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# 各式弱性微分性質與函數的量度性質

# Metrical properties of functions in terms of various forms of weak differentiability

林俊良

Chun-Liang Lin

指導教授：劉豐哲 博士

Advisor: Fon-Che Liu, Ph.D.

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# 目　錄

# 摘　要

依循 W. Stepanoff、H Whitney 及 H. Federer 的工作，我們研究函數與各種弱性微分有關的量度性質。綜合他們的工作，可知以下四個敘述的等價性：

(1) u 在 D 上幾乎處處幾近可微(approximately differentiable)；

(2) 給定 $\varepsilon>0$，存在一個定義在 $\mathbb{R}^n$ 上的連續可微函數 v，使得 u 與 v 相異點所成的集合的測度小於 $\varepsilon$；

(3) u 的一次差分的幾近上極限(approximate limsup)在 D 上幾乎處處有限；

(4) u 的一階幾近偏導數在 D 上幾乎處處存在。

接著，W. S. Tai 與 F. C. Liu 把這些結果推廣到更高階(非負整數)的弱性微分性質。我們更進一步地將其推廣到一般階(不限定為非負整數)，證明了以下定理：

**主要定理.** 對 $\gamma>0$，以下敘述是等價的：

(1) u 在 D 上擁有 $\gamma$ 階 Lusin 性質；

(2) u 在 D 上幾乎處處 $\gamma$ 階 Lipschitz 連續；

(3) u 在 D 上幾乎處處 $\gamma$ 階偏 Lipschitz 連續。

對於證明主要定理的重要工具—Whitney 擴張定理，我們也做了仔細的研究，附加上範數的估計，將定理重新敘述成更容易應用的型式。

# Abstract

Metrical properties of measurable functions in terms of various forms of weak differentiability are studied along a line suggested by works of W. Stepanoff, H. Whitney, and H. Federer which can be summarily described as stating that the following four statements are equivalent:

(1) $u$ is approximately differentiable a.e. on $D$.

(2) Given $\varepsilon > 0$, there is a $C^1$ function $v$ on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$.

(3) $ap \limsup_{y \to x} \frac{|u(y)-u(x)|}{|y-x|} < \infty$ for almost all $x \in D$.

(4) First order approximate partial derivatives of $u$ exist a.e. on $D$.

W. S. Tai and F. C. Liu then generalize the results to the situation involving higher (integral) order of weak differentiability. For a further generalization to fractional order, we prove the following theorem:
**Main Theorem.** For $\gamma > 0$, the following statements are equivalent:

(1) $u$ has Lusin property of order $\gamma$ on $D$.

(2) $u$ is approximately Lipschitz continuous of order $\gamma$ at almost every point of $D$.

(3) $u$ is partially approximately Lipschitz continuous of order $\gamma$ at almost all point of $D$.

Whitney's Extension Theorem, which is a main tool for the proof of the Main Theorem, is also given a detailed consideration and reformulated in a form with appropriate norm estimates. This form seems to be of a final touch and can be applied more effectively.

# 1 Introduction

Metrical properties of measurable functions defined on a measurable subset of $\mathbb{R}^n$ will be studied through approximate partial derivatives. Different from classical partial derivatives, approximate partial derivatives can be introduced for functions defined on a measurable set and stay the same when functions are redefined on sets of measure zero. In other words, the process of taking approximate partial derivatives is a more stable process than that of taking classical partial derivatives. This fact is important because the functions that arise naturally are usually those that are limit functions under certain limit processes, and therefore they might not be defined everywhere. Even though one starts with a class of functions which have certain regularity properties, one may end up after certain limit process with a much larger class of functions most of which do not enjoy the original regularity properties and has to satisfy oneself with weaker regularity properties for the enlarged class of functions. Our study of metrical properties of functions follows this line of thought when differentiability of functions are in view.

The most well-known example of this approach is the Lusin Theorem:

**Theorem 1.** *Suppose that $u$ is a measurable function defined on a measurable set $D \subset \mathbb{R}^n$, then for any $\varepsilon > 0$ there is a continuous function $v$ defined on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$.*

Here, we use $|A|$ to denote the Lebesgue measure of a subset $A$ of $\mathbb{R}^n$. It follows from Lusin theorem that every measurable function on $D$ is the limit a.e of a sequence of continuous functions. It is therefore expedient to weaken the concept of continuity in order to describe measurable functions. This leads to approximate continuity. A function $u$ defined on $D$ is called approximately continuous at $x \in D$ if there is a measurable subset $S$ of $D$ with density one at $x$ such that $u|_S$, the restriction of $u$ to $S$, is continuous at $x$. Using approximate continuity, Lusin theorem can be restated as follows.

**Theorem 2.** *A function $u$ defined on a measurable set $D \subset \mathbb{R}^n$ is measurable if and only if $u$ is approximately continuous almost everywhere on $D$.*

This means that starting from the class of continuous functions, we obtain the class of measurable functions by weakening the concept of continuity appropriately. Now as Whitney [16] has shown, if we start with the class of continuously differentiable ($C^1$) functions we shall arrive at the class of approximately differentiable functions which is related to the class of continuouosly differentiable functions in the same way as the class of measurable functions is related to the class of continuous functions through Lusin theorem. Actually, Whitney proved the following theorem:

**Theorem 3** ([16])**.** *Let $u$ be a measurable function on $D$. Then $u$ is approximately differentiable a.e. in $D$ if and only if for any $\varepsilon > 0$, there is a $C^1$ function $v$ defined on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$.*

A function $u$ on $D$ is called approximately differentiable at $x \in D$ if there is a measurable subset $S$ on $D$ with density one at $x$ such that $u|_S$ is totally differentiable at $x$, i.e. there is $d = (d_1, \cdots, d_n) \in \mathbb{R}^n$ such that

$$\lim_{y \in D, y \to x} \frac{u(y) - \{u(x) + d \cdot (y - x)\}}{|y - x|} = 0.$$

Note that $d$ is uniquely determined and $d_j$ is the approximate partial derivative of $u|_S$ at $x$ in $e_j$ direction if linear density of $S$ at $x$ in this direction is 1, where $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

A real number $l$ is called an approximate limit of a measurable function $f$ at $x \in D$ if the set $\{y \in D : |f(y) - l| < \varepsilon\}$ has density 1 at $x$ for every $\varepsilon > 0$. Since $l$ is unique when it exists, it is called the approximate limit of $f$ at $x$ and is denoted by $ap \lim_{y \to x} f(y)$. Observe that the approximate limit of $f$ at $x$ exists if and only if it exists when $f$ is restricted to a certain measurable subset of $D$. Because of this fact the symbol $ap \lim_{y \to x} f(y)$ applies whenever $f$ is defined on a measurable set with density 1 at $x$. Then a measurable function $u$ on $D$ is approximate differentiable at $x$ if there is $d \in \mathbb{R}^n$ such that $ap \lim_{y \to x} \frac{u(y) - \{u(x) + d \cdot (y - x)\}}{|y - x|} = 0$, and the approximate partial derivative $ap \frac{\partial u}{\partial x_j}(x)$ in direction $e_j$ of $u$ at $x$ is defined as $ap \lim_{h \to 0} \frac{u(x + he_j) - u(x)}{h}$ if it exists, and if confusion is not likely we denote it by the classical notation $\frac{\partial u}{\partial x_j}$.

It was proved by Stepanoff [14] that a measurable function $u$ on $D$ is approximately differentiable a.e. if and only if at almost every point in $D$, it has approximate partial derivatives in each coordinate direction. This is distinctly different from the classical case when differentiability and partial derivatives replace approximate differentiability and approximate partial derivatives respectively, and suggests that it might be convenient to consider approximate differentiability and approximate partial derivatives of higher order and to study how they are related to the differentiability of higher order in the classical sense. For this purpose we introduce first some terminologies to be used later. In the following definitions $u$ is a measurable function defined on a measurable set $D \subset \mathbb{R}^n$ and $k$ is a nonnegative integer.

**Definition 1.** *$u$ is said to have Lusin property of order $k$ if for any $\varepsilon > 0$, there is a $C^k$-function $v$ defined on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$.*

It is easy to see that if $u$ has Lusin property of order $k$, then for a.e. $x \in D$ there is a polynomial $T_x$ of order less than or equal to $k$ such that

$$ap \lim_{y \to x} \frac{|u(y) - T_x(y)|}{|y - x|^k} = 0. \tag{1.1}$$

If (1.1) holds, $u$ is called approximately Taylor-differentiable of order $k$ at $x \in D$ and the unique polynomial $T$ is called the approximate Taylor polynomial of $u$ at $x$. Note that approximate continuity and differentiability can be viewed as approximately Taylor-differentiability of order 0 and 1 respectively.

We have already defined first order approximate partial derivatives of $u$. Naturally, approximate partial derivatives of $u$ of higher order can be defined inductively, and they are measurable on wherever they are defined[12]. If confusion is not likely, we shall use the classical notation to denote approximate partial derivatives. $u$ is said to have unbiased approximate partial derivatives up to order $k$ at $x \in D$ if the approximate partial derivatives of $u$ of order less than or equal to $k$ are defined at $x$ and if all the mixed approximate partial derivatives do not depend on the order of taking the approximate partial derivatives.

**Definition 2.** *$u$ is called partially approximately Taylor-differentiable of order $k$ at $x \in D$ if $u$ has unbiased approximate partial derivatives up to order $k$ and if*

$$ap \lim_{y_i \to x_i} \frac{|\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i) - T^{(i)}_{\alpha,x}(y_i)|}{|y_i - x_i|^{k-|\alpha|}} = 0, \tag{1.2}$$

*for each $1 \leq i \leq n$ and $|\alpha| < k$, where*

$$T^{(i)}_{\alpha,x}(y_i) := \sum_{l=0}^{k-|\alpha|} \frac{1}{l!} \frac{\partial^{\alpha + le_i} u}{\partial x^{\alpha + le_i}}(x)(y_i - x_i)^l.$$

Note that (1.2) always holds true for $|\alpha| = k-1$ by the existence of approximate partial derivatives of $u$. What we really concern with are those $\alpha$'s with $|\alpha| \leq k-2$. Note also that if $u \in W^{k,1}_{loc}(\Omega)$, then $u$ is partially approximately Taylor-differentiable of order $k$ at a.e. $x \in \Omega$. Recall that for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, where each $\alpha_i$ is a nonnegative integer, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ (cf. p.2 in [17] for notations related to multi-index $\alpha$).

**Definition 3.** *For $k \in \mathbb{N}$, $u$ is called approximately Lipschitz continuous of order $k$ at $x \in D$ if there is a polynomial $P_x$ of order less than or equal to $k-1$ such that*

$$ap \limsup_{y \to x} \frac{|u(y) - P_x(y)|}{|y-x|^k} < \infty.$$

Here, the approximate limsup of a function $g$ at $x$ is the infimum of the real numbers $t$ such that the set $\{g > t\}$ has density 0 at $x$.

The following theorem in [6] and [8] generalized the results of Stepanoff, Whitney (Theorem 3) and H. Federer. ( When $k = 1$, the equivalence of (2) and (3) in Theorem 4 is the theorem 3.1.16 in [4] and the equivalence of (1) and (4) is the Stepanoff Theorem mentioned above. )

**Theorem 4** ([6],[8]). *For a function $u$ defined on $D$, the following four statements are equivalent:*

*(1) $u$ is approximately Taylor-differentiable of order $k$ at almost every point of $D$;*

*(2) $u$ has the Lusin type property of order $k$ on $D$;*

*(3) $u$ is approximately Lipschitz continuous of order $k$ at almost every point of $D$;*

*(4) $u$ is partially approximately Taylor-differentiable of order $k$ at almost every point of $D$.*

The equivalence of (1) and (2) in Theorem 4 is first stated and proved in [6] under the further assumption of the measurability of the coefficients of the approximate Taylor polynomials, while the equivalence of (1) and (4) is stated without proof in [6]. And the the equivalence of (1), (2), and (3) is proved in [5] with the byproduct that the coefficients of the approximate Taylor polynomial are measurable functions of $x$. Also note that in Theorem 4 the equivalence of (1) and (3) is a form of Rademacher phenomenon.

To describe and to prepare for our further extension of these works, we introduce first some definitions. Fix a finite number $\gamma > 0$, let $\mathring{\gamma}$ be the largest integer $< \gamma$ and write $\gamma = \mathring{\gamma} + \mu$, then $0 < \mu \leq 1$. A function $u$ on $D$ is called approximately Lipschitz continuous of order $\gamma$ at $x \in D$ if there is a polynomial $P_x(y)$ of order at most $\mathring{\gamma}$ and centered at $x$ such that

$$ap \limsup_{y \to x} \frac{|u(y) - P_x(y)|}{|y - x|^{\gamma}} < \infty. \tag{1.3}$$

Note that $P_x(y)$ is uniquely determined at each point $x \in D$ where it exists. A function $u$ on $D$ belongs to the class $\mathrm{Lip}_{ap}(\gamma, D)$ if $u$ is approximately Lipschitz continuous of order $\gamma$ at almost all points of $D$. Observe that for $\gamma > 0$ any function $u \in \mathrm{Lip}_{ap}(\gamma, D)$ is approximately Taylor-differentiable of order $\mathring{\gamma}$ a.e. on $D$ with $T_x = P_x$. Hence the coefficients of $P_x(y)$ are measurable functions of $x$.

For an open set $\Omega \subset \mathbb{R}^n$, we denote by $C^{\gamma}_{loc}(\Omega)$ the space of all those function $v \in C^{\mathring{\gamma}}(\Omega)$ such that $\frac{\partial^\alpha v}{\partial x^\alpha} \in C^{0,\mu}(K)$ for each compact set $K \subset \Omega$ when $|\alpha| = \mathring{\gamma}$. A function $u$ on $D$ is said to have Lusin property of order $\gamma$ if for every $\varepsilon > 0$ there is a function $v \in C^{\gamma}_{loc}(\mathbb{R}^n)$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$. The space of all functions on $D$ which have Lusin property of order $\gamma$ is denoted by $LC^{(\gamma)}(D)$. Note that, when $\gamma \in \mathbb{N}$, the definition here is equivalent to Definition 1 By Theorem 4 in [16] and Rademacher Theorem for differentiability a.e. of locally Lipschitz functions.

And similar to the definition above, for $\gamma > 0$, $u$ is said to be partially approximately Lipschitz continuous of order $\gamma$ if $u$ has unbiased approximate partial derivatives up to order $\mathring{\gamma}$, and each approximate partial derivative $\frac{\partial^\alpha u}{\partial x^\alpha}$, $|\alpha| \leq \mathring{\gamma}$, satisfies

$$ap \limsup_{y_i \to x_i} \frac{|\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i) - P^{(i)}_{\alpha,x}(y_i)|}{|y_i - x_i|^{\gamma - |\alpha|}} < \infty, \tag{1.4}$$

for each $1 \leq i \leq n$, where

$$P^{(i)}_{\alpha,x}(y_i) := \sum_{l=0}^{\mathring{\gamma} - |\alpha|} \frac{1}{l!} \frac{\partial^{\alpha + le_i} u}{\partial x^{\alpha + le_i}}(x)(y_i - x_i)^l.$$

Our purpose in this thesis is to generalize Theorem 4 to the following theorem:

**Theorem 5.** *For $\gamma > 0$, the following statements are equivalent:*

*(1)* $u \in LC^{(\gamma)}(D)$.

*(2)* $u \in Lip_{ap}(\gamma, D)$.

*(3) $u$ is partially approximately Lipschitz continuous of order $\gamma$ at almost all point of $D$.*

Theorem 5 (or Theorem 4) allows us to establish a generalization of Theorem 4 in [16], and the method of the proof implies an interesting consequence which contains in particular a substantial generalization of a theorem of Currier [3].

Preliminaries on measurability of sets related to density are given in §2. A discussion in detail of Whitney's Extension Theorem, which is the main tool to obtain the $C^k$-function while proving Theorem 5, is given in §3. We will prove Theorem 5 together with a remark that it implies a consequence alluded to above. §5 consists of some remarks and applications of Theorem 5.

## 2 Measurability of Sets

The following theorem guarantees the measurability of sets and functions appearing in this note while considering approximate limits. Let $E$ be a set in $\mathbb{R}^{n+m}$, define $E_x := \{y \in \mathbb{R}^m : (x, y) \in E\}$ for $x \in \mathbb{R}^n$. The open ball centered at $c$ with radius $r$ in a Euclidean space is denoted by $B_r(c)$ as usual.

**Lemma 1.** *Suppose that $E$ is a measurable set in $\mathbb{R}^{n+m}$, $f$ a measurable map from $D$ to $\mathbb{R}^m$. If $S$ is a set of positive numbers and $g$ a lower semi-continuous function on $S$, then the following sets $D_1$ and $D_2$ are measurable:*

$$\begin{aligned} D_1 &:= \{x \in D : \ |E_x \cap B_r(f(x))| \geq g(r) \ \forall r \in S\}, \\ D_2 &:= \{x \in D : \ |E_x \cap B| \geq g(r) \text{ for each ball } B \text{ containing } f(x) \\ &\qquad \text{with radius } r \in S\}. \end{aligned}$$

*Proof.* Fix $r > 0$. Since the set $\widetilde{E} := \{(x, y) \in E : x \in D, |y - f(x)| < r\}$ is measurable in $\mathbb{R}^{n+m}$, by Fubini Theorem, the function $x \mapsto |\widetilde{E}_x| = |E_x \cap B_r(f(x))|$ is measurable on $D$. Therefore, $\{x \in D : |E_x \cap B_r(f(x))| \geq g(r)\}$ is measurable in $\mathbb{R}^n$. To show that $D_1$ and $D_2$ are measurable, we first choose a countable dense subset $\widetilde{S}$ of $S$ and let $Q = B_1(0) \cap \mathbb{Q}^m$, where $\mathbb{Q}$

is the set of rational numbers. Then the lemma follows from the following expressions of $D_1$ and $D_2$:

$$D_1 = \cap_{r\in\widetilde{S}}\{x \in D : |E_x \cap B_r(f(x))| \geq g(r)\},$$
$$D_2 = \cap_{r\in\widetilde{S},q\in Q}\{x \in D : |E_x \cap B_r(f_{r,q}(x))| \geq g(r)\},$$

where $f_{r,q} := f + rq$ is also measurable on $D$ for each $r \in \widetilde{S}$ and $q \in Q$. It is obvious that the left-hand side is contained in the right-hand side of each of the expressions. To show the opposite direction for $D_1$, suppose that $x$ is a point in $D$ such that $|E_x \cap B_r(f(x))| \geq g(r)$ for all $r \in \widetilde{S}$. For any $r \in S$ we can choose a sequence $\{r_j\}_{j\in\mathbb{N}} \subset \widetilde{S}$ converging to $r$, then

$$|E_x \cap B_r(f(x))| = \lim_{j\to\infty} |E_x \cap B_{r_j}(f(x))| \geq \liminf_{j\to\infty} g(r_j) \geq g(r)$$

by the continuity of $|E_x \cap B_r(f(x))|$ in $r$. A similar argument for $D_2$ holds by continuity of $|E_x \cap B_r(f_{r,q}(x))|$ in $(r, q)$.

□

**Remarks.**

1. In the definition of $D_1$ and $D_2$ in Lemma 1, if $B_r(c)$ is replaced by cubes (or bounded sets of any given shape whose boundary is of measure zero) with dilation $r$ and translation $c$, the Lemma still holds true.

2. Lemma 1 is still valid if "$\geq$" replaced by "$\leq$" and if the upper semi-continuity of $g$ is assumed instead of lower semi-continuity.

Since it makes no difference to define a point of density by using balls or cubes either containing or centered at the point, when points of density are concerned, we have the freedom to choose one that simplifies arguments.

The following corollary illustrates how Lemma 1 applied in proving measurability of functions or sets.

**Corollary 1.** *Suppose that* $\{u_\alpha\}_{|\alpha|\leq k}$ *is a family of measurable functions on* $D$ *and* $P_x(y) := \sum_{|\alpha|\leq k} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha$*. Then* $f(x) := ap\limsup_{y\to x} \frac{|u(y)-P_x(y)|}{|y-x|^\gamma}$ *is a measurable function of* $x$*.*

*Proof.* For each $M > 0$, $f(x) < M$ if and only if the set $D_{x,p} := \{y \in D : |u(y) - P_x(y)| \leq (M - 1/p)|y - x|^\gamma\}$ has density 1 at $x$ for some $p \in \mathbb{N}$. Thus the set $\{x \in D : f(x) < M\}$ can be expressed as

$$\cup_{p\in\mathbb{N}} \cap_{q\in\mathbb{N}} \cup_{s\in\mathbb{N}} \{x \in D : \frac{|D_{x,p} \cap B_r(x)|}{|B_r(x)|} \geq (1 - \frac{1}{q}) \ \forall\ 0 < r < 1/s\}$$

which is measurable by applying Lemma 1 by taking $E = \{(x, y) \in D \times D : |u(y) - P_x(y)| \leq (M - 1/p)|y - x|^\gamma\}$, $f$ the identify map on $D$, $S = (0, 1/s)$, and $g(r) = (1 - 1/q)|B_r(0)|$ for each $p, q, s \in \mathbb{N}$. □

# 3 Whitney's Extension Theorem with Norm Estimates

In 1934, Whitney gave the necessary and sufficient condition of the existence of a $C^k$-extension of a function defined on a closed subset $F$ in $\mathbb{R}^n$. The extended function constructed by Whitney grows without bound from the closed set and hence there is no estimates of the extended function in terms of $C^k$-norm. By applying suitable cut-off methods, it is possible to give an estimate of $C^k$-norm of the extended function under Whitney's conditions without further assumptions which may not be necessary as in [13], [17].

Suppose that $k$ is a nonnegative integer, we shall first define $C^k$-functions on a closed subset $F$ of $\mathbb{R}^n$ in Whitney's sense. Consider a collection $\mathcal{U} = \{u_\alpha\}_{|\alpha|\leq k}$ of real functions on $F$. For such a collection of functions, define the corresponding Taylor type polynomial $T(\mathcal{U}, x; y)$ of order $k$ centered at $x \in F$ by

$$T(\mathcal{U}, x; y) = \sum_{|\alpha|\leq k} \frac{u_\alpha(x)}{\alpha!}(y - x)^\alpha.$$

Note that $D_y^\alpha T(\mathcal{U}, x; y) = \sum_{|\beta|\leq k-|\alpha|} \frac{u_{\alpha+\beta}(x)}{\beta!}(y-x)^\beta$. For convenience, $D_y^\alpha T(\mathcal{U}, x; y)$ will be denoted by $T_\alpha(\mathcal{U}, x; y)$ where $D_y^\alpha = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\partial y_2^{\alpha_2}\cdots\partial y_n^{\alpha_n}}$. By simple computation, for $x_1, x_2 \in F$ and $|\alpha| \leq k$,

$$T_\alpha(\mathcal{U}, x_2; y) - T_\alpha(\mathcal{U}, x_1; y) = \sum_{|\beta|\leq k-|\alpha|} \frac{u_{\alpha+\beta}(x_2) - T_{\alpha+\beta}(\mathcal{U}, x_1; x_2)}{\beta!}(y - x_2)^\beta. \tag{3.1}$$

And for such a family $\mathcal{U}$, we will denote for $0 \le s \le 1$

$$\begin{aligned} &M_1(\mathcal{U}, F) := \max_{|\alpha|\le k} \sup_{x\in F} |u_\alpha(x)|; \\ &M_{2,s}(\mathcal{U}, F) := \max_{|\alpha|\le k} \sup_{\substack{x,\, y \,\in\, F \\ x \,\neq\, y}} \tfrac{|u_\alpha(y) - T_\alpha(\mathcal{U},x;y)|}{|y-x|^{k+s-|\alpha|}}; \text{ and} \\ &M_s(\mathcal{U}, F) := M_1(\mathcal{U}, F) \vee M_{2,s}(\mathcal{U}, F). \end{aligned}$$

Note that

$$\begin{aligned} M_1(\mathcal{U}, F) &= \sup\nolimits_{x\in F, r>0} M_{1,x,r}(\mathcal{U}, F), \\ M_{2,s}(\mathcal{U}, F) &= \sup\nolimits_{x\in F, r>0} M_{2,s,x,r}(\mathcal{U}, F), \end{aligned}$$

where

$$\begin{aligned} &M_{1,x,r}(\mathcal{U}, F) := \sup\{|u_\alpha(y)| : y \in \overline{B_r(x)} \cap F, |\alpha| \le k\} \\ &M_{2,s,x,r}(\mathcal{U}, F) := \sup\{\tfrac{|u_\alpha(z) - T_\alpha(\mathcal{U},y;z)|}{|z-y|^{k+s-|\alpha|}} : y, z \in \overline{B_r(x)} \cap F, y \neq z, |\alpha| \le k\}, \end{aligned}$$

for $x \in F$ and $r > 0$. (Note that $M_{2,s,x,r}(\mathcal{U}, F) = 0$ if $\overline{B_r(x)} \cap F = \{x\}$ according to usual convention. )

Following Malgrange[11], a family $\mathcal{U} = \{u_\alpha\}_{|\alpha|\le k}$ of functions on $F$ will be called a $C^k$-jet on $F$ if

$$\lim_{\substack{y,\, z \,\to\, x \\ y,\, z \,\in\, F}} \frac{|u_\alpha(z) - T_\alpha(\mathcal{U}, y; z)|}{|z - y|^{k-|\alpha|}} = 0 \tag{3.2}$$

for $x \in F$ and all $\alpha$ with $|\alpha| \le k$. The limit here means that

$$\lim_{r\to 0} M_{2,0,x,r}(\mathcal{U}, F) = 0. \tag{3.3}$$

Note that a $C^k$-jet $\mathcal{U}$ is uniquely determined by $u_0$ on $\overline{intF}$. Thus when $F = \mathbb{R}^n$, we will denote $M_s(\mathcal{U}, \mathbb{R}^n)$ by $M_s(u_0)$.

Following Whitney, a function $u$ on $F$ is called a $C^k$-function if there is a $C^k$-jet $\mathcal{U} = \{u_\alpha\}_{|\alpha|\le k}$ on $F$ such that $u = u_0$, and in this case we say that $u$ is adapted to the $C^k$-jet $\mathcal{U}$. We note that in general a $C^k$-function on $F$ may be adapted to several $C^k$-jets, for we can change the values of $u_\alpha' s$, $0 < |\alpha| \le k$ at finitely many isolated points of $F$ without violating (3.2). Our purpose is to prove the following strengthened form of Whitney extension theorem.

**Theorem 6.** *A function $u$ defined on $F$ can be extended to a $C^k$-function $v$ on $\mathbb{R}^n$ if and only if $u$ is $C^k$ on $F$. Moreover, if $u$ is adapted to the $C^k$-jet*

*$\mathcal{U}$, for each $\varepsilon > 0$, $v$ can be chosen to be $C^\infty$ on $F^c$, $D^\alpha v \equiv u_\alpha$ on $F$ for $|\alpha| \leq k$, and suppv is contained in the $\varepsilon$-neighborhood of $F$. Furthermore, for each $0 \leq s \leq 1$, the estimate*

$$M_s(v) \leq CM_s(\mathcal{U}, F) \tag{3.4}$$

*holds for some constant $C$ depending only on $n$, $k$ and $\varepsilon$.*

**Remark:**
In fact, the condition (3.2) can be replaced by a little weaker one: for each $|\alpha| \leq k$,

$$\lim_{x,y\to z \text{ via } F} \frac{|u_\alpha(y) - T_{\alpha,x}(y)|}{|y-x|^{k-|\alpha|}} = 0$$

for $z \in \partial F \cap F'$; and

$$\lim_{y\to z \text{ via } F} \frac{|u_\alpha(y) - T_{\alpha,z}(y)|}{|y-z|^{k-|\alpha|}} = 0$$

for $z \in intF$.

We precede the proof by describing the method of defining the extended function $v$ outside $F$ and by giving some necessary estimates. The method of extension is basically that of Whitney[15] with some modifications due to Stein[13]. However, our norm estimate of $v$ is more explicit and refined.

### 3.1 Extension of $C^k$-functions on $F$

To define the values of $v$ outside $F$, we need the Whitney decomposition of $F^c$[15]. For a closed subset $F \subset \mathbb{R}^n$, there is a collection $\{Q_i\}_{i\in\mathbb{N}}$ of nonoverlapping closed cubes such that

($P_1$) $\cup_{i\in\mathbb{N}} Q_i = F^c$;

($P_2$) $diam(Q_i) \leq dist(Q_i, F) \leq 4diam(Q_i)$ for each $i \in \mathbb{N}$;

($P_3$) For each $Q_{i_0}$, the number of $Q_i$'s which intersect $Q_{i_0}$ is less than $12^n$ ($6^n$).

For each $i \in \mathbb{N}$, denote by $Q_i^*$ the closed cube with the same center as $Q_i$ with the length of its side $9/8$ times that of $Q_i$. According to ($P_2$) and ($P_3$), each $Q_{i_0}^*$ intersects less than $12^n$ $Q_i^*$'s. There is a $C^\infty$ partition of unity

$\{\phi_i\}_{i\in\mathbb{N}}$ of $F^c$ subordinate to $\{intQ_i^*\}_{i\in\mathbb{N}}$ such that for each multi-index $\alpha$, $x\in\mathbb{R}^n$ and $i\in\mathbb{N}$,

$$|D_\alpha\phi_i(x)|\le A_\alpha dist(x,F)^{-|\alpha|}$$

for some constant $A_\alpha$ depending only on $\alpha$ and $n$.

For each $i\in\mathbb{N}$, choose $\xi^i$ on $F$ such that $dist(Q_i,F)=dist(\xi^i,Q_i)$, and define $v:\mathbb{R}^n\to\mathbb{R}$ by

$$v(x):=\begin{cases} u(x) & \text{if } x\in F;\\ \sum\limits_{i\in\mathbb{N}}\phi_i(x)T(\mathcal{U},\xi^i;x) & \text{if } x\notin F.\end{cases}\tag{3.5}$$

**Remarks:**

1. Observe that for $x\in F$, $y\in F^c$, the following inequalities hold for those $i\in\mathbb{N}$ such that $y\in Q_i^*$:

$$|x-\xi^i|\le 4|y-x|,\tag{3.6}$$

and hence

$$|y-\xi^i|\le 5dist(y,F).\tag{3.7}$$

2. Note that $v$ depends on the choices of $\{\xi^i\}_{i\in\mathbb{N}}$.

3. Since $v$ is locally a finite sum of $C^\infty$-functions on $F^c$, $v$ is infinitely differentiable on $F^c$ and for any multi-index $\alpha$ and $x\in\mathbb{R}^n$,

$$\begin{aligned} D^\alpha v(x) &= \sum_{i\in\mathbb{N}} D^\alpha\left(\phi_i T(\mathcal{U},\xi^i;x)\right)\\ &= \sum_{i\in\mathbb{N}}\sum_{\beta\le\alpha}\tfrac{\alpha!}{\beta!(\alpha-\beta)!}D^{\alpha-\beta}\phi_i(x)T_\beta(\mathcal{U},\xi^i;x)\\ &= \sum_{i\in\mathbb{N},x\in Q_i^*}\sum_{\beta\le\alpha}\tfrac{\alpha!}{\beta!(\alpha-\beta)!}D^{\alpha-\beta}\phi_i(x)T_\beta(\mathcal{U},\xi^i;x)\end{aligned}$$

**Proposition 1.** *There exists a constant $C=C(n,k)$ such that for each $x\in F$, $y\in F^c$, $|\alpha|\le k$, $0\le s\le 1$ and $\xi\in F$ with $|y-\xi|=dist(x,F)$ we have the following estimates:*

*(a) For each $i$ with $x\in Q_i^*$,*

$$\frac{|T_\alpha(\mathcal{U},\xi^i;y)-T_\alpha(\mathcal{U},x;y)|}{|y-x|^{k+s-|\alpha|}}\le CM_{2,s,x,4|y-x|}(\mathcal{U},F)$$

*and*

$$\frac{T_\alpha(\mathcal{U},\xi^i;y)-T_\alpha(\mathcal{U},\xi;y)|}{|y-\xi|^{k+s-|\alpha|}}\le CM_{2,s,x,4|y-x|}(\mathcal{U},F).$$

*(b)*

$$\left|\sum_{i\in\mathbb{N}}\sum_{\beta\le\alpha,\beta\ne\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}D^{\alpha-\beta}\phi_i(y)T_\beta(\mathcal{U},\xi^i;y)\right|\le CM_{2,s,x,6|y-x|}(\mathcal{U},F)dist(y,F)^{k+s-|\alpha|}.$$

*Proof.* (a) If $y\in Q_i^*$, both $\xi^i$ and $\xi$ belong to $B_{4|y-x|}(x)$ by (3.6), thus by applying (3.1)

$$\begin{aligned}&|T_\alpha(\mathcal{U},\xi^i;y)-T_\alpha(\mathcal{U},x;y)|\\ \le\ &\sum_{|\beta|\le k-|\alpha|}M_{2,s,x,4|y-x|}(\mathcal{U},F)(4|y-x|)^{k+s-|\alpha+\beta|}|y-x|^{|\beta|}\\ \le\ &CM_{2,s,x,4|y-x|}(\mathcal{U},F)|y-x|^{k+s-|\alpha|},\end{aligned}$$

and similarly

$$|T_\alpha(\mathcal{U},\xi^i;y)-T_\alpha(\mathcal{U},\xi;y)|\le CM_{2,s,x,4|y-x|}(\mathcal{U},F)|y-\xi|^{k+s-|\alpha|}.$$

(b) Since $\sum_{i\in\mathbb{N}}\phi_i\equiv 1$ and it is locally a finite sum of smooth functions on $F^c$, for each multi-index $\alpha\ne 0$ and $y\in F^c$,

$$0=D^\alpha(\sum_{i\in\mathbb{N}}\phi_i)(y)=\sum_{i\in\mathbb{N}}D^\alpha\phi_i(y).$$

Hence,

$$\begin{aligned}&\left|\sum_{i\in\mathbb{N}}\sum_{\beta\le\alpha,\beta\ne\alpha}\tfrac{\alpha!}{\beta!(\alpha-\beta)!}D^{\alpha-\beta}\phi_i(y)T_\beta(\mathcal{U},\xi^i;y)\right|\\ =\ &\left|\sum_{\beta\le\alpha,\beta\ne\alpha}\tfrac{\alpha!}{\beta!(\alpha-\beta)!}\sum_{i\in\mathbb{N}}D^{\alpha-\beta}\phi_i(y)(T_\beta(\mathcal{U},\xi^i;y)-T_\beta(\mathcal{U},\xi;y))\right|\\ \le\ &\sum_{\beta\le\alpha,\beta\ne\alpha}\tfrac{\alpha!}{\beta!(\alpha-\beta)!}\sum_{i\in\mathbb{N}}A_{\alpha-\beta}dist(y,F)^{-|\alpha-\beta|}CM_{2,s,x,4|y-x|}(\mathcal{U},F)|y-\xi|^{k+s-|\beta|}\\ \le\ &CM_{2,s,x,4|y-x|}(\mathcal{U},F)dist(y,F)^{k+s-|\alpha|};\end{aligned}$$

in the inequality above, we have used (a) and the fact that $\sum_{i\in\mathbb{N}}D^{\alpha-\beta}\phi_i(y)=0$ if $\alpha\ne\beta$.

□

**Lemma 2.** *There is a constant $C = C(n,k)$ such that for $x \in F$, $y \in F^c$, $|\alpha| \le k$, $0 \le s \le 1$ and $\xi \in F$ satisfying $|y-\xi| = dist(y,F)$,*

$$\frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}, x; y)|}{|y-x|^{k+s-|\alpha|}} \le CM_{2,s,x,4|y-x|}(\mathcal{U}, F)$$

*and*

$$\frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}, \xi; y)|}{|y-\xi|^{k+s-|\alpha|}} \le CM_{2,s,x,4|y-x|}(\mathcal{U}, F)$$

*Proof.* Since $\sum_{i\in\mathbb{N}} \phi_i \equiv 1$ on $F^c$ and by using Proposition 1 at appropriate places, we have

$$\begin{aligned}
& |D^\alpha v(y) - T_\alpha(\mathcal{U}, x; y)| \\
\le\ & \sum_{i\in\mathbb{N}:yinQ_i^*} \phi_i(y)|T_\alpha(\mathcal{U}, \xi^i; y) - T_\alpha(\mathcal{U}, x; y)| + \left|\sum_{i\in\mathbb{N}} \sum_{\beta\le\alpha,\beta\ne\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y) T_\beta(\mathcal{U}, \xi^i; y)\right| \\
\le\ & \sum_{i\in\mathbb{N}:x\in Q_i^*} A_0 CM_{2,s,x,4|y-x|}(\mathcal{U}, F)|y-x|^{k+s-|\alpha|} + CM_{2,s,x,4dist(y,F)}(\mathcal{U}, F)|y-x|^{k+s-|\alpha|} \\
\le\ & CM_{2,s,x,4|y-x|}|y-x|^{k+s-|\alpha|},
\end{aligned}$$

and also

$$|D^\alpha v(y) - T_\alpha(\mathcal{U}, \xi; y)| \le CM_{2,s,x,4|y-x|}|y-\xi|^{k+s-|\alpha|}.$$

□

**Lemma 3.** *There exists a constant $C = C(n,k)$ such that for $y \in F^c$ and $0 \le s \le 1$,*

$$|D^\alpha v(y)| \le CM_s(\mathcal{U}, F)(1 + dist(y,F)^{k+s-|\alpha|})$$

*if $|\alpha| \le k$, and*

$$|D^\alpha v(y)| \le CM_{2,s}(\mathcal{U}, F) dist(y,F)^{s-1}$$

*if $|\alpha| = k+1$.*

*Proof.* For each $y \in F^c$ and $\xi \in F$ with $|y-\xi| = dist(y,F)$, if $i \in \mathbb{N}$ is such that $y \in Q_i^*$, then by (3.7)

$$\begin{aligned}
|T_\alpha(\mathcal{U}, \xi^i; y)| &\le \sum_{|\beta|\le k-|\alpha|} |u_{\alpha+\beta}(\xi^i)||y-\xi^i|^{|\beta|}/\beta! \\
&\le \sum_{|\beta|\le k-|\alpha|} M_1(\mathcal{U}, F)(5dist(y,F))^{|\beta|} \\
&\le CM_1(\mathcal{U}, F)(1 + dist(y,F)^{k+s-|\alpha|}).
\end{aligned}$$

Thus, for $|\alpha| \le k$, $|D^\alpha v(y)|$ is dominated by

$$
\begin{aligned}
& \sum_{i\in\mathbb{N}, y\in Q_i^*} \phi_i(y)|T_\alpha(\mathcal{U},\xi^i;y)| + |\sum_{i\in\mathbb{N},x\in Q_i^*}\sum_{\beta\le\alpha,\beta\ne\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y)T_\beta(\mathcal{U},\xi^i;y)| \\
\le\ & CA_0M_1(\mathcal{U},F)(1+dist(y,F)^{k+s-|\alpha|}) + CM_{2,s}(\mathcal{U},F)dist(x,F)^{k+s-|\alpha|} \\
\le\ & CM_s(\mathcal{U},F)(1+dist(y,F)^{k+s-|\alpha|}),
\end{aligned}
$$

where Proposition 1(b) has been involved.

Now for $|\alpha| = k+1$, since $D^\beta T(\mathcal{U},\xi^i;\cdot) \equiv 0$ for $|\beta| = k+1$ and $\sum_{i\in\mathbb{N}} D^\gamma\phi_i \equiv 0$ if $\gamma \ne 0$,

$$
\begin{aligned}
|D^\alpha v(y)| &= |\sum_{i\in\mathbb{N}}\sum_{\beta\le\alpha,|\beta|\le k} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y)T_\beta(\mathcal{U},\xi^i;y)| \\
&\le \sum_{\beta\le\alpha,|\beta|\le k} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} \sum_{i\in\mathbb{N}} |D^{\alpha-\beta}\phi_i(y)||T_\beta(\mathcal{U},\xi^i;y) - T_\beta(\mathcal{U},\xi;y)| \\
&\le \sum_{\beta\le\alpha,|\beta|\le k} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} \sum_{i\in\mathbb{N}} A_{\alpha-\beta} dist(y,F)^{-|\alpha-\beta|} CM_{2,s}(\mathcal{U},F)|y-\xi|^{k+s-|\beta|} \\
&\le CM_{2,s}(\mathcal{U},F)dist(y,F)^{s-1}.
\end{aligned}
$$

□

**Lemma 4.** *There exists a constant $C = C(n,k)$ such that for all $\alpha$ with $|\alpha| = k$, $y,z \in F^c$, and $0 \le s \le 1$,*

$$|D^\alpha v(z) - D^\alpha v(y)| \le CM_{2,s}(\mathcal{U},F)|z-y|^s.$$

*Proof.* Fix $y,z \in F^c$ with $y \ne z$, denote the segment connecting $y$ and $z$ by $L$.

In case $dist(L,F) > |z-y| > 0$, it follows from Lemma 3 that for all $w \in L \subset F^c$,

$$
\begin{aligned}
|\nabla(D^\alpha v)(w)| &\le CM_{2,s}(\mathcal{U},F)dist(w,F)^{s-1} \\
&\le CM_{2,s}(\mathcal{U},F)|x-y|^{s-1},
\end{aligned}
$$

thus

$$
\begin{aligned}
|D^\alpha v(z) - D^\alpha v(y)| &\le \int_0^1 |\nabla(D^\alpha v)((1-t)y+tz)||z-y|dt \\
&\le CM_{2,s}(\mathcal{U},F)\int_0^1 |z-y|^{s-1}|z-y|dt \\
&= CM_{2,s}(\mathcal{U},F)|z-y|^s.
\end{aligned}
$$

In case $dist(L,F) \le |z-y|$, choose $w \in L$ and $\xi \in F$ such that $|w-\xi| = dist(L,F) \le |z-y|$. Then

$$|y-\xi| \le |y-w| + |w-\xi| \le 2|z-y|$$

and

$$|z-\xi| \le |z-w| + |w-\xi| \le 2|z-y|.$$

Therefore,

$$\begin{aligned} |D^\alpha v(z) - D^\alpha v(y)| &\le |D^\alpha v(z) - u_\alpha(\xi)| + |u_\alpha(\xi) - D^\alpha v(y)| \\ &= |D^\alpha v(zy) - T_\alpha(\mathcal{U}, \xi; z)| + |T_\alpha(\mathcal{U}, \xi; y) - D^\alpha v(y)| \\ &\le CM_{2,s}(\mathcal{U}, F)|z-\xi|^s + CM_{2,s}(\mathcal{U}, F)|y-\xi|^s \\ &\le CM_{2,s}(\mathcal{U}, F)|z-y|^s, \end{aligned}$$

by Lemma 2. □

**Lemma 5.** *$v \in C^k(\mathbb{R}^n)$ and $D^\alpha v = u_\alpha$ on $F$ for $|\alpha| \le k$.*

*Proof.* For each $x \in F$,

$$\lim_{\substack{y \to x \\ y \in F^c}} \frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}, x; y)|}{|y-x|^{k-|\alpha|}} = 0 \tag{3.8}$$

by (3.3) and Lemma 2, and hence

$$\lim_{y \to x} \frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}, x; y)|}{|y-x|^{k-|\alpha|}} = 0. \tag{3.9}$$

We are going to show that $D^\alpha v$ exists and coincide with $u_\alpha$ on $F$ for $|\alpha| \le k$ . First note that $v = u = u_0$ in $F$ by the definition of $v$. Suppose that we have shown that $D^\alpha v = u_\alpha$ in $F$ for each $|\alpha| \le \tilde{k} < k$, then for each $|\alpha| \le \tilde{k}$, $|\beta| = 1$ and $z \in F$, it follows from (3.9) and the partial derivatives of $T_\alpha(\mathcal{U}, x; \cdot)$ that

$$\begin{aligned} & \lim_{h\to 0} \left|\tfrac{D^\alpha v(x+h\beta) - D^\alpha v(x)}{h} - u_{\alpha+\beta}(x)\right| \\ = & \lim_{h \to 0} \tfrac{|D^\alpha v(x+h\beta) - u_\alpha(x) - u_{\alpha+\beta}(x)h|}{|h|} \\ \le & \lim_{h\to 0} \left\{ \tfrac{|D^\alpha v(x+h\beta) - T_\alpha(\mathcal{U}, x; x+h\beta)|}{|(x+h\beta)-x|} + \tfrac{|T_\alpha(\mathcal{U}, x; x+h) - T_\alpha(\mathcal{U}, x; x) - T_{\alpha+\beta}(\mathcal{U}, x; x)h|}{|h|} \right\} \\ = & \ 0, \end{aligned}$$

which shows that $D^{\alpha+\beta} v$ exists and equals $u_{\alpha+\beta}$ at each $x \in F$. By repeating the process until $\tilde{k} = k-1$, we have the desired result. Finally, again from (3.9), the continuity of $D^\alpha v$ for $|\alpha| \le k$ follows from that the fact that for each $x \in F$

$$\begin{aligned} \lim_{y \to x} D^\alpha v(y) &= \lim_{y\to x} (D^\alpha v(y) - T_\alpha(\mathcal{U}, x; y)) + \lim_{x \to z} T_\alpha(\mathcal{U}, x; y) \\ &= u_\alpha(x) = D^\alpha v(x). \end{aligned}$$

□

**Proof of Theorem 6** The necessary part is obvious. For the other part, note that Lemma 5 shows the existence of a $C^k$-extension of $u$. To show the remaining part of Theorem 6, observe first that for any $\alpha$ with $|\alpha| < k$, $x, y \in \mathbb{R}^n$ and $0 \le s \le 1$,

$$\begin{aligned}
& |D^\alpha v(y) - \sum_{|\beta|\le k-|\alpha|} \frac{D^{\alpha+\beta}v(x)}{\beta!}(y-x)^\beta| \\
\le \quad & (k-|\alpha|) \sum_{|\gamma|=k,\gamma\ge\alpha} \frac{|y-x|^{k-|\alpha|}}{(\gamma-\alpha)!} \int_0^1 |D^\gamma v(x+(y-x)t) - D^\gamma v(x)|dt \\
\le \quad & (k-|\alpha|) \sum_{|\gamma|=k,\gamma\ge\alpha} \frac{|y-x|^{k-|\alpha|}}{(\gamma-\alpha)!} \int_0^1 CM_{2,s}(\mathcal{U},F)|(y-x)t|^s dt \\
= \quad & CM_{2,s}(\mathcal{U},F)|y-x|^{k+s-|\alpha|},
\end{aligned}$$

by Lemma 4 and Taylor's Theorem with integral remainder.

In order to arrive at estimate (3.4), we need a suitable cut-off function defined as below. Fix a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ with $supp\varphi \subset B(0,1)$, $\varphi \ge 0$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. Let $\psi = \varphi_{\varepsilon/3} * \chi_{F_{2\varepsilon/3}}$ where $F_r := \{x | dist(x,F) \le r\}$ for $r > 0$. Then $\psi \in C^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ on $F_{\varepsilon/3}$, $\psi \equiv 0$ outside $F_\varepsilon$, $0 \le \psi \le 1$, and for each $\alpha$ with $|\alpha| \le k+1$,

$$\|D^\alpha \psi\|_\infty \le C \max_{|\alpha|\le k+1} \|D^\alpha \varphi\|_\infty$$

for some constant $C = |B(0,1)|(1+(3/\varepsilon)^{k+1}) = C(n,k)\max\{1, (\frac{1}{\varepsilon})^{k+1}\} = C(n,k,\varepsilon)$.
(Note that $\max_{|\alpha|\le k+1} \|D^\alpha\varphi\|_\infty$ is a constant only depending on $n$ and $k$.)

Define $w = v\psi$, then $w \in C^k(\mathbb{R}^n)$, $w = v$ in $intF_{\varepsilon/3} \supset F$ and $suppw \subset supp\psi \subset F_\varepsilon$. This implies that $D^\alpha w = D^\alpha v = u_\alpha$ on $intF_{\varepsilon/3} \supset F$ for $|\alpha| \le k$ and $D^\alpha \equiv 0$ on $F_\varepsilon^c$ for all $\alpha$.

Moreover, for any $x \in F_\varepsilon \backslash F$ and $|\alpha| \le k$, by Lemma 2

$$\begin{aligned}
|D^\alpha w(x)| \quad & \le \quad \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}|D^\beta v(x)||D^{\alpha-\beta}\psi(x)| \\
& \le \quad \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} CM_s(\mathcal{U},F)(1+dist(x,F)^{k+s-|\beta|}) \\
& \le \quad \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} CM_s(\mathcal{U},F)(1+\varepsilon^{k+1}) \\
& \le \quad CM_s(\mathcal{U},F)
\end{aligned}$$

where $C$ is a constant only depends on $n, k$ and $\varepsilon$. $\therefore \max_{|\alpha|\le k} \|D^\alpha w\|_\infty \le CM_s(\mathcal{U},F)$.

Similarly, for $x \in F_\varepsilon \backslash F$ and $|\alpha| = k+1$ and $0 \le s \le 1$, by Lemma 3

$$\begin{aligned} |D^\alpha w(x)| &\le \sum_{\beta\le\alpha,\beta\ne\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!}|D^\beta v(x)||D^{\alpha-\beta}\psi(x)| + |D^\alpha w(x)||\psi(x)| \\ &\le \sum_{\beta\le\alpha,\beta\ne\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} CM_s(\mathcal{U},F)(1+dist(x,F)^{k+s-|\beta|}) \\ &\quad +CM_{2,s}(\mathcal{U},F)dist(x,F)^{s-1} \\ &\le CM_s(\mathcal{U},F)\{\sum_{\beta\le\alpha,\beta\ne\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!}(1+\varepsilon^{k+1}) + dist(x,F)^{s-1}\} \\ &\le CM_s(\mathcal{U},F)dist(x,F)^{s-1} \end{aligned}$$

because $1 = \varepsilon^{1-s}\varepsilon^{s-1} \le (1+\varepsilon)\varepsilon^{s-1} \le Cdist(x,F)^{s-1}$. Thus, for any $x \in F^c$ and $|\alpha| = k+1$,

$$|D^\alpha w(x)| \le CM_s(\mathcal{U},F)dist(x,F)^{s-1}.$$

Now, we are going to show that for any $x, y \in \mathbb{R}^n$ and $|\alpha| = k$,

$$|D^\alpha w(y) - D^\alpha w(x)| \le CM_s(\mathcal{U},F)|y-x|^s. \tag{3.10}$$

for some constant $C = C(n,k,\varepsilon)$. In case $|y-x| \ge 1$,

$$|D^\alpha w(y) - D^\alpha w(x)| \le 2\max_{|\alpha|\le k} \|D^\alpha w\|_\infty \le CM_s(\mathcal{U},F) \le CM_s(\mathcal{U},F)|y-x|^s.$$

For $|y-x| < 1$, denote $L := \{(1-t)x + ty : t \in [0,1]\}$. In case $dist(L,F) > |y-x| > 0$, $(L \subset F^c)$

$$\begin{aligned} &|D^\alpha w(y) - D^\alpha w(x)| \\ \le\ & \int_0^1 |\nabla(D^\alpha w)((1-t)x+ty)||y-x|dt \\ \le\ & \int_0^1 CM_s(\mathcal{U},F)dist((1-t)x+ty,F)^{s-1}|y-x|dt \\ \le\ & CM_s(\mathcal{U},F)\int_0^1 |y-x|^{s-1}|y-x|dt \\ =\ & CM_s(\mathcal{U},F)|y-x|^s. \end{aligned}$$

In case $dist(L,F) \le |y-x|$, choose $w \in L$ and $\xi \in F$ such that $|w-\xi| = dist(L,F) \le |y-x|$. It is easy to see that $|z-\xi| \le 2|y-x| < 2$ for all $z \in L$. Since for any $|\beta| \le k$ and $z \in L$,

$$|T_\beta(\mathcal{U},\xi;z)| \le \sum_{|\gamma|\le k-|\beta|} \frac{M_s(\mathcal{U},F)}{\gamma!}2^{|\gamma|} \le C(n,k)M_s(\mathcal{U},F),$$

and

$$|T_\beta(\mathcal{U},\xi;y) - T_\beta(\mathcal{U},\xi;x)| \le CM_s(\mathcal{U},F)|y-x|$$

by mean value theorem with the previous estimate. Therefore,

$$
\begin{aligned}
& |D^\alpha w(y) - D^\alpha w(x)| \\
\le\ & \sum_{\beta\le\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} |D^\beta v(y) D^{\alpha-\beta}\psi(y) - D^\beta v(x) D^{\alpha-\beta}\psi(x)| \\
\le\ & \sum_{\beta\le\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} \{|D^\beta v(y) - T_\beta(\mathcal{U},\xi;y)||D^{\alpha-\beta}\psi(y)| + |T_\beta(\mathcal{U},\xi;y)||D^{\alpha-\beta}\psi(y) - D^{\alpha-\beta}\psi(x)| \\
& + |T_\beta(\mathcal{U},\xi;y) - T_\beta(\mathcal{U},\xi;x)||D^{\alpha-\beta}\psi(x)| + |T_\beta(\mathcal{U},\xi;x) - D^\beta v(x)||D^{\alpha-\beta}\psi(x)|\} \\
\le\ & CM_s(\mathcal{U},F) \sum_{\beta\le\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} \{|y-\xi|^{k+s-|\beta|} + \|\nabla(D^{(\alpha-\beta)}\psi)\|_\infty |y-x| \\
& + |y-x| \max_{|\alpha|\le k} \|D^\alpha\psi\|_\infty + |x-\xi|^{k+s-|\beta|} \max_{|\alpha|\le k} \|D^\alpha\psi\|_\infty\} \\
\le\ & CM_s(\mathcal{U},F) \sum_{\beta\le\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} \{2^{k+1}|y-x|^{k+s-|\beta|} + \max_{|\alpha|\le k+1} \|D^\alpha\psi\|_\infty |y-x| \\
& + |y-x| \max_{|\alpha|\le k+1} \|D^\alpha\psi\|_\infty + 2^{k+1}|y-x|^{k+s-|\beta|} \max_{|\alpha|\le k+1} \|D^\alpha\psi\|_\infty\} \\
\le\ & CM_s(\mathcal{U},F)|y-x|^s.
\end{aligned}
$$

$\therefore$ For any $x, y \in \mathbb{R}^n$ and $|\alpha| = k$,

Finally, with (3.10), for any $x, y \in \mathbb{R}^n$ and $|\alpha| < k$,

$$
\begin{aligned}
& |D^\alpha w(y) - \sum_{|\beta|\le k-|\alpha|} \tfrac{D^{\alpha+\beta} w(x)}{\beta!}(y-x)^\beta| \\
\le\ & (k-|\alpha|) \sum_{|\gamma|=k, \gamma\ge\alpha} \tfrac{|y-x|^{k-|\alpha|}}{(\gamma-\alpha)!} \int_0^1 |D^\gamma v((1-t)x+ty) - D^\gamma v(x)| dt \\
\le\ & (k-|\alpha|) \sum_{|\gamma|=k, \gamma\ge\alpha} \tfrac{|y-x|^{k-|\alpha|}}{(\gamma-\alpha)!} \int_0^1 CM_s(\mathcal{U},F)|((1-t)x+ty) - x|^s dt \\
\le\ & CM_s(\mathcal{U},F)|y-x|^{k+s-|\alpha|}.
\end{aligned}
$$

These give the estimate (3.4).

**Remarks:**

1. In fact, $v$ can be chosen to be analytic on the union of $F^c$ and the set of isolated points of $F$ by choosing $\phi_i$ 's and $\varphi$ to be analytic on $F^c$ and the fact that $v$ equals to $T(\mathcal{U}, x; y)$ near $x$ for each isolated point $x$ of $F$.

2. Each $u_\alpha$ in a $C^k$-jet is continuous on $F$, hence when $F$ is compact, $M_1(\mathcal{U}, F)$ must be finite.

3. In case that $F$ is a general set, we can similarly define a $C^k$-jet on $F$ by assuming (3.2) holds for each $x \in \overline{F}$, and define a $C^k$-function on $F$ in the same way. Then Theorem 6 also holds.

4. From the proofs of Lemma 3, Lemma and Theorem 6, it follows that for $0 < s \le 1$, if for each $x \in F$, $M_{2,s,x,r}(\mathcal{U}, F)$ is finite for some $r > 0$, then each $D^\alpha v$, $|\alpha| = k$, is locally Hölder continuous with exponent $s$; And if

$$\lim_{r \to 0} M_{2,s,x,r}(\mathcal{U}, F) = 0$$

for each $x \in F$, then

$$\lim_{y \to x} \frac{|D^\alpha v(y) - D^\alpha v(x)|}{|y - x|^s} = 0$$

for each $x \in \mathbb{R}^n$ and $|\alpha| = k$.

## 3.2 $C^\infty$-functions on $F$

It is natural now to consider the $C^\infty$-functions on a closed subset $F$ of $\mathbb{R}^n$. Let $\mathcal{U} = \{u_\alpha\}$ be a collection of real functions on $F$, we say that $\mathcal{U}$ is a $C^\infty$-jet on $F$ if for each $k \in \mathbb{N}$, the subfamily $\mathcal{U}_k := \{u_\alpha\}_{|\alpha| \le k}$ is a $C^k$-jet on $F$; and a function $u$ is a $C^\infty$-function on $F$ if there is a $C^\infty$-jet $\mathcal{U}$ on $F$ such that $u = u_0$. Note that the definition of $C^\infty$-function on $F$ is essentially following the fact that $\cap_{k \in \mathbb{N}} C^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. We are going to show a similar result for $C^\infty$-functions on $F$ as Theorem 1.

**Theorem 7.** *A function $u$ defined on $F$ can be extended to a $C^\infty$-function $v$ on $\mathbb{R}^n$ if and only if $u$ is $C^\infty$ on $F$. Moreover, if $\mathcal{U}$ is a $C^\infty$-jet such that $u = u_0$, then for any $\varepsilon > 0$, $v$ can be chosen so that $D^\alpha v = u_\alpha$ on $F$ for each multi-index $\alpha$, suppv contained in the $\varepsilon$-neighborhood of $F$, and for each nonnegative integer $k$*

$$M_0(\{D^\alpha v\}_{|\alpha| \le k}, \mathbb{R}^n) \le C \max\{M_0(\mathcal{U}_k, F), \varepsilon\}$$

*for some constant $C = C(n, k, \varepsilon)$.*

To prove the theorem, for each $i \in \mathbb{N}$, let

$$k_i := \sup\{k \in \mathbb{N} : \max_{|\alpha| \le k} |u_\alpha(\xi^i)| < dist(Q_i, F)^{-1/2}\}$$

and define

$$v(x) := \begin{cases} u(x) & \text{if } x \in F \\ \sum_{i \in \mathbb{N}} \phi_i(x) T(\mathcal{U}_{k_i}, \xi^i; x) & \text{if } x \notin F. \end{cases}$$

Note that it is possible that $k_i = \infty$, in which case $\mathcal{U}_\infty = \mathcal{U}$ and the power series $T(\mathcal{U}_{k_i}, \xi^i, x)$ converges absolutely on $\mathbb{R}^n$ by the estimate

$$\sum \frac{|u_\alpha(\xi^i)|}{\alpha!}|x-\xi^i|^{|\alpha|} \leq dist(Q_i, F)^{-1/2} \sum \frac{|x-\xi^i|^{|\alpha|}}{\alpha!} = dist(Q_i, F)^{-1/2} e^{n|x-\xi^i|} < \infty.$$

Thus, $v \in C^\infty(F^c)$ and for each multi-index $\alpha$, and for $y \in F^c$,

$$D^\alpha v(y) = \sum_{i\in\mathbb{N}} \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y) T_\beta(\mathcal{U}_{k_i}, \xi^i; y).$$

Now, in reference to the proof of Theorem 6, to show that $v \in C^\infty(\mathcal{U})$ and $D^\alpha v = u_\alpha$ on $F$ for each multi-index $\alpha$, it suffices to show that for each $k \in \mathbb{N}$, $x \in F$ and $|\alpha| = k-1$,

$$\lim_{\substack{y \to x \\ y \in F^c}} \frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}_k, x; y)|}{|y-x|} = 0.$$

For this, we separate the numerator as follow:

$$\begin{aligned}
& |D^\alpha v(y) - T_\alpha(\mathcal{U}_k, x; y)| \\
= \; & |\sum_{i\in\mathbb{N}} \sum_{\beta\leq\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y) T_\beta(\mathcal{U}_{k_i}, \xi^i; y) - T_\alpha(\mathcal{U}_k, x; y)| \\
\leq \; & \textstyle\sum_{i\in\mathbb{N}} \phi_i(y) |T_\alpha(\mathcal{U}_k, \xi^i; y) - T_\alpha(\mathcal{U}_k, x; y)| \\
& + |\sum_{i\in\mathbb{N}} \sum_{\beta\leq\alpha, \beta\neq\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y) T_\beta(\mathcal{U}_k, \xi^i; y)| \\
& + |\sum_{i\in\mathbb{N}} \sum_{\beta\leq\alpha} \tfrac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\phi_i(y) (T_\beta(\mathcal{U}_{k_i}, \xi^i; y) - T_\beta(\mathcal{U}_k, \xi^i; y))| \\
:= \; & I + II + III.
\end{aligned}$$

From Proposition 1, we know that

$$I + II \leq C(n,k) M_{2,0,x,4|y-x|}(\mathcal{U}_k, F)|y-x|.$$

To estimate III, we need the following lemma which can be proved by induction on the dimension easily.

**Lemma 6.** *Suppose $t \geq 0$, $l$ is a nonnegative integer, and $\alpha$ denotes n-dimensional multi-index. Then*

$$\sum_{|\alpha|\geq l} \frac{t^{|\alpha|}}{\alpha!} \leq \frac{n^l t^l e^t}{l!} + \sum_{j=2}^{n-1} \frac{(j^{l+1} - (j-1)^{l+1}) t^{l+1} e^{(n+1-j)t}}{(l+1)!} + \frac{t^{l+1} e^{nt}}{(l+1)!}.$$

*Moreover, if we further suppose that* $0 \leq t \leq 1$, *then*

$$\sum_{l \leq |\alpha|} \frac{t^{|\alpha|}}{\alpha!} \leq \frac{n^{l+2} t^l e^{nt}}{l!}.$$

Since $\mathcal{U}_k$ is a $C^k$-jet on $F$, each $u_\alpha$ is locally bounded on $F$, therefore, $m_{k,x} := \{|u_\alpha(y)| : y \in B_1(x), |\alpha| \leq k\} < \infty$ for each $x \in F$. Hence, if $y \in F^c \cap B_r(x)$ where $r = \min\{1/5, 7(1+m_{k,z})^{-2}/8\}$, then for each $i \in \mathbb{N}$ such that $y \in Q_i^*$,

$$\begin{array}{ll} & |x - \xi^i| \leq 4|y-x| < 1 \\ \Rightarrow & |u_\alpha(\xi^i)| \leq m_{k,x} \quad \text{for all} \quad |\alpha| \leq k \\ \Rightarrow & \max\limits_{|\alpha| \leq k} |u_\alpha(\xi^i)| \leq m_{k,x} < (8|y-x|/7)^{-1/2} \leq dist(Q_i, F)^{-1/2} \\ \Rightarrow & k_i \geq k. \end{array}$$

Thus, III is dominated by

$$\begin{array}{ll} & \sum\limits_{i \in \mathbb{N}, y \in Q_i^*} \sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} A_{\alpha-\beta} dist(y,F)^{|\beta|-|\alpha|} \sum\limits_{k+1-|\beta| \leq |\gamma| \leq k_i - |\beta|} \frac{|u_{\beta+\gamma(\xi^i)}|}{\gamma!} |y - \xi^i|^{|\gamma|} \\ \leq & \sum\limits_{i \in \mathbb{N}, y \in Q_i^*} \sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} A_{\alpha-\beta} dist(y,F)^{|\beta|-|\alpha|} \sum\limits_{k+1-|\beta| \leq |\gamma| \leq k_i - |\beta|} \frac{1}{\gamma!} dist(Q_i, F)^{-1/2} |y - \xi^i|^{|\gamma|} \\ \leq & \sum\limits_{i \in \mathbb{N}, y \in Q_i^*} \sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} A_{\alpha-\beta} dist(y,F)^{|\beta|-|\alpha|} \sum\limits_{k+1-|\beta| \leq |\gamma|} \frac{1}{\gamma!} (5 dist(y,F))^{|\gamma|-1/2} \\ \leq & C \sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} dist(y,F)^{|\beta|-|\alpha|} \sum\limits_{k+1-|\beta| \leq |\gamma|} \frac{1}{\gamma!} (5 dist(y,F))^{|\gamma|-1/2} \\ \leq & C \sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} dist(y,F)^{|\beta|-|\alpha|} (5 dist(y,F))^{-1/2} \frac{n^4 (5 dist(y,F)^{k+1-|\beta|} e^{5n dist(y,F)}}{(k+1-|\beta|)!} \\ = & C(\sum\limits_{\beta \leq \alpha, \beta \neq \alpha} \frac{5^{k+1-|\beta|}}{(k+1-|\beta|)!}) dist(y,F)^{3/2} \\ \leq & C|y-x|^{3/2} \end{array}$$

Therefore,

$$\begin{array}{rcl} \frac{|D^\alpha v(y) - T_\alpha(\mathcal{U}_k, x; y)|}{|y-x|} & \leq & C(M_{2,0,x,4|y-x|}(\mathcal{U}_k, F) + |y-x|^{1/2}) \\ & \to & 0 \quad \text{as} \quad y \to x, y \in F^c. \end{array}$$

Now for $1 > \varepsilon > 0$, procede similarly as above, we have for $|\alpha| = k$ and $y \in F_{\varepsilon^2} \backslash F$,

$$\begin{array}{rcl} |D^\alpha v(y)| & \leq & |D^\alpha y(y) - u_\alpha(\xi)| + |u_\alpha(\xi)| \\ & \leq & C \max\{M_0(\mathcal{U}_k, F), \varepsilon\}. \end{array}$$

Then by Taylor's Theorem with integral remainder,

$$M_0(\{D^\alpha v\}_{|\alpha|\le k}, F_{\varepsilon^2}) \le C \max\{M_0(\mathcal{U}_k, F), \varepsilon\}.$$

Finally, by multiplying $v$ by a cut-off function as in the proof of Theorem 6, we get the extended function desired.

The following corollary is a straightforward application of Theorem 7.

**Corollary 2.** *The intersection of all $LC^{(k)}(D)$, $k \in \mathbb{N}$, is the set $LC^{(\infty)}(D)$ consisting of measurable functions $u$ on $D$ which have Lusin property of order $\infty$, that is, for each $\varepsilon > 0$, there is a $C^\infty$-function $v$ defined on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \ne v(x)\}| < \varepsilon$.*

*Proof.* It suffices to show that $\cap_k LC^{(k)}(D) \subset LC^{(\infty)}(D)$. Suppose that $u$ is a measurable function on $D$ such that for each $\varepsilon > 0$ and $k \in \mathbb{N}$, there is $v_k \in C^k(\mathbb{R}^n)$ satisfying $|E_k| := |\{x \in D : u(x) \ne v_k(x)\}| < \varepsilon 2^{-k}$. Note that all the approximate partial derivatives $\frac{\partial^\alpha u}{\partial x^\alpha} := u_\alpha$ of $u$ exist a.e. and, for $k \ge |\alpha|$, $u_\alpha = \frac{\partial^\alpha v_k}{\partial x^\alpha}$ a.e. on $D\backslash(\cap_k E_k)$ whose complement in $D$ has Lebesgue measure less than $\varepsilon$. Thus, we can choose a closed subset $F$ of $D\backslash(\cap_k E_k)$ with $|D\backslash F| < \varepsilon$ such that $u_\alpha = \frac{\partial^\alpha v_k}{\partial x^\alpha}$ on $F$ for each $k \ge |\alpha|$. We can conclude that $u \in LC^{(\infty)}(D)$ by applying Theorem 7 with the $C^\infty$-jet $\mathcal{U} := \{u_\alpha\}$. □

# 4 Proof of Theorem 5

We prove first a lemma which contains the key step in the proof of the implication (3) ⇒ (2) and will also be used to obtain similar results in the next section.

**Lemma 7.** *There is a positive constant $C_n$ such that, for each $M > 0$,*

$$ap \limsup_{y\to x} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma} < C_n M \tag{4.1}$$

*holds true for almost every point $x$ in the set $D_M^P$ consisting of the points $x \in D$ satisfying*

$$ap \limsup_{y_i \to x_i} \frac{|\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i) - P^{(i)}_{\alpha,x}(y_i)|}{|y_i - x_i|^{\gamma - |\alpha|}} < M$$

*for all $1 \le i \le n$, $|\alpha| \le \mathring{\gamma}$, where $P_x(y) := \sum_{|\alpha|\le\mathring{\gamma}} \frac{1}{\alpha!}\frac{\partial^\alpha u}{\partial x^\alpha}(x)(y-x)^\alpha$.*

*Proof.* For $1 \le i \le n$ and a set $A \subset \mathbb{R}^n$, let $\mathcal{J}_i = \{\alpha = (\alpha_1, \cdots, \alpha_n) : |\alpha| \le \mathring{\gamma}, \alpha_i = \alpha_{i+1} = \cdots = \alpha_n = 0\}$, $\pi_{i,x}(A) := \{y_i \in \mathbb{R} : x + (y_i - x_i)e_i \in A\}$ , and $D_{i,x} := \{y_i \in \pi_{i,x}(D) : |\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i) - P^{(i)}_{\alpha,x}(y_i)| \le M|y_i - x_i|^{\gamma - |\alpha|}\ \forall \alpha \in \mathcal{J}_i\}$. Fix $1 > \tau > 0$. We will recursively define a family $\{D_{l_1,\cdots,l_i} : l_1, \cdots, l_i \in \mathbb{N}, 1 \le i \le n\}$ of subsets of $D$ as follows. For $l_1 \in \mathbb{N}$, let $D_{l_1} = D^\tau_{l_1}$ be the collection of points $x \in D$ satisfying that $|D_{1,x} \cap [a, b]| \ge \tau(b - a)$ whenever $[a, b]$ is a closed interval containing $x_1$ and $b - a < l_1^{-1}$. $D_{l_1}$ is measurable by Lemma 1. Suppose that for $1 \le i \le n - 1$ and $l_1, \cdots, l_i \in \mathbb{N}$, a measurable subset $D_{l_1,\cdots,l_i}$ of $D$ is defined, then for $l_{i+1} \in \mathbb{N}$, let $D_{l_1,\cdots,l_{i+1}} = D^\tau_{l_1,\cdots,l_{i+1}}$ be the set of $x \in D_{l_1,\cdots,l_i}$ such that $|D_{i+1,x} \cap \pi_{i+1,x}(D_{l_1,\cdots,l_i}) \cap [a, b]| \ge \tau(b - a)$ whenever $x_{i+1} \in [a, b]$ and $b - a < l_{i+1}^{-1}$. Then each $\{D_{l_1,\cdots,l_i}\}_{l_i \in \mathbb{N}}$ is an increasing sequence of measurable sets by Lemma 1, and $|D^P_M \backslash \cup_{l_1,\cdots,l_n \in \mathbb{N}} D_{l_1,\cdots,l_n}| = 0$. Note that if $y = (y_1, \cdots, y_n) \in D$ satisfies $y_1 \in D_{1,x^{(1)}}$ and $y_i \in D_{i,x^{(i)}} \cap \pi_{i,x^{(i)}}(D_{l_1,\cdots,l_{i-1}})$ for each $2 \le i \le n$, then

$$
\begin{array}{rl}
& |u(y) - P_x(y)| \\
\le & \sum_{i=1}^n \sum_{\alpha \in \mathcal{J}_i} |\frac{\partial^\alpha u}{\partial x^\alpha}(x^{(i-1)}) - P^{(i)}_{\alpha,x^{(i)}}(y_i)||\frac{(y-x)^\alpha}{\alpha!}| \\
\le & \sum_{i=1}^n \sum_{\alpha \in \mathcal{J}_i} M|y_i - x_i|^{\gamma-|\alpha|}\frac{|y-x|^{|\alpha|}}{\alpha!} \\
\le & (\sum_{|\alpha| \le \mathring{\gamma}} \frac{1}{\alpha!})M|y - x|^\gamma \\
:= & C(n, \mathring{\gamma})M|y - x|^\gamma.
\end{array}
$$

where $x^{(i)} = (x_1, \cdots, x_i, y_{i+1}, \cdots, y_n)$ (note that $x^{(0)} = y$, $x^{(n)} = x$, and $x^{(i)} + (y_i - x_i)e_i = x^{(i-1)}$). Thus, for each $x \in D_{l_1,\cdots,l_n}$ and any rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ containing $x$ with $\max_{1 \le i \le n}(b_i - a_i) < \min_{1 \le i \le n} l_i^{-1}$,

$$
\begin{array}{rl}
& |\{y \in D^P_M : |u(y) - P_x(y)| \le C(n, \mathring{\gamma})M|y - x|^\gamma\} \cap R| \\
\ge & \int_{D_{n,x^{(n)}} \cap \pi_{n,x^{(n)}}(D_{l_1,\cdots,l_{n-1}}) \cap [a_n,b_n]} \cdots \int_{D_{2,x^{(2)}} \cap \pi_{2,x^{(2)}}(D_{l_1}) \cap [a_2,b_2]} \int_{D_{1,x^{(1)}} \cap [a_1,b_1]} dy_1 dy_2 \cdots dy_n \\
\ge & \tau^n(b_n - a_n) \cdots (b_1 - a_1) \\
= & \tau^n|R|,
\end{array}
$$

since $y_i \in \pi_{i,x^{(i)}}(D_{l_1,\cdots,l_{i-1}})$ implies $x^{(i-1)} \in D_{l_1,\cdots,l_{i-1}}$. Hence for each $x$ belonging $\cap_{t \in \mathbb{N}} \cup_{l_1,\cdots,l_n \in \mathbb{N}} D^{1-t^{-1}}_{l_1,\cdots,l_n}$, we have

$$ap \limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|^\gamma} \le C(n, \mathring{\gamma})M < e^n M := C_n M.$$

Thus (4.1) holds a.e., because the complement in $D^P_M$ of $\cap_{t \in \mathbb{N}} \cup_{l_1,\cdots,l_n \in \mathbb{N}}$ $D^{1-t^{-1}}_{l_1,\cdots,l_n}$ is a null set. □

We shall need the following lemma given by De Giorgi in [2].

**Lemma 8** ([2]). *Let $\sigma$ be a positive real number less than or equal the volume of unit ball in $\mathbb{R}^n$. Then there exists a constant $C$ depending only on $n$, $k$ and $\sigma$ such that for each $x \in \mathbb{R}^n$, $r > 0$ and any measurable set $E \subset B_r(x)$ with $|E| \geq \sigma r^n$, the inequality*

$$|D^\alpha p(x)| \leq \frac{C}{r^{n+|\alpha|}} \int_E |p(y)| dy$$

*holds for each polynomial $p$ with $\deg p \leq k$ and each muilti-index $\alpha$.*

We are going to prove Theorem 5 by showing that the sequence of implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) holds. The implication (1)$\Rightarrow$(3) follows from the fact that a measurable set has linear density 1 in each direction $e_i$ at almost all points of the set, and the following formula of Taylor remainder of a $C^{\mathring{\gamma}}$ function $v$:

$$\begin{aligned} & v(y) - \textstyle\sum_{|\alpha| \leq \mathring{\gamma}} \frac{1}{\alpha!} \frac{\partial^\alpha v}{\partial x^\alpha}(x)(y-x)^\alpha \\ = \; & \mathring{\gamma} \textstyle\sum_{|\alpha| = \mathring{\gamma}} \frac{(y-x)^\alpha}{\alpha!} \int_0^1 (1-t)^{\mathring{\gamma}-1} \left( \frac{\partial^\alpha v}{\partial x^\alpha}(x + (y-x)t) - \frac{\partial^\alpha v}{\partial x^\alpha}(x) \right) dt. \end{aligned}$$

The implication (3)$\Rightarrow$(2) follows from Lemma 7, if one notes that the set of points in $D$ where (1.3) holds can be expressed as the following union of sets:

$$\cup_{M \in \mathbb{N}} \{x \in D : ap \limsup_{y \to x} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma} < C_n M\},$$

and the set of points in $D$ where $u$ is partially approximately Lipschitz continuous of order $\gamma$ is also the union of $D_M^P$ over $M \in \mathbb{N}$.

For the implication (2) $\Rightarrow$ (1), we write $P_x(y) = \sum_{|\alpha| \leq \mathring{\gamma}} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha$ and fix $\varepsilon > 0$. There exists a sequence of compact sets $\{K_j\}_{j \in \mathbb{N}}$ in $\mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \cup_{j \in \mathbb{N}} K_j| < \varepsilon$ and $dist(K_{j_0}, \cup_{j \neq j_0} K_l) > 0$ for each $j_0 \in \mathbb{N}$. Thus by considering the intersection of $K_j$ and $D$ for each $j \in \mathbb{N}$, we may assume that $|D| < \infty$. There exists $M > 0$ such that the complement of $\{x \in D : ap \limsup_{y \to x} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma} < M\}$ in $D$ has measure strictly less than $\varepsilon$. Let $\sigma$ be the ratio of $|B_1(0) \cap B_1(e_1)|$ to $|B_1(0)|$ and $\tau = (2+\sigma)/4 \in (0,1)$ (note that $\sigma$ is a number depending only on $n$). For $l \in \mathbb{N}$, let

$$D_l := \{x \in D : |E_x \cap B_r(x)| \geq \tau |B_r(x)| \ \forall \ 0 < r < 1/l\},$$

where $E_x := \{y \in D : |u(y) - P_x(y)| \leq M|y-x|^\gamma\}$. $D_l$ is an increasing sequence of measurable subsets of $D$ by Lemma 1, and $|D \setminus \cup_{l \in \mathbb{N}} D_l| = 0$.

We can choose $l_0 \in \mathbb{N}$ and a compact subset $K$ of $D_{l_0}$ such that $|D\backslash K| < \varepsilon$. For $x_1, x_2 \in K$ with $0 < r := |x_1 - x_2| < 1/l_0$, consider the set $E := E_{x_1} \cap B_r(x_1) \cap E_{x_2} \cap B_r(x_2)$, then $|E| \geq (1-\sigma/2)|B_r(0)|$. By Lemma 8 with $k = \overset{\circ}{\gamma}$ and $p = P_{x_1} - P_{x_2}$,

$$\begin{aligned}
& |u_\alpha(x_1) - \textstyle\sum_{|\beta|\leq k-|\alpha|} \frac{u_{\alpha+\beta}(x_2)}{\beta!}(x_2 - x_1)^\beta| \\
= \; & |D_y^\alpha(P_{x_1}(y) - P_{x_2}(y))|_{y=x_1}| \\
\leq \; & \tfrac{C}{r^{n+|\alpha|}} \textstyle\int_E |P_{x_1}(y) - P_{x_2}(y)|dy \\
\leq \; & \tfrac{C}{r^{n+|\alpha|}} \{\textstyle\int_E |P_{x_1}(y) - u(y)|dy + \int_E |u(y) - P_{x_2}(y)|dy\} \\
\leq \; & \tfrac{CM}{r^{n+|\alpha|}} \{\textstyle\int_{B_r(x_1)} |x_1 - y|^\gamma dy + \int_{B_r(x_2)} |y - x_2|^\gamma dy\} \\
\leq \; & CM|x_1 - x_2|^{\gamma-|\alpha|},
\end{aligned}$$

for each $\alpha$ with $|\alpha| \leq \overset{\circ}{\gamma}$. Since $K$ is compact, we can cover $K$ by finitely many balls with radius $l_0^{-1}$, hence there is $\widetilde{M} > 0$ such that $|u_\alpha(x_1) - \sum_{|\beta|\leq k-|\alpha|} \frac{u_{\alpha+\beta}(x_2)}{\beta!}(x_2 - x_1)^\beta| \leq \widetilde{M}|x_1 - x_2|^{\gamma-|\alpha|}$ for any $x_1, x_2 \in K$. (2)⇒(1) then follows by Whitney's Extension Theorem, concluding the proof of Theorem 5.

**Remarks.**

1. Since the multi-indices $\alpha$ used in the proof of Lemma 7 are those in $\cup_{1\leq i\leq n}\mathcal{J}_i$, it follows that (2) holds under weaker assumptions than those stated in (3). For example, when $1 < \gamma \leq 2$, (1.3) holds a.e. on $D$ if and only if, for almost every $x \in D$, the approximate partial derivative $u_{e_i} := \frac{\partial u}{\partial x_i}$ exists for $1 \leq i \leq n$, and satisfies
$$ap \limsup_{y_i \to x_i} \frac{|u_{e_i}(x + (y_j - x_j)e_j) - u_{e_i}(x)|}{|y_j - x_j|^{\gamma-1}} < \infty$$
for all $1 \leq i < j \leq n$, and $u$ satisfies
$$ap \limsup_{y_i \to x_i} \frac{|u(x + (y_i - x_i)e_i) - u(x) - u_{e_i}(x)(y_i - x_i)|}{|y_i - x_i|^\gamma} < \infty$$
for each $1 \leq i \leq n$. In general, (1.3) holds a.e. on $D$ if and only if there is a permutation $\phi$ of $\{1, 2, \cdots, n\}$ such that for a.e. $x \in D$,

   (1) for each multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $|\alpha| \leq \overset{\circ}{\gamma}$, the approximate partial derivative
   $$u_\alpha := \frac{\partial^{\alpha_{\phi(n)}}}{\partial x_{\phi(n)}^{\alpha_{\phi(n)}}} \left( \frac{\partial^{\alpha_{\phi(n-1)}}}{\partial x_{\phi(n-1)}^{\alpha_{\phi(n-1)}}} \left( \cdots \left( \frac{\partial^{\alpha_{\phi(1)}} u}{\partial x_{\phi(1)}^{\alpha_{\phi(1)}}} \right) \right) \right)$$

exists; and

(2) for each $|\alpha| \leq \overset{\circ}{\gamma}$ with $\alpha_{\phi(n)} = 0$,

$$ap \limsup_{y_i \to x_i} \frac{|u_\alpha(x + (y_i - x_i)e_i) - P^{(i)}_{\alpha,x}(y_i)|}{|y_i - x_i|^{\gamma - |\alpha|}} < \infty \tag{4.2}$$

holds for $i = \phi(j)$ with $\alpha_{\phi(j)} = \alpha_{\phi(j+1)} = \cdots = \alpha_{\phi(n)} = 0$.

2. In the implication (2)⇒(1), the function $v$ for (1) can be chosen such that

$$M_\mu(v) \leq C M_\mu(\{u_\alpha\}, K),$$

where $k = \overset{\circ}{\gamma}$ in the definition of $M_\mu(v)$ and $M_\mu(\{u_\alpha\}, K)$.

For a measurable function $u$ on $D$, set $\omega_u(\lambda) = |\{x \in D : |u(x)| > \lambda\}|$ for $\lambda \geq 0$ and let $M_0(D)$ be the class of measurable functions $u$ on $D$ such that $\lim_{\lambda \to \infty} \omega_u(\lambda) = 0$. If $u \in M_0(D)$, the nonincreasing rearrangement $u^*$ of $u$ is defined as $u^*(t) = \sup\{\lambda : \omega_u(\lambda) > t\}$. With the notations defined above, we have the following corollary of Theorem 5.

**Corollary 3.** *Suppose that $u \in Lip_{ap}(\gamma, D)$ with the property that both $u$ and $L(x) := ap \limsup_{y \to x} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma}$ belong to $M_0(D)$. Then for each $\varepsilon > 0$, there is $v \in C^{\overset{\circ}{\gamma},\mu}(\mathbb{R}^n)$ whose norm is dominated by $Cu^*(\varepsilon/3) \vee L^*(\varepsilon/3)$ such that*

$$|\{x \in D : u(x) \neq v(x)\}| < \varepsilon,$$

*where $C$ is a constant depending only on $\gamma$ and $n$.*

*Proof.* For each $\varepsilon > 0$, let $\tilde{D} = \{x \in D : u(x) \leq u^*(\varepsilon/3), L(x) \leq L^*(\varepsilon/3)\}$. Then $|D \backslash \tilde{D}| \leq 2\varepsilon/3$. The corollary follows by applying Theorem 5 on $\tilde{D}$ and remark 2 above. □

# 5 Applications of Theorem 5 and Some Remarks

We now formulate a generalization of a theorem of Currier [3] which is a direct application of Theorem 5.

**Theorem 8.** *Suppose that $u$ is a measurable function defined on a measurable set $D \subset \mathbb{R}^2$ whose approximate partial derivatives $\frac{\partial u}{\partial x_1}$, $\frac{\partial u}{\partial x_2}$ exist a.e.*

*on $D$. If for $i = 1, 2$,*

$$ap\limsup_{y_i \to x_i} \frac{|u(x + (y_i - x_i)e_i) - u(x) - \frac{\partial u}{\partial x_i}(x)(y_i - x_i)|}{|y_i - x_i|^2} < \infty, \tag{5.1}$$

*holds, and*

$$ap\limsup_{y_2 \to x_2} \frac{|\frac{\partial u}{\partial x_1}(x_1, y_2) - \frac{\partial u}{\partial x_1}(x)|}{|y_2 - x_2|} < \infty, \tag{5.2}$$

*a.e. on $D$, then all the second order approximate partial derivatives of $u$ exist and $\frac{\partial}{\partial x_1}(\frac{\partial u}{\partial x_2}) = \frac{\partial}{\partial x_2}(\frac{\partial u}{\partial x_1})$ a.e. on $D$.*

Suppose now that $u$ is a function defined on a subset of $\mathbb{R}^2$ containing a segment $L := (x_1 - r, x_1 + r) \times \{x_2\}$ for some $r > 0$. If the classical first-order partial derivative $\frac{\partial u}{\partial x_1}$ exists at each point in $L$ and $\frac{\partial^2 u}{\partial x_1^2}$ exists at $x = (x_1, x_2)$, then there exists $0 < \delta < r$ such that

$$\left| \frac{\partial u}{\partial x_1}(y_1, x_2) - \frac{\partial u}{\partial x_1}(x) - \frac{\partial^2 u}{\partial x_1^2}(x)(y_1 - x_1) \right| < |y_1 - x_1|$$

whenever $|y_1 - x_1| \le \delta$. Thus, $\frac{\partial u}{\partial x_1}(\cdot, x_2)$ is bounded on $I := [x_1 - \delta, x_1 + \delta]$, and hence $u(\cdot, x_2)$ is absolutely continuous on $I$. Therefore,

$$\begin{aligned} & |u(y_1, x_2) - u(x) - \tfrac{\partial u}{\partial x_1}(x)(y_1 - x_1)| \\ \le \;& |\textstyle\int_{x_1}^{y_1} (\frac{\partial u}{\partial x_1}(t, x_2) - \frac{\partial u}{\partial x_1}(x) - \frac{\partial^2 u}{\partial x_1^2}(x)(t - x_1))dt| + \frac{1}{2}|\frac{\partial^2 u}{\partial x_1^2}(x)||y_1 - x_1|^2 \\ \le \;& \tfrac{1}{2}(1 + |\tfrac{\partial^2 u}{\partial x_1^2}(x)|)|y_1 - x_1|^2 \end{aligned}$$

for each $|y_1 - x_1| < \delta$. That is, (5.1) holds true at $x$ for $i = 1$. Similar results also hold for $i = 2$. Thus, the following corollary holds by Theorem 8 (note that (5.2) holds by the existence of the approximate partial derivative $\frac{\partial^2 u}{\partial x_2 \partial x_1}$).

**Corollary 4.** *Let $u(x_1, x_2)$ be defined on an open subset $\mathcal{O}$ of $\mathbb{R}^2$, and let the first-order partial derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ exist on $\mathcal{O}$. Let $D$ be the measurable subset of $\mathcal{O}$ on which the second-order partial derivatives $\frac{\partial^2 u}{\partial x_1^2}$, $\frac{\partial^2 u}{\partial x_2^2}$, and the approximate partial derivative $\frac{\partial^2 u}{\partial x_2 \partial x_1}$ exist. Then the approximate partial derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ exists and equals $\frac{\partial^2 u}{\partial x_2 \partial x_1}$ almost everywhere on $D$.*

Clearly, Corollary 4 includes the following theorem of Currier [3] as a special case.

**Theorem 9** ([3])**.** *Let $u(x_1, x_2)$ be defined on an open subset $\mathcal{O}$ of $\mathbb{R}^2$, and let the first partial derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ exist on $\mathcal{O}$. Let $D$ be a measurable subset of $\mathcal{O}$ on which the four second-order partial derivatives $\frac{\partial^2 u}{\partial x_1^2}$, $\frac{\partial^2 u}{\partial x_1 \partial x_2}$, $\frac{\partial^2 u}{\partial x_2 \partial x_1}$, $\frac{\partial^2 u}{\partial x_2^2}$ exist almost everywhere. Then $\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1}$ almost everywhere on $D$.*

Theorem 5 is now applied to prove a generalization of Theorem 4 in [16]. First we quote a Lemma from [16].

**Lemma 9.** *Let $\phi$ be a function defined on an interval $I \subset \mathbb{R}$ whose derivatives exist up to order $k$ and its $k$-th order derivative is bounded on $I$. Then*

$$\phi(t_2) = \sum_{l=0}^{k} \frac{1}{l!} \frac{d^l \phi(t_1)}{dt^l}(t_2 - t_1)^l + \frac{1}{(k-1)!} \int_{t_1}^{t_2} (t_2 - t)^{k-1} \left(\frac{d^k \phi(t)}{dt^k} - \frac{d^k \phi(t_1)}{dt^k}\right) dt$$

*for $t_1, t_2 \in I$.*

The following corollary is a generalization of Theorem 4 in [16] which is used to prove (1)⇒(2) and (3)⇒(2) by induction on $k$ in [6] and [8] respectively.

**Theorem 10.** *Suppose that $\mathcal{O}$ is an open set in $\mathbb{R}^n$, and $u$ is a function defined on $\mathcal{O}$ such that the partial derivatives $\frac{\partial^\alpha u}{\partial x^\alpha}$ exist up to order $\mathring{\gamma}$ at every point of $\mathcal{O}$ and for almost every $x \in \mathcal{O}$,*

$$M_{\alpha,i}(x) := \limsup_{y_i \Rightarrow x_i} \frac{|\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i) - \frac{\partial^\alpha u}{\partial x^\alpha}(x)|}{|y_i - x_i|^{\gamma - |\alpha|}} < \infty \qquad (5.3)$$

*for each $|\alpha| = \mathring{\gamma}$ and $1 \le i \le n$. Then $u \in LC^{(\gamma)}(\mathcal{O})$.*

*Proof.* According to Theorem 5, it is sufficient to prove that (1.4) $|\alpha| \le \mathring{\gamma}$ for almost every point in $\mathcal{O}$. Suppose $x$ is a point in $\mathcal{O}$ at which (5.3) holds. Then, for each $|\alpha| \le \mathring{\gamma}$ and $1 \le i \le n$, there exists $\delta > 0$ such that $x + (y_i - x_i)e_i \in \mathcal{O}$ and

$$\left| \frac{\partial^{\alpha + (\mathring{\gamma} - |\alpha|)e_i} u}{\partial x^{\alpha + (\mathring{\gamma} - |\alpha|)e_i}}(x + (y_i - x_i)e_i) - \frac{\partial^{\alpha + (\mathring{\gamma} - |\alpha|)e_i} u}{\partial x^{\alpha + (\mathring{\gamma} - |\alpha|)e_i}}(x) \right| < M_{\alpha,i}(x) + 1$$

whenever $|y_i - x_i| < \delta$. Thus, $\frac{\partial^\alpha u}{\partial x^\alpha}(x + (y_i - x_i)e_i)$ is a function on $y_i \in (x_i - \delta, x_i + \delta) := I$ whose derivatives exist up to order $\mathring{\gamma} - |\alpha|$ and its

$(\mathring{\gamma}-|\alpha|)$-th order derivative is bounded on $I$. By Lemma 9,

$$\begin{aligned} & |\tfrac{\partial^\alpha u}{\partial x^\alpha}(x+(y_i-x_i)e_i)-P^{(i)}_{\alpha,x}(y_i)| \\ = & \tfrac{1}{(\mathring{\gamma}-|\alpha|-1)!}|\int_{x_i}^{y_i}(y_i-t)^{\mathring{\gamma}-|\alpha|-1}(\tfrac{\partial^{\alpha+(\mathring{\gamma}-|\alpha|)e_i}u}{\partial x^{\alpha+(\mathring{\gamma}-|\alpha|)e_i}}(x+(t-x_i)e_i)-\tfrac{\partial^{\alpha+(\mathring{\gamma}-|\alpha|)e_i}u}{\partial x^{\alpha+(\mathring{\gamma}-|\alpha|)e_i}}(x))dt| \\ \leq & \tfrac{1}{(\mathring{\gamma}-|\alpha|-1)!}|\int_{x_i}^{y_i}|y_i-t|^{\mathring{\gamma}-|\alpha|-1}(M_{\alpha,i}(x)+1)dt| \\ = & \tfrac{M_{\alpha,i}(x)+1}{(\mathring{\gamma}-|\alpha|)!}|y_i-x_i|^{\mathring{\gamma}-|\alpha|} \end{aligned}$$

for $|y_i-x_i|<\delta$. That is, (1.4) holds at $x$. Thus $u$ is partially approximately Lipschitz continuous of order $\gamma$ at almost every point of $\mathcal{O}$, hence $u\in LC^{(\gamma)}(\mathcal{O})$. □

The following corollary is an application of Theorem 10 and Corollary 3, and it will be used to deal with the case when $\gamma$ is a positive integer (cf. [5]).

**Corollary 5.** *A $C^{k,1}$-function defined on an open set $\mathcal{O}\subset\mathbb{R}^n$ belongs to $LC^{(k+1)}(\mathcal{O})$. Moreover, for each $\varepsilon>0$, there is a $C^{k+1}$-function $v$ on $\mathbb{R}^n$ such that $|\{x\in\mathcal{O}:u(x)\neq v(x)\}|<\varepsilon$ and*

$$\|v\|_{C^{k+1}(\mathbb{R}^n)}\leq C\|u\|_{C^{k,1}(\mathcal{O})}.$$

*for some constant $C=C(n,k)$.*

Lien and Liu [5] provided a sufficient condition for a measurable function $u$ to have Lusin property of order $\gamma$. The result can be improved slightly by applying Corollary 4. We start the discussion by introducing some definitions from [5]. Suppose that $\mathcal{O}$ is an open set in $\mathbb{R}^n$ satisfying $A$-condition, that is, there is a constant $A>0$ such that $|\mathcal{O}(x,r)|\geq Ar^n$ for all $x\in\mathcal{O}$ and $0<r\leq 1$ where $\mathcal{O}(x,r):=\mathcal{O}\cap B(x,r)$. The space of all measurable function $u$ on $\mathcal{O}$ which is integrable on each bounded measurable subset of $\mathcal{O}$ will be denoted by $L^1_b(\mathcal{O})$.

**Definition 4.** *Let $\gamma$ to be a positive integer, and $\mathcal{L}^\gamma(\mathcal{O})$ be class of all those functions $u\in L^1_b(\mathcal{O})$ such that*

1. *for almost all $x\in\mathcal{O}$, there is a polynomial $T_x(\cdot)$ with degree $\leq\gamma$ satisfying*

$$[u]_\gamma(x):=\sup_{0<r\leq 1}r^{-\gamma}\frac{1}{\mathcal{O}(x,r)}\int_{\mathcal{O}(x,r)}|u(y)-T_x(y)|dy<+\infty;$$

*2. if we set*

$$\sigma_u(x) := [u]_\gamma(x) + \int_{\mathcal{O}(x,1)} |u(y)|dy,$$

*then $\sigma_u$ is in $M_0(\mathcal{O})$.*

Suppose $u \in \mathcal{L}^\gamma(\mathcal{O})$. For those $x \in \mathcal{O}$ at which 1 holds, denote

$$T_x(y) := \sum_{|\alpha| \le \gamma} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha.$$

Then for each $|\alpha| \le \gamma - 1$, $u_\alpha$ is uniquely defined and measurable. To see this, we need the following lemma of Calderon and Zygmund.

**Lemma 10.** *There exists $\phi \in C_0^\infty(\mathbb{R}^n)$ with supp$\phi \subset \overline{B(0,1)}$ such that for every polynomial $P$ on $\mathbb{R}^n$ of degree $\le \gamma$ and every $\varepsilon > 0$, $\phi_\varepsilon * P = P$, where $\phi_\varepsilon(x) := \varepsilon^{-n}(\varepsilon^{-1}x)$.*

Fix $x \in \mathcal{O}$, $u_\varepsilon := \phi_\varepsilon * u$ is defined for sufficiently small $\varepsilon > 0$. Then we have

$$D^\alpha u_\varepsilon(x) = D^\alpha T_x(x) + \int \varepsilon^{-n-|\alpha|} D^\alpha \phi(\frac{x-y}{\varepsilon})(u(y) - (T_x(y) - \sum_{|\alpha|=\gamma} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha))dy.$$

The above integral is dominated by

$$\begin{aligned} & C\varepsilon^{-n-|\alpha|} \int_{B(x,\varepsilon)} \{|u(y) - T_x(y)| + \sum_{|\alpha|=\gamma} \tfrac{|u_\alpha(x)|}{\alpha!} |y-x|^\gamma\} dy \\ \le \; & C\varepsilon^{-n-|\alpha|} \{[u]_\gamma(x)|B(x,\varepsilon)|\varepsilon^\gamma + \int_{B(x,\varepsilon)} (\sum_{|\alpha|=\gamma} \tfrac{1}{\alpha!}) \max_{|\alpha|=\gamma} |u_\alpha(x)|\varepsilon^\gamma dy\} \\ \le \; & C\varepsilon^{\gamma-|\alpha|} \to 0 \text{ as } \varepsilon \to 0 \text{ for } |\alpha| < \gamma. \end{aligned}$$

This shows that $u_\alpha(x) = D^\alpha T_x(x)$ is the limits of $D^\alpha u_\varepsilon(x)$ for $|\alpha| < \gamma$, hence is measurable and uniquely determined. In particular, $u(x) = T_x(x)$ at each Lebesgue point $x$ of $u$, i.e., $u = T_x = u_0$ a.e. on $\mathcal{O}$.

Note that from the definition of $\mathcal{L}^\gamma(\mathcal{O})$, $u_\alpha$, $|\alpha| = \gamma$, is not necessary measurable nor uniquely determined, for by adding a finite function on the highest coefficients, the assumption will not be violated. From now on , we suppose further that $u_\alpha$ is measurable for all $|\alpha| = \gamma$. Then $[u]_\gamma$ is approximately lower semi-continuous at $x \in \mathcal{O}$ which is a point of approximate continuity of all $u_\alpha$'s, therefore, $[u]_\gamma$ is measurable by a theorem of Kamke.

**Theorem 11.** *There exists a constant $C > 0$ depending only on $n$, $A$, and $\gamma$ such that for all $u \in \mathcal{L}^\gamma(\mathcal{O})$ whose $u_\alpha$ are measurable for all $|\alpha| = \gamma$, and $\varepsilon > 0$, there is a closed set $F_\varepsilon \subset \mathcal{O}$ and $v_\varepsilon \in C^\gamma(\mathbb{R}^n)$ such that*

1. $\mathcal{L}^n(\mathcal{O}\backslash F_\varepsilon) < 2\varepsilon$,
2. $v_\varepsilon \equiv u$, *and* $D^\alpha v_\varepsilon = u_\alpha$ *for* $|\alpha| \leq \gamma$ *on* $F_\varepsilon$, *and*

$$\max\{M_{1,v_\varepsilon}, M_{2,0,\varepsilon}\} \leq C\sigma_u^*(\varepsilon).$$

*Proof.* Fix $\varepsilon > 0$, let

$$\mathcal{O}_\varepsilon := \{x \in \mathcal{O} : \sigma_u(x) \leq \sigma_u^*(\varepsilon)\}.$$

Then $\mathcal{L}^n(\mathcal{O}\backslash\mathcal{O}_\varepsilon) \leq \varepsilon$. Choose a closed set $\tilde{F}_\varepsilon \subset \mathcal{O}_\varepsilon$ so that

$$\mathcal{L}^n(\mathcal{O}\backslash\tilde{F}_\varepsilon) < 2\varepsilon,$$

and $u = T_x = u_0$ on $F_\varepsilon$.

1. For $x \in \tilde{F}_\varepsilon$ and $|\alpha| \leq \gamma$, by Lemma 8, for some constant $C = C(n, A, \gamma)$,

$$\begin{aligned} |D^\alpha T_x(x)| &\leq C \int_{\mathcal{O}(x,1)} |T_x(y)|dy \\ &\leq C\{[u]_\gamma(x) + \int_{\mathcal{O}(x,1)} |u(y)|dy\} \\ &= C\sigma_u(x) \leq C\sigma_u^*(\varepsilon). \end{aligned}$$

   i.e., $M_1 \leq C\sigma_u^*(\varepsilon)$.

2. For $x, y \in \tilde{F}_\varepsilon$ and $|\alpha| \leq \gamma$. If $|x - y| < \frac{1}{2}$, we have

$$\begin{aligned} & |u_\alpha(y) - D^\alpha(T_x(y))| \\ = \ & |D_z^\alpha(T_y(z) - T_x(z))|_{z=y}| \\ \leq \ & \tfrac{C}{|y-x|^{n+|\alpha|}} \int_{\mathcal{O}(y,|y-x|)} |T_y(z) - T_x(z)|dy \\ \leq \ & \tfrac{C}{|y-x|^{n+|\alpha|}}\{\int_{\mathcal{O}(y,|y-x|)} |T_y(z) - u(z)|dz + \int_{\mathcal{O}(y,|y-x|)} |u(z) - T_x(z)|dy\} \\ \leq \ & \tfrac{C}{|y-x|^{n+|\alpha|}}\{\int_{\mathcal{O}(y,|y-x|)} |T_y(z) - u(z)|dz + \int_{\mathcal{O}(x,2|y-x|)} |u(z) - T_x(z)|dy\} \\ \leq \ & \tfrac{C}{|y-x|^{n+|\alpha|}}\{[u]_\gamma(y)|y-x|^\gamma\mathcal{L}^n(\mathcal{O}(y, |y-x|)) + [u]_\gamma(x)(2|y-x|)^\gamma\mathcal{L}^n(\mathcal{O}(x, 2|y-x|))\} \\ \leq \ & \tfrac{C}{|y-x|^{n+|\alpha|}}\{[u]_\gamma(y)|y-x|^\gamma\mathcal{L}^n(B(y, |y-x|)) + [u]_\gamma(x)(2|y-x|)^\gamma\mathcal{L}^n(B(x, 2|y-x|))\} \\ \leq \ & C|y-x|^{\gamma-|\alpha|}\{[u]_\gamma(y) + [u]_\gamma(x)\} \\ \leq \ & C|y-x|^{\gamma-|\alpha|}\sigma_u^*(\varepsilon). \end{aligned}$$

On the other hand, if $|y-x| \geq \frac{1}{2}$,

$$\begin{aligned} & |u_\alpha(y) - D^\alpha(T_x(y))| \\ \leq\ & |u_\alpha(y)| + \sum_{|\beta|\leq\gamma-|\alpha|} \tfrac{|u_{\alpha+\beta(x)}|}{\beta!}|y-x|^{|\beta|} \\ \leq\ & \sigma_u^*(\varepsilon)(1 + \sum_{|\beta|\leq\gamma-|\alpha|} |y-x|^{|\beta|}) \\ \leq\ & C\sigma_u^*(\varepsilon)(\tfrac{|y-x|^{\gamma-|\alpha|}}{|y-x|^{\gamma-|\alpha|}} + \sum_{|\beta|\leq\gamma-|\alpha|} \tfrac{|y-x|^{\gamma-|\alpha|}}{|y-x|^{\gamma-|\alpha|-|\beta|}}) \\ \leq\ & C\sigma_u^*(\varepsilon)(2^{\gamma-|\alpha|} + \sum_{|\beta|\leq\gamma-|\alpha|} 2^{\gamma-|\alpha|-|\beta|})|y-x|^{\gamma-|\alpha|} \\ \leq\ & C\sigma_u^*(\varepsilon)|y-x|^{\gamma-|\alpha|}. \end{aligned}$$

i.e., for each $z \in F_\varepsilon$ and $|\alpha| \leq \gamma - 1$,

$$\lim_{x,y\to z; x,y\in F_\varepsilon} \frac{|u_\alpha - D^\alpha(T_x(y))|}{|y-x|^{\gamma-1-|\alpha|}} = 0,$$

and $M_{2,1} \leq C\sigma_u^*(\varepsilon)$.

Thus, $u|_{F_\varepsilon}$ can be extended to a $C^{\gamma-1,1}$ function $\tilde{v}_\varepsilon$ on $\mathbb{R}^n$ with

$$\max\{M_{1,\tilde{v}_\varepsilon}, M_{2,1,\tilde{v}_\varepsilon}\} \leq C\sigma_u^*(x).$$

Finally, apply Corollary 4 to find $v_\varepsilon$ and $\hat{F}_\varepsilon$ with $\mathcal{L}^n(\mathbb{R}\setminus\hat{F}_\varepsilon) < 2t - \mathcal{L}^n(\mathcal{O}\setminus\tilde{F}_\varepsilon)$ and the estimate

$$\max\{M_{1,v_\varepsilon}, M_{2,1,v_\varepsilon}\} \leq C\sigma_u^*(x).$$

Simply let $F_\varepsilon = \tilde{F}_\varepsilon \bigcap \hat{F}_\varepsilon$ to conclude the proof. □

Finally, we give some remarks to conclude this thesis.

1. By modifying slightly the proof of Theorem 5, one can see that the following statements are equivalent:

   (1) For each $\varepsilon > 0$, there is a function $v \in C^\gamma_{loc}(\mathbb{R}^n)$ with

   $$\lim_{y\to x} \frac{|\frac{\partial^\alpha v}{\partial x^\alpha}(y) - \frac{\partial^\alpha v}{\partial x^\alpha}(x)|}{|y-x|^\mu} = 0$$

   for all $x \in \mathbb{R}^n$ and $|\alpha| = \mathring{\gamma}$ such that $|\{x \in D : u(x) \neq v(x)\}| < \varepsilon$.

(2) For almost all point $x \in D$, there exists a polynomial $P_x$ centered at $x$ with degree $\leq \overset{\circ}{\gamma}$ such that

$$ap \lim_{y \to x} \frac{|u(y) - P_x(y)|}{|y - x|^{\gamma}} = 0. \tag{5.4}$$

(3) For almost all point $x \in D$, the unbiased approximate partial derivatives of $u$ up to order $\overset{\circ}{\gamma}$ exist, and

$$ap \lim_{y_i \to x_i} \frac{|\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x + (y_i - x_i)e_i) - P^{(i)}_{\alpha,x}(y_i)|}{|y_i - x_i|^{\gamma - |\alpha|}} = 0 \tag{5.5}$$

for each $1 \leq i \leq n$ and $|\alpha| \leq \overset{\circ}{\gamma}$, where

$$P^{(i)}_{\alpha,x}(y_i) = \sum_{l=0}^{\overset{\circ}{\gamma} - |\alpha|} \frac{1}{l!} \frac{\partial^{\alpha + l e_i} u}{\partial x_i^l \partial x^{\alpha}}(x)(y_i - x_i)^l.$$

To illustrate the modification of the proof, we show (3)⇒(2) as follows. Since

$$\{x \in D : (5.4) \text{ holds at } x\} = \cup_{m \in \mathbb{N}} \{x \in D : ap \lim_{y \to x} \frac{|u(y) - P_x(y)|}{|y - x|^{\gamma}} < C_n/m\},$$

and

$$\{x \in D : (5.5) \text{ holds at } x \ \forall 1 \leq i \leq n, |\alpha| \leq \overset{\circ}{\gamma}\} = \cup_{m \in \mathbb{N}} D^P_{1/m},$$

the implication follows from Lemma 7, where $C_n$, $P_x(y)$, and $D^P_{1/m}$ are as in the lemma.

2. As we have indicated that statement (2) in Theorem 4 can be replaced by statement (2)′ in the remark following the theorem, it is easy to see that Theorem 5 reduces to Theorem 4 when $\gamma = k$, hence Theorem 5 is a generalization of Theorem 4 as we claim.

3. Ziemer proved a Rademacher type theorem (Theorem 3.8.1 in [17]) in $L^p$-context (see also [7]), the equivalence of (1) and (3) in Theorem 4 is the corresponding Rademacher type theorem in the context of approximate limit and approximate supremum.

## References


[1] A. P. Calderon, A. Zygmund, *Local Properties of Solutions of Elliptic Partial Differential Equations*, Studia Math. 20 (1961), 171-225.

[2] S. Campanato, *Proprietá di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa 18 (1964), 137-160.

[3] A. E. Currier, *Proof of the fundamental theorems on second order cross partial derivatives*, Trans. Amer. Math. Soc. 35 (1933), 245-253.

[4] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969.

[5] W. C. Lien and F. C. Liu, *Maximal mean estimates of Taylor remainder*, Bull. Inst. Math. Acad. Sinica, 29(2001), 79-97.

[6] F. C. Liu, *On a theorem of Whitney*, Bull. Inst. Math. Acad. Sinica 1 (1973), 63-70.

[7] F. C. Liu, *A Localized Lusin Theorem and a Rademacher Type Theorem*, Bull. Inst. Math. Acad. Sinica Vol. 3, No. 2 (2008), 243-253

[8] F. C. Liu, W. S. Tai, *Approximate Taylor polynomials and differentiation of functions*, Topol. Methods Nonlinear Anal. 3 (1994), 189-196.

[9] F. C. Liu, W. S. Tai, *Maximal Mean Steepness and Lusin Type Properties*, Ricerche di Matem., XLIII(1994), 365-384.

[10] F. C. Liu, W. S. Tai, *Lusin Properties and Interpolation of Sobolev Spaces*, Topo. Meth. Nonlin. Ana. Vol. 9 (1997), 163-177.

[11] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, 1966.

[12] S. Saks. *Theory of the integral*, Second Edition, Monografje Matemacyczne, Warszawalwow, 1937.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[14] W. Stepanoff, *Sur les conditions de l'existence de la différentielle totale*, Rec. Math. Soc. Math. Moscou 32 (1925) 511-526.

[15] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63-89.

[16] H. Whitney, *On totally differentiable and smooth functions*, Pacific J. Math. 1 (1951), 143-159.

[17] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, 1989.