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麥克斯威爾電磁系統的強唯一連續延拓性及其定量分析

Quantitative uniqueness estimate of strong unique continuation property for the Maxwell system with anisotropic media

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中文摘要

在這篇文章中，我們考慮非零解在時諧性麥克斯威爾系統的局部行為，其系統為非等向性的媒體。而我們主要得到的結果是此系統的強連續延拓性在某些條件之下將會成立，並且導出強連續延拓性的定量分析，也可以得到非零解趨近到零的速度。

我們主要運用到的工具為 *Carleman* 估計導出 Three-balls 不等式,再運用另一個 *Carleman* 估計以及 Three-balls 不等式推導出 Doubling 不等式，因此可得出強連續延拓性的定量分析。

中文關鍵字：卡勒門估計，麥克斯威爾系統，非等向性，強唯一連續延拓性

Abstract

In this article, we consider the local behavior of a non-trivial solution for the time-harmonic Maxwell system with anisotropic media. The main result of this article is the bound on the vanishing order of the solution of the Maxwell system, which is a quantitative estimate of the strong unique continuation property(SUCP). And the most important tool is *Carleman* estimate. Our strategy in the proof is to derive doubling inequality through three-balls inequality.

Key words: Carleman estimate, Maxwell system, anisotropic media, strong unique continuation

目　錄

Quantitative uniqueness estimate of strong unique continuation property for the Maxwell system with anisotropic media

1. Introduction

The Maxwell system is firstly mentioned by *James Clerk Maxwell* in the paper "On Physical Lines of Force" which is published in 1861. He derived it from *Gauss's law*, *Faraday's law* and *Ampre's circuital law*. Furthermore, he derived electromagnetic wave equations in 1865 and claim that light is an electromagnetic wave. In fact, he established the fundamental electrodynamics and had a significant impact on modern physics.

1. *Gauss's law*: The total electric flux coming out of a closed surface is equal to the total charge enclosed by that closed surface. It means that

$$\oiint \epsilon_0 E \cdot dA = Q$$

2. *Gauss law* for the Magnetic Fields: The total magnetic flux coming out of a closed surface is always zero. It means that

$$\oiint \mu_0 H \cdot dA = 0$$

3. *Faraday's law*: The line integral of electric field over a closed contour is equal to the time rate of change of the total magnetic flux that goes through any arbitrary surface that is bounded by the closed contour. It means that

$$\oint E \cdot ds = \frac{\partial}{\partial t} \iint \mu_0 H \cdot dA$$

4. *Ampre's circuital law*: The line integral of magnetic field over a closed contour is equal to the total current plus the time rate of change of the total electric

flux that goes through any arbitrary surface that is bounded by the closed contour. It means that

$$\oint H \cdot ds = \iint J \cdot dA + \frac{\partial}{\partial t} \iint \epsilon_0 E \cdot dA$$

where E is the electric field, H is the magnetic field, J is the current density, Q is the total charge, ϵ_0 is the permittivity of vacuum, and μ_0 is the permeability of vacuum.

Actually, Maxwell system can describe more complicated physical phenomenon in real life, so it has a general form, which is depending on the medium. Now we assume $J = 0$ to simplify the problem.

So we can define that $E = (E_1, E_2, E_3)$ is the electric field, $H = (H_1, H_2, H_3)$ is the magnetic field and ω is the frequency in a domain Ω. Denote the time-harmonic Maxwell system with anisotropic media

$$\begin{cases} curlE = -i\omega\mu H \\ curlH = i\omega\varepsilon E \end{cases} \quad \text{in } \Omega \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^3$ containing 0, $\omega \in \mathbb{C}\setminus\{0\}$,and $\varepsilon(x)$,$\mu(x)$ are two real symmetric matrix-valued and positive-definite functions in Ω satisfying the following property :

(a) $\varepsilon(0) = h\mu(0)$ where h is a constant.

(b) $\varepsilon,\mu \in \mathbf{C^2}\ (\Omega)$

We can reduce the Maxwell system to a weakly coupled second order elliptic system. Denote that

$$\gamma_{jl}^{k} = \begin{cases} 1 \ \ \text{, if (k,j,l) is an even permutation of (1,2,3)} \\ -1\text{, if (k,j,l) is an odd permutation of (1,2,3)} \\ 0 \ \ \text{, otherwise} \end{cases}$$

From the Maxwell system we can obtain that

$$\begin{cases} \partial_k E = \nabla E_k - i\omega\gamma^k \mu H \\ \partial_k H = \nabla H_k + i\omega\gamma^k \varepsilon E \end{cases}$$

By simple calculation and (1.1), we know that

$$\begin{cases} \operatorname{div}(curlH) = 0 \\ \operatorname{div}(curlE) = 0 \end{cases} \Rightarrow \begin{cases} \operatorname{div}(\varepsilon E) = 0 \\ \operatorname{div}(\mu H) = 0 \end{cases}$$

So we have that for $k = 1, 2, 3$, the following formulas is called (1.2)

$$\begin{cases} 0 = \partial_k div(\varepsilon E) = div(\varepsilon \nabla E_k) + div(\partial_k \varepsilon \cdot E - i\omega\varepsilon\gamma^k \mu H) \\ 0 = div(\mu \nabla H_k) + div(\partial_k \mu \cdot H + i\omega\mu\gamma^k \varepsilon E) \end{cases}$$

Now let $P(x, D) = \sum_{j,k} a_{jk}(x) D_j D_k$ be an elliptic operator in Ω such that $a_{jk}(0)$ is a symmetric and positive-definite matrix and $a_{jk}(x) \in \mathbf{C^2}\ (\Omega)$, so we can rewrite (1.2)

$$\begin{cases} P_1(x, D)E + 2\nabla E \cdot div\varepsilon + E \cdot \tilde{\varepsilon} - \sum_{k=1}^{3} div(i\omega\varepsilon\gamma^k \mu H) = 0 \\ P_2(x, D)H + 2\nabla H \cdot div\mu + E \cdot \tilde{\mu} + \sum_{k=1}^{3} div(i\omega\mu\gamma^k \varepsilon E) = 0 \end{cases}$$

where

$$P_1(x, D) = \sum_{i,j=1}^{3} \varepsilon_{ij}(x) D_i D_j \ ,\ P_2(x, D) = \sum_{k,l=1}^{3} \mu_{kl}(x) D_k D_l,$$

$$\tilde{\varepsilon} = \sum_{m,n=1}^{3} D_m D_n \varepsilon(x) \ ,\ \tilde{\mu} = \sum_{m,n=1}^{3} D_m D_n \mu(x)$$

So it implies that

$$\begin{cases} |P_1(x, D)E| \le \alpha_1 |E| + \alpha_2 |\nabla E| + \alpha_3 |\nabla H| \le \alpha_4 |U| + \alpha_5 |\nabla U| \\ |P_2(x, D)H| \le \beta_1 |H| + \beta_2 |\nabla H| + \beta_3 |\nabla E| \le \beta_4 |U| + \beta_5 |\nabla U| \end{cases} \tag{1.3}$$

where $U = (E, H)$ is the non-trivial solution for the (1.1), α_i, β_i are constants for $i = 1, 2, 3, 4, 5$, and by (1.3) we can assume that $M_1 = max\{\alpha_4, \alpha_5\}$ and $M_2 = max\{\beta_4, \beta_5\} \Rightarrow$

$$\begin{cases} |P_1(x, D)E| \le M_1 |U| + M_2 |\nabla U| \\ |P_2(x, D)H| \le M_1 |U| + M_2 |\nabla U| \end{cases} \tag{1.4}$$

2. The main theorems

Theorem 1 There exists a positive number $R_1 < 1$ such that if $0 < r_1 < r_2 < r_3 \leq R_0$ and $r_1/r_3 < r_2/r_3 < R_1$ then

$$\int_{|x|<r_2} |U|^2 dx \leq C \left(\int_{|x|<r_1} |U|^2 dx \right)^{\tau} \left(\int_{|x|<r_3} |U|^2 dx \right)^{1-\tau}$$

for $U = (E, H) \in (L^2(B_{R_0}))^6$ where $B_{R_0} \subset \Omega$ and U is the non-trivial solution for the (1.1), where C depend on r_1/r_3, r_2/r_3 ,$P_1(x, D)$ and $P_2(x, D)$ and $0 < \tau < 1$ is only depending on r_1/r_3, r_2/r_3.

And then we want to show the quantitative estimate of strong unique continuation property for the Maxwell system. The strong unique continuation means that

For all $U = (E, H) \in H^1_{loc}(\Omega)$ vanishes of infinite order at 0, then U=0 in Ω

Theorem 2 gives the upper bound on the vanishing order of the solution of the Maxwell system, and theorem 3 is the quantitative estimate of strong unique continuation property.

Theorem 2 If $U = (E, H) \in (L^2_{loc}(\Omega))^6$ is a non-trivial solution of Maxwell system,

then we can find a constant R_2 depending on $P_1(x, D)$, $P_2(x, D)$ and constant m_1 depending on $P_1(x, D)$, $P_2(x, D)$ and $||U||_{L^2(|x|<R_2^2)}/||U||_{L^2(|x|<R_2^4)}$ satisfying

$$\int_{|x|<R} |U|^2 dx \geq K R^{m_1}$$

where R is sufficient small and the constant K depending on R_2 , U.

Theorem 3 Let $U = (E, H) \in (L^2_{loc}(\Omega))^6$ be a non-trivial solution to the Maxwell system. Then there exists positive constant R_3 and C_3 depending on $P_1(x, D)$, $P_2(x, D)$ and m_1 such that if $0 < r \leq R_3$,

$$\int_{|x|<2r} |U|^2 dx \leq C_3 \int_{|x|<r} |U|^2 dx$$

where m_1 is the constant obtained in theorem 2.

3. Proofs

Proof of the theorem 1

First we denote that $\varphi_\beta = \varphi_\beta(|x|) = exp((\frac{\beta}{2})(log|x|)^2)$ and recall a Carleman estimate [1]:

Lemma For any $\beta > 0$ large enough. Let S be a small neighborhood of 0, and $u : S\backslash\{0\} \subset \Omega \to \mathbb{R}$ and that $u \in H^2(S\backslash\{0\})$ with compact support. Then we have

$$\beta^3 \int \varphi_\beta^2 |x|^{-n}|u|^2 dx + \beta \int \varphi_\beta^2 |x|^{-n+2}|\nabla u|^2 dx \le \tilde{C}_0 \int \varphi_\beta^2 |x|^{-n+4}|P(x,D)u|^2 dx \tag{1.5}$$

for some positive constant $\tilde{C}_0$ depending only on $P(x,D)$. Now ε, μ are $\mathbf{C^2}$ functions, and $U = (E,H) \in (L^2_{loc}(\Omega))^6$ then $U = (E,H) \in (H^1_{loc}(\Omega))^6$ [2] and using regularization, *Friedrich's Lemma* and ellipticity of $P(x,D)$. We can see that $U = (E,H) \in (H^2_{loc}(\Omega\backslash\{0\}))^6$.

Consider that $0 < r_1 < r_2 < R < 1$, $B_R \subset \Omega$ where R is a constant. Define a cut-off function $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \phi(x) \le 1$ and

$$\phi(x) = \begin{cases} 0, \text{ if } |x| \le \frac{r_1}{e} \\ 1, \text{ if } \frac{r_1}{2} \le |x| \le er_2 \\ 0, \text{ if } |x| \ge 3r_2 \end{cases}$$

where $exp(1) = e$. And it is easy to know that for all multiindex α and C_1,C_2 are constants.

$$\begin{cases} |D^\alpha \phi(x)| \le C_1 r_1^{-|\alpha|}, \ \forall \ \frac{r_1}{e} \le |x| \le \frac{r_1}{2} \\ |D^\alpha \phi(x)| \le C_2 r_2^{-|\alpha|}, \ \forall \ er_2 \le |x| \le 3r_2 \end{cases} \tag{1.6}$$

We assume $n = 3$ in the lemma because of the domain $\Omega \in \mathbb{R}^3$ and then apply (1.5) to ϕE and ϕH. Firstly, we consider ϕE and use (1.4),(1.5),(1.6) and Cauchy-Schwarz inequality. We obtain

$$\beta^3 \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-3}|E|^2 dx + \beta \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-1}|\nabla E|^2 dx$$
$$\le \beta^3 \int \varphi_\beta^2 |x|^{-3}|\phi E|^2 dx + \beta \int \varphi_\beta^2 |x|^{-1}|\nabla \phi E|^2 dx$$

$$
\begin{aligned}
&\leq \tilde{C}_0 \int \varphi_\beta^2 |x||P_1(x,D)(\phi E)|^2 dx \\
&\leq \tilde{C}_0 \Big\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|(2M_1^2|\phi U|^2 + 2M_2^2|\phi \nabla U|^2) dx \\
&+ \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^{-3}(C_1|U|^2 + C_2|x|^2|\nabla U|^2) dx \\
&+ \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^{-3}(C_1|U|^2 + C_2|x|^2|\nabla U|^2) dx \Big\} \\
&\leq \tilde{C}_1 \Big\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 (|x|^{-3}|U|^2 + |x|^{-1}|\nabla U|^2) dx \\
&+ \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^{-3}(|U|^2 + |x|^2|\nabla U|^2) dx \\
&+ \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^{-3}(|U|^2 + |x|^2|\nabla U|^2) dx \Big\} \\
&\leq \tilde{C}_2 \Big\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 (|x|^{-3}|U|^2 + |x|^{-1}|\nabla U|^2) dx \\
&+ r_1^{-3}\varphi_\beta^2(r_1/e) \int_{r_1/e<|x|<r_1/2} (|U|^2 + |x|^2|\nabla U|^2) dx \\
&+ r_2^{-3}\varphi_\beta^2(er_2) \int_{er_2<|x|<3r_2} (|U|^2 + |x|^2|\nabla U|^2) dx \Big\}
\end{aligned} \tag{1.7}
$$

where $\tilde{C}_1 = max\{2\tilde{C}_0 M_1^2, 2\tilde{C}_0 M_2^2, \tilde{C}_0 C_1, \tilde{C}_0 C_2\}$ and $\tilde{C}_1 e^3 = \tilde{C}_2$

We introduce a corollary in [3]

Corollary For $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$, we can show the following inequality

$$
\int_{a_1 r<|x|<a_2 r} ||x|^{|\alpha|} D^\alpha u|^2 dx \leq C' \int_{a_3 r<|x|<a_4 r} |u|^2 dx
$$

where C' is a constant independent of r and $|\alpha| \leq 2$.

So by the corollary, it implies (1.8)

$$
\begin{aligned}
\beta^3 \int_{r_1/2<|x|<er_2} & \varphi_\beta^2 |x|^{-3}|E|^2 dx + \beta \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-1}|\nabla E|^2 dx \\
&\leq \tilde{C}_3 \Big\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 (|x|^{-3}|U|^2 + |x|^{-1}|\nabla U|^2) dx \\
&+ r_1^{-3}\varphi_\beta^2(r_1/e) \int_{r_1/4<|x|<r_1} |U|^2 dx + r_2^{-3}\varphi_\beta^2(er_2) \int_{2r_2<|x|<4r_2} |U|^2 dx \Big\}
\end{aligned}
$$

where $\tilde{C}_1$,$\tilde{C}_2$,$\tilde{C}_3$ are independent of r_1,r_2.

And we have the same conclusion for the ϕH, so we obtained

$$
\beta^3 \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-3}|H|^2 dx + \beta \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-1}|\nabla H|^2 dx
$$

$$\le \tilde{C}_3\Big\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2(|x|^{-3}|U|^2 + |x|^{-1}|\nabla U|^2)dx$$
$$+r_1^{-3}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1}|U|^2dx + r_2^{-3}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2}|U|^2dx\Big\} \tag{1.9}$$

Therefore, we can combine the inequality (1.8) and (1.9) such that

$$\beta^3\int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-3}(|E|^2+|H|^2)dx + \beta\int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-1}(|\nabla E|^2+|\nabla H|^2)dx$$
$$= \beta^3\int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-3}|U|^2dx + \beta\int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-1}|\nabla U|^2dx$$
$$\le 2\tilde{C}_3\Big\{\int_{r_1/2<|x|<er_2}\varphi_\beta^2(|x|^{-3}|U|^2+|x|^{-1}|\nabla U|^2)dx$$
$$+r_1^{-3}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1}|U|^2dx + r_2^{-3}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2}|U|^2dx\Big\}$$

Now let $\beta_0 \ge 1$ and $\beta^3 \ge \beta \ge \beta_0 \ge 3\tilde{C}_3$, then we can get another inequality (1.10)

$$\int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-3}|U|^2dx + \int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-1}|\nabla U|^2dx$$
$$\le \tilde{C}_4\left\{r_1^{-3}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1}|U|^2dx + r_2^{-3}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2}|U|^2dx\right\}$$

where $\tilde{C}_4 = 1/\tilde{C}_3$, and it is easy to get that

$$r_2^{-3}\varphi_\beta^2(r_2)\int_{r_1/2<|x|<r_2}|U|^2dx \le \int_{r_1/2<|x|<r_2}\varphi_\beta^2|x|^{-3}|U|^2dx \le \int_{r_1/2<|x|<er_2}\varphi_\beta^2|x|^{-3}|U|^2dx$$
$$\le \tilde{C}_4\left\{r_1^{-3}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1}|U|^2dx + r_2^{-3}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2}|U|^2dx\right\}$$

Dividing the term $r_2^{-3}\varphi_\beta^2(r_2)$, we obtain

$$\int_{r_1/2<|x|<r_2}|U|^2dx \le \tilde{C}_4\Big\{(r_2/r_1)^3[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{r_1/4<|x|<r_1}|U|^2dx$$
$$+[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{2r_2<|x|<4r_2}|U|^2dx\Big\}$$
$$\le \tilde{C}_5\Big\{(r_2/r_1)^3[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{|x|<r_1}|U|^2dx$$
$$+(r_2/r_1)^3[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{|x|<4r_2}|U|^2dx\Big\} \tag{1.11}$$

where $\tilde{C}_5 = max\{\tilde{C}_4, 1\}$

By choosing such $\tilde{C}_5$, we know that

$$\tilde{C}_5(r_2/r_1)^3[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] > 1$$

for $0 < r_1 < r_2 \leq 1$.

Adding $\int_{|x|<r_1/2} |U|^2 dx$ to the both sides of (1.11) and $r_2 < 1/4$, and then we have

$$\begin{aligned}\int_{|x|<r_2} |U|^2 dx \leq \tilde{C}_5 \Big\{ &(r_2/r_1)^3 [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x|<r_1} |U|^2 dx \\ &+ (r_2/r_1)^3 [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{|x|<1} |U|^2 dx \Big\} \\ &+ \tilde{C}_5 (r_2/r_1)^3 [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x|<r_1/2} |U|^2 dx \\ &\leq 2\tilde{C}_5 \Big\{ (r_2/r_1)^3 [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x|<r_1} |U|^2 dx \\ &+ (r_2/r_1)^3 [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{|x|<1} |U|^2 dx \Big\}\end{aligned}$$

Assume $A = (\log r_1 - 1)^2 - (\log r_2)^2$, $B = -1 - 2\log r_2$, and $A > 0, B > 0$ by simple computation. Therefore, the above inequality becomes

$$\int_{|x|<r_2} |U|^2 dx \leq 2\tilde{C}_5 (r_2/r_1)^3 \Big\{ exp(A\beta) \int_{|x|<r_1} |U|^2 dx + exp(-\beta B) \int_{|x|<1} |U|^2 dx \Big\} \tag{1.12}$$

By standard argument, we consider two cases

Case1 : If $exp(A\beta_0) \int_{|x|<r_1} |U|^2 dx < exp(-\beta_0 B) \int_{|x|<1} |U|^2 dx$ and pick $\beta > \beta_0$ such that

$$exp(A\beta) \int_{|x|<r_1} |U|^2 dx = exp(-\beta B) \int_{|x|<1} |U|^2 dx$$

so we have the following important inequality

$$\begin{aligned}\int_{|x|<r_2} |U|^2 dx &\leq 4\tilde{C}_5 (r_2/r_1)^3 exp(A\beta) \int_{|x|<r_1} |U|^2 dx \\ &= 4\tilde{C}_5 (r_2/r_1)^3 \Big(exp(A\beta) \int_{|x|<r_1} |U|^2 dx \Big)^{\frac{B}{A+B}} \Big(exp(-\beta B) \int_{|x|<1} |U|^2 dx \Big)^{\frac{A}{A+B}} \\ &= 4\tilde{C}_5 (r_2/r_1)^3 \Big(\int_{|x|<r_1} |U|^2 dx \Big)^{\frac{B}{A+B}} \Big(\int_{|x|<1} |U|^2 dx \Big)^{\frac{A}{A+B}}\end{aligned}$$

Case2 : If $exp(A\beta_0) \int_{|x|<r_1} |U|^2 dx \geq exp(-\beta_0 B) \int_{|x|<1} |U|^2 dx$, then we have

$$\begin{aligned}\int_{|x|<r_2} |U|^2 dx &\leq \Big(\int_{|x|<1} |U|^2 dx \Big)^{\frac{B}{A+B}} \Big(\int_{|x|<1} |U|^2 dx \Big)^{\frac{A}{A+B}} \\ &\leq exp(\beta_0 B) \Big(\int_{|x|<r_1} |U|^2 dx \Big)^{\frac{B}{A+B}} \Big(\int_{|x|<1} |U|^2 dx \Big)^{\frac{A}{A+B}}\end{aligned}$$

By the arguments, we can take $\tilde{C}_6 = max\{exp(\beta_0 B), 4\tilde{C}_5 (r_2/r_1)^3\}$ and get that

$$\int_{|x|<r_2} |U|^2 dx \le \tilde{C}_6 \Big(\int_{|x|<r_1} |U|^2 dx \Big)^{\frac{B}{A+B}} \Big(\int_{|x|<1} |U|^2 dx \Big)^{\frac{A}{A+B}} \tag{1.13}$$

For the general case, we can assume that $R_1 \le 1/4$ and $0 < r_1 < r_2 < r_3 \le R_0$ with $r_1/r_3 < r_2/r_3 \le 1/4$.

By scaling, $\tilde{U}(y) = U(r_3 y)$, $\tilde{\varepsilon_{ij}}(y) = \varepsilon_{ij}(r_3 y)$, $\tilde{\mu_{ij}}(y) = \mu_{ij}(r_3 y)$ We can have the same conclusion by above argument and obtain

$$\int_{|y|<r_2/r_3} |\tilde{U}|^2 dx \le C \Big(\int_{|y|<r_1/r_3} |\tilde{U}|^2 dx \Big)^{\tau} \Big(\int_{|y|<1} |\tilde{U}|^2 dx \Big)^{1-\tau} \tag{1.14}$$

where $\tau = B/(A+B)$ and $C = max\{exp(\beta_0 B), 4\tilde{C}_5 (r_2/r_1)^3\}$

$$\begin{cases} A = \big(\log(r_1/r_3) - 1\big)^2 - \big(\log(r_2/r_3)\big)^2 \\ B = -1 - 2\log(r_2/r_3) \end{cases}$$

Providing $r_3 < 1$ and $\tilde{C}_5$ can be chosen independent of r_3. So undoing the change of variable of (1.14), we have

$$\int_{|x|<r_2} |U|^2 dx \le C \Big(\int_{|x|<r_1} |U|^2 dx \Big)^{\tau} \Big(\int_{|x|<r_3} |U|^2 dx \Big)^{1-\tau} \tag{1.15}$$

The proof is now complete.

And then we are going to prove that the Maxwell system have the strong unique continuation property, so we have to prove the two theorems by using theorem 1.

Proof of the theorem 2 and theorem 3

Without loss of generality, we can use the change of coordinates and property (a) to obtain that

$$P_1(0,D) = \sum_{i,j=1}^{3} \varepsilon(0) D_i D_j = \Delta$$

$$P_2(0,D) = \sum_{i,j=1}^{3} \mu(0) D_i D_j = \frac{P_1(0,D)}{h} = \frac{\Delta}{h}$$

So we recall another Carleman estimate[1] : For any $u \in H^2_{loc}(\mathbb{R}^n \backslash \{0\})$ with compact support and for any $m \in \{j + \frac{1}{2} | j \in \mathbb{N}\}$ we have that

$$\sum_{|\alpha|\le 2} \int m^{2-2|\alpha|} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 dx \le C \int |x|^{-2m+4-n} |\Delta u|^2 dx \tag{2.1}$$

where C only depends on the dimension n.

And from the previous description, we know that $U = (E,H) \in (H^2_{loc}(\Omega \backslash \{0\}))^6$, so we can use the Carleman estimate for U.

Define a cut-off function $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \chi(x) \le 1$ and

$$\chi(x) = \begin{cases} 0, \text{ if } |x| \le \dfrac{\delta}{3} \\ 1, \text{ if } \dfrac{\delta}{2} \le |x| \le \dfrac{(R_0+1)R_0 R}{4} = r_4 R \\ 0, \text{ if } |x| \ge 2 r_4 R \end{cases}$$

where $\delta \le R_0^2 R/4$, $R_0 > 0$ is a small number and it will be determined later, and R is sufficiently small satisfying $0 < R \le R_0$. Using the (2.1) for χE and χH. Now for χE, we can derive that

$$\begin{aligned} &\sum_{|\alpha|\le 2} m^{2-2|\alpha|} \int_{\delta/2\le|x|\le r_4 R} |x|^{-2m+2|\alpha|-3} |D^\alpha E|^2 dx \\ &\le \sum_{|\alpha|\le 2} m^{2-2|\alpha|} \int |x|^{-2m+2|\alpha|-3} |D^\alpha (\chi E)|^2 dx \\ &\le C \int |x|^{-2m+1} |\Delta(\chi E)|^2 dx \\ &\le C \int_{\delta/2\le|x|\le r_4 R} |x|^{-2m+1} |\Delta E|^2 dx + C \int_{|x|>r_4 R} |x|^{-2m+1} |\Delta(\chi E)|^2 dx \\ &+ C \int_{\delta/3\le|x|\le\delta/2} |x|^{-2m+1} |\Delta(\chi E)|^2 dx \end{aligned} \tag{2.2}$$

On the right hand side of (2.2), the first term we use the triangle inequality (2.3)

$$C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\Delta E-P_1(x,D)E+P_1(x,D)E|^2dx$$
$$\le C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\Delta E-P_1(x,D)E|^2dx$$
$$+C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|P_1(x,D)E|^2dx$$

and the first term of the right hand side of (2.3), we can find out that

$$C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\Delta E-P_1(x,D)E|^2dx$$
$$=C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|(P_1(0,D)-P_1(x,D))E|^2dx$$
$$=C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\sum_{i,j=1}^{3}(\varepsilon_{ij}(0)-\varepsilon_{ij}(x))D_iD_jE|^2dx$$
$$\le C\int_{\delta/2\le|x|\le r_4R}(|x|sup|\varepsilon'_{ij}(x)|)^2|x|^{-2m+1}\sum_{i,j=1}^{3}|D_iD_jE|^2dx$$
$$\le C'\sum_{\alpha=2}r_4^2R^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|D^\alpha E|^2dx \tag{2.4}$$

since $\varepsilon_{ij}(x)$ is $\mathbf{C^2}$-function and C,C' are constants.

So by (2.2),(2.3),(2.4) and (1.4) we obtain

$$\sum_{|\alpha|\le2}m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-3}|D^\alpha E|^2dx$$
$$\le C'\sum_{\alpha=2}r_4^2R^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|D^\alpha E|^2dx$$
$$+2CM_1^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|U|^2dx+2CM_2^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\nabla U|^2dx$$
$$+C\int_{|x|>r_4R}|x|^{-2m+1}|\Delta(\chi E)|^2dx+C\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+1}|\Delta(\chi E)|^2dx \tag{2.5}$$

And we can have the same argument for χH to get that

$$\sum_{|\alpha|\le2}m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-3}|D^\alpha H|^2dx$$
$$\le \tilde{C}'\sum_{\alpha=2}r_4^2R^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|D^\alpha H|^2dx$$
$$+2\tilde{C}M_1^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|U|^2dx+2\tilde{C}M_2^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\nabla U|^2dx$$
$$+\tilde{C}\int_{|x|>r_4R}|x|^{-2m+1}|\Delta(\chi H)|^2dx+\tilde{C}\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+1}|\Delta(\chi H)|^2dx \tag{2.6}$$

And then we can derive (2.7) from (2.5),(2.6)

$$\sum_{|\alpha|\le 2} m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-3}|D^\alpha U|^2dx$$
$$\le C''\sum_{\alpha=2} r_4^2R^2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|D^\alpha U|^2dx$$
$$+C_M\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|U|^2dx + C_M\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+1}|\nabla U|^2dx$$
$$+\hat{C}\int_{|x|>r_4R}|x|^{-2m+1}|\Delta(\chi U)|^2dx+\hat{C}\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+1}|\Delta(\chi U)|^2dx \quad (2.7)$$

where $C''=C'+\tilde{C}'$, $C_M=2((C+\tilde{C})M_1^2+(C+\tilde{C})M_2^2)$ and $\hat{C}=C+\tilde{C}$

By choosing

$$R=\frac{1}{m\sqrt{C''}} \qquad \text{and} \qquad r_4^2R^2=\frac{R_0^2(R_0+1)^2}{16m^2C''}$$

Choosing $R_0<1$ (such that $\dfrac{R_0^2(R_0+1)^2}{16}<\dfrac{1}{2}$) and $m=m(R_0)$ large enough such that

$$\sum_{|\alpha|\le 2} m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-3}|D^\alpha U|^2dx$$
$$\le 2C\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+1}|\Delta(\chi U)|^2dx+2C\int_{r_4R<|x|<2r_4R}|x|^{-2m+1}|\Delta(\chi U)|^2dx \quad (2.8)$$

The first three terms on right hand side of (2.7) is absorbed by the left hand side when the $|x|$ is small enough.

And then by the definition of χ, it is easy to obtain that for all multiindex α

$$\begin{cases} |D^\alpha\chi(x)|\le C_3\delta^{-|\alpha|}, & \forall\ \dfrac{\delta}{3}\le|x|\le\dfrac{\delta}{2} \\ |D^\alpha\chi(x)|\le C_4(r_4R)^{-|\alpha|}, & \forall\ r_4R\le|x|\le 2r_4R \end{cases} \quad (2.9)$$

where C_3,C_4 are constants. Now we provide $R_0\le 1/16$ such that $R_0^2\le r_4$, so by the corollary in theorem 1 and (2.9), we can derive (2.10) from (2.8)

$$m^2(2\delta)^{-2m-3}\int_{\delta/2\le|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{2\delta<|x|\le R_0^2R}|U|^2dx$$
$$\le\sum_{|\alpha|\le 2} m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-3}|D^\alpha U|^2dx$$
$$\le 2CC_3\sum_{|\alpha|\le 2}\delta^{-4+2|\alpha|}\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+1}|D^\alpha U|^2dx$$
$$+2CC_4\sum_{|\alpha|\le 2}(r_4R)^{-4+2|\alpha|}\int_{r_4R<|x|<2r_4R}|x|^{-2m+1}|D^\alpha U|^2dx$$
$$\le C_3'(\delta/3)^{-2m-3}\int_{|x|\le\delta}|U|^2dx+C_4'(r_4R)^{-2m-3}\int_{|x|\le R_0R}|U|^2dx \quad (2.10)$$

where C_3',C_4' are independent of R_0,R and m.

Adding $m^2(2\delta)^{2m-3}\int_{|x|<\delta/2}|U|^2dx$ to both sides of (2.10), we can get the left hand side

$$\begin{aligned}
&m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{2\delta<|x|\le R_0^2R}|U|^2dx\\
&=\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx\\
&+m^2(R_0^2R)^{-2m-3}\int_{2\delta<|x|\le R_0^2R}|U|^2dx\\
&\ge\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx\\
&+m^2(R_0^2R)^{-2m-3}\int_{2\delta<|x|\le R_0^2R}|U|^2dx\\
&=\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{|x|\le R_0^2R}|U|^2dx \quad (2.11)
\end{aligned}$$

Combine (2.10) and (2.11), it implies that

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{|x|\le R_0^2R}|U|^2dx\\
&\le(C_3'+m^2)(\delta/3)^{-2m-3}\int_{|x|\le\delta}|U|^2dx+C_4'(r_4R)^{-2m-3}\int_{|x|\le R_0R}|U|^2dx
\end{aligned}$$

We can rewrite $C_4'(r_4R)^{-2m-3}$ and get $m^2(R_0^2R)^{-2m-3}C_4'm^{-2}(\frac{R_0^2}{r_4})^{2m+3}$.

Therefore,

$$C_4'm^{-2}(\frac{R_0^2}{r_4})^{2m+3}=C_4'm^{-2}(\frac{4R_0}{R_0+1})^{2m+3}\le C_4'm^{-2}(4R_0)^{2m+3}\le exp(-2m)$$

For all $R_0\le 1/16$ and $m^2\ge C_4'$. Thus, we can derive

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}\int_{|x|\le R_0^2R}|U|^2dx\\
&\le(C_3'+m^2)(\delta/3)^{-2m-3}\int_{|x|\le\delta}|U|^2dx+m^2(R_0^2R)^{-2m-3}e^{-2m}\int_{|x|\le R_0R}|U|^2dx
\end{aligned}$$

The above argument is valid for all $m=m(R_0)=j+1/2$ with $j\in\mathbb{N}$, and m large enough. So we can assume a j_0 which depends on R_0, and $R_j=(\sqrt{C''}(j+1/2))^{-1}$. For all $j\ge j_0$, we have the following inequality (2.12)

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-3}\int_{|x|\le 2\delta}|U|^2dx+m^2(R_0^2R_j)^{-2m-3}\int_{|x|\le R_0^2R_j}|U|^2dx\\
&\le(C_3'+m^2)(\delta/3)^{-2m-3}\int_{|x|\le\delta}|U|^2dx+m^2(R_0^2R_j)^{-2m-3}e^{\frac{-2c}{R_j}}\int_{|x|\le R_0R_j}|U|^2dx
\end{aligned}$$

where $c=\dfrac{1}{\sqrt{C''}}$

We can observe that $R_{j+1}<R_j<2R_{j+1}$ for all $j\in\mathbb{N}$ by simple calculation. If we can find a R such that $R_{j+1}<R\le R_j$, it implies $R_0R_j\le 2R_{j+1}/16\le R_{j+1}$ since $R_0<1/16$ and then we have such relation $R_0R_j\le R_{j+1}<R<R_j<2R_{j+1}$. So we can get a conclusion

$$\begin{cases} \int_{|x|\le R_0^2 R} |U|^2 dx \le \int_{|x|\le R_0^2 R_j} |U|^2 dx \\ e^{\frac{-2c}{R_j}} \int_{|x|\le R_0 R_j} |U|^2 dx \le e^{\frac{-c}{R}} \int_{|x|\le R} |U|^2 dx \end{cases} \tag{2.13}$$

If there exists a $s \in \mathbb{N}$ such that

$$R_{j+1} < R_0^{2s} \le R_j \text{ for some } j \ge j_0 \tag{2.14}$$

It is just replacing R by R_0^{2s} on the above description. By (2.12) and (2.13), we can obtain (2.15)

$$\begin{aligned} &\frac{1}{2}m^2(2\delta)^{-2m-3} \int_{|x|\le 2\delta} |U|^2 dx + m^2(R_0^2 R_j)^{-2m-3} \int_{|x|\le R_0^{2s+2}} |U|^2 dx \\ &\le (C_3' + m^2)(\delta/3)^{-2m-3} \int_{|x|\le\delta} |U|^2 dx + m^2(R_0^2 R_j)^{-2m-3} exp(-cR_0^{-2s}) \int_{|x|\le R_0^{2s}} |U|^2 dx \end{aligned}$$

So now our goal is to find a appropriate s and R_0 to claim an inequality which is

$$exp(-cR_0^{-2s}) \int_{|x|\le R_0^{2s}} |U|^2 dx \le \frac{1}{2} \int_{|x|\le R_0^{2s+2}} |U|^2 dx \tag{2.16}$$

By theorem 1, we assume $r_1 = R_0^{2s+2}$,$r_2 = R_0^{2s}$ and $r_3 = R_0^{2s-2}$ where $s \ge 1$.
And $r_1/r_3 < r_2/r_3 \le R_0^2 \le 1/4$, then we divide $\left(\int_{|x|<R_0^{2s}} |U|^2 dx \right)^{1-\tau}$ to both sides of (1.15)

$$\left(\int_{|x|<R_0^{2s}} |U|^2 dx \right)^{\tau} \le \overline{C} \left(\int_{|x|<R_0^{2s+2}} |U|^2 dx \right)^{\tau} \left(\int_{|x|<R_0^{2s-2}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s}} |U|^2 dx \right)^{1-\tau}$$

and it implies (2.17)

$$\int_{|x|<R_0^{2s}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s+2}} |U|^2 dx \le \overline{C}^{1/\tau} \left(\int_{|x|<R_0^{2s-2}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s}} |U|^2 dx \right)^{a}$$

where $\overline{C} = max\{exp(\beta_0(-1 - 4logR_0)), 4\tilde{C}_5(R_0)^{-6}\}$

$a = \dfrac{1-\tau}{\tau} = \dfrac{(4logR_0 - 1)^2 - (2logR_0)^2}{-1 - 4logR_0}$ by definition of τ in the proof of theorem 1.

We can see that

$$\begin{cases} 1 < \overline{C} \le \tilde{C}_5 R_0^{-\beta_1} \\ 2 < a \le -5logR_0 \end{cases} \tag{2.18}$$

where $\beta_1 = max\{6, 4\beta_0\}$.

Because $exp(\beta_0(-1 - 4logR_0) = e^{-\beta_0} R_0^{-4\beta_0}$ and $\tilde{C}_5 \geq 1$, we have first inequality.

For the second inequality, we consider that

$$a = \frac{-4logR_0 + 2 - 1/4logR_0 + logR_0}{1/4logR_0 + 1} > -3logR_0 + 2 > 2$$
$$a < \frac{-3logR_0 - logR_0}{1 - 1/5} < -5logR_0$$

And we use (2.17) recursively

$$\int_{|x|<R_0^{2s}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s+2}} |U|^2 dx \leq \overline{C}^{1/\tau} \left(\int_{|x|<R_0^{2s-2}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s}} |U|^2 dx \right)^a$$
$$\leq \overline{C}^{\frac{a^{s-1}-1}{\tau(a-1)}} \left(\int_{|x|<R_0^{2}} |U|^2 dx \Big/ \int_{|x|<R_0^{4}} |U|^2 dx \right)^{a^{s-1}} \qquad (2.19)$$

For all $s \in \mathbb{N}$. And by definition of a we know that $\tau = 1/(a+1)$, from (2.18) we have

$$\frac{a^{s-1} - 1}{\tau(a-1)} = \frac{(a+1)(a^{s-1} - 1)}{a - 1} \leq 3a^{s-1}$$

Then we derive the following inequality from (2.19)

$$\int_{|x|<R_0^{2s}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s+2}} |U|^2 dx \leq \overline{C}^{3(-5logR_0)^{s-1}} \left(\int_{|x|<R_0^{2}} |U|^2 dx \Big/ \int_{|x|<R_0^{4}} |U|^2 dx \right)^{a^{s-1}}$$
$$\leq (\tilde{C}_5^3 R_0^{-3\beta_1})^{(-5logR_0)^{s-1}} \left(\int_{|x|<R_0^{2}} |U|^2 dx \Big/ \int_{|x|<R_0^{4}} |U|^2 dx \right)^{a^{s-1}}$$

Multiply $exp(-cR_0^{-2s})$ on both sides, we obtain (2.20)

$$exp(-cR_0^{-2s}) \int_{|x|<R_0^{2s}} |U|^2 dx \Big/ \int_{|x|<R_0^{2s+2}} |U|^2 dx$$
$$\leq exp(-cR_0^{-2s}) (\tilde{C}_5^3 R_0^{-3\beta_1})^{(-5logR_0)^{s-1}} \left(\int_{|x|<R_0^{2}} |U|^2 dx \Big/ \int_{|x|<R_0^{4}} |U|^2 dx \right)^{a^{s-1}}$$

Let $\kappa = -logR_0$, and compute $log(\tilde{C}_5^3 R_0^{-3\beta_1})^{(5\kappa)^{s-1}} = (5\kappa)^{s-1}(log\tilde{C}_5^3 + 3\beta_1\kappa)$.

So we can find out that if R_0 small enough, it means that κ sufficient large, then we have (2.21)

$$\frac{cR_0^{-2s}}{4} > (5\kappa)^{s-1}(log\tilde{C}_5^3 + 3\beta_1\kappa) \Rightarrow (\tilde{C}_5^3 R_0^{-3\beta_1})^{(5\kappa)^{s-1}} < exp(\frac{cR_0^{-2s}}{4}) < \frac{1}{2} exp(\frac{cR_0^{-2s}}{2})$$

The (2.21) holds for all $s \in \mathbb{N}$, and now we should fix R_0 such that (2.21) holds and the $m(R_0)$ and $j(R_0)$ are fixed as well. Fixing R_0, we can derive from (2.20)

$$exp(-cR_0^{-2s})\int_{|x|<R_0^{2s}}|U|^2dx \le \frac{1}{2}exp(\frac{-cR_0^{-2s}}{2})\Big(\int_{|x|<R_0^2}|U|^2dx\Big/\int_{|x|<R_0^4}|U|^2dx\Big)^{a^{s-1}} \times$$
$$\int_{|x|<R_0^{2s+2}}|U|^2dx$$

Our goal is (2.16), so coping with the term $\Big(\int_{|x|<R_0^2}|U|^2dx\Big/\int_{|x|<R_0^4}|U|^2dx\Big)^{a^{s-1}}$

If we have an estimate that

$$\Big(\int_{|x|<R_0^2}|U|^2dx\Big/\int_{|x|<R_0^4}|U|^2dx\Big)^{a^{s-1}} \le exp(cR_0^{-2s}/2) \qquad (2.22)$$

Then (2.16) is proved for appropriate s. So now we have to find s such that (2.22) holds.

Assume $N = \Big(\int_{|x|<R_0^2}|U|^2dx\Big/\int_{|x|<R_0^4}|U|^2dx\Big)$, take $loglog$ for (2.22), we have

$$log2 - log(ac) + loglogN \le s(-2logR_0 - loga)$$

By (2.18), we know $-2logR_0 - loga > 0$ for all $R_0 \le 1/16$, so we can define a number s_0 as

$$s_0 = min\{s \in \mathbb{N} | s \ge (log2 - log(ac) + loglogN)(-2logR_0 - loga)^{-1}\}$$

So the claim (2.16) holds for all $s \ge s_0$.

But now s should also be chosen to assure (2.14) holds.

Let s_1 be the smallest positive integer such that $R_0^{2s_1} \le R_{j_0}$, then we can find a $j_1 \in \mathbb{N}$ with $j_1 \ge j_0$ such that $R_{j_1+1} < R_0^{2s_1} \le R_{j_1}$. We now can define s_p depending on $P_1(x,D)$, $P_2(x,D)$ and N as

$$s_p = max\{s_0, s_1\}$$

For this s_p, (2.14),(2.21),(2.22) hold. Thus, we set $m = j_1 + \frac{1}{2}$ and $m_1 = 3 + 2m$ plus into (2.15).

$$(\frac{m_1-3}{8})^2(2\delta)^{-2(\frac{m_1-3}{2})-3}\int_{|x|\le 2\delta}|U|^2dx + (\frac{m_1-3}{8})^2(R_0^2R_{j_1})^{-2(\frac{m_1-3}{2})-3}\int_{|x|\le R_0^{2s_p+2}}|U|^2dx$$
$$\le (C_3' + (\frac{m_1-3}{2})^2)(\delta/3)^{-2(\frac{m_1-3}{2})-3}\int_{|x|\le\delta}|U|^2dx \qquad (2.23)$$

So consider the second term of the left hand side of (2.23)

$$\frac{1}{8}(m_1-3)^2(R_0^2R_{j_1})^{-m_1}\int_{|x|\le R_0^{2s_p+2}}|U|^2dx \le (C_3' + (\frac{m_1-3}{2})^2)(\delta/3)^{-m_1}\int_{|x|\le\delta}|U|^2dx$$

then it implies

$$\frac{(m_1-3)^2}{8C_3'+2(m_1-3)^2}(3R_0^2R_{j_1})^{-m_1}\int_{|x|\le R_0^{2s_p+2}}|U|^2dx \le (\delta)^{-m_1}\int_{|x|\le\delta}|U|^2dx \tag{2.24}$$

(2.24) is valid for all $\delta \le R_0^{2s_p+2}/4$ because of the definition of δ. So the proof of theorem 2 is complete with $R_2 = R_0$. And the first term of the left hand side of (2.23)

$$\frac{1}{8}(m_1-3)^2(2\delta)^{-m_1}\int_{|x|\le 2\delta}|U|^2dx \le (C_3'+(\frac{m_1-3}{2})^2)(\delta/3)^{-m_1}\int_{|x|\le\delta}|U|^2dx$$

then it implies

$$\int_{|x|\le 2\delta}|U|^2dx \le \frac{8C_3'+2(m_1-3)^2}{(m_1-3)^2}6^{m_1}\int_{|x|\le\delta}|U|^2dx \tag{2.25}$$

(2.25) is valid for all $\delta \le R_0^{2s_p+2}/4$ because of the definition of δ. So the proof of theorem 3 is complete with $R_3 = R_0^{2s_p+2}/4$ and $C_3 = \dfrac{8C_3'+2(m_1-3)^2}{(m_1-3)^2}6^{m_1}$.

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