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Del Pezzo 曲面之幾何

On Geometry of Del Pezzo Surfaces

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本論文係林金毅君（學號 D96221006）在國立臺灣大學數學學系、所完成之博士學位論文，於民國一〇三年〇一月廿七日承下列考試委員審查通過及口試及格，特此證明

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誌謝

「一年好景君須記，最是橙黃橘綠時」是蘇東坡的名句。對我而言，這果眞是須記的冬季。這篇博士論文正是經歷多年努力才得成熟的果實。然而要這顆果實的栽培並非我一人一力所能完成。謹在此感謝各方奧援。

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林金毅謹誌

于癸辰年除夕

摘要

本文介紹 del Pezzo 曲面之研究。早期的研究主要以光滑曲面爲對向，但近年則多考慮帶有奇點的曲面。因此第二章即討論各種奇點，始自第三章起正式定義 del Pezzo 曲面，介紹光滑曲面的分類。第四章介紹 Shokurov 發展的 complement 理論，並在第五章的 weighted complete intersection 中給出例子。第六章介紹凱勒—愛因斯坦距離和 del Pezzo 曲面的關係。第七章與第八章是作者的研究結果利用黎曼—羅赫定理計算尤拉示性數並得到一種特別的不消沒定理。

關鍵字：del Pezzo 曲面、奇點、complement、凱勒—愛因斯坦距離、不消沒定理

Abstract

The thesis in on the geometry of del Pezzo surfaces. Early researches focused on smooth surfaces, while recently surfaces with singularities have been mostly considered. Consequently, in Chapter 2, different types of singularities are first discussed, and then del Pezzo surfaces can be defined formally in Chapter 3. Research on smooth surfaces are also given there. In Chapter 4, we introduce the complement theory developed by Shokurov, and we give some examples of weighted complete intersection in Chapter 5. Chapter 6 is about the relation between Kähler-Einstein metrics and del Pezzo surfaces. In Chapter 7 and Chapter 8, we introduce our research result. We use Riemann-Roch theorem to calculated Euler characteristics, and then give a special type of nonvanishing theorem.

Keywords: del Pezzo surfaces, singularities, complement, Kähler-Einstein metrics, nonvanishing.

目錄 Contents

1 Introduction

Del Pezzo surfaces are projective surfaces with ample anticanonical line bundles. They have been a popular research topic for a long time since they are Fano and thus rationally connected, and also come out naturally from MMP. Smooth del Pezzo surfaces have been classified completely. Studies of singular del Pezzo surfaces remain an active research area and draw a lot of attention recently. There are several possible definition of a singular del Pezzo surface. By a singular del Pezzo surface, we mean a normal surface with at worst klt singularities such that the anticanonical divisor is nef and big. We remark that for surfaces, klt singularities are known to be equivalent to quotient singularities[14, 5.21]. In this paper, we shall focus on normal del Pezzo surfaces with only cyclic quotient singularities.

Given a singular del Pezzo surface X with $-K_X$ nef and big, it is natural to study the non-vanishing of anti-plurigenera $h^0(X, -mK_X)$. Shokurov [19] proved that there exists a uniform bound m_0 for any singular del Pezzo surfaces X such that $h^0(X, -mK_X) > 0$ for some $1 \leq m < m_0$. Nevertheless, there is no effective estimate of m_0. Thus, the nonvanishing of $h^0(X, -mK_X)$ for small value of m are still of great interest. It is easy to see that $h^0(X, -K_X) > 0$ for smooth del Pezzo surfaces, thanks to the Riemann-Roch formula. For singular surfaces, there is also a singular Riemann-Roch formula by Reid [18, III (8.6)]. By using the singular Riemann-Roch formula, Prokhorov and Verevkin [17, Cor. 1.3] showed that $h^0(X, -K_X) > 0$ if X has the Picard number $\rho = 1$, and contains exactly 5 singularities. Later, it was shown in [1] that it is impossible to have 5 singularities. On the other hand, there exists an example of weighted hypersurface $X_{256} \subseteq \mathbb{P}(13, 35, 81, 128)$ such that $h^0(X, -mK_X) = 0$, for $m \leq 12$, since $\mathcal{O}_X(-K_X) = \mathcal{O}(1)$ (cf. [4, Table 2.]). Therefore, one cannot expect the general effective bound to be very small.

We would like to draw the reader's attention to the recent study of singular Fano 3-folds. For $\mathbb{Q}$-Fano 3-folds with nef and big anticanonical divisors and with at worst terminal singularities, the work [5] of Meng Chen and Jungkai Chen shows $h^0(X, -6K_X) > 0$.

In this paper, we consider surfaces with singularities of type $\frac{1}{r}(1,1)$. There are two reasons for this. First of all, surfaces with singularities of type $\frac{1}{r}(1,1)$ have very nice combinatorial properties in the singular Riemann-Roch formula. We are able to derive an interesting type of non-vanishing.

Theorem 1.1 (Main Theorem)**.** *Suppose X is a del Pezzo surface with only singularities of the form $\frac{1}{r}(1,1)$. Then $h^0(X, -mK_X) > 0$ for $m = 1$ or 3.*

Moreover, given a surface with cyclic quotient singularities, we develop a partial resolution via particular choices of weighted blowups which we call L-blowups. L-blowups transform cyclic quotient singularities to singularities of the form $\frac{1}{r}(1,1)$. In principle, Euler characteristics $\chi(X, -mK_X)$ for small m are preserved under L-blowups. However, situation varies depending on types of singularities. In any event, we have

Proposition 1.2. *Let $Y \to X$ be an L-blowup. Then $\chi(X, -K_X) = \chi(Y, -K_Y)$.*

1.1 Notation and Conventions

In this paper, we always work over the complex number field $\mathbb{C}$.

Notation:

$\zeta = \zeta_r$: a primitive r-th root of unity in $\mathbb{C}$

μ_r: the group of all r-th roots of unity in $\mathbb{C}$

$e_1, e_2, \ldots, e_n$: the standard basis of $\mathbb{R}^n$

All schemes and varieties are assumed to be at least quasi-projective. For $\mathbb{Q}$-Cartier divisors, the intersection numbers are defined by extending the intersection number of Cartier divisors by $\mathbb{Q}$-linearity.

2 Singularities

2.1 Basic properties

We study singularities formally isomorphic to cyclic quotients of $\mathbb{A}^n$. First, we recall some facts about quotient varieties.

Given an affine variety $X = \mathrm{Spec}A$ with a finite group G action, we construct the affine quotient $Y = X^G = \mathrm{Spec}A^G$ and the natural quotient map $f : X \to Y$ induced by the inclusion $A^G \hookrightarrow A$. When X is merely quasi-projective, one covers X with G-invariant affine open sets, and finds the affine quotients can be glued together. It is easy to see that

Fact 2.1. *i) X is normal $\Longrightarrow Y$ is normal,*

ii) X is $\mathbb{Q}$-factorial $\Longrightarrow Y$ is $\mathbb{Q}$-factorial.

Conversely, given a normal variety Y, we may pick the normalization X of Y in a Galois extension over the function field of Y. Then $Y = X^G$, where G is the Galois group.

Definition 2.2. (Quotient singularities)

(i) Given $a_1, \ldots, a_n \in \mathbb{N}$, let μ_r acts on $\mathbb{A}^n$ by

$$\zeta \cdot (x_1, x_2, \ldots, x_n) \mapsto (\zeta^{a_1} x_1, \zeta^{a_2} x_2, \ldots, \zeta^{a_n} x_n)$$

. Denote the quotient X_0 by $\mathbb{A}^n/\mu_r$ or $\mathbb{A}^n/\frac{1}{r}(a_1, \ldots, a_n)$ to be the standard quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$. Usually we assume $\gcd(a_1, \ldots, a_n) = 1$.

(ii) Let $p \in X$ be a point. If locally near p, there is a map to a standard quotient singularity: $\varphi : (p \in X) \to (0 \in X_0)$ inducing formal isomorphism $\hat{\mathcal{O}}_{0,X_0} \to \hat{\mathcal{O}}_{p,X}$, we call $(p \in X)$ a quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$, or a $\frac{1}{r}(a_1, \ldots, a_n)$ point.

The morphism required in the definition is actually étale, that make it possible to pullback resolutions of standard quotient singularities to general ones.

Proposition 2.3. *Keep the notation as in Definition 2.2. φ is étale near p.*

Proof. Since étaleness is an open condition, it suffices to prove this for local homomorphism $\mathcal{O}_{0,X_0} \to \mathcal{O}_{p,X}$. By faithful flatness of completion, we only need that $\hat{\mathcal{O}}_{0,X_0} \to \hat{\mathcal{O}}_{p,X}$ is étale, which is an isomorphism by assumption, and hence the proof. □

The following proposition says that quotients of smooth varieties by μ_r actions indeed give rise to quotient singularities.

Proposition 2.4. *Let μ_r act on a smooth variety X, and $p \in X$ be an isolated fixed point. Let the quotient be $(\bar{p} \in \bar{X})$. Then it is a quotient singularity.*

Proof. We may assume $X = \mathrm{Spec} A$, an invariant open set, is affine. Since μ_r acts on the maximal ideal p, we get the eigenspace decomposition:

$$p = \bigoplus_{i=0}^{r-1} I_i$$

where the action is given by $\zeta \cdot s = \zeta^i s, \forall s \in I_i$.

We may choose $s_1, \dots s_n \in A$ satisfying $\zeta \cdot s_i = \zeta^{a_i} s_i$ so that they form a regular system of parameters of A_p, and obtain a morphism: $s : A \to \mathbb{A}^n$ defined by $x_1 = s_1, \dots, x_n = s_n$. s induces an isomorphism of completed local rings.

Let μ_r act on $\mathbb{A}^n$ by $\zeta \cdot (x_1, x_2, \dots, x_n) \mapsto (\zeta^{a_1} x_1, \zeta^{a_2} x_2, \dots, \zeta^{a_n} x_n)$. Then s is equivariant, inducing the natural map $\bar{s} : \bar{X} \to X_0$ between quotient varieties. We need to check that $\bar{s}$ also induces isomorphisms of completed local rings. This is done by Theorem 2.5 below. □

Theorem 2.5. *Suppose $A_1 \to A_2$ is a G-equivariant local morphism such that the maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ are G-invariant. Set $B_i = A_i^G$, for $i = 1, 2$. Then B_i is local. If the induced map $\hat{A}_1 \to \hat{A}_2$ is an isomorphism, then so is $\hat{B}_1 \to \hat{B}_2$.*

We prove Theorem 2.5 with a series of lemmata.

Lemma 2.6. *Suppose $B \subseteq A$ be a finite extension of noetherian rings. Let I be an ideal of B and $J_n = I^n A \cap B$. Then the I-adic and J_n-adic topologies are the same.*

Proof. Firstly, $I^n \subseteq J_n$ is trivial.

Conversely, $T = A \oplus (\bigoplus_{n=1}^{\infty} (IA)^n)$ is finite over $B \oplus (\bigoplus_{n=1}^{\infty} I^n)$, and hence so is the subalgebra $S = B \oplus (\bigoplus_{n=1}^{\infty} J_n)$ of T.

This implies $IJ_i = J_{i+1}$ for $i \geq N$ for some N. We have then $J_n = I^{n-N} J_N \subseteq I^{n-N}$. □

Lemma 2.6 holds in particular for $B = A^G$, where G is a finite group acting on A. For $p \in \mathrm{Spec} B$, $B_p = (A_p)^G$. Now we assume $(B, \mathfrak{n})$ local. A is then semilocal. Denote by $\mathfrak{m}$ its Jacobson radical.

Corollary 2.7. *i) In A, $\mathfrak{n}A$-adic and $\mathfrak{m}$-adic topologies are the same.*

ii) In B, $\mathfrak{n}$-adic and $\mathfrak{m}^n \cap B$-adic topologies are the same.

Proof. i) Note that $A/\mathfrak{n}A$ is finite over $B/\mathfrak{n}$, and hence an artin ring. $\sqrt{\mathfrak{n}A} = \mathfrak{m}$.

ii) By Lemma 2.6, $\mathfrak{n}$-adic, $(\mathfrak{n}A)^n \cap B$-adic topologies, and $\mathfrak{m}^n \cap B$-adic topology are the same. □

Lemma 2.8. *Let $\hat{B}$ be the $\mathfrak{n}$-adic completion of B, $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Then $\hat{A} = A \otimes_B \hat{B}$, and $\hat{B} \subseteq \hat{A}$ as topological subspace under $\hat{\mathfrak{n}}$-adic, and $\hat{\mathfrak{m}}$-adic topologies respectively.*

Proof. Since $A \otimes_B \hat{B}$ is the $\mathfrak{n}A$-adic completion of A, which is the same as the $\mathfrak{m}$-adic completion of A (c.f. Corollary 2.7). Since $\hat{B}$ is flat over B, we obtain the inclusion. □

Now assume that in Theorem 2.5, $|G| \in A^\times$ is a unit. This holds in particular, when A contains a field k, and char $k \not| |G|$.

Lemma 2.9. $\hat{B} = \hat{A}^G$.

Proof. Consider $\iota : B \hookrightarrow A$, and $\epsilon : A \to B$, which is defined by $\epsilon(x) = \frac{1}{|G|} \sum_{g \in G} g \cdot x$. Then $\epsilon \circ \iota =$id.

Tensoring with $\hat{B}$ over B, we obtain that the composition $\hat{B} \hookrightarrow \hat{A} \to \hat{B}$ is the identity map, and $x = \epsilon(x) \in \hat{B}$, for any $x \in \hat{A}^G$. □

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. The theorem follows from taking G-invariants of the induced map $\hat{A}_1 \to \hat{A}_2$. □

Remark 2.10. Singularities of normal surfaces are always isolated. We shall call a surface quotient singularity a $\frac{1}{r}(a, b)$ point, if the μ_r action is given by $\zeta \cdot (x, y) \mapsto (\zeta^a x, \zeta^b y)$.

Quotient singularities only occur on the locus where the group action is not free. First we state an application of Hurwitz formula.

Definition 2.11. Suppose that $X =$ Spec$A$, $Y =$ Spec$B$ are normal varieties and $f : X \to Y$ is a finite map.

For any height-1 prime p of A, and $q = p \cap B$ of B, the map f induces a local morphism of discrete valuation rings $B_q \to A_p$. Denote the local parameters of A_p and B_q by s and t respectively. We define the ramification index $e = e(p, q)$ to be the integer such that $t = us^e$ for some $u \in A_p^\times/$

Theorem 2.12. *Suppose $X =$ Spec$A$, $Y =$ Spec$B$ are normal varieties and $f : X \to Y$ is a finite map. Then $K_X = f^* K_Y + \sum_{ht(p)=1}(e(p, p \cap B) - 1)p$, and the sum is a finite sum.*

Proof. Denote the residue fields of A_p, B_q by $k(p)$, $k(q)$ respectively. Pick a transcendental basis $y_1, \dots, y_{n-1} \in B$ of $k(q)$ over $\mathbb{C}$, which is also a transcendental basis of $k(p)$ over $\mathbb{C}$. Consider the rational n-forms $\omega_1 = ds \wedge dy_1 \wedge dy_2 \dots \wedge dy_{n-1}$ and $\omega_2 = dt \wedge dy_1 \wedge dy_2 \dots \wedge dy_{n-1}$.

First we note that ω_1 is the local generator of ω_X at p. Indeed, by the second exact sequence of differential [10, II. 8.4], we have the following exact sequence

$$pA_p/(pA_p)^2 \to \Omega_{A_p/\mathbb{C}} \to \Omega_{\kappa(p)/\mathbb{C}} \to 0,$$

and thus $\Omega_{X,p}$ is generated by $ds, dy_1, \dots, dy_{n-1}$. Similarly, ω_2 is the local generator of ω_Y.

Fix a rational n-form ω on Y, we may also regard it as an n-form on X by pulling back. We find the coefficient c of q in K_Y satisfies $\omega = u_1 t^c \omega_1$, where $u_1 \in A^\times$. The coefficient of p in $f^* K_Y$ is ec, and the coefficient c' of p in K_X satisfies $\omega = u_2 s^{c'} \omega_2$, where $u_2 \in A^\times$.

Now we have

$$\begin{aligned} \omega_2 = &\; dt \wedge dy_1 \wedge dy_2 \dots \wedge dy_{n-1} \\ = &\; d(us^e) \wedge dy_1 \wedge dy_2 \dots \wedge dy_{n-1} \\ = &\; s^{e-1}(ue + sb)\omega_1 \\ = &\; s^{e-1} u' \omega_1. \end{aligned}$$

for some $b \in B, u' \in B_q^\times$.

We obtain $c' = ec + (e-1)$, which is the desired equality. □

Remark 2.13. In terms of divisors, write $K_X + D = f^*(K_Y + E)$, $D = \sum d_i D_i$, $E = \sum b_j E_j$. Suppose $f(D_i) = E_j$ for some i, j, then $d_i = b_j e - (e-1)$, i.e., $(1 - d_i) = e(1 - b_j)$.

Theorem 2.14. *Suppose G is a finite group acting on the normal variety X, and the action is free generically. Let $f : X \to Y$ be the quotient variety. Then*

$K_X = \pi^* K_Y + \sum(|G_D| - 1)D$, where G_D is the subgroup fixing D.

Proof. We may assume that $X = \mathrm{Spec} A$, $Y = \mathrm{Spec} B$ are affine, with fraction fields K, L respectively. Then $L = K^G$.

We only need to check that for any prime divisor D, corresponding to height-1 prime p of A, we have $e(p, q) = |G_D|$, where $q = p \cap B$. Consider the Dedekind domains $B_q \subseteq A_q = A \otimes_B B_q$.

Denote the subgroups $I_p = \{g \mid g(a) - a \in pA_q, \forall a \in A_q\}$ and $D_p = \{g \mid g(a) \in pA_q, \forall a \in pA_q\}$ of G. By Hilbert's ramification theory (cf. [15, I Prop.9.4]) and since in characteristic 0, the residue field extension $k(p)/k(q)$ is always separable, we know that $\begin{cases} r & := |\text{orbit of } p| = |G|/|D_p| \\ f & := [k(p) : k(q)] = \mathrm{Gal}(k(p)/k(q)) = |D_p|/|I_p| \\ e & = |I_p| \end{cases}$ But I_p is exactly the subgroup G_D fixing D, and hence the proof. □

Definition 2.15. Let G be a finite group acting on a variety X. Say the action is free if $\Phi : G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is a closed immersion.

The definition may seem strange at the first glance. However, the following lemma tells us that, it is the same as the intuitive definition. The advantage is that Definition 2.15 can be generalized straightforwardly to algebraic group actions on any schemes.

Lemma 2.16. *Let G be a finite group acting on a variety X. The action is free if and only if for any closed point $m \in X$, there is no non-identity element $g \in G$, such that $g \cdot m = m$.*

Proof. We may assume that $X = \operatorname{Spec} A$ is affine, and G acts on A by automorphisms $g_1, \ldots, g_k$.

First we observe that $g \cdot m = m$ if and only if m is g-invariant, and g acts on A/m as identity, which means $\{g(a) - a \mid a \in A\} \subseteq m$.

Also, Φ induces the ring homomorphism $\phi : A \otimes_{\mathbb{C}} A \to A^{|G|}$, defined by $a \otimes b \to (g_1(a)b, \ldots, g_k(a)b)$.

Suppose that ϕ is surjective. Then there are a_i, b_i such that $\sum_i a_i b_i = 1$, and $\sum_i g(a_i) b_i = 0$ for all non-identity elements g. We have , and hence $\{g(a) - a \mid a \in A\}$ generates A.

Conversely, we are given that $\{g(a) - a \mid a \in A\}$ generates A for all non-identity elements g. For $g \neq h$, we also have $\{g(a) - h(a) \mid a \in A\}$ generates A. That is, there are a_i, b_i such that $\sum_i g(a_i) b_i - \sum_i h(a_i) b_i = 1$. So the components corresponding to g and h of $\sum_i a_i \otimes b_i - \sum_i 1 \otimes h(a_i) b_i$ are 1 and 0 respectively. From this, we see that ϕ is a surjective ring homomorphism. □

Proposition 2.17. *Suppose G is a finite group acting on an integral scheme $X = \mathrm{Spec}A$ freely. Then A is locally free over $B = A^G$ of rank $|G|$. Moreover, A is étale over B.*

Proof. For $p \in \mathrm{Spec}\ B$, let k be the algebraic closure of the residue field $k(p)$. We observe that the G also acts freely on the geometric fiber $X_p = \mathrm{Spec}A'$, where $A' = A \otimes_B k$. Indeed, the surjective homomorphism $\phi : A \otimes_{\mathbb{C}} A \to A^{|G|}$ remains surjective after tensoring with k. Moreover, since we have the splitting $\epsilon : A \to B$ $\epsilon(a) = \frac{1}{|G|}\sum_{g\in G} g(a)$ in characteristics zero, tensoring with k gives we $A'^G = k$.

Now, by the structure theorem of artin rings, $A' = A_1 \times A_2 \times \ldots \times A_m$, and there is a canonically defined subring $A_0 = k \times \ldots \times k$ ($|G|$ copies) such that the composition $A_0 \hookrightarrow A' \to A'/\sqrt{0} \cong A_0$ is the identity map. G also acts on A_0 by permuting components. Since we must have $A_0^G = k$, G acts transitively on components. From this we see all A_i are isomorphic, and G acts on A' by permuting components as well. But then $A'^G = \{(x, x, \ldots, x)|x \in A_1\} = k$. We must have $A_1 = k$, and $A' = A_0$

Counting dimensions over k gives $\dim_k A' = |G|$. We find $\dim_{k(p)} A \otimes k(p) = |G|$ for all $p \in \mathrm{Spec}B$, and hence A is locally free over B of rank $|G|$. A being flat, we only need to check A over B is unramified, which can be checked on geometric fibers. But $A' = k \times \ldots \times k$, it is then obvious that $\Omega_{A'/k} = 0$. □

Remark 2.18. When no divisor of X is fixed by a non-identity element in G, we see the quotient map $\pi : X \to Y$ is étale in codimension 1, and the Hurwitz formula reads $K_X = \pi^* K_Y$.

2.2 Log singularities

Consider a variety X together with a $\mathbb{Q}$-divisor $D = \sum_i d_i D_i$, where $d_i \in \mathbb{Q}$ and D_i are prime divisors of X. If $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$), then we call D a boundary (resp. subboundary). If $K_X + D$ is $\mathbb{Q}$-Cartier, then we call (X, D) a log pair. We usually assume that X is normal, and D is a boundary unless stated otherwise. By Hironaka's desingularization theorem, there is a proper birational morphism $f : Y \to X$ such that Y is smooth, and the proper transform of $\sum D_i$ and the exceptional divisors are simple normal crossing (SNC, for short). Denote by $\tilde{D}_i$ the proper transform of D_i, and set $\tilde{D} = \sum d_i \tilde{D}_i$. We may write

$$K_Y + \tilde{D} = f^*(K_X + D) + \sum_j a(X, D; E_j) E_j$$

for some $a(X, D; E_j) \in \mathbb{Q}$. Here E_j are exceptional divisors of f.

For a non-exceptional divisor E, define

$$a(X, D; E) = \begin{cases} -d_i, & \text{if } E = \tilde{D}_i, \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that $a(X, D; E)$ is independent of log resolutions, and is called the discrepancy of E with respect to (X, D). We write X only instead of $(X, 0)$ if $D = 0$. Discrepancies make a nice measure of singularities of pairs as follows.

Definition 2.19. Given a log pair (X, D), we introduce the following:

i) (X, D) has only log canonical singularities or is log canonical, denoted lc, if $a(X, D; E) \geq -1$ for all E, and for all log resolutions.

ii) (X, D) has only Kawamata log terminal singularities or is klt if $a(X, D; E) > -1$ for all E, and for all log resolutions. In particular, $0 \leq d_i < 1$.

iii) (X, D) has only pure log terminal singularities or is plt if $a(X, D; E) > -1$ for all exceptional E, and for all log resolutions.

iv) The log canonical threshold (lct) of (X, D) is defined by

$$\operatorname{lct}(X, D) = \sup\{\lambda | (X, \lambda D) \text{ has only log canonical singularities.}\}$$

Remark 2.20. i) The definition for log canonical and klt singularities are actually independent of resolutions, which can be seen by using a common resolution.

ii) Being log canonical, klt, or plt is a local property. For an open cover $\{U_\alpha\}$ of X, (X, D) is log canonical (resp. klt, plt) if and only if $(U_\alpha, D|_{U_\alpha})$ is log canonical (resp. klt, plt) for all α. Similarly, $\operatorname{lct}(X, D) = \inf_\alpha \operatorname{lct}(U_\alpha, D|_{U_\alpha})$. It also makes sense to talk about lct at a point p, namely,

$$\operatorname{lct}_p = \sup\{\lambda | (U, \lambda D|_U) \text{ is lc, for some neighborhood } U \text{of } p\}.$$

We will need some facts of singularities of pairs. (cf.[13]).

Lemma 2.21. *Let (X, D) and (Y, E) be log pairs. Suppose $f : Y \to X$ is a proper birational morphism, and $K_Y + E = f^*(K_X + D)$ then (X, D) is log canonical (resp. klt) if and only if (Y, E) is log canonical (resp. klt)*

Proof. Take a log resolution of (Y, E), which is also a log resolution of (X, D). Then this follows directly by definition. $\square$

Lemma 2.22. *Let (X, D) and (Y, E) be log pairs. Suppose $f : Y \to X$ is finite étale, and $E = f^*D$ then (X, D) is log canonical (resp. klt) if and only if (Y, E) is log canonical (resp. klt). Moreover, $lct(X, D) = lct(Y, E)$.*

Proof. First we take a resolution $\pi : (Z, \tilde{D}) \to (X, D)$, and write

$$K_Z + \tilde{D} = \pi^*(K_X + D) + \sum_j a_j E_j$$

Do the base change of $\pi : (Z, \tilde{D}) \to (X, D)$ by f, and we obtain $\pi' : (W, \tilde{E}) \to (Y, E)$ a resolution. Pulling back the above equation by $f' : W \to Z$ gives

$$K_W + \tilde{E} = \pi'^*(K_Y + E) + \sum_j a_j f'^* E_j$$

Now that f'^*E_j cannot be multiple for f' is étale. So every discrepancy remains invariant under pullback. $\square$

In general, being log canonical is preserved by finite morphisms.

Lemma 2.23. *Let (X, D) and (Y, E) be log pairs. Suppose $f : Y \to X$ is a finite morphism such that $K_Y + E \equiv f^*(K_X + D)$. Then (X, D) is log canonical (resp. klt) if and only if (Y, E) is log canonical (resp. klt).*

Proof. First we prove the "if" part. Choose a log resolution $g : Z \to X$, and define W to be the normalization of a component of $Y \times_X Z$ dominating X .

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & X \end{tikzcd}$$

Write $K_Z + D' = g^*(K_X + D)$, where $D' = g_*^{-1}D - \sum a(X, D; D_j)D_j$. Pulling back via f' gives $f'^*(K_Z + D') = f'^*g^*(K_X + D) = g'^*(K_Y + E)$

Now write $K_W + E' = f'^*(K_Z + D')$. By Hurwitz formula, for a divisor E_i in E', with $f'(E_i) = D_j$ for some j, the coefficients satisfy $(1 + a(X, D, D_j)) = \frac{1}{e}(1 + a(Y, E, E_i)) \geq 0$ (resp. > 0), if (Y, E) is log canonical (resp. klt).

From this, for the "only if" part, we reduced to the case f is a Galois cover, i.e., X is a quotient variety of Y for some finite group action of G. We take now, besides W, Z, a G-equivariant log resolution $g' : W_1 \to W$, and let the quotient be $Z_1 = W^G$. Such resolution exists by the functorial construction under smooth morphisms. Replace W, Z by W_1, Z_1. Note that Z_1 is $\mathbb{Q}$-factorial, and we can calculate as before. For a divisor E_i in E' with $f'(E_i) = D_j$, Hurwitz formula gives $(1 + a(Y, E; E_i)) = e(1 + a(X, D; D_j)) \geq 0$ (resp. > 0), if (X, D) is log canonical (resp. klt). $\square$

Kawamata-Viehweg vanishing theorem is a useful theorem in birational geometry.

Theorem 2.24 (Kawamata-Viehweg Vanishing Theorem)**.** *Suppose X is smooth, H is an ample $\mathbb{Q}$-divisor. Then $H^i(X, K_X + \lceil H \rceil) = 0$ for $i > 0$.*

We also have the following relative version. [11, Remark 1-2-6]

Theorem 2.25 (Relative Kawamata-Viehweg Vanishing Theorem)**.** *Suppose (X, D) is klt, H is a $\mathbb{Q}$-divisor, and $K_X + D + H$ is an integral divisor.. If $f : X \to Y$ is a projective morphism such that H is f-nef and f-big then $Rf^i_* \mathcal{O}_X(K_X + D + H) = 0$ for $i > 0$.*

Proof. Case 1. X is smooth, H is f-ample, and $D = \lceil H \rceil - H$ is SNC.

We may assume that X, Y projective. Let L be an ample Cartier divisor on Y. Replace H by $H + f^*L$, we may assume that H ample by projection formula.

Consider the spectral sequence:

$$E_2^{i,j} = H^i(Y, Rf^j_*(\mathcal{O}_X(K_X + \lceil H \rceil) \otimes \mathcal{O}_Y(Y, mL))) \Rightarrow H^{i+j}(X, \mathcal{O}_X(K_X + \lceil H \rceil + mf^*L))$$

. By Serre vanishing, for m large, the spectral sequence degenerates as

$H^0(Y, Rf^j_*(\mathcal{O}_X(K_X + \lceil H \rceil) \otimes \mathcal{O}_Y(Y, mL))) = H^j(X, \mathcal{O}_X(K_X + \lceil H \rceil + mf^*L)) = 0$ for $j > 0$ by Theorem 2.24. So, $Rf^j_*(\mathcal{O}_X(K_X + \lceil H \rceil) = 0$.

Case2. General case.

By Lemma 2.26 below, which is a corollary of Kodaira lemma, we take a resolution $g : Z \to X$ of (X, D) such that $g^*H - \sum_j \delta_j F_j$ is $(f \circ g)$-ample for some $0 < \delta_j \ll 1$, and $\{F_j\}$, proper transform of D and exceptional divisors, are SNC.

We may apply Case 1 on g and $h = f \circ g$, to the divisor $H_1 = g^*H - \sum_j \delta_j F_j$. Then for $i > 0$,

$$R^i g_* \mathcal{O}_Z(K_Z + \lceil H_1 \rceil) = R^i h_* \mathcal{O}_Z(K_Z + \lceil H_1 \rceil) = 0.$$

Hence by spectral sequence,

$$0 = R^i h_* \mathcal{O}_Z(K_Z + \lceil H_1 \rceil) = R^i f_* g_* \mathcal{O}_Z(K_Z + \lceil H_1 \rceil).$$

On the other hand,

$$g_* \mathcal{O}_Z(K_Z + \lceil H_1 \rceil) = g_* \mathcal{O}_Z(\lceil K_Z + H_1 \rceil) = \mathcal{O}_X(K_X + D + H),$$

since $\lceil K_Z - g^*(K_X + D) \rceil$ is effective exceptional by the condition of being klt. $\square$

Lemma 2.26. *Suppose $f : X \to Y$ is a proper surjective morphism of normal varieties, and H is an f-nef and f-big divisor. Then there is a resolution $g : Z \to X$ such that $g^*H - \sum_j \delta_j F_j$ is $f \circ g$-ample for small $0 < \delta_j < 1$ and $\{F_j\}$, proper transform of D and exceptional divisors, are simple normal crossing.*

Remark 2.27. Being f-nef and f-big is a numerical property. We easily derive the form that for an integral divisor $D' \equiv K_X + D + H$, we have $R^i f_*(\mathcal{O}(D')) = 0$, for $i > 0$.

Here we introduce a special type of singularities called rational singularities.

Definition 2.28. X is said to have only rational singularities if for a resolution $f : Y \to X$, $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.

Remark 2.29. It is known that the definition of rational singularities is independent of resolutions.

Theorem 2.30. *A surface with only quotient singularities is klt and thus has only rational singularities.*

By Lemma 2.23, we see quotient singularities are klt. To prove klt singularities are rational, however, requires some work. We reproduce the proof in [14, Chap. 5] here.

First we recall a coherent sheaf $\mathcal{F}$ is CM (Cohen-Macaulay) if all its stalks $\mathcal{F}_p$ are CM modules. A scheme X is CM if the structure sheaf $\mathcal{O}_X$ is CM. Projective CM varieties can be characterized as follows:

Lemma 2.31. *[14, 5.72] For a projective variety X and an ample Cartier divisor D, X is CM if and only if $H^i(X, \mathcal{O}_X(-rD)) = 0$ for $i < n$ and large r.*

We have the following alternative characterization of rational singularities.

Proposition 2.32. *[13, 11.9] X has only rational singularities if and only if X is CM and for a resolution $f : Y \to X$, we have $f_*\omega_Y = \omega_X$.*

Proof of Theorem 2.30. Suppose (X, Δ) is klt, and we prove X has only rational singularities. Let $f : Y \to X$ be a resolution. Then it suffices to prove that $f_*\omega_Y = \omega_X$. Write $K_Y = f^*(K_X + \Delta) + E^+ - E^-$, where $E^+, E^- \geq 0$ are exceptional divisors without common components. Now $\lceil E^+ \rceil = K_Y - f^*(K_X + \Delta) + E^- + \{-E^+\}$, and $(Y, E^- + \{-E^+\})$ is klt for Y is smooth and $E^- + \{-E^+\}$ is SNC with coefficients in $[0, 1)$. By Kawamata-Viehweg vanishing theorem, $R^i f_*\mathcal{O}_Y(\lceil E^+ \rceil) = 0$ for $i > 0$.

For any ample Cartier divisor D, by Larey spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{O}_X(-D) \otimes R^q f_*\mathcal{O}_Y(\lceil E^+ \rceil)) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y(\lceil E^+ \rceil - f^*D),$$

we get $H^i(X, \mathcal{O}_X(-D)) \cong H^i(Y, \mathcal{O}_Y(\lceil E^+ \rceil - f^*D)$. Since this morphism factors through $H^i(Y, \mathcal{O}_Y(-f^*D)$, we get the injection:

$$H^i(X, \mathcal{O}_X(-D)) \hookrightarrow H^i(Y, \mathcal{O}_Y(-f^*D)).$$

By Serre duality [14, 5.71] and Kawamata-Viehweg vanishing theorem,

$$H^i(Y, \mathcal{O}_Y(-f^*D)) = H^{n-i}(Y, \omega_Y(f^*D)) = 0$$

for $i < n$, and thus $H^i(X, \mathcal{O}_X(-D)) = 0$. This implies X is CM. On the other hand, for $i = n$ Serre duality gives

$$H^0(Y, \omega_Y(f^*D)) = H^0(X, f_*\omega_Y(D))) \twoheadrightarrow H^i(X, \omega_X(D)),$$

which implies $f_*\omega_Y \to \omega_X$ is surjective, and hence an isomorphism. □

2.3 Toric varieties and singularities

It is easier to study cyclic quotient singularities as toric varieties. Here is a brief review.

Given a lattice $N \subseteq \mathbb{R}^n$, and a rational polyhedral cone σ, we consider the dual lattice $M = \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z}, \forall y \in N\}$ and the dual cone $\sigma^\vee = \{x \in \mathbb{R}^n | (x, y) \geq 0, \forall y \in \sigma\}$. Then we define $R_\sigma = k[X^x \mid x \in \sigma^\vee \cap M] \subseteq k[X] = k[X_1, \ldots, X_n]$ and the toric variety $X_\sigma = \text{Spec } R_\sigma$.

Example 2.33. The standard quotient singularity $\mathbb{A}^n/\mu_r$ of type $\frac{1}{r}(a_1, a_2, \ldots, a_n)$ is the toric variety defined by the lattice $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \ldots + \mathbb{Z}e_n + \mathbb{Z} \cdot \frac{1}{r}(a_1, a_2, \ldots, a_n)$ and σ the first orthant. We may assume $0 < a_i < r$.

For surfaces, let N be any lattice in $\mathbb{R}^2$, and σ a cone bounded by the rays l_1 and l_2. By a change of coordinates, we may assume σ the first quadrant, and write $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}\frac{1}{r}(1, b)$, where $\frac{1}{r}(1, b)$ is the nearest point of N to the y-axis in $(0, 1) \times (0, 1)$. We always get a cyclic quotient singularity on X_σ or X_σ is smooth.

Fact 2.34. *If $\sigma' \subseteq \sigma$ is a face, then the induced map $X_{\sigma'} \to X_\sigma$ is an open immersion.*

From this, we see that given a fan Σ, i.e., a collection of cones that closed under taking faces, we may paste all X_σ along the faces to get a variety X_Σ.

Example 2.35. Consider $N \subseteq \mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ a lattice and a cone defined by two rays bounding it. We construct a fan Σ defined by several rays $l_0 = e_1, l_1, \ldots, l_k = e_2$ in order. Let the cone bounded by l_{i-1} and l_i be σ_i, and the cone bounded byl_0 and l_k is σ. We say $l_1, \ldots, l_{k-1}$ subdivide σ into $\sigma_1, \ldots, \sigma_k$.

If $\sigma_i \cap N$ can be generated by only two elements as monoid for all i, then the corresponding X_{σ_i} and hence the entire X_Σ is smooth.

2.4 Resolution

A standard technique to resolve quotient singularities is weighted blowup. It is usually useful and convenient to introduce in terms of toric geometry.

2.4.1 Weighted blowups

Suppose $N \subseteq \mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ is a lattice, and a cone σ is assumed to be the first quadrant. Then N, σ defines a toric variety X_0. Consider a ray $l = \mathbb{R}_{>0}v$ in σ, such that $l \cap N = \mathbb{Z}_{\geq 0}v$, If l subdivides σ into two cones σ_1, σ_2, and the resulting fan defines a toric variety Y_0. Call $\pi : Y_0 \to X_0$ the standard weighted blowup along v. We see that a standard weighted blowup turns a standard quotient singularity in X_0 to other possibly nicer standard quotient singularities in Y_0.

In general, for a quotient singularity, locally we have an étale morphism $X \to X_0$,then we set $Y = Y_0 \times_{X_0} X$. In what follows, we omit the fibred product construction, and treat all quotient singularities as if they were standard. A weighted blowup turns a quotient singularity in X to other possibly nicer quotient singularities in Y of the same types as those of X_0 and Y_0.

To see this, observe that the formal fibres $Y \times \mathrm{Spec}\hat{\mathcal{O}}_{X,p} \to Y_0 \times \mathrm{Spec}\hat{\mathcal{O}}_{X_0,0}$ over $\mathrm{Spec}\hat{\mathcal{O}}_{X,p} \to \mathrm{Spec}\hat{\mathcal{O}}_{X_0,0}$ are isomorphic. In particular, the fibres are isomorphic. If $q \in Y$ is a point corresponding to $q_0 \in Y_0$ on the fibres. The morphism $\mathrm{Spec}\hat{\mathcal{O}}_{Y_0,q_0} \to Y_0$ factors through $Y_0 \times \mathrm{Spec}\hat{\mathcal{O}}_{X_0,0}$, and hence $\hat{\mathcal{O}}_{Y_0,q_0} \cong \hat{\mathcal{O}}_{Y,q}$. We see q and q_0 are either both smooth points or both quotient singularities of the same type.

If we pick several rays in σ, we may do weighted blowups in any order without affecting the final result, which is defined by the resulting fan of cones.

Proposition 2.36. *Suppose $P \in X$ is a $\frac{1}{r}(1,b)$ point, and we do a weighted blowup $\pi : Y \to X$ for P along $v = \frac{1}{r}(s,t)$. Then we get two possibly singular points: Q_1, a $\frac{1}{s}(1, \frac{-t+bs}{r})$ point, and Q_2, a $\frac{1}{t}(1, \frac{-s+\bar{b}t}{r})$ point, where $\bar{b}$ denotes the minimal positive integer such that $b\bar{b} \equiv 1 (mod\ r)$. Moreover,*

$$\begin{aligned} K_Y &= \pi^* K_X + (\tfrac{s+t}{r} - 1)E; \\ E^2 &= -\tfrac{r}{st}; \\ K_Y^2 - K_X^2 &= -\tfrac{r}{st}(\tfrac{s+t}{r} - 1)^2. \end{aligned}$$

where E is the exceptional divisor.

Remark 2.37. To be strict, intersection numbers are only defined for proper varieties. However, we may assume X and hence Y are proper. The above statement only depends on a neighborhood of P.

Proof. Let $\sigma_1 = \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} v$, $\sigma_2 = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} v$ be the two cones formed after the subdivision. Then we write $N = \mathbb{Z}e_1 + \mathbb{Z}v + \mathbb{Z}\frac{1}{r}(1,b) = \mathbb{Z}e_1 + \mathbb{Z}v + \mathbb{Z}\frac{1}{r}(\bar{b},1)$ with $\frac{1}{r}(1,b) = \frac{1}{s}v + \frac{-t+bs}{sr}e_2$ and $\frac{1}{r}(\bar{b},1) = \frac{1}{t}v + \frac{-s+\bar{b}t}{rt}e_1$. We find σ_i defines a quotient singularity Q_i of the asserted type for $i = 1,2$.

Denote by D_1, D_2, the invariant divisors on X associated to e_1, e_2, and $\tilde{D}_1, \tilde{D}_2, \tilde{D}_v = E$, the invariant divisors on Y associated to e_1, e_2, v respectively. Then we have locally (cf. [9, p.61,89])

$$\begin{aligned} K_X &= -D_1 - D_2 \\ K_Y &= -\tilde{D}_1 - \tilde{D}_2 - \tilde{D}_v \\ \pi^* D_i &= \tilde{D}_i + \langle e_i, v \rangle \tilde{D}_v \end{aligned}$$

We obtain that $K_Y = \pi^* K_X + (\frac{s+t}{r} - 1)E$.

Moreover, by computing the intersection number, we have $\tilde{D}_1.E = \frac{1}{t}$. By projection formula, we have $0 = \pi^* D_1.E = \tilde{D}_1.E + \frac{s}{r}E^2$, and thus $E^2 = -\frac{r}{st}$. Finally, $K_Y^2 - K_X^2 = -\frac{r}{st}(\frac{s+t}{r} - 1)^2$ follows from another application of projection formula. $\square$

Fact 2.38. *Let X be a proper toric surface. Suppose the open set X_σ is defined by the lattice $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}\frac{1}{r}(1, b) \subseteq \mathbb{R}^2$, and cone σ is the first quadrant. Denote by D_1, D_2, the invariant divisors on X associated to e_1, e_2. Then we have $D_1.D_2 = \frac{1}{r}$. (cf. [9, p.97])*

2.4.2 Hirzebruch-Jung continued fractions

Definition 2.39. Define $\langle u_n, \ldots, u_1 \rangle$ to be the upper left corner entry of the matrix

$$\begin{bmatrix} u_n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{n-1} & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} u_1 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is convenient to define $\langle \emptyset \rangle = 1$.

Lemma 2.40. *The following properties hold:*

i) $\langle u \rangle = u$, *and* $\langle u_n, \ldots, u_1 \rangle = u_n \langle u_{n-1}, \ldots, u_1 \rangle - \langle u_{n-2}, \ldots, u_1 \rangle$.

ii) If $u_1, \ldots, u_n \geq 2$, *then* $\langle u_n, \ldots, u_1 \rangle > \langle u_{n-1}, \ldots, u_1 \rangle$.

iii) $$\begin{bmatrix} u_n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{n-1} & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} u_1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \langle u_n, \ldots, u_1 \rangle & -\langle u_n, \ldots, u_2 \rangle \\ \langle u_{n-1}, \ldots, u_1 \rangle & -\langle u_{n-1}, \ldots, u_2 \rangle \end{bmatrix},$$
and $\langle u_{n-1}, \ldots, u_1 \rangle \langle u_n, \ldots, u_2 \rangle = \langle u_n, \ldots, u_1 \rangle \langle u_{n-1}, \ldots, u_2 \rangle + 1$.

iv)

$$\frac{\langle u_n, \ldots, u_1 \rangle}{\langle u_{n-1}, \ldots, u_1 \rangle} = u_n - \cfrac{1}{\cfrac{\langle u_{n-1}, \ldots, u_1 \rangle}{\langle u_{n-2}, \ldots, u_1 \rangle}} = u_n - \cfrac{1}{u_{n-1} - \cfrac{1}{u_{n-2} - \cfrac{1}{\ddots - \cfrac{1}{u_1}}}}.$$

v) $\langle u_n,\dots,u_1\rangle = \langle u_1,\dots,u_n\rangle$.

vi) Given relatively prime $r > b \in \mathbb{N}$, *one can write*

$$\frac{r}{b} = u_n - \cfrac{1}{u_{n-1} - \cfrac{1}{u_{n-2} - \cfrac{1}{\ddots - \cfrac{1}{u_1}}}},$$

where $u_i \geq 2$ *are integers. Then* $r = \langle u_n,\dots,u_1\rangle$, *and* $b = \langle u_{n-1},\dots,u_1\rangle$.

Proof. These are some what standard, we prove (v) for reader's convenience.

v) Note that

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} u & -1 \\ 1 & 0 \end{bmatrix}^{T} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} u & -1 \\ 1 & 0 \end{bmatrix}.$$

□

Remark 2.41. The continued fraction of $\frac{r}{b}$ above is called the Hirzebruch-Jung continued fraction. We refer the readers to [**?**, Notation 2.1] and [**?**, Sec. 2] for the introduction and proofs. We will sometimes call a $\frac{1}{r}(1,b)$ point a $\langle u_n,\dots,u_1\rangle$ point. Denote $r = \langle u_n,\dots,u_1\rangle$, $b = \langle u_{n-1},\dots,u_1\rangle$, and set $a = u_n$, $c = \langle u_{n-2},\dots,u_1\rangle$, $\bar{b} = \langle u_n,\dots,u_2\rangle$, and $k = \langle u_{n-1},\dots,u_2\rangle$ then we have $r = ab - c$,$1 + rk = b\bar{b}$, with $a \geq 2$, $0 \leq \bar{b}, c < b$.

Let $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}\frac{1}{r}(1,b)$, and the cone $\sigma = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$. The Newton polygon P is defined to be the convex hull of $N \cap \sigma \backslash \{0\}$. Write

$$P_0 = e_2, P_1 = (a_1,b_1),\dots,P_k = (a_k,b_k), P_{k+1} = e_1$$

with $0 < a_1 < \ldots < a_k < 1$, and $1 > b_1 > \ldots > b_k > 0$, to be the lattice points appeared on the boundary of P. We have a formula for (a_i,b_i).

Proposition 2.42. *If we denoter* $= \langle u_n,\dots,u_1\rangle$, *and* $b = \langle u_{n-1},\dots,u_1\rangle$, *then* $k = n$, and $(a_i,b_i) = (\frac{1}{r}\langle u_n,\dots,u_{i+1}\rangle, \frac{1}{r}\langle u_{n-i},\dots,u_1\rangle),$.

Proof. We prove it by induction on n. Firstly, it is clear that $(a_1,b_1) = \frac{1}{r}(1,b)$.

We find (a_2, b_2) is defined by $1 + b_2 = a_2 b$, with $b_2 < b_1$. In other words, $q = ra_2$ and $q' = rb_2$ satisfy $r = qb - q'$, so $q = u_n$, $q' = \langle u_{n-2}, \dots, u_1 \rangle$.

Now set $\sigma' = \mathbb{R}_{\geq 0} P_1 + \mathbb{R}_{\geq 0} e_1$, and we find $N = \mathbb{Z}P_1 + \mathbb{Z}e_1 + \mathbb{Z}P_2$, with $P_2 = \frac{1}{b} e_1 + \frac{q'}{b} P_1$.

The Newton polytope P' of $N \cap \sigma'$, which is the same as $P \cap \sigma'$ will have vertices $P_{i+1} = a'_i e_1 + b'_i P_1$, where $(a'_i, b'_i) = (\frac{1}{b} \langle u_{n-1}, \dots, u_{i+1} \rangle, \frac{1}{b} \langle u_{n-i}, \dots, u_1 \rangle)$.

So $(a_{i+1}, b_{i+1}) = (a'_i + \frac{1}{r} b'_i, \frac{b}{r} b_i) = (\frac{1}{r} \langle u_n, \dots, u_{i+1} \rangle, \frac{1}{r} \langle u_{n-i}, \dots, u_1 \rangle)$. $\square$

Moreover, we see that

$$\begin{cases} u_n P_1 & = P_0 + P_2, \\ u_{n-1} P_2 & = P_1 + P_3, \\ & \dots \\ u_1 P_n & = P_{n-1} + P_{n+1.} \end{cases}$$

2.4.3 Resolution and partial resolution

To resolve a $\frac{1}{r}(1, b)$ point, we consider $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z} \cdot \frac{1}{r}(1, b)$ and σ is the first quadrant. Keep the notations in Remark 2.41

Proposition 2.43. *If we do the weighted blowup along $P_1 = \frac{1}{r}(1, b)$, then we get only one possibly singular point .*

Proof. In the proof of Proposition 2.42, we find that σ' gives a $\frac{1}{b}(1, c)$ point. $\square$

Continue in this manner, we can do weighted blowups along P_i for $i = 1, \dots, n$. Denote the resulting cone by σ_i for $i = 0, \dots, n$. Then $\sigma_i \cap N = \mathbb{Z}_{\geq 0} P_i \oplus \mathbb{Z}_{\geq 0} P_{i+1}$. We see the minimal resolution of a $\frac{1}{r}(1, b)$ point can be obtained as the composition of weighted blowups along $P_1, \dots, P_k$. If instead, we consider the weighted blowup along $Q_1 = P_1 + P_2$, then we obtain $\pi_1 : X_1 \to X$, which we call a *simple L-blowup*. The following proposition shows what we get after a simple L-blowup.

Proposition 2.44. *If we do the weighted blowup along $Q_1 = P_1 + P_2$, then we get only two possibly singular points of types $\frac{1}{a+1}(1, 1)$ and $\frac{1}{b+c}(1, c)$.*

Proof. First we note $P_2 = \frac{1}{r}(a, c)$, $Q_1 = \frac{1}{r}(a + 1, b + c)$. Now σ is divided into σ'_1, σ'_2, and we may write $N = \mathbb{Z}e_2 \oplus \mathbb{Z}P_1 \oplus Q_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}P_2 \oplus \mathbb{Z}Q_1$with $P_1 = \frac{1}{a+1} Q_1 + \frac{1}{a+1} e_2$

and $P_2 = \frac{1}{b+c}e_1 + \frac{c}{b+c}Q_1$. We obtain singular points of types $\frac{1}{a+1}(1,1)$ and $\frac{1}{b+c}(1,c)$ corresponding to the cones σ_1' and σ_2' respectively. $\square$

Continue in this manner, we do further blowups along $Q_2 = P_2 + P_3, \dots, Q_{k-1} = P_{k-1} + P_k$ successively, . Denote the resulting cone by σ_i', $i = 1, \dots, n$. Then we may write $N = \mathbb{Z}P_0 \oplus \mathbb{Z}P_1 \oplus \mathbb{Z}Q_1 = \mathbb{Z}_{\geq 0}P_{n+1} \oplus \mathbb{Z}_{\geq 0}P_n \oplus \mathbb{Z}_{\geq 0}Q_{n-1}$ and $N = \mathbb{Z}_{\geq 0}Q_{i-1} + \mathbb{Z}_{\geq 0}P_i + \mathbb{Z}_{\geq 0}Q_i$ for $0 < i < n$. We then obtain n singular points of types

$$\frac{1}{u_n+1}(1,1), \frac{1}{u_{n-1}+2}(1,1), \frac{1}{u_{n-2}+2}(1,1), \dots, \frac{1}{u_2+2}(1,1), \frac{1}{u_1+1}(1,1)$$

corresponding to the cones $\sigma_0', \dots, \sigma_n'$ respectively. The map $Y_L \to X$ is defined to be an L-blowup, and we summarize the above discussion to the following.

Proposition 2.45. *For a surface X with cyclic quotient singularities, there exists an L-blowup*

$$Y_L = X_n \to X_{n-1} \to \dots \to X_1 \to X,$$

such that each $X_i \to X_{i-1}$ is a simple L-blowup and Y_L contains only singularities of type $\frac{1}{r}(1,1)$.

3 Del Pezzo Surfaces

Del Pezzo surfaces are the main subject in this paper. Here is a brief introduction.

Definition 3.1. i) [6] A del Pezzo surface (resp. generalized del Pezzo) X is a smooth surface with $-K_X$ ample (resp. nef and big).

ii) [16]A log del Pezzo surface is a log pair (X, D) with only log canonical singularities, such that $-(K_X + D)$ is nef and big.

Example 3.2. It is clear that $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ are del Pezzo surfaces.

Starting from $X_0 = \mathbb{P}^2$, we perform blowups at a point repeatedly. Say,

$$X_r \to X_{r-1} \to \dots \to X_0$$

Since $K_{X_i}^2 = K_{X_{i-1}}^2 - 1$, and $K_{X_0}^2 = 9$, we find $K_{X_r}^2 > 0$ if and only if $r \leq 8$. For a curve $C \cong \mathbb{P}^1$ in X_{i-1}, let $\tilde{C}$ be its proper transform in X_i. Then $-K_{X_i}.\tilde{C}$ remains positive (resp. nonnegative) if and only if the point blown up does not lie on C, or $C^2 > -1$ (resp. $C^2 > -2$).

Conversely, we want to find the condition to ensure $-K_{X_r}$ is ample (resp. nef and big). Firstly, we know $r \leq 8$ and we may not blow up a point on a (-1)-curve (resp. (-2)-curve). Now for a curve $C_0 \subset \mathbb{P}^2$ of degree $d \geq 3$, denote by C_i its proper transform in X_i. We only need to decide when $-K_{X_r}.C_r > 0$ (resp. ≥ 0)

If the point we blown up at step i is a point of multiplicity e_i of C_{i-1}. Then $C_i.K_{X_i} = C_{i-1}.K_X + e_i$, and $p_a(C_i) = p_a(C_{i-1}) - \frac{1}{2}e_i(e_i - 1)$. (cf [10, V. Prop. 3.3, Cor. 3.7])

We have $C_r.K_{X_r} = e_1 + \ldots + e_r - 3d$, and $p_a(C_r) = \frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_{i=1}^{r} e_i(e_i - 1) \geq 0$. Since the function $g(x) = x(x-1)$ is concave, let $s = e_1 + \ldots + e_r$. Then

$$(d-1)(d-2) \geq \sum_{i=1}^{r} e_i(e_i - 1) \geq s(\frac{s}{r} - 1) \geq s(\frac{s}{8} - 1)$$

Suppose $s \geq 3d$, we have $(d-1)(d-2) \geq 3d(\frac{3}{8}d - 1)$, and thus $d = 3, 4$. When $d = 4$, all inequalities take equality, and in particular $e_1 = \ldots = e_8 = 3/2$, which is absurd. When $d = 3$, the only case is $r = 8$, and e_i consists of $1, 1, 1, 1, 1, 1, 1, 2$, and $C_r.K_{X_r} = 0$.

In conclusion, $-K_{X_r}$ is ample if and only if $r \leq 8$, and we do not blow up any point on a (-1)-curves each time, nor do we blow up 7 points together with a singular point on a cubic curve. We say the points satisfying this condition to be "in general positions". Likewise, $-K_{X_r}$ is nef and big if and only if $r \leq 8$, and we do not blow up any point on a (-2)-curves each time. We say the satisfying this condition to be "in almost general positions".

Smooth surfaces with nef and big anticanonical bundle are classified as follows.

Theorem 3.3. *[21, 6, Proposition 0.4]*

i) A del Pezzo surface is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown up $1, \ldots, 8$ points in general positions.

ii) A generalized del Pezzo surface is a $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface F_2, or $\mathbb{P}^2$ blown up $1, \ldots, 8$ points in almost general positions.

We recall that the n-th Hirzebruch surface F_n is the projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ over $\mathbb{P}^1$.

Proof. By the following Lemma 3.4, X is rational. It is well known that the minimal rational surfaces are $\mathbb{P}^2$ and Hirzebruch surfaces. However, it is impossible for a del

Pezzo surface (resp. generalized del Pezzo surface) to contain a (-2)-curve (resp, (-3)-curve). We must have X is a blowup of $\mathbb{P}^2$ or F_2 itself in the case of generalized del Pezzo surfaces. $\square$

Lemma 3.4. *Let X be a generalized del Pezzo surface. Then X is rational.*

Proof. By Castelnuovo's Rationality Criterion [10, V.6.2], it suffices to prove the irregularity $q = H^1(X, \mathcal{O}_X) = 0$ and the plurigenus $P_2 = H^0(X, 2K_X) = 0$. Write $\mathcal{O}_X = K_X + (-K_X)$, and Kawamata-Viehweg vanishing theorem gives $q = 0$. On the other hand, if $D \in |2K_X|$, then $0 \geq D.K_X = 2K_X^2 > 0$, which is a contradiction. $\square$

Here are some examples of log del Pezzo surfaces.

Example 3.5. Let $X = F_n$ be the n-th Hirzebruch surface. Denote the negative curve by C, and a fiber by f. We have the intersection pairing given by $C^2 = -n$, $C.f = 1$, and $f^2 = 0$. The canonical divisor is $K_X \equiv -2C + (-2-n)f$.(cf. [10, V.2.11]) We find that $(X, (1-\frac{2}{n})C)$ is a log del Pezzo surface, and $(K_X+(1-\frac{2}{n})C)^2 = n + 4 + \frac{4}{n}$.

From the above example, we see that for log del Pezzo surfaces $(K_X + D)^2$ is unbounded. The problem seems to arise because $(X, (1-\frac{2}{n})C)$ becomes more and more singular, i.e., $1 - \frac{2}{n} \to 1$ as $n \to \infty$. Alexeev and Nikulin showed that if we only allow ϵ-klt singularities, then there is a bounded family for log surfaces.

Definition 3.6. For $\epsilon > 0$. Let (X, D) be a log pair, $D = \sum_i d_i D_i$. We call (X, D) is ϵ-klt if for all log resolutions we have $a(X, D; E_j) > -1+\epsilon$, for all E, in particular, $0 \leq d_i < 1 - \epsilon$.

Theorem 3.7. *[16]For $\epsilon > 0$, Let (X, D) be a ϵ-klt log surface such that $-(K_X + D)$ is nef. Then the class $\{X\}$ is bounded. Except for the case when $D = 0$ and $K_X \equiv 0$.*

In particular, the theorem works for log del Pezzo surfaces.

4 Complements on Log Surfaces

The notion of complements was introduced by Shokurov. It turns out to be a very useful tool in the study of Fano varieties. We recall some results of Shokurov. The material of this section are mainly from [16].

4.1 n-complement

Definition 4.1. Let X be a normal variety, and set $D = S + B$, where $S \geq 0$ and integral, $\lfloor B \rfloor \leq 0$, and S, B have no common components.

We call (X, D) n-complementary and (X, D^+) its n-complement if

i) $n(K_X + D^+) \sim 0$

ii) (X, D^+) has only log canonical singularities.

iii) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$

We see that when $D = S = B = 0$, the third condition is equivalent to $D^+ \geq 0$. Also, that $(X, 0)$ is n-complementary implies the nonvanishing, $h^0(X, -nK_X) > 0$.

Remark 4.2. (X, D) is n-complementary if and only if $\exists \bar{D} \in |-nK_X - nS - \lfloor (n+1)B \rfloor|$ such that $D^+ = S + \frac{1}{n}(\lfloor (n+1)B \rfloor + \bar{D})$, and (X, D^+) is lc.

For curves, we have a slight generalization.

Definition 4.3. Let X be a nodal curve, and set $D = S + B$, where $S \geq 0$ and integral, $\lfloor B \rfloor \leq 0$, and S, B have no common components.

We call (X, D) n-semi-complementary and (X, D^+) its n-semi-complement if

i) $n(K_X + D^+) \sim 0$

ii) (X, D^+) is slc, i.e. Supp $D^+ \cap \operatorname{Sing} X = \emptyset$, and all coefficients are ≤ 1

iii) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$

Then the following theorem is proven by classification.

Theorem 4.4. *[12, 19.4] Let X be a nodal connected curve, and D, a boundary divisor, is supported on smooth and compact part of X. Assume the degree of $-(K_X + D)$ is nonnegative on every compact component of X. Then $K_X + D$ is n-semi-complementary for some $n = 1, 2, 3, 4, 6$.*

We may induce semi-complements on a curve to complements on a log del Pezzo surface.

Theorem 4.5. *If (X, D) is a log del Pezzo surface which is not klt, then there exists a regular complement i.e., n-complement for some $n = 1, 2, 3, 4,$ or 6.*

To prove this, we need a series of preparation. First we introduce dlt singularities, which will be proven useful.

Definition 4.6. Suppose (X, D) is a log pair. We say it has only divisorial log terminal singularities or is dlt if there is a log resolution $f : Y \to X$ such that the exceptional locus consists of only divisors, and when we write

$$K_Y + \tilde{D} = f^*(K_X + D) + \sum_j a(X, D; E_j)E_j$$

we have $a(X, D; E_j) > -1$ for all exceptional divisors E_j.

We have a useful technique called dlt modification.

Proposition 4.7. *[16, 3.1.1]*

Let (X, D) be a log pair of dimension ≤ 3, and (X, D) is lc. Then there exist $g : X' \to X$ and a boundary D' such that

i) $K_{X'} + D' = g^*(K_X + D)$

ii) (X', D') is dlt

iii) X' is $\mathbb{Q}$-factorial, and if $\dim X = 2$, *we may assume X' is smooth.*

Remark 4.8. The proof involves the relative log minimal model program (LMMP) [14, 3.31], which is a procedure described as follows:

The input is a dlt pair (Z, Δ), with Z normal and $\mathbb{Q}$-factorial, and a projecive morphism $a : Z \to S$.

The output is also a dlt pair (Z', Δ'), with Z' normal and $\mathbb{Q}$-factorial and birational over S and either

i) $K_{Z'} + \Delta$ is nef over S, or

ii) There is a Fano contraction $Z' \to W$,$\dim W < \dim Z'$.

(Z, Δ) and (Z', Δ') is connected by a series of divisorial contractions and flips, whose inverses do not contract any divisors.

Proof. Take a log resolution $f : Y \to X$. Write $K_Y + D_Y = f^*(K_X + D) + E^+ - E^-$, where $D_Y = f_*^{-1}D_X$, and E^+,E^- are effective exceptional divisors with no common components. Then $D_Y + E^-$ is a boundary, hence $(Y, D_Y + E^-)$ is dlt. Apply LMMP to $(Y, D_Y + E^-)$ over X, we arrive at $g : X' \to X$, with X' normal $\mathbb{Q}$-factorial , such that (X', D') is dlt, and $K_X + D'$ is g-nef.

Set $h : Y \dashrightarrow X'$. We find $K_{X'} + D' = h_*(K_Y + D_Y + E^-) = h_*(f^*(K_X + D) + E^+) = g^*(K_X + D) + h_*E^+$, and thus h_*E^+ is g-nef.

Now $g_*(h_*E^+) = 0$, by the following lemma, we have $-h_*E^+ \geq 0$. But it is clear that $h_*E^+ \geq 0$, so $h_*E^+ = 0$. We get (i).

Now we consider the case $\dim X = 2$, and show X' is smooth. If $E^+ \neq 0$, then $E^{+2} < 0$ and there is an exceptional divisor E, with $E^+.E < 0$, and $E^2 < 0$. Then $K_Y + D_Y + E^-.E = E^+.E < 0$, and thus $K_Y.E < 0$. We find E is a (-1)-curve, and LMMP contracts such curves. The process then proceed with only smooth surfaces. □

Lemma 4.9. *[14, 3.39]*

Let $g : X' \to X$ be a proper birational morphism between normal varieties. Suppose $-B$ is a g-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $B \geq 0$ if and only if $g_*B \geq 0$.

By the following observation, we find we may assume (X, D) is dlt in the theorem.

Proposition 4.10. *Let $f : X \to Y$ be a birational map, and D be a subboundary. $K_X + D$ is n-complementary implies $K_Y + f(D)$ is n-complementary.*

Proof. Pick $f(D)^+ = f(D^+)$. We note that $n(K_X + D^+) \sim 0$ implies $n(K_Y + f(D^+)) \sim 0$. Thus $K_X + D^+ = f^*(K_Y + f(D^+))$. □

Now we introduce another ingredient called connectedness lemma.

Lemma 4.11 (Connectedness Lemma)**.** *[16, 2.3.1]*

Let $f : X \to Z$ be a contraction, i.e. $f_*\mathcal{O}_X = \mathcal{O}_Z$. Let (X, D) be a log pair such that $D \geq 0$ and $-(K_X + D)$ is f-big and f-nef. Let $g : Y \to X$ be a log resolution, and write

$$K_Y = g^*(K_X + D) + E^+ - E^-$$

where the coefficients of $E^- \geq 1$, the coefficients of $E^+ > -1$, and E^+, E^- have no common components.

Then Supp E^- is connected in a neighborhood of any fiber of $h = f \circ g$.

Proof. We find that $\lceil E^+ \rceil - \lfloor E^- \rfloor = K_Y - g^*(K_X + D) + \{-E^+\} + \{E^-\}$ is h-nef and h-big by assumption. By Kawamata-Viehweg vanishing theorem

$$R^1 h_* \mathcal{O}_Y(\lceil E^+ \rceil - \lfloor E^- \rfloor) = 0$$

From this we derive the surjectivity $h_*\mathcal{O}_Y(\lceil E^+ \rceil) \to h_*\mathcal{O}_{\lfloor E^- \rfloor}(\lceil E^+ \rceil)$.

Since a component E in E^+ is either g-exceptional, or the proper transform of a component of D, and in the latter case, $\lceil E^+ \rceil = 0$, we must have $\lceil E^+ \rceil$ is g-exceptional, and $h_* \mathcal{O}_Y(\lceil E^+ \rceil) = \mathcal{O}_Z$. Now near some fiber $h^{-1}(z)$ of $z \in Z$, we have $\mathcal{O}_{Z,z} = h_* \mathcal{O}_Y(\lceil E^+ \rceil)_{(z)} \twoheadrightarrow h_* \mathcal{O}_{\lfloor E^- \rfloor}(\lceil E^+ \rceil)_{(z)}$. Nevertheless, $h_* \mathcal{O}_{\lfloor E^- \rfloor}(\lceil E^+ \rceil)_{(z)}$ can not be a direct sum of two proper submodules, and Supp E^-=Supp $\lfloor E^- \rfloor$ is connected near $h^{-1}(z)$. $\square$

The following application is important.

Definition 4.12. Let (X, D) be a log pair. Call a subvariety $W \subset X$ a log canonical center, if there exist a log resolution $p : Y \to X$ and a divisor E (not necessarily exceptional) such that $a(X.D; E) \leq -1$, and $p(E) = W$. The union of all log canonical centers is call the locus of log canonical singularities, denoted $\mathrm{LCS}(X, D)$.

Remark 4.13. In the definition of LCS, we can get all log canonical centers in one log resolution.

Corollary 4.14. *Under the same assumptions as the theorem, LCS(X, D) is connected in a neighborhood of any fiber of f.*

We also need some knowledge of adjunction and inversion of adjunction. [12, Chapter 16]

Proposition 4.15. *Let X be normal, S be a reduced subscheme of codimension 1, and B be a $\mathbb{Q}$-divisor. Assume $(X, S+B)$ is log canonical in codimension two, then there is a naturally defined effective $\mathbb{Q}$-divisor $\mathit{Diff}_S(B)$ called the different such that $K_X + S + B|_S = K_S + \mathit{Diff}_S(B)$.*

Remark 4.16. Rigorous definition of the *different* may be found in [12, Chapter 16]. There, the different is defined as a $\mathbb{Q}$-Weil divisorial sheaf under very mild condition. However, showing the different a $\mathbb{Q}$-divisor, i.e., supporting outside the singularity of S, in this context is by classification.

Proposition 4.17 (Inversion of adjunction)**.** *[12, 17.6]*

Let $(X, S+B)$ be a log pair, S be an irreducible divisor, and $\lfloor B \rfloor = 0$. Then $K_X + S + B$ is plt near S if and only if $K_S + \mathit{Diff}(B)$ is klt.

Proof. Let $g : Y \to X$ be a log resolution, write $K_Y = g^*(K_X + S + B) + E^+ - E^-$, where the coefficients of $E^- \geq 1$, and the coefficients of $E^+ > -1$. By the Connectedness lemma with $f : X \to X$ the identity map, we have that E^- is connected near any fiber of g.

By adjunction, $K_{S'} = g^*(K_S + \mathrm{Diff}_S(B)) + (E^+ - E')|_{S'}$, where S' is the proper transform of S, and $E^- = S' + E'$.

By definition, $K_X + S + B$ is plt $\iff E' = 0$, and $K_S + \mathrm{Diff}_S(B)$ is klt $\iff E' \cap S' = \emptyset$. By connectedness of E^- we see they are equivalent. □

Theorem 4.5 is reduced to the following induction theorem.

Proposition 4.18. *Let (X, D) be a log surface, $f : X \to Z$,and $o \in Z$. Denote $S = \lfloor D \rfloor$, and $B = D - S$. Suppose*

i) $K_X + D$ is dlt

ii) $-(K_X + D)$ is f-nef and f-big

iii) $S \neq 0$ near $f^{-1}(o)$

If near $f^{-1}(o) \cap S$, there exists an n-semi-complement $K_S + \mathit{Diff}_S(B)^+$ of $K_S + \mathit{Diff}_S(B)$, then near $f^{-1}(o)$, there exists an n-complement $K_X + S + B^+$ of $K_X + S + B$. Moreover, $\mathit{Diff}_S(B)^+ = \mathit{Diff}_S(B^+)$.

Proof. Firstly, by classification, S is simple normal crossing, and X is smooth near the singularities of S. B does not pass through singularities of S, for otherwise it would be not lc. By Szabo's refinement of Hironaka's resolution theorem, we may take a log resolution $g : Y \to X$, such that $g_S = g|_S$ is an isomorphism. Write $K_Y + S_Y + A = g^*(K_X + S + B)$, where S_Y is the proper transform of S. There is a $\mathbb{Q}$-divisor $K_S + \mathrm{Diff}_S(B) = K_X + S + B|_S$ [12, 16.6], and $K_{S_Y} + \mathrm{Diff}_{S_Y}(A) = g_S^*(K_S + \mathrm{Diff}_S(B))$.

By assumption, there is an n-semi-complement $K_S + \mathrm{Diff}_S(B)^+$ of $K_S + \mathrm{Diff}_S(B)$, and hence $K_{S_Y} + \mathrm{Diff}_{S_Y}(A)^+$ of $K_{S_Y} + \mathrm{Diff}_{S_Y}(A)$. That is, there exists

$\Theta \in |nK_{S_Y} - \lfloor (n+1)\mathrm{Diff}_{S_Y}(A) \rfloor|$ such that $n\mathrm{Diff}_{S_Y}(A)^+ = \lfloor (n+1)\mathrm{Diff}_{S_Y}(A) \rfloor + \Theta$

Now $-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor = K_Y + \lceil -(n+1)(K_Y + S_Y + A) \rceil$, and by Kawamata-Viehweg vanishing theorem, we have $R^1 h_*(\mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor) = 0$, where $h = f \circ g$. We get the surjectivity: $H^0(Y, \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor) \to H^0(S_Y, \mathcal{O}_{S_Y}(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor)$.

So there exists $\Xi \in |-nK_Y - nS_Y - \lfloor (n+1)A \rfloor|$, such that $\Xi|_{S_Y} = \Theta$. Let $A^+ = \frac{1}{n}(\lfloor (n+1)A \rfloor + \Xi)$, and $B^+ = g_*A^+$. Then we get $K_X + S + B^+|_S = g_*(K_Y + S_Y + A^+)|_S = K_S + \mathrm{Diff}_S(B)^+$ is slc. Pushing forward $n(K_Y + S_Y + A^+) \sim 0$ by g gives $n(K_X + S + B^+) \sim 0$, in particular, $K_X + S + B \equiv 0$.

It suffices to show $(X, S + B^+)$ is lc. Suppose not, then $K_X + S + B + \alpha(B^+ - B)$ is also not lc for $\alpha < 1$ and near 1.

Now $-\{K_X + S + B + \alpha(B^+ - B)\} = -(1-\alpha)(K_X + S + B) - \alpha(K_X + S + B^+)$ which is f-nef and f-big by assumption. We apply the connectedness lemma and get $\mathrm{LCS}(X, S + B + \alpha(B^+ - B))$ is connected near $f^{-1}(o)$.

On the other hand, we want to prove $\mathrm{LCS}(X, S + B + \alpha(B^+ - B)) = S$ near $f^{-1}(o) \cap S$.

Near the singularities of S, we first note that B^+ does not appear there by construction, and $(X, S + \alpha(B^+ - B)) = (X, S)$ is plt. Indeed, $\mathrm{LCS}(X, S + B + \alpha(B^+ - B)) = S$ there.

Outside the singular locus of S, we have$(S, \mathrm{Diff}_S(B))$ is klt by adjunction and $(S, \mathrm{Diff}_S(B)^+)$ is lc, so $(S,\ \alpha\mathrm{Diff}_S(B)^+ + (1-\alpha)(\mathrm{Diff}_S(B))$ is klt. By inversion of adjunction, $(X, S+B+\alpha(B^+-B))$ is plt there. We see $\mathrm{LCS}(X, S+B+\alpha(B^+-B)) = S$ near $f^{-1}(o) \cap S$.

By the connectedness we just proved, we find $\mathrm{LCS}(X, S + B + \alpha(B^+ - B)) = S$ near $f^{-1}(o)$, which in turn implies $(X, S+B+\alpha(B^+-B))$ is plt, a contradiction. □

Remark 4.19. Szabo's theorems says for a variety X and a divisor D, one can get a log resolution by repeatedly blowing up smooth centers, and unlike Hironaka's result, can leave where X is smooth and D is simple normal crossing unchanged.

Theorem 4.20. *[16, 2.1.2 2.1.3]*

Let X be a normal surface, and C be a reduced curve. $K_X + C$ is dlt near P. Then near P,(X, C) is analytically isomorphic to

a) $(\mathbb{C}^2, \{x = 0\})/\frac{1}{m}(1, a)$, with $\gcd(a, m) = 1$ if (X, C) is plt. In this case C is irreducible and smooth.

b) $(\mathbb{C}^2, \{xy = 0\}$, if (X, C) is not plt. In this case C has two smooth components, and X is smooth at P.

5 Weighted Complete Intersection

Weighted complete intersections provide many examples of del Pezzo surfaces and surfaces of general type with cyclic quotient singularities. The materials in this section can be found in [8]. In this section, let k be a field.

5.1 Weighted projective space

Definition 5.1. The weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ of weights $a_0, \ldots, a_n$ is Proj $k[X_0, \ldots, X_n]$, where the grading is defined by $\deg X_i = a_i$, $i = 0, 1, 2, \ldots, n$.

Denote by U_i the basic open set $\{X_i \neq 0\} \subseteq \mathbb{P}$. We have $U_i = \mathrm{Spec}\{k[X_0, \ldots, X_n]_{X_i}\}_0$, where $\{\}_0$ means taking the degree zero part.

Construct the following morphism:

$$\tilde{p_i} : \mathrm{Spec}\ k[y_0, \ldots, \hat{y_i}, \ldots, y_n, t, \frac{1}{t}] \to \mathrm{Spec}\ k[X_0, \ldots, X_n]_{X_i},$$

defined by

$$X_l \mapsto \begin{cases} y_l t^{a_l}, & \text{if } i \neq l \\ t^{a_i}, & \text{if } i = l \end{cases}$$

Taking degree 0 part induces the morphism

$$p_i : \mathrm{Spec}\ k[y_0, \ldots, \hat{y_i}, \ldots, y_n] \to \mathrm{Spec}\ \{k[X_0, \ldots, X_n]_{X_i}\}_0.$$

We see p_i is a quotient maps from $\mathbb{A}^n \to U_i$, and moreover we have the following.

Proposition 5.2. *$\mathbb{P}(a_0, \ldots, a_n)$ is covered by the affine open set $U_i = \mathbb{A}^n/\mu_{a_i}$, with action $\zeta_{a_i} \cdot (x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto (\zeta_{a_i}^{a_0} x_0, \ldots, \widehat{\zeta_{a_i}^{a_i} x_i}, \ldots, \zeta_{a_i}^{a_n} x_n)$. In other words, it has exactly $n+1$ standard quotient singularities, each being of the type $\frac{1}{a_i}(a_1, \ldots, \hat{a_i}, \ldots, a_n)$.*

The quotient maps p_i can be glued together in the following fashion.

Proposition 5.3. *There is a natural map $p : \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}$*

Proof. Plug in $y_l = \frac{Y_l}{Y_i}$, we obtain $p_i' : \mathrm{Spec} k[Y_0, \ldots, Y_n]_{Y_i} \to \mathrm{Spec}\{k[X_0, \ldots, X_n]_{X_i}\}_0$ defined by $X_l \mapsto Y_l$. Glue all the p_i' together and we get $p : \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}$ □

Note that they are related by $p_i = p_i'|_{\{Y_i = 1\}}$, and p_i' can be represented by the projection $\{Y_i \neq 0\} \cong \{Y_i = 1\} \times (\mathbb{A}^1 \backslash \{0\}) \to \{Y_i = 1\}$ followed by p_i.

Proposition 5.4. *There is a covering $q : \mathbb{P}^n \to \mathbb{P}$.*

Proof. Define q_i : Spec $k[y_0, \ldots, \hat{y}_i, \ldots, y_n] \to$ Spec $\{k[X_0, \ldots, X_n]_{X_i}\}_0$, by $X_l \mapsto \begin{cases} y_l^{a_l}, & \text{if } i \neq l \\ 1, & \text{if } i = l \end{cases}$.

Write $\mathbb{A}^{n+1} =$ Spec $k[Y_0, \ldots, Y_n]$, and plug in $y_l = Y_l/Y_i$ for $l \neq i$, and $t = Y_i$. We obtain $\tilde{q}_i$: Spec $k[\frac{Y_0}{Y_i}, \ldots, \frac{\hat{Y}_i}{Y_i}, \ldots, \frac{Y_n}{Y_i}] \to$ Spec$\{k[X_0, \ldots, X_n]_{X_i}\}_0$. These morphisms can be glued together. □

Remark 5.5. We can also define q to be the map between Proj schemes induced by the graded injection: $k[X_0, \ldots, X_n] \to k[Y_0, \ldots, Y_n]$, $X_i \mapsto Y_i^{a_i}$. Note that the q constructed above is different from p.

Definition 5.6. If $\gcd(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = 1$ for $i = 0, 1, \ldots, n$. We say $\mathbb{P}(a_0, \ldots, a_n)$ is well formed.

Lemma 5.7. *p_i is étale in codimension 1 for all i if and only if $\mathbb{P}(a_0, \ldots, a_n)$ is well formed.*

Proof. We note $\frac{1}{r}(a_1, \ldots, a_n)$ group action has no divisor fixed by a non-identity element if and only if $\gcd(r, a_1, \ldots, \hat{a}_i, \ldots, a_n) = 1$ for $i = 1, 2, \ldots, n$. □

Furthermore, examining carefully, we see that if $\gcd(a_{i_0}, a_{i_1} \ldots, a_{i_s}) = d \neq 1$, then for the action of $\mu_{a_{i_0}}$ on the basic open set U_{i_0}, an order d element fixes the locus $\{X_{j_1} = \ldots = X_{j_{n-s}} = 0\}$, where $\{i_0, i_1, \ldots, i_s, j_1, \ldots, j_{n-s}\} = \{0, 1, \ldots, n\}$. Such loci are the only possible places that can be singular, called the singular strata.

Denote by U^0 the complement of all singular strata in $\mathbb{P}(a_0, \ldots, a_n)$. We find U^0 is covered by all $U_{i_0,\ldots,i_s} = \{X_{i_0} X_{i_1} \ldots X_{i_s} \neq 0\}$ with $\gcd(a_{i_0}, \ldots a_{i_s}) = 1$, and $p : p^{-1}(U_{i_0,\ldots,i_s}) \to U_{i_0,\ldots,i_s}$ is isomorphic to the first projection $U_{i_0,\ldots,i_s} \times (\mathbb{A}^1 \backslash 0) \to U_{i_0,\ldots,i_s}$ albeit not canonically. Indeed, p is induced by the inclusion $\{k[X_0, \ldots X_n]_{X_{i_0} X_{i_1} \ldots X_{i_s}}\}_0 \to k[X_0, \ldots X_n]_{X_{i_0} X_{i_1} \ldots X_{i_s}} = \{k[X_0, \ldots X_n]_{X_{i_0} X_{i_1} \ldots X_{i_s}}\}_0[T, \frac{1}{T}]$, where $T = \prod_{l=0}^{s} X_{i_l}^{v_{i_l}}$, provided we fix $v_{i_l} \in \mathbb{Z}$, $\sum_{l=0}^{s} v_{i_l} a_{i_1} = 1$. Note that if we choose v differently, T will differ by a multiple of a unit in $\{k[X_0, \ldots X_n]_{X_{i_0} X_{i_1} \ldots X_{i_s}}\}_0$. We see $p^{-1}(U^0) \to U^0$ is a $\mathbb{G}_m$-torsor, for $\mathbb{G}_m =$ Spec $k[u, \frac{1}{u}]$ acts naturally by $T \mapsto uT$.

Lemma 5.8. *Any $\mathbb{P}(a_0, \ldots, a_n)$ is isomorphic to some well formed $\mathbb{P}(a_0', \ldots, a_n')$.*

Proof. We only need the following observation $\mathbb{P}(a_0, a_1 \dots, a_n) \cong \mathbb{P}(a_0, qa_1, \dots, qa_n)$ for any $q \in \mathbb{N}$, $(q, a_0) = 1$. $\square$

5.2 Weighted complete intersection

A degree d form f_d in $k[X_0, \dots, X_n]$ is $f_d = \sum C_\alpha X^\alpha$ with $\sum_{i=0}^n a_i\alpha_i = d$. A general form f_d may be assumed such that all $C_\alpha \neq 0$. In a weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$, we may define a subvariety X by homogeneous ideal generated by such forms. We note that the above constructed p_i can be restricted to X.

Definition 5.9.

If $X \subseteq \mathbb{P}(a_0, \dots, a_n)$ a subvariety, such that $p^{-1}(X) \in \mathbb{A}^{n+1}\backslash\{0\}$ is smooth, call X quasi-smooth.

If X is defined by $c = \operatorname{codim} X$ forms in $k[X_0, \dots, X_n]$, call X a weighted complete intersection. Denoted by $X_{d_1,d_2,\dots,d_c}$if the defining forms $f_1, \dots, f_c$ are of degrees $d_1, d_2, \dots, d_c$. In particular, when $c = 1$, X_d is called a weighted hypersurface. We set $k[x_0, \dots, x_n] = k[X_0, \dots, X_n]/(f_1, \dots, f_c)$.

Suppose X is a weighted complete intersection of codimension c, X is called well formed if $\mathbb{P}(a_0, \dots, a_n)$ is well formed, and X contains no codimension $c+1$ singular stratum.

Proposition 5.10. *If X is well formed, then $p_i|_{p_i^{-1}(X\cap U_i)}$ is étale in codimension 1* .

Proof. This follows from that on each affine open set U_i, the action of μ_{a_i} is free in codimension 1. $\square$

From this we see if X is well formed and quasi-smooth, it has only cyclic quotient singularities. In this case, we have the following formula for the canonical divisor.

Proposition 5.11. *If $X = X_{d_1,d_2,\dots,d_c}$ is well formed and quasi-smooth, then $K_X = \mathcal{O}(\alpha)$, where $\alpha = \sum_{j=1}^c d_j - \sum_{i=0}^n a_i$ is called the amplitude.*

This proposition is proved by calculating the *Ext* definition of the dualizing sheaf in [7]. Here we provide a proof dealing directly with differential forms.

Proof. Let $V^0 = U^0 \cap X$, covered by $V_{i_0,\ldots,i_r} = U_{i_0,\ldots,i_r} \cap X$ with $\gcd(a_{i_0},\ldots,a_{i_r}) = 1$. By assumption, codim $X \backslash V^0 > 1$. Thus it suffices to consider K_{V^0}. We see as before that $p^{-1}(V_{i_0,\ldots,i_r}) \to V_{i_0,\ldots,i_r}$ is induced by $k[V_{i_0,\ldots,i_r}] \to k[V_{i_0,\ldots,i_r}][T, \frac{1}{T}]$, where T may differ by a multiple of $k[V_{i_0,\ldots,i_r}]^\times$. We proceed as follows.

i) For a line bundle L over V^0, and p^*L is trivial in $p^{-1}(V)$. Then $L = \mathcal{O}(m)$ for some $m \in \mathbb{Z}$. Indeed, suppose L is defined by the transition functions $g_{\mu\nu}$ over the open cover $\{V_\mu\}$, a refinement of $\{V_{i_0,\ldots,i_r}\}$. We may write $g_{\mu\nu} = h_\mu / h_\nu$ for $h_\mu \in (k[V_\mu][T_\mu, \frac{1}{T_\mu}])^\times$ since p^*L is trivial. Now that h_μ is a monomial $\tilde{h}_\mu T_\mu^{-m}$, where m is common for all μ. In this case $L = \mathcal{O}(m)$.

ii) We have the following nowhere vanishing regular differential $(n-c+1)$-form on $p^{-1}(V)$, which is smooth.

$$\omega_0 = \operatorname{sgn}\begin{pmatrix} 0 & \ldots & n-c & n-c+1 & \ldots & n \\ i_0 & \ldots & i_{n-c} & j_1 & \ldots & j_c \end{pmatrix} \frac{dX_{i_0} \wedge dX_{i_1} \wedge \ldots \wedge dX_{i_{n-c}}}{\frac{\partial(f_1,\ldots f_c)}{\partial(X_{j_1},\ldots X_{j_c})}}$$

On each V_μ, denote by $z_1,\ldots,z_{n-c}$ the regular system of parameters. We may write $\omega_0 = h_\mu T_\mu^{-\alpha} dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{n-c} \wedge \frac{dt}{t}$ with $h_\mu \in k[V_\mu]^\times$, and K_X is defined by the cocycle $g_{\mu\nu} = h_\mu T_\mu^{-\alpha} / h_\nu T_\nu^{-\alpha} \in k[V_\mu \cap V_\nu]^\times$. Therefore $K_X = \mathcal{O}_X(\alpha)$. □

Consider a weighted hypersurface X_d. Note that all degree d forms form a linear system $L(d)$ in $\mathbb{A}^{n+1}$, and by Bertini theorem, for general f_d, singularities of $p^{-1}(X_d)$ only lie in the base locus of $L(d)$. Since $L(d)$ is spanned by monomials, the base locus B is a union of coordinate k-planes $E_{i_1 i_2 \ldots i_{n-k}} = \{X_{i_1} = X_{i_2} = \ldots = X_{i_{n-k}} = 0\}$. We then require the gradient ∇f_d is nowhere vanishing on these k-planes except at origin. Write this condition explicitly, we have the following.

Proposition 5.12. *Let $X_d \subseteq \mathbb{P}(a_0, a_1, a_2, a_3)$ be a general degree d hypersurface.*

It is well formed if and only if $\gcd(a_0,\ldots,\hat{a}_i,\ldots,a_3) = 1$ for all i, and $\gcd(a_i,a_j) | d$ for all $i \neq j$

Suppose $d > a_0,\ldots,a_3$. Then X is quasi-smooth if and only if the following conditions are satisfied.

i) For all i, there exists j such that there is a monomial $X_i^{m_i} X_j$ in $L(d)$.

ii) For all $i \neq j$, there is a monomial $X_i^{m_i} X_j^{m_j}$ in $L(d)$, or there are monomials $X_i^{m_i} X_j^{m_j} X_k$ and $X_i^{m_i'} X_j^{m_j'} X_l$ in $L(d)$, where $\{i,j,k,l\} = \{0,1,2,3\}$.

iii) For all i, there exists a monomial not involving X_i in $L(d)$.

Proof. We keep the notation $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

i) The existence of X_i^m is equivalent to $E_{jkl} \nsubseteq B$. If not, then the existence of $X_i^m X_j$ with $j \neq i$ is equivalent to that $\partial_j f_d|_{E_{jkl}}$ is nowhere vanishing.

ii) The existence of $X_i^{m_i} X_j^{m_j}$ is equivalent to $E_{kl} \nsubseteq B$. If not, then the existence of $X_i^{m_i} X_j^{m_j} X_k$ and $X_i^{m_i'} X_j^{m_j'} X_l$ is equivalent to that $(\partial_k f_d, \partial_l f_d)|_{E_{kl}}$does not vanish unless $X_i = X_j = 0$. which is exclude by i).

iii) The existence of monomial not involving X_i is equivalent to $E_i \nsubseteq B$. $\square$

In [4] weighted hypersurfaces that are del Pezzo are classified. As an illustration, we show general weighted hypersurfaces in the Table 1. has n-complement.

For convenience, we denote the variables $\mathbb{P}(a_0 a_1, a_2, a_3) = \text{Proj } \mathbb{C}[w, x, y, z]$. Denote $U_0 = \{w \neq 0\}$, $p_0 : V_0 = \{w = 1\} \subseteq (\mathbb{A}^{n+1} \backslash 0) \to U_0$, and similarly for x, y, z. We also adopt the coefficient convention: e.g., a general equation $F = y^3 + x^2 z$ means $F = c_1 y^3 + c_2 x^2 z$ for general c_1, c_2.

Example 5.13. Let $X = X_{12n-9} \subseteq \mathbb{P}(1, 3n-2, 4n-3, 6n-5)$, $\alpha = -n$. It has a 3-complement.

Proof. The general equation for X is $F = y^3 + x^2 z + wz^2 +$ higher terms of w. Pick a section $w^2 x$ of $\mathcal{O}_X(-3K_X)$, and let D be this divisor. We show that $(X, \frac{1}{3}D)$ is lc.

Recall wellformedness of X implies the covering map p_i is finite étale in codimension 1 for all i. By Lemma 2.23, it suffices to prove the preimage $(\tilde{X}_i, \frac{1}{3}\tilde{D}_i)$ of $(X, \frac{1}{3}D)|_{U_i}$ is lc for all i. Also, by quasi-smoothness of X, $\tilde{X}_i$ is smooth.

Near a smooth point P of the reduced part of $\tilde{D}_i$, $(\tilde{X}_i, \frac{1}{3}\tilde{D}_i)$ is analytically isomorphic to $(\mathbb{A}^2, cL)$, where L is an axis, $c = \frac{1}{3}$ or $\frac{2}{3}$. It is lc here.

The only singular point is $P = (0, 0, 0, 1)$ of $\tilde{D}_3$ in V_3, where the tangent plane $T_P X$ is $\{w = 0\}$. We project $(\tilde{X}_3, \frac{1}{3}\tilde{D}_3)$ to $T_P X$. Since the projection is étale near P. It suffices to prove that $(\mathbb{A}^2, \frac{1}{3}(2C_1 + C_2))$ is lc. where $C_1 = \{y^3 + x^2 = 0\} = X \cap \{w = 0\}$, $C_2 = \{x = 0\}$. This can be shown by standard blowup resolution calculation.

Let $f : \tilde{X} \to X$ be the minimal log resolution of $(\mathbb{A}^2, C_1 + C_2)$, which can be obtained by repeatedly blowing up singular points. Denote by $\tilde{C}_1, \tilde{C}_2$ the proper transforms of C_1, C_2 respectively, and E_j be the exceptional divisors for $j = 1, 2, 3$.

Then

$$\begin{aligned}
K_{\tilde{X}} &= f^*K_X + E_1 + 2E_2 + 4E_3; \\
f^*C_1 &= \tilde{C}_1 + 2E_1 + 3E_2 + 6E_3; \\
f^*C_2 &= \tilde{C}_2 + E_1 + 2E_2 + 3E_3.
\end{aligned}$$

We find that $\mathrm{lct}(2C_1 + C_2) = \min\{\frac{2}{5}, \frac{3}{8}, \frac{5}{15}\} = \frac{1}{3}$. Therefore, $(\mathbb{A}^2, \frac{1}{3}(2C_1 + C_2))$ is lc. $\square$

Example 5.14. Let $X = X_{126n-81} \subseteq \mathbb{P}(7, 28n-18, 42n-27, 63n-44)$, $\alpha = -(7n-1)$. It has a 4-complement.

Proof. The general equation is $F = y^3 + yx^3 + wz^3 +$ higher terms of w. Pick a section w^2x of $\mathcal{O}_X(-4K_X)$, and let D be this divisor. We show that $(X, \frac{1}{4}D)$ is lc.

The only singular point is $P = (0,0,0,1)$ of $\tilde{D}_3$ in V_3. We project to $T_PX = \{w = 0\}$. Then similar to previous example, it is reduced to $(\mathbb{A}^2, \frac{1}{4}(2C_1 + C_2))$ is lc, where $C_1 = \{y^3 + yx^3 = 0\}$ and $C_2 = \{x = 0\}$. This is also shown by similar calculation. For convenience, denote $C_1 = C_3 + C_4$ with $C_3 = \{y = 0\}$ and $C_4 = \{y^2 + x^3 = 0\}$.

Let $f : \tilde{X} \to X$ be the minimal log resolution of $(\mathbb{A}^2, C_2 + C_3 + C_4)$. Denote by $\tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ the proper transforms of C_2, C_3, C_4 respectively, and E_j be the exceptional divisors for $j = 1, 2, 3$. Then

$$\begin{aligned}
K_{\tilde{X}} &= f^*K_X + E_1 + 2E_2 + 4E_3; \\
f^*C_2 &= \tilde{C}_2 + E_1 + 2E_2 + 3E_3; \\
f^*C_3 &= \tilde{C}_3 + E_1 + E_2 + 2E_3; \\
f^*C_4 &= \tilde{C}_4 + 2E_1 + 3E_2 + 6E_3.
\end{aligned}$$

We find that $\mathrm{lct}(C_2 + 2C_3 + 2C_4) = \min\{\frac{2}{7}, \frac{3}{10}, \frac{5}{19}\} > \frac{1}{4}$. Therefore, $(\mathbb{A}^2, \frac{1}{4}(2C_1 + C_2))$ is lc. $\square$

Example 5.15. Let $X = X_{24n-12} \subseteq \mathbb{P}(2, 6n-3, 8n-4, 12n-7)$, $\alpha = -2n$. It has a 4-complement.

Proof. The general equation is $F = x^4 + y^3 + wz^2 +$ higher terms of w. Pick a section w^2y of $\mathcal{O}_X(-4K_X)$, and let D be this divisor. We show that $(X, \frac{1}{4}D)$ is lc.

The only singular point is $P = (0,0,0,1)$ of $\tilde{D}_3$ in V_3. We project to $T_PX = \{w = 0\}$. Then it is reduced to $(\mathbb{A}^2, \frac{1}{4}(2C_1 + C_2))$ is lc, where $C_1 = \{y^3 + x^4 = 0\}$ and $C_2 = \{y = 0\}$. We calculate as follows.

Let $f : \tilde{X} \to X$ be the minimal log resolution of $(\mathbb{A}^2, C_1 + C_2)$. Denote by $\tilde{C}_1, \tilde{C}_2$ the proper transforms of C_1, C_2 respectively, and E_j be the exceptional divisors for $j = 1, 2, 3, 4$. Then

$$\begin{aligned} K_{\tilde{X}} &= f^*K_X + E_1 + 2E_2 + 4E_3 + 6E_4; \\ f^*C_1 &= \tilde{C}_1 + 3E_1 + 4E_2 + 8E_3 + 12E_4; \\ f^*C_2 &= \tilde{C}_2 + E_1 + 2E_2 + 3E_3 + 4E_4. \end{aligned}$$

We find that $\operatorname{lct}(2C_1 + C_2) = \min\{\frac{2}{7}, \frac{3}{10}, \frac{5}{19}, \frac{7}{28}\} = \frac{1}{4}$. Therefore, $(\mathbb{A}^2, \frac{1}{4}(2C_1 + C_2))$ is lc. $\square$

Example 5.16. Let $X = X_{8n+4} \subseteq \mathbb{P}(2, 2n+1, 2n+1, 4n+1)$, $\alpha = -1$. It has a 2-complement.

Proof. The general equation is $F = f_4(x, y) + wz^2 +$ higher terms of w, where f_4 is a homogeneous polynomial of degree 4. Pick a section w of $\mathcal{O}_X(-2K_X)$, and let D be this divisor. We show that $(X, \frac{1}{2}D)$ is lc.

The only singular point is $P = (0, 0, 0, 1)$ of $\tilde{D}_3$ in V_3. We project to $T_PX = \{w = 0\}$. Then it is reduced to $(\mathbb{A}^2, \frac{1}{4}(L_1 + L_2 + L_3 + L_4))$ is lc, where L_i is a line through the origin for $i = 1, 2, 3, 4$. For general F, L_i are distinct. We calculate as follows.

Let $f : \tilde{X} \to X$ be the minimal log resolution of $(\mathbb{A}^2, L_1 + L_2 + L_3 + L_4)$. Denote by $\tilde{L}_i$ the proper transform of L_i for $i = 1, 2, 3, 4$, and E be the exceptional divisor.

$$\begin{aligned} K_{\tilde{X}} &= f^*K_X + E; \\ f^*L_i &= \tilde{L}_i + E. \end{aligned}$$

We find that $\operatorname{lct}(L_1 + L_2 + L_3 + L_4) = \frac{2}{4} = \frac{1}{2}$. Therefore, $(\mathbb{A}^2, \frac{1}{4}(L_1 + L_2 + L_3 + L_4))$ is lc. $\square$

The followings are some examples from Table 2 of [4]. These examples are not n-complementary for $n \leq 6$ since $|-nK_X| = \emptyset$.

Example 5.17. Let $X = X_{256} \subseteq \mathbb{P}(13, 35, 81, 128)$, $\alpha = -1$. It has a 13-complement.

Proof. The general equation is $F = z^2 + x^5y + wy^3 +$ higher terms of w. Pick a section w of $\mathcal{O}_X(-13K_X)$, and let D be this divisor. We show that $(X, \frac{1}{13}D)$ is lc.

The only singular point is $P = (0, 0, 1, 0)$ of $\tilde{D}_2$ in V_2. We project to $T_PX = \{w = 0\}$. Then it is reduced to $(\mathbb{A}^2, \frac{1}{13}C)$ is lc, where $C = \{z^2 + x^5 = 0\}$.

Let $f : \tilde{X} \to X$ be the minimal log resolution of $(\mathbb{A}^2, C_1 + C_2)$. Denote by $\tilde{C}$ the proper transform of C, and E_j be the exceptional divisors for $j = 1, 2, 3, 4$. Then

$$\begin{aligned} K_{\tilde{X}} &= f^*K_X + E_1 + 2E_2 + 3E_3 + 6E_4; \\ f^*C &= \tilde{C} + 2E_1 + 4E_2 + 5E_3 + 10E_4. \end{aligned}$$

We find that $\mathrm{lct}(C) = \min\{\frac{2}{2}, \frac{3}{4}, \frac{4}{5}, \frac{7}{10}\} = \frac{7}{10}$. Therefore, $(\mathbb{A}^2, \frac{7}{10}C)$ and hence $(\mathbb{A}^2, \frac{1}{13}C)$ is lc. □

Example 5.18. Let $X = X_{76} \subseteq \mathbb{P}(11, 13, 21, 38)$, $\alpha = -7$. It has a 11-complement.

Proof. The general equation is $F = z^2 + xy^3 + wx^5 +$ higher terms of w. Pick a section w^7 of $\mathcal{O}_X(-11K_X)$, and let D be this divisor. We show that $(X, \frac{1}{11}D)$ is lc.

The only singular point is $P = (0, 1, 0, 0)$ of $\tilde{D}_1$ in V_1. We project to $T_PX = \{w = 0\}$. Then it is reduced to $(\mathbb{A}^2, \frac{7}{11}C)$ is lc, where $C = \{z^3 + y^2 = 0\}$. It is well-known that $\mathrm{lct}(C_1) = \frac{5}{6} > \frac{7}{11}$. We have $(\mathbb{A}^2, \frac{7}{11}C)$ is indeed lc. □

Remark 5.19. The weighted complete intersection $X_{6,6} \subseteq \mathbb{P}(2, 2, 3, 3, 3)$ is an example of well formed quasi-smooth del Pezzo surface.

6 Kähler-Einstein Metric

In Riemannian geometry, it has been a fundamental problem to find nice metrics on a manifold. In Kähler geometry, similarly, it has been an active research problem to determine the existence of Kähler-Einstein metrics. We recall the following definitions.

Definition 6.1.

i) Let (X, h) be a Hermitian complex manifold, Define $g = \mathrm{Re}h$ and $\omega =$ -Im h. If ω is a closed form, then call ω a Kähler form, g a Kähler metric, and X a Kähler manifold.

ii) Let (X, g) be a Riemannian manifold. If $R = kg$, where R is the Ricci tensor, and $k \in \mathbb{R}$ a constant then call g an Einstein metric, and X an Einstein manifold.

iii) When (X, h) is a Hermitian complex manifold such that g is both Kähler metric and an Einstein metric, g is called a Kähler-Einstein metric.

The problem has been solved for X with $c_1(X) < 0$ (resp. $= 0$). In this case, X always has a Kähler-Einstein metric, which is known as Aubin-Yau theorem (resp. Calabi-Yau theorem). However, for $c_1(X) > 0$, i.e., the Fano case, X does not always have a Kähler-Einstein metric. Thus, it remains an interesting question. For smooth surfaces, Tian solved this problem completely.

Theorem 6.2. *[21] A smooth del Pezzo surface X has a Kähler-Einstein metric if and only if X is not $\mathbb{P}^2$ blown up 1 or 2 points.*

Consequently, Matsushima's necessary condition for existence of Kähler-Einstein metric, i.e., the Lie algebra of $\mathrm{Aut}(X)$ is reductive, is also sufficient for surfaces. In the proof of the theorem α-invariant and its refinements are defined. Here we give the definition of α-invariant for example.

Definition 6.3. For an n-dimensional smooth projective manifold X with ample line bundle L define

$$\begin{aligned}\alpha(X,L) &= \sup\{\alpha > 0 \mid \exists C > 0 \text{ such that} \\ &\quad \int_X e^{-\alpha(\varphi-\sup_X \varphi)}\omega^n \leq C, \forall \varphi \in C^\infty(X,\mathbb{R}), \omega + i\partial\bar{\partial}\varphi > 0\}\end{aligned}$$

In particular, if X is Fano, we define $\alpha(X) = \alpha(X, -K_X)$.

There is an algebraic formula for $\alpha(X,L)$ involving log canonical threshold due to Demailly.(cf. [21, Appendix])

Theorem 6.4. $\alpha(X,L) = \inf_{m\in\mathbb{N}} \inf_{D\in|mL|} lct(X, \frac{1}{m}D)$

With α-invariants, there is a sufficient condition for existence of Kähler-Einstein metrics.

Theorem 6.5. *[20, Theorem 2.1.] If $\alpha(X) > \frac{n}{n+1}$, then X has a Kähler-Einstein metric.(cf. [21, 3.2])*

For orbifolds, there is a similar result.

Theorem 6.6. *[3] A Fano orbifold X of dimension n has a Kähler-Einstein metric if*

$$\alpha(X) := \inf_n \inf_{D\in|-mK_X|} lct(X, \frac{1}{n}D) > \frac{n}{n+1}$$

It is not easy to compute the global log canonical threshold $\alpha(X)$ [1], for it involves all divisors $\mathbb{Q}$-equivalent to $-K_X$.

Example 6.7. $\alpha(\mathbb{P}^2) = \frac{1}{3}$, $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{1}{2}$. We note that if X is smooth at P, but (X, D) is not lc at P then $\mathrm{mult}_P D > 1$. However, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ do have Kähler-Einstein metric, namely, the Fubini-Study metric.

In [2], Cheltsov proved that for a smooth degree d hypersurface X in $\mathbb{P}^n$, and $B \equiv rH$, where H is the hyperplane section. Then there is a lower bound for log canonical threshold:

$$\mathrm{lct}(X, B) \geq \min\{\frac{n-1}{rd}, \frac{1}{r}\}$$

In particular, if X is a cubic surface in $\mathbb{P}^3$, we have $\alpha(X) \geq \frac{2}{3}$.

7 Euler Characteristics

7.1 Singular Riemann-Roch Theorem

The Riemann-Roch formula plays the pivotal role in the study of geometry of nonsingular varieties. It is desirable to have a similar formula at least for mildly singular surface. For surfaces with only quotient singularities, there is a singular Riemann-Roch formula due to Reid (cf. [18]). To state the formula, we need to introduce Dedekind sums first.

Definition 7.1 (Dedekind Sum)**.** For a quotient singularity $\frac{1}{r}(a, b)$, define

$$\sigma_j = \sigma_j(\frac{1}{r}(a, b)) = \sum_{\zeta \in \mu_r \setminus \{1\}} \frac{\zeta^j}{(1-\zeta^a)(1-\zeta^b)}$$

Theorem 7.2 (Singular Riemann-Roch Theorem)**.** *[18]*

Let X be a surface with only cyclic quotient singularities, and D be a Weil divisor. Then

$$\chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D.(D - K_X) + \sum_{P \in X} c_P(D)$$

where $c_P(D) = \frac{1}{r}(\sigma_j - \sigma_0)$ for quotient singularity P of type $\frac{1}{r}(a, b)$ and j is the weight of cyclic group action on the sheaf $\mathcal{O}_X(D)$.

[1] Many papers including [3] use the notation "$\mathrm{lct}(X)$". However, it seems to cause confusion here, so we use "$\alpha(X)$" instead.

In particular,

$$\chi(X, -mK_X) = \chi(X, \mathcal{O}_X) + \frac{m(m+1)}{2} K_X^2 + \sum_{P \in X} c_P(-mK_X)$$

where $c_p(-mK_X) = \frac{1}{r}(\sigma_{-m(a+b)} - \sigma_0)$.

We set $\tau_m := (\sigma_{(m+1)j} - \sigma_{mj})/r$ and $\delta_m := \tau_{m+1} - \tau_m$. We have $\tau_m = -\tau_{-m}$ (see below). For simplicity, we denote

$$\varkappa_m := \chi(X, -mK_X),$$

and the difference and the second difference are given by

$$\begin{aligned} \Delta\varkappa_m &:= \varkappa_{m+1} - \varkappa_m \\ &= (m+1)K_X^2 + \textstyle\sum_{P \in X} \tau_{m+1}; \\ \Delta^2\varkappa_m &:= \Delta\varkappa_{m+1} - \Delta\varkappa_m \\ &= K_X^2 + \textstyle\sum_{P \in X} \delta_{m+1}. \end{aligned}$$

Given a quotient singularity $\frac{1}{r}(a, b)$, we let α (resp. $\bar{\alpha}$) be the smallest positive integer such that $a\alpha \equiv b \pmod r$ (resp. $(b\bar{\alpha} \equiv a (\mathrm{mod}\ r))$.

Lemma 7.3. *Keep the notation as above.*

i) $\tau_m = -\tau_{-m}$, *and* $\delta_m = \delta_{-(m+1)}$.

ii) $\tau_m = (1 + R^m) - m\frac{\alpha+\bar{\alpha}+2}{r}$, *where*

$$R^m = \frac{1}{2}\{\left\lfloor \frac{m(1+\alpha)}{r} \right\rfloor + \left\lfloor \frac{m(1+\alpha)-1}{r} \right\rfloor + \left\lfloor \frac{m(1+\bar{\alpha})}{r} \right\rfloor + \left\lfloor \frac{m(1+\bar{\alpha})-1}{r} \right\rfloor\}.$$

iii) $\delta_m = Z_m - \frac{\alpha+\bar{\alpha}+2}{r}$, *where*

$$\begin{aligned} Z_m(\frac{1}{r}(a,b)) &= \#\{0 \le j \le \alpha \mid ja \equiv -m(a+b)(mod\ r)\} + \\ &\quad \#\{1 \le j \le \bar{\alpha}+1 \mid jb \equiv -m(a+b)(mod\ r)\} \\ &= \#\{0 \le j \le \alpha \mid ja \equiv (m+1)(a+b)(mod\ r)\} + \\ &\quad \#\{1 \le j \le \bar{\alpha}+1 \mid jb \equiv (m+1)(a+b)(mod\ r)\}. \end{aligned}$$

$Z_m \in \{0, 1, 2\}$. *Moreover,* $R^{m+1} - R^m = Z_{m+1}$, $R^m \in \mathbb{Z}$.

Proof. i) By definition we have

$$\begin{aligned} r\tau_m &= \textstyle\sum_\zeta \frac{\zeta^{a+b}-1}{(1-\zeta^a)(1-\zeta^b)} \cdot \zeta^{m(a+b)} \\ &= \textstyle\sum_\zeta \frac{\zeta^{-(a+b)}-1}{(1-\zeta^{-a})(1-\zeta^{-b})} \cdot \zeta^{-m(a+b)} \\ &= \textstyle\sum_\zeta \frac{1-\zeta^{(a+b)}}{(1-\zeta^a)(1-\zeta^b)} \cdot \zeta^{-m(a+b)} = -r\tau_{-m}. \end{aligned}$$

ii) Denote a', b' the least positive integer such that $aa' \equiv bb' \equiv 1 (\text{mod } r)$. Then

$$\begin{aligned}
r\tau_m &= \textstyle\sum_\zeta \frac{\zeta^{a+b}-1}{(1-\zeta^a)(1-\zeta^b)} \cdot \zeta^{m(a+b)} \\
&= \textstyle\frac{-1}{2} \sum_\zeta \{ \frac{\zeta^{m(a+b)}(1+\zeta^a)}{1-\zeta^a} + \frac{\zeta^{m(a+b)}(1+\zeta^b)}{1-\zeta^b} \} \\
&= \textstyle\frac{-1}{2} \{ \sum_\zeta \frac{\zeta^{m(a+b)a'}(1+\zeta)}{1-\zeta} + \sum_\zeta \frac{\zeta^{m(a+b)b'}(1+\zeta)}{1-\zeta} \} \\
&= (1-r) - m(a+b)(a'+b') - 1 - rR'_m,
\end{aligned}$$

where $R'_m = \frac{1}{2}\{ \left\lfloor -\frac{m(a+b)a'}{r} \right\rfloor + \left\lfloor -\frac{m(a+b)a'+1}{r} \right\rfloor + \left\lfloor -\frac{m(a+b)b'}{r} \right\rfloor + \left\lfloor -\frac{m(a+b)b'+1}{r} \right\rfloor \}$.

We may assume $a = a' = 1$,and $b = \alpha, b' = \bar{\alpha}$. Let $bb' = rk + 1$. Then

$r\tau_m = -r - m(2 + b + b' + kr) - rR'_m$,

$$\begin{aligned}
R'_m &= \textstyle\frac{1}{2}\{ \left\lfloor -\frac{m(1+b)}{r} \right\rfloor + \left\lfloor -\frac{m(1+b)+1}{r} \right\rfloor + \left\lfloor -\frac{m(1+b)b'}{r} \right\rfloor + \left\lfloor -\frac{m1+b)b'+1}{r} \right\rfloor \} \\
&= \textstyle\frac{1}{2}\{ \left\lfloor -\frac{m(1+b)}{r} \right\rfloor + \left\lfloor -\frac{m(1+b)+1}{r} \right\rfloor + \left\lfloor -\frac{m(1+b')}{r} \right\rfloor + \left\lfloor -\frac{m(1+b')+1}{r} \right\rfloor \} - mk.
\end{aligned}$$

So we may rewrite $r\tau_m = -m(2 + b + b') - r(1 + R_m)$, where

$$R_m = \frac{1}{2}\{ \left\lfloor -\frac{m(1+b)}{r} \right\rfloor + \left\lfloor -\frac{m(1+b)+1}{r} \right\rfloor + \left\lfloor -\frac{m(1+b')}{r} \right\rfloor + \left\lfloor -\frac{m(1+b')+1}{r} \right\rfloor \}.$$

Also note $\tau_m = -\tau_{-m}$, we find $R_m + R_{-m} = -2$. Write

$$R^m = R_{-m} = \frac{1}{2}\{ \left\lfloor \frac{m(1+b)}{r} \right\rfloor + \left\lfloor \frac{m(1+b)-1}{r} \right\rfloor + \left\lfloor \frac{m(1+b')}{r} \right\rfloor + \left\lfloor \frac{m(1+b')-1}{r} \right\rfloor \},$$

and $r\tau_m = -m(2 + b + b') + r(1 + R^m)$.

iii)
$$\begin{aligned}
r\delta_m &= \textstyle\sum_\zeta \left(\frac{\zeta^{a+b}-1}{\zeta^a-1} \right) \left(\frac{\zeta^{a+b}-1}{\zeta^b-1} \right) \zeta^{m(a+b)} \\
&= \textstyle\sum_\zeta (1 + \zeta^a + \zeta^{2a} \ldots + \zeta^{\alpha a})(1 + \zeta^b + \zeta^{2b} \ldots + \zeta^{\bar{\alpha} b}) \zeta^{m(a+b)} \\
&= \textstyle\sum_\zeta (1 + \zeta^a + \zeta^{2a} \ldots + \zeta^{(\alpha-1)a})(1 + \zeta^b + \zeta^{2b} \ldots + \zeta^{(\bar{\alpha}-1)b}) \zeta^{m(a+b)} \\
&+ \textstyle\sum_\zeta (1 + \zeta^a + \zeta^{2a} \ldots + \zeta^{(\alpha-1)a}) \zeta^{m(a+b)+a} \\
&+ \textstyle (1 + \zeta^b + \zeta^{2b} \ldots + \zeta^{(\alpha^{-1}-1)b}) \zeta^{m(a+b)+b} + \sum_\zeta \zeta^{(m+1)(a+b)}
\end{aligned}$$

Now $(1 + \zeta^a + \zeta^{2a} \ldots + \zeta^{(\alpha-1)a})(1 + \zeta^b + \zeta^{2b} \ldots + \zeta^{(\bar{\alpha}-1)b}) = 1$.

Also note $\sum_\zeta \zeta^s = rz - 1$, where $z = \begin{cases} 0 & \text{, if } s \nmid r \\ 1 & \text{, if } s | r \end{cases}$.

So $r\delta_m = \sum_\zeta f(\zeta) \zeta^{m(a+b)}$, where

$$f(x) = 1 + (x^a + x^{2a} + \ldots + x^{\alpha a}) + (x^b + x^{2b} + \ldots + x^{\bar{\alpha} b}) + x^{a+b}$$

We have $r\delta_m = rZ_m - (\alpha + \bar{\alpha} + 2)$. The latter formula is obtained from $Z_m = Z_{-(m+1)}$. □

Remark 7.4. From $\tau_m = -\tau_{-m}$, we see directly $\chi(X, mK_X) = \chi(X, (1-m)K_X)$ for all m. This can be also derived from Serre duality as in [14, 5.27], which states for an klt log pair (X,Δ), and D any $\mathbb{Q}$-Cartier Weil divisor, X is CM and $H^i(X,\mathcal{O}_X(D))$ is dual to $H^{n-i}(X,\omega_X(-D))$ for all i.

For $m=0$, we have

$$Z_0(P)=\begin{cases}1, & \text{if } P \text{ is not a canonical singularity;}\\ 2, & \text{if } P \text{ is a canonical singularity.}\end{cases}$$

. From this we see canonical singularities do not affect Euler characteristics, and can be considered negligible afterward. Since $\tau_0=0$, $\varkappa_0-\varkappa_{-1}=\Delta\varkappa_{-1}=0$, $\delta_0=\tau_1$, we have

$$\Delta\varkappa_0=K_X^2+\sum_{P\in X}\delta_0(P)=K_X^2+\sum_{P\text{ is not canonical}}\frac{1}{r}(1-\alpha-\bar{\alpha}-2).$$

7.2 Euler characteristics under L-blowups

Given a surface X with cyclic quotient singularities, we let $Y\to X$ be its L-blowup. The purpose of this subsection is to compare Euler characteristics of these two surfaces.

Theorem 7.5. *Suppose $f:Y\to X$ is birational, both have only rational singularities . Then $\chi_0(Y)$=$\chi_0(X)$.*

Proof. Take a resolution $g:Z\to Y$, denote $h=f\circ g$, then $R^ig_*\mathcal{O}_Z=0, R^ih_*\mathcal{O}_Z=0$ for all $i>0$

By Larey spectral sequence, $R^pf_*R^qg_*\mathcal{O}_Z\Rightarrow R^nh_*\mathcal{O}_Z$, from this we see $R^if_*\mathcal{O}_Y=0$, for all $i>0$ □

Under general birational morphisms, $\varkappa_1$ may be changed. We prove that $\varkappa_1$ is preserved by L-blowups.

Theorem 7.6 (= Proposition 1.2). *If $f:Y\to X$ is an L-blowup, then $\chi(Y,-K_Y)=\chi(X,-K_X)$*

Proof. By induction, it suffices to prove this for a simple L-blowup. We thus assume that X has singularity P of type $\frac{1}{r}(1,b)$ and $\pi:Y\to X$ is a simple L-blowup so that Y has two singularities Q_1 and Q_2 of types $\frac{1}{a+1}(1,1)$ and $\frac{1}{b+c}(1,c)$ respectively, where $c=ab-r$.

The following calculation uses the identities of Hirzebruch-Jung continued fractions, as in Remark 2.41:

Denote

$$\begin{aligned} r &= \langle u_n, \ldots, u_1 \rangle, & a &= u_n, \\ b &= \langle u_{n-1}, \ldots, u_1 \rangle, & c &= \langle u_{n-2}, \ldots, u_1 \rangle, \\ \bar{b} &= \langle u_n, \ldots, u_2 \rangle, & k &= \langle u_{n-1}, \ldots, u_2 \rangle < b, \\ k_1 &= \langle u_{n-2}, \ldots, u_2 \rangle < c. \end{aligned}$$

Then we have

$$\begin{aligned} c &= ab - r, & b\bar{b} &= 1 + rk, \\ \bar{b} &= ak - k_1, & ck &= 1 + bk_1, \end{aligned}$$

Let c' is the least positive integer such that $cc' \equiv 1 \pmod{b+c}$. Then by $b + c = \langle u_{n-1} + 1, a_{n-2}, \ldots, u_1 \rangle$, $(k_1 + k)c = 1 + k_1(b + c)$. We find $c' = k_1 + k$. Combining these, we obtain

$$b\bar{b} + \bar{b}c - c'r = (1 + rk) + (ak - k_1)c - r(k + k_1) = 1 - rk_1 - ck_1 + a(1 + bk_1) = 1 + a. \quad (\dagger)$$

By singular Riemann-Roch:

$$\begin{aligned} \Delta\varkappa_0(Y) &= K_Y^2 + \delta_0(\frac{1}{a+1}(1,1)) + \delta_0(\frac{1}{b+c}(1,c)) + \sum_{P' \neq Q_1, Q_2} \delta_0(P') \\ \Delta\varkappa_0(X) &= K_X^2 + \delta_0(\frac{1}{r}(1,b)) + \sum_{P' \neq P} \delta_0(P') \end{aligned}$$

From this we see

$$\begin{aligned} \varkappa_1(Y) - \varkappa_1(X) &= \Delta\varkappa_0(Y) - \Delta\varkappa_0(X) \\ &= K_Y^2 - K_X^2 + 1 + \frac{b + \bar{b} + 2}{r} - \frac{c + c' + 2}{b + c} - \frac{4}{a+1}. \end{aligned}$$

By Proposition 2.36,

$$K_Y^2 - K_X^2 = -\frac{r}{(a+1)(b+c)}(\frac{a+1+b+c}{r} - 1)^2,$$

and it is reduced to prove

$$(a+1+b+c-r)^2 = (a+1)[(b+c)r + (b+c)(b+\bar{b}+2) - r(c+c'+2)] - 4(b+c)r. \quad (\ddagger)$$

Using $(\dagger)$ and $b + c + r = ab + b$,

$$\begin{aligned} \text{LHS of } (\ddagger) &= (a+1)^2 + (b+c+r)^2 + 2(a+1)(b+c-r) - 4(b+c)r \\ &= (a+1)^2 + (b+c+r)(ba+b) + 2(a+1)(b+c-r) - 4(b+c)r \\ &= (a+1)(a+1+bc+br+b^2+2b+2c-2r) - 4(b+c)r \\ &= \text{RHS of } (\ddagger) \end{aligned}$$

$\square$

Theorem 7.7. *Suppose X has a $\frac{1}{r}(1,b)$ point, and we blow up X along $\frac{1}{r}(s,t)$ to get Y. If*

$$\frac{m}{m+1} < \frac{s+t}{r} < \frac{m+1}{m},$$

then $\varkappa_{i,X} = \varkappa_{i,Y}$ for $i = 1, 2, \ldots, m$. In particular, if $s+t = r$, then $\varkappa_{i,X} = \varkappa_{i,Y}$ for all i.

Proof. By Proposition 2.36 $K_Y = \pi^* K_X + (\frac{s+t}{r} - 1)E$.

We claim that (Y, cE) is klt for $0 \leq c < 1$. By construction, it suffices to consider the toric singularity without loss of generality.

Let Z be the toric variety defined by a lattice $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z} \cdot \frac{1}{r}(1,b)$, the cone σ be the first quadrant, and E be the divisor corresponding to the x-axis. Recall the minimal resolution $f : W \to Z$ is obtained from blowing up along all vertices of the Newton polytope $P =$ convex hull of $N \cap \sigma$. We denote those vertices by $(0,1) = P_0 = (a_0, b_0), P_1 = (a_1, b_1), \ldots, P_k = (a_k, b_k), (1,0) = P_{k+1} = (a_{k+1}, b_{k+1})$, with $0 < a_1 < \ldots < a_k < 1$, and $1 > b_1 > \ldots > b_k > 0$. Then

$$K_W = f^* K_Z + \sum_{i=1}^{k} (a_i + b_i - 1) E_i$$

$$f^* E = \tilde{E} + \sum_{i=1}^{k} a_i E_i$$

where E_i is the exceptional divisor corresponding to P_i. Since $0 < a_i + b_i - 1 - c a_i < 1$, (Z, cE) is klt.

If $s + t \leq r$, write $-mK_Y = K_Y - (m+1)K_Y \equiv_f K_Y + ((m+1)(1 - \frac{s+t}{r})E$, then by Kawamata-Viehweg vanishing theorem, we have $R^i f_* \mathcal{O}_Y(-mK_Y) = 0$ for $i > 0$ if $(m+1)(1 - \frac{s+t}{r}) < 1$. In this case, $f_* \mathcal{O}_Y(-mK_Y) = \mathcal{O}_X(-mK_X)$ We find $\chi(Y, -mK_Y) = \chi(X, -mK_X)$.

If $s + t \geq r$, write$(m+1)K_Y = K_Y + mK_Y \equiv_f K_Y + m(\frac{s+t}{r} - 1)E)$, then by Kawamata-Viehweg vanishing theorem, we have $R^i f_* \mathcal{O}_Y((m+1)K_Y) = 0$ for $i > 0$ if $m(\frac{s+t}{r} - 1) < 1$. In this case, $f_* \mathcal{O}_Y((m+1)K_Y) = \mathcal{O}_X((m+1)K_X)$ We find $\chi(Y, (m+1)K_Y) = \chi(X, (m+1)K_X)$, and hence $\chi(Y, -mK_Y) = \chi(X, -mK_X)$ by Serre duality. $\square$

In the proof we used the following lemma. It is not a difficult result, but we do not know a proper reference. We include the proof for completeness.

Lemma 7.8. *Let $f : Y \to X$ be a birational morphism between normal varieties. Suppose K_X is $\mathbb{Q}$-Cartier, we write $K_Y \sim_{\mathbb{Q}} f^*K_X + E$, where $E = \sum a_{E_i} E_i$ is a $\mathbb{Q}$-divisor. Suppose $E \geq 0$ (resp. $-E \geq 0$). Then $f_*\mathcal{O}_Y(mK_Y) = \mathcal{O}_X(mK_X)$, for $m \geq 0$ (resp. $m \leq 0$)*

Proof. We may assume X is affine. We note that since X, Y are normal, there is $U \subseteq X$ such that $X \backslash U$ has codimension at least two, and $f|_{f^{-1}(U)}$ is an isomorphism. There is an natural inclusion $H^0(Y, \mathcal{O}_Y(mK_Y)) \hookrightarrow H^0(U, \mathcal{O}_U(mK_U)) = H^0(X, \mathcal{O}_X(mK_X))$ as subspaces of the space of rational sections of $\omega_{\kappa/k}^{\otimes m}$, where κ is the common function field of X and Y.

To prove equality, take a regular section $\omega \in H^0(X, \mathcal{O}_X(mK_X))$, and prove it in $H^0(Y, \mathcal{O}_Y(mK_Y))$. We focus on an exceptional prime divisor E_0 at a time. Now we may assume Y is affine, and E_0 corresponds to a height 1 prime p. We pick ω_0 as in the prove of Hurwitz formula. Then ω is regular at E_0 if and only if $\mathrm{ord}_p(\omega/\omega_0) \geq 0$.

Suppose $r \in \mathbb{N}$ such that rK_X is Cartier, and $rK_Y \sim f^*rK_X + rE$. We see $\omega^{\otimes r} \in H^0(X, \mathcal{O}_X(rmK_X))$ and thus $\mathrm{ord}_p(\omega^{\otimes r}/\omega_0^{\otimes r}) \geq mra_{E_0}$, i.e., $\mathrm{ord}_p(\omega/\omega_0) \geq ma_{E_0}$. From this we conclude that if $a_{E_0} \geq 0$ (resp. $a_{E_0} \leq 0$), then ω is regular at E_0 if $m \geq 0$ (resp. $m \leq 0$), and hence the proof. $\square$

8 Nonvanishing

Now let X be a surface with only quotient singularities $\{\frac{1}{r}(1,1)\}$. It is clear that

$$Z_m(\tfrac{1}{r}(1,1)) = \begin{cases} 1, & \text{if } m \equiv 0, \frac{r}{2}, \frac{r}{2}-1, -1 (\mathrm{mod}\ r) \\ 2, & \text{if } m \equiv \frac{r-1}{2} (\mathrm{mod} r) \\ 0, & \text{otherwise} \end{cases}$$

Let c_r be the number of singularities of type $\frac{1}{r}(1,1)$. From the singular Riemann-Roch theorem, we calculate the difference and second differences of $\varkappa$:

$$\Delta \varkappa_0 = K_X^2 + \sum_r c_r (1 - \frac{4}{r})$$

$$\Delta^2 \varkappa_m = K_X^2 + \sum_r c_r (Z_{m+1}(\frac{1}{r}(1,1)) - \frac{4}{r})$$

Written explicitly,

$$\begin{aligned}
\Delta^2\varkappa_0 &= K_X^2 + 2c_3 + c_4 - \sum_r c_r\frac{4}{r} \\
\Delta^2\varkappa_1 &= K_X^2 + c_3 + c_4 + 2c_5 + c_6 - \sum_r c_r\frac{4}{r} \\
\Delta^2\varkappa_2 &= K_X^2 + c_3 + c_4 + c_6 + 2c_7 + c_8 - \sum_r c_r\frac{4}{r} \\
\Delta^2\varkappa_3 &= K_X^2 + 2c_3 + c_4 + c_5 + c_8 + 2c_9 + c_{10} - \sum_r c_r\frac{4}{r} \ (*)
\end{aligned}$$

and so on. Combining the above formulae, we find

$$-2\varkappa_0 + \varkappa_1 + \varkappa_2 + \varkappa_3 - \varkappa_4 = 2\Delta\varkappa_0 - \Delta^2\varkappa_1 - \Delta^2\varkappa_2 = c_8 + 2\sum_{r\geq 9} c_r \geq 0. \qquad (**)$$

Therefore, we have the following:

Theorem 8.1. *If X is a surface with only $\frac{1}{r}(1,1)$ points, then*

$$-2\varkappa_0 + \varkappa_1 + \varkappa_2 + \varkappa_3 - \varkappa_4 \geq 0$$

Now we are able to prove the Main Theorem.

Proof of the Main Theorem. Suppose that X is a del Pezzo surface. Then $\varkappa_0 = 1$ and $\varkappa_m = h^0(-mK_X) \geq 0$ for $m \geq 0$ by Kawamata-Viehweg vanishing theorem. We also have $\varkappa_2 \leq \varkappa_4$. By 8.1, $\varkappa_1 + \varkappa_2 + \varkappa_3 - \varkappa_4 \geq 2$, and we get $\varkappa_1 + \varkappa_3 \geq 2$. □

Example 8.2. Examples of such X include the weighted projective space $\mathbb{P}(1,1,a)$, weighted complete intersections $X_{15} \subseteq \mathbb{P}(3,3,5,5)$ and $X_{6,6} \subseteq \mathbb{P}(2,2,3,3,3)$. In the latter cases $h^0(X,-K_X) = 0$, but $h^0(X,-3K_X) > 0$.

Using the same formulae, we see that del Pezzo with only $\frac{1}{r}(1,1)$ points have the following property.

Theorem 8.3. *If X is a del Pezzo surface with only $\frac{1}{r}(1,1)$ points, then $h^0(-mK_X) \geq 2$, for some $m \leq 5$.*

Proof. By $\varkappa_1 + \varkappa_3 \geq 2$, we have $h^0(-3K_X) = \varkappa_3 \geq 2$ if $h^0(-K_X) = 0$. We thus assume that $h^0(-K_X) > 0$. Hence we have $\varkappa_n > 0$ for all n. Suppose the contrary that $\varkappa_1 = \varkappa_2 = \varkappa_3 = \varkappa_4 = \varkappa_5 = 1$. Then by (**),

$$2 = \varkappa_1 + \varkappa_2 + \varkappa_3 - \varkappa_4 = 2 + c_8 + 2\sum_{r\geq 9} c_r.$$

So $c_r = 0$ for all $r \geq 8$. Now since $\Delta^2\varkappa_0 = \Delta^2\varkappa_1 = \Delta^2\varkappa_2 = \Delta^2\varkappa_3 = 0$. By (*), we have $c_5 = c_7 = 0$, and $c_3 = c_6$. But then

$$0 = \Delta\varkappa_0 = K_X^2 > 0,$$

a contradiction. □

We have the following generalization.

Theorem 8.4. *Suppose X is a surface with only singularities of types $\frac{1}{r}(1,1)$ and $\frac{1}{2s+1}(1,-2)$. Then*

$$\varkappa_2 + \varkappa_4 + \varkappa_6 \geq 1 + \varkappa_3 + \varkappa_7.$$

If moreover X is a del Pezzo surface, then $h^0(X, -mK_X) > 0$ for some $m = 2, 4$ or 6.

Proof. For odd number $r = 2s+1$,

$$Z_m(\frac{1}{r}(1, r-2)) = \begin{cases} 1, & \text{if } m \equiv 0, 2, 4, \ldots, r-1 (\mathrm{mod} r) \\ 2, & \text{if } m \equiv 1, 3, 5, \ldots, r-2 (\mathrm{mod} r) \end{cases}$$

Let c_r be the number of singularities of type $\frac{1}{r}(1,1)$, and d_s be the number of singularities of type $\frac{1}{2s+1}(1,-2)$. From the singular Riemann-Roch theorem, we calculate the difference and second differences of $\varkappa$:

$$\Delta\varkappa_0 = K_X^2 + \sum_r c_r(1 - \frac{4}{r}) + \sum_s d_s(1 - \frac{3s+1}{2s+1})$$

$$\Delta^2\varkappa_m = K_X^2 + \sum_r c_r(Z_{m+1}(\frac{1}{r}(1,1)) - \frac{4}{r}) + \sum_s d_s(Z_m(\frac{1}{2s+1}(1,-2)) - \frac{3s+1}{2s+1})$$

Denote $A = K_X^2 - \sum_r \frac{4}{r}c_r - \sum_s \frac{3s+1}{2s+1}d_s, d' = \sum_{s\geq 3} d_s$, we find in particular,

$$\begin{aligned}
\Delta\varkappa_0 &= A + \sum_r c_r + d_2 + d' \\
\Delta^2\varkappa_0 &= A + 2c_3 + c_4 + 2d_2 + 2d' \\
\Delta^2\varkappa_3 &= A + 2c_3 + c_4 + c_5 + c_8 + 2c_9 + c_{10} + d_2 + d' \\
\Delta^2\varkappa_4 &= A + c_3 + c_4 + c_5 + c_6 + c_{10} + 2c_{11} + c_{12} + d_2 + 2d' \\
\Delta^2\varkappa_5 &= A + c_3 + c_4 + c_6 + c_7 + c_{12} + 2c_{13} + c_{14} + 2d_2 + d'
\end{aligned}$$

which yields

$$2\Delta\varkappa_0 + \Delta^2\varkappa_0 - (\Delta^2\varkappa_3 + \Delta^2\varkappa_4 + \Delta^2\varkappa_5) = c_7 + c_8 + c_{14} + 2\sum_{r\geq 15} c_r \geq 0$$

Hence,

$$\varkappa_2 + \varkappa_4 + \varkappa_6 \geq 1 + \varkappa_3 + \varkappa_7$$

The last assertion follows by $\varkappa_m = h^0(X, -mK_X) \geq 0$ for $m \geq 0$. $\square$

In general, we want to construct similar inequalities for $\Delta^2\varkappa$ of the form

$$\sum_i \lambda_i \Delta^2 \varkappa_{m_i} \geq 0,$$

where $\sum_i \lambda_i = 0$ This is equivalent to $\sum_i \lambda_i Z_{m_i}(\frac{1}{r}(1,b) \geq 0$. However, we find for a $\frac{1}{r}(1,b)$ point,

$$Z_m(\frac{1}{r}(1,b)) = \begin{cases} 1, & \text{if } r|m(1+b) \\ 1, & \text{if } r|(m+1)(1+b) \\ 1_{\{(m+1)(\frac{1+b}{r})\}<\frac{1+b}{r}} + 1_{\{(m+1)(\frac{1+\bar{b}}{r})\}<\frac{1+\bar{b}}{r}}, & \text{otherwise.} \end{cases}$$

By using Hirzebruch-Jung continued fractions we find the set $\{(\frac{b}{r}, \frac{\bar{b}}{r}) \mid 0 < b < r, (r,b) = 1\}$ and hence $\{(\frac{1+b}{r}, \frac{1+\bar{b}}{r}) \mid 0 < b < r, (r,b) = 1\}$ is dense in $[0,1] \times [0,1]$. Indeed, given a rational point $(x,y) \in (0,1) \times (0,1)$, we represent them by Hirzebruch-Jung continued fractions as

$$x = \frac{\langle u_0, \ldots, u_{n-1} \rangle}{\langle u_0, \ldots, u_n \rangle}, y = \frac{\langle v_0, \ldots, v_{n'-1} \rangle}{\langle v_0, \ldots, v_{n'} \rangle}.$$

Consider $r_M = \langle u_0, \ldots, u_n, M, v_{n'}, \ldots, v_0 \rangle$, $b_M = \langle u_0, \ldots, u_n, M, v_{n'}, \ldots, v_1 \rangle$, and $\bar{b}_M = \langle v_1, \ldots, v_{n'}, M, u_n, \ldots, u_0 \rangle$. We have $(\frac{b_M}{r_M}, \frac{\bar{b_M}}{r_M}) \to (x,y)$ as $M \to \infty$.

Since for $0 < x < 1$, with $mx \notin \mathbb{Z}$, $\{mx\} < x$ if and only if

$$x \in (\frac{1}{m}, \frac{1}{m-1}) \cup (\frac{2}{m}, \frac{2}{m-1}) \cup \ldots \cup (\frac{m-2}{m}, \frac{m-2}{m-1}) \cup (\frac{m-1}{m}, 1)$$

This means for any finite sequence z_m of $0, 1, 2$ with $z_0 = 1$, there are infinitely many pairs (r,b) such that $Z_m(\frac{1}{r}(1,b)) = z_m$. It seems to prevent us from constructing similar inequalities.

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