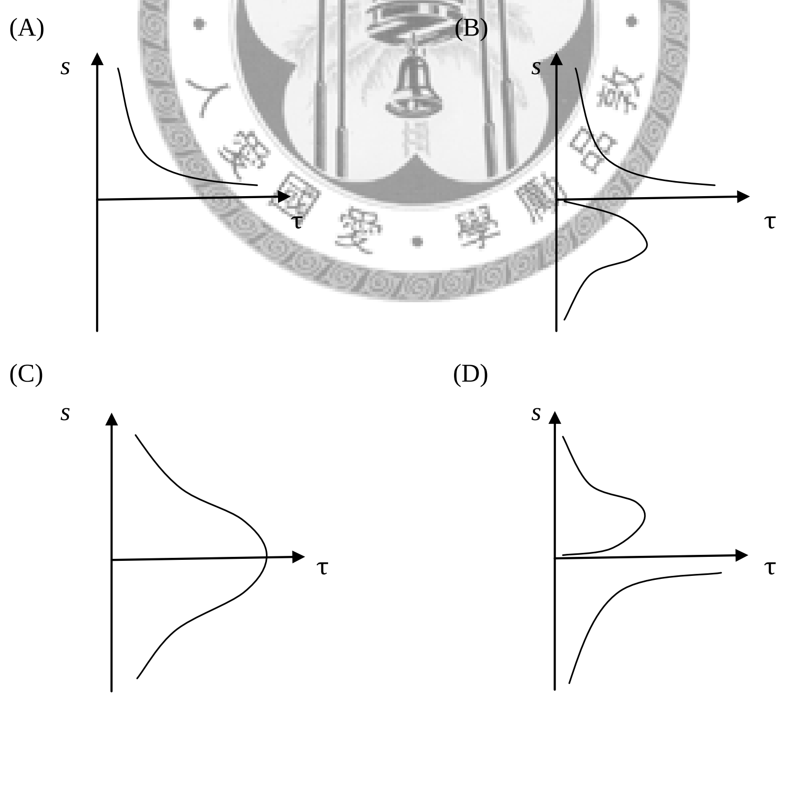

Figure 2.2: (A)$\gamma \geq \gamma_2$ (B)$\gamma_1 < \gamma < \gamma_2$ (C)$\gamma = \gamma_1$ (D)$\gamma_0 < \gamma < \gamma_1$

2.2.2 Monostable cases

For the monostable case, Ermentrout, Hastings and Troy [7] proved the existence of two standing pulses by using the shooting method. Later Dockery [6] obtained a similar result by a different approach: a geometric singular perturbation.

Theorem 2.2.3. ([6] and [7]) *Let* $0 < \gamma < \frac{4}{(1-\beta)^2}$. *If* δ/d *is sufficiently small, system* (2.2.1)-(2.2.2) *with* $c = 0$ *has least two nonconstant, bounded solutions satisfies the following:*
(i) $\lim_{|x|\to\infty}(u, u_x, v, v_x) = (0, 0, 0, 0)$*;*
(ii) $u(x)$ *and* $v(x)$ *have exactly one relative maximum in* $\mathbf{R}$ *which occurs at* $x = 0$.
Moreover, system (2.2.1)-(2.2.2) *with* $c = 0$ *has a continuum of periodic solutions.*

2.3 Standing waves in $\mathbf{R}^N$

For the higher dimension case $\Omega = \mathbf{R}^N$ and γ is large, symmetric standing waves were obtained by Reinecke et al. [38] and Wei et al. [44]. The system in $\mathbf{R}^N$ is that

$$\Delta u + f(u) - v = 0, \tag{2.3.1}$$
$$\Delta v + \epsilon(u - \gamma v) = 0, \tag{2.3.2}$$

where $\epsilon = \delta/d$.

Reinecke and Sweers [38] constructed a entire solution of system (2.3.1)-(2.3.2) by using solutions to the system (2.3.1)-(2.3.2) with Dirichlet boundary condition on the ball B_R and letting $R \to \infty$. The main theorem is stated as follows.

Theorem 2.3.1. ([38])*If* $\gamma > \max\{\frac{2}{\sqrt{\epsilon}} + \frac{1-\beta}{\epsilon}, \frac{9}{2\beta^2-5\beta+2} + \frac{2\beta^2-14\beta+11}{9\epsilon}\}$, *there exists a pair of positive solution* $(u, v) \in C^\infty(\mathbf{R}^N) \times C^\infty(\mathbf{R}^N)$ *for system* (2.3.1)-(2.3.2). *Moreover, u and v are radially symmetric, decreasing and satisfy* $p_1 < \max_{x\in\mathbf{R}^N} u(x) < p_2$ *and* $\max_{x\in\mathbf{R}^N} v(x) < \frac{p_2}{\gamma}$.

By a perturbation for δ, Wei and Winter [44] established the following existence.

Theorem 2.3.2. ([44]) *For all* $\alpha \in (0, \beta)$, *there exists a* $\epsilon_0 = \epsilon_0(\alpha, \beta)$ *such that for all* $0 < \epsilon < \epsilon_0$ *and* $\gamma = \frac{\alpha}{\epsilon}$ *system* (2.3.1)-(2.3.2) *has a unique standing wave* (u_ϵ, v_ϵ) *in* $\mathbf{R}^N$. *Moreover,* u_ϵ *and* v_ϵ *are radially symmetric.*

Chapter 3

Steady states on bounded domains and periodic solutions in $\mathbf{R}^N$

3.1 Introduction

In this chapter, we are interested in using a variational approach to study the steady states of (DFHN) on a bounded domain Ω in $\mathbf{R}^N$. Let $\epsilon = \delta/d$. The system we study is as follows:

$$\Delta u + f(u) - v = 0, \tag{3.1.1}$$

$$\Delta v + \epsilon(u - \gamma v) = 0, \tag{3.1.2}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \text{ or } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \tag{3.1.3}$$

where ν is the outer normal of Ω.

In [20], the nonexistence theorem (see Theorem 2.1.2) suggests us that Ω is sufficiently large and ϵ is small if we would like to look for a nontrivial solution as $\gamma \leq \gamma_1$. Some arguments in proving the existences of a minimizer and a mountain pass solution will be omitted if the proofs have showed in [20] and [19]. Our main theorem is stated as follows.

THEOREM 3.1.1. *Assume $\gamma \leq \gamma_1$. There exists $R_0 = R_0(\beta, N) > 0$ such that if $B_{R_0} \subset \Omega$, then we have the following existence result.*
(i) There exists $\epsilon_0 = \epsilon_0(\beta, N, \gamma, |\Omega|)$ so that for all $0 < \epsilon \leq \epsilon_0$ system (3.1.1)-(3.1.2) *with Dirichlet condition has two pair of classical solutions.*
(ii) There exist $k_1 = k_1(\beta, N, \gamma)$ and $\epsilon_1 = \epsilon_1(\beta, N, \gamma, \Omega)$ such that for all Ω satisfying $|\Omega| \geq k_1$ and for all $0 < \epsilon \leq \epsilon_1$, system (3.1.1)-(3.1.2) *with Neumann condition has two pair of classical solutions.*

In addition, we can construct the following existence theorems of periodic solutions in $\mathbf{R}^1$ by applying the above theorem to the domains $\Omega = [0, L_1]$.

COROLLARY 3.1.2. *Assume $\gamma \leq \gamma_1$ and $\Omega = \boldsymbol{R}^1$. Then there exists $\epsilon_2 = \epsilon_2(\beta, \gamma) > 0$ such that for all $0 < \epsilon < \epsilon_2$ system* (3.1.1)-(3.1.2) *has infinitely many periodic solutions.*

3.2 Proof of the main theorem

We first observe that from (3.1.2), v can be solved formally expressed in term of u. With v expressed in terms of u, system (3.1.1)-(3.1.2) is reduced to the single equation

$$\Delta u + f(u) - B[u] = 0, \tag{3.2.1}$$

where $v = B[u]$ is a solution of (3.1.2). In this section, we consider (3.2.1) under either the Dirichlet or Neumann condition. When the Dirichlet (resp., Neumann) problem of (3.2.1) is taken into account, we denote $v = B_0[u]$ (resp., $v = B_1[u]$) by the solution of (3.1.2). With no cause for ambiguity, we continue to denote $B[u]$ by replacing $B_0[u]$ or $B_1[u]$.

For Dirichlet (resp., Neumman) problem, we define the energy functional $\Phi_0[u] : H_0^1(\Omega) \to \mathbf{R}$((resp., $\Phi_1[u] : H^1(\Omega) \to \mathbf{R}$) by

$$\Phi_i[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega F(u) + \frac{1}{2}\int_\Omega uB_i[u], i = 0, 1 \tag{3.2.2}$$

where $F(u) = -\int_0^u f(s)ds = \frac{1}{4}u^4 - \frac{\beta+1}{3}u^3 + \frac{\beta}{2}u^2$. With no cause for ambiguity, we continue to denote $\Phi_0[u]$ or $\Phi_1[u]$ by $\Phi[u]$.

From the a priori estimate for a classical solution u, Klaasen and Mitidieri [20] modified $F(u)$ such that the growth of $F(u)$ is not greater than the quadratic function ku^2 for some $k > 0$ as $u \to \pm\infty$, i.e. for large $|u|$, we have

$$|F(u)| \le ku^2. \tag{3.2.3}$$

We denote the modification of $F(u)$ in [20] by $\tilde{F}(u)$. Let $\tilde{\Phi}[u]$ be the energy, replacing the term $F(u)$ by $\tilde{F}(u)$ in $\Phi[u]$. Then $\tilde{F}(u)$ and $\tilde{\Phi}[u]$ enjoy the properties that $\tilde{\Phi}[u]$ is weakly lower semicontinuous and $\int_\Omega \tilde{F}(u) = \frac{\beta}{2}\int_\Omega u^2 + o(\|u\|^2)$ at $u = 0$. The second statement benefits the geometry of $\tilde{\Phi}[u]$ in mountain pass theorem.

The following lemma asserts that $B[u]$ is a bounded operator in L^2 space and the operator norm of $B[u]$ tends to 0 as $\epsilon \to 0$, which is curial in showing that the term $\int_\Omega uB[u]$ in $\tilde{\Phi}[u]$ is small as ϵ is small.

Lemma 3.2.1. *Let $\lambda_{1,0}$ (resp., $\lambda_{1,1}$) be the first eigenvalue of $-\Delta$ with Dirichlet (resp., Neumman) condition on the $\partial\Omega$. Then $\|B_0[u]\|_2 \le \frac{\epsilon}{\lambda_{1,0}(\Omega)+\epsilon\gamma}\|u\|_2$ and $\|B_1[u] - \frac{\int_\Omega u}{\gamma|\Omega|}\|_2 \le \frac{\epsilon}{\lambda_{1,1}(\Omega)+\epsilon\gamma}\|u\|_2$.*

Proof. Let $\lambda_{n,0}$ (resp., $\lambda_{n,1}$) be the positive eigenvalue sequence of $-\Delta$ with Dirichlet (resp., Neumman) boundary condition and $w_{n,0}$ (resp., $w_{n,1}$) be the corresponding eigenfunction with $\|w_{n,i}\|_2 = 1$ for $i = 0, 1$. Then

$$B_0[u] = \Sigma_{n=1}^\infty \frac{\int_\Omega \epsilon u w_{n,0}}{\lambda_{n,0} + \epsilon\gamma} w_{n,0} \text{ and } B_1[u] = \frac{\int_\Omega u}{\gamma|\Omega|} + \Sigma_{n=1}^\infty \frac{\int_\Omega \epsilon u w_{n,1}}{\lambda_{n,1} + \epsilon\gamma} w_{n,1}.$$

Therefore, for $i = 1, 2$

$$\begin{aligned}&\|B_0[u]\|_2^2 \text{ or } \|B_1[u] - \frac{\int_\Omega u}{\gamma|\Omega|}\|_2^2 \\ &= \Sigma_{n=1}^\infty \left|\frac{\int_\Omega \epsilon u w_{n,i}}{\lambda_{n,i} + \epsilon\gamma}\right|^2 \le \frac{\epsilon^2}{(\lambda_{1,i} + \epsilon\gamma)^2}\Sigma_{n=1}^\infty \left|\int_\Omega u w_{n,i}\right|^2 \le \frac{\epsilon^2}{(\lambda_{1,i} + \epsilon\gamma)^2}\|u\|_2^2.\end{aligned}$$

□

Proof of theorem 3.1.1. By standard variational arguments, $\tilde{\Phi}[u]$ is weakly lower semicontinuous, coercive and bounded below. Therefore, $\tilde{\Phi}[u]$ attains a global minimum on $H_0^1(\Omega)$ or $H^1(\Omega)$. Next, we prove the minimizer is nontrivial if there exists a nontrivial test function $u_{R_0}(x)$ such that $\tilde{\Phi}[u_{R_0}] \leq 0$, where R_0 is determined later. For $R \geq 1$, define $u_R = u_R(x)$ by

$$u_R(x) = \begin{cases} \beta_0, & \text{if } x \in B_{R-1} \\ \beta_0(R - |x|), & \text{if } x \in C_R \\ 0, & \text{if } x \in \Omega - B_R \end{cases} \tag{3.2.4}$$

where $\beta_0 = \frac{2(1+\beta)}{3}$, B_R is a ball with radius R and $C_R = B_R - B_{R-1}$. Then, by Lemma 3.2.1,

$$\begin{aligned} \tilde{\Phi}_0[u_R] &= \frac{1}{2}\int_\Omega |\nabla u_R|^2 + \int_\Omega F(u_R) + \frac{1}{2}\int_\Omega u_R B_0[u_R] \\ &\leq \frac{\beta_0^2}{2}|C_R| + F(\beta_0)|B_{R-1}| + F(\beta)|C_R| + \frac{\beta_0^2\epsilon}{2(\lambda_{1,0}(\Omega)+\epsilon\gamma)}|B_R| \\ &= k(\beta)|C_R| + [F(\beta_0) + \frac{\beta_0^2\epsilon}{2(\lambda_{1,0}(\Omega)+\epsilon\gamma)}]|B_R|, \end{aligned}$$

where $k(\beta) = \frac{\beta_0^2}{2} - F(\beta_0) + F(\beta)$. To choose R_0 which depends only on β and N, we assume $\frac{\beta_0^2\epsilon}{2(\lambda_{1,0}(\Omega)+\epsilon\gamma)} \leq -\frac{F(\beta_0)}{2}$ or $\epsilon \leq \frac{\lambda_{1,0}(\Omega)}{2\gamma_1-\gamma}$. Here, we already use the assumption $\gamma \leq \gamma_1 = \frac{9}{2\beta^2-5\beta+2} = \frac{\beta_0^2}{-2F(\beta_0)}$. By the Rayleigh-Faber-Krahn inequality, that is, $\lambda_{1,0}(\Omega) \geq \lambda_{1,0}(B_\rho)$, it follows that $\lambda_{1,0}(\Omega) \geq \frac{\lambda_{1,0}(B_1)}{\rho^2}$, where $|B_\rho| = |\Omega|$. Therefore, we choose

$$\epsilon_0 = \frac{\lambda_{1,0}(B_1)|B_1|^{2/N}}{(2\gamma_1-\gamma)|\Omega|^{2/N}}. \tag{3.2.5}$$

It followews from the fact $|C_R| \leq |B_1|2^{N-1}R^{N-1}$, for all $0 < \epsilon \leq \epsilon_0$ that

$$\tilde{\Phi}_0[u_R] \leq |B_1|R^{N-1}\{k(\beta)2^{N-1} + \frac{F(\beta_0)}{2}R\}$$

Taking

$$R_0 = \frac{k(\beta)2^N}{-F(\beta_0)},$$

The proof of the existence of a minimizer of $\tilde{\Phi}_0$ is completed.

For the Neumann condition,

$$\begin{aligned} \frac{1}{2}\int_\Omega u_R B_1[u_R] &= \frac{1}{2}\int_\Omega u_R(B_1[u_R] - \frac{\int_\Omega u_R}{\gamma|\Omega|}) + \frac{(\int_\Omega u_R)^2}{2\gamma|\Omega|} \\ &\leq \frac{\beta_0^2\epsilon}{2(\lambda_{1,1}(\Omega)+\epsilon\gamma)}|B_R| + \frac{\beta_0^2|B_R|^2}{2\gamma|\Omega|} \end{aligned}$$

Similarly, $\tilde{\Phi}_1[u_{R_0}] \leq 0$ if we assume that

$$\frac{\beta_0^2 \epsilon}{2(\lambda_{1,1}(\Omega) + \epsilon\gamma)} + \frac{\beta_0^2 |B_{R_0}|}{2\gamma|\Omega|} \leq -\frac{F(\beta_0)}{2}.$$

A sufficient condition of the above inequality is

$$\frac{\beta_0^2 \epsilon}{2(\lambda_{1,1}(\Omega) + \epsilon\gamma)} \leq -\frac{F(\beta_0)}{4} \text{ and } \frac{\beta_0^2 |B_{R_0}|}{2\gamma|\Omega|} \leq -\frac{F(\beta_0)}{4} \tag{3.2.6}$$

Therefore, we can choose

$$\epsilon_1 \leq \frac{\lambda_{1,1}(\Omega)}{4\gamma_1 - \gamma} \text{ and } k_1 = \frac{|B_{R_0}|}{4\gamma_1\gamma}.$$

We also need to exclude the probability of constant solutions for Neumann problem. Let $(u, v) = (p, q)$ solve $u - \gamma v = 0$ and $f(u) - v = 0$. Then $\tilde{\Phi}(p) = \int_\Omega [F(p) + \frac{p^2}{2\gamma}] = \int_\Omega \frac{p^2}{4}[(p-\beta_0)^2 + \frac{2}{\gamma} - \frac{2}{\gamma_0}] \geq 0$, which shows that a minimizer of $\tilde{\Phi}_1$ is nonconstant. Employing the mountain pass theorem, the other solution can be obtained. Moreover, those solutions are C^2-functions. The detail of the existence of the minimizer and the mountain pass solution can be found in [20].

Proof of corollary 3.1.2. Let the domain $\Omega_1 = [0, L_1]$. Then $\lambda_{1,1}(\Omega) = \frac{\pi^2}{L_1^2}$. To chose ϵ independent of Ω_1, we estimate L_1 by using (3.2.6). Then $\frac{2R_0}{4\gamma_1\gamma} \leq L_1 \leq \frac{\pi}{\sqrt{\epsilon(4\gamma_1-\gamma)}}$. On the other hand, if Ω_1 contains a ball B_{R_0}, then $L_1 > 2R_0$. To ensure the existence of L_1, we let $\frac{\pi}{\sqrt{\epsilon(4\gamma_1-\gamma)}} > \max\{2R_0, \frac{2R_0}{4\gamma_1\gamma}\} := M$ or $\epsilon < \frac{\pi^2}{M^2(4\gamma_1-\gamma)} := \epsilon_2$. Then for all $0 < \epsilon < \epsilon_2$, choose L_1 such that $M < L_1 \leq \frac{\pi}{\sqrt{\epsilon(4\gamma_1-\gamma)}}$ Then the Neumann problem is solvable.

By even reflections with respect to the boundary of Ω, the domain of this solution can be extended to a larger one. Continuing in this manner, we obtain a solution in the whole domain $\mathbf{R}^1$. Since L_1 can be arbitrarily chosen in a interval, the system (3.1.1)-(3.1.2) has infinitely many solutions.

Chapter 4

Monostable-type solutions in $\mathbf{R}^N$

This chapter is concerned with monostable-type travelling wave solutions of (DFHN) in $\mathbf{R}^N$ for the two components u and v. By solving v in terms of u, this system can be reduced to a non-local single equation for u. When the diffusion coefficients in the system are equal, we construct travelling wave solutions for the non-local equation by the method of super- and subsolutions developed by Morita and Ninomiya [29]. Moreover, we propose a condition for γ, which is similar to the condition Reinecke and Sweers [38] used to transform (DFHN) into a quasimonotone system.

4.1 Introduction

In the present work, we are concerned with (DFHN) in $\mathbf{R}^N$ i.e.,

$$u_t = u_{\xi\xi} + \Delta_y u + f(y,u) - v, \tag{4.1.1}$$
$$v_t = dv_{\xi\xi} + \Delta_y v + \delta(u - \gamma v), \tag{4.1.2}$$

where $(\xi, y) \in \mathbf{R}^N = \mathbf{R}^1 \times \mathbf{R}^{N-1}$, $N \geq 2$, $\delta, \gamma > 0$ and $d \geq 0$. A typical example of $f(y,u)$ is $f(y,u) = u(1-u)(u-\beta)$ for $0 < \beta < \frac{1}{2}$. Throughout the chapter we assume that f is a C^2-function in u and f, f_u and f_{uu} are bounded in $\{(y,u)|y \in \Omega_y, |u| \leq K\}$ for some large constant $K > 0$. In addition, f satisfies (**H1**)-(**H5**).

The solutions of interest here are traveling wave solutions. Let $x = \xi - ct$, then travelling wave solutions of (4.1.1)-(4.1.2) satisfy

$$u_{xx} + cu_x + \Delta_y u + f(y,u) - v = 0, \tag{4.1.3}$$
$$dv_{xx} + cv_x + \Delta_y v + \delta(u - \gamma v) = 0. \tag{4.1.4}$$

Over the past decades, this system has been extensively studied. For instance, as $N = 1$, under different assumptions, system (4.1.3)-(4.1.4) admits standing pulses in [6], [7] and [22], infinitely many periodic solutions in [22], fronts, back waves in [17] and [21] and travelling pulses in [21]. For the higher dimension case $N \geq 2$, symmetric standing waves were established by Reinecke and Sweers [38] and Wei and Winter [44].

As $\gamma \to \infty$, if the solutions are assumed to be bounded, the equations (4.1.3)-(4.1.4) tend to the single equation

$$u_{xx} + cu_x + \Delta_y u + f(y,u) = 0. \tag{4.1.5}$$

Let $f(y,u)$ be a C^2 function $g(u)$ which has the property that for some $\theta \in (0,1)$ $g(0) = g(\theta) = g(1) = 0, g_u(0) < 0, g_u(\theta) > 0, g_u(1) < 0$, $g < 0$ on $(0,\theta)$ and $g > 0$ on $(\theta, 1)$. In addition to the planar waves, (4.1.5) admits other types of solutions, including travelling curved fronts ($N = 2$), conical shapes and pyramidal shapes ($N \geq 3$) in [14], [23], [33] and [41]. Moreover, Hamel and Roquejoffre [15] established travelling wave solutions of (4.1.5) in $\mathbf{R}^2$ which connect one unstable periodic solution at $x \to \infty$ $(-\infty)$ and one stable constant solution at $x \to -\infty$ (∞). On the other hand, travelling wave solutions of (4.1.5) in $\mathbf{R}^N$ connecting a unstable one-peak solution at $x \to \infty$ $(-\infty)$ and a stable constant solution $x \to -\infty$ (∞) were obtained by Morita and Ninomiya [29].

In this paper, we use the method of super- and subsolutions developed in [29]. Due to technical restriction, we assume $d = 1$. Since equation (4.1.4) is linear, v can be solved formally in terms of u. With v expressed in terms of u, system (4.1.3)-(4.1.4) is reduced to the non-local equation

$$\mathcal{F}[u] := u_{xx} + cu_x + \Delta_y u + f(y,u) - B_c[u] = 0, \tag{4.1.6}$$

where we denote v by $B_c[u] := \delta(-\frac{\partial^2}{\partial x^2} - c\frac{\partial}{\partial x} - \Delta_y + \delta\gamma)^{-1}u$. It is readily seen that if u is independent of x, then by the uniqueness theorem $B_c[u] = \delta(-\Delta_y + \delta\gamma)^{-1}u$. As $x \to \pm\infty$, the asymptotic behaviors of travelling wave solutions of (4.1.6) formally satisfy

$$\Delta_y u + f(y,u) - B_c[u] = 0, \tag{4.1.7}$$

where $B_c[u] = \delta(-\Delta_y + \delta\gamma)^{-1}u$. Our main purpose is to look for monostable-type travelling wave solutions $u(x,y)$ which connect a stable solution of (4.1.7) as $x \to -\infty$ (∞) and a unstable one as $x \to \infty$ $(-\infty)$. Without loss of generality, we may assume that $u(+\infty, y)$ is an unsatble solution. Throughout this paper, the following hypotheses are assumed.
(**H1**) There are two solutions $u_\pm(y)$ of (4.1.7) satisfying $u_-(y) \geq u_+(y)$. Moreover, there exist an eigenvalue $\mu > 0$ and its corresponding eigenfunction $\phi(y) > 0$ with $\max_{\{y\in\mathbf{R}^{N-1}\}} \phi(y) = 1$ and $\lim_{|y|\to\infty} \phi(y) = 0$ such that

$$\Delta_y \phi + f_u(y, u_+)\phi - B_c[\phi] = \mu\phi. \tag{4.1.8}$$

(**H2**) $u_-(y) \geq u_+(y) + \epsilon\phi(y)$ for some $\epsilon > 0$.
(**H3**) There exists no other solution $u(y)$ of (4.1.7) with the property $u_-(y) \geq u(y) \geq u_+(y)$.
(**H4**) For all small $\eta > 0$, there exist solutions $u_+^\eta(y)$ satisfying $\lim_{\eta\to 0} u_+^\eta(y) = u_+(y)$,

$$\Delta_y u_+^\eta + f(y, u_+^\eta) - B_c[u_+^\eta] + \eta = 0 \tag{4.1.9}$$

and

$$u_+^\eta(y) \geq u_+(y) + \frac{\eta}{M} \tag{4.1.10}$$

for some constant $M > 0$.
(**H5**)

$$\Delta_y \psi_i - (K_1 + \sqrt{\delta})\psi_i \leq 0, i = 1, 2, 3, \tag{4.1.11}$$

where

$$K_1 = -\min_{\{u_-(y)\geq u\geq u_+(y), y\in\mathbf{R}^{N-1}\}} f_u(y,u) > 0, \tag{4.1.12}$$

$\psi_1 = \phi$, $\psi_2 = u_- - u_+$ and $\psi_3 = u_+^\eta - u_+$.

To simplify the proof of the main theorem in this paper, we modify the nonlinear term $f(y,u)$ such that the minimum and maximum of $f_u(y,u)$ in $\{u(y)\in\mathbf{R}, y\in\mathbf{R}^{N-1}\}$ are the same as those in $\{u_-(y)\geq u\geq u_+(y), y\in\mathbf{R}^{N-1}\}$. For convenience, we still denote $f(y,u)$ for the new modification of f. Set

$$K^* := \max_{\{u_-(y)\geq u\geq u_+(y), y\in\mathbf{R}^{N-1}\}} f_u(y,u) > 0 \tag{4.1.13}$$

and let $K_2>0$ satisfy $K_2 + \frac{\delta}{\delta\gamma+K_2} = K^*$. We state the main theorem as follows.

THEOREM 4.1.1. *Assume $\gamma \geq \frac{2}{\sqrt{\delta}} + \frac{K_1+\mu}{\delta}$ and (**H1**)-(**H5**) hold. Then there exists $c^* = \max\{2\sqrt{\mu}, 2\sqrt{K_2}\} > 0$ such that for all $c\geq c^*$, system (4.1.3)-(4.1.4) admits a pair of smooth solutions (u^*,v^*) which satisfies $u_x^*\leq 0, v_x^*\leq 0$ and the boundary conditions $(u^*,v^*)(\pm\infty,y) = (u_\pm(y), v_\pm(y))$, where $v_\pm(y) = B_c[u_\pm(y)]$.*

Remark 1. In (**H1**), when the inequality $u_-(y)\geq u_+(y)$ is reversed,i.e., $u_-(y)\leq u_+(y)$, a result similar to Theorem 4.1.1 can be proved except that the inequalities $u_x^*\leq 0$ and $v_x^*\leq 0$ in Theorem 4.1.1 need to be replaced by $u_x^*\geq 0$ and $v_x^*\geq 0$ respectively.

Remark 2. In fact, (**H5**) can be weakened to the following assumption.

$$\Delta_y\psi_i - M_i\psi_i \leq 0, \text{ for some constants } M_i > 0. \tag{4.1.14}$$

This condition holds if $\Delta_y\psi_i$ does not decay faster than ψ_i as $|y|\to\infty$. In this case, if we choose $\gamma\geq \frac{1}{\sqrt{\delta}} + \frac{K_3+\mu}{\delta}$, where $K_3 = \max\{M_1, M_2, M_3, K_1+\sqrt{\delta}\}$, then a similar result can be proved.

It is not easy to find an example which satisfies assumptions (**H1**)-(**H5**) even for the case $f(y,u) = u(1-u)(u-\beta)$ since the stability of the radially symmetric solutions obtained in [38] and [44] has not yet been studied. However, we believe that for $\gamma\gg 1$ the structure of system (4.1.3)-(4.1.4) is similar to that of equation (4.1.6). Accordingly, we extend the result of theorem 2.1 in [29] to the one in Theorem 4.1.1.

4.2 Proof of the main theorem

To prove the Theorem 4.1.1, we use the super- and subsolutions constructed in [29]. By considering the following equation, we construct subsolutions of $\mathcal{F}[u]$. Let $w(x)$ satisfy

$$w_{xx} + cw_x + \mu w - w^2 = 0, \tag{4.2.1}$$
$$w(-\infty) = \mu, w(\infty) = 0. \tag{4.2.2}$$

For all $c\geq 2\sqrt{\mu}$, the above boundary value problem admits an unique solution $w(x)$ (up to a translation) which is strictly increasing in x. Subsolutions of $\mathcal{F}[u]$ are established as follows.

Lemma 4.2.1. *Let $\underline{U}(x,y) = u_+(y) + \sigma\phi(y)w(x)$. Then there exists $\sigma_1 > 0$ such that $\mathcal{F}[\underline{U}] \geq 0$ for all $0 < \sigma \leq \sigma_1$ and $c \geq 2\sqrt{\mu}$.*

Proof. Let $V := wB_c[\phi] - B_c[\phi w] \geq 0$, then $V \geq 0$. Indeed, it is easy to see that $B_c[\phi] \geq 0$ by the maximum principle and $\phi > 0$. A straightforward calculation gives

$$V_{xx} + cV_x + \Delta_y V - \delta\gamma V = -w(\mu - w)B_c[\phi] \leq 0. \tag{4.2.3}$$

Using the maximum principle, we obtain $V \geq 0$. Therefore by (**H1**)

$$\begin{aligned}
&\mathcal{F}[\underline{U}] \\
&= \sigma\phi(w_{xx} + cw_x) + (\Delta_y u_+ - B_c[u_+]) + \sigma w \Delta_y \phi + f(y, u_+ + \sigma\phi w) - \sigma B_c[\phi w] \\
&= \sigma\phi(w_{xx} + cw_x + \mu w) + f(y, u_+ + \sigma\phi w) - f(y, u_+) - f_u(y, u_+)\sigma\phi w + \sigma V \\
&\geq \sigma\phi w^2 + G,
\end{aligned}$$

where $G = f(y, u_+ + \sigma\phi w) - f(y, u_+) - f_u(y, u_+)\sigma\phi w$.
Let $M_1 = \min_{\{u_-(y) \geq u \geq u_+(y), y \in \mathbf{R}^{N-1}\}} f_{uu}(y,u)$. By choosing $\sigma \leq \frac{\epsilon}{\mu}$ and using (**H2**), we obtain $u_+ \leq u_+ + \sigma\phi w \leq u_+ + \epsilon\phi \leq u_-$. According to the mean value theorem, we have $G \geq 0$ if $M_1 \geq 0$ and $G \geq M_1\sigma^2\phi^2 w^2$ if $M_1 < 0$. Therefore $\mathcal{F}[\underline{U}] \geq 0$ if $\sigma \leq \sigma_1$, where $\sigma_1 = \frac{\epsilon}{\mu}$ as $M_1 \geq 0$ and $\sigma_1 = \min\{\frac{\epsilon}{\mu}, \frac{-1}{M_1}\}$ as $M_1 < 0$. The proof is completed. □

In what follows we construct supersolutions of $\mathcal{F}[u]$.

Lemma 4.2.2. *Let $Q(x) = e^{-\frac{c - \sqrt{c^2 - 4K_2}}{2}x}$ and $U^+(x,y) = u_+^\eta(y) + Q(x)$, where $K_2 > 0$ satisfies $K_2 + \frac{\delta}{\delta\gamma + K_2} = K^*$ and $c \geq 2\sqrt{K_2}$. Then $\mathcal{F}[U^+] < 0$.*

Proof. Note that $Q_{xx} + cQ_x + K_2 Q = 0$ and $0 < B_c[Q] < \infty$. Indeed, by the uniqueness theorem we have $B_c[Q(x)] = \delta(-\frac{\partial^2}{\partial x^2} - c\frac{\partial}{\partial x} + \delta\gamma)^{-1}Q$ and

$$B_c[Q] = \frac{\delta}{\sqrt{c^2 + 4\gamma\delta}} \int_{-\infty}^{+\infty} e^{-\frac{\sqrt{c^2+4\gamma\delta}}{2}|x-\xi| + \frac{c}{2}(\xi - x)} Q(\xi)d\xi = \frac{\delta}{\delta\gamma + K_2} Q(x).$$

It follows from (**H4**) that

$$\begin{aligned}
\mathcal{F}[U^+] &= (Q_{xx} + cQ_x) + (\Delta_y u_+^\eta - B_c[u_+^\eta]) + f(y, u_+^\eta + Q) - B_c[Q] \\
&= -K_2 Q + f(y, u_+^\eta + Q) - f(y, u_+^\eta) - \eta - B_c[Q] \\
&= \{-K_2 + f_u(y, u_+^\eta + \theta Q) - \frac{\delta}{\delta\gamma + K_2}\}Q - \eta \leq -\eta < 0,
\end{aligned}$$

where $0 \leq \theta \leq 1$. The last second inequality is due to

$$K_2 + \frac{\delta}{\delta\gamma + K_2} = \max_{\{u_-(y) \geq u \geq u_+(y), y \in \mathbf{R}^{N-1}\}} f_u(y,u).$$

We complete the proof of the lemma. □

Let

$$\mathcal{L}[u] = u_{xx} + cu_x + \Delta_y u - (K_1 + \mu + \sqrt{\delta})u, \tag{4.2.4}$$

where $K_1 = -\min_{\{u_-(y)\geq u\geq u_+(y), y\in\mathbf{R}^{N-1}\}} f_u(y,u) > 0$.

To show the existences of travelling wave solutions of (4.1.7), we use the following iteration process:

$$\begin{aligned} &u_n(x,y) = \mathcal{L}^{-1}(-f(u_{n-1}) + B_c[u_{n-1}] - (K_1 + \mu + \sqrt{\delta})u_{n-1}), n = 1, 2, \cdots, \\ &u_0(x,y) = \underline{U}. \end{aligned} \tag{4.2.5}$$

In the following lemma, we assert that the supersolutions of $\mathcal{F}$ are greater than or equal to the subsolutions of $\mathcal{F}$. Moreover, we show that both $U^+ - \underline{U}$ and $u_- - \underline{U}$ are supersolutions of $\mathcal{L}$, which is useful in the proof of iteration process.

Lemma 4.2.3. *Assume $\gamma \geq \frac{2}{\sqrt{\delta}} + \frac{K_1+\mu}{\delta}$ and let $\overline{U} := \min\{U^+(x,y), u_-(y)\}$. Then for all $\eta > 0$ there exists $\sigma_2 > 0$ depending on η such that for all $0 < \sigma \leq \sigma_2$ we have*

$$\overline{U} \geq \underline{U}, \mathcal{L}[U^+ - \underline{U}] \leq 0 \text{ and } \mathcal{L}[u_- - \underline{U}] \leq 0. \tag{4.2.6}$$

Proof. For the case $\overline{U} = u_-(y)$ we take $\sigma \leq \frac{\epsilon}{\mu}$, then

$$\overline{U} - \underline{U} = u_-(y) - u_+(y) - \sigma\phi(y)w(x) \geq u_-(y) - u_+(y) - \epsilon\phi(y) \geq 0. \tag{4.2.7}$$

The last inequality holds by (**H2**). On the other hand,

$$\mathcal{L}[u_- - \underline{U}] = \Delta_y(u_- - u_+) - (K_1 + \mu + \sqrt{\delta})(u_- - u_+) + A, \tag{4.2.8}$$

where $A = -\sigma\phi(w_{xx} + cw_x) + (K_1 + \mu + \sqrt{\delta})\sigma\phi w - \sigma w\Delta_y\phi$. According to (**H5**), $|A| \leq \sigma C\phi$ for some positive constant $C = C(\mu, \delta, K_1)$. By choosing $\sigma \leq \frac{\epsilon\mu}{C}$, we obtain

$$\begin{aligned} \mathcal{L}[u_- - \underline{U}] &\leq \Delta_y(u_- - u_+) - (K_1 + \sqrt{\delta})(u_- - u_+) - \mu(u_- - u_+) + \sigma C\phi &(4.2.9)\\ &\leq -\epsilon\mu\phi + \sigma C\phi \leq 0, &(4.2.10) \end{aligned}$$

which holds due to assumptions (**H2**) and (**H5**).

For the case $\overline{U} = u_+^\eta(y) + Q(x)$, given $\eta > 0$ we choose $\sigma \leq \frac{\eta}{\mu M}$ and use assumption (**H4**), then

$$\overline{U} - \underline{U} = u_+^\eta(y) + Q(x) - u_+(y) - \sigma\phi(y)w(x) \geq \frac{\eta}{M} - \sigma\mu \geq 0. \tag{4.2.11}$$

Moreover,

$$\begin{aligned} \mathcal{L}[U^+ - \underline{U}] = \Delta_y(u_+^\eta - u_+) - (K_1 + \mu + \sqrt{\delta})(u_+^\eta - u_+) + A + Q_{xx} + Q_x \\ - (K_1 + \mu + \sqrt{\delta})Q. \end{aligned}$$

It is readily seen that $Q_{xx} + Q_x - (K_1 + \mu + \sqrt{\delta})Q \leq 0$. By (**H4**) and (**H5**),

$$\mathcal{L}[U^+ - \underline{U}] \leq -\frac{\eta\mu}{M} + \sigma C \leq 0 \text{ if } \sigma \leq \frac{\eta\mu}{MC}.$$

Setting $\sigma_2 = \min\{\frac{\epsilon}{\mu}, \frac{\epsilon\mu}{C}, \frac{\eta}{\mu M}, \frac{\eta\mu}{MC}\}$, the lemma holds. □

To generalize the result of Theorem 2.1 in [29], the nonlocal term of (4.1.6) needs to be better estimated. More precisely, we pointwisely control $B_c[u]$ by the local term u such that the iterative sequence u_n is comparable with u_{n-1}.

LEMMA 4.2.4. *Let $u \in C^2(\boldsymbol{R}^N)$ be nonnegative and solve $u_{xx} + cu_x + \Delta_y u - au \leq 0$ for some constant a. Assume $\gamma \geq \frac{a}{\delta} + \frac{1}{b}$ for some b. Then $bu - B_c[u] \geq 0$.*

Proof. Let $v = B_c[u]$ and $U = bu - v$.Then $v \geq 0$ because of $u \geq 0$ and the maximum principle. Our main purpose is to claim $U \geq 0$. By the assumption of u and the definition of v, we have

$$U_{xx} + cU_x + \Delta_y U - \frac{ab+\delta}{b} U \leq -(\delta\gamma - a - \frac{\delta}{b})v \leq 0. \tag{4.2.12}$$

The last inequality follows from the hypothesis of γ and the nonnegativity of v. By the maximum principle, $U \geq 0$. □

As γ becomes large, we claim that the iterative sequence u_n is increasing.

LEMMA 4.2.5. *Assume $\gamma \geq \frac{2}{\sqrt{\delta}} + \frac{K_1+\mu}{\delta}$ and $c \geq c^* = \max\{2\sqrt{\mu}, 2\sqrt{K_2}\}$, then for all $\eta > 0$ and $0 < \sigma \leq \min\{\sigma_1, \sigma_2\}$ we have $u_{n,x} \leq 0$ and*

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq \overline{U}. \tag{4.2.13}$$

Proof. We first claim that $u_n \leq \overline{U}$ for all n. Indeed, by Lemma 4.2.3 and Lemma 4.2.4 (take $a = K_1 + \mu + \sqrt{\delta}$ and $b = \sqrt{\delta})$ we obtain

$$\sqrt{\delta}(U^+ - u_0) - B_c[U^+ - u_0] \geq 0. \tag{4.2.14}$$

Therefore Lemma 4.2.2 and Lemma 4.2.3 yield

$$\begin{aligned}\mathcal{L}[U^+ - u_1] &\leq -f(U^+) + B_c[U^+] + f(u_0) - B_c[u_0] - (K_1 + \mu + \sqrt{\delta})(U^+ - u_0)\\ &\leq \{-f_u(\theta U^+(1-\theta)u_0) - K_1\}(U^+ - u_0) \leq 0,\end{aligned}$$

where $0 \leq \theta \leq 1$. According to the maximum principle, $U^+ - u_1 \geq 0$. It follows form the proof of $U^+ - u_1 \geq 0$ that $u_- - u_1 \geq 0$. Therefore $u_1 \leq \overline{U}$. Continuing this process, we have $u_n \leq \overline{U}$ for all n by induction.

Next obvert that $\mathcal{L}[u_1 - u_0] = -\mathcal{F}[\underline{U}] \leq 0$ due to Lemma 4.2.1. By the maximum principle, $u_1 - u_0 \geq 0$. Applying Lemma 4.2.4 to $u_1 - u_0$, we have

$$\sqrt{\delta}(u_1 - u_0) - B_c[u_1 - u_0] \geq 0. \tag{4.2.15}$$

Therefore

$$\begin{aligned}\mathcal{L}[u_2 - u_1] &= -(f(u_1) - f(u_0)) + B_c[u_1 - u_0] - (K_1 + \mu + \sqrt{\delta})(u_1 - u_0)\\ &\leq \{-f_u(\theta u_1 + (1-\theta)u_0 - K_1\}(u_1 - u_0) - \sqrt{\delta}(u_1 - u_0) + B_c[u_1 - u_0]\\ &\leq 0,\end{aligned}$$

where $0 \leq \theta \leq 1$. Thus $u_2 \geq u_1$. By induction, the sequence of functions $\{u_n\}$ is nondecreasing. On the other hand, obvert that $u_{0,x} = \sigma\phi w_x < 0$. Therefore by (**H5**), we obtain

$$\mathcal{L}[-u_{0,x}] = \sigma\phi(\mu w_x - 2ww_x) - \sigma w_x \Delta_y \phi + (K_1 + \mu + \sqrt{\delta})\sigma\phi w_x \tag{4.2.16}$$

$$= -\sigma w_x\{\Delta_y \phi - (K_1 + \sqrt{\delta})\phi + (-2\mu + 2w)\phi\} \leq 0. \tag{4.2.17}$$

Using Lemma 4.2.4 again, we have

$$\sqrt{\delta}(-u_{0,x}) - B_c[-u_{0,x}] \geq 0 \tag{4.2.18}$$

and

$$\mathcal{L}[u_{1,x}] = -f_u(u_0)u_{0,x} + B_c[u_{0,x}] - (K_1 + \mu + \sqrt{\delta})u_{0,x} \geq 0. \tag{4.2.19}$$

Then $u_{1,x} \leq 0$ by the maximum principle. Inducting in n, we obtain $u_{n,x} \leq 0$. □

Proof of Theorem 4.1.1. By Lemma 4.2.5, we define $u^*(x, y) = \lim_{n\to\infty} u_n(x, y)$. Following the proof of theorem 2.1 in [29] , (**H2**) and (**H3**), for all $c \geq c^*$ we obtain that $u^*(x, y)$ is a smooth solution of (4.1.6), $u^*_x \leq 0$ and $u^*(\pm\infty, y) = u_\pm(y)$. Let $v^* = B_c[u^*]$, then $v^*_x = B_c[u^*_x] \leq 0$ by the maximum principle. We complete the proof of the theorem.

Chapter 5

Travelling waves in a cylinder for bistable cases

5.1 Introduction

In this chapter, we are concerned with (DFHN) with Dirichlet boundary condition in a cylinder Ω

$$u_t = u_{\xi\xi} + \Delta_y u + f(u) - v, \tag{5.1.1}$$
$$v_t = d(v_{\xi\xi} + \Delta_y v) + \delta(u - \gamma v), \tag{5.1.2}$$
$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{5.1.3}$$

where $(\xi, y) \in \Omega := \mathbf{R}^1 \times \Omega_y$ with Ω_y being a bounded C^{2,α_0} domain in $\mathbf{R}^{N-1}$, $d \geq 0$, $\delta, \gamma > 0$, and $f(u) = u(1-u)(u-\beta)$ for $0 < \beta < \frac{1}{2}$. We also consider this system with Neumann boundary condition in Section 5.7.

As $\gamma \to \infty$, if the solutions are assumed to be bounded, the equations (5.1.1)-(5.1.3) tend to the single equation

$$u_t = u_{\xi\xi} + \Delta_y u + f(u), \tag{5.1.4}$$
$$u|_{\partial\Omega} = 0, \tag{5.1.5}$$

which is a gradient system. For $N = 2$, the existence of travelling waves of (5.1.4) with boundary condition (5.1.5) were obtained by Gardner [9] when $\Omega_y = [0, L]$ and L is large. His result indicates that large Ω_y seems to be necessary for the existence of a travelling wave with the Dirichlet boundary condition. For higher dimension cases, existence results of travelling waves of (5.1.4)-(5.1.5) were obtained by Volpert, A. and Volpert, V [43], Heinze [11], and Lucia, Muratov and Novaga [25].

In this chapter, we are interested in using a variational approach to study the travelling front solution of (DFHN) and also interested in the higher dimension case $N > 1$. Let's first consider the case of a gradient system. For a gradient system, when the wave speed is zero, it is natural to consider the solution as a critical point of the corresponding energy of the system. However when the speed is not zero, how to use the variational method becomes a very subtle problem. Let c denote the wave speed. Assume $c > 0$. Heinze [11] made the change of variable $x = c(\xi - ct)$ and considered a minimization problem of a weighted energy with a constraint. According to his ingenious setting, a minimizer of the problem corresponds to a travelling front solution while the Lagrange multiplier of the constraint corresponds to the wave speed. We explain this more precisely by considering (5.1.4)-(5.1.5) as follows. Using the new variable $x = c(\xi - ct)$, a travelling wave solution of the single equation (5.1.4) satisfies

$$c^2(u_{xx} + u_x) + \Delta_y u + f(u) = 0. \tag{5.1.6}$$

Heinze considered the weighted energy, acting on the Sobolev space $\mathbf{H}$, which is the space $W^{1,2}$ with the weight $e^x\,dx$ (See section 5.2.)

$$S_c[u] := c^2 I_0[u] + J_0[u] := \frac{c^2}{2}\int_\Omega e^x u_x^2 + \frac{1}{2}\int_\Omega e^x \left(|\nabla_y u|^2 + F(u)\right),$$

where $F(u) = -\int_0^u f(s)ds = \frac{1}{4}u^4 - \frac{\beta+1}{3}u^3 + \frac{\beta}{2}u^2$. Let $\hat{u}$ be a nontrivially critical point of $S_c[u]$ for some $c > 0$. Denote $DS_c[\hat{u}]\phi$ be the Fréchet derivative of S_c at $\hat{u}$ acting on ϕ. Then the Euler-Lagrange equation $DI_0[\hat{u}]\phi + \frac{1}{c^2}DJ_0[\hat{u}]\phi = 0$ holds for all test function $\phi \in \mathbf{H}$. According to this, Heinze viewed $\hat{u}$ as a minimizer of $I_0[u]$ under the constraint $\mathcal{A}_k = \{u \in \mathbf{H} | J_0[u] = k\}$ and $\frac{1}{c^2}$ as the corresponding Lagrange multiplier. Also it is easy to verify that the corresponding Euler-Lagrange equation is (5.1.6). On the other hand, multiplying (5.1.6) by $u_x e^x$ and taking integration one obtains $c^2 I_0[u] + J_0[u] = 0$. This implies $J_0[u] < 0$ if u is a nontrivial solution of (5.1.6). By a suitable translation x_0, $J_0[u(x - x_0, y)] = -1$. Therefore Heinze chose $k = -1$ and solved the minimizing problem

$$\inf_{\mathcal{A}_{-1}} I_0[u]. \tag{5.1.7}$$

Moreover, he proved that $\mathcal{A}_{-1} \neq \emptyset$ is equivalent to

$$\inf_{w \in H_0(\Omega_y)} \int_{\Omega_y} \left(\frac{1}{2}|\nabla_y w|^2 + F(w)\right) < 0. \tag{5.1.8}$$

This condition also guarantees that the minimizer is nontrivial.

Later in a series of papers [24]-[31], Lucia, Muratov and Novaga further developed the variational approach and proved existence results via subtle ideas and techniques. In their approach (see [25]), the ξ variable is not scaled and the energy

$$\hat{S}_c[u] := \hat{I}_0[u] + \hat{J}_0[u] := \frac{1}{2}\int_\Omega e^{cz} u_z^2 + \frac{1}{2}\int_\Omega e^{cz}\left(|\nabla_y u|^2 + F(u)\right)$$

is considered on the Sobolev space with weight $e^{cz}\,dz$, where $z = \xi - ct$. In [25], they assumed that there is a $c^* > 0$ such that $\hat{S}_{c^*}[u] \leq 0$ for a nontrivial u, which is equivalent to Heinze's condition (5.1.8) (see proposition 6.2 in [25]). Then they minimized the energy $\hat{S}_{c*}[u]$ under the constraint $\hat{I}_0[u] = 1$. In their proof, to show the lower semi-continuity of $\hat{S}_c[u]$ is one of the most crucial steps for the existence of $\inf_{\{I_0[u]=1\}} \hat{S}_c[u]$. They proved a minimizer u_{c*} of $\hat{S}_{c^*}[u]$ under the constraint $\hat{I}_0[u] = 1$ can be achieved. By the scaling property of the equations, the travelling wave solution can be obtained as $u(x, y) = u_{c^*}(x\sqrt{1 - S_{c^*}[u_{c^*}]}, y)$ and the travelling wave speed equals $c = c^*\sqrt{1 - S_{c^*}[u_{c^*}]}$.

Lucia, Muratov and Novaga also gave a new criterion for the so called linear and nonlinear selection of mono-stable travelling waves as an application of their methods (see [24]).

For a gradient system, one disadvantage in applying these variational approaches comes from that one needs to assume the diffusion coefficients of all components are equal when the wave speed $c \neq 0$. On the other hand, the variational approaches have some advantages also. Besides they themselves provide very interesting and different viewpoints of the problem, these methods are more easily generalized to higher dimension cases, e. g. waves on cylindrical domains, and usually require only mild assumptions for the existence of a solution.

Although (DFHN) is not a gradient system, by replacing v by $\sqrt{\delta}v$, this system has the skew-gradient structure defined by Yanagida [45]. More precisely, under this replacement, (DFHN) becomes

$$u_t = \Delta u + f(u) - \sqrt{\delta}v, \tag{5.1.9}$$

$$v_t = d\Delta v + \sqrt{\delta}v - \gamma\delta v, \tag{5.1.10}$$

of which the steady states correspond to the critical points of the energy

$$S[u,v] = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - d|\nabla v|^2 + H(u,v)\right), \tag{5.1.11}$$

where $H(u,v) = F(u) + \sqrt{\delta}uv - \frac{1}{2}\gamma\delta v^2$. Restricted to the u direction, (5.1.9) is the gradient flow of the energy (5.1.11) while restricted in the v direction, (5.1.10) is the gradient flow of the minus of (5.1.11). Along the orbits of (5.1.9) and (5.1.10), the "u-part" of the energy (5.1.11) decreases and the "v-part" of the energy increases. Yanagida [45] called a system of reaction-diffusion equations with an energy like this a skew-gradient system. He developed a theory for a skew-gradient system and found that the correct notion in such a system corresponding to a minimizer in a gradient system should be a mini-maximizer. With this structure in mind, the authors feel curious about the following problem:

Question 1 Can variational methods be applied to find wave front or pulse solutions of a skew-gradient system? For example, applied to (DFHN)?

The setting of Heinze [11] and the setting of Lucia, Muratov and Novaga [24]-[31] in applying variational methods mentioned above are slightly different. The former uses the change of variables $x = c(\xi - ct)$ to scale out the factor c in the weight of the energy while the latter does not make any scaling in ξ. Although these two settings are almost parallel to each other, they posses different advantages and weakness in some subtle situations. We will comment on this later. Following their approaches, we assume that $d = 1$ and consider the following problem. Let us first assume $c > 0$ and make the change of variables $x = c(\xi - ct)$ as in [10]. Then a travelling wave solution of (5.1.9) and (5.1.10) satisfies

$$c^2(u_{xx} + u_x) + \Delta_y u + f(u) - \sqrt{\delta}v = 0, \tag{5.1.12}$$

$$c^2(v_{xx} + v_x) + \Delta_y v + \sqrt{\delta}u - \gamma\delta v = 0. \tag{5.1.13}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \tag{5.1.14}$$

To answer Question 1, we consider the weighted energy

$$\Psi_c[u,v] := \frac{c^2}{2}\int_\Omega e^x(u_x^2 - v_x^2) + \frac{1}{2}\int_\Omega e^x(|\nabla_y u|^2 - |\nabla_y v|^2) + \int_\Omega e^x H(u,v). \qquad (5.1.15)$$

Then as in the case of a gradient system, the first problem is to determine the speed c. In general, this is a difficult problem. Assuming that we know the value of c, it is easy to check that a suitable critical point of this energy is a solution of (5.1.12)-(5.1.14). Since (DFHN) is a skew-gradient system, it is expected that a mini-max approach should be proper to solve the problem. Unfortunately compared to a minimization problem, there are much less methods to solve a mini-maximization problem. Recently Chen-Hu [5] succeeded in applying a mini-max approach to solve (5.1.12)-(5.1.13) with $c = 0$ on a bounded domain in $\mathbf{R}^N$. In their study, the Sobolev space on which the energy is defined is decomposed into a "positive" space and a "negative" space, which both have infinite dimensions. It is a very interesting problem whether one can generalize their method to the case $c \neq 0$.

To further explore the existence problem, we can also consider another variational setting for the steady state of (DFHN), which has been used in many literatures. That is, to solve the equation for v first and substitute it in the u's equation. Then we obtain an equation with only one unknown function u and a non-local term. For our problem $c \neq 0$, we can solve (5.1.13) under the boundary condition (5.1.14) first and denote the solution

$$v = B_c[u] := \sqrt{\delta}\left(-c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\right) - \Delta_y + \gamma\delta\right)^{-1}[u]. \qquad (5.1.16)$$

Substituting this into (5.1.12), we obtain the non-local equation for u

$$c^2(u_{xx} + u_x) + \Delta_y u + f(u) - \sqrt{\delta}B_c[u] = 0. \qquad (5.1.17)$$

For this equation, we consider the weighted energy

$$\Phi_c^*[u] = \frac{1}{2}\int_\Omega e^x(c^2u_x^2 + |\nabla_y u|^2) + \int_\Omega e^x F(u) + \frac{\sqrt{\delta}}{2}\int_\Omega e^x u B_c[u], \qquad (5.1.18)$$

Fortunately the bilinear form of this non-local term in this energy is symmetric even $c \neq 0$. Therefore one can readily check that once the speed c is known, a suitable critical point of (5.1.18) corresponds to a solution of (5.1.12)-(5.1.14). This consideration leads to the question.

Question 2 Can variational methods be applied to find wave front or pulse solutions of a system with a non-local term? For example, applied to the non-local formulation (5.1.17).

In this chapter, the authors are concerned with finding travelling front solutions of (DFHN). The major part will focus on Question 2 for (DFHN) and rely on the non-local energy (5.1.18) to solve (5.1.12)-(5.1.14). As for Question 1, it seems more complicated to apply the energy $\Psi_c[u,v]$ to obtain a travelling wave solution. We have only partial answer for it. In section 5.6, after converting the Lagrange multiplier formulation of Heinze for the wave speed into a quotient form of energy, we use the energy $\Psi_c[u,v]$ to describe a mini-max formulation for the wave speed and show that a mini-maximizer, as in Yanagida's theory, of the speed functional is a solution of (5.1.12)-(5.1.14). However we do not know how to find a mini-maximizer in general when $c \neq 0$.

To study this problem, we refer to some papers written by Lucia, Muratov and Novaga [24]-[31]. They obtained many good viewpoint for gradient systems, with a local energy. The first two term of the energy $\Phi_c^*[u]$ is the same as their energy. Therefore the process of this paper is a little similar to [25]. However, we discuss our energy containing a nonlocal term, which causes that the boundedness of solution is not easy. Moreover, combining the advantages of [11] and [25], we obtain the travelling wave speed according to the minimal energy in subsection 5.3.2 In [25], travelling wave solutions are obtained by scaling a minimizer with negative energy. However, we show the existence of travelling wave by choosing a "good" minimizing sequence to approximate it. If some lemmas are similar to [25], we skip the proof.

For the existence of travelling waves to (DFHN), we need to assume that Ω_y is large enough. Our main theorem is as follows.

Theorem 5.1.1. *Assume $\gamma > \frac{9}{2\beta^2-5\beta+2}$. Then there exists $R_0 > 0$ such that* (5.1.12)-(5.1.14) *has a solution (u_0, v_0) with $c = c_0 > 0$ for some c_0 if Ω contains a ball with radius R_0. Moreover, (u_0, v_0) decays exponentially to 0 uniformly in y as $x \to +\infty$.*

To understand the behavior of u_0, v_0 as $x \to -\infty$, we need to investigate the equations (5.1.12)-(5.1.14) without x-coordinate, i.e.,

$$\Delta_y u + f(u) - \sqrt{\delta} v = 0 \text{ in } \Omega_y, \tag{5.1.19}$$

$$\Delta_y v + \sqrt{\delta} u - \gamma\delta v = 0 \text{ in } \Omega_y, \tag{5.1.20}$$

$$u|_{\partial\Omega_y} = v|_{\partial\Omega_y} = 0. \tag{5.1.21}$$

The above system is associated with the energy $E[u] := H_0^1(\Omega_y) \to \mathbf{R}$ defined by

$$E[u] := \frac{1}{2}\int_{\Omega_y} |\nabla_y u|^2 + \int_{\Omega_y} F(u) + \frac{\delta}{2}\int_{\Omega_y} u(-\Delta_y + \gamma\delta)^{-1}[u]. \tag{5.1.22}$$

Assume $\gamma > \frac{9}{2\beta^2-5\beta+2}$, Klaasen and Mitidieri [20] showed that $E[u]$ has at least two critical points if the domain Ω_y contains a "large" ball. One is a minimizer with negative energy and the other one derived from the Mountain Pass theorem has positive energy.

Due to the nonlocal term of (5.1.17), the asymptotic behavior of the solution (u_0, v_0) obtained in Theorem 5.1.1 as $x \to -\infty$ is much more complicated than the behavior of a gradient system. It seems (u_0, v_0) may not tend to a steady state satisfying (5.1.19)-(5.1.21) as $x \to -\infty$ in general. However, when $\gamma^2\delta > 1$, we can obtain the L^2-estimate for $u_{0,x}$ (see Lemma 5.5.3). Using this estimate and following the ideas of Proposition 6.6 and Corollary 6.8 in [25], we obtain the following two theorems.

Theorem 5.1.2. *Assume $\gamma^2\delta > 1$ and the assumptions in Theorem* 5.1.1 *hold. Let (u_0, v_0) be the solution in Theorem* 5.1.1. *Then there exists a sequence $x_n \to -\infty$ such that $\lim_{n\to+\infty}(u_0, v_0)(x_n, y)$ exists and solves* (5.1.19)-(5.1.21). *Moreover, if all critical points of $E[u]$ with negative energy are discrete in $H_0^1(\Omega_y)$, then the above limit is a full limit, that is, $\lim_{x\to-\infty}(u_0, v_0)(x, y)$ exists and $E[u_0(-\infty, y)] < 0$.*

THEOREM 5.1.3. *Let $N = 1$ and $\gamma^2\delta > 1$. Then $\gamma > \frac{9}{2\beta^2-5\beta+2}$ if and only if there exist $c_0 > 0$ and a pair of classical solutions (u_0, v_0) for* (5.1.12)-(5.1.13) *which satisfies $u_0 \in \boldsymbol{H}$, $u_{0,x} \in L^2(\Omega)$ $(u_0, v_0)(+\infty) = (0, 0)$ and $(u_0, v_0)(-\infty) = (p_2, q_2)$ (see figure 2).*

This chapter is organized as follows. In Section 5.2, we recall some Poincaré-type inequalities of weighted Sobolev spaces and investigate the properties of the non-local operator $B_c[u]$. Also the maximum principle on an unbounded cylinder is proved for equation (5.1.13). In Section 5.3 we define an energy functional associated with equations (5.1.12)-(5.1.13) and show the boundedness and low semicontinuity of the energy. Next the continuity of the minimal energy and the estimate of the travelling speed are obtained. Then, in Section 5.4 we claim the existences and some properties of minimizers with negative energy, which are chosen to approximate the travelling wave solutions. In Section 5.5, we establish the existence of the travelling wave. Moreover, the behavior of travelling wave as $x \to \pm\infty$ are investigated. Finally, in Section 5.6 we discuss the skew-gradient structure and Neumann problem of our system in Section 5.7.

5.2 Preliminaries

5.2.1 Basic properties of the weighted Sobolev space

Let $L^2_w = \{u | \|u\|^2_{L^2_w} := \int_\Omega e^x|u|^2 < \infty\}$ and $\mathbf{H}$ be the weighted Sobolev space, the completion of $C_0^\infty(\Omega)$ (the function space consisting of C^∞ functions with a compact support in Ω) with respect to the norm $\|u\|^2_{\mathbf{H}} = \|u\|^2_{L^2_w} + \|\nabla u\|^2_{L^2_w}$. The following lemma was proved by Lucia, Muratov, and Novaga [25].

LEMMA 5.2.1. *If $u(x, y) \in \boldsymbol{H}$, then*

$$\int_R^{+\infty}\int_{\Omega_y} e^x u^2 \leq 4\int_R^{+\infty}\int_{\Omega_y} e^x u_x^2, \tag{5.2.1}$$

$$\int_{\Omega_y} u^2(R, y)dy \leq e^{-R}\int_R^{+\infty}\int_{\Omega_y} e^x u_x^2, \tag{5.2.2}$$

for any $R \in \mathbf{R}$. Moreover,

$$\int_\Omega e^x u^2 \leq 4\int_\Omega e^x u_x^2. \tag{5.2.3}$$

Proof. By integration by parts with respect to x,

$$\begin{aligned}\int_R^{+\infty}\int_{\Omega_y} e^x u^2 &= -e^R\int_{\Omega_y} u^2(R, y) - 2\int_R^{+\infty}\int_{\Omega_y} e^x u u_x \\ &\leq 2\left(\int_R^{+\infty}\int_{\Omega_y} e^x u^2\right)^{1/2}\left(\int_R^{+\infty}\int_{\Omega_y} e^x u_x^2 dx\right)^{1/2},\end{aligned}$$

which gives (5.2.1). (5.2.2) follows from the estimate

$$\int_{\Omega_y} e^R u^2(R,y)dy = -\int_R^{+\infty}\int_{\Omega_y} \frac{\partial}{\partial x}(e^x u(x,y)^2)dx = -\int_R^{+\infty}\int_{\Omega_y} e^x\left(u^2+2uu_x\right)$$
$$= -\int_R^{+\infty}\int_{\Omega_y} e^x\left(u+u_x\right)^2 + \int_R^{+\infty}\int_{\Omega_y} e^x u_x^2 \le \int_R^{+\infty}\int_{\Omega_y} e^x u_x^2.$$

Finally, letting $R \to -\infty$ in (5.2.1), we obtain (5.2.3). $\square$

5.2.2 Non-local operator

The system of u and v can be reduced to one equation if we solve v, denoted by $v = B_c[u]$, in term of u from equation (5.1.13) and put it into (5.1.12). The dependence of $B_c[u]$ on γ is omitted. When $N = 1$, the operator B_c can be written as

$$B_c[u] = \frac{\sqrt{\delta}}{c\sqrt{c^2+4\gamma\delta}}\int_{-\infty}^{+\infty} e^{-\frac{\sqrt{1+4\gamma\delta/c^2}}{2}|x-\xi|+\frac{1}{2}(\xi-x)}u(\xi)d\xi. \tag{5.2.4}$$

For $N = 2$, we have (see example 7.3.2-7 in [36])

$$B_c[u] = \frac{\sqrt{\delta}}{cL}\int_0^L\int_{-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{1}{\sigma_n}\exp\left(-\frac{\sigma_n}{c}|x-\xi| + \frac{\xi-x}{2}\right)\sin\left(\frac{\pi n}{L}y\right)\sin\left(\frac{\pi n}{L}\eta\right)u(\xi,\eta)d\xi d\eta,$$

where $\sigma_n = \sqrt{\frac{\pi^2 n^2}{L^2}+\gamma\delta+\frac{c^2}{4}}$ and $\Omega_y = [0, L]$. In general, it is difficult to find a simple representation of $B_c[u]$ if $N \ge 3$. The following lemma is concerned with the existence and properties of operator B_c.

Lemma 5.2.2. *Assume $u \in L^2_w$. Then the following properties hold.*
(a) There exists a unique $v := B_c[u] \in \boldsymbol{H}$ which solves (5.1.13) *in the weak sense.*
(b) $\|B_c[u]\|_{\boldsymbol{H}} \le C_{\gamma,\delta,c}\|u\|_{L^2_w}$ for some constant $C_{\gamma,\delta,c}$ depending on γ, δ and c.
(c) $\int_\Omega e^x u B_c[u] \ge 0$.
(d) $\int_\Omega e^x u_1 B_c[u_2] = \int_\Omega e^x u_2 B_c[u_1]$ for $u_1, u_2 \in L^2_w$.
(e) If $u \in L^2_w \cap L^\infty(\Omega)$, then $B_c[u] \in C^{1,\alpha}_{loc}(\Omega)$ for all $0 < \alpha < 1$. Moreover, if the support of u is compact in Ω, $B_c[u] = O(1)e^{-x/2}$ as $x \to \pm\infty$ uniformly in y.
(f) If $u \in \boldsymbol{H} \cap L^\infty(\Omega)$, then $B_c[u] = O(1)e^{-x/2}$ as $x \to -\infty$ and $B_c[u] = O(1)e^{-x/(N+2)}$ as $x \to \infty$.

Proof. First, we prove (a). Let $w = e^{x/2}v$ and $g = e^{x/2}u$. Then (5.1.13) is equivalent to

$$c^2 w_{xx} + \Delta_y w - (\gamma\delta + c^2/4)w + \sqrt{\delta}g = 0, \tag{5.2.5}$$

where $g \in L^2(\Omega)$. We define an inner product in $H^1_0(\Omega)$ by

$$< w, \phi > := \int_\Omega c^2 w_x\phi_x + \nabla_y w\cdot\nabla_y\phi + (\gamma\delta + c^2/4)w\phi$$

and define $T_g(\phi) : H^1_0(\Omega) \to \mathbf{R}^1$ by $T_g(\phi) = \sqrt{\delta}\int_\Omega g\phi$. From the Riesz representation theorem, it follows that there exists one unique solution $w \in H^1_0(\Omega)$ solving $T_g(\phi) =< w, \phi >$ for all $\phi \in H^1_0(\Omega)$. This implies the statement (a).

From $T_g(w) =< w, w >$, (b) and (c) follow easily.

Let $w_i = e^{x/2}B_c[u_i]$ and $g_i = e^{x/2}u_i, i = 1, 2$. According to $T_{g_1}(w_2) =< w_1, w_2 >= T_{g_2}(w_1)$,we have (d).

To prove (e), let $x_1 < x_2 < x_3 < x_4$, $K_1 = (x_2, x_3) \times \Omega_y$ and $K_2 = (x_1, x_4) \times \Omega_y$. By (5.2.5), L^p-theory (see Theorem 8.12, Theorem 9.13 and Lemma 9.16 in [10]) and the Sobolev imbedding theorem, for all $0 < \alpha < 1$, we obtain

$$\|w\|_{C^{1,\alpha}(K_1)} \leq C(\|w\|_{L^2(K_2)} + \|g\|_{L^\infty(K_2)}), \tag{5.2.6}$$

where C depends on N, c, p, γ, δ and the geometry of K_1 and K_2. Therefore, $B_c[u] \in C^{1,\alpha}_{loc}(\Omega)$. Furthermore, if u has compact support then so does g. The right hand side of (5.2.6) is less than $C(\|w\|_{L^2(\Omega)} + \|g\|_{L^\infty(\Omega)})$, which is invariant when K_1 is translated along x-axis. This means $w \in L^\infty(\Omega)$ and $v(x, y) = e^{-x/2}w(x, y) = O(1)e^{-x/2}$ as $x \to \pm\infty$ uniformly in y.

Now we prove (f). Let $v = B_c[u]$. By (5.2.6),

$$\|v\|_{C^{1,\alpha}(K_1)} \leq C_1e^{-x/2}\|w\|_{C^{1,\alpha}(K_1)} \leq C_1C[e^{-x/2}\|w\|_{L^2(\Omega)} + \|u\|_{L^\infty(\Omega)}]. \tag{5.2.7}$$

Therefore $|\nabla v| = O(1)$ as $x \to \infty$ and $|\nabla v| = O(e^{-x/2})$ as $x \to -\infty$. Suppose $|v(x, y)| = \sigma > 0$. If $x > 0$, then since $|\nabla v| = O(1)$, there exists a neighborhood U of (x, y) with volume of the order σ^N on which $|v| > \sigma/2$. Hence

$$e^x\sigma^{N+2} \leq C\int_U e^xv^2 \leq C||v||^2_{L^2_w} < \infty$$

and

$$|v(x, y)| = \sigma = O(x/(N + 2)) \text{ as } x \to \infty.$$

Now we assume $x < 0$ and $\sigma > e^{-x/2}$. Then there exists a neighborhood U of (x, y) with volume $O(1)$ on which $|v| > \sigma/2$. Again by the boundedness of $||v||^2_{L^2_w}$, we conclude $v = O(e^{-x/2})$ as $x \to -\infty$. □

From Lemma 5.2.2(e), we know that v is bounded as $x \to \infty$ if the support of u is compact. This is not strong enough for our purpose. We will need the boundedness of v on the whole domain Ω in finding a travelling wave solution of (5.1.12)-(5.1.14). With the boundary condition (5.1.14), one may expect that the boundedness of v follows from the maximum principle if u is bound. However in general, this is not true in an unbounded domain if v does not satisfy suitable growth condition at infinity. The following lemma has the form suitable for our purpose. More results related to the maximum principle for second order elliptic equations on unbounded domains can be found in [42].

Lemma 5.2.3. *Let $u \in \boldsymbol{H}$ satisfy $\eta_1 \leq u \leq \eta_2$, where $\eta_1 \leq 0$ and $\eta_2 \geq 0$ are two constants. Assume $B_c[u] = O(1)e^{m_\pm x}$ as $x \to \pm\infty$ uniformly in y, where $m_+ \leq \dfrac{-1 + \sqrt{1 + 4\gamma\delta/c^2}}{2}$ and $m_- \geq \dfrac{-1 - \sqrt{1 + 4\gamma\delta/c^2}}{2}$. Then*

$$\frac{\eta_1}{\gamma\sqrt{\delta}} \leq B_c[u] \leq \frac{\eta_2}{\gamma\sqrt{\delta}}.$$

Proof. Let $h = B_c[u] - \frac{\eta_1}{\gamma\sqrt{\delta}}$. Then h satisfies $c^2(h_{xx}+h_x)+\Delta_y h-\gamma\delta h \le 0$ in the weak sense and $h|_{\partial\Omega} \ge 0$. We claim $h \ge 0$ in Ω and prove this by contradiction. Suppose $\inf_\Omega h < 0$. Let $\lambda > 0$ be the first eigenvalue of Laplace's operator $-\Delta$ on the unit ball $B_1(0) \subseteq \mathbf{R}^{N-1}$ with the Dirichlet boundary condition and $\phi(y)$ be the corresponding eigenfunction which is positive in $B_1(0)$ and satisfies $\sup \phi(y) = 1$. Let $\lambda_\pm = \frac{-1\pm\sqrt{1+4(\lambda\epsilon^2+\gamma\delta)/c^2}}{2}$ and $w(x,y) = (e^{\lambda_+ x} + e^{\lambda_- x})\phi(\epsilon y)$, where $\epsilon > 0$ is chosen such that $\Omega_y \subseteq B_{1/\epsilon}(0)$. Then we have $c^2(w_{xx}+w_x)+\Delta_y w-\gamma\delta w = 0$. By direct computation, the function $g = \dfrac{h}{w}$ satisfies

$$c^2(g_{xx}+g_x)+\Delta_y g+2[c^2\partial_x \log w \cdot g_x + \nabla_y \log w \cdot \nabla_y g)] \le 0.$$

Moreover, $g(x,y) \to 0$ as $x \to \pm\infty$ uniformly in y and $g|_{\partial\Omega} \ge 0$. Since g is continuous and we assume $\inf_\Omega h < 0$, g attains the negative minimum at some interior point (x_0, y_0). Now we choose a bounded subset Ω^* of Ω containing (x_0,y_0) such that $g|_{\partial\Omega^*}$ is greater than the minimum. Then the fact g has an interior minimum contradicts the maximum principle for the bounded domain Ω^*. Therefore $B_c[u] \ge \frac{\eta_1}{\gamma\sqrt{\delta}}$ must hold. The other inequality $B_c[u] \le \dfrac{\eta_2}{\gamma\sqrt{\delta}}$ follows from a similar argument. □

5.3 Variational approach

5.3.1 Boundedness and lower semicontinuity of the energy

Define the energy functional $\Phi_c^* : \mathbf{H} \to \mathbf{R}^1$ by

$$\begin{aligned}\Phi_c^*[u] &= \frac{1}{2}\int_\Omega e^x(c^2u_x^2+|\nabla_y u|^2)+\int_\Omega e^x F(u)+\frac{\sqrt{\delta}}{2}\int_\Omega e^x u B_c[u] \\ &:= \Phi_c^{(1)}[u]+\Phi_c^{(2)}[u]+\Phi_c^{(3)*}[u].\end{aligned} \tag{5.3.1}$$

Then the Euler-Lagrange equation of (5.3.1) is the equation (5.1.12) with $v = B_c[u]$. For a variational problem, we usually look for a minimizer of the energy functional. In our case, we have $\Phi_c^*[u(x-x_0,y)] = e^{x_0}\Phi_c^*[u(x,y)]$. Hence $\inf \Phi_c^* =$ either 0 or $-\infty$ in $\mathbf{H}$ for all $c > 0$ since x_0 can be passed to $\pm\infty$. For this reason, it is better to add some constraint in our problem to avoid the $-\infty$ minimum and bad minimizing sequences due to the translation. As in [25], we define the constraint

$$\mathcal{B} = \{u \in \mathbf{H} \mid \frac{1}{2}\int_\Omega e^x u_x^2 = 1\}. \tag{5.3.2}$$

If $u(x,y) \in \mathbf{H}$ is nontrival, then $u_x(x,y)$ is also nontrivial by Lemma 5.2.1. This implies there exists unique one x_0 such that $u(x-x_0,y) \in \mathcal{B}$. Moreover, Φ_c^* is bounded from below on $\mathcal{B}$. Indeed, from $F(u) \ge -\frac{2\beta^2-5\beta+2}{18}u^2$, Lemma 5.2.1 and Lemma 5.2.2 (c), it follows

$$\Phi_c^*[u] \ge c^2 - \frac{4(2\beta^2-5\beta+2)}{9} \text{ on } \mathcal{B}.$$

The non-local term $\Phi_c^{(3)*}$ causes additional difficulties. Due to this term, it is hard to show that the minimizers of Φ_c^* are bounded on Ω. To overcome this difficulty, we take a cut-off of u in $\Phi_c^{(3)*}$ while leave u in $\Phi_c^{(1)}$ and $\Phi_c^{(2)}$ unchanged. To do this, we consider the problem on a subset $\mathcal{C}$ of $\mathbf{H}$ as follows. In uv plane, choose a rectangle $ABCD$ such that $\overline{AB}$ ($\overline{CD}$) and $\overline{BC}$ ($\overline{DA}$) lay on the right (left) side of $\{v = \frac{u}{\gamma\sqrt{\delta}}\}$ and $\{v = \frac{f(u)}{\sqrt{\delta}}\}$, respectively, and such that $\overline{DA} \subset \{u = a_1 < 0\}$, $\overline{BC} \subset \{u = a_2 > 0\}$, $\overline{AB} \subset \{v = b_1 < 0\}$ and $\overline{CD} \subset \{v = b_2 > 0\}$ (See Figure 2.). Note that $b_2 > \frac{a_2}{\gamma\sqrt{\delta}}$ and $b_1 < \frac{a_1}{\gamma\sqrt{\delta}}$. By [3], this rectangle is an invariant set of the equations (5.1.1) and (5.1.2). Let $\overline{u} := \min\{\max\{u, a_1\}, a_2\}$ and

$$\mathcal{C} = \{u \in \mathbf{H} | a_1 \le u \le a_2, b_1 \le B_c[u] \le b_2\}. \tag{5.3.3}$$

By Lemma 5.2.2 (f) and Lemma 5.2.3,

$$b_1 < \frac{a_1}{\gamma\sqrt{\delta}} \le B_c[\bar{u}] \le \frac{a_2}{\gamma\sqrt{\delta}} < b_2, \tag{5.3.4}$$

$$\mathcal{C} = \{u \in \mathbf{H} | a_1 \le u \le a_2\} \text{ and } \bar{u} \in \mathcal{C} \text{ for } u \in H. \tag{5.3.5}$$

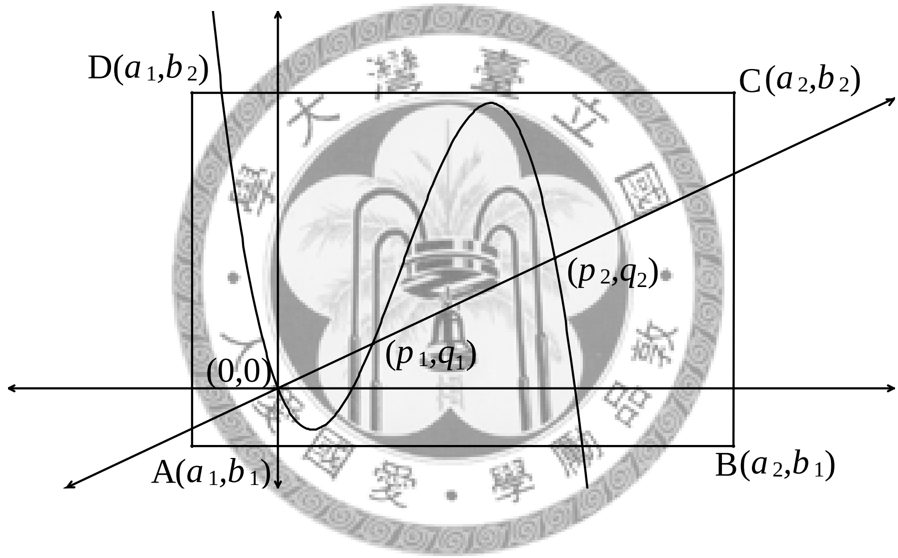

Figure 5.1: An invariant set of (5.1.1) and (5.1.2), where $0 < p_1 < p_2$ and p_1, p_2 solve $u^2 - (\beta + 1)u + (\beta + \frac{1}{\gamma}) = 0$.

We define the new cut-off energy $\Phi_c[u]$ by replacing $\Phi_c^{(3)*}[u]$ by

$$\Phi_c^{(3)}[u] := \frac{\sqrt{\delta}}{2} \int_\Omega e^x \overline{u} B_c[\overline{u}] = \Phi_c^{(3)*}[\bar{u}]. \tag{5.3.6}$$

That is,

$$\Phi_c[u] = \Phi_c^{(1)}[u] + \Phi_c^{(2)}[u] + \Phi_c^{(3)}[u].$$

$\Phi_c[u]$ and $\Phi_c^*[u]$ can be estimated by the same lower bound on $\mathcal{B}$, i.e.,

$$\Phi_c[u] \ge c^2 - \frac{4(2\beta^2 - 5\beta + 2)}{9}. \tag{5.3.7}$$

In the following sections, we will consider Φ_c instead of Φ_c^* on $\mathcal{B}$. As mentioned above, we have

$$b_1 < \frac{a_1}{\gamma\sqrt{\delta}} \le B_c[\bar{u}] \le \frac{a_2}{\gamma\sqrt{\delta}} < b_2$$

for $u \in H$. This makes the non-local term in Φ_c is easier to handle than the non-local term in Φ_c^*. Later we will show that the minimizer u_c of Φ_c we seek for satisfies $\bar{u}_c = u_c$ and is also a local minimizer of Φ_c^*. Now we show the weak lower semicontinuity (l.s.c.) of $\Phi_c[u]$.

LEMMA 5.3.1. *Let $u_n \to u$ weakly in $\boldsymbol{H}$. Then $\Phi_c[u] \le \liminf_{n\to\infty} \Phi_c[u_n]$.*

Proof. Note that $\Phi_c^{(1)}$, a part of the norm in $\mathbf{H}$, is weakly l.s.c.. By the proof of Proposition 5.5 of [25], we know that $\Phi_c^{(2)}$ is also weakly l.s.c.. Therefore, it suffices to show $\Phi_c^{(3)}[u]$ is weakly l.s.c..

First, we claim that $\Phi_c^{(3)*}[u]$ is weakly l.s.c. in L_w^2. This will follow if we can prove $\Phi_c^{(3)*}[u]$ is convex and l.s.c. in L_w^2. By Hölder's inequality and Lemma 5.2.2(b), $\Phi_c^{(3)*}[u]$ is l.s.c. in L_w^2. Indeed, let $\phi_n \to \phi$ strongly in L_w^2, then

$$\begin{aligned}\frac{2}{\sqrt{\delta}}\left|\Phi_c^{(3)*}[\phi_n] - \Phi_c^{(3)*}[\phi]\right| &\le \left|\int_\Omega e^x \phi_n B_c[\phi_n - \phi]\right| + \left|\int_\Omega e^x(\phi_n - \phi)B_c[\phi]\right| \\ &\le \|\phi_n\|_{L_w^2}\|B_c[\phi_n - \phi]\|_{L_w^2} + \|\phi_n - \phi\|_{L_w^2}\|B_c[\phi]\|_{L_w^2} \\ &\le C_{\gamma,\delta,c}(\|\phi_n\|_{L_w^2} + \|\phi\|_{L_w^2})\|\phi_n - \phi\|_{L_w^2}\end{aligned}$$

By choosing large n such that $\|\phi_n\|_{L_w^2} \le 2\|\phi\|_{L_w^2}$, the above inequality implies that $\Phi_c^{(3)*}[u]$ is continuous in L_w^2. To show the convexity of $\Phi_c^{(3)*}$, let ψ_0, ψ_1 be in L_w^2 and $0 \le k \le 1$. Then

$$\begin{aligned}&\frac{2}{\sqrt{\delta}}\left((1-k)\Phi_c^{(3)*}[\psi_0] + k\Phi_c^{(3)*}[\psi_1] - \Phi_c^{(3)*}[(1-k)\psi_0 + k\psi_1]\right) \\ &= k(1-k)\int_\Omega e^x(\psi_0 - \psi_1)B_c[\psi_0 - \psi_1] \ge 0.\end{aligned}$$

Here we have used the linearity of $B_c[u]$ and Lemma 5.2.2(c). Hence $\Phi_c^{(3)*}[u]$ is convex in L_w^2. This together with the l.s.c. in L_w^2 implies that $\Phi_c^{(3)*}[u]$ is weakly l.s.c. in L_w^2.

Next, we show $\overline{u_n} \to \overline{u}$ weakly in $\mathbf{H}$. Suppose there are a bounded linear functional $h : \mathbf{H} \to \mathbf{R}^1$, a number $\epsilon > 0$ and a subsequence u_{n_k} of u_n such that

$$|h(\overline{u_{n_k}}) - h(\overline{u})| \ge \epsilon. \tag{5.3.8}$$

From $u_{n_k} \to u$ weakly in $\mathbf{H}$, we obtain that after passing to a subsequence, $\overline{u_{n_k}} \to \overline{u}$ a.e. in Ω and $\sup_k \|\overline{u_{n_k}}\|_{\mathbf{H}} \le \sup_k \|u_{n_k}\|_{\mathbf{H}} < \infty$. Consequently, there is a subsequence $u_{n_{k_j}}$ of u_{n_k} such that $\overline{u_{n_{k_j}}} \to u^*$ weakly in $\mathbf{H}$ and a.e. in Ω for some $u^* \in \mathbf{H}$. This implies that $u^* = \overline{u}$ a.e. and $\overline{u_{n_{k_j}}} \to \overline{u}$ weakly in $\mathbf{H}$, which contradicts to (5.3.8). The weak l.s.c. of $\Phi_c^{(3)}$ in $\mathbf{H}$ follows from the property $\bar{u}_n \to \bar{u}$ weakly in $\mathbf{H}$ and the (weak) l.s.c. of $\Phi_c^{(3)*}[u]$ in L_w^2. □

5.3.2 Continuity of minimal energy

From the lower bound (5.3.7) of $\Phi_c[u]$ for $u \in \mathcal{B}$, we obtain

$$\mu_c := \inf_{\mathcal{B}} \Phi_c > -\infty.$$

In this section, we show that μ_c is continuous in c. To do this, we need to prove the uniform continuity of $\Phi_c[u]$ in c.

Lemma 5.3.2. *Let $c_1 \le c_2$ and C be positive and fixed. For $c, c' \in [c_1, c_2]$ and $\|u\|_{\boldsymbol{H}} \le C$,*

$$|\Phi_c[u] - \Phi_{c'}[u]| \le M|c - c'|, \tag{5.3.9}$$

where M is a constant depending only on c_1, c_2, C, γ and δ.

Proof. Let $v = B_c[\bar{u}]$. Obviously, $|\Phi_c^{(1)}[u] - \Phi_{c'}^{(1)}[u]| \le M_1|c - c'|$ for some M_1 and $|\Phi_c^{(2)}[u] - \Phi_{c'}^{(2)}[u]| = 0$. On the other hand, differentiating equation (5.1.13) and arguing as in the proof of Lemma 5.2.2, we obtain $v_{z_i} \in \mathbf{H}$ and

$$\|v_{z_i}\|_{\mathbf{H}} \le C_{\gamma,\delta,c}\|\bar{u}_{z_i}\|_{L^2_w}, i = 1, 2, \cdots n, \tag{5.3.10}$$

where $(z_1, z_2, \cdots z_n) = (x, y)$.

Consider the following two equations.

$$c^2(v_{xx} + v_x) + \Delta_y v + \sqrt{\delta}\bar{u} - \gamma\delta v = 0,$$
$$c'^2(v'_{xx} + v'_x) + \Delta_y v' + \sqrt{\delta}\bar{u} - \gamma\delta v' = 0.$$

Setting $V = v - v'$, we obtain

$$c^2(V_{xx} + V_x) + \Delta_y V - \gamma\delta V = -(c^2 - c'^2)(v'_{xx} + v'_x).$$

Using the argument in Lemma 5.2.2 again, we arrive at

$$\|V\|_{\mathbf{H}} \le \frac{C_{\gamma,\delta,c}}{\sqrt{\delta}}|c^2 - c'^2|\|v'_{xx} + v'_x\|_{L^2_w}.$$

This together with (5.3.10) implies that $\|B_c[\bar{u}] - B_{c'}[\bar{u}]\|_{L^2_w} \le M_2|c - c'|$ for some M_2. By Hölder's inequality, $|\Phi_c^{(3)}[u] - \Phi_{c'}^{(3)}[u]| \le M|c - c'|$ for some M. The proof is complete. □

Theorem 5.3.3. *μ_c is continuous in $c \in (0, +\infty)$.*

Proof. For $c > 0$, let $c_k \in [c/2, 2c]$ and $c_k \to c$. We show that $\limsup_{k\to\infty} \mu_{c_k} = \mu_c = \liminf_{k\to\infty} \mu_{c_k}$. Suppose $\limsup_{k\to\infty} \mu_{c_k} > \mu_c$. Then there is a subsequence k_l such that

$$\mu_{c_{k_l}} \ge \mu_c + 3\sigma, \tag{5.3.11}$$

where $\sigma := \frac{1}{4}(\limsup_{k\to\infty} \mu_{c_k} - \mu_c) > 0$. By the definition of μ_c, we choose $u \in \mathcal{B}$ so that

$$\mu_c \ge \Phi_c[u] - \sigma. \tag{5.3.12}$$

By Lemma 5.3.2, as l is large,

$$\Phi_c[u] \geq \Phi_{c_{k_l}}[u] - \sigma \geq \mu_{c_{k_l}} - \sigma \tag{5.3.13}$$

Putting (5.3.11) to (5.3.13) together, we obtain a contradiction. On the other hand, assume $\limsup_{k\to\infty} \mu_{c_k} < \mu_c$. Then there is a subsequence k_j such that

$$\mu_c \geq \mu_{c_{k_j}} + 3\eta, \tag{5.3.14}$$

where $\eta := \frac{1}{4}(\mu_c - \limsup_{k\to\infty} \mu_{c_k}) > 0$. For each k_j, we can choose $u_j \in \mathcal{B}$ so that

$$\mu_{c_{k_j}} \geq \Phi_{c_{k_j}}[u_j] - \eta. \tag{5.3.15}$$

By Lemma 5.3.2, as j is large,

$$\Phi_{c_{k_j}}[u_j] \geq \Phi_c[u_j] - \eta \geq \mu_c - \eta. \tag{5.3.16}$$

Combining (5.3.14) to (5.3.16), we obtain a contradiction again. The proof for $\liminf \mu_{c_k} = \mu_c$ is similar. We omit it here. $\square$

5.3.3 Estimates for the travelling speed

In this subsection, we character the travelling wave speed c by the minimal energy μ_c. The following lemma indicates that a travelling wave solution has zero energy.

Lemma 5.3.4. *Suppose that $u \in L^\infty(\Omega) \cap C^2(\Omega) \cap \boldsymbol{H}$ solves* (5.1.12)-(5.1.14) *with $v = B_c[u]$ for some $c \neq 0$. Then $\Phi_c^*[u] = 0$.*

Proof. If we multiple equation (5.1.12) by $e^x u_x$ and integrate it over Ω. By some integrations by parts, we obtain

$$\begin{aligned}
0 &= \int_\Omega e^x u_x\{c^2(u_{xx} + u_x) + \Delta_y u + f(u) - \sqrt{\delta}B_c[u]\} \\
&= \int_{\Omega_y} e^x\{\frac{1}{2}c^2u_x^2 - F(u) - \frac{1}{2}|\nabla_y u|^2 - \frac{\sqrt{\delta}}{2}uB_c[u]\}|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\int_{\partial\Omega_y} e^x u_x \nabla_y u \cdot \nu_y \\
&\quad - \int_\Omega e^x\{\frac{1}{2}c^2u_x^2 - F(u) - \frac{1}{2}|\nabla_y u|^2 - \frac{\sqrt{\delta}}{2}uB_c[u]\} + \int_\Omega c^2 e^x u_x^2 \\
&= \Phi_c^*[u],
\end{aligned}$$

where ν_y is the outer normal of $\partial\Omega_y$. Here we have used the assumption of u, $B_c[u_x] = (B_c[u])_x$ and Lemma 5.2.2 (d). The proof is complete. $\square$

The above lemma implies that if μ_c is realized by some travelling wave solution $u \in \mathcal{B} \cap \mathcal{C}$, then $\mu_c = \Phi_c[u] = 0$. Therefore it is crucial to search for the roots of μ_c. In the following lemma, we prove that $\mu_c > 0$ if $0 < c \gg 1$ and $\mu_c < 0$ if $c \ll 1$. Then by the intermediate value theorem for continuous functions, μ_c has at least a root. At the same time, this lemma also obtains a lower bound and an upper bound of the travelling wave speeds of the variational solutions.

Lemma 5.3.5. *Let $\mu_c := \inf_{\mathcal{B}} \Phi_c$. Then*
(a) $\mu_c > 0$ if $c > c_{\max} := \frac{2}{3}\sqrt{2\beta^2 - 5\beta + 2}$.
(b) For $\gamma > \frac{9}{2\beta^2-5\beta+2}$, there exist $c_{\min} > 0$ and $R_0 > 0$ such that if Ω contains a ball B_{R_0} with radius R_0, then $\mu_c < 0$ for all $0 < c < c_{\min}$.

Proof. (a) follows from (5.3.7). (b) is proven as follows. In [20], Klaasen and Mitidieri showed that there exist $R_0 > 0$ and $u \in C_0^{0,1}(\Omega)$ with $u = p_2$ on $B_{R_0-1}(0)$, $u = p_2(R_0 - |(x,y)|)$ on $B_{R_0}(0) - B_{R_0-1}(0)$, $u \equiv 0$ on $\Omega - B_{R_0}(0)$ such that

$$\frac{1}{2}\int_{B_{R_0}} |\nabla u|^2 + \int_{B_{R_0}} F(u) + \frac{\sqrt{\delta}}{2}\int_{B_{R_0}} uB_0[u] < 0,$$

where $B_0[u] = \sqrt{\delta}(-\Delta + \gamma\delta)^{-1}u$. This together with Lebesgue's dominated convergence theorem implies that there is a small $c_{\min} > 0$ so that for $0 < c < c_{\min}$,

$$I := \frac{1}{2}\int_{B_{R_0}} e^{cx}|\nabla u|^2 + \int_{B_{R_0}} e^{cx}F(u) + \frac{\sqrt{\delta}}{2}\int_{B_{R_0}} e^{cx}uB_c^*[u] < 0,$$

where $B_c^*[u] = \sqrt{\delta}(-\Delta - c\partial_x + \gamma\delta)^{-1}u$. Let $\eta_c^*(x,y) = u(x/c, y)$. Then $\Phi_c[\eta_c^*] = cI < 0$. We further let $\eta_c(x,y) = \eta_c^*(x + x_0, y)$ and choose x_0 to make $\eta_c \in \mathcal{B}$. Then $\Phi_c[\eta_c] = e^{-x_0}\Phi_c[\eta_c^*] < 0$ also holds. From $\eta_c \in C_0^{0,1}(\Omega)$, Lemma 5.2.2$(e)$ and Lemma 5.2.3, it follows $0 \le B_c[\eta_c] \le \frac{p_2}{\gamma\sqrt{\delta}} = q_2$ (See Figure 2). Therefore, $\eta_c \in \mathcal{C}$. Consequently, $\mu_c < 0$ for $0 < c < c_{\min}$. $\square$

Remark. If $N = 1$, we can prove the above lemma directly. Let $\eta_c^*(x) = p_2$ if $x \le 0$ and $\eta_c^*(x) = p_2 e^{-\lambda x}$ if $x > 0$, where $\lambda > \frac{-1+\sqrt{1+4\gamma\delta/c^2}}{2}$ and $\lambda = O(\frac{1}{c^2})$. By direct calculation, $\Phi_c[\eta_c^*] = F(p_2) + \frac{p_2^2}{2\gamma} + o(c)$ when c is small enough. Since $F(p_2) + \frac{p_2^2}{2\gamma} < 0$ is equivalent to $\gamma > \frac{9}{2\beta^2-5\beta+2}$, we can make $\Phi_c[\eta_c^*] < 0$ when c is small.

5.4 Existences and properties of minimizers with negative energy

Theorem 5.4.1. *If $\mu_c < 0$ for some c, then there exists $u_c \in \mathcal{B}$ such that $\Phi_c[u_c] = \mu_c$.*

Proof. Choose a minimizing sequence $u_n \in \mathcal{B}$ with $\Phi_c[u_n] < 0$. Obviously, $\int_\Omega e^x|u_n|^2 \le 8$ by $u_n \in \mathcal{B}$ and Lemma 5.2.1. On the other hand, by $F(u) \ge -\frac{2\beta^2-5\beta+2}{18}u^2$ and Lemma 5.2.2(c), we obtain

$$\begin{aligned}\int_\Omega e^x|\nabla_y u_n|^2 &= 2\Phi_c[u_n] - c^2\int_\Omega e^x u_{n,x}^2 - \sqrt{\delta}\int_\Omega e^x \overline{u_n}B_c[\overline{u_n}] - 2\int_\Omega e^x F(u_n)\\ &\le \frac{8(2\beta^2 - 5\beta + 2)}{9}.\end{aligned}$$

Therefore, u_n is uniformly bounded in $\mathbf{H}$. Consequently, u_n converges weakly to some $u \in \mathbf{H}$, up to a subsequence. By Lemma 5.3.1, $\Phi_c[u] \le \mu_c < 0$ and u is non-trivial. Note that

$$0 < \frac{1}{8}\int_\Omega e^x u^2 \le \frac{1}{2}\int_\Omega e^x u_x^2 \le \liminf_{n\to\infty}\frac{1}{2}\int_\Omega e^x u_{n,x}^2 = 1.$$

By translating u in x-coordinate, there is a $x_0 \geq 0$ such that $\psi(x,y) = u(x-x_0,y) \in \mathcal{B}$. We claim that $u, \psi \in \mathcal{C}$. Indeed, let $g_n = e^{x/2}\overline{u_n}$, $g = e^{x/2}\overline{u}$, $w_n = e^{x/2}B_c[\overline{u_n}]$ and $w = e^{x/2}B_c[\overline{u}]$. Note that $g_n \rightharpoonup g$ in $L^2_w(\Omega)$ and $\overline{u_n} \rightharpoonup \overline{u}$ in $\mathbf{H}$. (by Lemma 5.3.1). By Lemma 5.2.2(b),

$$\|w_n\|_{H^1_0(\Omega)} \leq C_{\gamma,\delta,c}\|\overline{u_n}\|_{L^2_w(\Omega)} \leq C_{\gamma,\delta,c} \sup_n \|u_n\|_{\mathbf{H}} < \infty.$$

Therefore, up to a subsequence, $w_n \rightharpoonup \tilde{w}$ in $H^1_0(\Omega)$ for some $\tilde{w} \in H^1_0(\Omega)$. Moreover,

$$\int_\Omega \left[c^2 w_{n,x}\phi_x + \nabla_y w_n \cdot \nabla_y \phi + (\gamma\delta + c^2/4)w\phi\right] - \sqrt{\delta}\int g_n\phi = 0, \tag{5.4.1}$$

where ϕ is a test function in $H^1_0(\Omega)$. Let $n \to \infty$ in (5.4.1). Then we obtain

$$\int_\Omega \left[c^2 \tilde{w}_x\phi_x + \nabla_y \tilde{w} \cdot \nabla_y \phi + (\gamma\delta + c^2/4)\tilde{w}\phi\right] - \sqrt{\delta}\int g\phi = 0.$$

By the uniqueness of w, we have $\tilde{w} = w$ and $w_n \rightharpoonup w$ in H^1_0. It implies that $w_n \to w$ a.e. in Ω and $B_c[\overline{u_n}] \to B_c[\overline{u}]$ a.e. in Ω. Therefore, $u, \psi \in \mathcal{C}$ since $u_n \in \mathcal{C}$. From

$$\mu_c \leq \Phi_c[\psi] = e^{x_0}\Phi_c[u] \leq \Phi_c[u] \leq \mu_c,$$

we derive $x_0 = 0$ and $\Phi_c[u] = \mu_c$. This u denoted by u_c is what we seek for. □

The function u_c in theorem 5.4.1 may attain the boundary of $\mathcal{C}$. If this case happens, then u_c satisfies the Euler-Lagrange equation corresponding to Φ_c, but may not satisfy the Euler-Lagrange equation (5.1.12)-(5.1.14) corresponding to Φ_c^*. The following lemma is one of the crucial steps which shows that $a_1 < u_c < a_2$ and the Euler-Langrange equations corresponding to Φ_c and to Φ_c^* are the same. In the following lemma, $a > b$ denotes "the essential sup of $b - a > 0$".

LEMMA 5.4.2. *Let $v_c = B_c[u_c]$. Then u_c and v_c are $C^{2,\alpha_0}(\bar{\Omega})$ and satisfy for each $(x,y) \in \Omega$, $a_1 < u_c(x,y) < a_2$ and $\frac{a_1}{\gamma\sqrt{\delta}} \leq v_c(x,y) \leq \frac{a_2}{\gamma\sqrt{\delta}}$. Moreover, u_c is a solution of*

$$(c^2 - \mu_c)(u_{c,xx} + u_{c,x}) + \Delta_y u_c + f(u_c) - \sqrt{\delta}v_c = 0. \tag{5.4.2}$$

Proof. First, we claim $a_1 \leq u_c \leq a_2$. Suppose that $S := \{(x,y) \in \Omega | u_c < a_1 \text{ or } u_c > a_2\}$ has positive measure. Then $\Phi_c^{(1)}[\overline{u}] < \Phi_c^{(1)}[u]$. Since $F(u)$ is decreasing on $(-\infty, a_1]$ and increasing on $[a_2, +\infty)$, $\Phi_c^{(2)}[\overline{u}] \leq \Phi_c^{(2)}[u]$. Consequently, $\Phi_c[\overline{u}] < \Phi_c[u]$, which contradicts $\Phi_c[u] = \mu_c$. Therefore S has measure zero and for a suitable representation of u_c, $a_1 \leq u_c \leq a_2$ at each point of Ω. Moreover, from Lemma 5.2.3 and Lemma 5.2.2 (f), it follows $\frac{a_1}{\gamma\sqrt{\delta}} \leq v_c(x,y) \leq \frac{a_2}{\gamma\sqrt{\delta}}$.

Next, we obtain a variational inequality for u_c. For given $0 \leq \phi \in C^1_0(\Omega)$. Note that $u_c + \epsilon\phi \in \mathcal{C}$ if ϵ is small. Indeed, by Lemma 5.2.2(e) and Lemma 5.2.3,

$$|B_c[\overline{u_c + \epsilon\phi} - \overline{u_c}]| \leq \frac{\epsilon\|\phi\|_\infty}{\gamma\sqrt{\delta}}.$$

Therefore

$$\frac{a_1 - \epsilon\|\phi\|_\infty}{\gamma\sqrt{\delta}} \le B_c[\overline{u_c + \epsilon\phi}] \le \frac{a_2 + \epsilon\|\phi\|_\infty}{\gamma\sqrt{\delta}}.$$

Choose $\epsilon < \min\{\frac{a_1 - \gamma\sqrt{\delta}b_1}{\|\phi\|_\infty}, \frac{\gamma\sqrt{\delta}b_2 - a_2}{\|\phi\|_\infty}\}$, then $u_c + \epsilon\phi \in \mathcal{C}$.

Note that $u_c \in \mathcal{B}$ is a minimizer of the energy $\Psi_c[u] := \dfrac{\Phi_c[u]}{1/2\int_\Omega e^x u_x^2}$ on $\mathbf{H}$. Therefore we have the variational inequality

$$\begin{aligned} 0 &\le \lim_{\epsilon\to 0^+} \frac{\Psi_c[u_c + \epsilon\phi] - \Psi_c[u_c]}{\epsilon} \\ &= \int_\Omega e^x((c^2 - \mu_c)u_{c,x}\phi_x + \nabla_y u_c \nabla_y \phi) - \int_\Omega e^x f(u_c)\phi + \sqrt{\delta}\int_\Omega e^x B_c[u_c]\chi_E\phi, \end{aligned} \tag{5.4.3}$$

where χ_E is the characteristic function on $E = \{(x,y) \in \Omega | a_1 \le u_c(x,y) < a_2\}$. The last term of (5.4.3) is obtained as follows. Let $g_\epsilon = \dfrac{\overline{u_c + \epsilon\phi} - \overline{u_c}}{\epsilon}$. Then $|g_\epsilon| \le \phi$, $\lim_{\epsilon\to 0^+} g_\epsilon = \chi_E\phi$ a.e. and

$$\lim_{\epsilon\to 0^+} \frac{\Phi_c^{(3)}[u_c + \epsilon\phi] - \Phi_c^{(3)}[u_c]}{\epsilon} = \frac{\sqrt{\delta}}{2}\lim_{\epsilon\to 0^+}\int_\Omega e^x(g_\epsilon B_c[\overline{u_c + \epsilon\phi}] + \overline{u_c}B_c[g_\epsilon]). \tag{5.4.4}$$

According to Lemma 5.2.2 (f) and Lemma 5.2.3, $\lim_{\epsilon\to 0^+} B_c[\overline{u_c + \epsilon\phi}] = B_c[\overline{u_c}]$. By Lemma 5.2.2 ($d$), the integrand of (5.4.4) is bounded by $2\phi\cdot\max\{-b_1, b_2\}$. By Lebesgue's dominated convergence theorem, we obtain the last term of (5.4.3).

Let $U := \{z \in \Omega | \exists 0 < r_z \le 1, \operatorname{essinf}_{B_{r_z/2}(z)\cap\Omega} u_c = a_1\}$. We use weak Harnack's inequality for supersolutions in [10] (Theorem 8.18) to show that $U = \emptyset$. Let $w = u_c - a_1 \ge 0$. From the choice of the invariant set $\mathcal{C}$, it follows

$$-f(u_c) + \sqrt{\delta}B_c[u_c]\chi_E \le -f(w + a_1) + \sqrt{\delta}b_2 = -f(a_1) + \sqrt{\delta}b_2 + wg < wg, \tag{5.4.5}$$

where $g = w^2 + (3a_1 - 1 - \beta)w + 3a_1^2 - 2(\beta+1)a_1 + \beta$ is in $L^\infty(\Omega)$. This together with (5.4.3) leads to

$$\int_\Omega e^x((c^2 - \mu_c)w_x\phi_x + \nabla_y w\nabla_y\phi + gw\phi) \ge 0. \tag{5.4.6}$$

If $z \in U$, by weak Harnack's inequality,

$$r_z^{-N/p}\|w\|_{L^p(B_{r_z}(z))} \le C\operatorname{essinf}_{B_{r_z/2}(z)} w = 0$$

for $1 \le p < N/(N-2)$. Therefore $w \equiv 0$ on $B_{r_z}(z)$. We conclude from this that U is relatively open in Ω. On the other hand, we can show that U is relatively closed in Ω. Indeed, if there is a $z \in \partial U \cap \Omega$, then by the definition of U, we have $z \in U$. The above argument shows that either $U = \Omega$ or $U = \emptyset$. If $U = \Omega$, then it contradicts to $u|_{\partial\Omega} = 0$ for $N \ge 2$ and $u(+\infty) = 0$ for $N = 1$. Thus U is empty and the essential inf of u_c is greater than a_1 on any compact set.

Replacing the test function ϕ by $-\phi$ in (5.4.3), we can use similar argument to obtain the other variational inequality, which implies the essential sup of u_c is less than a_2 on any compact set. Together with the lower bound estimate "$u_c > a_1$", it implies that the sign of ϕ can be taken arbitrarily and (5.4.3) becomes an equality. That is, u_c is a weak solution of (5.4.2). Since u_c and v_c are bounded, by the regularity theory of elliptic equations, we conclude that u_c and v_c are $C^{2,\alpha_0}(\bar{\Omega})$, and $a_1 < u_c < a_2$ has pointwise meaning. The proof is complete. $\square$

Following the proof of the Proposition 3.3 in [25], we can derive further properties of u_c.

Lemma 5.4.3. *Let u_c be the minimizer in Theorem 5.4.1. Assume $c_1 \leq c \leq c_2$ for some constants $c_1, c_2 > 0$, then*
(a) $u_c \in C^{2,\alpha}_{loc}(\Omega) \cap C^{1,\alpha}_{loc}(\overline{\Omega})$.
(b) ∇u_c is uniformly continuous and bounded with its uniform continuity and sup norm depending on c_1 and c_2 but being independent of c.
(c) For all $(x, y) \in \Omega$, $|u_c(x, y)| \leq Ce^{-\lambda x}$ for some constant $C > 0$ and $\lambda > 0$, depending on c_1 and c_2 but being independent of c.

5.5 Existence of travelling solution

Theorem 5.5.1. *Assume $\gamma > \dfrac{9}{2\beta^2 - 5\beta + 2}$ and Ω contains a ball with a sufficient large radius R_0. Then there exist $c_0 > 0$ and $u_0 \in \mathcal{B}$ satisfying $c_{\min} \leq c_0 \leq c_{\max}$ and $\Phi_{c_0}[u_0] = \mu_{c_0} = 0$.*

Proof. By Theorem 5.3.3 and Lemma 5.3.5, there exists $c_0 > 0$ such that $c_{\min} \leq c_0 \leq c_{\max}$, $\mu_{c_0} = 0$ and $\mu_c < 0$ for $c_{\min} \leq c < c_0$. We choose a sequence c_n satisfying $\mu_{c_n} < 0$, $c_0 > c_n > c_0/2$ and $\lim_{n\to\infty} c_n = c_0$. By Theorem 5.4.1, there exist $u_n \in \mathcal{B}$ such that $\Phi_{c_n}[u_n] = \mu_{c_n}$. Let α be the minimal positive root of $F(u)$. Then $x_n := \max\{x | u_n(x, y) = \alpha \text{ for some } y \in \Omega_y\}$ is well-defined due to the continuity and the decay of u_n (See Lemma 5.4.3).

We claim that x_n is bounded. Because u_n decays uniformly, x_n is bounded above. On the other hand, if x_n is not bounded below, we have $x_n \to -\infty$ up to a subsequence. We write the energy functional as following.

$$\begin{aligned}\mu_{c_n} - \int_{-\infty}^{x_n}\int_{\Omega_y} e^x F[u_n] =& \frac{c_n^2}{2}\int_\Omega u_{n,x}^2 + \frac{1}{2}\int_\Omega e^x(|\nabla_y u_n|^2 + \sqrt{\delta} u_n B_{c_n}[u_n]) \\ &+ \int_{x_n}^{+\infty}\int_{\Omega_y} e^x F[u_n] \qquad (5.5.1)\end{aligned}$$

Note that the last two terms of (5.5.1) are positive from Lemma 5.2.2(c) and definition of x_n. Therefore the right hand side of (5.5.1) is greater than c_n^2. However, the left hand side of (5.5.1) converges to 0 because $\|u_{n_k}\|_\infty$ is uniformly bounded. This is a contradiction. So x_n is bounded.

Defining $w_n(x, y) = u_n(x+x_n, y)$, we have $w_n(0, y) = \alpha$ for some y. From Lemma 5.4.3(b), ∇w_n is uniformly bounded in Ω. By the Arzela-Ascoli theorem, w_n converges uniformly to some $w_0 \in C^0(\Omega)$ on any compact subset of Ω. Therefore, $w_0(0, y) = \alpha$ for some y.

Consequently, w_0 is nontrivial. On the other hand, w_n is uniformly bounded in $\mathbf{H}$ because of the boundedness of x_n and $u_n \in \mathcal{B}$ (See Theorem 5.4.1.). Consequently, $w_n \rightharpoonup w_0$ weakly in $\mathbf{H}$, up to a subsequence. By Lemma 5.3.1,

$$\Phi_{c_0}[w_0] \leq \liminf_{n\to\infty} \Phi_{c_0}[w_n] \leq 0.$$

The last inequality comes from $\Phi_{c_n}[u_n] < 0$ and Lemma 5.3.2. By translating the x-coordinate of w_0, we obtain a $u_0 \in \mathcal{B}$ (see the proof of Theorem 5.4.1) and $\Phi_{c_0}[u_0] \leq 0 = \mu_{c_0}$. Therefore $\Phi_{c_0}[u_0] = 0$. □

Remark. If $N = 1$, we can show this theorem by considering any minimizing sequences of μ_{c_0} because a function $u \in \mathbf{H}$ is continuous and decays exponentially.

The minimizer u_0 has the same properties of u_c with $\mu_c < 0$ in Lemma 5.4.2 and Lemma 5.4.3. We state in the following without proof.

LEMMA 5.5.2. *Let u_0 be the minimizer obtained in Theorem* 5.5.1 *and $v_0 := B_{c_0}[u_0]$. Then*
(a) $u_0 \in C^{2,\alpha}_{loc}(\Omega) \cap C^{1,\alpha}_{loc}(\overline{\Omega})$, $\nabla u_0 \in L^\infty(\Omega)$ and ∇u_0 is uniformly continuous.
(b) For all each $(x, y) \in \Omega$, $a_1 < u_0(x, y) < a_2$ and $\frac{a_1}{\gamma\sqrt{\delta}} \leq v_0(x, y) \leq \frac{a_2}{\gamma\sqrt{\delta}}$.
(c) For all $(x, y) \in \Omega$, $|u_0(x, y)| \leq Ce^{-\lambda x}$ for some constant $C > 0$ and $\lambda > 0$.
(d) u_0 and v_0 solve

$$c_0^2(u_{0,xx} + u_{0,x}) + \Delta_y u_0 + f(u_0) - \sqrt{\delta} v_0 = 0. \tag{5.5.2}$$

The minimizer obtained in Theorem 5.5.1 automatically lies in L^2_w. To understand the asymptotic behavior as $x \to -\infty$, we also need to study the L^2 norm of the derivative of u_0 in x. For a gradient system, it is easier to claim that the derivative of a minimizer is in $L^2(\Omega)$. However, for a skew-gradient system, it is much difficult to prove this. The key observation of the following lemma is to recognize that the condition $\gamma^2\delta > 1$ plays an important role.

LEMMA 5.5.3. *If $\gamma^2\delta > 1$, then $u_{0,x}, v_{0,x} \in L^2(\Omega)$ and*

$$\int_\Omega v_{0,x}^2 \leq \frac{1}{\gamma^2\delta} \int_\Omega u_{0,x}^2. \tag{5.5.3}$$

Moreover, $\lim_{x\to-\infty} u_{0,x}(x, y) = \lim_{x\to-\infty} v_{0,x}(x, y) = 0$ *uniformly in y.*

Proof. In this proof, we simply denote u_0, v_0 and c_0 by u, v and c respectively. We integrate the subtraction of (5.1.12) multiplied u_x from (5.1.13) multiplied by v_x over $I_{LR} := [-L, R] \times \Omega_y$ for $L, R > 0$. By integration by parts, we obtain

$$\begin{aligned} c^2 \int_{I_{LR}} (u_x^2 - v_x^2) = &- \int_{\Omega_y} \left[\frac{c^2}{2}(u_x^2 - v_x^2) - \frac{1}{2}(|\nabla_y u|^2 - |\nabla_y v|^2) + H(u, v) \right]_{x=-L}^{x=R} \\ &- \int_{\partial\Omega_y} \int_{-L}^{R} \left[u_x \frac{\partial u}{\partial \nu_y} - v_x \frac{\partial v}{\partial \nu_y} \right] \end{aligned} \tag{5.5.4}$$

Here $H(u,v) = F(u) + \sqrt{\delta}uv - \frac{\gamma\delta}{2}v^2$. The last term of (5.5.4) vanishes by the boundary condition. Since $u, v, \nabla u$ and ∇v are uniformly bounded independently of R and L, (5.5.4) implies

$$\int_{I_{LR}} u_x^2 \leq C + \int_{I_{LR}} v_x^2, \tag{5.5.5}$$

where C is a constant independent of R and L. Since $u \in C^{2,\alpha}_{loc}(\Omega)$, Schauder's theory and equation (5.1.13) imply $v \in C^{4,\alpha}_{loc}(\Omega)$. Therefore we can differentiate (5.1.13) with respect to x to obtain

$$c^2(v_{xxx} + v_{xx}) + \Delta_y v_x + \sqrt{\delta}u_x - \gamma\delta v_x = 0. \tag{5.5.6}$$

Since $\int_{\Omega_y} v_x^2$ is bounded in x, there are two positive increasing sequences $R_n \to \infty$ and $L_n \to \infty$ such that

$$\int_{\Omega_y} v_x(-L_n, y)v_{xx}(-L_n, y)dy = \int_{\Omega_y} \frac{1}{2}(v_x^2)_x(-L_n, y)dy \to 0 \text{ as } n \to \infty.$$

and

$$\int_{\Omega_y} v_x(R_n, y)v_{xx}(R_n, y)dy \to 0 \text{ as } n \to \infty.$$

We multiply (5.5.6) by v_x and integrate it over $I_{L_nR_n}$. Then

$$\sqrt{\delta}\int_{I_{L_nR_n}} u_x v_x - \gamma\delta\int_{I_{L_nR_n}} v_x^2 = -c^2\int_{\Omega_y} v_x v_{xx}|_{L_n}^{R_n} - \int_{\Omega_y}\frac{c^2}{2}v_x^2(R_n, y) \tag{5.5.7}$$

$$+\frac{c^2}{2}\int_{\Omega_y} v_x^2(-L_n, y) + c^2\int_{I_{L_nR_n}} v_{xx}^2 - \int_{\partial\Omega_y}\int_{-L_n}^{R_n} v_x\frac{\partial v_x}{\partial \nu_y} + \int_{I_{L_nR_n}} |\nabla_y v_x|^2 \tag{5.5.8}$$

By the choice of R_n, L_n and the behavior of v as $x \to \infty$ (Lemma 5.2.2 (b) and Lemma 5.5.2), the right hand side of (5.5.7) approaches to 0. On the other hand, (5.5.8) is nonnegative since the boundary term on $\partial\Omega_y \times [-L_n, R_n]$ vanishes. If v_{xx} is not identically zero, then (5.5.8) is greater than a positive number when n is large. If v_{xx} is identically zero, then v_x is a constant equaling zero and (5.5.7) must be zero. Therefore, for large n,

$$\sqrt{\delta}\int_{I_{L_nR_n}} u_x v_x - \gamma\delta\int_{I_{L_nR_n}} v_x^2 \geq 0$$

By Hölder's inequality, this leads to

$$\int_{I_{L_nR_n}} v_x^2 \leq \frac{1}{\gamma^2\delta}\int_{I_{L_nR_n}} u_x^2. \tag{5.5.9}$$

Combining (5.5.5) and (5.5.9), we obtain for large n

$$\int_{I_{L_nR_n}} u_x^2 \leq \left(1 - \frac{1}{\gamma^2\delta}\right)^{-1} C \tag{5.5.10}$$

Here we have used $\gamma^2\delta > 1$. Therefore $u_x \in L^2(\Omega)$ by taking $n \to \infty$. By (5.5.9), $v_x \in L^2(\Omega)$. Letting $n \to \infty$ in (5.5.7) and (5.5.8), we obtain $\sqrt{\delta}\int_\Omega u_x v_x - \gamma\delta\int_\Omega v_x^2 \geq 0$ and conclude that (5.5.3) holds. Since $u_x, v_x \in L^2(\Omega)$ are uniformly continuous in Ω (see Lemma 5.5.2), $\lim_{x\to-\infty} u_x(x,y) = \lim_{x\to-\infty} v_x(x,y) = 0$ uniformly in y. $\square$

The lemma above indicates that the limit behaviors of u_0 and v_0 as $x \to -\infty$ are related when $\gamma^2\delta > 1$. Since the travelling speed $c_0 > 0$, we know that u_0 prefers the state at $x = -\infty$ than the state at $x = +\infty$. Therefore, if $u_0(-\infty, y)$ exists, $E[u_0(-\infty, y)]$ should be less than $E[u_0(\infty, y)] = 0$ and be negative. Indeed, by Lemma 5.5.3 and taking $R \to +\infty$ in (5.5.4), we have

$$\begin{aligned} E[u_0(-\infty,y)] &= \int_{\Omega_y} [\frac{1}{2}(|\nabla_y u_0(-\infty,y)|^2 - |\nabla_y v_0(-\infty,y)|^2) + H(u_0,v_0)(-\infty,y)] \\ &= -c_0^2(\int_\Omega u_{0,x}^2 - v_{0,x}^2) < 0. \end{aligned}$$

Proof of Theorem 5.1.1. The existence of (u_0, v_0) follows from Theorem 5.5.1. The decay behavior of (u_0, v_0) at ∞ follows from Lemma 5.5.2 and Lemma 5.2.2 (b). The proof is complete.

Proof of Theorem 5.1.2. The existence of (u_0, v_0) follows from Theorem 5.5.1. (or Theorem 5.1.1). The asymptotic behavior of (u_0, v_0) at $-\infty$ follows from Lemma 5.5.3, Lemma 5.5.2, $E[u_0(-\infty, y)] < 0$ and the argument in the proof of Corollary 6.8 in [25]. The proof is complete.

Proof of Theorem 5.1.3. For $N = 1$ and $\gamma > \dfrac{9}{2\beta^2 - 5\beta + 2}$, there are three constant steady states satisfying $H(0,0) = 0$, $H(p_1, q_1) > 0$ and $H(p_2, q_2) < 0$. Therefore, Theorem 5.1.2 implies the sufficient part of Theorem 5.1.3. For the necessary part, we argue by contradiction. Assume $\gamma \leq \frac{9}{2\beta^2-5\beta+2}$, then $H(u, \frac{u}{\gamma}) = \frac{u^2}{12}[3u^2 - 4(\beta+1)u + 6(\beta + \frac{1}{\gamma})] \geq 0$. This means three constant steady states have nonnegative energy. However, if a travelling wave (u_0, v_0) exists, then $H(p_2, q_2) = E[u_0(-\infty, y)] < 0$. This leads to a contradiction.

5.6 Skew-gradient structure

According to the skew-gradient structure it is natural to consider another approach to solve (5.1.12)-(5.1.14), that is, finding critical points of the strongly indefinite functional $\Psi_c[u,v] := c^2 I[u,v] + J[u,v]$ on $\mathbf{H} \times \mathbf{H}$, where

$$I[u,v] = \frac{1}{2}\int_\Omega e^x(u_x^2 - v_x^2)$$

and

$$J[u,v] = \frac{1}{2}\int_\Omega e^x(|\nabla_y u|^2 - |\nabla_y v|^2) + \int_\Omega e^x H(u,v).$$

This variational problem does not have a minimizer since the gradient term of v in the energy has a minus sign. Therefore one needs to apply a mini-max theory to study such a problem. When the domain Ω is bounded, a steady state was obtained by Chen and Hu in [5] via the critical point theory developed in [1]. Unfortunately it is not easy to apply the method in [5] to the travelling wave problem with non-zero speed on a cylinder. On the other hand, from Heinze's viewpoint in [11], the travelling wave problem can be viewed as a variational problem under a constraint with its wave speed corresponding to the Lagrange multiplier. Instead of solving the constraint problem proposed by Heinze, we consider in this section a slightly different variational problem, that is, the critical points of the quotient energy

$$K[u,v] = \frac{J[u,v]}{I[u,v]} \text{ on } \mathbf{H} \times \mathbf{H}. \tag{5.6.1}$$

The functional $K[u,v]$ is not well-defined when the denominator vanishes. However we notice that (u_0, v_0) obtained in Theorem 5.1.2 satisfies $I[u_0, v_0] > 0$. See the proof of Theorem 5.6.2 below for this. Therefore we consider the case $I > 0$ in this section. Recall that (u^*, v^*) is called a local mini-maximizer of $K[u,v]$ (see [45]) if u^* is a local minimizer of $K[u, v^*]$ and v^* is a local maximizer of $K[u^*, v]$. Our first result asserts that a local mini-maximizer of K corresponds to a travelling wave solution of (5.1.12)-(5.1.14) while the value $-K$ of it corresponds to the square of its speed.

Theorem 5.6.1. *Assume that $(u^*, v^*) \in \boldsymbol{H} \times \boldsymbol{H}$ is a local mini-maximizer of $K[u,v]$ with $I[u^*, v^*] > 0$ and $J[u^*, v^*] < 0$. Then*
(a) (u^*, v^*, c^*) solves (5.1.12)-(5.1.14) weakly, where $c^{*2} = -K[u^*, v^*]$;
(b) u^* is a nontrivially local minimizer of $\Phi^*_{c^*}[u]$ with $\Phi^*_{c^*}[u^*] = 0$.

Proof. (a) can be easily obtained by first variation of $K[u,v]$ with respect to u and v in a neighborhood of u^* and v^*. Next, we show (b). By $I[u^*, v^*] > 0$, u^* is nontrivial. We observe that u^* is a local minimizer of $K[u, v^*]$ if and only if u^* is a local minimizer of $\Psi_{c^*}[u, v^*]$. Indeed, by the definition of c^*, $\Psi_{c^*}[u^*, v^*] = 0$. Choose a neighborhood G of u^* such that $I[u, v^*] \geq 0$ for all $u \in G$ and u^* is a minimizer of $K[u, v^*]$ in G. Then $K[u, v^*] \geq K[u^*, v^*] = -c^{*2}$ in G, equivalently, $\Psi_{c^*}[u, v^*] \geq 0$ in G. Therefore u^* is a local minimizer of $\Psi_{c^*}[u, v^*]$. Now we show a relation between $\Phi^*_{c^*}$ and Ψ_{c^*}. By v^*'s equation,

$$\int_\Omega e^x (c^{*2} v_x^{*2} + |\nabla_y v^*|^2 + \gamma\delta v^{*2}) = \sqrt{\delta} \int_\Omega e^x u^* v^*.$$

Therefore

$$\begin{aligned} \Phi^*_{c^*}[u] &= \Psi_{c^*}[u, v^*] + \frac{\sqrt{\delta}}{2} \int_\Omega e^x u B_{c^*}[u] + \frac{\sqrt{\delta}}{2} \int_\Omega e^x u^* v^* - \sqrt{\delta} \int_\Omega e^x u v^* \\ &= \Psi_{c^*}[u, v^*] + \frac{\sqrt{\delta}}{2} \int_\Omega e^x (u - u^*) B_{c^*}[u - u^*], \end{aligned} \tag{5.6.2}$$

where we have used Lemma 5.2.2 for the last equality. Therefore $\Phi^*_{c^*}[u^*] = \Psi_{c^*}[u^*, v^*] = 0$ and $\Phi^*_{c^*}[u] \geq 0$ locally due to Lemma 5.2.2(c). The theorem is proven. $\square$

Conversely, under an extra condition, we have the following result.

THEOREM 5.6.2. *Let u_0 and c_0 be obtained in Theorem* 5.5.1 *and let* $v_0 = B_{c_0}[u_0]$. *Assume* $\gamma^2\delta > 1$ *and for all nontrivial* $\phi \in \boldsymbol{H}$,

$$\int_\Omega e^x(c_0^2\phi_x^2 + |\nabla_y\phi|^2) + \int_\Omega e^x(3u_0^2 - 2(\beta+1)u_0 + \beta)\phi^2 > 0.$$

Then (u_0, v_0) *is a local mini-maximizer of* $K[u,v]$ *with* $I[u_0,v_0] > 0$ *and* $J[u_0,v_0] < 0$.

Proof. First we claim $I[u_0, v_0] > 0$. Equation (5.5.6) and the uniqueness of $B_{c_0}[u_{0,x}]$ yield $v_{0,x} = B_{c_0}[u_{0,x}]$. Following the proof of Lemma 5.2.2, we obtain

$$\int_\Omega c_0^2|(e^{x/2}v_{0,x})_x|^2 + |\nabla_y e^{x/2}v_{0,x}|^2 + (\gamma\delta + c_0^2/4)e^x v_{0,x}^2 = \sqrt{\delta}\int_\Omega e^x u_{0,x}v_{0,x}.$$

It is readily proved that $(\int_\Omega e^x v_{0,x}^2)^{1/2} \le \frac{\sqrt{\delta}}{\gamma\delta + c_0^2/4}(\int_\Omega e^x u_{0,x}^2)^{1/2}$. This implies $I[u_0,v_0] > 0$ since $\gamma^2\delta > 1$. By (5.6.2),

$$\Psi_{c_0}[u, v_0] = \Phi^*_{c_0}[u] - \frac{\sqrt{\delta}}{2}\int_\Omega e^x(u-u_0)B_{c_0}[u-u_0].$$

It follows from this relation that $\Psi_{c_0}[u_0,v_0] = \Phi^*_{c_0}[u_0] = 0$ and $J[u_0,v_0] < 0$. For all $\phi \in \mathbf{H}$, a straightforward computation gives

$$\begin{aligned} D_u\Psi_{c_0}[u_0,v_0]\phi &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\Psi_{c_0}[u_0+\epsilon\phi, v_0] \\ &= D_u\Phi^*_{c_0}[u_0]\phi - \frac{\sqrt{\delta}}{2}\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\int_\Omega e^x\epsilon\phi B_{c_0}[\epsilon\phi] = 0 \end{aligned}$$

and

$$\begin{aligned} D_{uu}\Psi_{c_0}[u_0,v_0]\phi &= \lim_{\epsilon\to 0}\frac{d^2}{d\epsilon^2}\Psi_{c_0}[u_0+\epsilon\phi, v_0] \\ &= D_{uu}\Phi^*_{c_0}[u_0]\phi - \sqrt{\delta}\int_\Omega e^x\phi B_{c_0}[\phi] \\ &= \int_\Omega e^x(c_0^2\phi_x^2 + |\nabla_y\phi|^2) + \int_\Omega e^x(-f_u(u))\phi^2 \\ &= \int_\Omega e^x(c_0^2\phi_x^2 + |\nabla_y\phi|^2) + \int_\Omega e^x(3u_0^2 - 2(\beta+1)u_0 + \beta)\phi^2 > 0. \end{aligned}$$

Therefore u_0 is a local minimizer of $\Psi_{c_0}[u, v_0]$. Equivalently, u_0 is a local minimizer of $K[u, v_0]$. Since $\Psi_{c_0}[u_0, v]$ is a concave functional and v_0 is a critical point of $\Psi_{c_0}[u_0, v]$, v_0 is a local maximizer of $\Psi_{c_0}[u_0, v]$. This implies v_0 is a local maximizer of $K[u_0, v]$. The proof is complete. □

5.7 Neumann problem

In the section, we consider the travelling wave equations (5.1.12) and (5.1.13) with Neumann condition. The function space $\mathbf{H}$ we used in the previous sections also need to be changed. Let $\mathbf{H}_N$ be the weighted Sobolev space, the completion of $C^\infty(\Omega)$ with respect to the norm $\|u\|^2_{\mathbf{H}_N} = \|u\|^2_{L^2_w} + \|\nabla u\|^2_{L^2_w}$. Following the idea of Proposition 6.3 in [25], we obtain the following result which indicates that the variational approach derive planar waves for system (5.1.12)-(5.1.13) with Neumann condition.

THEOREM 5.7.1. *Suppose $\hat{u}(x,y) \in \boldsymbol{H}_N \bigcap C^1_{loc}(\Omega)$ is a nontrivial minimizer of $\Phi^*_c[u]$ in $\boldsymbol{H}_N$ for some $c = \hat{c} > 0$, then $\hat{u}$ depends only on x.*

Proof. We first claim $\Phi^*_{\hat{c}}[u] \geq 0$ for all $u \in \mathbf{H}_N$. It is sufficient to show $\Phi^*_{\hat{c}}[\hat{u}] = 0$. Indeed, $\Phi^*_{\hat{c}}[\hat{u}] \leq \Phi_{\hat{c}}[0] = 0$. If $\Phi^*_{\hat{c}}[\hat{u}] < 0$, then $\Phi^*_{\hat{c}}[\hat{u}(x-a,y)] = e^a \Phi^*_{\hat{c}}[\hat{u}(x,y)] \to -\infty$ as $a \to \infty$, which contradicts to the existence of the minimizer of $\Phi^*_{\hat{c}}[u]$. Consequently, $\Phi^*_{\hat{c}}[\hat{u}] = 0$. Let $\hat{v}(x,y) = B_{\hat{c}}[\hat{u}(x,y)]$ and

$$m(y) = \int_{\mathbf{R}^1} e^x \Big(\frac{\hat{c}^2}{2}\hat{u}_x(x,y)^2 + |\nabla_y \hat{u}(x,y)|^2 + F(\hat{u}(x,y)) + \frac{\sqrt{\delta}}{2}\hat{u}(x,y)\hat{v}(x,y)\Big)dx.$$

Therefore $\int_{\Omega_y} m(y) = \Phi^*_{\hat{c}}[\hat{u}] = 0$. Next, we show that $m(y) = 0$ for all $y \in \Omega_y$. If no, then there exists a $y_1 \in \Omega_y$ with $m(y_1) < 0$ by the continuity of $m(y)$ and $\int_{\Omega_y} m(y) = 0$. Note that $\hat{u}(x,y_1) \in \mathbf{H}_N$ and $\hat{v}(x,y_1) = B_{\hat{c}}[\hat{u}(x,y_1)]$ by the uniqueness of $B_{\hat{c}}[\hat{u}(x,y_1)]$. Therefore a straightforward computation gives

$$\begin{aligned}
0 &\leq \Phi^*_{\hat{c}}[\hat{u}(x,y_1)] \\
&= |\Omega_y| \int_{\mathbf{R}^1} e^x \Big(\frac{\hat{c}^2}{2}\hat{u}_x(x,y_1)^2 + F(\hat{u}(x,y_1)) + \frac{\sqrt{\delta}}{2}\hat{u}(x,y_1)B_{\hat{c}}[\hat{u}(x,y_1)]dx \\
&= |\Omega_y|(m(y_1) - \int_{\mathbf{R}^1} e^x |\nabla_y \hat{u}(x,y_1)|^2 dx) < 0.
\end{aligned}$$

Consequently, $m(y) \equiv 0$. We argue that u depends only on x by a contradiction. If there exists a $y_2 \in \Omega_y$ with $\int_{\mathbf{R}^1} e^x |\nabla_y \hat{u}(x,y_2)|^2 dx > 0$, then a similar calculation yield

$$0 \leq \Phi^*_{\hat{c}}[\hat{u}(x,y_2)] = -|\Omega_y| \int_{\mathbf{R}^1} e^x |\nabla_y \hat{u}(x,y_2)|^2 dx < 0.$$

In conclusion, $|\nabla_y \hat{u}(x,y)| = 0$ for all $(x,y) \in \Omega$. The proof is completed. □

5.8 Appendix

In this section, we list all the energies mentioned above.
(1) The gradient energy of $c^2(u_{xx} + u_x) + \Delta_y u + f(u) = 0$:

$$S_c[u] = c^2 I_0[u] + J_0[u] = \frac{c^2}{2}\int_\Omega e^x u_x^2 + \frac{1}{2}\int_\Omega e^x \left(|\nabla_y u|^2 + F(u)\right)$$

(2) The gradient energy of $u_{zz} + cu_z + \Delta_y u + f(u) = 0$:

$$\hat{S}_c[u] = \hat{I}_0[u] + \hat{J}_0[u] = \frac{1}{2}\int_\Omega e^{cz} u_z^2 + \frac{1}{2}\int_\Omega e^{cz}\left(|\nabla_y u|^2 + F(u)\right)$$

(3) The nonlocal gradient energies of $c^2(u_{xx} + u_x) + \Delta_y u + f(u) - \sqrt{\delta}B_c[u] = 0$:

(*a*) The non-cut-off energy:

$$\begin{aligned}
\Phi^*_c[u] &= \Phi^{(1)}_c[u] + \Phi^{(2)}_c[u] + \Phi^{(3)*}_c[u] \\
&= \frac{1}{2}\int_\Omega e^x(c^2 u_x^2 + |\nabla_y u|^2) + \int_\Omega e^x F(u) + \frac{\sqrt{\delta}}{2}\int_\Omega e^x u B_c[u]
\end{aligned}$$

(b) The cut-off energy:

$$\Phi_c[u] = \Phi_c^{(1)}[u] + \Phi_c^{(2)}[u] + \Phi_c^{(3)}[u]$$
$$= \Phi_c^{(1)}[u] + \Phi_c^{(2)}[u] + \frac{\sqrt{\delta}}{2}\int_\Omega e^x \overline{u} B_c[\overline{u}]$$

(c) The limit energy as $x \to \pm\infty$:

$$E[u] = \frac{1}{2}\int_{\Omega_y} |\nabla_y u|^2 + \int_{\Omega_y} F(u) + \frac{\delta}{2}\int_{\Omega_y} u(-\Delta_y + \gamma\delta)^{-1}[u]$$

(4) The skew-gradient flow of $u_t = \Delta u + f(u) - \sqrt{\delta}v$ and $v_t = d\Delta v + \sqrt{\delta}v - \gamma\delta v$:

$$S[u,v] = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - d|\nabla v|^2 + H(u,v)\big)$$

(5) The skew-gradient energies of

$$c^2(u_{xx} + u_x) + \Delta_y u + f(u) - \sqrt{\delta}v = 0,$$
$$c^2(v_{xx} + v_x) + \Delta_y v + \sqrt{\delta}u - \gamma\delta v = 0.$$

(a) The strongly indefinite energy:

$$\Psi_c[u,v] = c^2 I[u,v] + J[u,v]$$
$$= c^2\{\frac{1}{2}\int_\Omega e^x(u_x^2 - v_x^2)\} + \frac{1}{2}\int_\Omega e^x(|\nabla_y u|^2 - |\nabla_y v|^2) + \int_\Omega e^x H(u,v)$$

(b) The quotient energy:

$$K[u,v] = \frac{J[u,v]}{I[u,v]}$$

Bibliography

[1] Benci, V. and Rabinowitz, P. H., *Critical point theorems for indefinite functionals*, Invent. Math. 52 (1979), no. 3, 241-273.

[2] Carpenter, G. A., *A geometric approach to singular perturbation problems with applications to nerve impulse equations*, J. Differential Equations 23 (1977), no. 3, 335-367.

[3] Chueh, K. N., Conley, C. C. and Smoller, J. A., *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. 26 (1977), no. 2, 373-392.

[4] Casten, R. G., Cohen, H. and Lagerstrom, P. A., *Perturbation analysis of an approximation to the Hodgkin-Huxley theory*, Quart. Appl. Math. 32 (1974/75), 365-402.

[5] Chen, C.-N. and Hu, X., *Stability criteria for reaction-diffusion systems with skew-gradient structure*, Comm. Partial Differential Equations 33 (2008), no. 2, 189-208.

[6] Dockery, J. D., *Existence of standing pulse solutions for an excitable activator-inhibitory system*, J. Dynam. Differential Equations 4 (1992), no. 2, 231-257.

[7] Ermentrout, G. B., Hastings, S. P. and Troy, W. C., *Large amplitude stationary waves in an excitable lateral-inhibitory medium*, SIAM J. Appl. Math. 44 (1984), no. 6, 1133-1149.

[8] FitzHugh, R., *Impulse and physiological states in models of nerve membrans*, Biophysics J. 1, 445-466 (1961).

[9] Gardner, R., *Existence of multidimensional travelling wave solutions of an initial-boundary value problem*, J. Differential Equations 61 (1986), no. 3, 335-379.

[10] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order.*, Springer-Verlag, Berlin, 1983.

[11] Heinze, S., *A Variational Approach to Traveling Waves.*, Preprint 85, Max Planck Institute for Mathematical Sciences, 2001.

[12] Hodgkin, A. L. and Huxley, A. F., *A quantitative description of membrane current and its application to conduction and excitation in nerve.*, J. Physiol. 117 (1952), 500-544 .

[13] Hagberg, A. and Meron, E., *From labyrinthine patterns to spiral turbulence.*, Phys. Rev. Lett. 72 (1994), 2494-2497.

[14] Hamel, F., Monneau, R. and Roquejoffre, J.-M., *Existence and qualitative properties of multidimensional conical bistable fronts*, Disc. Cont. Dyn. Systems 13 (2005), 1069-1096.

[15] Hamel, F. and Roquejoffre, J.-M., *Heteroclinic connections for multidimensional bistable reaction-diffusion equations*, Discrete Contin. Dyn. Syst. Ser. S 4 (2011), no. 1, 101-123.

[16] Ikeda, H., *Singular pulse wave bifurcations from front and back waves in bistable reaction-diffusion systems*, Methods Appl. Anal. 3 (1996), no. 3, 285-317.

[17] Ikeda, H., Mimura, M. and Nishiura, Y., *Global bifurcation phenomena of travelling wave solutions for some bistable reaction-diffusion systems*, Nonlinear Anal. 13 (1989), no. 5, 507-526.

[18] Keener, J. P., *Waves in excitable media*, SIAM J. Appl. Math. 39 (1980), no. 3, 528-548.

[19] Klaasen, G. A., *Stationary spacial patterns for a reaction-diffusion system with an excitable steady state*, Rocky Mountain J. Math. 16 (1986), no. 1, 119-127.

[20] Klaasen, G. A. and Mitidieri, E., *Standing wave solutions for a system derived from the FitzHugh-Nagumo equations for nerve conduction*, SIAM J. Math. Anal. 17 (1986), no. 1, 74-83.

[21] Kokubu, H., Nishiura, Y. and Oka, H., *Heteroclinic and homoclinic bifurcations in bistable reaction diffusion systems*, J. Differential Equations 86 (1990), no. 2, 260-341.

[22] Klaasen, G. A. and Troy, W. C., *Stationary wave solutions of a system of reaction-diffusion equations derived from the FitzHugh-Nagumo equations*, SIAM J. Appl. Math. 44 (1984), no. 1, 96-110.

[23] Kurokawa, Y. and Taniguchi, M., *Multi-dimensional pyramidal travelling fronts in the Allen-Cahn equations*, Proc. Roy. Soc. Edinburgh Sect. A 141 (2011), no. 5, 1031-1054.

[24] Lucia, M., Muratov, C. B. and Novaga, M., *Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium*, Comm. Pure Appl. Math. 57 (2004), no. 5, 616-636.

[25] Lucia, M., Muratov, C. B. and Novaga, M., *Existence of traveling waves of invasion for Ginzburg-Landau-type problems in infinite cylinders*, Arch. Ration. Mech. Anal. 188 (2008), no. 3, 475-508.

[26] Matsuzawa, H., *Asymptotic profiles of variational solutions for a FitzHugh-Nagumo-type elliptic system*, Differential Integral Equations 16 (2003), no. 8, 897-926.

[27] Malevanets, A. and Kapral, R., *Microscopic model for FitzHugh-Nagumo dynamics*, Phy. Rev. E 55 (1997), no. 5, 897-926.

[28] Muratov, C. B., *A global variational structure and propagation of disturbances in reaction-diffusion systems of gradient type*, Discrete Contin. Dyn. Syst. Ser. B4 (2004), no. 4, 867-892.

[29] Morita, Y., Ninomiya, H., *Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space*,Bull. Inst. Math. Acad. Sin. (N.S.) 3 (2008), no. 4, 567-584.

[30] Muratov, C. B. and Novaga, M., *Front propagation in infinite cylinders. I. A variational approach*, Commun. Math. Sci. 6 (2008), no. 4, 799-826.

[31] Muratov, C. B. and Novaga, M., *Front propagation in infinite cylinders. II. The sharp reaction zone limit*, Calc. Var. Partial Differential Equations 31 (2008), no. 4, 521-547.

[32] Nagumo, J.S., Arimoto, S. and Yoshizawa, S., *An active pulse transmission line simulating nerve axon*, Proc. IRE 50, 2061-2071 (1962).

[33] Ninomiya, H., Taniguchi, M., *Existence and global stability of traveling curved fronts in the Allen-Cahn equations*, J. Differential Equations 213 (2005), no. 1, 204-233.

[34] Ohta, T., Mimura, M. and Kobayashi R., *Higher-dimensional localized patterns in excitable media*, Physica D, 34 (1989), 115-144.

[35] Ohta, T. and Nakazawa H., *Self-organization in an excitable reaction-diffusion system. II. Reduction to a coupled oscillator*, Phys. Rev. A 45 (1992), 5504-5511.

[36] Polyanin, A. D., *Handbook of linear partial differential equations for engineers and scientists*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[37] Petrich, D. M. and Goldstein, R. E., *Nonlocal contour dynamics model for chemical front motion*, Phys. Rev. Lett. 72 (1994), 1120-1123.

[38] Reinecke, C. and Sweers, G., *A positive solution on* $\boldsymbol{R}^N$ *to a system of elliptic equations of FitzHugh-Nagumo type*, J. Differential Equations 153 (1999), no. 2, 292-312.

[39] Reinecke, C. and Sweers, G., *Existence and uniqueness of solutions on bounded domains to a FitzHugh-Nagumo type elliptic system*, Pacific J. Math. 197 (2001), no. 1, 183-211.

[40] Rinzel, J. and Terman, D., *Propagation phenomena in a bistable reaction-diffusion system*, SIAM J. Appl. Math. 42 (1982), no. 5, 1111-1137.

[41] Taniguchi, M., *Traveling fronts of pyramidal shapes in the Allen-Cahn equations*, SIAM J. Math. Anal. 39 (2007), no. 1, 319-344.

[42] Vitolo, A., *On the Phragmén-Lindelöf principle for second-order elliptic equations*, J. Math. Anal. Appl. 300(2004), no. 1, 244-259.

[43] Volpert, A. and Volpert, V., *Existence of multidimensional travelling waves and systems of waves*, Comm. Partial Differential Equations 26(2001), no. 3-4, 421-459.

[44] Wei, J. and Winter, M., *Standing waves in the FitzHugh-Nagumo system and a problem in combinatorial geometry*, Math. Z. 254 (2006), no. 2, 359-383.

[45] Yanagida, E., *Mini-maximizers for reaction-diffusion systems with skew-gradient structure*, J. Differential Equations, 179(2002), no. 1, 311-335.