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碩士論文

Department of Physics
College of Science
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Master Thesis

對 von Kármán 旋轉流的進一步研究

Further Investigation on von Kármán's Swirling Flow

林蔚廷

Wei-Ting Lin

指導教授：陳義裕博士

Advisor: Yih-Yuh Chen, Ph.D.

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Acknowledgement

我要感謝我的指導老師陳義裕教授，在碩士修業過程中，教了我許多事情。三年前，我對研究，對物理，都可說是門外漢，現在如果有一點點心得，很大的原因是因為聽老師上課，與老師討論而得來的。在做研究的過程中，每次討論他都給了我啟發性的建議，他總能從我想不到的角度切入問題。然而，研究還是常常窒礙難行，我於是學到，「研究總是這樣的」，我想這也是他教給我很重要的一課，而這似乎是任何物理課本都沒有教的。篇幅所限，不知如何表達對老師的感激之情，僅能止筆於此。

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中文摘要

我們比較了 von Kármán 方程式的數值解與線性近似的結果，發現某些 von Kármán 方程式的解的特性，能夠被其線性化方程所表現。為了更進一步驗證這個結果，我們採用 amplitude equation。為了解決 damping 項造成的影響，我們提出一個新的方法。此方法雖然 amplitude 部份近似的很好，但 phase 部份相當差，有待進一步的研究。

Abstract

By the comparison between the numerical results and the linearized solution, we show some solutions of von Kármán's equation have global feature that can be captured by the linearized solutions, although there are some shifts in amplitude and phase.

To show the validity of this approach and polish the approximation, we use the amplitude equation approach, which can tune the amplitude and phase of the linearized solutions to make them fit. We encounter difficulty in the course of finding the amplitude equation of damping systems. To solve this problem, we propose an idea to deal with the exponential growth of the zero-th order solution. Although the phase of the approximate solution deviates from the numerical results very much, the amplitude fits well. Further investigation is under way.

Chapter 1

Introduction

From hurricanes to eddies in water tubs, phenomena involving rotating fluids are ubiquitous. They are not only interesting in themselves but also important in both science and engineering. As a devastating example, we note that typhoons and tornados have been causing great damages every year, and so the study of related topics clearly is of utmost importance in meteorology. A better understanding of the structure and the dynamics of these big swirls will help us better equipped to deal with the potential damage they can cause. Another good example is concerned with electric fans. The study of rotating flows may improve the efficiency of fans, which is of course important in engineering. For more details, one may refer to [4] and [10] for more information.

One of the important topics in rotating flows is concerned with rotating disk systems. Many engineering applications contain disk-like structures such as hard disks and flywheels. Because all these devices work in the air or in some fluid, the study of rotating flows with rotating disks is inevitable.

To immerse a disk into a fluid in a cylindrical container is perhaps one of the simplest examples involving rotating disks. The effect of the rotating disk is similar to stirring tea in a cup. This seemingly easy daily task actually exhibits an interesting fluid dynamic phenomenon which has become the famous tea leaf problem[1]; and this problem had been solved qualitatively by Einstein [6]. However, the quantitative analysis is more difficult. Indeed, up to now we still have no analytical solution for it.

In analogy to the tea leaf problem, we can do a slightly varied version of the experiment, this time replacing the tea leaves with another fluid. Thus, we have two immiscible fluids with different densities in the cylindrival container. When the disk is rotating, one can see the interface between the two layers of fluid to deform into many interesting shapes when the rotating speed of the disk is varied. This problem, like many other fluid problems, is difficult to analyze.

To get some insight into this problem, we may investigate the problem with a simpler geometry. One example is to consider an infinite disk rotating in a space full of fluid. In other words, we have effectively removed the sidewall and the bottom of the cylindrical container, and make the size of the system infinite. This idealized case may be thought of as an approximation to a large enough system which can be more easily susceptible to mathematical analysis. As it turns out, this is indeed the case, and the first person to tackle it is none other than the famous von Kármán [11]. In 1921 he managed to find an analytical solution to this problem using similarity principles. Because of the assumed form of the similarity solution, the original governig equations, a set of coupled partial differential equations, are reduced to a system of ordinary differential equations, which are much easier to deal with.

[1] http://en.wikipedia.org/wiki/Tea_leaf_paradox

The same problem was once again picked up by Cochran in 1934 [5] , who corrected some of the numerical errors contained in [11] and did a numerical integration of the equations derived by von Kármán.

In 1940, Bödewadt consider a similar problem with opposite boundary conditions. Bödewadt's disk is stationary and the fluid at infinity is rotating, while von Kármán's disk is rotating and the fluid at infinity is stationary. In a sense, then, Bödewadt's system is more closely related to the tea leaf problem.

Generalizing the original von Kármán problem a bit, another important study was done by Batchelor when he extended these one-disk problems to two disks [1]. The similarity solution still holds for the two-disk problems. However, Batchelor derived his conclusion only from physical intuition and continuity of solutions, leaving the analytical part mostly untouched.

Interestingly, however, Stewartson did not agree with Batchelor's conclusion about the flow between a rotating disk and a stationary disk [9]. Instead, he derived another conclusion from both theoretical argument and experiment. The two conclusions, which had been derived respectively by two famous figures in fluid dynamics, are conflicting.

Eventually, the conflict was settled down by numerical results. In 1968 Mellor, Chapple and Stokes showed the multiplicity of the solutions to this problem [7]. For high Reynolds numbers, their research indicated that solutions of the two types are both possible. (That is, the solutions are not unique, and which solution can be realized then depends on how one reaches the final configuration.)

To make the problem more complicated and interesting, some researchers further consider suction or injection at the disk surface so that the z-component of the velocity can be nonzero at the disk. For more, the readers are referred to an excellent review article by Zandbergen and Dijkstra [12].

In this thesis, we will investigate the simplest possible scenario of the one-disk and two-disk problems. By comparing with numerical results, we show that these nonlinear similarity solutions contain features which are prominently present in the solutions to the linearized equations. We also discuss an attempt to use the amplitude equation approach to show why the behavior of some solutions of the nonlinear systems is dominated by that of the linearized systems. The structure of the thesis is as follows: After this brief introduction, we give a simple review of the solution to the original von Kármán approach in Chapter 2. Then in Chapter 3 we consider the linearized solution to von Kármán's equations and show how it compares to the numerical solution to the full set of von Kármán's equations. Chapter 4 begins with the observation of why an amplitude equation approach might be able to explain the adequacy of using our linearized solution to approximate the full nonlinear solution, then we discuss a simplified system which is meant to illustrate the point we are trying to get across, together with the difficulties we have encountered. Chapter 5 summarizes our investigation.

Chapter 2

A review of von Kármán's swirling solution

In this chapter we give a brief review of the similarity solution first proposed by von Kármán, which actually can be applied to problems consisting of one or two infinite disks. We also present some numerical results of these equations which will serve as a benchmark for our analysis of Chapter 3.

2.1 von Kármán's similarity solution

Consider the viscous incompressible fluid in the region $z > 0$, with an infinite disk on the plane $z = 0$ (Fig. 2.1). The disk rotates along z-axis at angular velocity Ω, while as $z \to \infty$, the fluid has $u_\theta \to 0$. With the assumption that the solution has azimuthal symmetry and if we consider only time-independent solutions, Navier-Stokes equation and the incompressibility condition in cylindrical coordinates

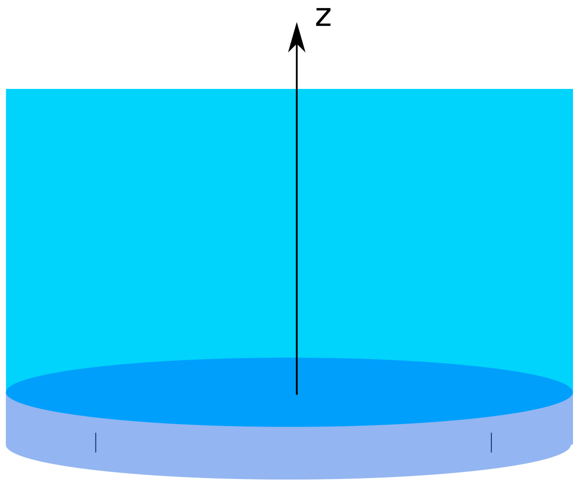

Figure 2.1: An infinite disk with fluid.

become

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right), \tag{2.1}$$

$$u_r\frac{\partial u_\theta}{\partial r}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}=\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{u_\theta}{r^2}\right), \tag{2.2}$$

$$u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right), \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z}=0. \tag{2.4}$$

In the above, ν is the kinematic viscosity, and P is the pressure in the fluid. Though axial symmetry alone has already helped simplifying the equations a lot, this by itself does not make finding the solutions any easier.

To make further progress, von Kármán proposed a similarity solution in 1921, which reduces the coupled partial differential equations with both r and z dependence into coupled ordinary differential equations depending on just one single variable z. This significantly simplifies the whole problem. Following von Kármán, let

$$u_r = r\Omega f(\zeta), \quad u_\theta = r\Omega g(\zeta), \quad u_z = \sqrt{\nu\Omega}h(\zeta), \tag{2.5}$$

where

$$\zeta = \sqrt{\frac{\Omega}{\nu}} z$$

is a rescaled height parameter. After substituting this ansatz into Eqns. (2.1)-(2.4), the equations become

$$f^2 + hf'^2 = -\frac{1}{\rho\Omega^2}\frac{1}{r}\frac{\partial P}{\partial r} + f'', \tag{2.6}$$

$$2fg + hg' = g'', \tag{2.7}$$

$$hh' = -\frac{1}{\rho\nu\Omega}\frac{\partial P}{\partial \zeta} + h'', \tag{2.8}$$

$$2f + h' = 0. \tag{2.9}$$

In conformity to the usual practice, we will impose the no-slip boundary condition on the disk. We also assume that the radial and angular velocity both vanish at infinity. (Thus, rotation is assumed to be strong only near the "driving" disk whereas the fluid far away is only "sucked in" along the axial direction and is not rotating at all.) Therefore,

$$u_r = 0, \quad u_\theta = r\Omega, \quad u_z = 0 \quad \text{at } z = 0,$$

$$u_r = 0, \quad u_\theta = 0, \quad \text{as } z \to \infty.$$

Note that we do not specify the axial velocity at infinity, as it will have to be determined self-consistently afterwards so that mass conservation can be satisfied for given value of Ω. Then the boundary conditions for f, g, and h are

$$f = 0, \quad g = 1, \quad h = 0 \quad at\ \zeta = 0, \tag{2.10}$$

$$f \to 0, \quad g \to 0 \quad as\ \zeta \to \infty. \tag{2.11}$$

Integrating Eqn. (2.8), we get

$$-\frac{1}{\rho\nu\Omega}P = \frac{1}{2}h^2 - h' + \Pi(r)$$

for some function $\Pi(r)$ yet to be determined. Substituting this expression into Eqn. (2.6), we get

$$f^2 + hf'^2 - f'' = \frac{\nu}{\Omega}\frac{1}{r}\frac{d\Pi}{dr}. \tag{2.12}$$

Since the left-hand-side of the above equation is a function of z only, wheras the right-hand-side depends on r only, they each must be equal to some constant. But in view of the boundary condition Eqn. (2.11), we have

$$f \to 0, \quad f' \to 0, \quad , f'' \to 0, \quad g \to 0 \quad as\ \zeta \to \infty.$$

So the constant is 0.

Hence we have

$$f^2 + hf'^2 = f'' \tag{2.13}$$

$$2fg + hg' = g'' \tag{2.14}$$

$$2f + h' = 0 \tag{2.15}$$

Before attempting any solutions, we note in passing that we now have three equations involving three unknown functions f, g, and h, and there are five boundary conditions for this 5th order system, so that everything is self-consistent up to this point.

Although this system still can not be solved analytically, one can numerically solve it using computers (Fig. 2.2). In finding the numerical solution, we have used a user-friendly fortran package called BVP_SOLVER-2 [3] to solve boundary value problems.

From the numerical result, we see that there is a radially outward flow near the disk. By the continuity condition, this tells us that that there is a suction towards the disk, and fluid far away from the disk is sucked in, as already mentioned before.

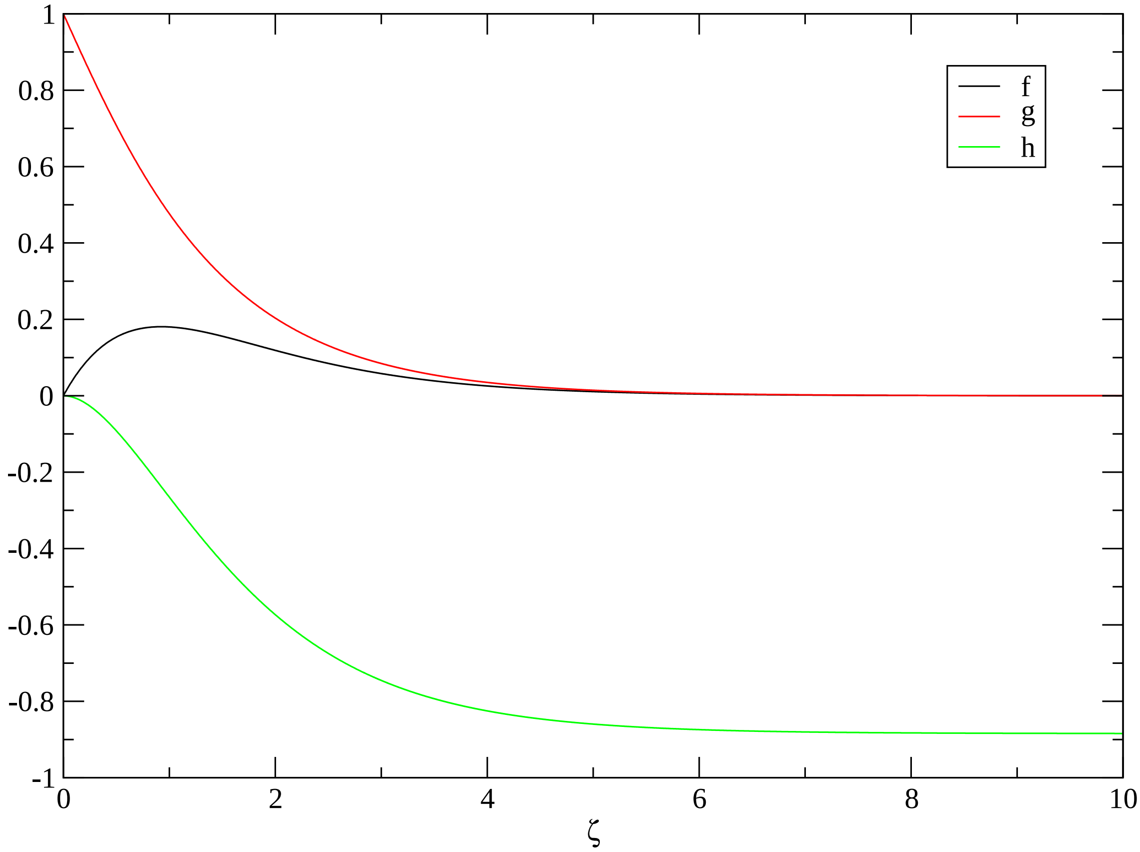

Figure 2.2: The numerical result of von Kármán's equations.

Thus, the disk acts like a suction fan, which rotates and throws the fluid nearby radially out, whose action then induces a suction to drive the fluid in axially.

2.2 Rotating fluid with a stationary disk

Obviously, the magic of von Kármá's similarity solution is the assumption made in Eqn. (2.5). This also points the way to generalizing it to other systems that resemble von Kármán's. Here we consider a rotating fluid with a stationary disk.

We assume

$$u_r = 0, \quad u_\theta = 0, \quad u_z = 0 \quad \text{at } z = 0, \tag{2.16}$$

$$u_r = 0, \quad u_\theta = r\Omega, \quad \text{as } z \to \infty. \tag{2.17}$$

In other words, the fluid exhibits solid rotation far away from a stationary disk. Using the same similarity form of Eqn. (2.5), the boundary conditions become

$$f = 0, \quad g = 0, \quad h = 0 \quad at\ \zeta = 0, \tag{2.18}$$

$$f \to 0, \quad g \to 1 \quad as\ \zeta \to \infty. \tag{2.19}$$

Similarly, after imposing Eqn. (2.19) to Eqn. (2.12), we get

$$f^2 + hf'^2 = f'' - 1 \tag{2.20}$$

$$2fg + hg' = g'' \tag{2.21}$$

$$2f + h' = 0 \tag{2.22}$$

From the numerical result (Fig. 2.3), we observe that there is a radially inward flow near the stationary disk this time. This solution then reminds one of the famous tea cup problem: when we stir the tea (to make it a rotating fluid), tea leaves near

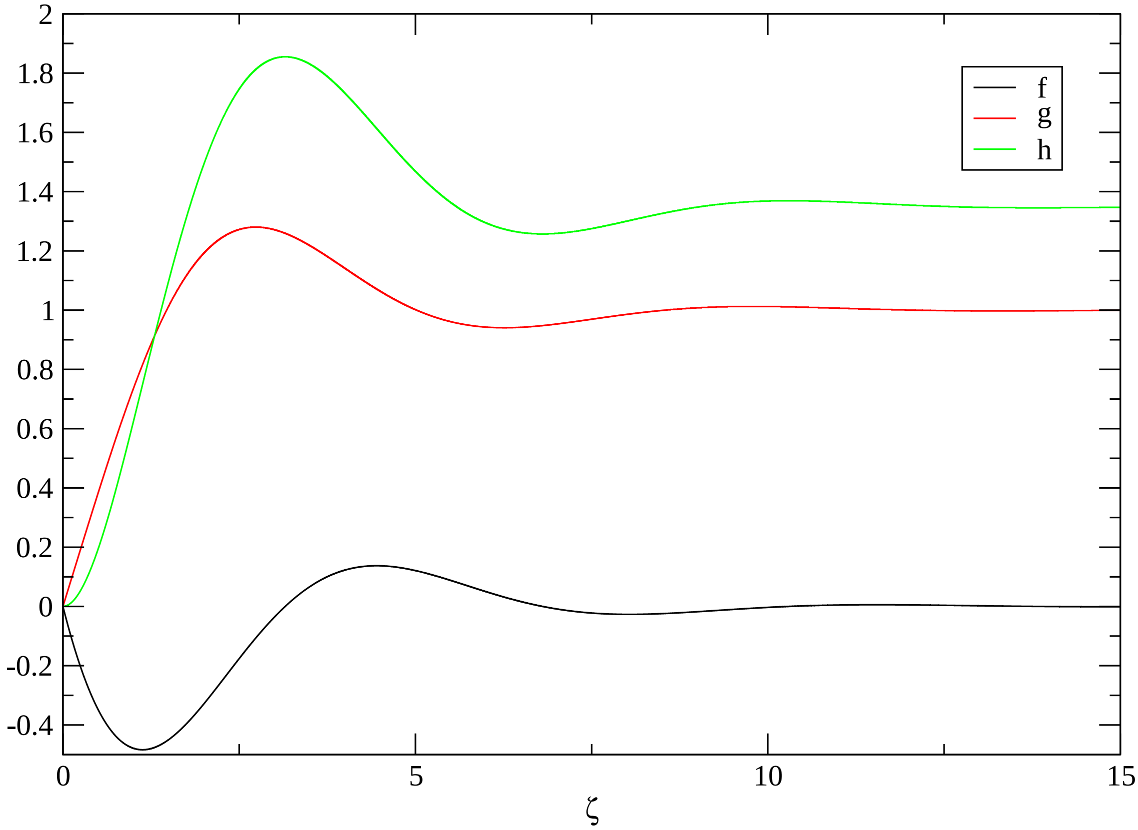

Figure 2.3: The numerical result of the rotating fluid with a stationary disk.

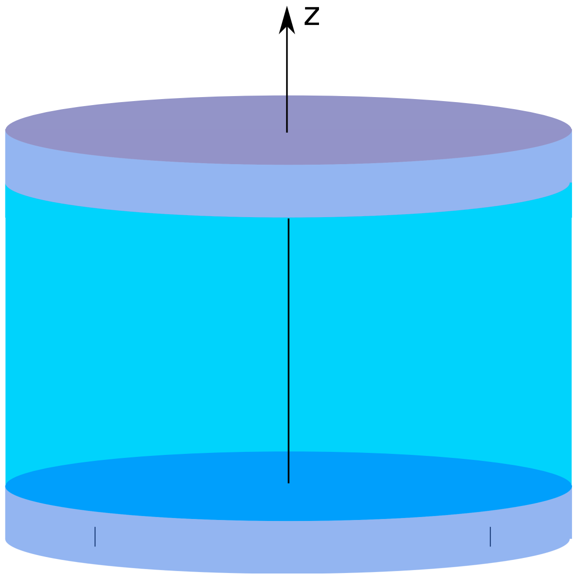

Figure 2.4: Two infinite disks.

the bottom of the cup tend to aggregate near the center of the cup, which is a result of the radial inward flow, whereas the fluid near the central column will rise (just to satisfy the cntinuity equation).

2.3 The two-disk problem

In the two examples we discussed above, the fluid extends infinitely in the positive z-direction. In fact, the similarity solution still works even if we place another disk at $z = d$ plane (Fig. 2.4). This makes the system finite along the z-axis and so slightly more realistic, although it is still infinite in the r-direction.

Assume the upper disk at $z = d$ has an angular velocity Ω, and the lower one at $z = 0$ has an angular velocity of Ω'. The boundary conditions are

$$u_r = 0, \quad u_\theta = r\Omega, \quad u_z = 0 \quad \text{at } z = d, \tag{2.23}$$

$$u_r = 0, \quad u_\theta = r\Omega', \quad u_z = 0 \quad \text{at } z = 0. \tag{2.24}$$

Using the ansatz of Eqn. (2.5) we once again get Eqns. (2.6)-(2.9). And the boundary conditions become

$$f = 0, \quad g = 1, \quad h = 0 \quad \text{at } \zeta = d, \tag{2.25}$$

$$f = 0, \quad g = \frac{\Omega'}{\Omega}, \quad h = 0 \quad \text{at } \zeta = 0. \tag{2.26}$$

Before going on, a remark is in order: Though we can still argue as before that each side of the equality sign of Eqn. (2.12) must separately equal some constant, the numerical value of this constant can not be directly read off from the current boundary conditions, because we don't know f' and f'' at either boundary.

To progress further, we may eliminate the pressure using cross differentiation. (This is equivalent to taking the curl of the Navier-Stokes equation.) For instance, Differentiating Eqn. (2.8) with respect to r gives us an expression for $\frac{\partial^2 P}{\partial r \partial \zeta}$, and a similar trick differentiating Eqn. (2.6) with respect to ζ yields yet another expression for the same $\frac{\partial^2 P}{\partial r \partial \zeta}$. Then, we may use the two results to eliminate $\frac{\partial^2 P}{\partial r \partial \zeta}$. The result is

$$2ff' + h'f' + hf'' - 2gg' = f''',$$

or

$$(2f + h')f' + hf'' - 2gg' = f'''.$$

The first term of the above equation is zero by the continuity equation (Eqn. (2.9)). So, finally, we get

$$hf'' - 2gg' = f''' \tag{2.27}$$

$$2fg + hg' = g'' \tag{2.28}$$

$$2f + h' = 0. \tag{2.29}$$

Unlike the one-disk problem, various different combinations of d and Ω'/Ω might result in different classes of solutions. Here we consider a particular case with

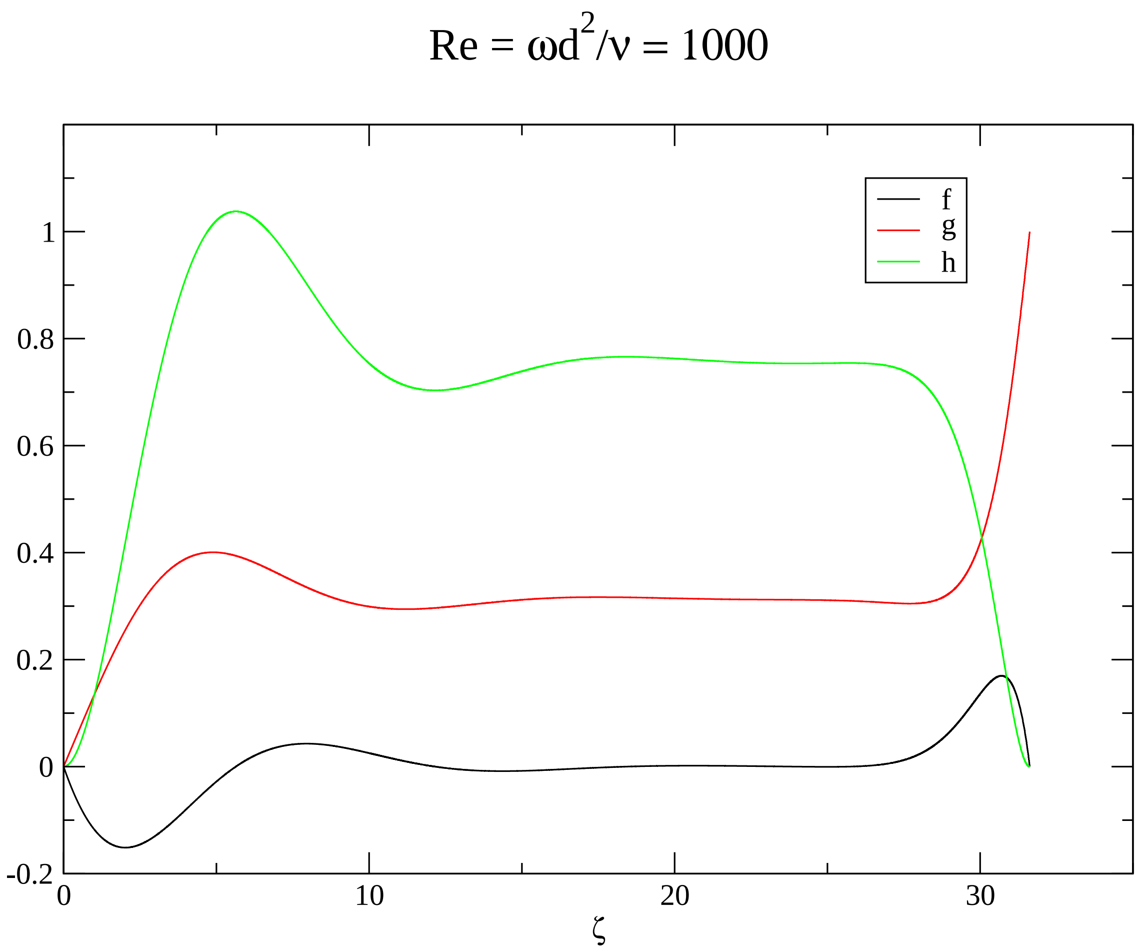

Figure 2.5: The numerical result of the rotating fluid within a stationary disk and a rotating disk.

$\Omega' = 0$. For this system, we can define the Reynolds number to be $Re \equiv \Omega d^2/\nu$. Since $\zeta = \sqrt{\Omega/\nu}z$, $z = d$ corresponds to $\zeta = \sqrt{\Omega d^2/\nu} = \sqrt{Re}$. The numerical result of $Re = 1000$ is shown in Fig. 2.5. Before ending this brief review, we must emphasize once again that this solution is just one of several possible solutions. In fact, there exists many solutions for a given Reynolds number [7].

Chapter 3

Linearized Solution

Inspecting the coupled differential equations of von Kármán's, one does not have a clear idea of how an analytical solution can be sought directly. However, after playing around with the system parameters from the numerical results, a persistent feature eventually emerges: For regimes we have studied, the behavior of the solutions does not look complicated at all. Encouraged by this observation, we decided to see if there is any approximate way to look at things.

As an illustration, we found that, as long as the Reynolds number is not too small, f, g, and h are almost constants in the domain except for regions near the boundaries. This then suggests the possibility of treating the variables as some constants plus a suitable perturbation. As such, we may attempt to linearize the equations and study their behavior. As a first test case, we try it on the one-disk problem in which the Reynolds number may be thought of as being infinite (because there is no physical finite length scale to enter the definition of Reynolds number).

3.1 The one-disk problem

Let g_∞ and h_∞ be the asymptotic values of g and h as $\zeta \to \infty$. Then the equations in Sections 2.1 and 2.2 can be written as

$$f^2 + hf'^2 = f'' - g_\infty^2 \tag{3.1}$$

$$2fg + hg' = g'' \tag{3.2}$$

$$2f + h' = 0 \tag{3.3}$$

Now, we may use Eqn. (3.3) to eliminate f in all the equations above. This yields

$$h''' = -\frac{(h')^2}{2} + hh''^2 - 2g_\infty^2, \tag{3.4}$$

$$g'' = -h'g + hg'. \tag{3.5}$$

Let

$$h \equiv h_\infty + h_1,$$

$$g \equiv g_\infty + g_1,$$

and assume h_1 and g_1 are small. After substituting these expressions into Eqns. (3.4) and (3.5) and neglect higher order terms, the linearized system is

$$h_1''' = h_\infty h_1'' + 4g_\infty g_1 \tag{3.6}$$

$$g_1'' = -g_\infty h_1' + h_\infty g_1'. \tag{3.7}$$

The second equation of the above can be integrated to yield

$$g_1' = -g_\infty h_1 + h_\infty g_1 + c_0, \tag{3.8}$$

where c_0 is a constant.

Since Eqn. (3.6) can be recast into

$$g_1 = \frac{1}{4g_\infty}(h_1''' - h_\infty h_1''),$$

we may substitute this equation into Eqn. (3.8) to obtain

$$h_1^{(4)} - 2h_\infty h_1^{(3)} + h_\infty^2 h_1^{(2)} = -4g_\infty^2 h_1 + 4g_\infty c_0. \tag{3.9}$$

This is a linear inhomogeneous differential equation whose solution can be derived from standard method. Thus, let $h_1 = e^{\lambda\zeta}$ and then the associated characteristic equation is

$$\lambda^4 - 2h_\infty\lambda^3 + h_\infty^2\lambda^2 = -4g_\infty^2 \tag{3.10}$$

$$\Rightarrow \quad \lambda^2(\lambda - h_\infty)^2 = -4g_\infty^2. \tag{3.11}$$

In von Kármán's original problem one has $g_\infty = 0$, while in the problem consisting of a rotating fluid with a stationary disk, $g_\infty = 1$. In either case, h_∞ is determined by numerical results.

3.1.1 Von Kármán's problem ($g_\infty = 0$, $h_\infty < 0$)

Since $g_\infty = 0$,

$$\lambda = 0, \quad 0, \quad h_\infty, \quad h_\infty.$$

Hence

$$h_1 = Ae^{h_\infty\zeta} + B\zeta e^{h_\infty\zeta} + C\zeta + D.$$

Since $h_1 \to 0$ as $\zeta \to \infty$, $C = 0$ and $D = 0$. We should also note that $h_\infty < 0$ so that the solution eventually exhibits exponential decay as ζ tends to infinity. At the other boundary $\zeta = 0$, $h = 0$ and $h' = -2f = 0$, so $h_1(0) = -h_\infty$ and $h_1'(0) = 0$.

Hence $A = -h_\infty$ and $B = h_\infty^2$, and then

$$h = h_\infty - h_\infty e^{h_\infty \zeta} + h_\infty^2 \zeta e^{h_\infty \zeta}.$$

Since $f = -\frac{h'}{2}$, we also have

$$f = -\frac{1}{2} h_\infty^3 \zeta e^{h_\infty \zeta}.$$

From Eqn. (3.8) and the fact that $g_\infty = 0$ we obtain

$$g_1' = h_\infty g_1 + c_0, \tag{3.12}$$

$$\Rightarrow \quad g_1 = g_0 e^{h_\infty \zeta} - \frac{c_0}{h_\infty}, \tag{3.13}$$

where g_0 is a constant. Since $g_1 \to 0$ as $\zeta \to \infty$, we have $c_0 = 0$. Then, by $g = g_\infty + g_1$ and $g_\infty = 0$ we have $g = g_0 e^{h_\infty \zeta}$. By the boundary condition $g(0) = 1$ we finally arrive at

$$g = e^{h_\infty \zeta}.$$

We can compare this approximate solution with the numerical result. From Fig. 3.1, we see that the linearized solution has captured the qualitative behavior of the actual solution amazingly well.

3.1.2 Rotating fluid with a stationary disk

Since $g_\infty = 1$, we immediately have

$$\lambda(\lambda - h_\infty) = \pm 2i,$$

$$\Rightarrow \quad \lambda = \frac{h_\infty}{2}\left(1 \pm \sqrt{1 \pm \frac{8i}{h_\infty^2}}\right).$$

Since $h_1 \to 0$ as $\zeta \to \infty$, λ must have a negative real part, so the admissible λ's are $\frac{h_\infty}{2}(1 - \sqrt{1 + \frac{8i}{h_\infty^2}})$ and $\frac{h_\infty}{2}(1 + \sqrt{1 - \frac{8i}{h_\infty^2}})$, which are complex conjugate to each other.

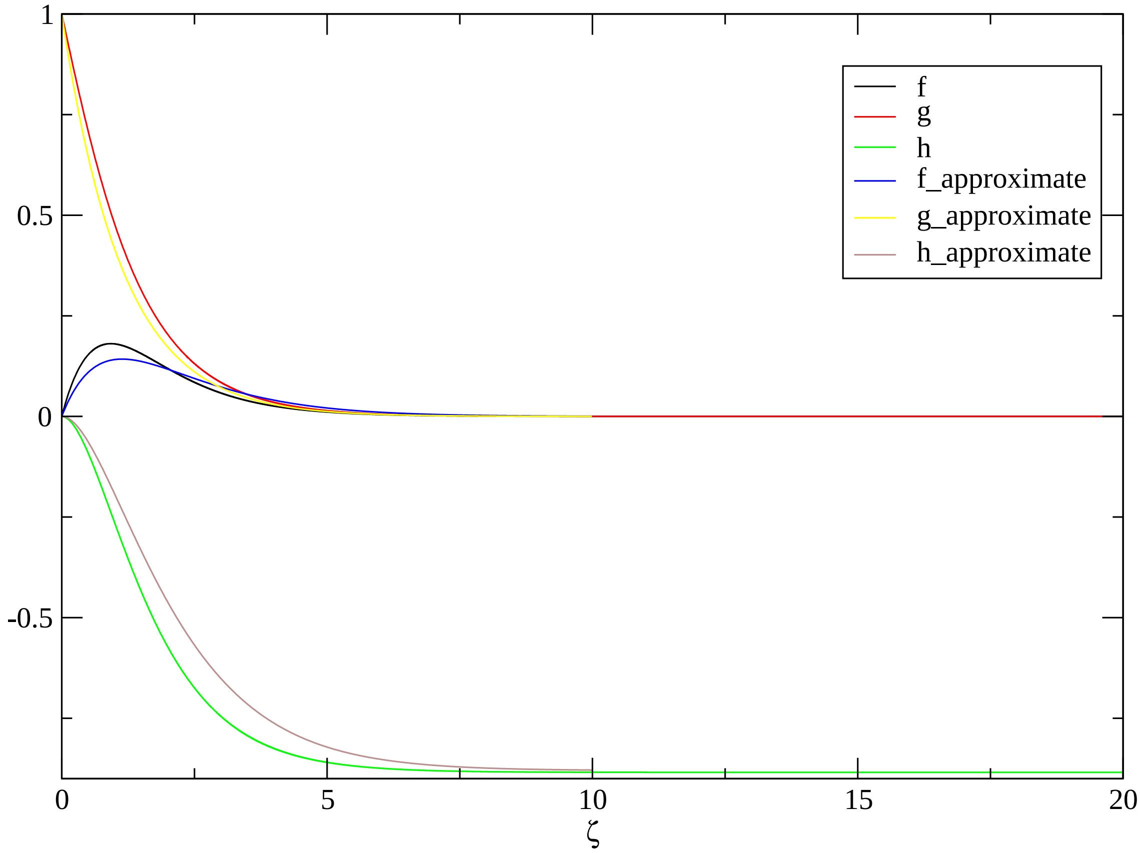

Figure 3.1: The numerical result of von Kármán's equations and the approximate solution with $h_\infty = -0.88$.

Let the two admissible λ's be $\alpha + \beta i$ and $\alpha - \beta i$, where α and β are real and $\alpha < 0$. Then from Eqn. (3.9),

$$h_1 = Ae^{(\alpha+\beta i)\zeta} + Be^{(\alpha-\beta i)\zeta}.$$

Note that $c_0 = 0$ since $h_1 \to 0$ as $\zeta \to \infty$.

By $h(0) = h_\infty + h_1(0) = 0$ and $h'(0) = h_1'(0) = -2f(0) = 0$, we can use them to to get the values of A and B. And then

$$h = h_\infty + Ae^{(\alpha+\beta i)\zeta} + Be^{(\alpha-\beta i)\zeta}$$

and

$$f = -\frac{h'}{2} = -\frac{A}{2}(\alpha+\beta i)e^{(\alpha+\beta i)\zeta} - \frac{B}{2}(\alpha-\beta i)e^{(\alpha-\beta i)\zeta}.$$

From Eqn. (3.8),

$$g_1 = \frac{-A}{\alpha - h_\infty + \beta i}e^{(\alpha+\beta i)\zeta} + \frac{-B}{\alpha - h_\infty - \beta i}e^{(\alpha-\beta i)\zeta} + g_0 e^{h_\infty \zeta},$$

where g_0 is a constant. But we do not want g_1 to diverge as $\zeta \to \infty$, and this dictates that $g_0 = 0$.

But now comes a weak point of this approximation: Here we observe that the boundary condition $g(0) = 0$ has not been imposed, and yet all of the integration constants have been determined. What this means is that the boundary condition $h_1(\infty) = 0$ is too strong a requirement for our purpose, because it has killed more than one integration constants in one stroke! In Fig. 3.2 we compare the numerical result with this less-than-perfect approximation to check how poor it is.

From the figure we readily see that, surprisingly, although the boundary condition of g is not satisfied, the global behavior of the approximate solution is not too bad at all. All qualitative features of the numerical result are correctly captured by this approximation.

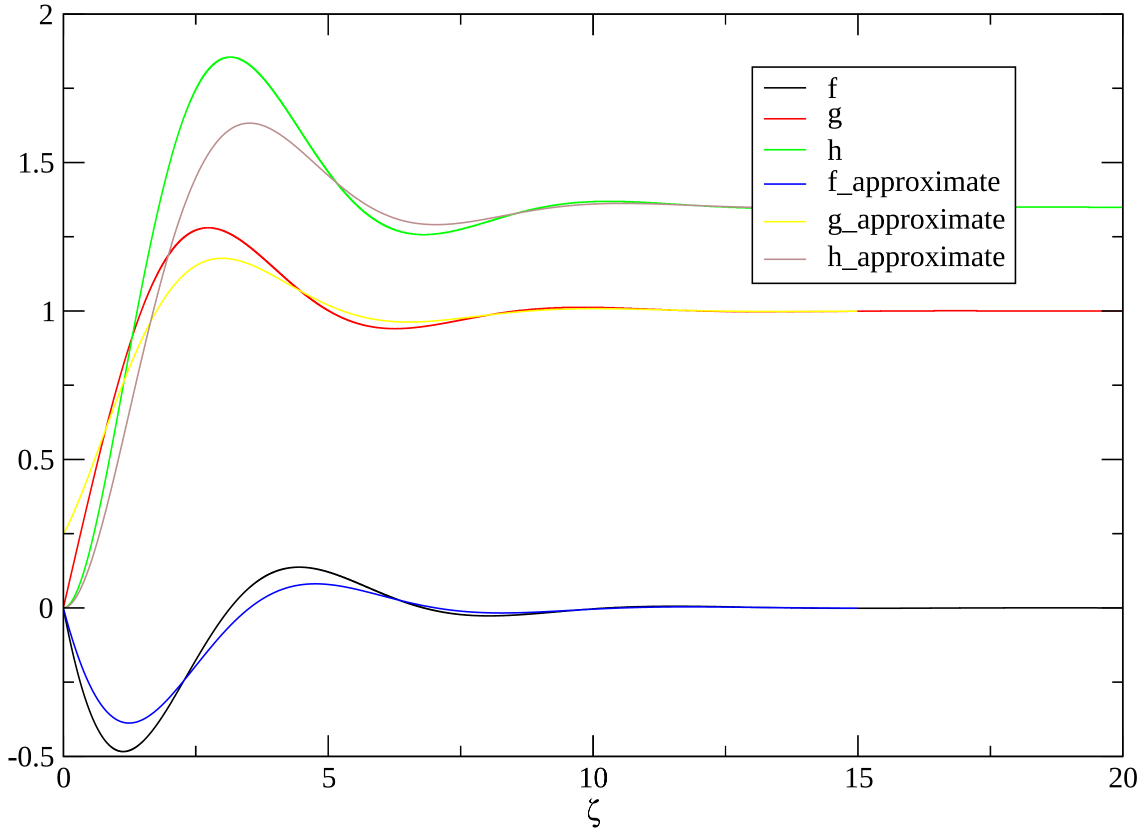

Figure 3.2: The numerical result of von Kármán's equations and the approximate solution with $h_\infty = 1.35$.

3.2 The two-disk problem

Let g_0 and h_0 be two constants which play the role of g_∞ and h_∞. Recall from Chapter 2 that the equations governing this problem are Eqns. (2.27)-(2.29). After reducing the equations, we get

$$h^{(4)} = hh^{(3)} + 4gg^{(1)} \tag{3.14}$$

$$g^{(2)} = -h^{(1)}g + hg^{(1)}. \tag{3.15}$$

The linearization method is the same as in the one-disk problem of the previous section. Thus, let

$$h = h_0 + h_1 \tag{3.16}$$

$$g = g_0 + g_1 \tag{3.17}$$

and assume h_1 and g_1 are much smaller than h_0 and g_0, we arrive at the following linearized system

$$h_1^{(4)} = h_0 h_1^{(3)} + 4g_0 g_1^{(1)}, \tag{3.18}$$

$$g_1^{(2)} = -g_0 h_1^{(1)} + h_0 g_1^{(1)}. \tag{3.19}$$

Writing Eqn. (3.18) as

$$g_1^{(1)} = \frac{h_1^{(4)} - h_0 h_1^{(3)}}{4g_0} \tag{3.20}$$

and substituting it into Eqn. (3.19) we have

$$h_1^{(5)} - 2h_0 h_1^{(4)} + h_0^2 h_1^{(3)} = -4g_0^2 h_1^{(1)}. \tag{3.21}$$

It is not surprising that this equation has the same structure as the corresponding one in the previous section.

Similarly, the characteristic roots are

$$\lambda = 0, \quad \frac{h_0}{2}(1 \pm \sqrt{1 \pm \frac{8g_0 i}{h_0^2}}).$$

We denote the two complex conjugate pairs by γ_1, $\bar{\gamma_1}$, γ_2, $\bar{\gamma_2}$, where $\Re(\gamma_1) < 0$, and $\Re(\gamma_2) > 0$. Hence

$$h_1 = Ae^{\gamma_1 \zeta} + Be^{\bar{\gamma_1}\zeta} + Ce^{\gamma_2 \zeta} + De^{\bar{\gamma_2}\zeta} + E,$$

where A, B, C, D, and E are constants to be determined by the boundary conditions. Substituting this h into Eqn. (3.20) and integrating the equation, we obtain an explicit expression of g with another integration constant F.

Now we specialize to the particular case in which the lower disk is stationary. The boundary conditions are

$$h(0) = 0, \quad h(\sqrt{\frac{\Omega d^2}{\nu}}) = 0, \tag{3.22}$$

$$h'(0) = -2f(0) = 0, \quad h'(\sqrt{\frac{\Omega d^2}{\nu}}) = -2f(\sqrt{\frac{\Omega d^2}{\nu}}) = 0, \tag{3.23}$$

$$g(0) = 0, \quad g(\sqrt{\frac{\Omega d^2}{\nu}}) = 1. \tag{3.24}$$

These six boundary conditions can determine all the six integration constants. Because this is just a direct solution of six linearly coupled algebraic equations, we will simply avoid the complicated-looking analytical expression and proceed directly to a comparison of the numerical result and this approximation. This is shown in Fig. 3.3. In the above, we have chosen $h_0 = 0.75$ and $g_0 = 0.3$, which are approximately the values of h and g in the middle region in the numerical result. The qualitative features of the numerical result and the approximate solution are the same. But there is a numerical shift: the values of the numerical result are larger than the approximate solution, which isn't unlike what was observed in the

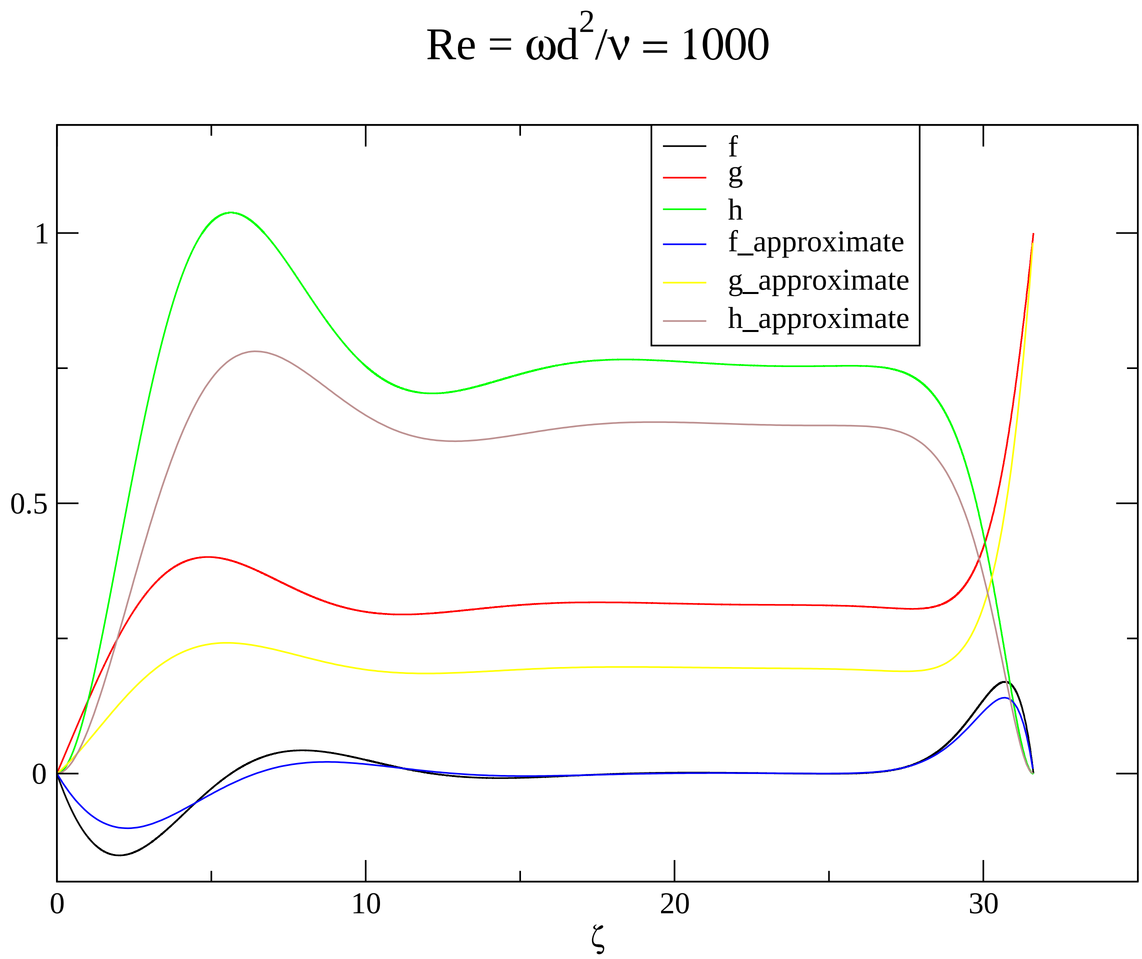

Figure 3.3: The numerical result of two-disk problem and the approximate solution with $h_0 = 0.75$ and $g_0 = 0.3$.

previous case. Having seen the qualitative as well as the quantitative success of this simple approximation, we will discuss in the next chapter whether there is a method to correct the shortcoming of our approximation and in what sense it is good.

Chapter 4

The amplitude equation approach

In the last chapter we compared the numerical results with those predicted by the linearized equations. It is obvious from the comparison that the approximate solutions exhibit the major feature of the numerical solutions except that the numerical fit is not perfect. Specifically, we found that, from a peek of Fig. 3.2, the oscillation phenomenon of the approximate solution is smaller in magnitude than the numerical solution. This by itself suggests that an amplitude equation approach, in which one assumes that the actual solution can be well approximated by the linear solution but with a modification in its (usually) complex amplitude which typically is slowly varying in space and time, may be the next best thing one can do in analysis. The other two cases we discussed before similarly suggest like treatment. The equation satisfied by this amplitude is then termed the amplitude equation.

Mathematically, there are different ways to derive the amplitude equation, and here we focus particularly on the multiple scale analysis. Introducing an amplitude into our system has the merit of helping us understand why our linear solution can well-approximate the full nonlinear solution. However, in trying to complete this program, we have encountered certain technical difficuty which still begs for clarification. In spite of all this, we feel that we are still on the right track.

In the following, we will first review the multiple scale analysis, and then explain how the idea may be tailored to fit our problem.

4.1 A brief review of the multiple scale analysis

Here we briefly review the multiple scale analysis via an example. For more details, one can refer to books on perturbation methods, such as [2] and [8]. The example we will use as our illustration is the Duffing equation:

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0, \tag{4.1}$$

where ϵ is a small number.

As a tentative solution, one can try using a regular perturbation method. Thus, assuming

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots$$

and substituting it into Eqn. (4.1) and collecting terms of the same power of ϵ, we get

$$\epsilon^0 : \quad \ddot{x_0} + \omega_0^2 x_0 = 0$$

$$\epsilon^1 : \quad \ddot{x_1} + \omega_0^2 x_1 + x_0^3 = 0$$

and so on. The zero-th order equation gives

$$x_0(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}.$$

After substituting x_0 into the ϵ^1-equation, we can solve for x_1. But we immediately find out that x_0^3 contributes $e^{i\omega t}$ and $e^{-i\omega t}$ terms, which are secular terms in the equation for x_1. The secular terms make ϵx_1 grow in time, which usually is an undesired feature or artifact.

The problem originates from the fact that the frequency of the nonlinear perturbation is in resonance with the natural frequency of the equation for x_1. Since physically we know the frequency must be altered by the presence of the cubic term, one must use one way or another to account for this shift. Indeed, this is exactly the major idea behind the amplitude equation approach. Thus, we try to change the originally constant amplitudes A and B into $A(t)$ and $B(t)$, which turn out to be something like $e^{i\omega_1 t}$ so that the frequency of the solution is effectively shifted to $\omega_0 + \omega_1$. Note that though we have used the term "amplitude," its meaning is not restricted to a revision of only the amplitude in its traditional sense. Rather, because it is complex, the phase is also changed.

Furthermore, since the perturbation is small, the correction in frequency should be also small, so we can write ω_1 as $\epsilon\omega_1$. That means the change in phase is a long-term effect. In other words, this phenomenon belongs in a longer time scale, so instead of $\epsilon\omega_1$ we put ϵ and t together and have $A(\epsilon t)$ and $B(\epsilon t)$ instead of $A(t)$ and $B(t)$.

More generally, we let

$$x = x(T_0, T_1, T_2, \cdots) = x_0(T_0, T_1, T_2, \cdots) + \epsilon x_1(T_0, T_1, T_2, \cdots) + \cdots, \qquad (4.2)$$

where

$$T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \quad \cdots .$$

Then

$$\begin{aligned}\frac{d}{dt} &= \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + \cdots \\ &= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots . \end{aligned} \tag{4.3}$$

In these new variables, we can handle phenomena of different scales. Substituting Eqns. (4.2) and (4.3) into Eqn. (4.1), we have

$$\epsilon^0 : \frac{\partial^2 x_0}{\partial T_0^2} + \omega_0^2 x_0 = 0 \tag{4.4}$$

$$\epsilon^1 : \frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 + 2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} + x_0^3 = 0 \tag{4.5}$$

and so on.

From Eqn. (4.4), we have

$$x_0 = A_0(T_1, T_2, \cdots)e^{i\omega_0 T_0} + B_0(T_1, T_2, \cdots)e^{-i\omega_0 T_0}.$$

Substituting x_0 into Eqn. (4.5) we have

$$\begin{aligned}&\frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 + 2(i\omega_0\frac{\partial A_0}{\partial T_1}e^{i\omega_0 T_0} - i\omega_0\frac{\partial B_0}{\partial T_1}e^{-i\omega_0 T_0}) \\ &+ (A_0^3 e^{i3\omega_0 T_0} + 3A_0^2 B_0 e^{i\omega_0 T_0} + 3A_0 B_0^2 e^{-i\omega_0 T_0} + B_0^3 e^{-i3\omega_0 T_0}) = 0.\end{aligned}$$

Note that the $3A_0^2 B_0 e^{i\omega_0 T_0}$ and $3A_0 B_0^2 e^{-i\omega_0 T_0}$ are the cause of secular terms, i.e., the resonance terms. But now we have the extra terms $2i\omega_0\frac{\partial A_0}{\partial T_1}e^{i\omega_0 T_0}$ and $-2i\omega_0\frac{\partial B_0}{\partial T_1}e^{-i\omega_0 T_0}$. If we *require* (from hindsight) that

$$\begin{aligned}2i\omega_0\frac{\partial A_0}{\partial T_1}e^{i\omega_0 T_0} + 3A_0^2 B_0 e^{i\omega_0 T_0} &= 0, \\ -2i\omega_0\frac{\partial B_0}{\partial T_1}e^{-i\omega_0 T_0} + 3A_0 B_0^2 e^{-i\omega_0 T_0} &= 0,\end{aligned}$$

then all the resonance terms are eliminated. This, then, is the basic idea behind the multiple scale analysis: By introducing new long time scales one may kill the secular terms to save the day. The above two equations are the amplitude equations we were looking for, which determine A_0 and B_0. Note the $e^{i\omega_0 T_0}$ and $e^{-i\omega_0 T_0}$ are cancelled out. So

$$2i\omega_0 \frac{\partial A_0}{\partial T_1} + 3A_0^2 B_0 = 0 \tag{4.6}$$

$$-2i\omega_0 \frac{\partial B_0}{\partial T_1} + 3A_0 B_0^2 = 0. \tag{4.7}$$

After solving this system, we find out

$$A_0 \propto e^{i\frac{3C^2}{8\omega_0}T_1} = e^{i\frac{3C^2}{8\omega_0}\epsilon t}$$

$$B_0 \propto e^{-i\frac{3C^2}{8\omega_0}T_1} = e^{-i\frac{3C^2}{8\omega_0}\epsilon t},$$

where C^2 is an integration constant. So we get a correction on frequency, which tends to 0 as $\epsilon \to 0$. Usually simply one stops at this point as the lowest order correction to the original nonlinear problem, though the method clearly can be carried through to higher orders.

To summarize, the scheme of this approach is assuming a slowly varying amplitude, which has to be determined in the next order approximation by some solvability conditions. In this example, the solvability condition is to eliminate the resonace terms. From the solvability condition, we can get the amplitude equation and then solve for the amplitude.

4.2 Damping and negative damping systems

The multiple scale analysis works very well for the example we illustrated, but it can not be applied directly to our problem. The main trouble is that our system has both positive and negative damping. This is because two of the characteristic values of Eqn. (3.21) have a negative real part, and two others have a positive real part. The difficulty can be easily seen this way: Because the resonance terms in the example come from the cancellation of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$, for example $e^{i\omega_0 t} \cdot e^{i\omega_0 t} \cdot e^{-i\omega_0 t} = e^{i\omega_0 t}$, so if there are real parts, then $e^{r+i\omega_0 t} \cdot e^{r+i\omega_0 t} \cdot e^{r-i\omega_0 t} \neq e^{r+i\omega_0 t}$, and there is no resonance terms, which at first sight might be hailed as a good news while in reality it is not, because now one loses insight as to how the solvability condition might be imposed. Not only that, the nonlinear terms also give rise to stronger positive and/ or negative damping, which must be seriously dealt with. Because of this difficulty, we decided to take a detour and investigate the problem using a simplified model equation.

To illustrate the problem more clearly, therefore, we consider the following example:

$$\ddot{x} + (\epsilon x^2 - 1)\dot{x} + x = 0. \tag{4.8}$$

The unperturbed system ($\epsilon = 0$) has characteristic values with a positive real part, so it exhibits negative damping. When x is small, this linear approximation (i.e. $\epsilon = 0$) is good. But x grows exponentially in time till it saturates (shifting from negative damping to positive damping), we must take the ϵx^2 term into account to make the approximation good when t is large. Therefore, we need an amplitude which can cancel out the exponential growth.

Copying what was done in the previous section, we let

$$x = x(T_0, T_1, T_2, \cdots) = x_0(T_0, T_1, T_2, \cdots) + \epsilon x_1(T_0, T_1, T_2, \cdots) + \cdots ,$$

where

$$T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \quad \cdots .$$

Then, with

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots .$$

we get

$$\begin{aligned}\epsilon^0 :& \frac{\partial^2 x_0}{\partial T_0^2} - \frac{\partial x_0}{\partial T_0} + x_0 = 0 \\ \epsilon^1 :& \frac{\partial^2 x_1}{\partial T_0^2} + 2\frac{\partial^2 x_0}{\partial T_1 \partial T_0} + x_0^2\frac{\partial x_0}{\partial T_0} - \frac{\partial x_1}{\partial T_0} - \frac{\partial x_0}{\partial T_1} + x_1 = 0\end{aligned}$$

and so on. To simplify the equations, let $L \equiv \frac{\partial^2}{\partial T_0^2} - \frac{\partial}{\partial T_0} + 1$, then the above equations become

$$\begin{aligned}\epsilon^0 :& Lx_0 = 0 \\ \epsilon^1 :& Lx_1 + 2\frac{\partial^2 x_0}{\partial T_1 \partial T_0} + x_0^2\frac{\partial x_0}{\partial T_0} - \frac{\partial x_0}{\partial T_1} = 0.\end{aligned}$$

The solution of the zeroth order equation is

$$x_0 = A_0(T_1, \cdots)e^{\lambda T_0} + B_0(T_1, \cdots)e^{\bar{\lambda} T_0},$$

where $\lambda = \frac{1+\sqrt{3}i}{2}$. Then the first order equation becomes

$$\begin{aligned}&Lx_1 + 2(\frac{\partial A_0}{\partial T_1}\lambda e^{\lambda T_0} + \frac{\partial B_0}{\partial T_1}\bar{\lambda} e^{\bar{\lambda} T_0}) + (A_0^2 e^{2\lambda T_0} + 2A_0 B_0 e^{(\lambda+\bar{\lambda})T_0} \\ &+B_0^2 e^{2\bar{\lambda} T_0})(A_0\lambda e^{\lambda T_0} + B_0\bar{\lambda} e^{\bar{\lambda} T_0}) - (\frac{\partial A_0}{\partial T_1} e^{\lambda T_0} + \frac{\partial B_0}{\partial T_1} e^{\bar{\lambda} T_0}) = 0.\end{aligned}$$

Note after expanding the product, the only resonance terms are $2(\frac{\partial A_0}{\partial T_1}\lambda e^{\lambda T_0} + \frac{\partial B_0}{\partial T_1}\bar{\lambda} e^{\bar{\lambda} T_0})$ and $-(\frac{\partial A_0}{\partial T_1} e^{\lambda T_0} + \frac{\partial B_0}{\partial T_1} e^{\bar{\lambda} T_0})$, so the amplitude equations are $\frac{\partial A_0}{\partial T_1} = 0$ and $\frac{\partial B_0}{\partial T_1} = 0$. But then the amplitude is just a constant, which can not cancel out the exponential growth.

The point is that even with a straightforward expansion, there is no resonance in this non-conservative system. This means we can not blindly apply the original solvability condition to this problem.

4.2.1 A possible approach

How do we remedy the difficulty discussed above? Though we have not yet completely solved this problem, it is believed that we may be on the right track following the main idea of the approach expounded below.

To motivate the idea, we note that the trouble comes from nonlinear terms, which makes the behavior of the solutions differ in the small amplitude regime from the large amplitude regime. We may view the nonlinear terms as causing a sort of "frequency-amplitude interaction." For example, in the Duffing equation the frequency would be corrected by a small factor depending on the amplitude. When the zeroth order solution not only oscillates but also grows or decays exponentially, the small parameter ϵ, which is related to the amplitude, presumably should also be affected. In other words, *the small parameter itself should change with time, too.* This means adopting the slow time scale ϵt in the original oscillation problems like the Duffing equation is no longer valid for our problem. Instead, we should try to use a nonlinear time scale τ, which is to be determined when we carry out the next order perturbation calculation.

So let us explicitly work it out for Eqn. (4.8). Let $x = x(t, \tau)$, where τ is another time scale to be determined in the next order approximation. The differential operator becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\tau}{dt}\frac{\partial}{\partial \tau}.$$

We assume $\frac{d\tau}{dt}$ is of order ϵ. And we do not consider $\frac{d^2\tau}{dt^2}$ term. So

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\frac{d\tau}{dt}\frac{\partial^2}{\partial \tau \partial t} + (\frac{d\tau}{dt})^2\frac{\partial^2}{\partial \tau^2}.$$

Then the zeroth order solution is

$$x_0(t,\tau) = A(\tau)e^{\lambda t} + B(\tau)e^{\bar{\lambda}t}, \quad \lambda = \frac{1+\sqrt{3}i}{2}. \tag{4.9}$$

The first order equation is

$$\epsilon\frac{\partial^2 x_1}{\partial t^2} + 2\frac{d\tau}{dt}\frac{\partial^2 x_0}{\partial\tau\partial t} + [\epsilon(x_0+\epsilon x_1)^2 - 1](\frac{\partial x_0}{\partial t} + \epsilon\frac{\partial x_1}{\partial t} + \frac{d\tau}{dt}\frac{\partial x_0}{\partial \tau}) + \epsilon x_1 = 0.$$

After expansion and taking only ϵ and $\frac{d\tau}{dt}$ terms, we obtain the following equation:

$$\epsilon\frac{\partial^2 x_1}{\partial t^2} + 2\frac{d\tau}{dt}\frac{\partial^2 x_0}{\partial\tau\partial t} + \epsilon x_0^2\frac{\partial x_0}{\partial t} - \epsilon\frac{\partial x_1}{\partial t} - \frac{d\tau}{dt}\frac{\partial x_0}{\partial\tau} + \epsilon x_1 = 0$$

Substituting Eqn. (4.9) into this equation, we get

$$\begin{aligned}&\epsilon(\frac{\partial^2 x_1}{\partial t^2} - \frac{\partial x_1}{\partial t} + x_1) + \frac{d\tau}{dt}(2A'\lambda e^{\lambda t} + 2B'\bar{\lambda}e^{\bar{\lambda}t} - A'e^{\lambda t} - B'e^{\bar{\lambda}t})\\ &+ \epsilon(A^3\lambda e^{3\lambda t} + A^2B(2\lambda+\bar{\lambda})e^{(2\lambda+\bar{\lambda})t} + AB^2(\lambda+2\bar{\lambda})e^{(\lambda+2\bar{\lambda})t} + B^3\bar{\lambda}e^{3\bar{\lambda}t}) = 0.\end{aligned}$$

Now, in analogy to the usual solvability condition, we *impose* the following restriction: All terms containing $e^{i\Im(\lambda)t}$ and $e^{-i\Im(\lambda)t}$ must be eliminated. (Note that we use $\Im(z)$ to denote the imaginary part of z, and $\Re(z)$ to denote the real part of z.) Collecting terms with $e^{i\Im(\lambda)t}$ and $e^{-i\Im(\lambda)t}$, we get

$$\begin{aligned}&\frac{d\tau}{dt}(2\lambda-1)A'e^{\Re(\lambda)t} + \epsilon(2\lambda+\bar{\lambda})A^2Be^{3\Re(\lambda)t} = 0\\ &\frac{d\tau}{dt}(2\bar{\lambda}-1)B'e^{\Re(\lambda)t} + \epsilon(2\bar{\lambda}+\lambda)AB^2e^{3\Re(\lambda)t} = 0.\end{aligned}$$

Note here A' and B' denote $\frac{dA}{d\tau}$ and $\frac{dB}{d\tau}$ respectively.

The amplitude equations contain the variable t explicitly. Since we want a clean amplitude equation, which depends only on the other time scale τ, we let

$$\frac{d\tau}{dt} = \epsilon e^{2\Re(\lambda)t}. \tag{4.10}$$

Integrating this equation and requiring $\tau(0) = 0$, we have

$$\tau = \frac{\epsilon}{2\Re(\lambda)}(e^{2\Re(\lambda)t} - 1) \tag{4.11}$$

Note that as $\Re(\lambda) \to 0$, $\tau \to \epsilon t$. So the system without damping is just a degenerate case.

Then we have

$$(2\lambda - 1)A' + (2\lambda + \bar{\lambda})A^2B = 0$$

$$(2\bar{\lambda} - 1)B' + (2\bar{\lambda} + \lambda)AB^2 = 0.$$

This system can be solved analytically. Substituting λ, we get

$$A' + (\frac{1}{2} - i\frac{\sqrt{3}}{2})A^2B = 0$$
$$B' + (\frac{1}{2} + i\frac{\sqrt{3}}{2})AB^2 = 0.$$

Multiply B to the first equation, A to the second equation and then add them together to get

$$\frac{dAB}{d\tau} + A^2B^2 = 0.$$

Hence

$$AB(\tau) = \frac{1}{\tau + C}$$

So the equations become

$$A' + (\frac{1}{2} - i\frac{\sqrt{3}}{2})(\frac{1}{\tau + C})A = 0$$
$$B' + (\frac{1}{2} + i\frac{\sqrt{3}}{2})(\frac{1}{\tau + C})B = 0.$$

These are linear equations and can be easily integrated. Then we have

$$\log(A) + (\frac{1}{2} - i\frac{\sqrt{3}}{2})\log(\tau + C) + C_1 = 0$$
$$\log(B) + (\frac{1}{2} + i\frac{\sqrt{3}}{2})\log(\tau + C) - C_1 = 0$$

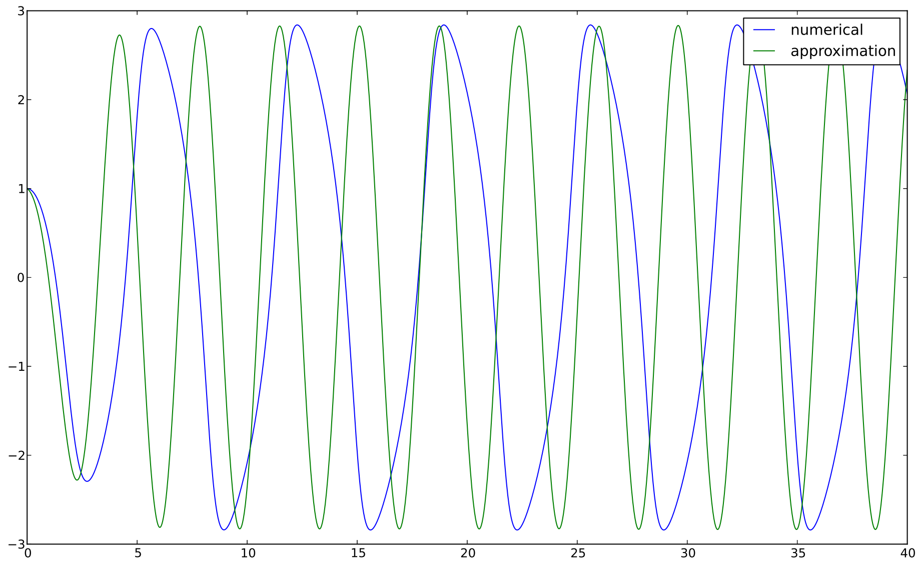

Figure 4.1: The numerical result and the approximation with $\epsilon = 0.5$.

We can compare this approximation with the numerical result. From Fig. 4.1, it is obvious that the approximation is *not* all that good. However, if we concentrate only on the behavior of the amplitude, then the result is seen to be amazingly good. For one thing, the exponentially growing amplitude of the zeroth order solution is cancelled out almost totally, and the amplitude of the approximation agrees with the numerical result very well.

Since we have got the amplitude right at the very least, there must be some ingredient in our approach that has got things right. But what is it? This is the part that we have not been able to crack. Encouraged by this coincidence in the good fit of the amplitude, we are now actively investigating a possible modification of our approach. Though this endeavour is quite far away from our original fluid mechanics problem of the swirling flow, it does seem to be a worthy detour in our research.

Chapter 5

Conclusion

By comparing the numerical results and the linearized solution, we show that the overall feature of the similarity solution of von Kármán's swirling flow can be captured by the linearized solution. The minor discrepancy in the amplitude and the phase shift presumably can be accounted for by the nonlinearity intrinsic to the problem.

To better understand why the linear solution works so well for the nonlinear system, we suspected that an amplitude equation approach could save the day. This conviction led us to the study of a much simplified model equation for which the idea could be developed and put to the test. In the new perturbation scheme we considered, we have modified the traditional multiple scale analysis into a form that allows a variable long time scale, which does reduce to the method of two-timing for oscillators containing a weak cubic restoring force. However, our method suffers from the fact that the computed oscillation is way off from the actual solution in phase, even though the amplitude matches rather well. This suggests that we may be on the right track, and further polishing of the idea might bring us a much closer agreement between the theory and the actual solution. Further investigation along this line is currently under way.

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