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均曲率流的拉格拉奇自同構解

Lagrangian Self-Similar Solutions for Mean Curvature Flow

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誌　謝

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摘　要

在這篇論文裡, 我們推廣Colding 和 Minicozzi 的工作, 將收縮超曲面的F-穩定性推廣到高餘維。我們推導了F-泛函的一次變分和二次變分。對於一般高餘維, 我們發現了一個關於F-穩定性的等價條件。使用這等價條件, 我們能對F-穩定的乘積收縮解做分類以及證明 Anciaux 所造的拉格拉奇收縮解是 F-不穩定。

Abstract

In this thesis, we generalize Colding and Minicozzi's work on the stability of self-shrinkers in the hypersurface case to higher co-dimensional cases. The 1st and 2nd variation formulae of the F-functional are derived and an equivalent condition to the stability in general codimension is found. Using the equivalent condition, we can classify F-stable product self-shrinkers and show that the Lagrangian self-shrinkers given by Anciaux are F- unstable.

Contents

Chapter 1

Introduction

Let Σ be an n-dimensional manifold and X be an isometric immersion of Σ in $\mathbb{R}^m$. Mean curvature flow of X is a family of immersions $X_t : \Sigma \to \mathbb{R}^m$ which satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t} X_t(x)\right)^{\perp} & = H(x,t) \\ X_0 & = X \end{cases},$$

where $H(x,t)$ is the mean curvature vector of $X_t(\Sigma)$ at $X_t(x)$ and $\perp$ denotes the projection of a vector into the normal space of $X_t(\Sigma)$. Mean curvature flow of a submanifold in a Riemannian manifold can also be defined similarly. Because the mean curvature vector points in the direction in which the area decreases most rapidly, mean curvature flow is thus a canonical way to construct minimal submanifolds. It also improves the geometric properties of an object along the flow (e.g., see [7]).

A submanifold Σ in $\mathbb{R}^m$ is called a *self-shrinker* if its position vector $X : \Sigma \to \mathbb{R}^m$ satisfies

$$H = -\frac{1}{2} X^{\perp}.$$

The terminology comes from the fact that $\sqrt{1-t}X(\Sigma)$ is a solution of mean curvature flow, i.e., a self-shrinker evolves homothetically along mean curvature flow in a shrinking way. Moreover, self-shrinkers describe all possible central blow-up limits of a finite-time singularity of the mean curvature flow. This follows from Huisken's monotonicity formula [8], and its generalization to type II singularity by Ilmanen [11] and White [18]. Singularities will occur in general along mean curvature flow and are obstacles to continue the flow. It is therefore an important issue to understand singularities and the candidates of their blow-up limits, self-shrinkers.

Standard sphere $\mathbb{S}^n(\sqrt{2n})$ and cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ are simple examples of self-shrinkers in $\mathbb{R}^m$. Abresch and Langer [4] found all immersed closed self-shrinkers in the plane. For higher dimensional complete hypersurface case, Huisken [9] classified all self-shrinkers with nonnegative mean curvature and bounded geometry. The bounded geometry condition is later weakened to polynomial volume growth by Colding and Minicozzi in [6]. On the other hand, many other different co-dimension one self-shrinkers are found (e.g., see [2]), and a classification of all self-shrinkers is not expected. Our understanding on self-shrinkers in higher co-dimension is even more limited. Smoczyk obtained a classification for self-shrinkers with parallel principal normal $\nu \equiv H/|H|$ and bounded geometry in [16]. In addition to above researches, a nice condition, Lagrangian condition, gives us another possibility to realize self-shrinkers for higher co-dimensional cases.

A half-dimensional submanifold L in a symplectic manifold, equipped with a closed nondegenerate differential 2-form ω, is called a Lagrangian submanifold if $\omega|_L = 0$. Especially, the standard complex Euclidean space $\left(\mathbb{C}^n, \omega = \sum_{j=1}^{n} dx_j \wedge dy_j\right)$ is a trivial symplectic manifold. In this thesis, our ambient space is the standard Euclidean space $\mathbb{R}^m$ or $\mathbb{C}^n$. Fortunately, the Lagrangian condition is preserved by mean curvature flow. It was proved by Smoczyk in [15]. In $\mathbb{C}^n$, various different families of Lagrangian self-shrinkers are constructed in [1], [14] and [12].

Adapted from the back heat kernel introduced by Huisken in [8], Colding and Minicozzi [6] defined a functional F by

$$F(\Sigma, x, t) = \frac{1}{\sqrt{4\pi t}^n} \int_\Sigma e^{\frac{-|X-x|^2}{4t}} d\mu. \tag{1.0.1}$$

for any submanifold $X : \Sigma^n \to \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$ and $t > 0$. For the first variation of F, the critical points (Σ, x_0, t_0) of F are the surfaces shrinking to x_0 at the time t_0 along mean curvature flow. Especially, they are self-shrinkers when $x_0 = 0$ and $t_0 = 1$. The entropy functional $\lambda = \lambda(\Sigma)$ of Σ is defined by

$$\lambda(\Sigma) = \sup_{x \in \mathbb{R}^{n+1}, t>0} F(\Sigma, x, t).$$

The entropy λ has three main properties. First, it is positive and is invariant under dilations, translations and rotations of Σ. Secondly, it is non-increasing along mean curvature flow. Thirdly, its critical points are self-shrinkers. However, it is not smooth with respect to the variable Σ. A self-shrinker is *entropy-stable* if it is a local minimum for the entropy functional. From the study of Colding and Minicozzi [6], the singular models of a "perturbed" mean curvature flow must be an entropy-stable self-shrinker. Moreover, they also showed that shrinking spheres, cylinders and planes are the only stable self-shrinkers under mean curvature flow. In their proof, one can see that an F-unstable self-shrinker which does not split a line must be entropy-unstable. This also holds for higher co-dimensional case. So it is worthy to discuss the F-stability of self-shrinkers of higher codimensions. Here F-stable indicates that for every compactly supported smooth variation Σ_s with $\Sigma_0 = \Sigma$, there exist variations x_s of 0 and t_s of 1 such that $\frac{\partial^2}{\partial s^2} F(\Sigma_s, x_s, t_s) \geq 0$ at $s = 0$.

In this thesis, we intend to generalize Colding and Minicozzi's work [6] to higher co-dimensional cases. The domain of the functional F is now (Σ, x, t) for $\Sigma^n \subset \mathbb{R}^m$, $x \in \mathbb{R}^m$ and $t > 0$. Colding and Minicozzi's classification on stable self-shrinkers in co-dimension one is first to conclude that the mean curvature function h is the first eigenvalue of an elliptic operator, it then implies $h \geq 0$, and Huisken's classification of self-shrinkers with nonnegative h will lead to the conclusion. Although the counter part of Huisken's result in higher co-dimension is still not available, we can also pin down the stability of self-shrinkers in higher co-dimension to the mean curvature vector being the first vector-valued eigenfunction for an elliptic system. More precisely, the equivalent condition of stabilities is as in the following Theorem 3.2.1.

Theorem 3.2.1. *Suppose $\Sigma \subset \mathbb{R}^m$ is an n-dimensional smooth closed self-shrinker, $H = -\frac{X^\perp}{2}$. The following statements are equivalent:*

(i) Σ is F-stable.

(ii) $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu \geq 0$ for any admissible vector field V, namely, a smooth vector field V which satisfies

$$\int_\Sigma \langle V, H\rangle e^{-\frac{|X|^2}{4}} d\mu = 0 \quad \textit{and} \quad \int_\Sigma \langle V, y^\perp\rangle e^{-\frac{|X|^2}{4}} d\mu = 0$$

for all constant vector $y \in \mathbb{R}^m$, where $L^\perp V = \Delta^\perp V + \langle A_{ij}, V\rangle g^{ki} g^{jl} A_{kl} + \frac{V}{2} - \frac{1}{2}\nabla^\perp_{X^\top} V$ is a second order operator and A_{ij} is the second fundamental form as definition in (2.1.1)*, and $\nabla^\perp$ is the normal connection of Σ.*

From the standard spectrum theory for unbounded domain, it is natural to consider the set $\mathcal{H}_0^1(\Sigma)$, which is the closure of the collection of all smooth normal vector fields with compact support with respect to the norm $||\cdot||_{1,e}$. See the definition of $||\cdot||_{1,e}$ in (2.1.2). We can also find the following equivalent condition for the stability of F in the complete noncompact case.

Theorem 3.2.2. *Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete noncompact self-shrinker, $H = -\frac{X^\perp}{2}$. Suppose that the second fundamental form A of Σ is of polynomial growth and Σ has polynomial volume growth. The following statements are equivalent:*

(i) Σ is F-stable.

(ii) $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu \geq 0$ for any admissible vector field V in $\mathcal{H}_0^1(\Sigma)$.

Let $\Sigma_1^{n_1} \subset \mathbb{R}^{m_1}$ and $\Sigma_2^{n_2} \subset \mathbb{R}^{m_2}$ be two smooth complete self-shrinkers, it is easy to see that $\Sigma = \Sigma_1 \times \Sigma_2$ is also a self-shrinker in $\mathbb{R}^{m_1+m_2}$. Conversely, consider a self-shrinker $\Sigma \subset \mathbb{R}^{m_1+m_2}$, if Σ can be expressed as $\Sigma_1^{n_1} \times \Sigma_2^{n_2}$ for smooth $\Sigma_1^{n_1} \subset \mathbb{R}^{m_1}$ and $\Sigma_2^{n_2} \subset \mathbb{R}^{m_2}$, then both $\Sigma_1^{n_1}$ and $\Sigma_2^{n_2}$ are self-shrinkers. Such Σ is called a product self-shrinker in the thesis. In Chapter 4, we prove

Theorem 1.0.1 *The n-plane is the only complete smooth F-stable product self-shrinker in $\mathbb{R}^m$ provided that its volume and second fundamental form are of polynomial growth.*

In general, the F-stability of higher co-dimensional self-shrinkers is not clear.

Hence we need to impose certain structure on the submanifold. In this thesis, we introduce the Lagrangian condition on self-shrinkers and discuss the F-stability of the example constructed by Anciaux in [1]. The self-shrinkers are Lagrangian submanifolds in $\mathbb{C}^n$, which are expressed as $\gamma(s)\psi(\sigma)$, where $\psi : M^{n-1} \to \mathbb{S}^{2n-1}$ is a minimal Legendrian immersion and γ satisfies the system of ordinary differential equations (5.1.1). Because the F-functional is infinite on the complete noncompact Lagrangian examples constructed by Anciaux in [1], we will only discuss the closed cases. That is, the corresponding curves γ are closed and the immersions $\psi : M \to \mathbb{S}^{2n-1}$ are closed. We employ the equivalent condition Theorem 3.2.1 to investigate the F-stability of this example.

Theorem 5.2.1. *If Anciaux's example is closed and has dimension* ≥ 2, *then it is* F*-unstable.*

Since Anciaux's examples are Lagrangian submanifolds in $\mathbb{C}^n$, it is thus natural to study whether Anciaux's example is furthermore F-unstable under the restricted Lagrangian variations. We have the following

Theorem 5.3.1. *If Anciaux's example is closed and has dimension* $n = 2$ *or* $n \geq 7$, *then it is* F*-unstable under Lagrangian variations.*

For dimensions between 2 and 7, this theorem still holds under a technical assumption on γ. Details can be found in Chapter 5. Besides self-shrinkers, in the last chapter, we also study expanding and translating Lagrangian graphs under certain conditions on symmetry.

Chapter 2

The 1st and 2nd variation formulae of F

2.1 Notation and Preliminaries

Let $X : \Sigma^n \to \mathbb{R}^m$ be a smooth isometric immersion of a submanifold of codimension $m - n$. The Riemannian metric g_{ij} on Σ is induced by the standard metric of $\mathbb{R}^m$. $\overline{\nabla}$ and ∇ denote the connection of the ambient space and Σ, respectively. If $\{e_i\}$ and $\{e_\alpha\}$ are orthonormal frames for the tangent bundle $T\Sigma$ and the normal bundle $N\Sigma$, respectively, then the coefficients of the second fundamental form and the mean curvature vector are defined to be

$$A_{ij} = A_{ij}^{\alpha} e_\alpha \equiv \langle \overline{\nabla}_{e_i} e_j, e_\alpha \rangle e_\alpha \qquad (2.1.1)$$
$$\text{and} \quad H = H^\alpha e_\alpha \equiv A_{ii},$$

where by convention we are summing over repeated indices. In general, $A^{B,C}$ and $H^{B,C}$ denote the second fundamental form and mean curvature of the submanifold B which is contained in C, respectively. When C is (complex) Euclidean space, $A^{B,C}$ and $H^{B,C}$ are simplified as A^B (or A) and H^B (or H), respectively. In particular, when Σ is a hypersurface, the mean curvature vector H and the second fundamental form reduce to the function $h = -\langle H, \mathbf{n} \rangle$ and the 2-tensor $h_{ij} = -\langle A_{ij}, \mathbf{n} \rangle$, respectively. Here $\mathbf{n}$ is the unit outer normal vector of Σ. Given a normal vector field V in the space of cross sections $\Gamma(N\Sigma)$, in terms of local coordinates $\{x_1, ..., x_n\}$, a $(2,0)-$tensor $\langle A, V \rangle$ is written as $\langle A_{ij}, V\rangle dx_i dx_j$ and the norm of the (2,0)-tensor is defined to be

$$|\langle A, V\rangle|^2 = \sum_{i,j,k,l=1}^{n} g^{ik}g^{jl}\langle A_{ij}, V\rangle\langle A_{kl}, V\rangle.$$

Definition 1 Let Σ be a submanifold in $\mathbb{R}^m$ and $B_r(0)$ be the geodesic ball in $\mathbb{R}^m$ with radius r. Σ is said to have *polynomial volume growth* if there are constants C_1, C_2 and $k \in \mathbb{N}$ so that for all $r \geq 0$

$$Vol(B_r(0) \cap \Sigma) \leq C_1 r^k + C_2.$$

Definition 2 A normal vector field V (or the second fundamental form A) of Σ is of *polynomial growth* if there are constants C_1, C_2 and $k \in \mathbb{N}$ so that for all $r \geq 0$

$$|V| \leq C_1 r^k + C_2 \quad (\text{or} \quad |A| \leq C_1 r^k + C_2) \quad \text{on} \quad B_r(0) \cap \Sigma.$$

For any smooth normal vector fields V and W in the space of cross sections $\Gamma(N\Sigma)$, its weighted L^2 inner product, denoted as $\langle V, W\rangle_e$, is defined to be $\int_\Sigma \langle V, W\rangle e^{-\frac{|X|^2}{4}} d\mu$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^m$. The weighted L^2 norm $||V||_e$ is induced by the weighted L^2 inner product $\langle V, V\rangle_e^{\frac{1}{2}}$. The space $(\Gamma(N\Sigma), \langle\cdot,\cdot\rangle_e)$ is called the *weighted L^2 inner product space.* For $V \in \Gamma(N\Sigma)$, we have the norm

$$||V||_{1,e} = (\int_\Sigma |V|^2 e^{-\frac{|X|^2}{4}} d\mu)^{1/2} + (\int_\Sigma |\nabla^\perp V|^2 e^{-\frac{|X|^2}{4}} d\mu)^{1/2}. \qquad (2.1.2)$$

Let $N_c(\Sigma)$ be the collection of all smooth normal vector fields in $\Gamma(N\Sigma)$ with compact support and denote the space $\mathcal{H}_0^1(\Sigma)$ as the closure of $N_c(\Sigma)$ with respect to the norm $||\cdot||_{1,e}$.

Definition 3 A submanifold Σ in $\mathbb{R}^m$ is called a *self-similar solution* if

$$H = \alpha X^\perp + T^\perp$$

for some constant $\alpha \in \mathbb{R}$ and some constant vector $T \in \mathbb{R}^m$, where X is the position vector. Σ is called a *self-shrinker* if $T = 0$ and $\alpha = -\frac{1}{2}$, a *self-expander* if $T = 0$ and $\alpha = 1$. When $\alpha = 0$, i.e., $H = T^\perp$, Σ is called a *translating solution.*

In Chapter 5 and 6, our ambient space is always the complex Euclidean space $\mathbb{C}^n$ with coordinates $z_j = x_j + \sqrt{-1}y_j$, the standard symplectic form $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$, and the standard almost complex structure J with $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$. A *Lagrangian submanifold* is an n-dimensional submanifold in $\mathbb{C}^n$ on which the symplectic form ω vanishes. On a Lagrangian submanifold L, the Lagrangian angle $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$ can be defined by the relation that $dz_1 \wedge \cdots \wedge dz_n|_L = e^{i\theta}\mathrm{Vol}_L$ and the mean curvature vector H is given by

$$H = J\nabla\theta$$

where ∇ is the gradient on L.

2.2 The first variation formula of F

Colding and Minicozzi derived the first and second variation formulae of the F-functional of a hypersurface in [6]. These can be generalized to higher co-dimensional cases by similar calculation. We derive the first variation formula of F in the following Theorem.

Theorem 2.2.1 *Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional complete manifold with polynomial volume growth. Suppose that $\Sigma_s \subset \mathbb{R}^m$ is a normal variation of Σ, x_s, t_s are variations of x_0 and t_0, and*

$$\frac{\partial \Sigma_s}{\partial s} = V, \quad \frac{dx_s}{ds} = y, \quad \text{and} \quad \frac{dt_s}{ds} = \tau,$$

where V has compact support. Then

$$\frac{\partial}{\partial s}F(\Sigma_s, x_s, t_s) = \frac{1}{\sqrt{4\pi t_s}^n}\int_{\Sigma_s}\Big(-\langle V, H_s + \frac{X_s - x_s}{2t_s}\rangle + \tau(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s}) + \frac{\langle X_s - x_s, y\rangle}{2t_s}\Big)e^{\frac{-|X_s - x_s|^2}{4t_s}}d\mu, \tag{2.2.1}$$

where X_s is the position vector of Σ_s and H_s is its mean curvature vector.

Proof. From the first variation formula for area, we know that

$$\frac{\partial}{\partial s}(d\mu) = -\langle H_s, V\rangle d\mu. \tag{2.2.2}$$

The variation of the weight $\frac{1}{\sqrt{4\pi t_s}^n} e^{-|X_s - x_s|^2/4t_s}$ have terms coming from the variation of X_s, the variation of x_s and the variation of t_s, respectively. Using the following equations

$$\begin{aligned} \frac{\partial}{\partial t_s} \log\big((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\big) &= \frac{-n}{2t_s} + \frac{|X_s - x_s|^2}{4t_s^2}, \\ \frac{\partial}{\partial x_s} \log\big((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\big) &= \frac{X_s - x_s}{2t_s} \\ \text{and} \quad \frac{\partial}{\partial X_s} \log\big((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\big) &= -\frac{X_s - x_s}{2t_s}, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{\partial}{\partial s} \log\big((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\big) \\ = &-\frac{\langle X_s - x_s, V\rangle}{2t_s} + \tau\Big(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s}\Big) + \frac{1}{2t_s}\langle X_s - x_s, y\rangle. \end{aligned}$$

Combining this with (2.2.2) gives (2.2.1). □

Definition 4 We will call (Σ, x_0, t_0) a *critical point* of F if it is critical with respect to all normal variations which have compact support in Σ and all variations in x and t.

From the definition of F in (1.0.1), we have $F(\Sigma, x, t) = F(\frac{\Sigma - x}{\sqrt{t}}, 0, 1)$ and it is easy to see the following property:

$$\begin{aligned} &(\Sigma, x_0, t_0) \text{ is a critical point of } F \text{ if and only if } \Big(\frac{\Sigma - x_0}{\sqrt{t_0}}, 0, 1\Big) \\ &\text{is a critical point of } F. \end{aligned} \tag{2.2.3}$$

Therefore, we only consider the case $x_0 = 0$, $t_0 = 1$. In the case of hypersurfaces, Colding and Minicozzi proved that $(\Sigma, 0, 1)$ is a critical point of F if Σ satisfies that $h = \frac{\langle X, \mathbf{n}\rangle}{2}$. Their result, when written in the vector form $H = -\frac{X^\perp}{2}$, also holds for higher co-dimensional cases. The proof needs following propositions.

Proposition 2.2.2 *If $\Sigma \subset \mathbb{R}^m$ is an n-dimensional complete submanifold with $H = -\frac{X^\perp}{2}$, then*

$$\mathcal{L}X_i = -\frac{1}{2}X_i \quad and$$
$$\mathcal{L}|X|^2 = 2n - |X|^2. \tag{2.2.4}$$

Here X_i is the i-th component of the position vector X, i.e., $X_i = \langle X, \partial_i \rangle$ and the linear operator $\mathcal{L}f = \Delta f - \frac{1}{2}\langle X, \nabla f \rangle = e^{\frac{|X|^2}{4}} div(e^{\frac{-|X|^2}{4}} \nabla f)$.

Proposition 2.2.3 *If $\Sigma \subset \mathbb{R}^m$ is an n-dimensional complete submanifold, with polynomial volume growth, and $H = -\frac{X^\perp}{2}$, then*

$$\int_\Sigma X e^{\frac{-|X|^2}{4}} d\mu = \overrightarrow{0} = \int_\Sigma X|X|^2 e^{\frac{-|X|^2}{4}} d\mu \quad and$$
$$\int_\Sigma (|X|^2 - 2n) e^{\frac{-|X|^2}{4}} d\mu = 0. \tag{2.2.5}$$

Moreover, if $W \in \mathbb{R}^m$ is a constant vector, then

$$\int_\Sigma \langle X, W \rangle^2 e^{-\frac{|X|^2}{4}} d\mu = 2 \int_\Sigma |W^\top|^2 e^{-\frac{|X|^2}{4}} d\mu. \tag{2.2.6}$$

These propositions were proved by Colding and Minicozzi in the case of hypersurfaces (see Lemma 3.20 and Lemma 3.25 in [6]). We omit the proofs here because the argument is similar. Combining (2.2.1), (2.2.3) and (2.2.5), we get

Proposition 2.2.4 *For any $x_0 \in \mathbb{R}^m$, $t_0 \in \mathbb{R}^+$, (Σ, x_0, t_0) is a critical point of F if and only if $H = -\frac{(X - x_0)^\perp}{2t_0}$.*

2.3 The general second variation formula of F

Theorem 2.3.1 *Let Σ be an n-dimensional complete manifold with polynomial volume growth. Suppose that Σ_s is a normal variation of Σ, x_s, t_s are variations of x_0 and t_0, and*

$$\frac{\partial \Sigma_s}{\partial s} = V, \quad \frac{dx_s}{ds} = y, \quad \frac{dt_s}{ds} = \tau, \quad \frac{d^2 x_s}{ds^2} = y', \quad and \quad \frac{d^2 t_s}{ds^2} = \tau',$$

where V has compact support. Then

$$
\begin{aligned}
&\frac{\partial^2 F}{\partial s^2}(\Sigma, x_0, t_0)\\
=&\frac{1}{\sqrt{4\pi t_0}^n}\int_\Sigma e^{-\frac{|X-x_0|^2}{4t_0}}\Big\{-\langle V, L^\perp_{x_0,t_0}V\rangle+\frac{\langle X-x_0,V\rangle}{t_0^2}\tau+\frac{\langle V,y\rangle}{t_0}\\
&-\frac{(|X-x_0|^2-nt_0)\tau^2}{2t_0^3}-\frac{|y|^2}{2t_0}-\frac{\tau\langle X-x_0,y\rangle}{t_0^2}\\
&+\Big(-\langle V,H+\frac{X-x_0}{2t_0}\rangle+\tau(\frac{|X-x_0|^2}{4t_0^2}-\frac{n}{2t_0})+\langle\frac{X-x_0}{2t_0},y\rangle\Big)^2\\
&-\langle\overline{\nabla}^\perp_V V,H+\frac{X-x_0}{2t_0}\rangle+\tau'(\frac{|X-x_0|^2}{4t_0}-\frac{n}{2t_0})+\frac{\langle X-x_0,y'\rangle}{2t_0}\Big\}d\mu,
\end{aligned}
\tag{2.3.1}
$$

where $L^\perp_{x_0,t_0}V=\Delta^\perp V+\langle A_{ij},V\rangle g^{ki}g^{jl}A_{kl}+\frac{V}{2t_0}-\frac{1}{2t_0}\nabla^\perp_{(X-x_0)^\top}V$ and A_{ij} is the second fundamental form as definition in (2.1.1).

Proof. Apply one more derivative on equation (2.2.1), it gives

$$
\begin{aligned}
&\frac{\partial^2 F}{\partial s^2}(\Sigma, x_0, t_0)\\
=&\frac{1}{\sqrt{4\pi t_0}^n}\int_\Sigma e^{-\frac{|X-x_0|}{4t_0}}\Big\{-\langle V,\frac{\partial}{\partial s}(H_s+\frac{X_s-x_s}{2t_s})\Big|_{s=0}\rangle\\
&+\tau\frac{\partial}{\partial s}(\frac{|X_s-x_s|^2}{4t_s^2}-\frac{n}{2t_s})\Big|_{s=0}+\langle\frac{\partial}{\partial s}(\frac{X_s-x_s}{2t_s})\Big|_{s=0},y\rangle\\
&+\Big(-\langle V,H+\frac{X-x_0}{2t_0}\rangle+\tau(\frac{|X-x_0|^2}{4t_0^2}-\frac{n}{2t_0})+\langle(\frac{X-x_0}{2t_0}),y\rangle\Big)^2\\
&-\langle V',(H+\frac{X-x_0}{2t_0})\rangle+\tau'(\frac{|X-x_0|^2}{4t_0^2}-\frac{n}{2t_0})+\langle(\frac{X-x_0}{2t_0}),y'\rangle\Big\}d\mu.
\end{aligned}
\tag{2.3.2}
$$

Similar to the derivation of the second variation formula for the area, we have

$$\langle(\frac{\partial H_s}{\partial s}),V\rangle=\langle\Delta^\perp V+\langle A_{ij},V\rangle g^{ki}g^{jl}A_{kl},V\rangle. \tag{2.3.3}$$

On the other hand, since $[V,(\frac{X-x_0}{2t_0})^\top]$ is tangent to Σ_s, it follows that

$$\langle\overline{\nabla}^\top_V V,\frac{X-x_0}{2t_0}\rangle=-\langle V,\overline{\nabla}_V(\frac{X-x_0}{2t_0})^\top\rangle=-\langle V,\overline{\nabla}_{(\frac{X-x_0}{2t_0})^\top}V\rangle. \tag{2.3.4}$$

Using $\frac{\partial X_s}{\partial s} = V$, $\frac{dt_s^{-1}}{ds} = -\tau t_s^{-2}$ and $\frac{dx_s}{ds} = y$, we simplify

$$
\begin{aligned}
&-\langle V, \frac{\partial}{\partial s}(H_s + \frac{X_s - x_s}{2t_s})\Big|_{s=0}\rangle - \langle V', H + \frac{X - x_0}{2t_0}\rangle \\
=&-\langle V, \frac{\partial H_s}{\partial s}\Big|_{s=0}\rangle - \langle V, \frac{\partial}{\partial s}(\frac{X_s - x_s}{2t_s})\Big|_{s=0}\rangle - \langle \overline{\nabla}_V^{\perp} V, H + \frac{X - x_0}{2t_0}\rangle - \langle \overline{\nabla}_V^{\top} V, \frac{X - x_0}{2t_0}\rangle \\
=&-\langle V, L_{x_0,t_0}^{\perp} V\rangle - \langle \overline{\nabla}_V^{\perp} V, H + \frac{X - x_0}{2t_0}\rangle + \langle V, \frac{y}{2t_0}\rangle + \frac{\tau}{2t_0^2}\langle V, X - x_0\rangle,
\end{aligned}
$$

where the second equality is from (2.3.3), (2.3.4), and the definition of $L_{x_0,t_0}^{\perp}$. The second term in (2.3.2) is given by

$$
\begin{aligned}
\frac{\partial}{\partial s}(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s})\Big|_{s=0} &= \frac{\langle X - x_0, V - y\rangle}{2t_0^2} - \frac{\tau|X - x_0|^2}{2t_0^3} + \frac{n\tau}{2t_0^2} \\
&= \frac{\langle X - x_0, V\rangle}{2t_0^2} - \frac{|X - x_0|^2 - nt_0}{2t_0^3}\tau - \frac{\langle X - x_0, y\rangle}{2t_0^2}.
\end{aligned}
$$

For the third term in (2.3.2), observe that

$$
\langle \frac{\partial}{\partial s}(\frac{X_s - x_s}{2t_s})\Big|_{s=0}, y\rangle = \langle \frac{V}{2t_0}, y\rangle - \frac{|y|^2}{2t_0} - \frac{\tau}{2t_0^2}\langle X - x_0, y\rangle.
$$

Combining these gives the theorem. □

2.4 The second variation at a critical point

For convenience, from now on we denote $D^2_{(V,y,\tau)}F$ as $\frac{\partial^2 F}{\partial s^2}(\Sigma, 0, 1)$ in (2.3.1). When $(\Sigma, 0, 1)$ is a critical point of F, we have $H = -\frac{X^{\perp}}{2}$, the second variation formula of F at the point can be simplified as the following equation (2.4.1).

Theorem 2.4.1 *Let Σ be a complete manifold with polynomial volume growth. Suppose that Σ_s is a normal variation of Σ, x_s, t_s are variations of $x_0 = 0$ and $t_0 = 1$, and*

$$
\frac{\partial \Sigma_s}{\partial s}\Big|_{s=0} = V, \quad \frac{dx_s}{ds}\Big|_{s=0} = y, \quad \frac{dt_s}{ds}\Big|_{s=0} = \tau,
$$

where V has compact support. If $(\Sigma, 0, 1)$ is a critical point of F, then

$$
\begin{aligned}
&D^2_{(V,y,\tau)}F \\
=&\frac{1}{\sqrt{4\pi}^n}\int_{\Sigma}\Big(-\langle V, L^{\perp}V\rangle - 2\tau\langle H, V\rangle - \tau^2|H|^2 + \langle V, y\rangle - \frac{1}{2}|y^{\perp}|^2\Big)e^{-\frac{|X|^2}{4}}d\mu. \quad (2.4.1)
\end{aligned}
$$

Here the operator $L^{\perp} = L^{\perp}_{0,1}$*, and*

$$L^{\perp}V = \Delta^{\perp}V - \frac{1}{2}\nabla^{\perp}_{X^{\top}}V + \langle A_{ij}, V\rangle g^{ki}g^{jl}A_{kl} + \frac{V}{2}. \tag{2.4.2}$$

Proof. Since $(\Sigma, 0, 1)$ is a critical point of F, by (2.2.1) we have that

$$H = -\frac{X^{\perp}}{2}. \tag{2.4.3}$$

It follows from (2.2.5) that

$$\int_{\Sigma} Xe^{\frac{-|X|^2}{4}}d\mu = \overrightarrow{0} = \int_{\Sigma} X|X|^2e^{\frac{-|X|^2}{4}}d\mu \quad \text{and} \quad \int_{\Sigma}(|X|^2 - 2n)e^{\frac{-|X|^2}{4}}d\mu = 0. \tag{2.4.4}$$

Theorem 2.3.1 (with $x_0 = 0$ and $t_0 = 1$) gives

$$\begin{aligned} D^2_{(V,y,\tau)}F = (4\pi)^{-\frac{n}{2}} \int_{\Sigma}\Big(&- \langle V, L^{\perp}V\rangle + \tau\langle X, V\rangle + \langle V, y\rangle - \frac{(|X|^2 - n)\tau^2}{2} - \frac{|y|^2}{2} \\ &- \tau\langle X, y\rangle + \{\tau(\frac{|X|^2}{4} - \frac{n}{2}) + \langle\frac{X}{2}, y\rangle\}^2\Big)e^{-\frac{|X|^2}{4}}d\mu, \end{aligned}$$

where we use (2.4.3) and (2.4.4) to conclude the vanishing of a few terms in (2.3.1). Note that y is a constant vector and τ is a constant. Squaring out the last term of $D^2_{(V,y,\tau)}F$ and using (2.4.3) and (2.4.4) again leads to

$$\begin{aligned} D^2_{(V,y,\tau)}F =&(4\pi)^{-\frac{n}{2}} \int_{\Sigma}\Big(-\langle V, L^{\perp}V\rangle - 2\tau\langle H, V\rangle + \langle V, y\rangle - \frac{|y|^2}{2} \\ &+ \tau^2(\frac{|X|^2}{4} - \frac{n}{2})^2 + \frac{1}{4}\langle X, y\rangle^2 - \frac{(|X|^2 - n)\tau^2}{2}\Big)e^{-\frac{|X|^2}{4}}d\mu. \end{aligned}$$

Using the equality (2.2.4) and Stokes' theorem, we have that

$$\int_{\Sigma}\tau^2(\frac{|X|^2}{4} - \frac{n}{2})^2e^{-\frac{|X|^2}{4}}d\mu = \int_{\Sigma}\tau^2\frac{|X^{\top}|^2}{4}e^{-\frac{|X|^2}{4}}d\mu.$$

Combining (2.2.5) and (2.2.6), the second variation $D^2_{(V,y,\tau)}F$ can be further simplified as

$$\frac{1}{\sqrt{4\pi}^n}\int_{\Sigma}\Big(-\langle V, L^{\perp}V\rangle - 2\tau\langle H, V\rangle - \tau^2|H|^2 + \langle V, y\rangle - \frac{1}{2}|y^{\perp}|^2\Big)e^{-\frac{|X|^2}{4}}d\mu.$$

□

In [6], Colding and Minicozzi defined the following concept.

Definition 5 A critical point $(\Sigma, 0, 1)$ of F is *F-stable* if for every compactly supported smooth variation Σ_s with $\Sigma_0 = \Sigma$ and $\frac{\partial \Sigma_s}{\partial s}\big|_{s=0} = V$, there exist variations x_s of 0 and t_s of 1 such that $D^2_{(V,y,\tau)} F \geq 0$, where $y = \frac{dx_s}{ds}\big|_{s=0}$ and $\tau = \frac{dt_s}{ds}\big|_{s=0}$.

Remark 1 *When Σ is fixed, i.e. $V = 0$, from* (2.4.1)*, we can see that the second variation formula of F is nonpositive under any variations of x_s and t_s.*

Chapter 3

An equivalent condition for F-stability

3.1 Vector-valued eigenfunctions and eigenvalues of $L^{\perp}$

Let $X : \Sigma \to \mathbb{R}^m$ be a closed self-shrinker. Recall that the second order operator $L^{\perp}$ is defined by

$$L^{\perp}V = \Delta^{\perp}V - \frac{1}{2}\nabla^{\perp}_{X^{\top}}V + \langle A_{ij}, V\rangle g^{ki}g^{jl}A_{kl} + \frac{V}{2}$$

for $V \in \Gamma(N\Sigma)$. Therefore, we have the following

$$\int_{\Sigma}\langle -L^{\perp}V, W\rangle e^{-\frac{|X|^2}{4}}d\mu = \int_{\Sigma}\Big(\langle \nabla^{\perp}V, \nabla^{\perp}W\rangle - \langle A_{ij}, V\rangle\langle A_{kl}, W\rangle g^{ik}g^{jl} - \frac{1}{2}\langle V, W\rangle\Big)e^{-\frac{|X|^2}{4}}d\mu$$

for $V, W \in \Gamma(N\Sigma)$. It is easy to see that the operator $L^{\perp}$ is self-adjoint in the weighted L^2 inner product space. From the standard spectrum theory, the operator $-L^{\perp}$ has distinct real eigenvalues $\{\mu_i\}$ such that $\mu_1 < \mu_2 \leq \mu_3 \leq ... \to +\infty$. We have the following proposition.

Proposition 3.1.1 *Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete self-shrinker, $H = -\frac{X^{\perp}}{2}$. Then the mean curvature vector H and the normal part $y^{\perp}$ of a constant vector field y are vector-valued eigenfunctions of $L^{\perp}$ with*

$$L^{\perp}H = H \quad \textit{and} \quad L^{\perp}y^{\perp} = \frac{1}{2}y^{\perp}. \tag{3.1.1}$$

Proof. Fix $p \in \Sigma$ and choose an orthonormal frame $\{e_i\}$ such that $\nabla_{e_i} e_j(p) = 0$, $g_{ij} = \delta_{ij}$ in a neighborhood of p. Using $H = -\frac{1}{2}X^\perp$, we have

$$\nabla^\perp_{e_i} H = \nabla^\perp_{e_i}(-\frac{1}{2}X^\perp) = \frac{1}{2}\nabla^\perp_{e_i}(\langle X, e_j\rangle e_j - X) = \frac{1}{2}\langle X, e_j\rangle A_{ij}. \tag{3.1.2}$$

In the second equality of (3.1.2), we used $X^\top = \langle X, e_j\rangle e_j$. Taking another covariant derivative at p, it gives

$$\begin{aligned}\nabla^\perp_{e_k}\nabla^\perp_{e_i} H &= \frac{1}{2}(\nabla_{e_k}\langle X, e_j\rangle)A_{ij} + \frac{1}{2}\langle X, e_j\rangle \nabla^\perp_{e_k} A_{ij} \\ &= \frac{1}{2}A_{ik} + \frac{1}{2}\langle X, A_{kj}\rangle A_{ij} + \frac{1}{2}\langle X, e_j\rangle \nabla^\perp_{e_j} A_{ik},\end{aligned} \tag{3.1.3}$$

where we used (3.1.2), $\nabla_{e_k} e_j(p) = 0$, and the Codazzi equation in the last equality. Taking the trace of (3.1.3) and using $H = -\frac{1}{2}X^\perp$, we conclude that

$$\Delta^\perp H = \frac{1}{2}H - \langle H, A_{ij}\rangle A_{ij} + \frac{1}{2}\nabla^\perp_{X^\top} H.$$

Therefore,

$$L^\perp H = \Delta^\perp H - \frac{1}{2}\nabla^\perp_{X^\top} H + \langle A_{ij}, H\rangle A_{ij} + \frac{1}{2}H = H.$$

For a constant vector y in $\mathbb{R}^m$, the covariant derivative of $y^\perp$ is

$$\nabla^\perp_{e_i} y^\perp = \nabla^\perp_{e_i}(y - \langle y, e_j\rangle e_j) = -\langle y, e_j\rangle A_{ij}. \tag{3.1.4}$$

Taking another covariant derivative at p, it gives

$$\begin{aligned}\nabla^\perp_{e_k}\nabla^\perp_{e_i} y^\perp &= -(\nabla_{e_k}\langle y, e_j\rangle)A_{ij} - \langle y, e_j\rangle \nabla^\perp_{e_k} A_{ij} \\ &= -\langle y, A_{kj}\rangle A_{ij} - \langle y, e_j\rangle \nabla^\perp_{e_j} A_{ki},\end{aligned} \tag{3.1.5}$$

by $\nabla_{e_k} e_j(p) = 0$ and the Codazzi equation. Taking the trace of (3.1.5) and using (3.1.2), (3.1.4), we conclude that

$$\begin{aligned}\Delta^\perp y^\perp &= -\langle y, A_{ij}\rangle A_{ij} - \langle y, e_j\rangle \nabla^\perp_{e_j} H \\ &= -\langle y^\perp, A_{ij}\rangle A_{ij} - \frac{1}{2}\langle y, e_j\rangle\langle X, e_i\rangle A_{ij} \\ &= -\langle y^\perp, A_{ij}\rangle A_{ij} + \frac{1}{2}\langle X, e_i\rangle \nabla^\perp_{e_i} y^\perp \\ &= -\langle y^\perp, A_{ij}\rangle A_{ij} + \frac{1}{2}\nabla^\perp_{X^\top} y^\perp.\end{aligned}$$

Therefore,

$$L^{\perp}y^{\perp} = \Delta^{\perp}y^{\perp} - \frac{1}{2}\nabla^{\perp}_{X^{\top}}y^{\perp} + \langle A_{ij}, y^{\perp}\rangle A_{ij} + \frac{1}{2}y^{\perp} = \frac{1}{2}y^{\perp}.$$

□

Remark 2 *At the same time, Andrews, Li, and Wei also showed the same result independently. See Proposition 5.1 in [5].*

For hypersurface case, we have the following immediate corollary.

Corollary 3.1.2 (Theorem 5.2 in [6]) *Let $\Sigma \subset \mathbb{R}^{n+1}$ be an n-dimensional smooth complete self-shrinker, $h = \frac{\langle X, \mathbf{n}\rangle}{2}$. Then the mean curvature function $h = -\langle H, \mathbf{n}\rangle$ and the normal part $\langle y, \mathbf{n}\rangle$ of a constant vector field y are eigenfunctions of L with $Lh = h$ and $L\langle y, \mathbf{n}\rangle = \frac{1}{2}\langle y, \mathbf{n}\rangle$, where $Lf = \Delta f - \frac{1}{2}\langle X, \nabla f\rangle + |\langle A_{ij}, n\rangle|^2 f + \frac{1}{2}f$.*

3.2 An equivalent condition

In the following theorems, we give an equivalent condition for $F(\Sigma, 0, 1)$ to be stable. Roughly speaking, Σ is F-stable if and only if the H and $y^{\perp}$ are the only eigenvectors of $L^{\perp}$ with positive eigenvalues for any constant vector y in $\mathbb{R}^m$. It is inspired by the proof of Lemma 4.23 of Colding and Minicozzi in [6].

Theorem 3.2.1 *Suppose $\Sigma \subset \mathbb{R}^m$ is an n-dimensional smooth closed self-shrinker, $H = -\frac{X^{\perp}}{2}$. The following statements are equivalent:*

(i) Σ is F-stable.

(ii) $\int_{\Sigma}\langle V, -L^{\perp}V\rangle e^{-\frac{|X|^2}{4}}d\mu \geq 0$ for any admissible vector field V, namely, a smooth normal vector field V which satisfies

$$\int_{\Sigma}\langle V, H\rangle e^{-\frac{|X|^2}{4}}d\mu = 0 \quad \textit{and} \quad \int_{\Sigma}\langle V, y^{\perp}\rangle e^{-\frac{|X|^2}{4}}d\mu = 0. \tag{3.2.1}$$

for all constant vector $y \in \mathbb{R}^m$.

Proof. $(i) \Rightarrow (ii)$ Assume the contrary that there is an admissible vector field V satisfying $\int_\Sigma \langle V, -L^\perp V\rangle e^{\frac{-|X|^2}{4}} d\mu < 0$. For any real value τ and constant vector y in $\mathbb{R}^m$, using (2.4.1), we have

$$
\begin{aligned}
&D^2_{(V,y,\tau)}F \\
=&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma \left(-\langle V, L^\perp V\rangle - 2\tau\langle H, V\rangle - \tau^2|H|^2 + \langle V, y\rangle - \frac{1}{2}|y^\perp|^2\right) e^{-\frac{|X|^2}{4}} d\mu \\
=&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma \left(-\langle V, L^\perp V\rangle - \tau^2|H|^2 - \frac{1}{2}|y^\perp|^2\right) e^{-\frac{|X|^2}{4}} d\mu \\
<&0.
\end{aligned}
$$

This contradicts the stability of F.

$(ii) \Rightarrow (i)$ From the standard spectrum theory, a smooth normal vector field can be decomposed as $aH + z^\perp + V_0$, where aH and $z^\perp$ are the projections of V to H and $\{y^\perp | y \in \mathbb{R}^m\}$ with respect to weighted L^2 inner product, respectively. Note that V_0 is an admissible vector field. For any real value τ and constant vector $y \in \mathbb{R}^m$, by plugging the decomposition of V into (2.4.1), we have

$$
\begin{aligned}
&D^2_{(V,y,\tau)}F \\
=&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma \left(-\langle V, L^\perp V\rangle - 2\tau\langle H, V\rangle - \tau^2|H|^2 + \langle V, y\rangle - \frac{1}{2}|y^\perp|^2\right) e^{-\frac{|X|^2}{4}} d\mu \\
=&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma \Big(- a^2|H|^2 - \frac{1}{2}|z^\perp|^2 - \langle V_0, L^\perp V_0\rangle - 2\tau a|H|^2 - \tau^2|H|^2 \\
&+ \langle z^\perp, y^\perp\rangle - \frac{1}{2}|y^\perp|^2\Big) e^{-\frac{|X|^2}{4}} d\mu \\
\geq&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma \left(-|H|^2(a+\tau)^2 - \frac{1}{2}|z^\perp - y^\perp|^2\right) e^{-\frac{|X|^2}{4}} d\mu,
\end{aligned}
$$

where the condition (ii) is used in the last inequality. Choosing $\tau = -a$ and $y = z$, it gives $D^2_{(V,z,-a)}F \geq 0$. That is, Σ is F-stable. □

Recall that $\mathcal{H}^1_0(\Sigma)$ is the closure of $N_c(\Sigma)$ with respect to the norm $||\cdot||_{1,e}$, where $N_c(\Sigma)$ is the collection of all smooth normal vector fields with compact support. If Σ is a smooth self-shrinker with polynomial volume growth and its second fundamental form A is of polynomial growth, it is easy to see that $|H|_e$ and $|y^\perp|_e$ are finite and belong to $\mathcal{H}_0^1(\Sigma)$. For any $V \in \mathcal{H}_0^1(\Sigma)$, the integral $\int_\Sigma \left(|\nabla^\perp V|^2 - |\langle A, V\rangle|^2 - \frac{1}{2}|V|^2\right) e^{-\frac{|X|^2}{4}} d\mu$ is finite. Assume that Σ has no boundary, we have

$$\langle V, -L^\perp V\rangle_e = \int_\Sigma \left(|\nabla^\perp V|^2 - |\langle A, V\rangle|^2 - \frac{1}{2}|V|^2\right) e^{-\frac{|X|^2}{4}} d\mu, \tag{3.2.2}$$

and can also find the following equivalent condition for the stability of F in the complete noncompact case.

Theorem 3.2.2 *Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete noncompact self-shrinker, $H = -\frac{X^\perp}{2}$. Suppose that the second fundamental form A of Σ is of polynomial growth and Σ has polynomial volume growth. The following statements are equivalent:*

(i) Σ is F-stable.

(ii) $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu \geq 0$ for any admissible vector field V in $\mathcal{H}_0^1(\Sigma)$.

Remark 3 *The condition of the second fundamental form can be weakened to that the integral $\int_\Sigma |A|^2 e^{-\frac{|X|^2}{4}} d\mu$ is finite. Admissible vector fields are characterized by* (3.2.1).

Proof of Theorem 3.2.2. $(i) \Rightarrow (ii)$ Assume the contrary that there is an admissible vector field V in $\mathcal{H}_0^1(\Sigma)$ satisfying $\langle V, -L^\perp V\rangle_e < 0$. Here V may not have a compact support. For $j \in \mathbb{N}$, consider smooth functions $\phi_j : \mathbb{R}^+ \bigcup\{0\} \to \mathbb{R}$ that satisfy $0 \leq \phi_j \leq 1$, $\phi_j \equiv 1$ on $[0, j)$, $\phi_j \equiv 0$ outside $[0, j+2)$ and $|\phi_j'| \leq 1$. Define cutoff functions $\psi_j(X) = \phi_j(\rho(X))$, $X \in \Sigma$, where $\rho(X)$ is the distance function from a fixed point $p \in \Sigma$ to X with respect to the metric g_{ij}. Let $V_j(X) = \psi_j(X)V(X)$, then we have

$$\begin{aligned}
|\nabla^\perp V_j|^2 &= \sum_{i=1}^n |(\nabla_{e_i}\psi_j)V + \psi_j \nabla_{e_i}^\perp V|^2 \\
&\leq 2|\nabla\psi_j|^2|V|^2 + 2|\psi_j|^2|\nabla^\perp V|^2 \\
&\leq 2|V|^2 + 2|\nabla^\perp V|^2.
\end{aligned}$$

Here $\{e_i\}$ is an orthonormal basis for $T_X\Sigma$. Since the second fundamental form A of Σ is of polynomial growth and $V \in \mathcal{H}_0^1(\Sigma)$, the weighted L^2 inner product $\langle V, -L^\perp V\rangle_e$ is finite. Using the dominant convergence theorem and the admissible condition, it follows that

$$\lim_{j\to\infty}\langle V_j, -L^\perp V_j\rangle_e = \langle V, -L^\perp V\rangle_e \text{ and } \lim_{j\to\infty}\langle V_j, H\rangle_e = \lim_{j\to\infty}\langle V_j, y^\perp\rangle_e = 0.$$

For any small positive ϵ, choose a sufficiently large j such that

$$\begin{aligned}
&\langle V_j, -L^\perp V_j\rangle_e < \frac{1}{2}\langle V, -L^\perp V\rangle_e < 0,\\
&|\langle V_j, H\rangle_e| < \epsilon |H|_e, \quad \text{and} \quad \max_{|y^\perp|_e=1} |\langle V_j, y^\perp\rangle_e| < \epsilon.
\end{aligned}$$

For any real value τ and constant vector y in $\mathbb{R}^m$, we get

$$\begin{aligned}
&D^2F_{(V_j,y,\tau)}\\
=&\frac{1}{\sqrt{4\pi}^n}\int_\Sigma\left(-\langle V_j, L^\perp V_j\rangle - 2\tau\langle H, V_j\rangle - \tau^2|H|^2 + \langle V_j, y^\perp\rangle - \frac{1}{2}|y^\perp|^2\right)e^{-\frac{|X|^2}{4}}d\mu\\
<&\frac{1}{\sqrt{4\pi}^n}\left(-\frac{1}{2}\langle V, L^\perp V\rangle_e + 2\tau\epsilon|H|_e - \tau^2|H|_e^2 + \epsilon|y^\perp|_e - \frac{1}{2}|y^\perp|_e^2\right)\\
=&\frac{1}{\sqrt{4\pi}^n}\left(-\frac{1}{2}\langle V, L^\perp V\rangle_e + \epsilon^2 - (\tau|H|_e - \epsilon)^2 + \frac{1}{2}\epsilon^2 - \frac{1}{2}(|y^\perp|_e - \epsilon)^2\right).
\end{aligned}$$

Choosing $\epsilon^2 < \frac{1}{10}\langle V, L^\perp V\rangle_e$, we get $D^2F_{(V_j,y,\tau)} < 0$ for every τ and y. This contradicts the stability of F.

$(ii) \Rightarrow (i)$ From the standard spectrum theory, a compactly supported smooth normal vector field V can be decomposed as $aH + z^\perp + V_0$, where V_0, is an admissible vector field. Because V, H, and $z^\perp$ belong to $\mathcal{H}_0^1(\Sigma)$, which is a Hilbert space, V_0 belongs to $\mathcal{H}_0^1(\Sigma)$, too. The remaining part of the proof is essentially the same as the proof of $(ii) \Rightarrow (i)$ in Theorem 3.2.1. □

Chapter 4

Classification of stable product self-shrinkers

In this chapter, we discuss the F-stability of product self-shrinkers.

4.1 For compact case

Corollary 4.1.1 *Suppose $\Sigma_i^{n_i} \subset \mathbb{R}^{m_i}$, $i = 1, 2$, are smooth closed self-shrinkers which satisfy $H_i = -\frac{X_i^\perp}{2}$. Here X_i are the position vectors of Σ_i. Then $\Sigma_1 \times \Sigma_2 \subset \mathbb{R}^{m_1+m_2}$ is a self-shrinker and is F-unstable.*

Proof. The mean curvature H of $\Sigma_1 \times \Sigma_2$ is equal to $(H_1, H_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and $\Sigma_1 \times \Sigma_2$ is a self-shrinker because $H_1 = -\frac{X_1^\perp}{2}$ and $H_2 = -\frac{X_2^\perp}{2}$. To prove this corollary, by Theorem 3.2.1, it suffices to construct an admissible vector field V such that $\int_{\Sigma_1 \times \Sigma_2} \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Let $V = (aH_1, bH_2)$, where a and b would be chosen later. Note that V does not completely vanish since Σ_1 and Σ_2 are closed submanifolds in Euclidean spaces. The first integral in the admissible condition is

$$\begin{aligned}
&\int_{\Sigma_1 \times \Sigma_2} \langle V, H\rangle e^{-\frac{|X|^2}{4}} d\mu \\
=&\int_{\Sigma_1} \int_{\Sigma_2} (a|H_1|^2 + b|H_2|^2) e^{-\frac{|X_1|^2}{4}} e^{-\frac{|X_2|^2}{4}} d\mu_2 d\mu_1 \\
=&a \int_{\Sigma_1} |H_1|^2 e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} e^{-\frac{|X_2|^2}{4}} d\mu_2 + b \int_{\Sigma_1} e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} |H_2|^2 e^{-\frac{|X_2|^2}{4}} d\mu_2.
\end{aligned}$$

We can choose a and b to be nonzero constants such that $\int_{\Sigma_1\times\Sigma_2}\langle V, H\rangle e^{-\frac{|X^2|}{4}}d\mu = 0$. Recall that H_i and $y_i^\perp$ are eigenfunctions of $L_i^\perp$ with respect to distinct eigenvalues for $y_i \in \mathbb{R}^{m_i}$, where $L_i^\perp$ is the operator corresponding to Σ_i, $i = 1, 2$. Hence the second integral in the admissible condition is

$$\begin{aligned}
&\int_{\Sigma_1\times\Sigma_2}\langle V, y^\perp\rangle e^{-\frac{|X|^2}{4}}d\mu\\
=&a\int_{\Sigma_1}\langle H_1, y_1\rangle e^{-\frac{|X_1|^2}{4}}d\mu_1\int_{\Sigma_2}e^{-\frac{|X_2|^2}{4}}d\mu_2 + b\int_{\Sigma_1}e^{-\frac{|X_1|^2}{4}}d\mu_1\int_{\Sigma_2}\langle H_2, y_2\rangle e^{-\frac{|X_2|^2}{4}}d\mu_2\\
=&0
\end{aligned}$$

for $y = (y_1, y_2) \in \mathbb{R}^{m_1+m_2}$. Therefore V is an admissible vector field and the weighted L^2 inner product $\langle V, -L^\perp V\rangle_e$ can be computed as

$$\begin{aligned}
&\int_{\Sigma_1\times\Sigma_2}\langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}}d\mu\\
=&\int_{\Sigma_1\times\Sigma_2}\langle (aH_1, bH_2), -(aH_1, bH_2)\rangle e^{-\frac{|X|^2}{4}}d\mu\\
=&-a^2\int_{\Sigma_1}|H_1|^2 e^{-\frac{|X_1|^2}{4}}d\mu_1\int_{\Sigma_2}e^{-\frac{|X_2|^2}{4}}d\mu_2 - b^2\int_{\Sigma_1}e^{-\frac{|X_1|^2}{4}}d\mu_1\int_{\Sigma_2}|H_2|^2 e^{-\frac{|X_2|^2}{4}}d\mu_2\\
<&0.
\end{aligned}$$

Then $\Sigma_1 \times \Sigma_2$ is F-unstable. □

4.2 For noncompact case

In [6], Colding and Minicozzi proved that $\mathbb{S}^{n-1}\times\mathbb{R}$ is F-unstable. In this section, we use analogous their idea to get the following

Corollary 4.2.1 *Let $\Sigma_i^{n_i} \subset \mathbb{R}^{m_i}$, $i = 1, 2$, be two smooth complete self-shrinkers with $H_i = -\frac{X_i^\perp}{2}$. Here X_i is the position vector of Σ_i. Suppose that each Σ_i has polynomial volume growth and the second fundamental form of each Σ_i is of polynomial growth. Then $\Sigma_1 \times \Sigma_2 \subset \mathbb{R}^{m_1+m_2}$ is a self-shrinker and is F-unstable except the plane $\mathbb{R}^{n_1+n_2}$.*

Proof. The mean curvature H of $\Sigma_1 \times \Sigma_2$ is equal to $(H_1, H_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and $\Sigma_1 \times \Sigma_2$ is a self-shrinker because $H_1 = -\frac{X_1^\perp}{2}$ and $H_2 = -\frac{X_2^\perp}{2}$. Since the smooth minimal self-shrinker must be a plane through 0 (Corollary 2.8 in [6]), only two cases, $\Sigma_1 \times \Sigma_2$ with $H_1 \neq 0, H_2 \neq 0$ or $\Sigma_1 \times \mathbb{R}^{n_2}$ with $H_1 \neq 0$, need to be considered. In the first case, the proof is similar to the proof of Corollary 4.1.1. In the second one, by Theorem 3.2.1, it suffices to construct an admissible vector field V satisfying $\int_{\Sigma_1 \times \mathbb{R}^{n_2}} \langle V, -L^{\perp} V \rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Set $V = s(H_1, 0)$, where s is the coordinate function corresponding to the first coordinates in $\mathbb{R}^{n_2}$. Using the fact that $\int_{\mathbb{R}^{n_2}} s e^{-\frac{|X_2|^2}{4}} d\mu_2 = 0$, the first integral in the admissible condition is

$$\int_{\Sigma_1 \times \mathbb{R}^{n_2}} \langle V, H \rangle e^{-\frac{|X|^2}{4}} d\mu = \int_{\mathbb{R}^{n_2}} s e^{-\frac{|X_2|^2}{4}} d\mu_2 \int_{\Sigma_1} |H_1|^2 e^{-\frac{|X_1|^2}{4}} d\mu_1 = 0$$

and the second integral in the admissible condition is

$$\int_{\Sigma_1 \times \mathbb{R}^{n_2}} \langle V, y^{\perp} \rangle e^{-\frac{|X|^2}{4}} d\mu = \int_{\mathbb{R}^{n_2}} s e^{-\frac{|X_2|^2}{4}} d\mu_2 \int_{\Sigma_1} \langle H_1, y_1^{\perp} \rangle e^{-\frac{|X_1|^2}{4}} d\mu_1 = 0$$

for $y = (y_1, y_2) \in \mathbb{R}^{m_1+m_2}$. Therefore, the smooth normal vector field $V = s(H_1, 0)$ is an admissible vector field and belongs to $\mathcal{H}_0^1(\Sigma)$. Using the fact $L^{\perp} H = H$, it implies that the vector field V is a eigenvector of $L^{\perp}$ with respect to eigenvalue $\frac{1}{2}$ and the weighted L^2 inner product $\langle V, -L^{\perp} V \rangle_e = \int_{\Sigma_1 \times \mathbb{R}^{n_2}} -\frac{1}{2} |V|^2 e^{-\frac{|X|^2}{4}} d\mu < 0$. Then $\Sigma_1 \times \mathbb{R}^{n_2}$ is F-unstable. □

Proof of Theorem 1.0.1. Combining Corollary 4.1.1 and 4.2.1, this complete the proof of Theorem 1.0.1. □

Chapter 5

The unstability of Anciaux's examples

5.1 Anciaux's examples

Let $\langle\langle \cdot, \cdot \rangle\rangle = \sum_{i=1}^{n} dz_i \otimes d\overline{z}_i$ be the standard Hermitian form on $\mathbb{C}^n$, where $z_i = x_i + \sqrt{-1}y_i$, $i = 1, ..., n$ are the standard complex coordinates. The standard Riemannian metric is $\langle \cdot, \cdot \rangle = \mathrm{Re}\langle\langle \cdot, \cdot \rangle\rangle = \sum_{i=1}^{n}(dx_i^2 + dy_i^2)$ and the symplectic form is $\omega(\cdot, \cdot) = -\mathrm{Im}\langle\langle \cdot, \cdot \rangle\rangle = \sum_{i=1}^{n} dx_i \wedge dy_i$. We have $\omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle$, where J is the standard almost complex structure $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$.

Recall that an immersion ψ from a manifold M of dimension $(n-1)$ into $\mathbb{S}^{2n-1}$ is said to be *Legendrian* if $\alpha|_{\psi(M)} = 0$ for the contact 1-form $\alpha(\cdot) = \omega(X^M, \cdot)$, where X^M is the position vector and ω is the standard symplectic form on $\mathbb{C}^n$. Moreover, $d\alpha = 2\omega$ and $\langle Jy, z \rangle = \omega(y, z) = \frac{1}{2}d\alpha(y, z) = 0$, $\langle JX^M, y \rangle = \omega(X^M, y) = \alpha(y) = 0$ for all y, $z \in T\psi(M)$. It means that y, Jz, X^M, and JX^M are mutually orthogonal with respect to the standard metric g for any y, $z \in T\psi(M)$. When ψ is a minimal immersion, the complex scalar product $\gamma\psi$ of a smooth regular curve $\gamma : I \to \mathbb{C}^*$ and ψ is a Lagrangian submanifold in $\mathbb{C}^n$, i.e., $\omega|_{\gamma\psi} \equiv 0$. This was observed by Anciaux in [1]. Indeed, he proved by following Lemma.

Lemma 5.1.1 (Anciaux [1]) *Let $\psi : M \to \mathbb{S}^{2n-1}$ be a minimal Legendrian immersion for $n \geq 2$ and $\gamma : I \to \mathbb{C}^*$ be a smooth regular curve parameterized by the* *arclength s. Then the following immersion*

$$\begin{aligned}\gamma * \psi : I \times M &\to \mathbb{C}^n \\ (s,\sigma) &\to \gamma(s)\psi(\sigma)\end{aligned}$$

is Lagrangian. Moreover, $\gamma * \psi$ satisfies the self-shrinker equation

$$H + \frac{1}{2}(\gamma * \psi)^{\perp} = 0$$

if and only if γ satisfies the following system of ordinary differential equations:

$$\begin{cases} r'(s) &= \cos(\theta - \phi), \\ \theta'(s) - \phi'(s) &= (\frac{r}{2} - \frac{n}{r})\sin(\theta - \phi), \end{cases} \tag{5.1.1}$$

where the curve γ is denoted as $r(s)e^{i\phi(s)}$ and θ is the angle of the tangent and the x-axis. From (5.1.1), *we have a conservation law*

$$r^n e^{-\frac{r^2}{4}} \sin(\theta - \phi) = E, \tag{5.1.2}$$

where $0 < E \le E_{max} = (\frac{2n}{e})^{n/2}$ is a constant determined by the initial data $(r(s_0), \theta(s_0) - \phi(s_0))$.

5.2 The unstability for general variations

Recall that F-functional is infinite on the complete noncompact Anciaux's Lagrangian. We will only discuss the closed cases.

Theorem 5.2.1 *Anciaux's closed example as described in Lemma* 5.1.1 *is F-unstable.*

To prove the result, we first set up the notations and derive a few Lemmas. For a fixed point $p \in \Sigma = \gamma * \psi(I \times M)$, it can be represented by $\gamma(s_0)q$ for some $s_0 \in I$ and $q \in \psi(M)$. Choose a local normal coordinate system $x^1, ..., x^{n-1}$ at q. Denote $u_s = \frac{\partial X}{\partial s} = \gamma' X^M$, $e_i = \frac{\partial X^M}{\partial x^i}$, and $u_i = \frac{\partial X}{\partial x^i} = \gamma e_i$ for $i = 1, ..., n-1$, where X^M is the position vector of $\psi(M)$ and $X = \gamma X^M$. The matrix $(g_{\alpha\beta})$ of the induced metric of Σ with respect to the basis $u_1, ..., u_{n-1}, u_s$ is

$$g_{ss} = 1, \quad g_{js} = g_{sj} = 0, \quad g_{jk} = r^2 h_{jk}, \quad \text{and} \quad h_{jk}(q) = \delta_{jk} \tag{5.2.1}$$

for $j,k=1,...,n-1$. The Levi-Civita connections on Σ and $\psi(M)$ are denoted by ∇ and ∇^M, respectively. Define

$$N_0=\{V|V=J(\gamma w),\ w\in\Gamma(T\psi(M))\}.$$

For $V\in N_0$, the operator $\langle V,-L^\perp V\rangle_e$ can be simplified as below.

Lemma 5.2.2 *Assume that Σ is a closed Lagrangian self-shrinker as in Lemma* 5.1.1 *and $V\in N_0$ is represented by $J(\gamma w)$. The second fundamental forms of Σ in $\mathbb{C}^n$ and $\psi(M)$ in $\mathbb{S}^{2n-1}$ are denoted by A^Σ and $A^{M,\mathbb{S}}$, respectively. We have*

$$(i)\ |\langle A^\Sigma,V\rangle|^2=|\langle A^{M,\mathbb{S}},Jw\rangle|^2+2\sin^2(\theta-\phi)|w|^2, \tag{5.2.2}$$

$$(ii)\ |\nabla^\perp V|^2=|\nabla^M w|^2+2\cos^2(\theta-\phi)|w|^2, \tag{5.2.3}$$

$$\begin{aligned}(iii)\ \langle V,-L^\perp V\rangle_e=&-\int_\gamma\left(\frac{1}{2}r^2-2+4\sin^2(\theta-\phi)\right)e^{\frac{-r^2}{4}}r^{n-1}ds\int_M|w|^2d\mu_M\\ &+\int_\gamma e^{\frac{-r^2}{4}}r^{n-1}ds\int_M\left(|\nabla^M w|^2-|\langle A^{M,\mathbb{S}},Jw\rangle|^2\right)d\mu_M.\end{aligned} \tag{5.2.4}$$

Proof. (i) For $V\in N_0$, it can be represented by $J(\gamma w)$ for some vector field $w\in\Gamma(T\psi(M))$. Using $\gamma\overline{\gamma}=r^2$ and $\gamma'\overline{\gamma}=re^{i(\theta-\phi)}$, we conclude that

$$\begin{aligned}\langle A^\Sigma_{kl},V\rangle&=\mathrm{Re}\langle\langle\gamma\frac{\partial^2X^M}{\partial x^k\partial x^l},J(\gamma w)\rangle\rangle=r^2\mathrm{Re}\langle\langle A^M_{kl},Jw\rangle\rangle=r^2\langle A^{M,S}_{kl},Jw\rangle,\\ \langle A^\Sigma_{ks},V\rangle&=\mathrm{Re}\langle\langle\gamma'\frac{\partial X^M}{\partial x^k},J(\gamma w)\rangle\rangle=r\sin(\theta-\phi)\langle e_k,w\rangle,\\ \langle A^\Sigma_{ss},V\rangle&=\mathrm{Re}\langle\langle\gamma''X^M,J(\gamma w)\rangle\rangle=\mathrm{Re}(\gamma''\overline{\gamma}\langle\langle X^M,Jw\rangle\rangle)=0\end{aligned} \tag{5.2.5}$$

for $k,l=1,..,n-1$. Here the second equalities of the second and third equations of (5.2.5) are followed by the fact that e_k, Jw, X^M, and JX^M are mutually orthogonal. Combining (5.2.1) and (5.2.5), it gives

$$\begin{aligned}|\langle A^\Sigma,V\rangle|^2=&\sum_{k,l=1}^{n-1}\langle A^\Sigma_{kl},V\rangle^2\frac{1}{r^4}+2\sum_{k=1}^{n-1}\langle A^\Sigma_{ks},V\rangle^2\frac{1}{r^2}+\langle A^\Sigma_{ss},V\rangle^2\\ =&|\langle A^{M,\mathbb{S}},Jw\rangle|^2+2\sin^2(\theta-\phi)|w|^2\quad\text{at } p.\end{aligned}$$

(ii) Since Σ is Lagrangian, $\{Ju_\alpha\}_{\alpha=1,...,n-1,s}$ is an orthogonal basis at p for the normal bundle. We will calculate the normal projection of $(\nabla^\perp_{u_\alpha}J(\gamma w))_{\alpha=1,...,n-1,s}$ on Ju_j

and Ju_s. Using the property that w, Je_k, X^M, and JX^M are mutually orthogonal, $\gamma\overline{\gamma} = r^2$ and $\gamma'\overline{\gamma} = re^{i(\theta-\phi)}$, we conclude that

$$\begin{aligned}
&\langle \nabla^{\perp}_{u_k} J(\gamma w), Ju_j\rangle = \mathrm{Re}\langle\langle i\gamma \tfrac{\partial}{\partial x^k} w, i\gamma e_j\rangle\rangle = r^2\langle \nabla^M_{e_k} w, e_j\rangle\\
&\langle \nabla^{\perp}_{u_k} J(\gamma w), Ju_s\rangle = -\mathrm{Re}\langle\langle i\gamma w, \tfrac{\partial}{\partial x^k} i\gamma' X^M\rangle\rangle = -r\cos(\theta-\phi)\langle w, e_k\rangle\\
&\langle \nabla^{\perp}_{u_s} J(\gamma w), Ju_j\rangle = \mathrm{Re}\langle\langle i\gamma' w, i\gamma e_j\rangle\rangle = r\cos(\theta-\phi)\langle w, e_j\rangle\\
&\langle \nabla^{\perp}_{u_s} J(\gamma w), Ju_s\rangle = \mathrm{Re}\langle\langle i\gamma' w, i\gamma' X^M\rangle\rangle = 0.
\end{aligned} \tag{5.2.6}$$

From (5.2.6), it follows that

$$\begin{aligned}
&|\nabla^{\perp} V|^2 = \langle \nabla^{\perp}_{u_\alpha} J(\gamma w), \nabla^{\perp}_{u_\beta} J(\gamma w)\rangle g^{\alpha\beta}\\
=&\sum_{k=1}^{n-1}\langle \nabla^{\perp}_{u_k} J(\gamma w), \nabla^{\perp}_{u_k} J(\gamma w)\rangle \frac{1}{r^2} + \langle \nabla_{u_s} J(\gamma w), \nabla_{u_s} J(\gamma w)\rangle\\
=&\left(\sum_{j,k=1}^{n-1}\langle \nabla^{\perp}_{u_k} J(\gamma w), \frac{Ju_j}{r}\rangle^2 + \sum_{k=1}^{n-1}\langle \nabla^{\perp}_{u_k} J(\gamma w), Ju_s\rangle^2\right)\frac{1}{r^2} + \sum_{j=1}^{n-1}\langle \nabla^{\perp}_{u_s} J(\gamma w), \frac{Ju_j}{r}\rangle^2\\
=&\sum_{j,k=1}^{n-1}\langle \nabla^M_{e_k} w, e_j\rangle^2 + \sum_{j=1}^{n-1} 2\cos^2(\theta-\phi)\langle w, e_j\rangle^2\\
=&|\nabla^M w|^2 + 2\cos^2(\theta-\phi)|w|^2.
\end{aligned}$$

(iii) Plugging (5.2.2) and (5.2.3) into (3.2.2), and using $e^{\frac{-|X|^2}{4}} d\mu_\Sigma = e^{-\frac{r^2}{4}} r^{n-1} ds d\mu_M$, we get

$$\begin{aligned}
&\langle V, -L^{\perp} V\rangle_e\\
=&\int_\Sigma \left(|\nabla^{\perp} V|^2 - |\langle A^\Sigma, V\rangle|^2 - \frac{1}{2}|V|^2\right) e^{-\frac{|X|^2}{4}} d\mu_\Sigma\\
=&\int_\gamma\int_M \Big(|\nabla^M w|^2 + 2\cos^2(\theta-\phi)|w|^2 - \left(|\langle A^{M,\mathbb{S}}, Jw\rangle|^2 + 2\sin^2(\theta-\phi)|w|^2\right)\\
&-\frac{1}{2}r^2|w|^2\Big) e^{-\frac{r^2}{4}} r^{n-1} d\mu_m ds\\
=&-\int_\gamma \left(\frac{1}{2}r^2 - 2 + 4\sin^2(\theta-\phi)\right) e^{\frac{-r^2}{4}} r^{n-1} ds \int_M |w|^2 d\mu_M\\
&+\int_\gamma e^{\frac{-r^2}{4}} r^{n-1} ds \int_M \left(|\nabla^M w|^2 - |\langle A^{M,\mathbb{S}}, Jw\rangle|^2\right) d\mu_M.
\end{aligned}$$

Thus (iii) is proved. □

To further simplify $\langle V, -L^{\perp} V\rangle_e$, we now derive some integral properties of the curve γ.

Lemma 5.2.3 *Let $\gamma : I \to \mathbb{C}^*$ be a closed smooth regular curve parameterized by the arclength s satisfying* (5.1.1). *That is, $\gamma * \psi$ in Lemma* 5.1.1 *define a closed self-shrinker. Then one has*

$$\int_\gamma (\frac{1}{2}r^2 - n)r^{n-1}e^{-\frac{r^2}{4}}ds = 0 \tag{5.2.7}$$

and

$$\int_\gamma (\frac{1}{2r^2} - \frac{n}{r^4})r^{n-1}e^{-\frac{r^2}{4}}ds = -\int_\gamma \frac{4\cos^2(\theta-\phi)}{r^4}r^{n-1}e^{-\frac{r^2}{4}}ds. \tag{5.2.8}$$

Remark 4 *The equality* (5.2.7) *is used to simplify* (5.2.4) *while the equality* (5.2.8) *is used to simplify* (5.3.4) *for Lagrangian variation.*

Proof. Equation (5.2.7) follows from the simplification of equation (2.2.5) and $\int_M d\mu_M \neq 0$. Indeed, equation (2.2.5) becomes

$$0 = \int_\gamma \int_M (r^2 - 2n)e^{-\frac{r^2}{4}}r^{n-1}d\mu_M ds = \int_\gamma (r^2 - 2n)e^{-\frac{r^2}{4}}r^{n-1}ds \int_M d\mu_M.$$

Recall that the linear operator $\mathcal{L}f = \Delta f - \frac{1}{2}\langle X, \nabla f\rangle = e^{\frac{|X|^2}{4}} div(e^{-\frac{|X|^2}{4}}\nabla f)$ in Proposition 2.2.2. It gives

$$\int_\Sigma \mathcal{L}(\frac{1}{|X|^2})e^{-\frac{|X|^2}{4}}d\mu_\Sigma = \int_\Sigma div(e^{-\frac{|X|^2}{4}}\nabla \frac{1}{|X|^2})d\mu_\Sigma = 0 \tag{5.2.9}$$

since $\partial\Sigma = \emptyset$. On the other hand, using equation (2.2.4) and $\nabla|X|^2 = 2X^\top$ gives

$$\mathcal{L}(\frac{1}{|X|^2}) = \frac{-\mathcal{L}|X|^2}{|X|^4} + \frac{2|\nabla|X|^2|^2}{|X|^6} = \frac{-2n + |X|^2}{|X|^4} + \frac{8|X^\top|^2}{|X|^6}. \tag{5.2.10}$$

Combining (5.2.9), (5.2.10), and using $|X^\top| = \mathrm{Re}\left(re^{i(\phi-\theta)}\right) = r\cos(\theta-\phi)$, one has

$$\begin{aligned} 0 &= \int_\gamma \int_M (\frac{-2n + r^2}{r^4} + \frac{8r^2\cos^2(\theta-\phi)}{r^6})e^{-\frac{r^2}{4}}r^{n-1}d\mu_M ds \\ &= \int_\gamma (\frac{-2n + r^2}{r^4} + \frac{8r^2\cos^2(\theta-\phi)}{r^6})e^{-\frac{r^2}{4}}r^{n-1}ds \int_M d\mu_M. \end{aligned}$$

Then it gets the equation (5.2.8) immediately since $\int_M d\mu_M \neq 0$. □

Next, we want to find a vector field w_0 in $\Gamma(T\psi(M))$ with nice special properties that will be needed in proving Theorem 5.2.1 and Theorem 5.3.1.

Lemma 5.2.4 *Let $\psi : M^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be a minimal Legendrian immersion. Then there exists a nonzero vector field w_0 in $\Gamma(T\psi(M))$ satisfying*

$$\frac{\int_M |\nabla^M w_0|^2 - |\langle A^{M,\mathbb{S}}, Jw_0\rangle|^2 d\mu}{\int_M |w_0|^2 d\mu} \leq 1 \quad \text{and} \quad \langle \nabla^M_x w_0, y\rangle = \langle \nabla^M_y w_0, x\rangle \tag{5.2.11}$$

for any $x, y \in T\psi(M)$.

Remark 5 *The condition $\langle \nabla^M_x w_0, y\rangle = \langle \nabla^M_y w_0, x\rangle$ implies that $\frac{1}{r^2}J(\gamma w_0)$ induces a Lagrangian variation.*

Proof. Define

$$f(y) = \int_M |\nabla^M y|^2 - |\langle A^{M,\mathbb{S}}, Jy\rangle|^2 d\mu$$

for $y \in \Gamma(T\psi(M))$. Let $E_1, ..., E_{2n}$ be the standard basis for $\mathbb{C}^n$ with $E_{\alpha+n} = JE_\alpha$ for $\alpha = 1, ..., n$. We claim that there exists a β_0 in $\{1, ..., 2n\}$ such that $w_0 = E^\top_{\beta_0}$ is a nonzero vector field satisfying $f(w_0) \leq \int_M |w_0|^2 d\mu$, where $E^\top_{\beta_0}$ is the projection of E_{β_0} into the tangent space of $\psi(M)$. For fixed $q \in \psi(M)$, choose a local normal coordinate system $x^1, ..., x^{n-1}$ at q. Denote $\partial_j = \frac{\partial}{\partial x^j}$. We have

$$\langle \nabla^M_{\partial_k}(E^\top_\beta), \partial_j\rangle = \langle \frac{\partial}{\partial x^k}(E_\beta - E^\perp_\beta), \partial_j\rangle = -\langle \frac{\partial}{\partial x^k} E^\perp_\beta, \partial_j\rangle = \langle E_\beta, A^M_{jk}\rangle, \tag{5.2.12}$$

where $E^\perp_\beta$ is the normal part of E_β. Since the map ψ is a Legendrian immersion into $\mathbb{S}^{2n-1}$, the span $\{\partial_1, ..., \partial_{n-1}, X^M\}$ is a Lagrangian plane in $\mathbb{C}^n$. It gives

$$A^M_{kj} = A^{M,\mathbb{S}}_{kj} + \langle A^M_{kj}, X^M\rangle X^M = A^{M,\mathbb{S}}_{kj} - \delta_{kj} X^M \quad \text{at } q \tag{5.2.13}$$

and the second fundamental form $A^{M,\mathbb{S}}_{jk}$ of the submanifold $\psi(M)$ in $\mathbb{S}^{2n-1}$ is orthogonal JX^M because that

$$\langle A^{M,\mathbb{S}}_{kj}, JX^M\rangle = \langle \frac{\partial}{\partial x^k}(\partial_j), JX^M\rangle = -\langle \partial_j, J\partial_k\rangle = 0.$$

Since ∂_l and X^M are orthogonal, we have $(JA^{M,\mathbb{S}})^\top = JA^{M,\mathbb{S}}$. Recall that ψ is a minimal immersion in $\mathbb{S}^{2n-1}$ and hence $H^{M,\mathbb{S}} = 0$. Combining the equations (5.2.12) and (5.2.13), the first term of $f(E_\beta^\top)$ can be simplified as

$$
\begin{aligned}
|\nabla^M(E_\beta^\top)|^2 &= \sum_{j,k=1}^{n-1} |\langle E_\beta, A_{kj}^{M,\mathbb{S}}\rangle - \langle E_\beta, \delta_{kj} X^M\rangle|^2 \\
&= |\langle E_\beta, A^{M,\mathbb{S}}\rangle|^2 - 2\langle E_\beta, H^{M,\mathbb{S}}\rangle\langle E_\beta, X^M\rangle + (n-1)\langle E_\beta, X^M\rangle^2 \\
&= |\langle E_\beta, A^{M,\mathbb{S}}\rangle|^2 + (n-1)\langle E_\beta, X^M\rangle^2 \quad \text{at} \quad q. \qquad (5.2.14)
\end{aligned}
$$

Using the equality $(JA^{M,\mathbb{S}})^\top = JA^{M,\mathbb{S}}$, the second term of $f(E_\beta^\top)$ can be simplified as

$$
\langle A^{M,\mathbb{S}}, J(E_\beta^\top)\rangle = -\langle JA^{M,\mathbb{S}}, E_\beta^\top\rangle = -\langle JA^{M,\mathbb{S}}, E_\beta\rangle = \langle A^{M,\mathbb{S}}, JE_\beta\rangle. \qquad (5.2.15)
$$

Combining (5.2.14) and (5.2.15), it gives

$$
f(E_\alpha^\top) = \int_M \left(|\langle E_\alpha, A^{M,\mathbb{S}}\rangle|^2 + (n-1)\langle E_\alpha, X^M\rangle^2 - |\langle E_{\alpha+n}, A^{M,\mathbb{S}}\rangle|^2\right) d\mu, \qquad (5.2.16)
$$

$$
f(E_{\alpha+n}^\top) = \int_M \left(|\langle E_{\alpha+n}, A^{M,\mathbb{S}}\rangle|^2 + (n-1)\langle E_{\alpha+n}, X^M\rangle^2 - |\langle E_\alpha, A^{M,\mathbb{S}}\rangle|^2\right) d\mu \qquad (5.2.17)
$$

for $\alpha = 1, ..., n$. Summing (5.2.16) and (5.2.17) over $\alpha = 1, ..., n$ gives

$$
\sum_{\alpha=1}^{n} \left(f(E_\alpha^\top) + f(E_{\alpha+n}^\top)\right) = (n-1)\sum_{\beta=1}^{2n} \int_M \langle E_\beta, X^M\rangle^2 d\mu = (n-1)\int_M d\mu \qquad (5.2.18)
$$

since $|X^M| = 1$.

On the other hand, we have $\sum_{\beta=1}^{2n} |E_\beta^\top|^2 = \sum_{\beta=1}^{2n}\sum_{j=1}^{n-1} \langle E_\beta, \partial_j\rangle^2 = \sum_{j=1}^{n-1} |\partial_j|^2 = n-1$ at q because $\partial_1, ..., \partial_{n-1}$ is an orthonormal basis for $T_q\psi(M)$. Plugging it into (5.2.18), we get

$$
\sum_{\beta=1}^{2n} \int_M \left(|\nabla^M(E_\beta^\top)|^2 - |\langle A^{M,\mathbb{S}}, J(E_\beta^\top)\rangle|^2\right) d\mu = \sum_{\beta=1}^{2n} \int_M |E_\beta^\top|^2 d\mu.
$$

Therefore, there exists a β_0 in $\{1, .., 2n\}$ such that $E_{\beta_0}^\top$ is a nonzero vector field and

$$
\int_M \left(|\nabla^M(E_{\beta_0}^\top)|^2 - |\langle A^{M,\mathbb{S}}, J(E_{\beta_0}^\top)\rangle|^2\right) d\mu \le \int_M |E_{\beta_0}^\top|^2 d\mu.
$$

Which is the inequality in (5.2.11). Using (5.2.12), $\langle E_{\beta_0}, A_{jk}^M\rangle$ is symmetric for j, k, it follows that the vector field $w_0 = E_{\beta_0}^\top$ satisfies both conditions in (5.2.11). □

Now we are ready to proved Theorem 5.2.1:

Proof of Theorem 5.2.1**.** By Theorem 3.2.1, it suffices to construct an admissible vector field V satisfying $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Assume $V = J(\gamma w)$, where $w \in \Gamma(T\psi(M))$ would be chosen later. Because H is parallel to Ju_s (see [1], p.40), the first integral $\int_\Sigma \langle V, H\rangle e^{-\frac{|X|^2}{4}} d\mu$ in the admissible condition is equal to zero. The second integral in the admissible condition is

$$\int_\Sigma V e^{-\frac{|X|^2}{4}} d\mu = i \int_\gamma \gamma e^{-\frac{r^2}{4}} r^{n-1} ds \int_M w d\mu_M.$$

Recall that the construction of γ in [1] is made by $m > 1$ pieces $\Gamma_1, ..., \Gamma_m$ which each corresponds one period of curvature function. Every piece Γ_i is the same as Γ_1 up to a rotation. Suppose the rotation index of γ is l. Then we have

$$\begin{aligned}\int_\gamma \gamma e^{-\frac{r^2}{4}} r^{n-1} ds &= \sum_{j=1}^{m} \int_{\Gamma_j} e^{-\frac{r^2}{4}} r^n e^{i\phi} ds \\ &= \int_{\Gamma_1} e^{-\frac{r^2}{4}} r^n e^{i\phi} (1 + e^{i\frac{2l\pi}{m}} + ... + e^{i\frac{(m-1)l}{m}\cdot 2\pi}) ds = 0\end{aligned}$$

since $1 + e^{i\frac{2\pi}{m}} + ... + e^{i\frac{(m-1)l}{m}\cdot 2\pi} = 0$ for $m > 1$. Therefore, the second integral in the admissible condition is equal to zero.

For the case $n \geq 3$, we choose $w = w_0$ satisfying (5.2.11) and $V_0 = J(\gamma w_0)$. Plugging the first inequality of (5.2.11) into (5.2.4), the weighted L^2 inner product $\langle V_0, -L^\perp V_0\rangle_e$ becomes

$$\begin{aligned}&\int_\Sigma \langle V_0, -L^\perp V_0\rangle e^{-\frac{|X|^2}{4}} d\mu \\ \leq& -\int_\gamma \left(\frac{1}{2} r^2 - 3 + 4\sin^2(\theta - \phi)\right) e^{\frac{-r^2}{4}} r^{n-1} ds \int_M |w_0|^2 d\mu_M \\ =& -\int_\gamma \Big(\big(n - 3 + 4\sin^2(\theta - \phi)\big)\Big) e^{-\frac{r^2}{4}} r^{n-1} ds \int_M |w_0|^2 d\mu_M \\ <&0.\end{aligned}$$

We use (5.2.7) to conclude the equality above.

For the case $n = 2$, the only minimal Legendrian curves in $\mathbb{S}^3$ are great circles. They are totally geodesic in $\mathbb{S}^3$. Therefore, the weighted L^2 inner product $\langle V, -L^\perp V\rangle_e$

can be simplified as

$$\begin{aligned}
&\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu \\
=& \int_\gamma e^{-\frac{r^2}{4}} r\Big(\int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1} w|^2 - \Big(\frac{1}{2}r^2 - 2 + 4\sin^2(\theta-\phi)\Big)|w|^2 d\mu_{\mathbb{S}^1}\Big) ds. \\
=& \int_\gamma e^{-\frac{r^2}{4}} r\Big(\int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1} w|^2 - 4\sin^2(\theta-\phi)|w|^2 d\mu_{\mathbb{S}^1}\Big) ds.
\end{aligned}$$

Here we use (5.2.7) again to get the last equality. Finally, by choosing w to be the tangent vector of the great circle, which is a parallel vector field, we can make the weighted L^2 inner product negative. □

5.3 The unstability for Lagrangian variations

Since Anciaux's examples are Lagrangian, it is natural to investigate whether these examples are still unstable under the more restricted lagrangian variations. That is, the variations come from the deformations of Lagrangian submanifolds. A simple calculation shows that a variational field V induces a Lagrangian variation if and only if the one form defined by $\alpha_V = \omega(V, \cdot)$ is closed. That is,

$$\langle \nabla^\perp_X V, JY\rangle = \langle \nabla^\perp_Y V, JX\rangle, \tag{5.3.1}$$

where $\nabla^\perp$ is the normal connection on $N\Sigma$ and $X,\ Y \in T\Sigma$. We have the following

Theorem 5.3.1 *Let Σ be an n-dimensional closed Lagrangian self-shrinker as in Lemma* 5.1.1. *Then Σ is F-unstable under Lagrangian variations for the following cases*

(i) $n = 2$ *or* $n \geq 7$,

(ii) $2 < n < 7$, *and* $E \in [\sqrt{\frac{7-n}{8}} E_{max}, E_{max}]$,

where E and E_{max} are described in (5.1.2).

For $V \in N_0$, V is not a Lagrangian variation since $\langle \nabla^{\perp}_{u_s} V, Ju_j \rangle \neq \langle \nabla^{\perp}_{u_j} V, Ju_s \rangle$. Hence, instead of using N_0, we need a new set N_1 as follows:

$$N_1 = \{V | V = \frac{1}{r^2} J(\gamma w), \text{ where } w \in \Gamma(T\psi(M)) \text{ satisfies}$$
$$\langle \nabla^M_x w, y \rangle = \langle \nabla^M_y w, x \rangle, \text{ for all } x, y \in T\psi(M)\}.$$

For $V \in N_1$, we claim that V satisfies the equation (5.3.1). That is, it induces a Lagrangian variation. Suppose $V = \frac{1}{r^2} J(\gamma w)$. Noting that $\gamma' = e^{i\theta}$ and $\langle V, Ju_s \rangle = 0$, we have

$$\begin{aligned}
\langle \nabla^{\perp}_{u_s} V, Ju_j \rangle &= -\frac{2r'}{r^3} \langle J(\gamma w), J(\gamma e_j) \rangle + \frac{1}{r^2} \langle J(\gamma' w), J(\gamma e_j) \rangle \\
&= -\frac{\cos(\theta - \phi)}{r} \langle w, e_j \rangle, \\
\langle \nabla^{\perp}_{u_j} V, Ju_s \rangle &= -\langle V, \nabla^{\perp}_{u_j} Ju_s \rangle = -\frac{1}{r^2} \langle J(\gamma w), J(\gamma' e_j) \rangle = -\frac{\cos(\theta - \phi)}{r} \langle w, e_j \rangle, \\
\langle \nabla^{\perp}_{u_k} V, Ju_j \rangle &= \frac{1}{r^2} \langle \frac{\partial}{\partial x_k} J(\gamma w), J(\gamma e_j) \rangle = \langle \nabla^M_{e_k} w, e_j \rangle \\
&= \langle \nabla^M_{e_j} w, e_k \rangle = \langle \nabla^{\perp}_{u_j} V, Ju_k \rangle.
\end{aligned}$$

This proves the claim.

For $V \in N_1$, the operator $\langle V, -L^{\perp} V \rangle_e$ can be simplified as in Lemma 5.3.2.

Lemma 5.3.2 *Assume that Σ is a closed Lagrangian self-shrinker as in Lemma 5.1.1 and $V \in N_1$ is represented by $\frac{1}{r^2} J(\gamma w)$. The second fundamental forms of Σ in $\mathbb{C}^n$ and of $\psi(M)$ in $\mathbb{S}^{2n-1}$ are denoted by A^{Σ} and $A^{M,\mathbb{S}}$, respectively. We have*

$$(i)\ |\langle A^{\Sigma}, V \rangle|^2 = \frac{1}{r^4} |\langle A^{M,\mathbb{S}}, Jw \rangle|^2 + \frac{1}{r^4} \sin^2(\theta - \phi) |w|^2, \tag{5.3.2}$$

$$(ii)\ |\nabla^{\perp} V|^2 = \frac{1}{r^4} |\nabla^M w|^2 + \frac{2\cos^2(\theta - \phi)}{r^4} |w|^2, \tag{5.3.3}$$

$$\begin{aligned}
(iii)\ \langle V, -L^{\perp} V \rangle_e = &-\int_{\gamma} \left(\frac{1}{2} r^2 - 2 + 4\sin^2(\theta - \phi) \right) e^{\frac{-r^2}{4}} r^{n-5} ds \int_M |w|^2 d\mu_M \\
&+ \int_{\gamma} e^{\frac{-r^2}{4}} r^{n-5} ds \int_M \left(|\nabla^M w|^2 - |\langle A^{M,\mathbb{S}}, Jw \rangle|^2 \right) d\mu_M.
\end{aligned} \tag{5.3.4}$$

Proof. (i) For $V \in N_1$, there exist $V_0 \in N_0$ such that V can be represented by $\frac{1}{r^2}V_0 = \frac{1}{r^2}J(\gamma w)$. Using the equation (5.2.2), we have

$$|\langle A^{\Sigma}, V\rangle|^2 = \frac{1}{r^4}|\langle A^{\Sigma}, V_0\rangle|^2 = \frac{1}{r^4}|\langle A^{M,\mathbb{S}}, Jw\rangle|^2 + \frac{1}{r^4}\sin^2(\theta-\phi)|w|^2.$$

(ii) Using the equations (5.3.1) and (5.2.6), we have

$$\begin{aligned}
\langle \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w), Ju_j\rangle &= \frac{1}{r^2}\langle \nabla^{\perp}_{u_k}J(\gamma w), Ju_j\rangle = \langle \nabla^{M}_{e_k}w, e_j\rangle \\
\langle \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w), Ju_s\rangle &= \frac{1}{r^2}\langle \nabla^{\perp}_{u_k}J(\gamma w), Ju_s\rangle = -\frac{1}{r}\cos(\theta-\phi)\langle w, e_j\rangle \qquad (5.3.5)\\
\langle \nabla^{\perp}_{u_s}\frac{1}{r^2}J(\gamma w), Ju_s\rangle &= \frac{-2r'}{r^3}\langle J(\gamma w), J(\gamma X^M)\rangle + \frac{1}{r^2}\langle \nabla^{\perp}_{u_s}J(\gamma w), Ju_s\rangle = 0.
\end{aligned}$$

Combining with (5.3.5) and (5.3.1), it gives

$$\begin{aligned}
&|\nabla^{\perp}V|^2 = \langle \nabla^{\perp}_{u_\alpha}\frac{1}{r^2}J(\gamma w), \nabla^{\perp}_{u_\beta}\frac{1}{r^2}J(\gamma w)\rangle g^{\alpha\beta}\\
=&\sum_{k=1}^{n-1}\langle \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w), \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w)\rangle\frac{1}{r^2} + \langle \nabla_{u_s}\frac{1}{r^2}J(\gamma w), \nabla_{u_s}\frac{1}{r^2}J(\gamma w)\rangle\\
=&\left(\sum_{j,k=1}^{n-1}\langle \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w), \frac{Ju_j}{r}\rangle^2 + \sum_{k=1}^{n-1}\langle \nabla^{\perp}_{u_k}\frac{1}{r^2}J(\gamma w), Ju_s\rangle^2\right)\frac{1}{r^2}\\
+&\sum_{j=1}^{n-1}\langle \nabla^{\perp}_{u_s}\frac{1}{r^2}J(\gamma w), \frac{Ju_j}{r}\rangle^2\\
=&\frac{1}{r^4}\sum_{j,k=1}^{n-1}\langle \nabla^{M}_{e_k}w, e_j\rangle^2 + \frac{1}{r^4}\sum_{j=1}^{n-1}2\cos^2(\theta-\phi)\langle w, e_j\rangle^2\\
=&\frac{1}{r^4}|\nabla^M w|^2 + \frac{2\cos^2(\theta-\phi)}{r^4}|w|^2.
\end{aligned}$$

(iii) Plugging (5.3.2) and (5.3.3) into (3.2.2), and using $e^{\frac{-|X|^2}{4}}d\mu_\Sigma = e^{-\frac{r^2}{4}}r^{n-1}dsd\mu_M$, we get

$$
\begin{aligned}
&\langle V, -L^\perp V\rangle_e \\
=&\int_\Sigma \left(|\nabla^\perp V|^2 - |\langle A^\Sigma, V\rangle|^2 - \frac{1}{2}|V|^2\right) e^{-\frac{|X|^2}{4}} d\mu_\Sigma \\
=&\int_\gamma \int_M \Big(|\nabla^M w|^2 + 2\cos^2(\theta-\phi)|w|^2 - \big(|\langle A^{M,\mathbb{S}}, Jw\rangle|^2 + 2\sin^2(\theta-\phi)|w|^2\big) \\
&- \frac{1}{2}r^2|w|^2\Big) e^{-\frac{r^2}{4}} r^{n-5} d\mu_M ds \\
=&-\int_\gamma \left(\frac{1}{2}r^2 - 2 + 4\sin^2(\theta-\phi)\right) e^{\frac{-r^2}{4}} r^{n-5} ds \int_M |w|^2 d\mu_M \\
&+\int_\gamma e^{\frac{-r^2}{4}} r^{n-5} ds \int_M \left(|\nabla^M w|^2 - |\langle A^{M,\mathbb{S}}, Jw\rangle|^2\right) d\mu_M.
\end{aligned}
$$

Thus (iii) is proved. □

Proof of Theorem 5.3.1. By Theorem 3.2.1, it suffices to construct an admissible Lagrangian variation V satisfying $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Assume $V = \frac{1}{r^2}J(\gamma w) \in N_1$, where $w \in \Gamma(T\psi(M))$ will be chosen later. Similar to the proof of Theorem 5.2.1, V is an admissible Lagrangian variation.

We now further specify V, so that $\int_\Sigma \langle V, -L^\perp V\rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. When $n \geq 3$, we choose $w = w_0$ satisfying (5.2.11). Then $V_1 = \frac{1}{r^2}J(\gamma w_0)$ is in N_1. From (5.2.11) and (5.3.4), the weighted L^2 inner product $\langle V_1, -L^\perp V_1\rangle_e$ becomes

$$
\begin{aligned}
&\int_\Sigma \langle V_1, -L^\perp V_1\rangle e^{-\frac{|X|^2}{4}} d\mu \\
\leq& -\int_\gamma \left(\frac{1}{2}r^2 - 3 + 4\sin^2(\theta-\phi)\right) e^{\frac{-r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M \\
=& -\int_\gamma \Big(\big(n - 3 + 4\sin^2(\theta-\phi) - 4\cos^2(\theta-\phi)\big)\Big) e^{-\frac{r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M, \\
=& -\int_\gamma \Big(\big(n - 7 + 8\sin^2(\theta-\phi)\big)\Big) e^{-\frac{r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M,
\end{aligned}
$$

where (5.2.8) is used to conclude the first equality. Thus it suffices to show that $f(s) = n - 7 + 8\sin^2(\theta - \phi)$ is nonnegative and positive at some point. For $n \geq 7$, the function f is clearly nonnegative and positive somewhere. Since $\sin(\theta - \phi) \geq \frac{E}{E_{max}}$ from (5.1.2), $f(s)$ is nonnegative and positive somewhere for $E \in [\sqrt{\frac{7-n}{8}}E_{max}, E_{max}]$.

In the case $n = 2$, the only minimal Legendrian curves in $\mathbb{S}^3$ are great circles which are totally geodesic. Choosing w_1 to be the tangent vector of the great circle, we have $|\nabla^{\mathbb{S}^1} w_1| = 0$ and $|w_1| = 1$. The vector field $V_1 = \frac{1}{r^2} J(\gamma w_1)$ gives a Lagrangian variation and the weighted L^2 inner product $\langle V_1, -L^\perp V_1\rangle_e$ in (5.3.4) can be simplified as

$$\begin{aligned}
&\int_\Sigma \langle V_1, -L^\perp V_1\rangle e^{-\frac{|X|^2}{4}} d\mu \\
=&-\int_\gamma \left(\frac{1}{2}r^2 - 2 + 4\sin^2(\theta-\phi)\right) e^{-\frac{r^2}{4}} r^{-3} ds \int_{\mathbb{S}^1} |w|^2 d\mu_{\mathbb{S}^1} \\
=&-2\pi \int_\gamma \left(\frac{1}{2}r^2 + 2\left(\sin^2(\theta-\phi) - \cos^2(\theta-\phi)\right)\right) e^{-\frac{r^2}{4}} r^{-3} ds
\end{aligned}$$

Using (5.2.8), it follows that

$$\int_\gamma \frac{1}{2} r^2 e^{-\frac{r^2}{4}} r^{-3} ds = \int_\gamma 2\left(\sin^2(\theta-\phi) - \cos^2(\theta-\phi)\right) e^{-\frac{r^2}{4}} r^{-3} ds.$$

Therefore, $\langle V_1, -L^\perp V_1\rangle_e = -2\pi \int_\gamma r^2 e^{-\frac{r^2}{4}} r^{-3} ds < 0$, and concludes the Lagrangian unstability in Theorem 5.3.1. $\square$

Chapter 6

Self-similar Lagrangian graph

By a result of Harvey and Lawson [10], locally an n-dimensional Lagrangian submanifold L can be described explicitly as the graph of a function over a tangent plane. That is

$$L = \{(x_1, x_2, .., x_n) + i\nabla f) \in \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n | \ f : \Omega \subset \mathbb{R}^n \to \mathbb{R}\}.$$

A Lagrangian L is called a *Lagrangian graph* with respect to f if the domain of the function f is $\mathbb{R}^n$.

6.1 Expanding Lagrangian graph

In this section, we discuss the expanding symmetric Lagrangian graph with respect to f, that is, $H = F^{\perp}$ and the function f satisfies $f(x_1, ..., x_n) = f(r)$, where $r^2 = \sum\limits_{i=1}^{n} x_i^2$. It can be simplified as

$$L = \{(r + gi)\sigma_{n-1} \in \mathbb{C}^n | g(r) = f'(r), \sigma_{n-1} \in \mathbb{S}^{n-1}\}. \tag{6.1.1}$$

The type is a special case of Anciaux's examples in [1]. In terms of the spherical coordinate system $\theta_1, ..., \theta_{n-1}, r$, let $F_r = \frac{\partial F}{\partial r} = (1 + g'i)\sigma_{n-1}$, $F_j = \frac{\partial F}{\partial \theta_j} = (r + gi)u_j$ for $j = 1, ..., n-1$, where F is the position vector of L and u_j's are tangent vectors on $\mathbb{S}^{n-1}$. Then

$$\det \begin{pmatrix} F_r \\ F_1 \\ \vdots \\ F_{n-1} \end{pmatrix}_{\mathbb{C}} = (1+g'i)(r+gi)^{n-1}$$

and it gives

$$\theta = \arg(1+g'i) + (n-1)\arg(r+gi), \tag{6.1.2}$$

where θ is the Lagrangian angle of L. We have the following theorem.

Theorem 6.1.1 *Assume a smooth expanding symmetric Lagrangian graph L doesn't contain the origin, then L is asymptotic to a cone C, where $C = \{\omega(s)\sigma_{n-1} | \sigma_{n-1} \in \mathbb{S}^{n-1}, \omega(s) \in \mathbb{C}$ and $\omega(s)$ is a line in $\mathbb{R}^2\}$.*

Proof. In terms of the spherical coordinate system, we have

$$\theta_r = \frac{g''}{1+(g')^2} + (n-1)\frac{rg'-g}{r^2+g^2}. \tag{6.1.3}$$

On the other hand, since $H = F^\perp$ and $H = J\nabla\theta$, it gives

$$\theta_r = \langle \nabla\theta, F_r \rangle = \langle F, JF_r \rangle = -rg' + g. \tag{6.1.4}$$

Combining (6.1.3) and (6.1.4), we get

$$\frac{g''}{1+(g')^2} = (g - rg')\left(1 + \frac{n-1}{r^2+g^2}\right). \tag{6.1.5}$$

Let $h = g - rg'$. Directly integrating from (6.1.5), we have $h(r) = h(0)e^{-\int_0^r s\varphi(s)ds}$ where $\varphi(s) = (1+|g'(s)|^2)(1+\frac{n-1}{s^2+g^2(s)})$. Since L does not contain the origin, $h(0)$ is positive. The function h is decreasing and positive on $[0.a)$ for some $a > 0$. The domain $[0,a)$ can be extended to $[0,\infty)$. If not there is a number $b \in \mathbb{R}^+$ such that $h(b) = 0$. Combining the definition of φ, $|F| \neq 0$, and the fact that g is defined on $[0,\infty)$, the limit of $\varphi(s)$ is infinite as s approaches to b^-. This is a contradiction for $h = g - rg'$. Using both the above properties of the function h, the functions $\frac{g}{r}$ and

g' are decreasing and increasing, respectively. Because $\frac{g}{r} - g' = \frac{h}{r} > 0$ and use the completeness of the real numbers, the limits of $\frac{g}{r}$ and g' exist as $r \to \infty$ and both the limits are equal. Take $m = \lim\limits_{r \longrightarrow \infty} \frac{g}{r} = \lim\limits_{r \longrightarrow \infty} g'$, we have $g' \nearrow m$ and $\frac{g}{r} \searrow m$. Since the function $g - mr$ is positive and decreasing, the limit of $g - mr$ exists as $r \to \infty$. That is, g has an asymptote $y = mx + c$. Then L is asymptotic to C. □

6.2 Translating Lagrangian graph

Now we discuss the translating Lagrangian graph L with respect to f. We have the following proposition.

Proposition 6.2.1 *Assume that f is symmetric on $\mathbb{R}^n$, i.e., $f(x_1, ..., x_n) = f(r)$. Then the translating Lagrangian graph L with respect to f is the plane $\mathbb{R}^n$.*

Proof. Choose $T = (0, ..., 1)$ without loss of generality. Since the Lagrangian angle θ is only depend on r from the equation (6.1.2), it gives

$$0 = \langle \nabla \theta, F_j \rangle = \langle H, JF_j \rangle = \langle T, JF_j \rangle = -g\langle T, Ju_j \rangle$$

for all $0 < j < n$, where u_j's are tangent vectors on $\mathbb{S}^{n-1}$ and $g(r) = f'(r)$. Therefore f is a constant function on $\mathbb{R}^n$ and L is the Euclidean plane $\mathbb{R}^n$. □

Next, we consider a more complex case for $f(x_1, ..., x_{n-1}, y) = f(r, y)$ and $r^2 = \sum\limits_{k=1}^{n-1} x_k^2$. The translating Lagrangian graph L can be written as

$$\begin{aligned} L &= \{(x_1, x_2, .., x_{n-1}, y) + i\nabla f) \in \mathbb{C}^n | \ f = f(r, y) \in \mathbb{R}, r^2 = \sum_{k=1}^{n-1} x_k^2\} \\ &= \{((r + f_r i)\sigma_{n-2}, y + f_y i) \in \mathbb{C}^n | \sigma_{n-2} \in \mathbb{S}^{n-2}\} \end{aligned} \tag{6.2.1}$$

In terms of the spherical coordinate system $\theta_1, ..., \theta_{n-2}, r, y$, let

$$\begin{cases} F_y &= \frac{\partial F}{\partial y} = (if_{ry}\sigma_{n-2}, 1 + if_{yy}) \\ F_r &= \frac{\partial F}{\partial r} = ((1 + f_{rr} i)\sigma_{n-2}, if_{yr}) \\ F_j &= \frac{\partial F}{\partial \theta_j} = ((r + f_r i)u_j, 0) \end{cases} \tag{6.2.2}$$

for $j = 1, ..., n-2$, where F is the position vector of L and u_j's are tangent vectors on $\mathbb{S}^{n-2}$. Then

$$\det \begin{pmatrix} F_1 \\ \vdots \\ F_{n-2} \\ F_r \\ F_y \end{pmatrix}_{\mathbb{C}} = (r + if_r)^{n-2}[(1 + f_{rr}i)(1 + f_{yy}i) + f_{ry}^2]$$

$$= (r + if_r)^{n-2}[(1 + f_{ry}^2 - f_{rr}f_{yy}) + i(f_{rr} + f_{yy})]$$

and it gives

$$\theta = (n-2)\arg(r + if_r) + \arg[(1 + f_{ry}^2 - f_{rr}f_{yy}) + i(f_{rr} + f_{yy})], \tag{6.2.3}$$

where θ is the Lagrangian angle of L. We have the following the theorem.

Theorem 6.2.2 *Fix $n \geq 3$, assume the Lagrangian graph L defined as above is translating and the the translating vector $T = (0, ..., 0, i)$. Then*

$$L = \{((r + f_r i)\sigma_{n-2}, (\theta + C) + f_\theta i) \in \mathbb{C}^n | \sigma_{n-2} \in \mathbb{S}^{n-2}\}$$

for some constant C. Moreover, if $f(r, y) = G(r) + P(y)$, then L is the product of $(n-1)$ special Lagrangian L' and Grim Reaper Γ. i.e., $L = L' \times \Gamma \subset \mathbb{C}^{n-1} \times \mathbb{C}$.

Remark 6 *The latter still holds under the translating vector $T = (0, ..., 0, z)$ for all $z \in \mathbb{C}$. This example is the same as the example constructed in [12].*

Proof. Using (6.2.2) and directly computing, it gives

$$\theta_r = \langle \nabla\theta, F_r \rangle = \langle H, JF_r \rangle = \langle T, JF_r \rangle = 0,$$

$$\theta_j = \langle \nabla\theta, F_j \rangle = \langle H, JF_j \rangle = \langle T, JF_j \rangle = 0,$$

$$\theta_y = \langle \nabla\theta, F_y \rangle = \langle H, JF_y \rangle = \langle T, JF_y \rangle = 1.$$

Therefore, the Lagrangian angle θ is equal to $\theta + C$ for some constant C. When $f(r, y) = G(r) + P(y)$, we let $g(r) = f_r$ and $p(y) = f_y$. From (6.2.3), the Lagrangian angle θ can be simplified as the following equation

$$\begin{aligned}\theta &= \arg(1 + p'(y)i) + \arg(1 + g'(r)i) + (n-2)\arg(r + g(r)i) \\ &= \alpha(y) + \beta(r),\end{aligned}$$

where $\alpha(y) = \arg(1+p'(y)i)$ and $\beta(r) = \arg(1+g'(r)i)+(n-2)\arg(r+g(r)i)$. Since $\theta_y = 1$, it gives that $\beta(r)$ is constant and $\alpha(y) = y + c$. Directly computing, the function $p(y) = \ln\sec(y+c) + c_1$. That is, the Lagrangian graph L is the product of $(n-1)$ special Lagrangian graph L' and Grim Reaper Γ which α and β are Lagrangian angles of L' and Γ, respectively. $\square$

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