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利用邊界測量重建彈性物體中的未知物

Reconstruction of Unknown Inclusions in an Elastic Medium by Boundary Measurements

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本論文係關汝琳（學號 D97221002）在國立臺灣大學數學系、所完成之博士學位論文，於民國 102 年 6 月 5 日承下列考試委員審查通過及口試及格，特此證明

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去年，王振男老師也鼓勵並推薦我申請千里馬計畫，前往美國加州訪問加州大學爾灣分校及華盛頓大學的 Uhlmann 教授。Uhlmann 教授雖然已經是地位崇高的教授，但他對我仍是非常友善。這次的訪美經驗，讓我對這個世界又有了多一層的認識。在訪美期間，Uhlmann 教授也給予我許多幫助。不但邀請我參加不同的學術活動，也給了我研究的新方向。雖然訪問 Uhlmann教授的時間並不是太長，但他給我的幫助卻一直持續，非常感謝他。

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中文摘要

這篇博士論文討論的是如何重建彈性物體中未知物的形狀與位置。我們考慮以下的反問題。有一個彈性物質位於Ω，$\Omega \subset \mathbb{R}^n$，$n = 2, 3$。假設這個彈性物質中有一個未知的物體位在$D$，$D \subset\subset \Omega$，並且此未知的物體與背景已知的彈性物質有相當差異的彈性性質。那麼我們要如何重新建構這個未知物的形狀與位置？我們所使用的方法是包圍法(enclosure-type methods)。包圍法是一個只利用邊界測量來建構內部未知物的方法，它是由Ikehata最先提出的[10,12]。所以它是一個非侵入性的探測方法。利用非侵入的方法探測物體的內部是一個很重要的議題，因爲它可以被當成一個安全的醫療診斷的工具。在第二章的部份，我們會在數學上解釋包圍法的想法與相關的結果。

包圍法已被應用在許多不同的數學模型中，如[14,15,16,23, 24,29,36,40]。其中一個包圍法中主要的探測工具就是複幾何光學解Complex geometrical optics (CGO) solutions。我們在這篇論文中把包圍法推廣應用在time-harmonic彈性波上。在我們所討論的數學模型中最大的困難是time-harmonic彈性波中有一個零階項。這個零階項的估計會影響我們如何去應用包圍法。我們可以參考這篇survey paper [40]。在第三章及第四章中，我們分別討論兩種不同的未知物：可滲透的未知物與不可滲透的未知物。在第三章中我們只考慮二維情形，並採用CGO solutions with complex polynomial phases作爲主要的探測工具。關於第三章可滲透的情形，先前一些文章也有討論過類似的問題，如[23,29]。在[23,29]中作者們給了一些邊界平滑性的假設。在這一章中，我們修改並推廣[29]中的做法，在二維的情形下將邊界的平滑性從Lipschitz降爲連續。在第四章不可滲透的情形中，二維跟三維的情形都有考慮進去。在第四章中三維的情形下，我們採用了CGO solutions with linear phases作爲主要探測工具。這個探測工具只能探測出未知物的convex hull。

關鍵字：反問題、包圍法(enclosure method)、time-harmonic彈性波方程組、複幾何光學解、可滲透的、不可滲透的

Abstract

The goal of this dissertation is to discuss how to reconstruct the shape and location of the unknown inclusions in an elastic body. We consider the following inverse problem. There is an elastic body occupying Ω, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Assume there is an unknown inclusion in Ω, which is denoted by D, and assume the Lamé coefficients of D is definitely different from those of the background material. Then how do we reconstruct the shape and location of the unknown inclusions? The method we will use is the so-called enclosure-type method. The enclosure-type method is a method of constructing inclusions only from boundary measurements, which is initiated by Ikehata [10,12]. Therefore it is a non-invasive reconstruction method. Utilizing non-invasive methods to detect the internal information of subjects is a very important issue, because they are probably proposed as a safe diagnostic tool in medical imaging. In the second chapter of this thesis, we will mathematically explain the idea of the enclosure method and discuss the related results.

The enclosure-type methods have been applied to many different mathematical models. See [14,15,16,23,24,29,36,40] for reference. One of the main probing tools of the enclosure method is complex geometrical optics (CGO) solutions. In this thesis, we extend and apply the enclosure-type methods to the time-harmonic elastic waves. The most difficult point for this model is the presence of zeroth order term in time-harmonic elastic waves. The estimate of the zeroth order term will influence on how to apply the enclosure method. We can see the survey paper [40] for Helmholtz-type equations. In chapter 3 and 4, we discuss the following two cases respectively: penetrable inclusions and impenetrable inclusions.

In Chapter 3, we only consider the two dimension case and adopt the CGO solutions with polynomial phases as a main probing tool. In the previous research similar to our problem, such as [23,29], the authors gave some regularity

assumptions on the boundary of inclusion D. In this chapter, we modify the approach of [29] and reduce the regularity assumption on ∂D from Lipschitz continuity to continuity. In Chapter 4, the impenetrable case, two and three dimension are considered. In three dimension, CGO solutions with linear phases are adopted as the probing tool. However using such probing tool, only the convex hull of D can be detected.

Keywords: inverse problems, enclosure method, time-harmonic elastic waves, complex geometrical optics (CGO) solutions, penetrable, impenetrable

Contents

Chapter 1

Introduction

Inverse boundary value problems is a field of discussing the inverse problems of partial differential equations. Since A. P. Calderón published his pioneering work "On an inverse boundary value problem" in 1980s, inverse boundary value problems have become a popular field in mathematics. The problem Calderón proposed is: whether we can determine the conductivity of an electrical material by measuring the voltage and current on its boundary. More precisely, by applying a specific voltage density on the boundary, there corresponds a current which can also be measured on the boundary. This correspondence is the so-called Direct-to-Neumann map. And the question is whether this map uniquely determines the conductivity distribution of the whole material. This problem led to the development of the Electrical Impedance Tomography (EIT), which is designed as a safe and low cost device for medical diagnosis as well as many other applications [37, 1, 8, 5, 17]. The problem then gives rise to the general idea of gaining informations from boundary data, which applies to many other kind of physical settings. Moreover, the questions gradually evolve from theoretical determinations to practical reconstructions. That is, not only being satisfied by knowing the boundary data will determine the material property, a great deal of researches now make efforts to give concrete algorithms.

In this thesis, what we mainly discuss is how to reconstruct unknown inclusions in a known background from the boundary information. This non-invasive method that we use in this thesis is a type of enclosure method initiated by Ikehata. We would give a more detailed description about the enclosure method and its related results in Chapter 2. Besides, we will apply the enclosure-type method to the time-harmonic elastic wave equations. A crucial point in this model is the presence of the zeroth order term. See [39] for reference. In Chapter 3 and 4, we will discuss our main results about penetrable and impenetrable unknown inclusions respectively.

In Chapter 3, we only consider the two dimension case and adopt the CGO solutions with polynomial phases as a main probing tool. In the previous research similar to our problem, such as [23,29], the authors gave some regularity assumptions on the boundary of inclusion D. In [23], the authors assumed the regularity of ∂D is C^2. And later in [29], the authors reduced the regularity assumption on ∂D to Lipschitz. In this chapter, we modify the approach of [29] and reduce again the regularity assumption on ∂D to continuity. In Chapter 4, the impenetrable case, two and three dimension are considered. In three dimension, CGO solutions with linear phases are adopted as the probing tool. However using such probing tool, only the convex hull of D can be detected. In the impenetrable case, we only assume the regularity of ∂D is C^2. And in the final chapter, some open problems and future work will be mentioned.

Chapter 2

The enclosure-type method: a reconstruction method of unknown inclusions

The enclosure method is a method to reconstruct unknown inclusions in a known background, which is initiated by Ikehata. See Ikehata's survey paper [12] for reference. The main tool of this reconstruction method is the following two (which will be defined later): the indicator functional and the complex geometrical optics (CGO) solutions. The idea of these two tools can be tracked back to the Calderón's work. Therefore in this chapter, we will start from the Calderón problem and then show how these two tools work.

2.1 Calderón's foundational paper

Since Calderón published his pioneering work "On an inverse boundary value problem" in 1980s [6], his work has influenced deeply the development of the inverse boundary value problems. Here we just briefly introduce the problem in [6] and emphasize the influence on the enclosure method.

The problem he concerned is a very interesting and meaningful problem: how to reconstruct conductivity of an unknown object by using boundary measurements? Precisely, let us consider that a conducting material with unknown conductivity occupy a bounded domain Ω. Then the conductivity and the voltage of the material are governed by the conductivity equations

$$\nabla \cdot (\gamma \nabla u) = 0, \quad \text{in} \quad \Omega, \tag{2.1.1}$$

If the conductivity γ is known, then when applying a voltage $u|_{\partial\Omega} = f$ on the boundary we can measure the corresponding current $\gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ on the boundary, where ν is the outer normal of $\partial\Omega$. This correspondence is the so-called Dirichlet-to-Neumann map (or called voltage-to-current map), which is given by

$$\Lambda_\gamma(f) = \big(\gamma \frac{\partial u}{\partial \nu}\big)\big|_{\partial\Omega}.$$

The inverse problem Calderón concerned can be therefore stated mathematically as determining γ from the knowledge of Λ_γ. This problem is not easy to deal with directly, so from the divergence theorem, Calderón consider the following nonlinear map

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 d\mathbf{x} = \int_{\partial\Omega} \Lambda_\gamma(f) \bar{f} d\mathbf{s},$$

where $d\mathbf{s}$ denotes the surface measure, and u solves (2.1.1) with Dirichlet boundary condition $u|_{\partial\Omega} = f$. Then the inverse problem becomes to determine γ from Q_γ. Calderón proved that the map Q_γ is analytic in [6] and the Fréchet derivative of it at γ_0 is injective in γ, when γ_0 is a constant. That means the linearization of the map from γ to Q_γ is injective at constant conductivities. Moreover, he also gave an approximation formula to reconstruct a conductivity which is close to a constant conductivity.

Calderón's work has a deep influence in the development of inverse boundary value problems. The idea of Q_γ is widely applied to subsequent inverse problems. For example, the indicator functional, which is a key tool in the next section (enclosure-type method), is one application of Q_γ. Besides, in [6], Calderón took the special harmonic functions $u = e^{\mathbf{x}\cdot(\boldsymbol{\rho} + i\boldsymbol{\rho}^\perp)}$ as a helper in order to show the injectivity of the linearized map, where $\boldsymbol{\rho} \in \mathbb{C}^n$ with $\boldsymbol{\rho} \cdot \boldsymbol{\rho}^\perp = 0$. This is the origin of the *complex geometrical optics* solutions, which we will discuss later in the next section. Subsequently, the class of CGO solutions becomes a very important tool in studying inverse problems. For more details of the development in inverse problems, we refer the survey paper [37].

By extending the brand new idea Calderón proposed, we can not only determine the conductivities within the subjects from boundary information, but also reconstruct the unknown inclusions within a subject. The enclosure method is the one example we want to discuss. The situation which the enclosure method can be applied to is as follow: the subject contains unknown inclusions, of which the conductivity is unknown and apparently different from that of the background. The enclosure method is not only a theoretical identification method for unknown inclusions, but provides a reconstruction algorithm for drawing the unknown inclusions. In the next section, we will describe the idea of the enclosure method carefully.

2.2 The idea of the enclosure method

To describe the idea of the enclosure method clearly, we take the following case as an example. Suppose the subject we concern is a conducting material, which is governed by the conductivity equation. We suppose the subject occupies the domain $\Omega \subset \mathbb{R}^n$ and the unknown inclusion occupied $D \subset \mathbb{R}^n$ with $D \subset\subset \Omega$. Here we consider the simplest case: the known conductivity γ_0 of the subject (without the unknown inclusions) is 1. And we denote by γ the total conductivity of the subject with the unknown inclusions. Therefore we have the following two conductivity equations:

$$\triangle u_0 = 0 \text{ in } \Omega \tag{2.2.2}$$

and

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \tag{2.2.3}$$

where $\gamma = 1 + \chi_D \gamma_D$. χ_D is the characteristic function of domain D and γ_D is difference between the conductivities within D and within $\Omega \setminus D$. u and u_0 denote the voltages of the situation with and without unknown inclusions D respectively. Then we can define the Dirichlet-to-Neumann maps for (2.2.2) and (2.2.3) as follows: for a Dirichlet boundary condition $f \in H^{1/2}(\partial\Omega)$

$$\begin{aligned} \Lambda_\gamma(f) &= \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} \\ \Lambda_{\gamma_0}(f) &= \gamma_0 \frac{\partial u_0}{\partial \nu}\bigg|_{\partial\Omega}, \end{aligned} \tag{2.2.4}$$

where ν is the outer normal of $\partial\Omega$.

Now it is ready to introduce the idea of the enclosure method. There are two main tools in this method: the indicator functional and a sequence of special functions (in fact they will be called CGO solutions in next section).

First we introduce the indicator functional, of which the idea comes from Q_γ in the previous section: for a Dirichlet boundary condition $f \in H^{1/2}(\Omega)^n$, $n = 2, 3$,

$$E(f) = \int_{\partial\Omega} \big(\Lambda_\gamma(f) - \Lambda_{\gamma_0}(f)\big) \cdot f ds. \tag{2.2.5}$$

Roughly speaking, it measures, for a given voltage on the boundary, the difference between currents or energies corresponding to the situations with and without D. Moreover, we can easily deduce that

$$E(f) \approx C \int_D |\nabla u_0|^2 d\mathbf{x},$$

for some constant C independent of f and u_0, where u_0 is the solution of Laplace equation (2.2.2) with $u_0|_{\partial\Omega} = f$.

On the other hand, we observe that for any $h > 0$, $\omega, \omega^\perp \in \mathbb{S}^{n-1}$ with $\omega \cdot \omega^\perp = 0$,

$$e^{\frac{1}{h}(\omega\cdot\mathbf{x} + i\omega^\perp\cdot\mathbf{x})}$$

is a solution of Laplace equation. Notice that it is also the special function Calderón proposed in [6]. Thus set, for any numbers d,

$$u_{0,d,h} = e^{-\frac{d}{h}} e^{\frac{1}{h}(\omega\cdot\mathbf{x} + i\omega^\perp\cdot\mathbf{x})},$$

then we have $\triangle\, u_{0,d,h} = 0$.

Now let $f_{0,d,h} = u_{0,d,h}|_{\partial\Omega}$ and take $f_{0,d,h}$ into the indicator functional E. Then we have

$$\begin{aligned} E(f_{0,d,h}) &\approx C \int_D |\nabla u_{0,d,h}|^2 d\mathbf{x} \\ &\approx C' \frac{1}{h^2} \int_D e^{\frac{2}{h}(\omega\cdot\mathbf{x}-d)} d\mathbf{x} \end{aligned}$$

for some constants C, C' independent of h. Then we can deduce as follows: for convenience, we let

$$\rho(\mathbf{x}) = \omega \cdot \mathbf{x} + i\omega^{\perp} \cdot \mathbf{x}$$

be the phase function and denote the real part of ρ by $Re(\rho)$. We have

1. If $D \cap \overline{\{\mathbf{x} : Re(\rho) - d > 0\}} = \emptyset$, then it means

$$\omega \cdot \mathbf{x} - d < 0,\ \forall \mathbf{x} \in D.$$

   Therefore we have

$$E(f_{0,d,h}) \to 0, \text{ as } h \searrow 0.$$

2. If $D \cap \{\mathbf{x} : Re(\rho) - d > 0\} \neq \emptyset$, then it means there exist an open set $U \subset D$ such that

$$\omega \cdot \mathbf{x} - d \geq 0, \text{ if } \mathbf{x} \in U.$$

   Therefore we have

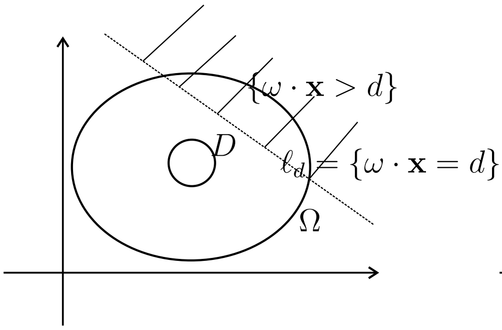
$$E(f_{0,d,h}) \to \infty, \text{ as } h \searrow 0.$$

Now, denote the level set of the real part of the phase function ρ at t by ℓ_t. That is,

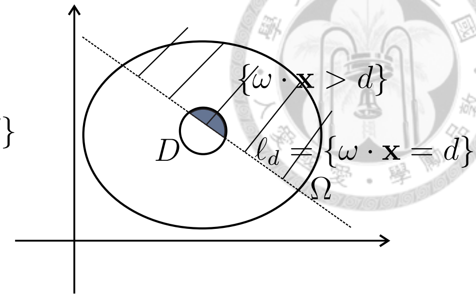
$$\ell_t := \{\mathbf{x} : Re(\rho(\mathbf{x})) = t\}.$$

And let

$$\Gamma_d := \overline{\bigcup_{d<t<\infty} \ell_t}.$$



(a) $D \cap \overline{\{\mathbf{x} : Re(\rho) - d > 0\}} = \emptyset$



(b) $D \cap \{\mathbf{x} : Re(\rho) - d > 0\} \neq \emptyset$

Then from the above deduction we can conclude that if we choose these $f_{0,d,h}$ as our testing data, then the limiting behavior of $E(f_{0,d,h})$ will indicate whether Γ_d intersects the unknown inclusion D. By varying d, we can theoretically find which level set ℓ_d just touches the unknown inclusion. Hence we the following conclusion:

the limiting behavior of $E(f_{0,d,h})$ will indicate which level set ℓ_d just touches the unknown inclusion D.

Hence we call the functional E the indicator functional. Actually, in [11] Ikehata called $E(f_{0,d,h})$ the indicator function.

Now we summarize the above idea as follows. First we establish the indicator functional E. Next, we will construct a suitable sequence of test boundary data $\{f_{d,h}\}_{h\to 0}$ such that, by taking in such a boundary data, the limiting behavior of $E(f_{d,h})$ will indicate which specific hyperplane just touches D. Technically, we choose the special solutions Calderón proposed as the testing data and the specific hyperplane is the level curve of the real part of the phase function ρ. And then we can prove that the limiting behavior of $E(f_{d,h})$ has a sharp difference between the cases of $\bar{D} \cap \Gamma_d = \emptyset$ and $\bar{D} \cap \Gamma_d \neq \emptyset$.

By performing such procedure repeatedly from different directions, that is adopting different ω, we can collect more and more planes touching ∂D, and find out the location and shape of D. It looks like one uses the planes to enclose the unknown inclusion, hence the name enclosure method.

2.3 Complex geometrical optics solutions and some related results

Since Ikehata proposed the idea of the enclosure method, there have been many results of extending the idea to many other kind of physical settings. In the following we try to show how to extend the idea to different physical settings and the related results.

Remember that the two main tools in the enclosure method are indicator functional and a sequence of suitable special functions. In different mathematical models, it should be not difficult to define similar corresponding indicator functionals. However it is not easy to find a suitable sequence of functions as test data. We try to extend the idea of the above Calderón type functions $e^{\frac{1}{h}x\cdot(\omega+i\omega^{\perp})}$. To do this, we notice that the above Calderón type functions are harmonic functions and they have "*complex phases*". So one idea of constructing a suitable test data is to find solutions of the corresponding model having the form

$$e^{i\frac{1}{h}\boldsymbol{\rho}(\mathbf{x})}(a(\mathbf{x})+R_h(\mathbf{x}))$$

with a "*complex phase*" function $\boldsymbol{\rho}$, where $R_h \ll a$ as $h \to 0^+$. The solutions with this form are the so-called "*complex geometrical optics solutions*"(CGO solutions). This is the second tool in enclosure-type method.

There are some results for proving the existence of CGO solutions for various mathematical models, for example [31, 32, 16, 26, 27, 9, 35]. And these articles also show that CGO solutions are useful in inverse boundary value problems. In particular, CGO solutions usually play the important role of the probing utility in enclosure type method, see for example [10, 12, 14, 15, 28, 9, 33, 35, 40, 23, 29].

From line phases to general phases

In Ikehata's early works, he used the Calderón type harmonic function $e^{x\cdot(\omega+i\omega^{\perp})}$ to construct the test data. So, as mentioned in the previous section, it looks like one uses lines (planes) to enclose the obstacle (and hence the name). As a consequence a connected inclusion is required to be convex for a complete identification, and in general only its convex hull can be determined. One can refer to the survey paper [13] for detailed explanation and early development of this theory. In [28], [24] and [9], the authors utilize the complex spherical wave solutions and some concave parts of unknown inclusions can be determined. In [35], due to the complex structure, the authors proposed a framework of constructing CGO solutions with general phases for some elliptic systems in two-dimension. It means this work provides more choices of phase functions of CGO solutions in $2D$. They also gave a concrete example: the CGO solutions with complex polynomial phases. In the same paper they also applied CGO solutions with complex polynomial phases to conductivity equations, and then inclusions with more general shapes can be determined. This type of CGO solutions were later applied to other equations, for example [36] for static elastic systems and [23] for Helmholtz equations.

Non-Laplacian leading term

The mathematical models we have mentioned above are almost equations or systems with the Laplacian as the leading order term or which can be reduced to the equations with the Laplacian as the leading term. To deal with more general cases, we consider the equations (or systems) with non-Laplacian leading order term. However, the anisotropy of non-Laplacian leading term prevents us from constructing CGO solutions by traditional methods. As a result, the authors in [25] proposed another type of CGO solutions, called "*oscillating-decaying solutions*". These oscillating-decaying solutions are also useful in inverse problems, especially in detecting unknown inclusions.

Chapter 3

Reconstruction of penetrable inclusions

In this chapter we consider the inverse problem of reconstructing penetrable unknown inclusions in a plane elastic body by boundary measurements. In [34] and [36], the same problem is considered in the context of elastostatics. In the present work we shall consider the situation when time-harmonic waves are applied.

We use Ikehata's enclosure method to reconstruct penetrable unknown inclusions in a plane elastic body in time-harmonic waves. Complex geometrical optics solutions with complex polynomial phases are adopted as the probing utility. In a situation similar to ours, due to the presence of a zeroth order term in the equation, some technical assumptions need to be assumed in early researches. In a recent work of Sini and Yoshida, they succeeded in abandoning these assumptions by using a different idea to obtain a crucial estimate. In particular the boundaries of the inclusions need only to be Lipschitz. In this work we apply the same idea to our model. It's interesting that, with more careful treatment, we find the boundaries of the inclusions can in fact be assumed to be only continuous.

The content of this chapter comes from [19].

3.1 Introduction

3.1.1 Mathematical model

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain (open connected set) occupied by our object, which consists of an elastic body as background and some unknown inclusions therein. For simplicity we assume Ω has $\mathcal{C}^\infty$ boundary. The background elastic body will be assumed to be homogeneous and isotropic with Lamé constants denoted by λ_0 and μ_0. Denote the region of unknown inclusions by D. D is an open subset of Ω with $\bar{D} \subset \Omega$. The inclusions are also assumed to be isotropic but may be inhomogeneous. Denote the differences between the Lamé coefficients of the inclusions and the background by λ_D and μ_D, which are assumed to be in $L^\infty(\Omega)$, with $\lambda_D = \mu_D = 0$ on $\Omega \setminus \bar{D}$. So the Lamé coefficients λ and μ of the whole object on Ω are given by

$$\lambda = \lambda_0 + \lambda_D \quad \text{and} \quad \mu = \mu_0 + \mu_D.$$

For simplicity we also assume our object has unit density. Now, consider we send a time-harmonic elastic wave with time dependence e^{ikt} into Ω. By singling out the space part we have the displacement field $\mathbf{u}$, which is a two-component vector-valued function, satisfying

$$\nabla \cdot (\sigma(\mathbf{u})) + k^2 \mathbf{u} = 0 \quad \text{in } \Omega. \tag{3.1.1}$$

Here, for any displacement field $\mathbf{v}$ (which we will assumed to be a column vector), $\sigma(\mathbf{v})$ is the corresponding stress tensor, which is represented by a 2×2 matrix:

$$\sigma(\mathbf{v}) = \lambda(\nabla \cdot \mathbf{v}) I_2 + 2\mu\epsilon(\mathbf{v}),$$

where I_2 is the 2×2 identity matrix and $\epsilon(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ denotes the infinitesimal strain tensor. Note that for $\mathbf{v} = (v_1, v_2)^T$, $\nabla \mathbf{v}$ denotes the 2×2 matrix whose j-th row is ∇v_j for $j = 1, 2$. And for a 2×2 matrix function A, $\nabla \cdot A$ denotes the column vector whose j-th component is the divergence of the j-th row of A for $j = 1, 2$.

For $D = \emptyset$, that is for the case with no inclusion, the corresponding displacement field will usually be denoted by $\mathbf{u}_0$, which satisfies

$$\nabla \cdot (\sigma_0(\mathbf{u}_0)) + k^2 \mathbf{u}_0 = 0 \quad \text{in } \Omega, \tag{3.1.2}$$

where

$$\sigma_0(\mathbf{v}) = \lambda_0(\nabla \cdot \mathbf{v}) I_2 + 2\mu_0 \epsilon(\mathbf{v})$$

for any displacement field $\mathbf{v}$. Accordingly, we will use $\sigma_D(\mathbf{v})$ to denote $\sigma(\mathbf{v}) - \sigma_0(\mathbf{v})$, i.e.

$$\sigma_D(\mathbf{v}) = \lambda_D(\nabla \cdot \mathbf{v}) I_2 + 2\mu_D \varepsilon(\mathbf{v}).$$

We assume λ_0, μ_0 and λ, μ satisfy the conditions

$$\begin{aligned} &\lambda_0 + 2\mu_0 > 0,\ \mu_0 > 0,\ \text{and} \\ &\lambda + 2\mu > 0,\ \mu > 0 \quad \text{on } \Omega, \end{aligned} \tag{3.1.3}$$

which ensure respectively that $-\nabla \cdot \sigma_0$ and $-\nabla \cdot \sigma$ are strongly elliptic operators. In particular the two operators both have at most countably many Dirichlet eigenvalues. As a consequence, we can readily choose (and will choose) $k \in \mathbb{R}$ so that k^2 is neither an eigenvalue of $-\nabla \cdot \sigma_0$ nor an eigenvalue of $-\nabla \cdot \sigma$. In this situation, the Dirichlet boundary value problems corresponding to (3.1.1) and (3.1.2) have unique solutions (see e.g. Ch.4 of [21]). Thus we can define the Dirichlet-to-Neumann maps Λ_D and $\Lambda_\emptyset$, both from $H^{\frac{1}{2}}(\partial\Omega)^2$ to $H^{-\frac{1}{2}}(\partial\Omega)^2$, by

$$\Lambda_D \mathbf{f} = \sigma(\mathbf{u})\boldsymbol{\nu}|_{\partial\Omega} \quad \text{and} \quad \Lambda_\emptyset \mathbf{f} = \sigma_0(\mathbf{u}_0)\boldsymbol{\nu}|_{\partial\Omega}, \tag{3.1.4}$$

where $\boldsymbol{\nu}$ is the unit outer normal on $\partial\Omega$ and $\mathbf{u}$ and $\mathbf{u}_0$ solve respectively (3.1.1) and (3.1.2) with Dirichlet boundary data $\mathbf{f}$. The goal is to determine the unknown inclusions from the knowledge of Λ_D and $\Lambda_\emptyset$.

3.1.2 The method and improvement

We will utilize the enclosure-type method to reconstruct the unknown inclusions. In this method, complex geometrical optics (CGO) solutions usually play the important role of the probing utility. Technically, the phases of these CGO solutions influence what kind of shapes of inclusions we can detect. In this work we will also apply CGO solutions with complex polynomial phases to our problem, of which the governing equations are the Helmholtz type elastic systems (3.1.1) and (3.1.2). These special solutions are first proposed in [35, 23].

A crucial point in our problem, as in [12, 24, 23], is the presence of the zeroth order term. Due to this, some technical assumptions are needed in early researches. In particular ∂D is assumed to be $\mathcal{C}^2$. However in the recent work [29] of Sini and Yoshida, by using a different idea to obtain a crucial estimate, they succeeded in abandoning these technical assumptions, and in particular ∂D can be only Lipschitz. In this paper, we apply the same idea to our model. With more careful treatment, we find the boundaries of the inclusions can in fact be assumed to be only continuous. More detailed discussions are given in the remark after our main theorem, Theorem 3.4.1.

In the following we give a sketch of this chapter as well as a rough idea of the whole process of the enclosure method. In section 2, we introduce a functional E on $H^{\frac{1}{2}}(\partial\Omega)^2$, which will be called the indicator functional for our model in Chapter 3. And then we give an upper bound and a lower bound of E, which play central roles in the proof of the main theorem. In fact, we will construct a family $\mathbf{f}_{d,h} \in H^{\frac{1}{2}}(\partial\Omega)^2$ as input data into E, and the limiting behavior of the output data, for various d, will indicate the location of ∂D. The construction of $\mathbf{f}_{d,h}$ is based on the construction of CGO solutions for (3.1.2), which is given in section 3. By using the Helmholtz decomposition and the Vekua transform, this construction is much the same as in [23]. The main theorem concerning the limiting behavior of E on $\mathbf{f}_{d,h}$, as well as discussions on the implication, the idea of proof and our improvement, are given in section 4.

3.2 The indicator functional

In this section we introduce the functional E on $H^{\frac{1}{2}}(\partial\Omega)$ defined by

$$E(\mathbf{f}) = \int_{\partial\Omega} [(\Lambda_D - \Lambda_\emptyset)\mathbf{f}] \cdot \bar{\mathbf{f}} ds,$$

where the Dirichlet-to-Neumann maps Λ_D and $\Lambda_\emptyset$ are defined in (3.1.4). E will be called the indicator functional (according to Ikehata's indicator function [11]), which plays a central role in the enclosure method. Intuitively, it measures, for a fixed Dirichlet boundary data, the difference between the tractions corresponding to the situations with and without D.

Now let $\mathbf{u}$ and $\mathbf{u}_0 \in H^1(\Omega)^2$ satisfy (3.1.1) and (3.1.2) respectively with the same boundary condition $\mathbf{f} \in H^{\frac{1}{2}}(\partial\Omega)^2$, and let $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$. The goal in this section is to prove Lemma 3.2.2, which gives a lower bound and an upper bound of $E(\mathbf{f})$ in terms of $\mathbf{u}_0$ and $\mathbf{w}$. To this end, we first give two identities. Note that we use $|A|$ to denote $\left(\sum_{i,j} a_{ij}^2\right)^{1/2}$ for a matrix $A = (a_{ij})$.

Lemma 3.2.1. *We have the following two identities:*

$$\begin{aligned} E(\mathbf{f}) = &\int_D \left\{ (\lambda_D + \mu_D) \left|\nabla \cdot \mathbf{u}_0\right|^2 + 2\mu_D \left| \epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0) I_2 \right|^2 \right\} dx \\ &- \int_\Omega \left\{ (\lambda + \mu) \left|\nabla \cdot \mathbf{w}\right|^2 + 2\mu \left| \epsilon(\mathbf{w}) - \frac{1}{2}(\nabla \cdot \mathbf{w}) I_2 \right|^2 \right\} dx \\ &+ \int_\Omega k^2 \left|\mathbf{w}\right|^2 dx; \end{aligned} \tag{3.2.1}$$

$$\begin{aligned} E(\mathbf{f}) = &\int_D \left\{ (\lambda_D + \mu_D) \left|\nabla \cdot \mathbf{u}\right|^2 + 2\mu_D \left| \epsilon(\mathbf{u}) - \frac{1}{2}(\nabla \cdot \mathbf{u}) I_2 \right|^2 \right\} dx \\ &+ \int_\Omega \left\{ (\lambda_0 + \mu_0) \left|\nabla \cdot \mathbf{w}\right|^2 + 2\mu_0 \left| \epsilon(\mathbf{w}) - \frac{1}{2}(\nabla \cdot \mathbf{w}) I_2 \right|^2 \right\} dx \\ &- \int_\Omega k^2 \left|\mathbf{w}\right|^2 dx. \end{aligned} \tag{3.2.2}$$

Lemma 3.2.2. *Assume that the Lamé coefficients λ_0, μ_0 and λ, μ satisfy the strong convexity condition, that is*

$$\lambda_0 + \mu_0,\ \mu_0 > 0 \quad \textit{and} \quad \lambda + \mu,\ \mu > 0,$$

then we have the following upper bound and lower bound of $E(\mathbf{f})$:

$$\begin{aligned}
E(\mathbf{f}) \le & \int_D (\lambda_D + \mu_D) \left|\nabla \cdot \mathbf{u}_0\right|^2 dx \\
& + 2\int_D \mu_D \left|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0) I_2\right|^2 dx + \int_\Omega k^2 \left|\mathbf{w}\right|^2 dx; \\
E(\mathbf{f}) \ge & \int_D \frac{(\lambda_D + \mu_D)(\lambda_0 + \mu_0)}{\lambda + \mu} \left|\nabla \cdot \mathbf{u}_0\right|^2 dx \\
& + 2\int_D \frac{\mu_D \mu_0}{\mu} \left|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0) I_2\right|^2 dx - \int_\Omega k^2 \left|\mathbf{w}\right|^2 dx.
\end{aligned}$$

Proof. The upper bound of $E(\mathbf{f})$ follows immediately from (3.2.1) (by omitting the second integral). On the other hand, from (3.2.2) we have

$$\begin{aligned}
E(\mathbf{f}) \ge & \int_D \left\{ (\lambda_D + \mu_D)|\nabla \cdot \mathbf{u}|^2 + 2\mu_D \left|\epsilon(\mathbf{u}) - \frac{1}{2}(\nabla \cdot \mathbf{u}) I_2\right|^2 \right\} dx \\
& + \int_D \left\{ (\lambda_0 + \mu_0)|\nabla \cdot \mathbf{w}|^2 + 2\mu_0 \left|\epsilon(\mathbf{w}) - \frac{1}{2}(\nabla \cdot \mathbf{w}) I_2\right|^2 \right\} dx \qquad (3.2.3) \\
& - \int_\Omega k^2 |\mathbf{w}|^2 dx.
\end{aligned}$$

And the lower bound follows from the following two identities, of which the verifications are straightforward (by using $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$).

(i)

$$\begin{aligned}
& (\lambda_D + \mu_D) \left|\nabla \cdot \mathbf{u}\right|^2 + (\lambda_0 + \mu_0) \left|\nabla \cdot \mathbf{w}\right|^2 \\
& = \left(\sqrt{\lambda + \mu} \nabla \cdot \mathbf{u} - \frac{\lambda_0 + \mu_0}{\sqrt{\lambda + \mu}} \nabla \cdot \mathbf{u}_0 \right)^2 + \frac{(\lambda_D + \mu_D)(\lambda_0 + \mu_0)}{\lambda + \mu} \left|\nabla \cdot \mathbf{u}_0\right|^2.
\end{aligned}$$

(ii)

$$\begin{aligned}
& 2\mu_D \left|\epsilon(\mathbf{u}) - \frac{1}{2}(\nabla \cdot \mathbf{u}) I_2\right|^2 + 2\mu_0 \left|\epsilon(\mathbf{w}) - \frac{1}{2}(\nabla \cdot \mathbf{w}) I_2\right|^2 \\
& = \sum_{i,j} \left| \sqrt{2\mu} b_{ij} - \frac{2\mu_0}{\sqrt{2\mu}} b_{ij}^0 \right|^2 + \frac{2\mu_D \mu_0}{\mu} \left|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0) I_2\right|^2,
\end{aligned}$$

where

$$(b_{ij}) := \epsilon(\mathbf{u}) - \frac{1}{2}(\nabla \cdot \mathbf{u}) I_2 \quad \text{and} \quad (b_{ij}^0) := \epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0) I_2.$$

□

For completeness we give the proof of Lemma 3.2.1 in the following. Before doing so, note that we have the following basic formulae:

$$\nabla \cdot (\sigma(\mathbf{u})\mathbf{v}) = (\nabla \cdot \sigma(\mathbf{u})) \cdot \mathbf{v} + tr(\sigma(\mathbf{u})\nabla \mathbf{v}); \tag{3.2.4}$$

$$tr(\sigma(\mathbf{u})\nabla \mathbf{v}) = tr(\sigma(\mathbf{v})\nabla \mathbf{u}). \tag{3.2.5}$$

Here $tr(\cdot)$ is the trace of matrices. And

$$tr(\sigma(\mathbf{u})\nabla \bar{\mathbf{u}}) = (\lambda + \mu)|\nabla \cdot \mathbf{u}|^2 + 2\mu \left| \epsilon(\mathbf{u}) - \frac{1}{2}(\nabla \cdot \mathbf{u})I_2 \right|^2. \tag{3.2.6}$$

These formulae are easy to check and we shall omit the proof. Also note that we have similar formulae with σ replaced by σ_0, σ_D, etc.

Now we give the proof of Lemma 3.2.1

Proof of Lemma 3.2.1. First note that $\int_{\partial\Omega} \Lambda_D \mathbf{f} \cdot \bar{\mathbf{f}} ds$ and $\int_{\partial\Omega} \Lambda_\emptyset \mathbf{f} \cdot \bar{\mathbf{f}} ds$ are real. In fact, by definition we have

$$\int_{\partial\Omega} \Lambda_D \mathbf{f} \cdot \bar{\mathbf{f}} ds = \int_{\partial\Omega} (\sigma(\mathbf{u})\boldsymbol{\nu}) \cdot \bar{\mathbf{u}} ds = \int_{\partial\Omega} (\sigma(\mathbf{u})^T \bar{\mathbf{u}}) \cdot \boldsymbol{\nu} dx.$$

By divergence theorem and (3.2.4) we then get

$$\begin{aligned}\int_{\partial\Omega} \Lambda_D \mathbf{f} \cdot \bar{\mathbf{f}} ds &= \int_\Omega (\nabla \cdot \sigma(\mathbf{u})) \cdot \bar{\mathbf{u}} dx + \int_\Omega tr(\sigma(\mathbf{u})\nabla \bar{\mathbf{u}}) dx \\ &= \int_\Omega -k^2 \mathbf{u} \cdot \bar{\mathbf{u}} dx + \int_\Omega tr(\sigma(\mathbf{u})\nabla \bar{\mathbf{u}}) dx,\end{aligned} \tag{3.2.7}$$

which is real. Similarly $\int_{\partial\Omega} \Lambda_\emptyset \mathbf{f} \cdot \bar{\mathbf{f}} ds$ is real.

Since $\mathbf{u}$ and $\mathbf{u}_0$ both equal $\mathbf{f}$ on $\partial\Omega$, similar to (3.2.7) we have

$$\int_{\partial\Omega} \Lambda_D \mathbf{f} \cdot \bar{\mathbf{f}} ds = \int_\Omega -k^2 \mathbf{u} \cdot \bar{\mathbf{u}}_0 dx + \int_\Omega tr(\sigma(\mathbf{u})\nabla \bar{\mathbf{u}}_0) dx; \tag{3.2.8}$$

$$\int_{\partial\Omega} \Lambda_\emptyset \mathbf{f} \cdot \bar{\mathbf{f}} ds = \int_\Omega -k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}} dx + \int_\Omega tr(\sigma_0(\mathbf{u}_0)\nabla \bar{\mathbf{u}}) dx. \tag{3.2.9}$$

Take complex conjugation of (3.2.8) and by (3.2.5) we get

$$\int_{\partial\Omega} \Lambda_D \mathbf{f} \cdot \bar{\mathbf{f}} ds = \int_\Omega -k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}} dx + \int_\Omega tr(\sigma(\mathbf{u}_0)\nabla \bar{\mathbf{u}}) dx. \tag{3.2.10}$$

Then subtract (3.2.9) from (3.2.10) we obtain

$$E(\mathbf{f}) = \int_\Omega tr(\sigma_D(\mathbf{u}_0)\nabla \bar{\mathbf{u}}) dx. \tag{3.2.11}$$

On the other hand,

$$\int_\Omega k^2\mathbf{w}\cdot\bar{\mathbf{w}}dx = \int_\Omega (k^2\mathbf{u}-k^2\mathbf{u}_0)\cdot\bar{\mathbf{w}}dx = -\int_\Omega \nabla\cdot(\sigma(\mathbf{u})-\sigma_0(\mathbf{u}_0))\cdot\bar{\mathbf{w}}dx.$$

Note that $\mathbf{w}\in H_0^1(\Omega)^2$, thus integration by parts gives

$$k^2\int_\Omega |\mathbf{w}|^2\,dx = \int_\Omega tr\left[(\sigma(\mathbf{u})-\sigma_0(\mathbf{u}_0))\nabla\bar{\mathbf{w}}\right]dx. \tag{3.2.12}$$

Now, substituting $\mathbf{u}=\mathbf{w}+\mathbf{u}_0$ into the right-hand side of (3.2.12), and by (3.2.11), we get

$$k^2\int_\Omega |\mathbf{w}|^2dx = \int_\Omega tr(\sigma(\mathbf{w})\nabla\bar{\mathbf{w}})dx - \int_\Omega tr(\sigma_D(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_0)dx + E(\mathbf{f}). \tag{3.2.13}$$

And the first identity (3.2.1) follows from (3.2.6).

Similarly, by substituting $\mathbf{u}_0=\mathbf{u}-\mathbf{w}$ into the right-hand side of (3.2.12) we will obtain (3.2.2). □

3.3 The testing boundary data

In this section we construct the boundary data to be input into E for detecting the location of ∂D. For this purpose, we first introduce the CGO solutions with complex polynomial phases.

3.3.1 CGO solutions with complex polynomial phases

We are to construct CGO solutions with complex polynomial phases to

$$\nabla\cdot\sigma_0(\mathbf{v})+k^2\mathbf{v}=\mathbf{0}\quad(\text{in }\mathbb{R}^2). \tag{3.3.1}$$

Suppose that $\mathbf{v}\in\mathcal{C}^\infty(\mathbb{R}^2)^2$ satisfies the above eqaution. By Helmholtz decomposition, we can write

$$\mathbf{v}=\nabla\varphi+\nabla^\perp\psi$$

for some smooth scalar functions φ and ψ, where $\nabla^\perp\psi:=(-\partial_2\psi,\partial_1\psi)^T$ (and here we also regard $\nabla\varphi$ as a column vector). Then φ and ψ satisfy

$$\nabla((\lambda_0+2\mu_0)\Delta\varphi+k^2\varphi)+\nabla^\perp(\mu_0\Delta\psi+k^2\psi)=0.$$

Let $k_1 = \left(\frac{k^2}{\lambda_0+2\mu_0}\right)^{1/2}$ and $k_2 = \left(\frac{k^2}{\mu_0}\right)^{1/2}$. From the above equation it's easy to see that conversely for any φ and $\psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying

$$\begin{cases} \triangle\, \varphi + k_1^2\varphi &= 0 \\ \triangle\, \psi + k_2^2\psi &= 0, \end{cases} \tag{3.3.2}$$

$\mathbf{v} = \nabla\varphi + \nabla^\perp\psi$ is a solution to (3.3.1). Moreover, if φ and ψ are CGO solutions to (3.3.2), then $\mathbf{v}$ is a CGO solution to (3.3.1).

It is not difficult to construct CGO solutions to (3.3.2) by using the Vekua transform, which transforms a harmonic function to a solution to a Helmholtz equation. Precisely, for any real constant ω, the Vekua transform T_ω associated with ω is defined as follows:

$$T_\omega(u)(\mathbf{x}) = u(\mathbf{x}) - \int_0^1 u(t\mathbf{x})\frac{\partial}{\partial t}\left\{J_0(\omega|\mathbf{x}|\sqrt{1-t})\right\} dt$$

for a function u, where J_0 is the zero order Bessel function of the first kind. If u is a harmonic function, then $T_\omega(u)$ satisfies

$$\triangle\,(T_\omega(u)) + \omega^2(T_\omega(u)) = 0.$$

This formula is derived by I. N. Vekua. One can refer to [38] for details and other related results.

In the following we adopt the same idea as in [36] and [23] to construct CGO solutions with complex polynomial phases. Given $N \in \mathbb{N}$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$, let $\rho = \rho_{N,\beta}$ be the function on $\mathbb{R}^2$ defined by

$$\rho(\mathbf{x}) = \beta(x_1 + ix_2)^N, \tag{3.3.3}$$

which, by regarding $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, is a complex polynomial. Then we define

$$\Gamma = \Gamma_{N,\beta} := \left\{r(\cos\theta, \sin\theta) : r > 0, |\theta - \theta_\mathbf{0}| < \frac{\pi}{2N}\right\}, \tag{3.3.4}$$

the open cone with axis $\theta = \theta_\mathbf{0}$ and open angle π/N, where $\theta_\mathbf{0}$ is such that $\beta = e^{-iN\theta_\mathbf{0}}$. Let $\tau = \tau_{N,\beta} := Re\{\rho_{N,\beta}\}$. Note that in Γ we have

$$\tau(\mathbf{x}) = r^N \cos N(\theta - \theta_\mathbf{0}) > 0,$$

where $\mathbf{x} = r(\cos\theta, \sin\theta)$.

Now for any constant $h > 0$, $e^{\frac{\rho}{h}}$ is harmonic (since it is holomorphic by regarding $\mathbb{R}^2$ as $\mathbb{C}$), and hence

$$\varphi = \varphi_h := T_{k_1}(e^{\frac{\rho}{h}}) \quad \text{and} \quad \psi = \psi_h := T_{k_2}(e^{\frac{\rho}{h}})$$

satisfy (3.3.2). Moreover, φ_h and ψ_h are CGO solutions. In fact, we can write

$$\varphi_h(\mathbf{x}) = e^{\frac{\rho(\mathbf{x})}{h}}(1 + R_{h,1}(\mathbf{x})) \quad \text{and} \quad \psi_h(\mathbf{x}) = e^{\frac{\rho(\mathbf{x})}{h}}(1 + R_{h,2}(\mathbf{x})) \qquad (3.3.5)$$

with $R_{h,l}$ ($l = 1, 2$) satisfying the following estimates in Γ:

$$\begin{aligned} &|R_{h,l}| \le h\frac{k_l^2|\mathbf{x}|^2}{4\tau(\mathbf{x})}; \\ &\left|\frac{\partial R_{h,l}(\mathbf{x})}{\partial x_j}\right| \le \frac{Nk_l^2|\mathbf{x}|^{N+1}}{4\tau(\mathbf{x})} + h\frac{k_l^2|x_j|}{2\tau(\mathbf{x})},\ j = 1, 2. \end{aligned} \qquad (3.3.6)$$

These estimates are established in [23, Lemma 2.1]. In this study we will also need estimates of the second derivatives of $R_{h,l}$, which are not hard to derive in the same manner as the derivation of (3.3.6) given in [23]. Actually, by repeatedly applying the following well-known recurrence formulae

$$\frac{d}{dt}(tJ_1) = tJ_0(t),\ \frac{dJ_0(t)}{dt} = -J_1(t),\ \forall t \ge 0,$$

where J_1 is the Bessel function of the first kind of order 1, and using the basic estimates

$$|J_1(t)| \le \frac{t}{2},\ |J_0(t)| \le 1,\ \forall t \ge 0,$$

the verification of the following estimates are direct (although somewhat lengthy):

$$\begin{aligned} \left|\frac{\partial^2 R_{h,l}(\mathbf{x})}{\partial x_i \partial x_j}\right| \le \frac{1}{h}&\left(\frac{k_l^2 N^2|\mathbf{x}|^{2N}}{4\tau_N(\mathbf{x})}\right) \\ &+ \left(\frac{k_l^2 N(N-1)|\mathbf{x}|^N}{4\tau_N(\mathbf{x})} + \frac{k_l^2 N|\mathbf{x}|^{N-1}(|x_i| + |x_j|)}{2\tau_N(\mathbf{x})}\right) \\ &+ h\left(\frac{k_l^4|x_i||x_j|}{4\tau_N(\mathbf{x})} + \frac{k_l^2\delta_{ij}}{2\tau_N(\mathbf{x})}\right) \end{aligned} \qquad (3.3.7)$$

in Γ, for $1 \le l, i, j \le 2$, where δ_{ij} is the Kronecker delta.

Let $diam(\Omega)$ denote the diameter of Ω. By (3.3.6) and (3.3.7) there exists a constant $C_R = C_R(\lambda_0, \mu_0, k, N, \beta, diam(\Omega)) > 0$ such that for any $0 < h \leq 1$, $1 \leq l, i, j \leq 2$ and $\mathbf{x} \in \Gamma \cap \Omega$,

$$\begin{aligned} |R_{h,l}(\mathbf{x})| &\leq h \frac{C_R}{\tau_N(\mathbf{x})}; \\ \left| \frac{\partial R_{h,l}(\mathbf{x})}{\partial x_j} \right| &\leq \frac{C_R}{\tau_N(\mathbf{x})}; \\ \left| \frac{\partial^2 R_{h,l}(\mathbf{x})}{\partial x_i \partial x_j} \right| &\leq \frac{1}{h} \frac{C_R}{\tau_N(\mathbf{x})}. \end{aligned} \tag{3.3.8}$$

Now $\mathbf{v} = \mathbf{v}_h := \nabla \varphi_h + \nabla^{\perp} \psi_h$ is a CGO solution to (3.3.1). $\mathbf{v}_h$ can be written down explicitly as follows:

$$\mathbf{v}_h(\mathbf{x}) = e^{\frac{\rho(\mathbf{x})}{h}} \begin{pmatrix} Q_{h,1}(\mathbf{x}) \\ Q_{h,2}(\mathbf{x}) \end{pmatrix},$$

where

$$\begin{aligned} Q_{h,1}(\mathbf{x}) = &\left[\frac{1}{h} \frac{\partial \rho(\mathbf{x})}{\partial x_1} (1 + R_{h,1}(\mathbf{x})) + \frac{\partial R_{h,1}(\mathbf{x})}{\partial x_1} \right] \\ &- \left[\frac{1}{h} \frac{\partial \rho(\mathbf{x})}{\partial x_2} (1 + R_{h,2}(\mathbf{x})) + \frac{\partial R_{h,2}(\mathbf{x})}{\partial x_2} \right] \end{aligned} \tag{3.3.9}$$

and

$$\begin{aligned} Q_{h,2}(\mathbf{x}) = &\left[\frac{1}{h} \frac{\partial \rho(\mathbf{x})}{\partial x_2} (1 + R_{h,1}(\mathbf{x})) + \frac{\partial R_{h,1}(\mathbf{x})}{\partial x_2} \right] \\ &+ \left[\frac{1}{h} \frac{\partial \rho(\mathbf{x})}{\partial x_1} (1 + R_{h,2}(\mathbf{x})) + \frac{\partial R_{h,2}(\mathbf{x})}{\partial x_1} \right]. \end{aligned} \tag{3.3.10}$$

Thus for $0 < h \leq 1$ and $i = 1, 2$, from (3.3.8) we have the following estimates for $Q_{h,i}$ in $\Gamma \cap \Omega$:

$$\begin{aligned} |Q_{h,i}(\mathbf{x})| &\leq \frac{2N|\mathbf{x}|^{N-1}}{h} \left(1 + h \frac{C_R}{\tau(\mathbf{x})} \right) + \frac{2C_R}{\tau(\mathbf{x})} \\ &\leq \frac{\tilde{C}_R}{h} + \frac{\tilde{C}_R}{\tau(\mathbf{x})}, \end{aligned} \tag{3.3.11}$$

and for $j = 1, 2$

$$\begin{aligned} \left| \frac{\partial Q_{h,i}(\mathbf{x})}{\partial x_j} \right| &\leq \frac{2N|\mathbf{x}|^{N-2}}{h} \left[(N + |\mathbf{x}|) \frac{C_R}{\tau(\mathbf{x})} + N \right] + \frac{2C_R}{\tau(\mathbf{x})} \\ &\leq \frac{\tilde{C}_R}{h} \left(1 + \frac{1}{\tau(\mathbf{x})} \right) + \frac{\tilde{C}_R}{\tau(\mathbf{x})}. \end{aligned} \tag{3.3.12}$$

where $\tilde{C}_R = \tilde{C}_R(\lambda_0, \mu_0, k, N, \beta, diam(\Omega)) > 0$ is a constant.

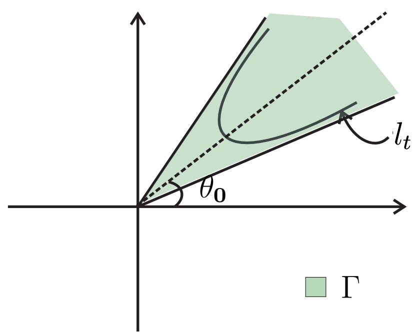
3.3.2 The testing boundary data

Note that from the discussion above the CGO solutions $\mathbf{v}_h$ are *controllable* in $\Gamma \cap \Omega$. In the following we go on to follow the idea in [36] and [23] to modify $\mathbf{v}_h$ into a family of functions localized in Γ.

From the idea of enclosure method the previous chapter stated, we know the level curve of real part of the phase function is very important. So let, for $t > 0$,

$$\ell_t := \{\mathbf{x} \in \Gamma : \tau(\mathbf{x}) = \frac{1}{t}\}, \tag{3.3.13}$$

the level curve of τ in Γ at $\frac{1}{t}$.



In fact, any level curve of $\tau = \tau_{N,\beta}$ has N branches, and the cone $\Gamma = \Gamma_{N,\beta}$ just contains one branch with the two edges of Γ being the asymptotes of that branch. On the other hand, we here choose one cone (or branch) which intersects the subject Ω. Also note that when t is larger, the curve ℓ_t is closer to the origin. (We refer to Figure 2 in [23] for an illustration.) Then for $d > 0$ let

$$\Gamma_d = \Gamma_{N,\beta,d} := \overline{\bigcup_{0<t<d} \ell_t}. \tag{3.3.14}$$

Note that for $d_1 > d_2 > 0$ we have $\Gamma_{d_2} \subset \Gamma_{d_1}$.

In the following we fix an $\varepsilon > 0$ and a compact interval $J \subset (0, \infty)$. Let $\{\phi_d\}_{d \in J}$ be a family of smooth cut-off functions such that

(i) $0 \leq \phi_d(\mathbf{x}) \leq 1$,

(ii) $\phi_d(\mathbf{x}) = 1$ (resp. 0) for $\mathbf{x} \in \Gamma_{d+\varepsilon}$ (resp. $\mathbf{x} \in \mathbb{R}^2 \setminus \Gamma_{d+2\varepsilon}$), and

(iii) for some $C_\phi > 0$, we have $|\partial^{\boldsymbol{\alpha}}_{\mathbf{x}} \phi_d(\mathbf{x})| \le C_\phi$ for each multiindex $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le 2$, for each $\mathbf{x} \in \Omega$ and for each $d \in J$.

The existence of such family $\{\phi_d\}$ is obvious and we omit a precise construction.

Now let

$$\mathbf{p}_{d,h}(\mathbf{x}) := \phi_d(\mathbf{x}) e^{-\frac{1}{hd}} \mathbf{v}_h \in \mathcal{C}^\infty(\mathbb{R}^2)^2. \tag{3.3.15}$$

It is the traces of these $\mathbf{p}_{d,h}$ on $\partial\Omega$ that will be the testing data to be input into E. In fact, we will see that the behavior of $E(\mathbf{p}_{d,h}|_{\partial\Omega})$ as $h \to 0^+$ tells whether Γ_d intersects D or not. Now note that although $\mathbf{p}_{d,h}$ is *controllable* from the discussion above, it is no longer a solution to (3.3.1). However, to get information from $E(\mathbf{p}_{d,h}|_{\partial\Omega})$ we will need estimates related to the solution of (3.3.1) with boundary condition $\mathbf{p}_{d,h}|_{\partial\Omega}$. But indeed for small h controllability of $\mathbf{p}_{d,h}$ gives controllability of the true solution of (3.3.1) with the same boundary condition. We explain this precisely in the following.

Let $\mathbf{u}_{0,d,h}$ satisfy

$$\begin{cases} \nabla \cdot \sigma_0(\mathbf{u}_{0,d,h}) + k^2 \mathbf{u}_{0,d,h} = 0 & \text{in } \Omega \\ \mathbf{u}_{0,d,h} = \mathbf{p}_{d,h}|_{\partial\Omega} & \text{on } \partial\Omega. \end{cases} \tag{3.3.16}$$

And let

$$\mathbf{w}_h = \mathbf{p}_{d,h} - \mathbf{u}_{0,d,h}, \tag{3.3.17}$$

then $\mathbf{w}_h$ satisfies

$$\begin{cases} \nabla \cdot \sigma_0(\mathbf{w}_h) + k^2 \mathbf{w}_h = \nabla \cdot \sigma_0(\mathbf{p}_{d,h}) + k^2 \mathbf{p}_{d,h} & \text{in } \Omega \\ \mathbf{w}_h = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3.18}$$

Let

$$\mathbf{g}_h = \nabla \cdot \sigma_0(\mathbf{p}_{d,h}) + k^2 \mathbf{p}_{d,h}, \tag{3.3.19}$$

then we have the following lemma.

Lemma 3.3.1. *There exists positive constants C_1 and C (depending on $\Omega, \lambda_0, \mu_0, k$) such that for $0 < h \leq 1$ and $d \in J$*

$$\|\mathbf{w}_h\|_{H^1(\Omega)^2} \leq C_1 \|\mathbf{g}_h\|_{L^2(\Omega)^2} \leq \frac{C}{h^2} e^{-\frac{1}{h}(\frac{1}{d} - \frac{1}{d+\varepsilon})}.$$

In particular, there exists $0 < h_0 < 1$ such that for $0 < h < h_0$ and $d \in J$, there is a positive constant $C' = C'(\Omega, \lambda_0, \mu_0, k)$ such that

$$\|\mathbf{w}_h\|_{H^1(\Omega)^2} \leq C_1 \|\mathbf{g}_h\|_{L^2(\Omega)^2} \leq C' e^{-\frac{1}{h}(\frac{1}{d} - \frac{1}{d+\varepsilon})}.$$

Proof. Note that in this paper for any two vectors $\mathbf{a}$ and $\mathbf{b}$, we define $\mathbf{a} \otimes \mathbf{b}$ to be the matrix whose ij-th entry is $a_i b_j$.

That $\|\mathbf{w}_h\|_{H^1(\Omega)^2} \leq C_1 \|\mathbf{g}_h\|_{L^2(\Omega)^2}$ for some C_1 is classical. So we need only to estimate $\|\mathbf{g}_h\|_{L^2(\Omega)^2}$.

Since $\mathbf{p}_{d,h}(\mathbf{x}) = \phi_d(\mathbf{x}) e^{-\frac{1}{hd}} \mathbf{v}_h$, we have

$$\begin{aligned}
\mathbf{g}_h &= e^{-\frac{1}{hd}} \Big\{ \lambda_0 \nabla(\nabla \cdot (\phi_d \mathbf{v}_h)) + \mu_0 \nabla \cdot (\nabla(\phi_d \mathbf{v}_h) + (\nabla(\phi_d \mathbf{v}_h))^T) + k^2 \phi_d \mathbf{v}_h \Big\} \\
&= e^{-\frac{1}{hd}} \Big\{ \lambda_0 \big[\nabla(\nabla \phi_d \cdot \mathbf{v}_h) + \nabla \phi_d (\nabla \cdot \mathbf{v}_h) \big] \\
&\qquad + \mu_0 \nabla \cdot \big[\mathbf{v}_h \otimes \nabla \phi_d + \nabla \phi_d \otimes \mathbf{v}_h \big] \\
&\qquad + \mu_0 (\nabla \mathbf{v}_h + (\nabla \mathbf{v}_h)^T) \nabla \phi_d \\
&\qquad + \phi_d [\nabla \cdot \sigma_0(\mathbf{v}_h) + k^2 \mathbf{v}_h] \Big\}.
\end{aligned}$$

Because $\nabla \cdot \sigma_0(\mathbf{v}_h) + k^2 \mathbf{v}_h = 0$ and $\nabla \phi_d = 0$ outside $\Gamma_{d+2\varepsilon} \setminus \Gamma_{d+\varepsilon}$, we have

$$\|\mathbf{g}_h\|_{L^2(\Omega)^2} \leq C_g e^{-\frac{1}{hd}} \|\mathbf{v}_h\|_{H^1((\Gamma_{d+2\varepsilon} \setminus \Gamma_{d+\varepsilon}) \cap \Omega)^2} \tag{3.3.20}$$

for some positive constant $C_g = C_g(\lambda_0, \mu_0, C_\phi)$.

By (3.3.11), for $\mathbf{x} \in (\Gamma_{d+2\varepsilon} \setminus \Gamma_{d+\varepsilon}) \cap \Omega$,

$$\begin{aligned}
|\mathbf{v}_h(\mathbf{x})| &= e^{\frac{\tau(\mathbf{x})}{h}} \sqrt{|Q_{h,1}(\mathbf{x})|^2 + |Q_{h,2}(\mathbf{x})|^2} \\
&\leq \sqrt{2} e^{\frac{\tau(\mathbf{x})}{h}} \left\{ \frac{\tilde{C}_R}{h} + \frac{\tilde{C}_R}{\tau(\mathbf{x})} \right\} \leq \frac{C'_R}{h} e^{\frac{\tau(\mathbf{x})}{h}}
\end{aligned}$$

for some positive constant $C'_R = C'_R(\lambda_0, \mu_0, k, diam\Omega)$. Hence we have the following estimate:

$$\|\mathbf{v}_h\|_{L^2((\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega)^2} \leq \frac{C'_R}{h}\left(\int_{(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega} e^{\frac{2\tau(\mathbf{x})}{h}}dx\right)^{\frac{1}{2}}.$$

Similarly by (3.3.11) and (3.3.12) we have

$$\|\nabla\mathbf{v}_h\|_{L^2((\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega)^2} \leq \frac{C''_R}{h^2}\left(\int_{(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega} e^{\frac{2\tau(\mathbf{x})}{h}}dx\right)^{\frac{1}{2}}.$$

Since

$$\int_{(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega} e^{\frac{2\tau(\mathbf{x})}{h}}dx \leq |(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega| e^{\frac{2}{h}\frac{1}{d+\varepsilon}},$$

we have

$$\|\mathbf{g}_h\|_{L^2(\Omega)^2} \leq C_g e^{-\frac{1}{hd}}\|\mathbf{v}_h\|_{H^1((\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})\cap\Omega)^2} \leq \frac{C}{h^2}e^{-\frac{1}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})}, \qquad (3.3.21)$$

where C depends only on λ_0, μ_0, k and Ω.

□

3.4 The main theorem for the reconstruction of unknown inclusions

We now come to considering our inverse problem of reconstructing D. For the main theorem we make the following three assumptions (in addition to those already made in the introduction) throughout this section.

1. We assume $\nabla\cdot\sigma_0$ and $\nabla\cdot\sigma$ satisfy the strong convexity condition (but not only the strong elliptic condition (3.1.3)):

$$\lambda_0 + \mu_0 > 0,\ \mu_0 > 0;$$
$$\lambda + \mu > 0,\ \mu > 0 \quad \text{on } \Omega.$$

   Thus, in particular, Lemma 3.2.2 applies.

2. $(\lambda_D + \mu_D)\mu_D \geq 0$ on D.

3. For any $\mathbf{y} \in \partial D$, there exists a ball $B_r(\mathbf{y})$ such that one of the following jump conditions holds:

$$\begin{aligned} &\text{(i)} \quad \mu_D(\mathbf{x}) > r, \quad \lambda_D(\mathbf{x}) + \mu_D(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in B_r(\mathbf{y}) \cap D; \\ &\text{(ii)} \quad \mu_D(\mathbf{x}) < -r, \quad \lambda_D(\mathbf{x}) + \mu_D(\mathbf{x}) \leq 0, \quad \forall \mathbf{x} \in B_r(\mathbf{y}) \cap D. \end{aligned} \tag{3.4.1}$$

Now assume the origin $\mathbf{0}$ is outside $\bar{\Omega}$.[1] As in section 3, in the following we fix an $N \in \mathbb{N}$, a $\beta \in \mathbb{C}$ with $|\beta| = 1$, an $\varepsilon > 0$, and a compact interval $J \subset (0, \infty)$. And recall the definition of ρ, Γ, ℓ_t, Γ_d, and $\mathbf{p}_{d,h}$ in (3.3.3), (3.3.4), (3.3.13), (3.3.14), and (3.3.15) respectively. Also recall that we use τ to denote $Re(\rho)$. Let

$$s_* := \begin{cases} \sup\limits_{\mathbf{x} \in D \cap \Gamma} \tau(\mathbf{x}), & \text{if } D \cap \Gamma \neq \emptyset \\ 0\,, & \text{if } D \cap \Gamma = \emptyset. \end{cases}$$

Note that $D \cap \Gamma \neq \emptyset$ if and only if $s_* > 0$, and in this situation ℓ_{1/s_*} is a curve just touching ∂D, i.e. $\ell_{1/s_*} \cap \bar{D} = \ell_{1/s_*} \cap \partial D \neq \emptyset$.

For notational simplicity let $\mathbf{f}_{d,h} := \mathbf{p}_{d,h}|_{\partial\Omega}$. Recall that $\mathbf{u}_{0,d,h}$ satisfies

$$\begin{cases} \nabla \cdot \sigma_0(\mathbf{u}_{0,d,h}) + k^2 \mathbf{u}_{0,d,h} = 0 & \text{in } \Omega \\ \mathbf{u}_{0,d,h} = \mathbf{f}_{d,h} & \text{on } \partial\Omega. \end{cases}$$

Similarly let $\mathbf{u}_{d,h}$ be the solution when the inclusion D exists:

$$\begin{cases} \nabla \cdot \sigma(\mathbf{u}_{d,h}) + k^2 \mathbf{u}_{d,h} = 0, & \text{in } \Omega \\ \mathbf{u}_{d,h} = \mathbf{f}_{d,h} & \text{on } \partial\Omega. \end{cases}$$

[1]In general, for $\mathbf{a} = (a_1, a_2)^T$ a point outside $\bar{\Omega}$, we should use $\rho = \beta((x_1 - a_1) + i(x_2 - a_2))^N$, and similar modifications of Γ, Γ_d, etc., and there is a similar result as Theorem 3.4.1. However as we can always set the coordinates so that $\mathbf{0} \notin \bar{\Omega}$ in practice, such consideration is not needed.

Now let $\mathbf{w}_{d,h} = \mathbf{u}_{d,h} - \mathbf{u}_{0,d,h}$. We have the following two inequalities from Lemma 3.2.2:

$$E(\mathbf{f}_{d,h}) \leq \int_D (\lambda_D + \mu_D)|\nabla \cdot \mathbf{u}_{0,d,h}|^2 dx + 2\int_D \mu_D \left|\epsilon(\mathbf{u}_{0,d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{u}_{0,d,h})I_2\right|^2 dx + k^2\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}; \tag{3.4.2}$$

$$E(\mathbf{f}_{d,h}) \geq \int_D \frac{(\lambda_0 + \mu_0)(\lambda_D + \mu_D)}{\lambda + \mu}|\nabla \cdot \mathbf{u}_{0,d,h}|^2 dx + 2\int_D \frac{\mu_0\mu_D}{\mu} \left|\epsilon(\mathbf{u}_{0,d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{u}_{0,d,h})I_2\right|^2 dx - k^2\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}. \tag{3.4.3}$$

They are the key to the following main theorem of this paper.

Theorem 3.4.1. *For $d \in J$ and $h > 0$ small enough, the following conclusions hold:*

(A) *If $\bar{D} \cap \Gamma_d = \emptyset$, then*

$$|E(\mathbf{f}_{d,h})| \leq Ch^{-4}e^{-\frac{2}{h}(\frac{1}{d}-s_d)}$$

for some $C > 0$ independent of h, where $s_d = \max(\frac{1}{d+\varepsilon}, s_*) < \frac{1}{d}$.

(B) *If $D \cap \Gamma_d \neq \emptyset$ and D has continuous boundary, then there exists a constant δ, $0 < \delta < s_* - \frac{1}{d}$, such that*

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-3}e^{\frac{2}{h}(s_* - \frac{1}{d} - \delta)}$$

for some $C > 0$ independent of h.

(B′) *If $\bar{D} \cap \Gamma_d \neq \emptyset$ and D has $C^{0,\alpha}$ boundary for $\frac{1}{3} < \alpha \leq 1$, then*

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-3+\frac{1}{\alpha}}e^{\frac{2}{h}(s_* - \frac{1}{d})}$$

for some $C > 0$ independent of h.

Before going into the proof of Theorem 3.4.1, we give two remarks.

Remark 3.4.1.

1. From Theorem 3.4.1, we have the following conclusions. In (A), since $s_d < \frac{1}{d}$, $|E(\mathbf{f}_{d,h})|$ tends to zero as h tends to zero. On the other hand, in (B) and in (B′), since $s_* \geq \frac{1}{d}$, $|E(\mathbf{f}_{d,h})|$ tends to infinity as h tends to zero. In particular, from (A) and (B), we have

$$s_* = \inf \left\{ \frac{1}{d} : \lim_{h \to 0^+} |E(\mathbf{f}_{d,h})| = 0 \right\}. \tag{3.4.4}$$

   Hence, although we don't know the limiting behavior of $E(\mathbf{f}_{d,h})$ when Γ_d just touches ∂D, we can reconstruct ∂D in principle. (Of course, due to the geometric nature of Γ_d, in fact only "detectable" points can be reconstructed. An explanation of this point can be found in [35, Corollary 5.4]. Also see [36] or [23] for a reconstruction algorithm, which is easily modified to be suited for our case. We omit such discussions in this paper.) From this point of view, almost no regularity assumption on ∂D is essential in the reconstruction. Nevertheless, for a complete characterization of the limiting behavior of $E(\mathbf{f}_{d,h})$, we include (B′) in our theorem, while for this purpose more regularity assumption has to be made.

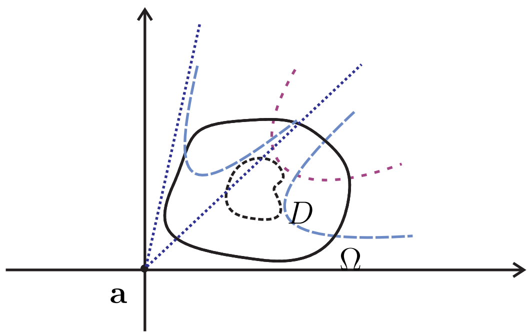


Figure 3.1: The reconstruction algorithm is to find more and more touching curves

2. We will use (3.4.2) and (3.4.3) to prove Theorem 3.4.1. Roughly speaking we have better knowledge of $\mathbf{u}_{0,d,h}$ than $\mathbf{w}_{d,h}$, and the crucial step is to give an appropriate control of $\|\mathbf{w}_{d,h}\|_{L^2(\Omega)}$ in terms of $\mathbf{u}_{0,d,h}$. For this

purpose, in the corresponding parts of early researches, e.g. [12, 24, 23], some technical assumptions (precisely, positivity of the relative curvature and finiteness of the number of touching points of ℓ_{1/s_*} (or say Γ_{1/s_*}) and ∂D) have to be made. In particular ∂D is usually assumed to be $\mathcal{C}^2$. (In order to apply CGO solutions with complex polynomial phases, even more technicalities are involved. For example, in [23, Lemma 3.7], the authors proposed an estimate which is based on a rather technical result in [20].) In [29], Sini and Yoshida came up with a totally different method to control $\|\mathbf{w}_{d,h}\|_{L^2(\Omega)}$ (while they did not adopt CGO solutions with complex polynomial phases). Precisely, they proposed (in our terminology)

$$\|\mathbf{w}_{d,h}\|_{L^2(\Omega)} \leq C\|\mathbf{u}_{0,d,h}\|_{W^{1,p}(D)} \tag{3.4.5}$$

for some $p < 2$, which was proved by using an L^p regularity estimate of Meyers and the Friedrichs' inequality. In this way the technical assumptions on the touching point are no more needed and ∂D can be assumed to be only Lipschitz. Inspired by this result, we tried to adopt their idea in our situation. We find it's interesting that, with more careful treatment, we find the boundaries of the inclusions can in fact be assumed to be only continuous. Moreover, we find in the case of Γ_d just touching ∂D, the regularity assumption on ∂D can be reduced to be $C^{0,\alpha}$ for any $\alpha \in (\frac{1}{3}, 1]$.

To save notation, in the remaining of this chapter we will freely use C to denote a constant, which may represent different values at different places.

The following lemma is just (3.4.5), we give the proof here for the sake of completeness.

Lemma 3.4.2. *There exist constants $C > 0$ and $1 \leq q_0 < 2$ such that for $q_0 < q \leq 2$,*

$$\|\mathbf{w}\|_{L^2(\Omega)} \leq C\|\nabla \mathbf{u}_0\|_{L^q(D)},$$

whenever $\mathbf{u}$ and $\mathbf{u}_0 \in H^1(\Omega)^2$ satisfy (3.1.1) *and* (3.1.2) *respectively, $\mathbf{u}$ and $\mathbf{u}_0$ have the same traces on $\partial\Omega$, and $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$.*

Proof. Let $\mathbf{q}$ be the element in $H_0^1(\Omega)^2$ satisfying

$$\nabla \cdot (\sigma(\mathbf{q})) + k^2\mathbf{q} = \bar{\mathbf{w}} \quad \text{in } \Omega.$$

Then, by taking inner product with $\mathbf{w}$ and integration by parts, we have

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2 dx &= -\int_\Omega tr(\sigma(\mathbf{q})\nabla\mathbf{w})dx + k^2\int_\Omega \mathbf{q}\cdot\mathbf{w}dx \\ &= -\int_\Omega tr(\sigma(\mathbf{w})\nabla\mathbf{q})dx + k^2\int_\Omega \mathbf{q}\cdot\mathbf{w}dx. \end{aligned} \tag{3.4.6}$$

On the other hand, note that

$$\nabla\cdot\sigma(\mathbf{w}) + k^2\mathbf{w} = -\nabla\cdot\sigma_D(\mathbf{u}_0),$$

which, by taking inner product with $\mathbf{q}$ and integration by parts, gives

$$-\int_\Omega tr(\sigma(\mathbf{w})\nabla\mathbf{q})dx + k^2\int_\Omega \mathbf{w}\cdot\mathbf{q}dx = \int_\Omega tr(\sigma_D(\mathbf{u}_0)\nabla\mathbf{q})dx. \tag{3.4.7}$$

From (3.4.6) and (3.4.7) we get

$$\int_\Omega |\mathbf{w}|^2 dx = \int_\Omega tr(\sigma_D(\mathbf{u}_0)\nabla\mathbf{q})dx.$$

Then by Hölder's inequality we have for any $1 \le p \le \infty$

$$\int_\Omega |\mathbf{w}|^2 dx \le \|\sigma_D(\mathbf{u}_0)\|_{L^q(D)}\|\nabla\mathbf{q}\|_{L^p(\Omega)}, \tag{3.4.8}$$

where q is the conjugate exponent of p.

Now let $\mathbf{Q} = \mathbf{q}$. By definition of $\mathbf{q}$ we have

$$\begin{cases} \nabla\cdot(\sigma(\mathbf{Q})) = \bar{\mathbf{w}} - k^2\mathbf{q} & \text{in } \Omega \\ \mathbf{Q} = 0 & \text{on } \Omega. \end{cases}$$

Then by [22, Theorem 1], there exist $p_0 > 2$ such that for each $2 \le p < p_0$,

$$\|\nabla\mathbf{q}\|_{L^p(\Omega)} = \|\nabla\mathbf{Q}\|_{L^p(\Omega)} \le C\left\{\|\mathbf{q}\|_{L^2(\Omega)} + \|\mathbf{w}\|_{L^2(\Omega)}\right\} \tag{3.4.9}$$

for some $C = C(k,\lambda,\mu) > 0$. Note that also by definition of $\mathbf{q}$, we have $\|\mathbf{q}\|_{L^2(\Omega)} \le C\|\mathbf{w}\|_{L^2(\Omega)}$ for some $C = C(k,\lambda,\mu) > 0$ (see e.g. [7, Section 6.2, Theorem 6]). So from (3.4.9) we have

$$\|\nabla\mathbf{q}\|_{L^p(\Omega)} \le C\|\mathbf{w}\|_{L^2(\Omega)} \tag{3.4.10}$$

for some $C = C(k, \lambda, \mu) > 0$. Combining (3.4.8) and (3.4.10), we have

$$\|\mathbf{w}\|_{L^2(\Omega)}^2 \leq C\|\nabla \mathbf{u}_0\|_{L^q(D)^2}\|\mathbf{w}\|_{L^2(\Omega)}$$

for some $C = C(k, \lambda, \mu) > 0$ and $2 \leq p < p_0$, and therefore

$$\|\mathbf{w}\|_{L^2(\Omega)} \leq C\|\nabla \mathbf{u}_0\|_{L^q(D)}$$

for some $C = C(k, \lambda, \mu) > 0$ and $q_0 < q \leq 2$, where $1 \leq q_0 < 2$ is the conjugate exponent of p_0. □

Remark 3.4.2. Remember that we assume Ω has a smooth boundary for simplicity. In fact, it is so assumed only to allow direct application of the L^p estimates in [22]. In other words, the regularity condition on $\partial\Omega$ really required is just that guarantees the validity of (3.4.9).

To make the proof of Theorem 3.4.1 more concise, some computational results are collected in the following lemma.

Lemma 3.4.3. *For $d \in J$, we have the following conclusions.*

(i) *There exsists a constant $C > 0$ such that for $q > 0$ and $0 < h \leq 1$, we have*

$$\|\nabla \mathbf{p}_{d,h}\|_{L^q(D)} \leq Ch^{-2}\left(\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau(\mathbf{x})-\frac{1}{d})}dx\right)^{\frac{1}{q}}. \tag{3.4.11}$$

In particular, since $s_* \geq \tau(\mathbf{x})$ for $\mathbf{x} \in D \cap \Gamma_{d+2\varepsilon}$, we have

$$\|\nabla \mathbf{p}_{d,h}\|_{L^q(D)} \leq Ch^{-2}e^{\frac{1}{h}(s_*-\frac{1}{d})}$$

for some $C > 0$ independent of h.

(ii) *There exist positive constants c and C such that, for $0 < h \leq 1$ and for any open set U with $D \cap \Gamma_{d+\varepsilon} \cap U \neq \emptyset$, we have*

$$\begin{aligned}&\left\|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{p}_{d,h})I_2\right\|_{L^2(D\cap\Gamma_{d+\varepsilon}\cap U)}^2 \\ &\quad\geq (ch^{-4} - Ch^{-2})\int_{D\cap\Gamma_{d+\varepsilon}\cap U} e^{\frac{2}{h}(\tau(\mathbf{x})-\frac{1}{d})}dx.\end{aligned} \tag{3.4.12}$$

(iii) *There exist constants* $C > 0$ *and* $q_0 < 2$ *such that for each* $q_0 < q \leq 2$ *and* $0 < h \ll 1$*, we have*

$$\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)} \leq Ce^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})} + Ch^{-4}\left(\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau(\mathbf{x})-\frac{1}{d})}dx\right)^{2/q}. \tag{3.4.13}$$

Proof. (i) Remember that

$$\mathbf{p}_{d,h} = (p^1_{d,h}, p^2_{d,h})^T = \phi_d e^{\frac{1}{h}(\rho(\mathbf{x})-\frac{1}{d})}(Q_{h,1}, Q_{h,2})^T,$$

where $Q_{h,1}$ and $Q_{h,2}$ are defined in (3.3.9) and (3.3.10) respectively. Then by definition of ϕ_d (in page 23), for $\mathbf{x} \in D \setminus \Gamma_{d+2\varepsilon}$, we have $\mathbf{p}_{d,h}(\mathbf{x}) = 0$. On the other hand, by (3.3.11) and (3.3.12), we have for $\mathbf{x} \in D \cap \Gamma_{d+2\varepsilon}$ and $0 < h \leq 1$,

$$\begin{aligned}
|\nabla \mathbf{p}_{d,h}(\mathbf{x})|^2 &= \sum_{j,l=1,2}\left|\frac{\partial p^j_{d,h}}{\partial x_l}\right|^2 \\
&= \sum_{j,l=1,2} e^{\frac{2}{h}(\tau(\mathbf{x})-\frac{1}{d})}\left|\frac{\partial \phi_d(\mathbf{x})}{\partial x_l}Q_{h,j}(\mathbf{x})\right. \\
&\qquad \left. + \phi_d(\mathbf{x})\left[\frac{1}{h}\frac{\partial \rho(\mathbf{x})}{\partial x_l}Q_{h,j}(\mathbf{x}) + \frac{\partial Q_{h,j}(\mathbf{x})}{\partial x_l}\right]\right|^2 \\
&\leq Ce^{\frac{2}{h}(\tau(\mathbf{x})-\frac{1}{d})}h^{-4},
\end{aligned} \tag{3.4.14}$$

for some positive constant C independent of h. Since $s_* = \sup_{\mathbf{x}\in D\cap\Gamma} \tau(\mathbf{x})$ and $|\nabla \mathbf{p}_{d,h}|^q = (|\nabla \mathbf{p}_{d,h}|^2)^{q/2}$, we have for $0 < h \leq 1$

$$\|\nabla \mathbf{p}_{d,h}\|_{L^q(D)} \leq Ch^{-2}\left(\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau(\mathbf{x})-\frac{1}{d})}dx\right)^{\frac{1}{q}} \leq Ce^{\frac{1}{h}(s_*-\frac{1}{d})}h^{-2},$$

for some positive constant C independent of h.

(ii) We can compute $\frac{\partial p^j_{d,h}}{\partial x_l}$ directly by (3.3.9) and (3.3.10) for $\mathbf{x} \in D \cap \Gamma_{d+2\varepsilon}$:

$$\begin{aligned}
\frac{\partial p^1_{d,h}(\mathbf{x})}{\partial x_l} &= e^{-\frac{1}{hd}}e^{\frac{\rho(\mathbf{x})}{h}}\left(\frac{\partial \phi_d(\mathbf{x})}{\partial x_l}Q_{h,1} + \frac{1}{h}\phi_d\frac{\partial \rho(\mathbf{x})}{\partial x_l}Q_{h,1} + \phi_d\frac{\partial Q_{h,1}(\mathbf{x})}{\partial x_l}\right) \\
&= e^{-\frac{1}{hd}}e^{\frac{\rho(\mathbf{x})}{h}}\left(\frac{1}{h^2}\left(\frac{\partial \rho(\mathbf{x})}{\partial x_1} - \frac{\partial \rho(\mathbf{x})}{\partial x_2}\right)\frac{\partial \rho(\mathbf{x})}{\partial x_l}\phi_d(\mathbf{x}) + I_{h^{-1}}\right)
\end{aligned}$$

and

$$\frac{\partial p_{d,h}^2(\mathbf{x})}{\partial x_l} = e^{-\frac{1}{hd}} e^{\frac{\rho(\mathbf{x})}{h}} \left(\frac{\partial \phi_d(\mathbf{x})}{\partial x_l} Q_{h,2} + \frac{1}{h} \phi_d \frac{\partial \rho(\mathbf{x})}{\partial x_l} Q_{h,2} + \phi_d \frac{\partial Q_{h,2}(\mathbf{x})}{\partial x_l} \right)$$
$$= e^{-\frac{1}{hd}} e^{\frac{\rho(\mathbf{x})}{h}} \left(\frac{1}{h^2} \left(\frac{\partial \rho(\mathbf{x})}{\partial x_1} + \frac{\partial \rho(\mathbf{x})}{\partial x_2} \right) \frac{\partial \rho(\mathbf{x})}{\partial x_l} \phi_d(\mathbf{x}) + I'_{h^{-1}} \right),$$

where

$$\begin{aligned} I_{h^{-1}} = {} & \frac{1}{h^2} \phi_d \frac{\partial \rho}{\partial x_l} \left(\frac{\partial \rho}{\partial x_1} R_{h,1} - \frac{\partial \rho}{\partial x_2} R_{h,2} \right) \\ & + \frac{1}{h} \frac{\partial \phi_d}{\partial x_l} \left[\frac{\partial \rho}{\partial x_1} (1 + R_{h,1}) - \frac{\partial \rho}{\partial x_2} (1 + R_{h,2}) \right] \\ & + \frac{1}{h} \phi_d \left[\frac{\partial \rho}{\partial x_l} \left(\frac{\partial R_{h,1}}{\partial x_1} - \frac{\partial R_{h,2}}{\partial x_2} \right) + \frac{\partial \rho}{\partial x_1} \frac{\partial R_{h,1}}{\partial x_l} - \frac{\partial \rho}{\partial x_2} \frac{\partial R_{h,2}}{\partial x_l} \right] \\ & + \frac{1}{h} \phi_d \left[\frac{\partial^2 \rho}{\partial x_l \partial x_1} (1 + R_{h,1}) - \frac{\partial^2 \rho}{\partial x_l \partial x_2} (1 + R_{h,2}) \right] \\ & + \phi_d \left(\frac{\partial^2 R_{h,1}}{\partial x_l \partial x_1} - \frac{\partial^2 R_{h,2}}{\partial x_l \partial x_2} \right) + \frac{\partial \phi_d}{\partial x_l} \left(\frac{\partial R_{h,1}}{\partial x_1} - \frac{\partial R_{h,2}}{\partial x_2} \right); \end{aligned}$$

$$\begin{aligned} I'_{h^{-1}} = {} & \frac{1}{h^2} \phi_d \frac{\partial \rho}{\partial x_l} \left(\frac{\partial \rho}{\partial x_2} R_{h,1} + \frac{\partial \rho}{\partial x_1} R_{h,2} \right) \\ & + \frac{1}{h} \frac{\partial \phi_d}{\partial x_l} \left[\frac{\partial \rho}{\partial x_2} (1 + R_{h,1}) + \frac{\partial \rho}{\partial x_1} (1 + R_{h,2}) \right] \\ & + \frac{1}{h} \phi_d \left[\frac{\partial \rho}{\partial x_l} \left(\frac{\partial R_{h,1}}{\partial x_2} + \frac{\partial R_{h,2}}{\partial x_1} \right) + \frac{\partial \rho}{\partial x_2} \frac{\partial R_{h,1}}{\partial x_l} + \frac{\partial \rho}{\partial x_1} \frac{\partial R_{h,2}}{\partial x_l} \right] \\ & + \frac{1}{h} \phi_d \left[\frac{\partial^2 \rho}{\partial x_l \partial x_2} (1 + R_{h,1}) + \frac{\partial^2 \rho}{\partial x_l \partial x_1} (1 + R_{h,2}) \right] \\ & + \phi_d \left(\frac{\partial^2 R_{h,1}}{\partial x_l \partial x_2} + \frac{\partial^2 R_{h,2}}{\partial x_l \partial x_1} \right) + \frac{\partial \phi_d}{\partial x_l} \left(\frac{\partial R_{h,1}}{\partial x_2} + \frac{\partial R_{h,2}}{\partial x_1} \right). \end{aligned}$$

By (3.3.8), for any $\mathbf{x} \in D \cap \Gamma_{d+2\varepsilon}$ and $0 < h \leq 1$,

$$|I_{h^{-1}}(\mathbf{x})|, |I'_{h^{-1}}(\mathbf{x})| \leq C h^{-1}$$

for some positive constant C independent of h.

Then we have for $\mathbf{x} \in D \cap \Gamma_{d+2\varepsilon}$ and $0 < h \leq 1$

$$\begin{aligned}
2&\left|\epsilon\left(\mathbf{p}_{d,h}\right) - \frac{1}{2}(\nabla \cdot \mathbf{p}_{d,h}) I_2\right|^2 \\
&\geq \left|\frac{\partial p^1_{d,h}}{\partial x_1} - \frac{\partial p^2_{d,h}}{\partial x_2}\right|^2 \\
&\geq \left|e^{\frac{1}{h}(\rho - \frac{1}{d})}\phi_d h^{-2}\left[\frac{\partial \rho}{\partial x_1}\left(\frac{\partial \rho}{\partial x_1} - \frac{\partial \rho}{\partial x_2}\right) - \frac{\partial \rho}{\partial x_2}\left(\frac{\partial \rho}{\partial x_2} + \frac{\partial \rho}{\partial x_1}\right)\right]\right|^2 \\
&\quad - \left|e^{\frac{1}{h}(\rho - \frac{1}{d})}\left(I_{h^{-1}} - I'_{h^{-1}}\right)\right|^2 \\
&\geq e^{\frac{2}{h}(\tau - \frac{1}{d})}(c\phi_d^2 h^{-4} - 2Ch^{-2})
\end{aligned}$$

for some positive constants c, C independent of h. Then (ii) of this lemma is valid.

(iii) By Lemma 3.4.2, there exist constants $C > 0$ and $1 \leq q_0 < 2$ such that

$$\|\mathbf{w}_{d,h}\|_{L^2(\Omega)} \leq C\|\nabla \mathbf{u}_{0,d,h}\|_{L^q(D)}$$

for each $q_0 < q \leq 2$. Therefore replacing $\mathbf{u}_{0,d,h}$ by $\mathbf{p}_{d,h} - \mathbf{w}_h$ and applying Hölder's inequality, we have

$$\begin{aligned}
\|\mathbf{w}_{d,h}\|_{L^2(\Omega)} &\leq C\left\{\|\nabla \mathbf{w}_h\|_{L^q(D)} + \|\nabla \mathbf{p}_{d,h}\|_{L^q(D)}\right\} \\
&\leq C\left\{\|\nabla \mathbf{w}_h\|_{L^2(D)} + \|\nabla \mathbf{p}_{d,h}\|_{L^q(D)}\right\} \\
&\leq C\left\{\|\nabla \mathbf{w}_h\|_{H^1(D)} + \|\nabla \mathbf{p}_{d,h}\|_{L^q(D)}\right\}.
\end{aligned}$$

Then by Lemma 3.3.1 and (3.4.11), (3.4.13) follows.

□

Now we give the proof of the main theorem.

Proof of Theorem 3.4.1. (A) By (3.4.3), we have

$$\begin{aligned}
-E(\mathbf{f}_{d,h}) \leq &\int_D \frac{-(\lambda_0 + \mu_0)(\lambda_D + \mu_D)}{\lambda + 2\mu}|\nabla \cdot \mathbf{u}_{0,d,h}|^2 dx \\
&+ 2\int_D \frac{-\mu_0 \mu_D}{\mu}\left|\epsilon(\mathbf{u}_{0,d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{u}_{0,d,h}) I_2\right|^2 dx \\
&+ \int_\Omega k^2 |\mathbf{w}_{d,h}|^2 dx.
\end{aligned}$$

Therefore, together with (3.4.2), we have

$$|E(\mathbf{f}_{d,h})| \leq C\left\{\|\nabla \mathbf{u}_{0,d,h}\|^2_{L^2(D)} + \|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}\right\}$$

for some positive constant C independent of h. Therefore from Lemma 3.3.1 and Lemma 3.4.3, we have, by choosing $q = 2$, the following estimate:

$$\begin{aligned}|E(\mathbf{f}_{d,h})| &\leq C\left\{\|\nabla \mathbf{w}_h\|^2_{L^2(\Omega)} + \|\nabla \mathbf{p}_{d,h}\|^2_{L^2(D)} + \|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}\right\} \\ &\leq C\left\{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})} + h^{-4}e^{\frac{2}{h}(s_*-\frac{1}{d})}\right\}.\end{aligned}$$

Therefore for $0 < h \leq 1$,

$$|E(\mathbf{f}_{d,h})| \leq C\left(h^{-4}e^{\frac{2}{h}(s_d-\frac{1}{d})}\right),$$

where $s_d = \max(\frac{1}{d+\varepsilon}, s_*)$. Moreover we notice that $\bar{D}\cap\Gamma_d = \emptyset$ implies $s_* < \frac{1}{d}$, and the conslusion (A) follows.

(B) We first consider case (i) of (3.4.1) and prove the conclusion (B) from (3.4.3).

Suppose $D\cap\Gamma_d \neq \emptyset$, then $s_* > \frac{1}{d} \geq \frac{1}{d+\varepsilon}$ since D is open. Therefore for any $\mathbf{y} \in \partial D\cap\Gamma_{d+\varepsilon}$, each neighborhood $U_{\mathbf{y}}$ of $\mathbf{y}$ satisfies $D\cap\Gamma_{d+\varepsilon}\cap U_{\mathbf{y}} \neq \emptyset$. By the assumption (i) of (3.4.1), for each $\mathbf{y} \in \partial D$, there exists $r_{\mathbf{y}}$ such that

$$\mu_D(\mathbf{x}) > r_{\mathbf{y}}, \quad \lambda_D + \mu_D \geq 0, \quad \forall \mathbf{x} \in B_{r_{\mathbf{y}}} \cap D. \tag{3.4.15}$$

Set $K := \partial D \cap \{\tau = s_*\} = \partial D \cap \ell_{1/s_*}$. It's easy to see that $K \neq \emptyset$. Since K is compact and is contained in $\cup_{y\in K} B_{r_{\mathbf{y}}}(\mathbf{y})$, there exsits $N \in \mathbb{N}$ such that $K \subset \cup_{j=1}^N B_{r_j}(\mathbf{y}_j)$, where $r_{\mathbf{y}_j}$ is abbreviated to r_j. Let $D_R = D \setminus \cup_{j=1}^N B_{r_j}(\mathbf{y}_j)$, then it is easy to see that there exists $\delta' > 0$ such that

$$\tau(\mathbf{x}) \leq s_* - \delta' \quad \text{in} \quad D_R.$$

Therefore for $q_0 < q \leq 2$ we have

$$\begin{aligned}\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx &\leq \int_{D_R} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx + \sum_{j=1}^{N}\int_{B_{r_j}(\mathbf{y}_j)\cap D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx \\ &\leq Ce^{\frac{q}{h}(s_*-\frac{1}{d}-\delta')} + N\int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx\end{aligned}$$

for some $\mathbf{y}_* \in \{\mathbf{y}_j\}_{j=1}^N$ and $r_* \in \{r_j\}_{j=1}^N$ such that

$$\int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx = \max_{j=1,\ldots,N}\left(\int_{B_{r_j}(\mathbf{y}_j)\cap D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx\right).$$

Moreover, we can compute more finely that

$$\begin{aligned}&\int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx \\ &= \int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx + \int_{B_{r_*}(\mathbf{y}_*)\cap D\cap(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx \\ &\leq \int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx + Ce^{\frac{q}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}.\end{aligned}$$

Therefore by combining the above inequalities, we have

$$\begin{aligned}\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx \leq C&\int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx \\ &+ Ce^{\frac{q}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})} + Ce^{\frac{q}{h}(s_*-\frac{1}{d}-\delta')}.\end{aligned}$$

Set

$$A_{q,*,h} := \int_{B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx.$$

Now we come back to (3.4.3), from Lemma 3.3.1 we have for $0 < h \ll 1$

$$\begin{aligned}E(\mathbf{f}_{d,h}) \geq C&\left\{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx - \|\mathbf{w}_h\|^2_{H^1(\Omega)}\right\} \\ &- k^2\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)} \\ \geq C&\left(\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx\right) \\ &\times\left(1 - \frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}}{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}\right. \\ &\qquad\left. - \frac{\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}}{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}\right).\end{aligned} \tag{3.4.16}$$

In the following we estimate each term separately.

First, by Lemma 3.4.3 we can compute

$$\begin{aligned}
&\frac{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h})-\frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}{\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}}\\
&\quad\geq C\frac{A_{2,*,h}(c-Ch^{-2})}{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}+h^{-4}\left(\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{q}{h}(\tau-\frac{1}{d})}dx\right)^{2/q}} \qquad (3.4.17)\\
&\quad\geq C\frac{A_{2,*,h}(ch^{-4}-Ch^{2})}{(A_{q,*,h})^{2/q}+e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}+e^{\frac{2}{h}(s_*-\frac{1}{d}-\delta')}},
\end{aligned}$$

for $q_0 < q \leq 2$.

Now we need to compute $A_{q,*,h}$ carefully. For $\mathbf{y}_* \in \ell_{1/s_*}\cap\partial D$, we consider the following change of coordinates as in [23]. First, let $\mathcal{T}$ be the composition of the following two rigid motions: i) translate $\mathbf{y}_*$ to the origin, and ii) rotate so that the unit inward normal of $\mathcal{T}(\Gamma_{1/s_*})$ at the origin is the vector $(0,1)^T$. Then set $\mathbf{z} = (z_1(\mathbf{x}), z_2(\mathbf{x}))^T = \mathcal{T}(\mathbf{x})$ and $\boldsymbol{\xi} = (\xi_1(\mathbf{z}), \xi_2(\mathbf{z}))^T = \Xi(\mathbf{z})$, where

$$\Xi(\mathbf{z}) = \begin{pmatrix} z_1 \\ \tau(\mathcal{T}^{-1}\mathbf{z}) - s_* \end{pmatrix}.$$

Then $\Xi\circ\mathcal{T}$ gives a $\mathcal{C}^2$ diffeomorphism in a neighborhood $U_{\mathbf{y}_*}$ of $\mathbf{y}_*$. Geometrically, under the transformation $\Xi\circ\mathcal{T}$ the point $\mathbf{y}_*$ becomes the origin of the new frame $(\xi_1,\xi_2)^T$, ξ_1-axis coincides with the curve ℓ_{1/s_*}, and the positive direction of ξ_2-axis coincides with the unit inward normal of $\mathcal{T}(\Gamma_{1/s_*})$ at $\mathbf{y}_*$.

We do the above change of coordinates, then we have $\Xi\circ\mathcal{T}(\mathbf{y}_*) = 0$ and

$$\begin{aligned}
&Ce^{\frac{q}{h}(s_*-\frac{1}{d})}\left(\int_{\Xi\circ\mathcal{T}(B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon})} e^{\frac{q}{h}\xi_2}d\boldsymbol{\xi}\right)\\
&\leq A_{q,*,h} \leq Ce^{\frac{q}{h}(s_*-\frac{1}{d})}\left(\int_{\Xi\circ\mathcal{T}(B_{r_*}(\mathbf{y}_*)\cap D\cap\Gamma_{d+\varepsilon})} e^{\frac{q}{h}\xi_2}d\boldsymbol{\xi}\right). \qquad (3.4.18)
\end{aligned}$$

Since ∂D is continuous, $\Xi\circ\mathcal{T}(\partial D)$ is also continuous and is able to be parametrized by a continuous function near $\boldsymbol{\xi} = \mathbf{0}$ under a suitable rotation. So, we consider a rotation $\widetilde{\mathcal{T}}$ with $\widetilde{\mathcal{T}}(\boldsymbol{\xi}) = \tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \tilde{\xi}_2)^T$ such that $\widetilde{\mathcal{T}}(\Xi \circ \mathcal{T}(\partial D))$ can be parametrized by $f_*(\tilde{\xi}_1)$ near $\tilde{\boldsymbol{\xi}} = 0$ with $f_*(\mathbf{0}) = \mathbf{0}$.

Actually, we can choose $\widetilde{\mathcal{T}}$ such that

$$\xi_2 = (\sin\theta)\tilde{\xi}_1 + (\cos\theta)\tilde{\xi}_2 \quad \text{with} \quad |\theta| < \frac{\pi}{2},$$

because $\Xi \circ \mathcal{T}(D) \subset \{\xi_2 \leq 0\}$ and D is open. Let $a = \sin\theta$ and $b = \cos\theta$, then $b > 0$. Without loss of generality, we assume $\widetilde{\mathcal{T}}(\Xi \circ \mathcal{T}(\partial D))$ can be parametrized by $f_*(\tilde{\xi}_1)$ in $\tilde{\xi}_1 < diam(\widetilde{\mathcal{T}}(\Xi \circ \mathcal{T}(B_{r_*}(\mathbf{y}_*) \cap D \cap \Gamma_{d+\varepsilon})))$. Set $\widetilde{U} = \widetilde{\mathcal{T}}(\Xi \circ \mathcal{T}(B_{r_*}(\mathbf{y}_*) \cap D \cap \Gamma_{d+\varepsilon}))$.

Here we note that f_* is continuous in $\widetilde{U}$ since we assume ∂D has continuous boundary, and $\widetilde{U} \subset \{a\tilde{\xi}_1 + b\tilde{\xi}_2 \leq 0\}$ since $\Xi \circ \mathcal{T}(D) \subset \{\xi_2 \leq 0\}$.

Now it's easy to see that there exist positive constants $\delta_1, \delta_2, \delta_1', \delta_2'$ independent of h with $\delta_2 < \delta_2'$ such that

$$\begin{aligned}
&\int_{-\delta_1'}^{\delta_1}\int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} \\
&\leq \int_{\Xi \circ \mathcal{T}(B_{r_*}(\mathbf{y}_*) \cap D \cap \Gamma_{d+\varepsilon})} e^{\frac{q}{h}\xi_2} d\boldsymbol{\xi} = \int_{\widetilde{U}} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} \\
&\leq \int_{-\delta_1'}^{\delta_1}\int_{-\delta_2'}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} \\
&\quad = \int_{-\delta_1'}^{\delta_1}\int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} + \int_{-\delta_1'}^{\delta_1}\int_{-\delta_2'}^{-\delta_2} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}}.
\end{aligned}$$

Since $\widetilde{U} \subset \{a\tilde{\xi}_1 + b\tilde{\xi}_2 \leq 0\}$,

$$\delta_2 \leq f_*(\tilde{\xi}_1) \leq -\frac{a}{b}\tilde{\xi}_1 \qquad \text{in} \quad \widetilde{U}.$$

Therefore, we can compute directly and obtain that

$$\begin{aligned}
&\int_{-\delta_1'}^{\delta_1}\int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} \\
&\leq \int_{\Xi \circ \mathcal{T}(B_{r_*}(\mathbf{y}_*) \cap D \cap \Gamma_{d+\varepsilon})} e^{\frac{q}{h}\xi_2} d\boldsymbol{\xi} \\
&\quad = \int_{-\delta_1'}^{\delta_1}\int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}} + (\delta_1' + \delta_1)(\delta_2' - \delta_2).
\end{aligned} \tag{3.4.19}$$

In order to make the computation clear, we set

$$B_{q,*,h} = \int_{-\delta_1'}^{\delta_1} \int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}(a\tilde{\xi}_1 + b\tilde{\xi}_2)} d\tilde{\boldsymbol{\xi}}. \tag{3.4.20}$$

By combining (3.4.17), (3.4.18) and (3.4.19), we have

$$\begin{aligned}
&\frac{\int_D \frac{\mu_0\mu_D}{\mu} \left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}{\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}} \\
&\quad \geq C\frac{e^{\frac{2}{h}(s_* - \frac{1}{d})}B_{2,*,h}(c - Ch^2)}{e^{\frac{2}{h}(s_* - \frac{1}{d})}(B_{q,*,h} + C)^{2/q} + e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - \frac{1}{d})} + e^{\frac{2}{h}(s_* - \frac{1}{d} - \delta')}} \\
&\quad = C\frac{B_{2,*,h}(c - Ch^2)}{(B_{q,*,h})^{2/q} + C + e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - s_*)} + e^{\frac{2}{h}(-\delta')}}.
\end{aligned} \tag{3.4.21}$$

Now we compute $B_{q,*,h}$ more carefully. We note that since f_* is continuous near $\tilde{\boldsymbol{\xi}} = 0$, for all $0 < \delta < \min(\delta_2, s_* - \frac{1}{d+\varepsilon}, \delta')$, there exists $0 < \delta_1'' < \min(\delta_1, \delta_1')$ such that

$$|f_*(\tilde{\xi}_1)| = |f_*(\tilde{\xi}_1) - f_*(0)| < \delta, \quad \forall \tilde{\xi}_1 \in (-\delta_1'', \delta_1'').$$

Therefore,

$$\begin{aligned}
B_{q,*,h} &\geq \int_{-\delta_1''}^{\delta_1''} e^{\frac{q}{h}a\tilde{\xi}_1} \left(\int_{-\delta_2}^{f_*(\tilde{\xi}_1)} e^{\frac{q}{h}b\tilde{\xi}_2} d\tilde{\xi}_2\right) d\tilde{\xi}_1 \\
&= \int_{-\delta_1''}^{\delta_1''} e^{\frac{q}{h}a\tilde{\xi}_1} \frac{h}{qb}\left(e^{\frac{q}{h}bf_*(\tilde{\xi}_1)} - e^{-\frac{q}{h}\delta_2}\right) d\tilde{\xi}_1 \\
&\geq \int_{-\delta_1''}^{\delta_1''} e^{\frac{q}{h}a\tilde{\xi}_1} \frac{h}{qb}\left(e^{-\frac{q}{h}\delta} - e^{-\frac{q}{h}\delta_2}\right) d\tilde{\xi}_1 \\
&\geq \frac{h}{qb} e^{-\frac{q}{h}b\delta}(1 - e^{-\frac{q}{h}(\delta_2 - \delta)}) \int_0^{\delta_1''} e^{\frac{q}{h}a\tilde{\xi}_1} d\tilde{\xi}_1.
\end{aligned}$$

Then for $0 < h \ll 1$, we obtain the following estimate

$$B_{q,*,h} \geq C\frac{h}{qb} e^{-\frac{q}{h}b\delta}, \tag{3.4.22}$$

for all $0 < \delta < \min(\delta_2, s_* - \frac{1}{d+\varepsilon}, \delta')$ and for some C independent of h. Moreover, we observe that for $0 < h \ll 1$

$$\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - s_*)}}{B_{2,*,h}} \leq Ch^{-1} e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - s_* + \delta)}$$

and

$$\frac{e^{-\frac{2}{h}\delta'}}{B_{2,*,h}} \leq Ch^{-1}e^{\frac{1}{h}(-\delta'+\delta)}.$$

Then (3.4.21) becomes the following estimate

$$\frac{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}{\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}} \geq C\frac{B_{2,*,h}}{(B_{q,*,h})^{2/q}},$$

for $q_0 < q \leq 2$ and $0 < h \ll 1$.

Actually, we can directly compute and use the Hölder inequality to obtain that for $q_0 < q \leq 2$,

$$\begin{aligned}
(B_{q,*,h})^{2/q} &= \left[\int_{-\delta_1'}^{\delta_1} e^{\frac{q}{h}a\tilde{\xi_1}}\frac{h}{qb}\left(e^{\frac{q}{h}bf_*(\tilde{\xi_1})} - e^{-\frac{q}{h}b\delta_2}\right)d\tilde{\xi_1}\right]^{2/q}\\
&= \left(\frac{h}{qb}\right)^{2/q}\left[\int_{-\delta_1'}^{\delta_1} e^{\frac{q}{h}a\tilde{\xi_1}}e^{\frac{q}{h}bf_*(\tilde{\xi_1})}\left(1 - e^{-\frac{q}{h}b(\delta_2+f_*(\tilde{\xi_1}))}\right)d\tilde{\xi_1}\right]^{2/q}\\
&\leq C\left(\frac{h}{qb}\right)^{2/q}\int_{-\delta_1'}^{\delta_1} e^{\frac{2}{h}a\tilde{\xi_1}}e^{\frac{2}{h}bf_*(\tilde{\xi_1})}\left(1 - e^{-\frac{q}{h}b(\delta_2+f_*(\tilde{\xi_1}))}\right)^{2/q}d\tilde{\xi_1}.
\end{aligned}$$

Since $\delta_2 + f_*(\tilde{\xi_1}) \geq 0$ for $\tilde{\xi_1} \in [-\delta_1', \delta_1]$, we have

$$0 < e^{-\frac{q}{h}(\delta_2+f_*(\tilde{\xi_1}))} \leq 1$$

and therefore for $q \leq 2$

$$\left(1 - e^{-\frac{q}{h}(\delta_2+f_*(\tilde{\xi_1}))}\right)^{2/q} \leq 1 - e^{-\frac{2}{h}(\delta_2+f_*(\tilde{\xi_1}))}.$$

Hence we obtain for $q_0 < q \leq 2$

$$\begin{aligned}
(B_{q,*,h})^{2/q} &\leq C\left(\frac{h}{qb}\right)^{2/q}\int_{-\delta_1'}^{\delta_1} e^{\frac{2}{h}a\tilde{\xi_1}}e^{\frac{2}{h}bf_*(\tilde{\xi_1})}\left(1 - e^{-\frac{2}{h}(\delta_2+f_*(\tilde{\xi_1}))}\right)d\tilde{\xi_1}\\
&= C\left(\frac{h}{qb}\right)^{2/q}\left(\frac{2b}{h}\right)\int_{-\delta_1'}^{\delta_1}\int_{-\delta_2}^{f_*(\tilde{\xi_1})} e^{\frac{2}{h}(a\tilde{\xi_1}+b\tilde{\xi_2})}d\tilde{\boldsymbol{\xi}}\\
&= C\left(\frac{h}{qb}\right)^{2/q}\left(\frac{2b}{h}\right)B_{2,*,h},
\end{aligned}$$

and then the most difficult part of the proof of Theorem 3.4.1 can be concluded that for $0 < h \ll 1$ and $q_0 < q < 2$

$$\frac{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx}{\|\mathbf{w}_{d,h}\|^2_{L^2(\Omega)}} \geq Ch^{1-\frac{2}{q}}, \tag{3.4.23}$$

for some constant C independent of h.

Back to (3.4.16), by (3.4.23) we have for $0 < h \ll 1$ and $q_0 < q < 2$

$$E(\mathbf{f}_{d,h}) \geq C\left(\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx\right) \times \left(1 - \frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}}{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx} - h^{(\frac{2}{q}-1)}\right). \tag{3.4.24}$$

By direct computation we have

$$\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}}{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx} \leq Ce^{\frac{2}{h}(\frac{1}{d+\varepsilon}-s_*)},$$

therefore

$$\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}}{\int_D \frac{\mu_0\mu_D}{\mu}\left|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{p}_{d,h})I_2\right|^2 dx} = o(1). \tag{3.4.25}$$

Hence by using Lemma 3.4.3 and by computing directly from (3.4.24) and (3.4.25), we have for $0 < h \ll 1$

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-4}A_{2,*,h} \geq Ch^{-4}e^{\frac{2}{h}(s_*-\frac{1}{d})}B_{2,*,h}. \tag{3.4.26}$$

Therefore by (3.4.22), for all $0 < \delta < \min(\delta_2, s_* - \frac{1}{d+\varepsilon}, \delta')$ and for $0 < h \ll 1$ we have

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-3}e^{\frac{2}{h}(s_*-\frac{1}{d}-\delta)}, \tag{3.4.27}$$

for some constant C independent of h. Choose δ such that $\delta < s_* - \frac{1}{d}$, then the proof of (B) is complete.

For case (ii) of (3.4.1), instead of using (3.4.3), we shall consider the negative of (3.4.2):

$$-E(\mathbf{f}_{d,h}) \geq \int_D -(\lambda_D + \mu_D)|\nabla\cdot\mathbf{u}_{0,d,h}|^2 dx - 2\int_D \mu_D\left|\epsilon(\mathbf{u}_{0,d,h}) - \frac{1}{2}(\nabla\cdot\mathbf{u}_{0,d,h})I_2\right|^2 dx - \int_\Omega k^2|\mathbf{w}_{d,h}|^2 dx.$$

And a similar argument will also give (3.4.27).

(B′) As in (B), we will only prove case (i) of (3.4.1) by (3.4.3), and case (ii) of (3.4.1) can be treated similarly by using the negative of (3.4.2). Suppose that $\bar{D} \cap \Gamma_d \neq \emptyset$ and D has $C^{0,\alpha}$ boundary. Since $\bar{D} \cap \Gamma_d \neq \emptyset$, $s_* \geq \frac{1}{d} > \frac{1}{d+\varepsilon}$ and $K = \partial D \cap \ell_{1/s_*} \neq \emptyset$.

In fact, we have proved in (B) that $s_* - \frac{1}{d+\varepsilon} > 0$ and continuity of ∂D ensure (3.4.23) holds. So (3.4.23) also holds under this assumption of (B′) and therefore (3.4.26) also holds.

However, since D has $C^{0,\alpha}$ boundary, we have the better estimate than (3.4.22). Without loss of generality, we assume $f_*(\tilde{\xi_1})$ is $C^{0,\alpha}$ for $\tilde{\xi_1} \in [-\delta_1, \delta_1']$. Then there exists a positive constant L such that for $\tilde{\xi_1} \in [-\delta - 1, \delta_1']$

$$|f_*(\tilde{\xi_1})| = |f_*(\tilde{\xi_1}) - f_*(0)| \leq L|\tilde{\xi_1}|^\alpha.$$

Therefore we can compute directly as follows:

$$\begin{aligned}
B_{2,*,h} &= \int_{-\delta_1'}^{\delta_1} \int_{-\delta_2}^{f_*(\tilde{\xi_1})} e^{\frac{2}{h}(a\tilde{\xi_1}+b\tilde{\xi_2})} d\tilde{\boldsymbol{\xi}} \\
&\geq \int_{-\delta_1'}^{0} e^{\frac{2a}{h}\tilde{\xi_1}} \left(\int_{-\delta_2}^{f_*(\tilde{\xi_1})} e^{\frac{2b}{h}\tilde{\xi_2}} d\tilde{\xi_2} \right) d\tilde{\xi_1} \\
&= \int_{-\delta_1'}^{0} e^{\frac{2a}{h}\tilde{\xi_1}} \frac{h}{2b} \left(e^{\frac{2b}{h} f_*(\tilde{\xi_1})} - e^{\frac{-2b\delta_2}{h}} \right) d\tilde{\xi_1} \\
&\geq \frac{h}{2b} \left(\int_{-\delta_1'}^{0} e^{\frac{2}{h}(a\tilde{\xi_1} - bL|\tilde{\xi_1}|^\alpha)} - e^{\frac{2a}{h}\tilde{\xi_1}} e^{\frac{-2b\delta_2}{h}} d\tilde{\xi_1} \right) \\
&= \frac{h}{2b} \left(\int_{0}^{\delta_1'} e^{-\frac{2}{h}(a\tilde{\xi_1} + bL\tilde{\xi_1}^\alpha)} - e^{-\frac{2a}{h}\tilde{\xi_1}} e^{\frac{-2b\delta_2}{h}} d\tilde{\xi_1} \right).
\end{aligned}$$

Without loss of generality, we assume that $0 < \delta_1 < 1$. Since $0 < \alpha \leq 1$, we have

$$\begin{aligned}
\int_0^{\delta_1'} e^{\frac{-2}{h}(a\tilde{\xi_1} + bL\tilde{\xi_1}^\alpha)} d\tilde{\xi_1} &\geq \int_0^{\delta_1'} e^{\frac{-2}{h}(a+bL)\tilde{\xi_1}^\alpha} d\tilde{\xi_1} \\
&= h^{\frac{1}{\alpha}} \int_0^{\frac{\delta_1'}{h^{1/\alpha}}} e^{-2(a+bL)\tilde{\xi_1}^\alpha} d\tilde{\xi_1}.
\end{aligned}$$

Then by computing directly, we have

$$B_{2,*,h} \geq \frac{h}{2b}\left\{h^{\frac{1}{\alpha}}\int_0^{\frac{\delta_1'}{h^{1/\alpha}}} e^{-2(a+bL)\tilde{\xi_1}^{\alpha}}d\tilde{\xi}_1 - he^{\frac{-2\delta_2}{h}}\int_0^{\frac{\delta_1'}{h}} e^{-2a\tilde{\xi}_1}d\tilde{\xi}_1\right\}.$$

Since

$$\int_0^{\frac{\delta_1'}{h^{1/\alpha}}} e^{-2(a+bL)\tilde{\xi_1}^{\alpha}}d\tilde{\xi}_1 \to \int_0^{\infty} e^{-2(a+bL)\tilde{\xi}_1^{\alpha}}d\tilde{\xi}_1 < \infty \quad \text{as} \quad h \to 0^+$$

and

$$\int_0^{\frac{\delta_1}{h}} e^{-2a\tilde{\xi}_1}d\tilde{\xi}_1 \to \int_0^{\infty} e^{-2a\tilde{\xi}_1}d\tilde{\xi}_1 < \infty \quad \text{as} \quad h \to 0^+,$$

we have

$$B_{2,*,h} \geq Ch^{1+\frac{1}{\alpha}} \quad \text{for} \quad 0 < h \ll 1.$$

Then by (3.4.26)

$$E(\mathbf{f}_{d,h}) \geq Ce^{\frac{2}{h}(s_*-\frac{1}{d})}h^{-4}h^{1+\frac{1}{\alpha}} = Ce^{\frac{2}{h}(s_*-\frac{1}{d})}h^{-3+\frac{1}{\alpha}}$$

for each $0 < h \ll 1$. If $\alpha > \frac{1}{3}$, then even when $s_* = \frac{1}{d}$, $|E(\mathbf{f}_{d,h})|$ tends to infinity as h tends to zero.

□

3.5 Remarks

In this work, we succeed in applying the enclosure-type method and CGO solutions to the theory of reconstructing unknown inclusions in time-harmonic elastic waves. At the same time, we observe that only Lipschitz assumption on the boundary regularity is needed. This is a very low regularity assumption on the boundary. Besides, we also find the effect of CGO solutions is taken in the enclosure-type method by utilizing the relationship between the reflection solution $\mathbf{w}$ and the solution $\mathbf{u}_0$ of (3.1.2), for example Lemma 3.4.2. Actually this relationship is the key point in reflecting the regularity assumption on the boundary. In other words, if we can have sharper description of this relation, we may reduce the assumptions on the boundary regularity.

Chapter 4

Reconstruction of impenetrable inclusions

4.1 Introduction

In this chapter, we consider the following inverse problem: reconstructing the shape and location of an impenetrable obstacle or cracks in an elastic body by using boundary measurements. We also use the enclosure-type method introduced in Chapter 2 to do the work of reconstructing the unknown obstacles. In the following we give the precise mathematical model of this inverse problem.

4.1.1 Mathematical model and some notations

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n = 2, 3$. For simplicity, we assume Ω has C^∞ boundary. In this chapter we suppose that the elastic subject occupies Ω, and there is an impenetrable and sound-hard obstacles, which is denoted by D with $D \subset\subset \Omega$. In our assumptions, for simplicity, we consider the elastic subject is isotropic and homogeneous with Lamé constants λ and μ. λ and μ are real numbers.

We send a time-harmonic elastic wave with time dependence e^{ikt} into Ω in order to detect the unknown D. By singling out the space part, the displacement field $\mathbf{u}$ satisfying

$$\begin{cases} \nabla \cdot \sigma(\mathbf{u}) + k^2 \mathbf{u} = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \sigma(\mathbf{u})\nu = 0 \quad \text{on} \quad \partial D, \end{cases} \tag{4.1.1}$$

when D is not empty. Here σ denotes the stress tensor of the elastic body, that is

$$\sigma(\mathbf{v}) = \lambda \nabla \cdot \mathbf{v} I_n + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \tag{4.1.2}$$

for any vector field $\mathbf{v}$ and I_n denotes the $n \times n$ identity matrix.

If $D = \emptyset$, that is there is no unknown obstacles or cracks, then we denote the corresponding displacement field by $\mathbf{u}_0$ which satisfies

$$\nabla \cdot \sigma(\mathbf{u}_0) + k^2 \mathbf{u}_0 = 0 \quad \text{in} \quad \Omega. \tag{4.1.3}$$

In the following when saying k^2 is not a Dirichlet eigenvalue of problem (4.1.1), it means the following corresponding homogeneous problem has only one trivial solution:

$$\begin{cases} \nabla \cdot \sigma(\mathbf{v}) + k^2 \mathbf{v} = 0, & \text{in} \quad \Omega \setminus \bar{D} \\ \sigma(\mathbf{v})\nu = 0, & \text{on} \quad \partial D \\ \mathbf{v} = 0, & \text{on} \quad \partial \Omega. \end{cases}$$

Similarly, when saying k^2 not a Dirichlet eigenvalue of the problem (4.1.3), it means the following corresponding homogeneous problem has only one trivial solution:

$$\begin{cases} \nabla \cdot \sigma(\mathbf{v}) + k^2 \mathbf{v} = 0, & \text{in} \quad \Omega \\ \mathbf{v} = 0, & \text{on} \quad \partial \Omega. \end{cases}$$

We note that if k^2 is neither an eigenvalue of problem (4.1.1) nor (4.1.3), then given any Dirichlet boundary condition $\mathbf{f}$ on $\partial\Omega$, problem (4.1.1) and (4.1.3) have the only one solution respectively. Therefore under this assumption on k^2, we can define the Dirichlet-to-Neumann operators Λ_D and $\Lambda_\emptyset$ from $H^{\frac{1}{2}}(\partial\Omega)^n$ to $H^{-\frac{1}{2}}(\partial\Omega)^n$ as follows: for any Dirichlet boundary condition $\mathbf{f} \in H^{\frac{1}{2}}(\partial\Omega)^n$,

$$< \Lambda_D \mathbf{f}, \mathbf{g} >= \int_{\Omega \setminus \bar{D}} tr(\sigma(\mathbf{u}) \nabla \mathbf{v}) d\mathbf{x} - k^2 \int_{\Omega \setminus \bar{D}} \mathbf{u} \cdot \mathbf{v} d\mathbf{x}$$

for any $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^n$ and some $\mathbf{v} \in H^1(\Omega \setminus \bar{D})^n$ satisfying $\mathbf{v}|_{\partial\Omega} = \mathbf{g}$, where $\mathbf{u}$ solves (4.1.1) with Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{f}$.

Similarly, for any Dirichlet boundary condition $\mathbf{f} \in H^{\frac{1}{2}}(\partial\Omega)^n$,

$$< \Lambda_\emptyset \mathbf{f}, \mathbf{g} >= \int_\Omega tr(\sigma(\mathbf{u}_0)\nabla \mathbf{v}) d\mathbf{x} - \int_\Omega \mathbf{u}_0 \cdot \mathbf{v} d\mathbf{x}$$

for any $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)$ and some $\mathbf{v} \in H^1(\Omega)$ satisfying $\mathbf{v}|_{\partial\Omega} = \mathbf{g}$, where $\mathbf{u}_0$ solves (4.1.3) with Dirichlet boundary condition $\mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$.

4.1.2 A remark on regularity assumptions

To establish an applicable algorithm, the regularity results for the reflection solution $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$ is crucial, where $\mathbf{u}$ and $\mathbf{u}_0$ are solutions of (4.1.1) and (4.1.3) with the same Dirichlet boundary conditions respectively. Precisely, what kind of regularity results of $\mathbf{w}$ we can get, will influence what kind of regularity assumptions on the boundary we can give. Actually, in [19] the regularity assumptions on ∂D can be reduced to $C^{0,\alpha}$, $\alpha > 1/3$ or even only continuity. However, the situation in the impenetrable case is totally different. It is hard to assume the regularity of ∂D is less than “Lipschitz”. This is because we can not obtain “good” estimate for $\mathbf{w}$ when the regularity of ∂D is less than Lipschitz.

In this chapter, in order to emphasize the influence of the reflected solution, we just assume the regularity of ∂D is C^2.

4.2 The corresponding indicator functional

In our reconstruction method, we hope to detect the unknown D but not to invade the elastic body occupying Ω. So all the information we can obtain only comes from $\partial\Omega$.

As mentioned before, we note that if we assume k^2 is neither the Dirichlet eigenvalue of problem (4.1.1) nor (4.1.3), then Λ_D and $\Lambda_\emptyset$ are well-defined. Then we can use Λ_D and $\Lambda_\emptyset$ to deal with our problem.

From now on, assume we can obtain all information such as follows: if we set a displacement $\mathbf{f}$ on $\partial\Omega$, we can measure the corresponding tractions $\sigma(\mathbf{u})\nu$ and $\sigma(\mathbf{u}_0)\nu$ on $\partial\Omega$ associated to (4.1.1) and (4.1.3) respectively. This means Λ_D

and $\Lambda_\emptyset$ are known. In other words, this assumption can let us make the most of Λ_D and $\Lambda_\emptyset$.

Next, we consider the following corresponding indicator functional E defined as follows: for any $\mathbf{f} \in H^{\frac{1}{2}}(\partial\Omega)^n$

$$E(\mathbf{f}) :=< (\Lambda_\emptyset - \Lambda_D)\mathbf{f}, \bar{\mathbf{f}} > .$$

It measures the difference between the energies corresponding to the situation with and without the impenetrable inclusion D. And it is one of the main tools in the enclosure-type method. To show how E comes into effect in the impenetrable case, we need to understand it more and establish useful identity in terms of the *known* term $\mathbf{u}_0$.

Here for a matrix A, we denote the trace of A by $tr(A)$.

Lemma 4.2.1. *For any* $\mathbf{f} \in H^{\frac{1}{2}}(\partial\Omega)^n$*,*

$$E(\mathbf{f}) = \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{w})\nabla\bar{\mathbf{w}}) - \int_{\Omega\setminus\bar{D}} k^2\mathbf{w}\cdot\bar{\mathbf{w}} + \int_D tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_0) - \int_D k^2\mathbf{u}_0\cdot\bar{\mathbf{u}}_0$$

where $\mathbf{u}$ *and* $\mathbf{u}_0$ *are the corresponding solutions to (4.1.1) and (4.1.3) with the boundary condition* $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$*, and* $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$*.*

Proof. The proof is simple. First, we notice that, due to the definition and property of σ, for any vector fields $\mathbf{v}$ and $\tilde{\mathbf{v}}$

$$tr(\sigma(\mathbf{v})\nabla\tilde{\mathbf{v}}) = tr(\sigma(\tilde{\mathbf{v}})\nabla\mathbf{v}). \tag{4.2.4}$$

and

$$tr(\sigma(\mathbf{v})\nabla\bar{\mathbf{v}}) \quad \text{is real.} \tag{4.2.5}$$

Since $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$, we have

$$< \Lambda_D\mathbf{f}, \bar{\mathbf{f}} > = \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{u})\nabla\bar{\mathbf{u}}) - \int_{\Omega\setminus\bar{D}} k^2\mathbf{u}\cdot\bar{\mathbf{u}} \tag{4.2.6}$$

$$= \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{u})\nabla\bar{\mathbf{u}}_0) - \int_{\Omega\setminus\bar{D}} k^2\mathbf{u}\cdot\bar{\mathbf{u}}_0. \tag{4.2.7}$$

From (4.2.6) and (4.2.5), we know $< \Lambda_D \mathbf{f}, \bar{\mathbf{f}} >$ is real. Therefore taking complex conjugate of (4.2.7) and using the formula (4.2.4), we have

$$< \Lambda_D \mathbf{f}, \bar{\mathbf{f}} >= \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}) - \int_{\Omega\setminus\bar{D}} k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}. \tag{4.2.8}$$

Similarly,

$$< \Lambda_\emptyset \mathbf{f}, \bar{\mathbf{f}} >= \int_{\Omega} tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}) - \int_{\Omega} k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}_\mathbf{0} \tag{4.2.9}$$

is also real. By subtracting (4.2.8) from (4.2.9), we obtain the energy difference functional E

$$\begin{aligned} E(\mathbf{f}) = &- \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{w}}) + \int_{\Omega\setminus\bar{D}} k^2 \mathbf{u}_0 \cdot \bar{\mathbf{w}} \\ &+ \int_D tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}) - \int_D k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}_\mathbf{0}, \end{aligned} \tag{4.2.10}$$

where $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$.

On the other hand, by subtracting (4.2.6) from (4.2.9) we have

$$\begin{aligned} E(\mathbf{f}) = &- \int_{\Omega\setminus\bar{D}} tr\Big(\sigma(\mathbf{u})\nabla\bar{\mathbf{u}} - \sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) + \int_{\Omega\setminus\bar{D}} k^2(\mathbf{u}\cdot\bar{\mathbf{u}} - \mathbf{u}_0\cdot\bar{\mathbf{u}}_\mathbf{0}) \\ &+ \int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) - \int_D k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}_\mathbf{0} \\ = &- \int_{\Omega\setminus\bar{D}} tr\Big(\sigma(\mathbf{w})\nabla\bar{\mathbf{w}} + \sigma(\mathbf{w})\nabla\bar{\mathbf{u}}_\mathbf{0} + \sigma(\mathbf{u_0})\nabla\bar{\mathbf{w}}\Big) \\ &+ \int_{\Omega\setminus\bar{D}} k^2(\mathbf{w}\cdot\bar{\mathbf{w}} + \mathbf{w}\cdot\bar{\mathbf{u}}_\mathbf{0} + \mathbf{u}_0\cdot\bar{\mathbf{w}}) \\ &+ \int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) - \int_D k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}_\mathbf{0}. \end{aligned}$$

By using formula (4.2.4) again, we get

$$tr\Big(\sigma(\mathbf{w})\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) = tr\Big(\sigma(\bar{\mathbf{u}}_\mathbf{0})\nabla\mathbf{w}\Big).$$

Therefore,

$$\begin{aligned} E(f) = &- \int_{\Omega\setminus\bar{D}} tr\Big(\sigma(\mathbf{w})\nabla\bar{\mathbf{w}}\Big) + \int_{\Omega\setminus\bar{D}} k^2 \mathbf{w}\cdot\bar{\mathbf{w}} \\ &+ \int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) - \int_D k^2 \mathbf{u}_0 \cdot \bar{\mathbf{u}}_\mathbf{0} \\ &- \int_{\Omega\setminus\bar{D}} 2Re\Big\{ tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{w}}\Big) - k^2\mathbf{u}_0\cdot\bar{\mathbf{w}} \Big\} \end{aligned} \tag{4.2.11}$$

From (4.2.10), we have

$$
\begin{aligned}
&-\int_{\Omega\setminus\bar{D}} tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{w}}\Big) + \int_{\Omega\setminus\bar{D}} k^2\mathbf{u}_0\cdot\bar{\mathbf{w}} \\
&= E(f) - \int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) + \int_D k^2\mathbf{u}_0\cdot\bar{\mathbf{u}}_\mathbf{0}
\end{aligned}
$$

and is real. Therefore we can deduce (4.2.11) becomes

$$
\begin{aligned}
E(f) = &-\int_{\Omega\setminus\bar{D}} tr\Big(\sigma(\mathbf{w})\nabla\bar{\mathbf{w}}\Big) + \int_{\Omega\setminus\bar{D}} k^2\mathbf{w}\cdot\bar{\mathbf{w}} \\
&+ \int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) - \int_D k^2\mathbf{u}_0\cdot\bar{\mathbf{u}}_\mathbf{0} \\
&+ 2E(f) - 2\int_D tr\Big(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_\mathbf{0}\Big) + 2\int_D k^2\mathbf{u}_0\cdot\bar{\mathbf{u}}_\mathbf{0}.
\end{aligned}
$$

Hence the conclusion of this lemma holds. $\square$

From the Lemma 4.2.1, it is easy to obtain the following simple upper bound and lower bound of the indicator functional E.

Corollary 4.2.2. *For $n = 2, 3$, assume that λ, μ are constants which satisfy the strongly convexity condition, that is,*

$$
\lambda + \frac{2}{n}\mu > 0, \quad \textit{and} \quad \mu > 0.
$$

For $\mathbf{f} \in H^{1/2}(\partial\Omega)^n$, we have the following two inequalities:

$$
|E(\mathbf{f})| \leq C\|\mathbf{w}\|^2_{H^1(\Omega\setminus\bar{D})^n} + C\|\mathbf{u}_0\|^2_{H^1(D)^n}
$$

for some constant C dependent only on λ, μ and

$$
\begin{aligned}
|E(\mathbf{f})| \geq (\lambda + \frac{2}{n}\mu)\|\nabla\cdot\mathbf{u}_0\|^2_{L^2(D)^n} + 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{n}(\nabla\cdot\mathbf{u}_0)I_n\|^2_{L^2(D)^n} \\
- k^2\|\mathbf{u}_0\|^2_{L^2(D)^n} - k^2\|\mathbf{w}\|^2_{L^2(\Omega\setminus\bar{D})^n},
\end{aligned}
$$

where I_n denotes the $n \times n$ identity matrix and $\mathbf{u}$ and $\mathbf{u}_0$ are the corresponding solutions to (4.1.1) and (4.1.3) with the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$, and $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$.

Proof. From Lemma (4.2.1), it is easy to see the first inequality immediately follows.

To see the second inequality, we recall the following formula: for any $\mathbf{v} \in H^1$, we have

$$tr(\sigma(\mathbf{v})\nabla\bar{\mathbf{v}}) = (\lambda + \frac{2}{n}\mu)|\nabla \cdot \mathbf{v}|^2 + 2\mu|\epsilon(\mathbf{v}) - \frac{1}{n}(\nabla \cdot \mathbf{v})I_n|^2. \qquad (4.2.12)$$

Then by strongly convexity condition, we have

$$tr(\sigma(\mathbf{w})\nabla\mathbf{w}) \geq 0$$

and therefore from Lemma (4.2.1),

$$E(\mathbf{f}) \geq \int_D tr(\sigma(\mathbf{u}_0)\nabla\bar{\mathbf{u}}_0) - \int_D k^2\mathbf{u}_0 \cdot \bar{\mathbf{u}}_0 - \int_{\Omega\setminus\bar{D}} k^2\mathbf{w} \cdot \bar{\mathbf{w}}.$$

By applying (4.2.12) again, we obtain the second result of this corollary. □

4.3 The regularity results of reflected solution

We call $\mathbf{w}$ the reflected solution corresponding to problem (4.1.1) with $\mathbf{u}_0$ if $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$, where $\mathbf{u}$ satisfies problem (4.1.1) with boundary condition $\mathbf{u}_0$ on $\partial\Omega$, that is,

$$\begin{cases} \nabla \cdot \sigma(\mathbf{u}) + k^2\mathbf{u} = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \sigma(\mathbf{u})\nu = 0 \quad \text{on} \quad \partial D \\ \mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

From the inequalities in the previous section, we know the key point of estimating the $E(\mathbf{f})$ is to estimate reflected solution $\mathbf{w}$, because $\mathbf{u}_0$ is a known solution.

So, let us see which equation $\mathbf{w}$ (corresponding to problem (4.1.1) with $\mathbf{u}_0$) satisfies:

$$\begin{cases} \nabla \cdot (\sigma(\mathbf{w})) + k^2\mathbf{w} = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \sigma(\mathbf{w})\nu = -\sigma(\mathbf{u}_0)\nu, & \text{on} \quad \partial D \\ \mathbf{w} = 0, & \text{on} \quad \partial\Omega. \end{cases} \qquad (4.3.13)$$

From a standard proof, we can obtain the existence and uniqueness of the problem (4.3.13), and furthermore a simple regularity result of $\mathbf{w}$. The following theorem tell us this.

Theorem 4.3.1. *Assume that D and Ω are two Lipschitz domain with $D \subset\subset \Omega$ in $\mathbb{R}^n$, $n = 2, 3$. Let $\mathbf{f} \in (H^1(\Omega \setminus \bar{D})^n)^*$, $\mathbf{g} \in H^{-1/2}(\partial D)^n$. We consider the following problem*

$$\begin{cases} \nabla \cdot \sigma(\mathbf{u}) + k^2\mathbf{u} = \mathbf{f}, & \textit{in} \quad \Omega \setminus \overline{D} \\ \sigma(\mathbf{u})\nu = \mathbf{g}, & \textit{on} \quad \partial D \\ \mathbf{u} = 0, & \textit{on} \quad \partial\Omega, \end{cases} \tag{4.3.14}$$

where σ has the form (4.1.2) with Lamé constants λ and μ which satisfy $\lambda + 2\mu > 0$ and $\mu > 0$. Now we also assume that k^2 is not a Dirichlet eigenvalue of (4.3.14). Then there exists the unique weak solution $\mathbf{u} \in H^1(\Omega \setminus \bar{D})^n$ to (4.3.14). That is, $\mathbf{u}$ satisfies

$$\int_{\partial D} \mathbf{g} \cdot \boldsymbol{\psi} - \int_{\Omega \setminus \bar{D}} tr(\sigma(\mathbf{u})\nabla\boldsymbol{\psi}) + \int_{\Omega \setminus \bar{D}} k^2\mathbf{u} \cdot \boldsymbol{\psi} = < \mathbf{f}, \boldsymbol{\psi} >,$$

for any $\boldsymbol{\psi} \in H^1(\Omega \setminus \bar{D})^n$ with $\boldsymbol{\psi}|_{\partial\Omega} = 0$. Moreover, we have

$$\|\mathbf{u}\|_{H^1(\Omega \setminus \bar{D})^n} \leq C\{\|\mathbf{f}\|_{(H^1(\Omega \setminus \bar{D})^n)^*} + \|\mathbf{g}\|_{H^{-1/2}(\partial D)^n}\},$$

where C is a positive constant independent of $\mathbf{u}$, $\mathbf{f}$ and $\mathbf{g}$.

Notice that from now on c and C denote constants, which may represent different values at different places.

Proof. We refer to [21, Theorem 4.10] for the standard proof. Nevertheless, we write down it here for the sake of convenience.

In order to deal with the problem within $\Omega \setminus \overline{D}$, a special function space is needed to be defined and discussed. Set a subspace of $H^1(\Omega \setminus \overline{D})^n$

$$\mathcal{H}_D^1 := \{\mathbf{u} \in H^1(\Omega \setminus \overline{D})^n : \mathbf{u}|_{\partial\Omega} = 0\}$$

with the inner product

$$(\mathbf{u},\mathbf{v})_{\mathcal{H}^1_D} := (\mathbf{u},\mathbf{v})_{H^1(\Omega\setminus\bar{D})^n}$$

for $\mathbf{u},\mathbf{v} \in \mathcal{H}^1_D$. It is a Hilbert space. Clearly, the inclusion $\mathcal{H}^1_D \subset L^2(\Omega \setminus \bar{D})^n$ is compact. Moreover, it is also easy to see that $\mathcal{H}^1_D$ is dense in $L^2(\Omega \setminus \bar{D})^n$, since

$$H^1_0(\Omega \setminus \bar{D})^n \subset \mathcal{H}^1_D \subset L^2(\Omega \setminus \bar{D})^n$$

and $H^1_0(\Omega \setminus \bar{D})$ is dense in $L^2(\Omega \setminus \bar{D})$. Then $L^2(\Omega \setminus \bar{D})^n$ acts as a pivot space for $\mathcal{H}^1_D$. The definition of a pivot space can be found in [21, p. 44]. (Let H and V be two Hilbert spaces. We say H acts as a pivot space on V, if V is a closed dense subspace of H with $\|\cdot\|_H \leq \|\cdot\|_V$. Then we can write $V \subseteq H \subseteq V^*$ and say H acts as a pivot space of V.)

Next, let B be a bilinear form on $\mathcal{H}^1_D \times \mathcal{H}^1_D$ defined as follows:

$$B(\mathbf{u},\mathbf{v}) := \int_{\Omega\setminus\bar{D}} tr(\sigma(\mathbf{u})\nabla\mathbf{v}) - \int_{\Omega\setminus\bar{D}} k^2\mathbf{u}\cdot\mathbf{v},$$

for $\mathbf{u},\mathbf{v} \in \mathcal{H}^1_D$. Since $\lambda + 2\mu, \mu > 0$, we have

$$Re\{B(\mathbf{u},\mathbf{u})\} \geq c\|\mathbf{u}\|^2_{\mathcal{H}^1_D} - C\|\mathbf{u}\|^2_{L^2(\Omega\setminus\bar{D})}.$$

Therefore, B is coercive on $\mathcal{H}^1_D$ (with respect to the pivot space $L^2(\Omega \setminus \bar{D})^n$). Consider the linear operator A from $\mathcal{H}^1_D$ to $(\mathcal{H}^1_D)^*$ defined as follows:

$$< A\mathbf{u},\mathbf{v} >:= B(\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in \mathcal{H}^1_D.$$

Then from Theorem 2.34 in [21] A is a Fredholm operator with zero index. From Theorem 2.27 in [21], the results of Fredholm Alternative holds. By assumption, 0 is not a Dirichlet eigenvalue of (4.3.14), therefore A has a bounded inverse A^{-1} (see [21, Corollary 2.2]). Define $F \in (\mathcal{H}^1_D)^*$ as follows:

$$F(\boldsymbol{\psi}) := \int_{\partial D} \mathbf{g}\cdot\boldsymbol{\psi} d\mathbf{s} - < f, \boldsymbol{\psi} > .$$

Then

$$\|\mathbf{u}\|_{\mathcal{H}^1_D} = \|A^{-1}F\|_{\mathcal{H}^1_D} \leq C\|F\|_{(\mathcal{H}^1_D)^*}.$$

Moreover, we obtain the following estimate by computing directly

$$\|F\|_{(\mathcal{H}^1_D)^*} \leq \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial D)^n} + \|f\|_{(H^1(\Omega\setminus\bar{D}))^*}.$$

Hence the conclusion of this theorem follows. □

We apply Theorem 4.3.1 to the problem (4.3.13), then we can immediately obtain the following lemma.

Lemma 4.3.2. *Let* $\mathbf{u}_0$ *solves (4.1.3). The reflected solution* $\mathbf{w}$ *corresponding to (4.1.1) with* $\mathbf{u}_0$ *has the following estimate:*

$$\begin{aligned}\|\mathbf{w}\|_{H^1(\Omega\setminus\bar{D})} &\leq C_1\|\sigma(\mathbf{u}_0)\nu\|_{H^{-1/2}(\partial D)^n}\\ &\leq C_2\|\mathbf{u}_0\|_{H^1(D)^n}\end{aligned}$$

for some constant C_1, C_2 *independent of* $\mathbf{w}$ *and* $\mathbf{u}_0$.

Therefore from Corollary 4.2.2 we get

$$|E(\mathbf{f})| \leq C\|\mathbf{u}_0\|^2_{H^1(D)^n} \leq C\|\mathbf{u}_0\|^2_{H^1(\Omega)^n}.$$

In other words, $|E(\mathbf{f})|$ has a *known* upper bound in terms of $\mathbf{u}_0$.

But how about the lower bound? Evidently, the result (4.3.2) is not enough. We must seek a better regularity result for $\mathbf{w}$ if it is possible.

Theorem 4.3.3. *Assume that* ∂D *is of class* C^2 *and* $\mathbf{u}_0$ *solves (4.1.3). Let* $\mathbf{w}$ *be the reflected solution corresponding to (4.1.1) with* $\mathbf{u}_0$*. Then we have the following estimate:*

$$\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})^n} \leq C\|\mathbf{u}_0\|_{H^{-t}(\partial D)^n} \tag{4.3.15}$$

for $\frac{1}{2} \leq t \leq \frac{3}{2}$ *and some positive constant* C *independent of* $\mathbf{w}$*,* $\mathbf{u}_0$ *and* t.

Proof. As in [23] or [11], consider the following special function $\mathbf{p} \in H^1(\Omega \setminus \bar{D})^n$ satisfying

$$\begin{cases} \nabla\cdot\big(\sigma(\mathbf{p})\big) + k^2\mathbf{p} = \bar{\mathbf{w}}, & \text{in} \quad \Omega\setminus\bar{D} \\ \sigma(\mathbf{p})\nu = 0, & \text{on} \quad \partial D \\ \mathbf{p} = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.3.16}$$

It means that for any $\varphi \in H^1(\Omega \setminus \bar{D})$ with $\varphi|_{\partial\Omega} = 0$, i.e. $\varphi \in \mathcal{H}^1_D$, we have

$$-\int_{\Omega\setminus\bar{D}} tr\big(\sigma(\mathbf{p})\nabla\varphi\big)d\mathbf{x} + \int_{\Omega\setminus\bar{D}} k^2\mathbf{p}\cdot\varphi d\mathbf{x} = \int_{\Omega\setminus\bar{D}} \bar{\mathbf{w}}\cdot\varphi d\mathbf{x}.$$

Therefore, we can obtain the following identity by substitute φ by $\mathbf{w}$:

$$\int_{\Omega\setminus\bar{D}} |\mathbf{w}|^2 d\mathbf{x} = -\int_{\Omega\setminus\bar{D}} tr\big(\sigma(\mathbf{p})\nabla\mathbf{w}\big)d\mathbf{x} + \int_{\Omega\setminus\bar{D}} k^2\mathbf{p}\cdot\mathbf{w}d\mathbf{x}.$$

Moreover, since $\mathbf{w}$ satisfies (4.3.13) and (4.2.4), we have

$$\int_{\Omega\setminus\bar{D}} |\mathbf{w}|^2 d\mathbf{x} = -\int_{\partial D} \sigma(\mathbf{w})\nu\cdot\mathbf{p}d\mathbf{s} = \int_{\partial D} \sigma(\mathbf{u}_0)\nu\cdot\mathbf{p}d\mathbf{s}. \qquad (4.3.17)$$

This identity tells us that we can estimate $\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})}$ by estimating $\sigma(\mathbf{u}_0)\nu|_{\partial D}$ and $\mathbf{p}|_{\partial D}$.

Since ∂D is of class C^2, from [McLean Theorem 4.18] we have the following regularity results for $\mathbf{p}$:

$$\begin{aligned}\|\mathbf{p}\|_{H^2(\Omega\setminus\bar{D})^n} &\le C\|\mathbf{p}\|_{H^1(\Omega\setminus\bar{D})^n} + C\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})^n}\\ &\le C\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})^n}.\end{aligned}$$

Therefore, by trace theorem, we obtain

$$\|\mathbf{p}\|_{H^{\frac{3}{2}}(\partial D)^n} \le C\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})^n}.$$

and the conclusion (4.3.15) follows.

□

Lemma 4.3.4. *Let $\mathbf{u}_0$ be any solution to (4.1.3), then for $\frac{1}{2} \le t \le \frac{3}{2}$*

$$\|\sigma(\mathbf{u}_0)\nu\|_{H^{-t}(\partial D)^n} \le C\|\mathbf{u}_0\|_{H^{-t+\frac{3}{2}}(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and t.

Proof. Since $\mathbf{u}_0$ satisfies (4.1.3), for any $\varphi \in H^1(D)^n$ we have

$$\int_{\partial D} \sigma(\mathbf{u}_0)\nu\cdot\varphi d\mathbf{s} - \int_D tr\big(\sigma(\mathbf{u}_0)\nabla\varphi\big)d\mathbf{x} + \int_D k^2\mathbf{u}_0\cdot\varphi d\mathbf{x} = 0$$

Thus, for $t > \frac{1}{2}$ we can regard $\sigma(\mathbf{u}_0)\nu$ as an element of $H^{-t}(\partial D)^n$ by

$$< \sigma(\mathbf{u}_0), \boldsymbol{\phi} >= \int_D tr\big(\sigma(\mathbf{u}_0)\nabla\boldsymbol{\varphi}\big)d\mathbf{x} - \int_D k^2\mathbf{u}_0 \cdot \boldsymbol{\varphi}$$

for any $\boldsymbol{\phi} \in H^t(\partial D)^n$ and $\boldsymbol{\varphi} \in H^{t+1/2}(D)^n$ with $\boldsymbol{\varphi}|_{\partial D} = \boldsymbol{\phi}$. Therefore, for any $\boldsymbol{\phi} \in H^t(\partial D)^n$

$$| < \sigma(\mathbf{u}_0), \boldsymbol{\phi} > | \leq \|\sigma(\mathbf{u}_0)\|_{H^{\frac{1}{2}-t}(D)^n} \|\nabla\boldsymbol{\varphi}\|_{H^{t-\frac{1}{2}}(D)^n} + \|\mathbf{u}_0\|_{L^2}\|\boldsymbol{\varphi}\|_{L^2(D)^n}$$

for any $\boldsymbol{\varphi}|_{\partial D} = \boldsymbol{\phi}$. Then for $\frac{1}{2} \leq t \leq \frac{3}{2}$

$$| < \sigma(\mathbf{u}_0, \boldsymbol{\phi}) > | \leq C\|\mathbf{u}_0\|_{H^{3/2-t}(D)^n}\|\boldsymbol{\varphi}\|_{H^{t+1/2}(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and t. Hence,

$$\|\mathbf{u}_0\|_{H^{-t}(\partial D)^n} \leq C\|\mathbf{u}_0\|_{H^{3/2-t}(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and t. □

Combining the above two theorem, we immediately obtain the following estimate.

Corollary 4.3.5. *Assume that D is of class C^2. Let $\mathbf{u}_0$ be a solution to (4.1.3) and $\mathbf{w}$ be the reflected solution corresponding (4.1.1) with $\mathbf{u}_0$. Then we have for* $\frac{1}{2} \leq t \leq \frac{3}{2}$

$$\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})} \leq C\|\mathbf{u}_0\|_{H^{3/2-t}(D)^n}$$

for some constant C independent of $\mathbf{u}_0$, $\mathbf{w}$ and t.

Remark 4.3.1. From the proof of Theorem (4.3.3), we know the regularity result of $\mathbf{p}$ help us obtain the sharper estimate for $\|\mathbf{w}\|_{L^2(\Omega\setminus\bar{D})^n}$, where $\mathbf{p}$ is the solution to (4.3.16). In [29], the authors consider the similar problem for the case of the Helmholtz equation. They adopt another method to prove the similar result as (4.3.5) when D has only Lipschitz boundary. Therefore it should be possible to extend the case that D has Lipschitz boundary to Lamé systems by imitating their method.

4.4 Reconstruction in 2D by using CGO solutions with complex polynomial phases

In this section, we reconstruct the unknown D in two dimension by using CGO solutions to (4.1.3) with complex polynomial phases.

4.4.1 CGO solutions with complex polynomial phases in 2D and the testing data

We use the CGO solutions constructed in the previous chapter. Precisely, given $N \in \mathbb{N}$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$, we consider the following complex polynomial, which is defined in (3.3.3),

$$\rho(\mathbf{x}) = \rho_{N,\beta}(\mathbf{x}) = \beta(x_1 + ix_2)^N,$$

where $\mathbf{x} = (x_1, x_2)^T$.

Then the CGO solution $\mathbf{v}_h$ with a parameter h is defined as

$$\mathbf{v}_h = \nabla(T_{k_1}(e^{\frac{\rho}{h}})) + \nabla^{\perp}(T_{k_2}(e^{\frac{\rho}{h}})),$$

where T_ω is the Vekua transform associated to ω, defined in the previous chapter. Here T_ω can transform a harmonic function to a solution of Helmholtz equation $(\triangle + \omega^2)v = 0$.

On the other hand, we also use the same testing boundary data as in the previous chapter. Precisely, for $h > 0$ and $d > 0$, define the testing data $\mathbf{p}_{h,d}$

$$\mathbf{p}_{h,d} := \phi_d e^{-\frac{1}{hd}} \mathbf{v}_h,$$

where ϕ_d is defined in section 3.3.2.

Recall that although $\mathbf{p}_{d,h}$ is not a solution to (4.1.3), it is *close* to the real solution $\mathbf{u}_{0,d,h}$ which satisfies (4.1.3) with $\mathbf{u}_{0,d,h}|_{\partial\Omega} = \mathbf{p}_{d,h}|_{\partial\Omega}$. Lemma 3.3.1 proves that.

4.4.2 Reconstruction of the unknown D

Throughout this section, we additionally assume that the Lamé constants λ and μ satisfy the strongly convexity:

$$\lambda + \mu > 0, \quad \mu > 0,$$

and that the unknown region D has C^2 boundary. The strongly convexity condition implies we have the following two key inequalities from Corollary 4.2.2:

$$|E(\mathbf{f})| \leq C\|\mathbf{w}\|^2_{H^1(\Omega\setminus\bar{D})^n} + C\|\mathbf{u}_0\|^2_{H^1(D)^n} \tag{4.4.18}$$

for some positive constant C depending only on λ and μ, and

$$|E(\mathbf{f})| \geq 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla\cdot\mathbf{u}_0)I_n\|^2_{L^2(D)^n} - k^2\|\mathbf{u}_0\|^2_{L^2(D)^n} - k^2\|\mathbf{w}\|^2_{L^2(\Omega\setminus\bar{D})^n}, \tag{4.4.19}$$

where I_n denotes the $n\times n$ identity matrix. On the other hand, the boundary regularity of D implies that Corollary 4.3.5 holds.

From now on we fix a compact interval $J \subset (0,\infty)$, $\varepsilon > 0$, $N \in \mathbb{N}$ and $\beta \in \mathbb{C}$. These parameters are set in Section 4.1 and 4.2 to construct the CGO solutions to (4.1.3) and corresponding testing data. Let s_* be the value of the level curve of τ which touches unknown D, that is,

$$s_* = \begin{cases} \sup\limits_{\mathbf{x}\in D\cap\Gamma} \tau(\mathbf{x}), & \text{if} \quad D\cap\Gamma \neq \emptyset \\ \qquad 0, & \text{if} \quad D\cap\Gamma = \emptyset. \end{cases}$$

Then we can reconstruct the unknown D by using the following main theorem. Remark that the procedure of reconstructing D is the same as in [19].

Theorem 4.4.1. *For $d \in J$ and $h > 0$ small enough, the following conclusions hold:*

(A) If $\bar{D}\cap\Gamma_d = \emptyset$, then

$$|E(\mathbf{f}_{d,h})| \leq Ch^{-4}e^{\frac{2}{h}(s_d-\frac{1}{d})},$$

for $0 < h \ll 1$ and some positive constant C independent of h, where $s_d = max(\frac{1}{d+\varepsilon}, s_*) < \frac{1}{d}$

(B) If $\bar{D} \cap \Gamma_d \neq \emptyset$ and D has C^2 boundary, then

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-2} e^{\frac{2}{h}(s_* - \frac{1}{d})}$$

for $0 < h \ll 1$ and some positive constant C independent of h.

In order to simplify the proof of Theorem 4.4.1, we give the following lemma.

Lemma 4.4.2. *For $d \in J$,*

$$\|\mathbf{p}_{d,h}\|_{L^2(D)^2} \leq Ch^{-1} \Big(\int_{D \cap \Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau - \frac{1}{d})} d\mathbf{x} \Big)^{\frac{1}{2}}$$

for some positive constant C independent of h.

Proof. The result of this lemma is obtained by direct computing. From the definition of $\mathbf{p}_{d,h}$, we have

$$\|\mathbf{p}_{d,h}\|^2_{L^2(D)^2} \leq \int_{D \cap \Gamma_{d+2\varepsilon}} |\phi_d e^{\frac{1}{h}(\rho - \frac{1}{d})} \mathbf{Q}_h|^2 d\mathbf{x},$$

where $\mathbf{Q}_h = (Q_{h,1}, Q_{h,2})^T$. By the estimate (3.3.11) for $Q_{h,i}$, $i = 1, 2$, we obtain

$$\begin{aligned}\|\mathbf{p}_{d,h}\|^2_{L^2(D)^2} &\leq C \int_{D \cap \Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau - \frac{1}{d})} (\frac{1}{h^2} + \frac{1}{\tau^2(\mathbf{x})}) d\mathbf{x} \\ &\leq Ch^{-2} \int_{D \cap \Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau - \frac{1}{d})} d\mathbf{x}\end{aligned}$$

for some positive constant C independent of h. □

proof of Theorem 4.4.1. (A) Suppose that $\bar{D} \cap \Gamma_d = \emptyset$. By (4.4.18) and Lemma 4.3.2, for any solution $\mathbf{u}_0$ to (4.1.3) with Dirichlet boundary condition $\mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$, we have

$$|E(\mathbf{f})| \leq C \left\{ \|\mathbf{u}_0\|^2_{H^1(D)^n} + \|\sigma(\mathbf{u}_0)\nu\|^2_{H^{-\frac{1}{2}}(\partial D)^n} \right\}$$

for some positive constant C independent of $\mathbf{u}_0$ and $\mathbf{f}$. Then by trace theorem, we have

$$|E(\mathbf{f})| \leq C \|\mathbf{u}_0\|_{H^1(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and $\mathbf{f}$. For given $d \in J$, let $\mathbf{f}_{d,h} = \mathbf{p}_{d,h}|_{\partial\Omega}$, then by substituting $\mathbf{f}_{h,d}$ for $\mathbf{f}$,

$$|E(\mathbf{f}_{h,d})| \leq C\|\mathbf{u}_{0,d,h}\|^2_{H^1(D)^n}$$

for some positive constant C independent of h, where $\mathbf{u}_{0,d,h}$ is the solution to (4.1.3) with Dirichlet boundary condition $\mathbf{u}_{0,d,h}|_{\partial\Omega} = \mathbf{f}_{d,h}$.

From Lemma 3.3.1, we obtain

$$\begin{aligned}|E(\mathbf{f}_{h,d})| &\leq C\big\{\|\mathbf{u}_{0,d,h} - \mathbf{p}_{d,h}\|^2_{H^1(D)^n} + \|\mathbf{p}_{d,h}\|^2_{H^1(D)^n}\big\}\\ &\leq C\big\{e^{-\frac{2}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})} + \|\mathbf{p}_{d,h}\|^2_{H^1(D)^n}\big\}\end{aligned}$$

for some positive constant C independent of h. By Lemma 4.4.2 and Lemma 3.4.3, we obtain

$$\begin{aligned}|E(\mathbf{f}_{h,d})| &\leq C\big\{e^{-\frac{1}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})} + h^{-2}\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}\\ &\qquad + Ch^{-4}\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}\big\}\\ &\leq C\big\{e^{-\frac{2}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})} + (h^{-2}+h^{-4})e^{\frac{2}{h}(s_*-\frac{1}{d})}|\Omega|\big\}\end{aligned}$$

for come positive constant C independent of h.

Since $\bar{D} \cap \Gamma_d = \emptyset$,

$$s_d = max(s_*, \frac{1}{d+\varepsilon}) < \frac{1}{d}.$$

Therefore we have

$$|E(\mathbf{f}_{d,h})| \leq Ch^{-4}e^{\frac{2}{h}(s_d-\frac{1}{d})}$$

for some positive constant C independent of h.

(B) Suppose that $\bar{D} \cap \Gamma_d \neq \emptyset$, then $s_* \geq \frac{1}{d}$. Since $\partial D \in C^2$, by (4.4.19) and Corollary 4.3.5, we have

$$|E(\mathbf{f})| \geq 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla\cdot\mathbf{u}_0)I_n\|^2_{L^2(D)^n} - k^2\|\mathbf{u}_0\|^2_{L^2(D)^n} - C\|\mathbf{u}_0\|^2_{H^{\frac{3}{2}-t}(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$, $\mathbf{f}$ and t, where $\mathbf{u}_0$ is the solution to (4.1.3) with Dirichlet boundary condition $\mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$ and $\frac{1}{2} \leq t \leq \frac{3}{2}$. Therefore

$$|E(\mathbf{f})| \geq 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{2}(\nabla \cdot \mathbf{u}_0)I_n\|^2_{L^2(D)^n} - C\|\mathbf{u}_0\|^2_{L^2(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and $\mathbf{f}$. By substituting $\mathbf{f}_{d,h}$ for $\mathbf{f}$, we have

$$|E(\mathbf{f}_{d,h})| \geq 2\mu\|\epsilon(\mathbf{u}_{0,d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{u}_{0,d,h})I_n\|^2_{L^2(D)^n} - C\|\mathbf{u}_{0,d,h}\|^2_{L^2(D)^n}$$

for some positive constant C independent of h. From Lemma 3.3.1, we have

$$\begin{aligned}|E(\mathbf{f}_{d,h})| \geq{}& 2\mu\|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{p}_{d,h})I_n\|^2_{L^2(D)^n} \\ &- 2\mu\|\epsilon(\mathbf{w}_h) - \frac{1}{2}(\nabla \cdot \mathbf{w}_h)I_n\|^2_{L^2(D)^n} - C\|\mathbf{p}_{d,h}\|^2_{L^2(D)^n} - C\|\mathbf{w}_h\|^2_{L^2(D)^n} \\ \geq{}& 2\mu\|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{p}_{d,h})I_n\|^2_{L^2(D)^n} - C\|\mathbf{p}_{d,h}\|^2_{L^2(D)^n} - C\|\mathbf{w}_h\|^2_{H^1(D)^n} \\ \geq{}& 2\mu\|\epsilon(\mathbf{p}_{d,h}) - \frac{1}{2}(\nabla \cdot \mathbf{p}_{d,h})I_n\|^2_{L^2(D)^n} - C\|\mathbf{p}_{d,h}\|^2_{L^2(D)^n} - Ce^{-\frac{2}{h}(\frac{1}{d} - \frac{1}{d+\varepsilon})}\end{aligned}$$

for some positive constant C independent of h.

Since $\bar{D} \cap \Gamma_d \neq \emptyset$, $s_* \geq \frac{1}{d}$ and $\bar{D} \cap \Gamma_{d+\varepsilon} \neq \emptyset$. By Lemma 3.4.3 and Lemma 4.4.2, for any open set U with $D \cap \Gamma_{d+\varepsilon} \cap U \neq \emptyset$, we have

$$|E(\mathbf{f}_{d,h})| \geq 2\mu(ch^{-4} - Ch^{-2}) \int_{D\cap\Gamma_{d+\varepsilon}\cap U} e^{\frac{2}{h}(\tau - \frac{1}{d})} d\mathbf{x} \tag{4.4.20}$$

$$- Ch^{-2} \int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau - \frac{1}{d})} d\mathbf{x} - Ce^{-\frac{2}{h}(\frac{1}{d} - \frac{1}{d+\varepsilon})} \tag{4.4.21}$$

for some positive constants c, C independent of h.

Let

$$K := \partial D \cap \{\tau = s_*\} = \partial D \cap \ell_{1/s_*}.$$

It is easy to see $K \neq \emptyset$. Since K is compact and contained in $\cup_{\mathbf{y}\in K} B_{r_{\mathbf{y}}}(\mathbf{y})$, there exists $N_0 \in \mathbb{N}$ such that $K \subset \cup_{1\leq j\leq N_0} B_{r_{\mathbf{y}_j}}(\mathbf{y}_j)$. Here we can assume each $r_{\mathbf{y}}$ satisfies ∂D can be parametrized by a C^2 function in $B_{r_{\mathbf{y}}}(\mathbf{y})$.

Let $D_R = D \setminus \big(\cup_{1\le j\le N_0} B_{r_{\mathbf{y}_j}}(\mathbf{y}_j)\big)$, then it is also easy to see that there exists small constant $\delta > 0$ such that

$$\tau \le s_* - \delta \quad \text{in} \quad D_R.$$

Therefore by similar computation of Theorem 3.4.1, we have

$$\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x} \le \int_{D_R} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x} + \int_{D\cap(\Gamma_{d+2\varepsilon}\setminus\Gamma_{d+\varepsilon})} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x} + \sum_{j=1}^{N_0}\int_{D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_j}}(\mathbf{y}_j)} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}$$

Let $\mathbf{y}_* \in \cup_{j=1}^{N_0}\{\mathbf{y}_j\}$ satisfy

$$\int_{D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_*}}(\mathbf{y}_*)} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x} = \max_{j=1,\dots,N_0}\bigg(\int_{D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_j}}(\mathbf{y}_j)} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}\bigg).$$

Then we have

$$\int_{D\cap\Gamma_{d+2\varepsilon}} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x} \le Ce^{\frac{2}{h}(s_*-\delta-\frac{1}{d})} + Ce^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})} + N_0\int_{D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_*}}(\mathbf{y}_*)} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}$$

Let

$$A_* := \int_{D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_*}}(\mathbf{y}_*)} e^{\frac{2}{h}(\tau-\frac{1}{d})}d\mathbf{x}.$$

Then (4.4.20) becomes

$$\begin{aligned}|E(\mathbf{f}_{d,h})| &\ge 2\mu(ch^{-4} - Ch^{-2})A_* - Ce^{-\frac{2}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})} \\ &\quad - Ch^{-2}\Big\{e^{\frac{2}{h}(s_*-\delta-\frac{1}{d})} + e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})} + A_*\Big\} \\ &\ge 2\mu h^{-4}A_*\bigg\{c - Ch^2 - Ch^4\frac{e^{-\frac{2}{h}(\frac{1}{d}-\frac{1}{d+\varepsilon})}}{A_*} \\ &\quad - Ch^2\frac{e^{\frac{2}{h}(s_*-\delta-\frac{1}{d})}}{A_*} - Ch^2\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon}-\frac{1}{d})}}{A_*}\bigg\}\end{aligned}$$

Since

$$A_* \le |D\cap\Gamma_{d+\varepsilon}\cap B_{r_{\mathbf{y}_*}}(\mathbf{y}_*)|e^{\frac{2}{h}(s_*-\frac{1}{d})},$$

we have

$$\frac{e^{\frac{2}{h}(s_* - \delta - \frac{1}{d})}}{A_*} \leq Ce^{-\frac{2\delta}{h}}$$

and

$$\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - \frac{1}{d})}}{A_*} \leq Ce^{\frac{2}{h}(\frac{1}{d+\varepsilon} - s_*)}.$$

Therefore we obtain

$$\frac{e^{\frac{2}{h}(s_* - \delta - \frac{1}{d})}}{A_*} \to 0, \quad \text{as} \quad h \to 0^+.$$

and

$$\frac{e^{\frac{2}{h}(\frac{1}{d+\varepsilon} - \frac{1}{d})}}{A_*} \to 0, \quad \text{as} \quad h \to 0^+$$

since $s_* \geq \frac{1}{d} > \frac{1}{d+\varepsilon}$. Hence for h small enough, we have

$$E(\mathbf{f}_{d,h}) \geq Ch^{-4} A_*.$$

From (3.4.26), we have

$$A_* \geq Ce^{\frac{2}{h}(s_* - \frac{1}{d})} B_{2,*,h}$$

for some positive constant C independent of h, where $B_{2,*,h}$ satisfies the following estimate, see the detail in the proof of the case (B′) of Theorem 3.4.1, for $\frac{1}{3} < \alpha \leq 1$

$$B_{2,*,h} \geq Ch^{1+\frac{1}{\alpha}}, \quad \text{for} \quad 0 < h \ll 1$$

for some positive constant C independent of h. Since $\partial D \in C^2$, we can choose $\alpha = 1$ and obtain

$$E(\mathbf{f}_{d,h}) \geq Ch^{-4} e^{\frac{2}{h}(s_* - \frac{1}{d})} h^2 = Ch^{-2} e^{\frac{2}{h}(s_* - \frac{1}{d})}$$

for $0 < h \ll 1$ and some positive constant C independent of h.

□

4.5 Reconstruction in 3D by using CGO solutions with linear phases

In this section, we reconstruct the shape and the location of the unknown D in three dimension by using CGO solution to (4.1.3) with linear phases.

4.5.1 CGO solutions to (4.1.3) with linear phases

We want to construct the CGO solutions to (4.1.3) in $\mathbb{R}^3$. Since the Lamé coefficients λ and μ are constants. we can reduce (4.1.3) to two Helmholtz equations by Helmholtz decomposition. Suppose $\mathbf{v} \in C^\infty(\mathbb{R}^3)^3$ is a solution to (4.1.3), then by Helmholtz decomposition there exist a smooth scalar function φ and smooth vector field $\boldsymbol{\psi}$ such that

$$\mathbf{v} = \nabla\varphi + \nabla \times \boldsymbol{\psi}.$$

Therefore φ and $\boldsymbol{\psi}$ satisfy

$$\nabla\left[(\lambda + 2\mu) \triangle \varphi + k^2\varphi\right] + \nabla \times \left[\mu \triangle \boldsymbol{\psi} + k^2\boldsymbol{\psi}\right] = 0.$$

Conversely, if we can construct CGO solutions φ and $\boldsymbol{\psi}$ to

$$\begin{cases} \triangle \varphi + k_1^2\varphi = 0 \\ \triangle \boldsymbol{\psi} + k_2^2\boldsymbol{\psi} = 0, \end{cases} \tag{4.5.22}$$

where $k_1 = \frac{k}{\sqrt{\lambda+2\mu}}$ and $k_2 = \frac{k}{\sqrt{\mu}}$, we can construct CGO solutions to (4.1.3) by defining solutions as $\mathbf{v} = \nabla\varphi + \nabla \times \boldsymbol{\psi}$.

Similar to section 4, we fix a compact interval J. Now choose $\boldsymbol{\omega}, \boldsymbol{\omega}^\perp \in S^2$ with $\boldsymbol{\omega} \cdot \boldsymbol{\omega}^\perp = 0$ and a constant vector $\mathbf{a} \in \mathbb{R}^\mathbf{3}$. For $d \in J$ and small parameter $h > 0$, we define

$$\varphi_{d,h} = e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega}-d)+i\sqrt{\frac{1}{h^2}+k_1^2}\mathbf{x}\cdot\boldsymbol{\omega}^\perp} = e^{\rho_{1,h}} \tag{4.5.23}$$

and

$$\boldsymbol{\psi}_{d,h} = e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega}-d)+i\sqrt{\frac{1}{h^2}+k_2^2}\mathbf{x}\cdot\boldsymbol{\omega}^\perp}\mathbf{a} = e^{\rho_{2,h}}\mathbf{a}, \tag{4.5.24}$$

where, for $j = 1, 2$,

$$\rho_{j,h}(\mathbf{x}) = \frac{1}{h}(\mathbf{x} \cdot \boldsymbol{\omega} - d) + i\sqrt{\frac{1}{h^2} + k_j^2}\mathbf{x} \cdot \boldsymbol{\omega}^\perp,$$

then $\varphi_{d,h}$ and $\boldsymbol{\psi}_{d,h}$ satisfy (4.5.22) respectively. Therefore we can define $\mathbf{v}_{d,h} = \nabla\varphi_{d,h} + \nabla \times \boldsymbol{\psi}_{d,h}$ and have

$$\mathbf{v}_{d,h} = e^{\rho_{1,h}}\left(\frac{1}{h}\boldsymbol{\omega} + i\sqrt{\frac{1}{h^2} + k_1^2}\boldsymbol{\omega}^{\perp}\right) + e^{\rho_{2,h}}\left(\frac{1}{h}\boldsymbol{\omega} + i\sqrt{\frac{1}{h^2} + k_2^2}\boldsymbol{\omega}^{\perp}\right) \times \mathbf{a} \tag{4.5.25}$$

and

$$\nabla\mathbf{v}_{d,h} = e^{\rho_{1,h}}\left(\frac{\boldsymbol{\omega}}{h} + i\sqrt{\frac{1}{h^2} + k_1^2}\boldsymbol{\omega}^{\perp}\right) \otimes \left(\frac{\boldsymbol{\omega}}{h} + i\sqrt{\frac{1}{h^2} + k_1^2}\boldsymbol{\omega}^{\perp}\right) \tag{4.5.26}$$

$$+ e^{\rho_{2,h}}\left(\left(\frac{\boldsymbol{\omega}}{h} + i\sqrt{\frac{1}{h^2} + k_2^2}\boldsymbol{\omega}^{\perp}\right) \times \mathbf{a}\right) \otimes \left(\frac{\boldsymbol{\omega}}{h} + i\sqrt{\frac{1}{h^2} + k_2^2}\boldsymbol{\omega}^{\perp}\right), \tag{4.5.27}$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the matrix with jk-th entry $a_j b_k$ for any two vector $\mathbf{a} = (a_1, ..., a_n)^T$ and $\mathbf{b} = (b_1, ..., b_n)^T$. Then $\mathbf{v}_{d,h}$ satisfies (4.1.3). Moreover we have the following estimates for $|\mathbf{v}_{d,h}|$ and $|\nabla\mathbf{v}_{d,h}|$. Here for any matrix A with entries a_{ij}, we denote its absolute value by $|A|$ and $|A|^2 = \sum_{i,j} |a_{ij}|^2$.

Lemma 4.5.1. *There exists a vector* $\mathbf{a} \in \mathbb{R}^3$ *such that there exists* $h_* > 0$*, for all small* $h < h_*$*,*

$$|\mathbf{v}_{d,h}| \leq \frac{C}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\omega - d)}, \tag{4.5.28}$$

$$|\nabla\mathbf{v}_{d,h}| \leq \frac{C}{h^2} e^{\frac{1}{h}(\mathbf{x}\cdot\omega - d)} \tag{4.5.29}$$

and

$$|\mathbf{v}_{d,h}| \geq \frac{c}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\omega - d)}, \tag{4.5.30}$$

$$|\nabla\mathbf{v}_{d,h}| \geq \frac{c}{h^2} e^{\frac{1}{h}(\mathbf{x}\cdot\omega - d)} \tag{4.5.31}$$

for some positive constants c, C *independent of* h*.*

Proof. From (4.5.25), we have

$$\begin{aligned} \mathbf{v}_{d,h} &= \frac{1}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} \Big\{ e^{i\sqrt{\frac{1}{h^2} + k_1^2}(\mathbf{x}\cdot\boldsymbol{\omega}^{\perp})} \big(\boldsymbol{\omega} + i\sqrt{1 + h^2k_1^2}\boldsymbol{\omega}^{\perp}\big) \\ &\qquad + e^{i\sqrt{\frac{1}{h^2} + k_2^2}(\mathbf{x}\cdot\boldsymbol{\omega}^{\perp})} \big(\big(\boldsymbol{\omega} + i\sqrt{1 + h^2k_2^2}\boldsymbol{\omega}^{\perp}\big) \times \mathbf{a}\big) \Big\} \\ &= \frac{1}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} A_1, \end{aligned}$$

where

$$A_1 = \Big\{ e^{i\sqrt{\frac{1}{h^2}+k_1^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\big(\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big) \\ + e^{i\sqrt{\frac{1}{h^2}+k_2^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\big((\boldsymbol{\omega} + i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp) \times \mathbf{a}\big)\Big\}.$$

It is easy to see that $|A_1|$ is bounded, therefore we obtain the upper bound of $|\mathbf{v}|$, (4.5.28). Now consider $\mathbf{a} = \boldsymbol{\omega} + i\boldsymbol{\omega}^\perp$. Then

$$(\boldsymbol{\omega} + i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp) \times \mathbf{a} = i(1 - \sqrt{1+h^2k_2^2})(\boldsymbol{\omega} \times \boldsymbol{\omega}^\perp) \tag{4.5.32}$$

Since

$$|1 - \sqrt{1+h^2k_2^2}| \to 0, \quad \text{as} \quad h \to 0,$$

there exists $0 < h_0 < 1$ such that

$$|(\boldsymbol{\omega} + i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp) \times \mathbf{a}| < 1, \quad \forall \quad 0 < h < h_0.$$

Therefore for $0 < h < h_0$

$$\begin{aligned} |A_1| \geq& |\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp| - |(\boldsymbol{\omega} + i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp) \times \mathbf{a}| \\ \geq& \ |\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp| - 1. \end{aligned}$$

Since

$$|\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp|^2 = 2 + h^2k_1^2 \geq 2,$$

we obtain for all $0 < h < h_0 < 1$

$$|\mathbf{v}_{d,h}| \geq \frac{c}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)}$$

for some positive constant c independent of h. In particular, $c = \sqrt{2} - 1 > 0$ in the case of $\mathbf{a} = \boldsymbol{\omega} + i\boldsymbol{\omega}^\perp$.

To see the case of $|\nabla \mathbf{v}_{d,h}|$, recall (4.5.26) to obtain

$$\begin{aligned} \nabla \mathbf{v}_{d,h} =& \frac{1}{h^2} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} \Big\{ e^{i\sqrt{\frac{1}{h^2}+k_1^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\big(\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big) \otimes \big(\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big) \\ &+ e^{i\sqrt{\frac{1}{h^2}+k_2^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\Big[\big(\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big) \times \mathbf{a}\Big] \otimes \big(\boldsymbol{\omega} + i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\Big\} \\ =& \frac{1}{h^2} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} A_2, \end{aligned}$$

where

$$
\begin{aligned}
A_2 = {}& e^{i\sqrt{\frac{1}{h^2}+k_1^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big) \\
&+ e^{i\sqrt{\frac{1}{h^2}+k_2^2}(\mathbf{x}\cdot\boldsymbol{\omega}^\perp)}\Big[\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\times\mathbf{a}\Big]\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big).
\end{aligned}
$$

It is also easy to see that $|A_2|$ is bounded and (4.5.29) follows. Now again consider $\mathbf{a}=\boldsymbol{\omega}+i\boldsymbol{\omega}^\perp$, then by (4.5.32) we have

$$
\begin{aligned}
|A_2| \geq {}& \Big|\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\Big| \\
&- \Big|\Big(1-\sqrt{1+h^2k_2^2}\Big)(\boldsymbol{\omega}\times\boldsymbol{\omega}^\perp)\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\Big|.
\end{aligned}
$$

Notice that, by compute directly, for all $h>0$, we have

$$
\begin{aligned}
&\Big|\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\Big| \\
&= \Big|\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_1^2}\boldsymbol{\omega}^\perp\big)\Big|^2 = 2+h^2k_2^2 \geq 2. \qquad (4.5.33)
\end{aligned}
$$

Since

$$
|1-\sqrt{1+h^2k_2^2}| \to 0, \quad \text{as} \quad h\to 0
$$

and

$$
|(\boldsymbol{\omega}\times\boldsymbol{\omega}^\perp)\otimes\big(\boldsymbol{\omega}+i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp\big)| < \infty, \quad \forall \quad 0<h<1,
$$

there exists $0<h_1<1$ such that for $0<h<h_1$

$$
\Big|\Big(1-\sqrt{1+h^2k_2^2}\Big)(\boldsymbol{\omega}\times\boldsymbol{\omega}^\perp)\otimes\Big(\boldsymbol{\omega}+i\sqrt{1+h^2k_2^2}\boldsymbol{\omega}^\perp\Big)\Big| \leq 1. \qquad (4.5.34)
$$

Combine (4.5.33) and (4.5.34), we obtain

$$
|A_2| \geq 2-1 = 1, \quad \forall \quad 0<h<h_1.
$$

Therefore

$$
|\nabla \mathbf{v}_{d,h}| \geq \frac{c}{h^2}e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega}-d)}, \quad \forall \quad 0<h<h_1
$$

for some positive constant c independent of h. In particular, $c=1$ in the case of $\mathbf{a}=\boldsymbol{\omega}+i\boldsymbol{\omega}^\perp$.

Then we complete the proof by choosing $h_* = \min(h_0, h_1)$. □

4.5.2 The testing boundary data and Reconstruction of the unknown D

As the previous section in two dimension, we also additionally assume the Lamé constant satisfy the strongly convexity condition throughout this section in three-dimension, that is,

$$\lambda + \frac{2}{3}\mu > 0, \quad \mu > 0,$$

and the unknown region D has C^2 boundary. Then (4.4.18), (4.4.19) and Corollary 4.3.5 follows. Now in order to apply the enclosure-type method, we need to choose appropriate testing boundary data. For a fixed compact interval $J \subset (0, \infty)$, let the testing boundary data $\mathbf{f}_{d,h}$ on $\partial\Omega$, for $d \in J$ and small $h > 0$, be as follows:

$$\mathbf{f}_{d,h} = \mathbf{v}_{d,h}|_{\partial\Omega}.$$

For $d \in J$, define the testing region

$$\Gamma_d = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \boldsymbol{\omega} \geq d\right\},$$

and denote by $s_* = \sup_{\mathbf{x} \in D} \tau_{\boldsymbol{\omega}}(\mathbf{x})$ the value of the level curve of $\tau_{\boldsymbol{\omega}}(\mathbf{x}) = \mathbf{x} \cdot \boldsymbol{\omega}$, which just touches the unknown region D.

Then we reconstruct the unknown D by using the following theorem. The procedure of reconstruction of the unknown region D is the same as in Chapter 3.

Theorem 4.5.2. *For $d \in J$ and $h > 0$ small enough, the following conclusion hold:*

(A) If $\bar{D} \cap \Gamma_d = \emptyset$, then

$$|E(\mathbf{f}_{d,h})| \leq C h^{-4} e^{\frac{2}{h}(s_* - d)},$$

for $0 < h \ll 1$ and some positive constant C independent of h. Note that in this case, $s_* < d$. Therefore

$$|E(\mathbf{f}_{d,h})| \to 0, \quad \text{as} \quad h \to 0.$$

(B) If $\bar{D} \cap \Gamma_d \neq \emptyset$ and D has C^2 boundary, then

$$|E(\mathbf{f}_{d,h})| \geq Ch^{-1} e^{\frac{2}{h}(s_* - d)},$$

for $0 < h \ll 1$ and some positive constant C independent of h. Note that in this case, $s_* \geq d$. Therefore

$$|E(\mathbf{f}_d)| \to \infty, \quad \textit{as} \quad h \to 0.$$

Proof. (A) Again by using Corollary 4.2.2 and Lemma 4.3.2, we obtain

$$|E(\mathbf{f})| \leq C \|\mathbf{u}_0\|_{H^1(D)^n}$$

for some positive constant C independent of $\mathbf{u}_0$ and $\mathbf{f}$, where $\mathbf{u}_0$ is the solution of (4.1.3) with the Dirichlet boundary condition $\mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$. Therefore we have

$$|E(\mathbf{f}_{d,h})| \leq C \|\mathbf{v}_{d,h}\|^2_{H^1(D)^n}$$

for some positive constant C independent of h. From (4.5.28) and (4.5.29), we have

$$\begin{aligned}
\|\mathbf{v}_{d,h}\|^2_{H^1(D)^n} &= \int_D |\mathbf{v}_{d,h}|^2 d\mathbf{x} + \int_D |\nabla \mathbf{v}_{d,h}|^2 d\mathbf{x} \\
&\leq \int_D \left(\frac{C}{h} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} \right)^2 d\mathbf{x} + \int_D \left(\frac{C}{h^2} e^{\frac{1}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} \right)^2 d\mathbf{x} \\
&\leq C|D|\left(\frac{1}{h^2} + \frac{1}{h^4}\right) e^{\frac{2}{h}(s_* - d)}.
\end{aligned}$$

Therefore for $0 < h \ll 1$,

$$|E(\mathbf{f}_{d,h})| \leq Ch^{-4} e^{\frac{2}{h}(s_* - d)}$$

for some positive constant C independent of h. Note that in the case of $\bar{D} \cap \Gamma_d = \emptyset$, $s_* < d$.

(B) Since D has C^2 boundary, Corollary 4.3.5 can be applied. Therefore by using Corollary 4.2.2 again and modifying the proof of (B) of Theorem

4.4.1 to 3-dimension case, we can obtain

$$
\begin{aligned}
|E(\mathbf{f})| \geq (\lambda + \frac{2}{3}\mu)\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3} + 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{3}(\nabla \cdot \mathbf{u}_0)I_n\|^2_{L^2(D)^3} \\
- C\|\mathbf{u}_0\|^2_{L^2(D)^3}
\end{aligned}
$$

for some positive constant C independent of $\mathbf{u}_0$ and $\mathbf{f}$, where $\mathbf{u}_0$ satisfies (4.1.3) with Dirichlet boundary condition $\mathbf{u}_0|_{\partial\Omega} = \mathbf{f}$. Due to the strongly convexity condition $\lambda + \frac{2}{3}\mu > 0$, there exists a small constant δ with $0 < \delta < 1$ such that

$$\lambda + \frac{2}{3}\mu\delta \geq 0.$$

Therefore the above inequality becomes

$$
\begin{aligned}
|E(\mathbf{f})| \geq (\lambda + \frac{2}{3}\mu\delta)\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3} + 2\mu\|\epsilon(\mathbf{u}_0) - \frac{1}{3}(\nabla \cdot \mathbf{u}_0)I_n\|^2_{L^2(D)^3} \\
+ \frac{2}{3}\mu(1-\delta)\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3} - C\|\mathbf{u}_0\|^2_{L^2(D)^3}.
\end{aligned}
$$

We notice that for any vector field $\mathbf{v} \in H^1$ in three dimension,

$$|\epsilon(\mathbf{v}) - \frac{1}{3}(\nabla \cdot \mathbf{v})I_3|^2 = |\epsilon(\mathbf{v})|^2 - \frac{1}{3}|\nabla \cdot \mathbf{v}|^2.$$

Hence, by $\lambda + \frac{2}{3}\mu\delta \geq 0$, we obtain

$$
\begin{aligned}
|E(\mathbf{f})| &\geq (\lambda + \frac{2}{3}\mu\delta)\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3} + 2\mu\|\epsilon(\mathbf{u}_0)\|^2_{L^2(D)^3} \\
&\quad - \frac{2}{3}\mu\delta\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3} - C\|\mathbf{u}_0\|^2_{L^2(D)^3} \\
&\geq 2\mu\delta\big(\|\epsilon(\mathbf{u}_0)\|^2_{L^2(D)^3} - \frac{1}{3}\|\nabla \cdot \mathbf{u}_0\|^2_{L^2(D)^3}\big) \\
&\quad + 2\mu(1-\delta)\|\epsilon(\mathbf{u}_0)\|^2_{L^2(D)^n} - C\|\mathbf{u}_0\|^2_{L^2(D)^3} \\
&= 2\mu\delta\|\epsilon(\mathbf{u}_0) - \frac{1}{3}(\nabla \cdot \mathbf{u}_0)I_3\|^2_{L^2(D)^3} \\
&\quad + 2\mu(1-\delta)\|\epsilon(\mathbf{u}_0)\|^2_{L^2(D)^3} - C\|\mathbf{u}_0\|^2_{L^2(D)^3} \\
&\geq 2\mu(1-\delta)\|\epsilon(\mathbf{u}_0)\|^2_{L^2(D)^3} - C\|\mathbf{u}_0\|^2_{L^2(D)^3}.
\end{aligned}
$$

By the following Korn's second inequality (see for instance Theorem 10.2 of [21]),

$$\|\epsilon(\mathbf{v})\|^2_{L^2(D)^3} \geq c\|\nabla\mathbf{v}\|^2_{L^2(D)^3} - C\|\mathbf{v}\|^2_{L^2(D)^3}$$

for some positive constants c, C independent of $\mathbf{v} \in H^1(D)^3$, we obtain the lower bound for $|E(\mathbf{f})|$

$$|E(\mathbf{f})| \geq c\|\nabla \mathbf{u}_0\|^2_{L^2(D)^3} - C\|\mathbf{u}_0\|^2_{L^2(D)^3}$$

for some positive constants c, C independent of $\mathbf{f}$ and $\mathbf{u}_0$.

Now for $d \in J$ and $h > 0$, let $\mathbf{f} = \mathbf{f}_{d,h}$, we have

$$|E(\mathbf{f}_{d,h})| \geq c\|\nabla \mathbf{v}_{d,h}\|^2_{L^2(D)^3} - C\|\mathbf{v}_{d,h}\|^2_{L^2(D)^3}.$$

for some positive constants c, C independent of h. From the (4.5.28) and (4.5.31) of Lemma 4.5.1, we can compute directly that

$$|E(\mathbf{f}_{d,h})| \geq \left(\frac{c}{h^4} - \frac{C}{h^2}\right) \int_D e^{\frac{2}{h}(\mathbf{x}\cdot\boldsymbol{\omega} - d)} d\mathbf{x}. \tag{4.5.35}$$

Now we need to compute the right hand side of the above inequality. Let $\mathbf{x}_0$ be a point lying on $\partial D \cap \ell_{s_*}$, where ℓ_{s_*} is a plane defined as

$$\ell_{s_*} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \boldsymbol{\omega} = s_*\} = \partial \Gamma_{s_*}.$$

Let

$$\mathbf{y} = R(\mathbf{x} - \mathbf{x}_0),$$

where R is the rotation transformation such that $R\boldsymbol{\omega} = (0, 1)^T$. Since ∂D is C^2, $\partial(R(D - \mathbf{x}_0))$ is also C^2. Therefore near $\mathbf{y} = 0$, it is easy to see there exists another C^2 function f_* such that $f_*(\mathbf{0}) = 0$ and there exist non-negative constants $\delta_1, \delta_1', \delta_2, \delta_2'$ and δ_3 satisfying for $\mathbf{y}' \in (-\delta_1, \delta_1') \times (-\delta_2, \delta_2')$,

$$(-\delta_1, \delta_1') \times (-\delta_2, \delta_2') \times (-\delta_3, f_*(\mathbf{y}')) \subset R(D - \mathbf{x}_0),$$

where $\mathbf{y} = (\mathbf{y}', y_3)^T$.

Sometimes f_* can be $\partial(R(D - \mathbf{x}_0))$ if $\partial(R(D - \mathbf{x}_0))$ can be parametrized by $(\mathbf{y}', f_*(\mathbf{y}'))$ and f_* is a C^2 function.

Moreover, since f_* is C^2 with $f_*(\mathbf{0}) = 0$, we have

$$|f_*(\mathbf{y}')| \leq M|\mathbf{y}'| \leq M(|y_1| + |y_2|)$$

for some positive constant M dependent only on ∂D, where $\mathbf{y}' = (y_1, y_2)^T$.

On the other hand, under this change of coordinates, we can see

$$\mathbf{x} \cdot \boldsymbol{\omega} = y_3 + s_*.$$

Now we can compute that

$$\begin{aligned}
\int_D e^{\frac{2}{h}(\mathbf{x}\cdot\boldsymbol{\omega}-d)} d\mathbf{x} &\geq \int_{-\delta_1}^{\delta_1'} \int_{-\delta_2}^{\delta_2'} \int_{-\delta_3}^{f_*(\mathbf{y}')} e^{\frac{2}{h}(y_3+s_*-d)} d\mathbf{y} \\
&\geq \frac{h}{2} e^{\frac{2}{h}(s_*-d)} \int_0^{\delta_1'} \int_0^{\delta_2'} \big(e^{\frac{2}{h}f_*(\mathbf{y}')} - e^{-\frac{2}{h}\delta_3}\big) d\mathbf{y}' \\
&\geq \frac{h}{2} e^{\frac{2}{h}(s_*-d)} \bigg\{ \int_0^{\delta_1'} \int_0^{\delta_2'} e^{-\frac{2}{h}M(|y_1|+|y_2|)} d\mathbf{y}' - Ce^{-\frac{2}{h}\delta_3} \bigg\} \\
&\geq \frac{h}{2} e^{\frac{2}{h}(s_*-d)} \bigg\{ \big(\frac{h}{2M}\big)^2 \big(1 - e^{-\frac{2M}{h}\delta_1'}\big)\big(1 - e^{-\frac{2M}{h}\delta_1'}\big) \\
&\qquad\qquad - Ce^{-\frac{2}{h}\delta_3} \bigg\}.
\end{aligned}$$

Therefore for $0 < h \ll 1$, we have

$$\int_D e^{\frac{2}{h}(\mathbf{x}\cdot\boldsymbol{\omega}-d)} d\mathbf{x} \geq Ch^3 e^{\frac{2}{h}(s_*-d)}$$

for some positive constant C independent of h. From (4.5.35), we obtain for any $0 < h \ll 1$,

$$\begin{aligned}
|E(\mathbf{f})| &\geq \big(\frac{c}{h} - Ch\big) e^{\frac{2}{h}(s_*-d)} \\
&\geq ch^{-1} e^{\frac{2}{h}(s_*-d)}
\end{aligned}$$

for some positive constants c, C independent of h.

□

Chapter 5

Future work

5.1 Maxwell's equations with anisotropic coefficients

The enclosure-type method have been applied to the isotropic time-harmonic Maxwell's equations [40, 18]. The regularity assumption on ∂D in [40] is C^2 and recently the assumption on ∂D is reduced to Lipschitz in [18]. However, the reconstruction problem for the "*anisotropic*" time-harmonic Maxwell's equation is still open. Our next work is to solve this reconstruction problem by applying the enclosure-type method.

In the above papers, the CGO solutions for the isotropic time-harmonic Maxwell's equations are needed. They are chosen as the test data in the enclosure-type method. But it is difficult to obtain the CGO solutions for the anisotropic time-harmonic Maxwell's equations. We try to construct another kind of special solutions, oscillating-decaying solutions which are proposed in [25], as a substitute.

To construct the oscillating-decaying solutions of anisotropic time-harmonic Maxwell's equations, our plan is to reduce the anisotropic Maxwell's equations to strongly elliptic systems. Recently we have already completed part of the plan.

5.2 Reconstruction of coefficients of anisotropic time-harmonic Maxwell's equations by using internal measurements

As is mentioned in the first chapter, the problem of reconstructing coefficients from knowledge of the Dirichlet-to-Neumann map led to the development of the Electrical Impedance Tomography (EIT). EIT is a medical imaging technique which have been applied to lung imaging and breast imaging. However, as a diagnostic tool in medical imaging, although EIT has the advantage of high contrast on imaging, the spatial resolution is low. Some methods combining various physical effects are then developed to improve the quality of acquired images. These ideas generally give rise to the so-called Hybrid Inverse Problems.

Mathematically, these hybrid inverse problems or hybrid imaging are usually separated into two inverse problems [2]. The first step is to obtain some internal data from the boundary. These internal data have high resolution but are not clear enough to distinguish whether there is something different, such as cancer, in normal tissue. So in the second step, we try to use the internal data obtaining from the first step to reconstruct the high-contrastive coefficients. The photo-acoustic and thermal-acoustic tomography are examples of hybrid imaging. See [30, 3, 4]. My future plan is to extend the second step of [3, 4] to the case of anisotropic time-harmonic Maxwell's equations. We believe the oscillating-decaying solutions of anisotropic Maxwell's equations should be useful for our problem.

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