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柯西黎曼 Li-Yau-Hamilton 不等式及其應用

CR Li-Yau-Hamilton Inequality and its Applications

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摘 要

這篇文章包含三大部分，第一部分證明矩陣形式的 Li-Yau-Hamilton Harnack 不等式。第二部份延續第一部分的工作，推廣至(1,1)-form 形式的 Li-Yau-Hamilton Harnack 不等式。第三部份將應用這不等式証明柯西黎曼上的 Gap 定理。

關鍵字：擬埃爾米特，Li-Yau-Hamilton，Gap 定理，Harnack 不等式

1. Abstract

In the first part of thesis, we first derive the CR analogue of matrix Li-Yau-Hamilton inequality for a positive solution to the CR heat equation in a closed pseudohermitian $(2n+1)$-manifold with nonnegative bisectional curvature and bitorsional tensor. We then obtain the CR Li-Yau gradient estimate in a standard Heisenberg group. Finally, we extend the CR matrix Li-Yau-Hamilton inequality to the case of Heisenberg groups. As a consequence, we derive the Hessian comparison property in the standard Heisenberg group.

In the second part, we study the CR Lichnerowicz-Laplacian heat equation deformation of $(1,1)$-tensors on a complete strictly pseudoconvex CR $(2n+1)$-manifold and derive the linear trace version of Li-Yau-Hamilton inequality for positive solutions of the CR Lichnerowicz-Laplacian heat equation. We also obtain a nonlinear version of Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation coupled with the CR Yamabe flow and trace Harnack inequality for the CR Yamabe flow.

In the last part, by applying a linear trace Li-Yau-Hamilton inequality for a positive $(1,1)$-form solution of the CR Hodge-Laplace heat equation and monotonicity of the heat equation deformation, we obtain an optimal gap theorem for a complete strictly pseudocovex CR $(2n+1)$-manifold with nonnegative pseudohermitian bisectional curvature and vanishing torsion. We prove that if the average of the Tanaka-Webster scalar curvature over a ball of radius r centered at some point o decays as $o\left(r^{-2}\right)$, then the manifold is flat.

Contents

2. Introduction

We briefly introduce our works and results.

In the seminal paper [LY], P. Li and S.-T. Yau established the parabolic Li-Yau Harnack estimate for the positive solution $u(x,t)$ of the time-independent heat equation

$$\frac{\partial u\left(x,t\right)}{\partial t}=\Delta u\left(x,t\right) \tag{2.1}$$

in a complete Riemannian l-manifold with nonnegative Ricci curvature. Here Δ is the Laplace-Beltrami operator. Later in [H2], Richard Hamilton extended the Li-Yau estimate to the full matrix version of the Hessian estimate of u under the stronger assumptions that M is Ricci parallel and of nonnegative sectional curvature. Furthermore, Hamilton ([H1]) proved the matrix Harnack inequality for solutions to the Ricci flow

$$\frac{\partial g_{ij}\left(x,t\right)}{\partial t}=-2R_{ij}\left(x,t\right) \tag{2.2}$$

when the curvature operator is nonnegative. This inequality is called the “Li-Yau-Hamilton” type estimates. Since then, there are many additional works in this direction which cover various different geometric evolution equations such as the mean curvature flow ([H2]), the Kähler-Ricci flow ([Ca]), the Yamabe flow ([C]), etc.

On the other hand, the Kähler-Ricci curvature $(1,1)$-tensor of a Kähler-Ricci flow solution satisfies a Lichnerowicz-Laplacian heat equation. In general, the Hodge-Laplacian heat equation on symmetric (p,p)-tensors is a geometrically interesting system and has been extensively studied since the original works of Hodge and Kodaira ([Mo] and references therein). For instances, we refer to the Lichnerowicz-Laplacian heat equation on $(1,1)$-tensors and the Hodge-Laplacian heat equation on (p,p)-tensors as in [NN].

Along this line with method of Li-Yau gradient estimate, H.-D. Cao and S.-T. Yau ([CY]) studied the heat equation

$$\frac{\partial u\left(x,t\right)}{\partial t}=Lu(x,t) \tag{2.3}$$

in a closed l-manifold with a positive measure and a subelliptic operator with respect to the sum of squares of vector fields $L=\sum_{i=1}^{h}X_i^2-Y$, $h\leq l$ with $Y=\sum_{i=1}^{h}c_iX_i$ where $X_1,\ X_2,\ ...,X_h$ are smooth vector fields satisfying Hörmander's condition : the vector fields together with their commutators up to finite order span the tangent space at every point of M. Suppose that $[X_i,[X_j,X_k]]$ can be expressed as linear combinations of $X_1,\ X_2,\ ...,X_h$ and their brackets $[X_1,X_2],...,[X_{l-1},X_h]$. They showed that the gradient estimate for the positive solution $u(x,t)$ of (2.3) on $M\times[0,\infty)$.

In the first part of this paper, we focus on the CR Li-Yau-Hamilton type gradient estimate for the positive solution $u(x,t)$ of the CR heat equation

$$\frac{\partial u\left(x,t\right)}{\partial t}=\Delta_b u\left(x,t\right). \tag{2.4}$$

and for the positive symmetric $(1,1)$-form $\eta(x,t)$ of CR Lichnerowicz-Laplacian heat equation

$$\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}=4\left[\Delta_b\eta_{\alpha\bar{\beta}}+2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}-(R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma}}+R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}})\right]. \tag{2.5}$$

On the other hand, we are also interested in the coupled heat equation. Let $\theta(t)$ be a family of smooth contact forms and $J(t)$ be a family of CR structures on $(M^3, J_0, \mathring{\theta})$ with $J(0)=J_0$ and $\theta(0)=\mathring{\theta}$. In the paper of [CKW], we consider the following torsion flow which is the CR analogue of the Hamilton Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}J=2A_{J,\theta}, \\ \frac{\partial}{\partial t}\theta=-2R\theta. \end{cases} \tag{2.6}$$

on $M\times[0,T)$ with $J(t)=i\theta^1\otimes Z_1-i\theta^{\bar{1}}\otimes Z_{\bar{1}}$ and $A_{J,\theta}(t)=A_{11}\theta^1\otimes Z_{\bar{1}}+A_{\bar{1}\bar{1}}\theta^{\bar{1}}\otimes Z_1$. In particular if the initial torsion is vanishing, the torsion flow (2.6) is equivalent to the CR Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t}\theta\left(t\right)=-2R\left(t\right)\theta\left(t\right), \\ \theta\left(0\right)=\mathring{\theta}, \end{cases} \tag{2.7}$$

in a closed CR 3-manifold. We will study the Li-Yau-Hamilton inequality for the coupled CR Yamabe flow as well.

Finally, we recall some definitions as followings.

Definition 2.1. *Let (M,J,θ) be a closed pseudohermitian 3-manifold. We call a CR structure J spherical if Cartan curvature tensor Q_{11} vanishes identically. Here*

$$Q_{11}=\frac{1}{6}W_{11}+\frac{i}{2}WA_{11}-A_{11,0}-\frac{2i}{3}A_{11,\bar{1}1}.$$

Note that (M,J,θ) is called a closed spherical pseudohermitian 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical pseudohermitian 3-manifold (M,J,θ) is locally CR equivalent to $(\mathbf{S}^3,\widehat{J},\widehat{\theta})$.

Definition 2.2. *Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold with $\xi=\ker\theta$. A piecewise smooth curve $\gamma:[0,1]\to M$ is said to be a Legendrian curve if $\dot{\gamma}(\tau)\in\xi$ whenever $\dot{\gamma}(\tau)$ exists. The length of γ is then defined by*

$$l(\gamma)=\int_0^1(\langle\ \dot{\gamma}(\tau),\ \dot{\gamma}(\tau)\ \rangle_{L_\theta})^{\frac{1}{2}}d\tau.$$

The Carnot-Carathéodory distance d_{cc} between two points $p,q\in M$ is defined by

$$d_{cc}(p,q)=inf\{l(\gamma)|\ \gamma\in C_{p,q}\},$$

where $C_{p,q}$ is the set of all Legendrian curves which join p and q.

2.1. **CR Li-Yau Gradient Estimate and Harnack Inequality.** Let u be the positive solution of (2.4) and denote

$$f\left(x,t\right)=\ln u\left(x,t\right).$$

Then $f\left(x,t\right)$ satisfies the equation

$$\left(\Delta_b-\frac{\partial}{\partial t}\right)f\left(x,t\right)=-\left|\nabla_b f\left(x,t\right)\right|^2. \tag{2.8}$$

We observe that one of difficulties is to deal with CR Bochner formula which involving a term $\langle J\nabla_b f,\ \nabla_b f_0\rangle$ that has no analogue in the Riemannian case. In order to overcome this difficulty, we introduce a new scalar Harnack quantity

$$F(x,t,\ a,\ c) = t\left(|\nabla_b f|^2(x) + af_t + ctf_0^2(x)\right), \tag{2.9}$$

with $f = \ln u$ by adding an extra term tf_0^2 to $|\nabla_b f|^2 + af_t$ which was appeared in Li-Yau estimate ([LY]). Then one can derive CR versions of Li-Yau gradient estimates and classical Harnack inequality.

Theorem 2.1. *([CKL1]) Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$2Ric(X,X) - (n-2)Tor(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If $u(x,t)$ is the positive solution of

$$\left(\Delta_b - \frac{\partial}{\partial t}\right)u(x,t) = 0$$

with

$$[\Delta_b,\ \mathbf{T}]\,u = 0$$

on $M \times [0,\ \infty)$. Then $f(x,t) = \ln u(x,t)$ satisfies the following subgradient estimate

$$\left[|\nabla_b f|^2 - (1+\frac{3}{n})f_t + \frac{n}{3}t(f_0)^2\right] < \frac{(\frac{9}{n}+6+n)}{t}. \tag{2.10}$$

When the manifold is complete noncompact, the proof of CR Li-Yau gradient estimate (2.11) relies on the CR sub-Laplacian comparison property and the extra u_0-growth property with $|u_0| \leq \frac{C}{t}u$ that has no analogue in the Riemannian case. However, both properties holds in a standard Heisenberg group $\mathbf{H}^n$ which is flat and vanishing torsion. Then we are able to derive the following CR Li-Yau gradient estimate on $\mathbf{H}^n$.

Theorem 2.2. *([CFTW]) Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (2.4) on $\boldsymbol{H}^n \times [0,\infty)$. Let $\varphi = \ln u$, for any $\alpha < -1$, then there exists a positive constant C depending on α such that*

$$|\nabla_b\varphi|^2 + \alpha\varphi_t + t\varphi_0^2 \leq \tfrac{C}{t}. \tag{2.11}$$

By applying Theorem 2.2, we have the following CR Liouville-type theorem for a positive pseudoharmonic function u on $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$ which recaptured the Liouville theorem due to Chang-Kuo-Tie [CKT] and Koranyi and Stanton ([KS]) by a different method.

Corollary 2.3. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive smooth function with $\Delta_b u = 0$, then $u(x,t)$ is constant. That is, there does not exist any positive nonconstant pseudoharmonic function in $\boldsymbol{H}^n$.*

By using the method of CR Li-Yau gradient estimate ([LY], [CKL1]) and CR Bochner formula, we derive a CR gradient estimate and CR Harnack inequality for the positive solution of the CR heat equation (2.4) in $(2n+1)$-dimensional Heisenberg group.

Corollary 2.4. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (2.4) on $\boldsymbol{H}^n \times [0,\infty)$, we have the Harnack inequality*

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \left(\frac{t_2}{t_1}\right)^C \exp\left(\frac{d_c(x_1,x_2)^2}{2(t_2-t_1)}\right) \tag{2.12}$$

for any x_1, x_2 in $\boldsymbol{H}^n$ and $0 < t_1 < t_2 < \infty$, where $d_c(x_1,x_2)$ is the Carnot-Carathéodory distance between x_1 and x_2.

As a consequence of Corollary 2.4 and [CY], we have the following upper bound estimate for the heat kernel of (2.4).

Corollary 2.5. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group and $H(x,y,t)$ be the heat kernel of (2.4) on $M \times [0,\infty)$. Then for some constant $\delta > 1$ and $0 < \epsilon < 1$, $H(x,y,t)$ satisfies the estimate*

$$H(x,y,t) \leq C(\epsilon)^{\delta} V^{-\frac{1}{2}}(B_x(\sqrt{t}))V^{-\frac{1}{2}}(B_y(\sqrt{t})) \exp\left(-\frac{d_{cc}^2(x,y)}{(4+\epsilon)t}\right) \tag{2.13}$$

with $C(\epsilon) \to \infty$ as $\epsilon \to 0$.

Once we have the upper bound estimate for the heat kernel and the sub-laplacian comparison property, then by applying the arguments of Li-Tam as in [LT] or [Li], we have the following mean value inequality.

Corollary 2.6. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group and g be subsolution of the CR heat equation such that*

$$\left(\frac{\partial}{\partial t} - \Delta_b\right) g\,(x,t) \leq 0.$$

Then for some constant C depend on δ, τ, η, such that $0 < \delta < 1$, $0 < \tau < T$, $0 < \eta < \frac{1}{2}$, the following inequality holds for any $\rho > 2\sqrt{T}$,

$$\sup_{B_p((1-\delta)\rho)\times[\tau,T]} g \leq C \int_{(1-\eta)\tau}^{T} \int_{B_p(\rho)} g\,(y,s)\,dyds. \tag{2.14}$$

2.2. **CR Matrix Li-Yau-Hamilton Inequality.** Let $u(x,t)$ be the positive solution of the CR heat equation (2.4). For the CR Li-Yau gradient estimate as in the paper [CKL1], we observe that one of difficulties is to deal with CR Bochner formula which involving a term $\langle J\nabla_b f,\ \nabla_b f_0\rangle$ that has no analogue in the Riemannian case. In order to overcome this difficulty, we introduce a new scalar Harnack quantity $F = t[|\nabla_b f|^2 + \alpha f_t + t f_0^2]$ with $f = \ln u$ by adding an extra term $t f_0^2$ to $|\nabla_b f|^2 + \alpha f_t$ which was appeared in Li-Yau estimate ([LY]).

Now we want to find the right quantity for the CR matrix Li-Yau-Hamilton inequality. By comparing the Harnack quantity in [CN] in the case of Kähler manifolds, we define the matrix Harnack quantity

$$N_{\alpha\bar{\beta}} = \frac{1}{2}(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) + 2\frac{u}{t}h_{\alpha\bar{\beta}} - b\frac{u_\alpha u_{\bar{\beta}}}{u} - at\frac{|u_0|^2}{u}h_{\alpha\bar{\beta}} \tag{2.15}$$

by adding an extra term $F := -at\frac{|u_0|^2}{u}h_{\alpha\bar{\beta}}$ in which positive constants a and b to be determined later (say $a = \frac{1}{24}$and $b = \frac{1}{4}$).

Definition 2.3. ([GL]) *Define the purely holomorphic Hessian operator* $P_{\alpha\bar{\beta}}$ *:*

$$P_{\alpha\bar{\beta}}\varphi := -2i(A_{\alpha\gamma}\varphi_{\bar{\gamma}})_{\bar{\beta}}$$

and the purely holomorphic Poisson operator Q *:*

$$Q\varphi := h^{\alpha\bar{\beta}}(P_{\alpha\bar{\beta}}\varphi) = -2i(A_{\alpha\gamma}\varphi_{\bar{\gamma}})_{\bar{\alpha}}$$

for any smooth function φ*. Note that* $P_{\alpha\bar{\beta}}\varphi = 0 = Q\varphi$ *for any smooth function* φ *if* $A_{\alpha\beta} = 0$ *on* M*.*

Then, based on the following key estimate, we have Theorem 2.7.

Lemma 2.1. *Let* $u(x,t)$ *be the positive solution of the CR heat equation (2.4). Then* $\frac{1}{2}(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha})$ *satisfies the following :*

$$\frac{1}{2}\left(\frac{\partial}{\partial t} - \Delta_b\right)\left(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}\right) = 2R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_{\gamma\bar{\delta}} - R_{\alpha\bar{\delta}}u_{\delta\bar{\beta}} - R_{\delta\bar{\beta}}u_{\alpha\bar{\delta}} + C_{\alpha\bar{\beta}},$$

where

$$\begin{aligned} C_{\alpha\bar{\beta}} &:= i\left(A_{\gamma\delta,\bar{\delta}}u_{\bar{\gamma}} - A_{\bar{\gamma}\bar{\delta},\delta}u_{\gamma}\right)h_{\alpha\bar{\beta}} + i\left(A_{\gamma\delta}u_{\bar{\gamma}\bar{\delta}} - A_{\bar{\gamma}\bar{\delta}}u_{\delta\gamma}\right)h_{\alpha\bar{\beta}} \\ &\quad + in\left(A_{\bar{\gamma}\bar{\beta}}u_{\alpha\gamma} - A_{\alpha\gamma}u_{\bar{\gamma}\bar{\beta}}\right) + in\left(A_{\bar{\gamma}\bar{\beta},\alpha}u_{\gamma} - A_{\gamma\alpha,\bar{\beta}}u_{\bar{\gamma}}\right) \\ &:= -(\operatorname{Re}Qu)h_{\alpha\bar{\beta}} + n(\operatorname{Re}P_{\alpha\bar{\beta}}u). \end{aligned}$$

Note that $trC_{\alpha\bar{\beta}} = h^{\alpha\bar{\beta}}C_{\alpha\bar{\beta}} = 0$*. In particular we have* $C_{1\bar{1}} = 0$ *for* $n = 1$*. In addition if the positive solution* u *satisfies* $P_{\alpha\bar{\beta}}u = 0$ *which is the case when the torsion is vanishing, then* $u_{\alpha\bar{\beta}}$ *satisfies the following CR Lichnerowicz-Laplacian heat equation* ([CCF]) *:*

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)u_{\alpha\bar{\beta}} = 2R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_{\gamma\bar{\delta}} - R_{\alpha\bar{\delta}}u_{\delta\bar{\beta}} - R_{\delta\bar{\beta}}u_{\alpha\bar{\delta}}.$$

Hence we have the following CR analogue of matrix Li-Yau-Hamilton inequality for any positive solution u to (2.4).

Theorem 2.7. ([CFTW]) *Let* M *be a closed pseudohermitian* $(2n+1)$*-manifold with nonnegative bisectional curvature and nonnegative bi-torsional tensor. Let* u *be the positive solution of the CR heat equation (2.4). In addition if the positive solution* u *satisfies the purely holomorphic Hessian operator* $P_{\alpha\bar{\beta}}u = 0$*. Then*

$$(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) + \frac{1}{2}[(u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}}V_\alpha) + uV_\alpha V_{\bar{\beta}}] - \frac{t}{12}\frac{|u_0|^2}{u}h_{\alpha\bar{\beta}} + \frac{4}{t}uh_{\alpha\bar{\beta}} \geq 0 \tag{2.16}$$

for $t > 0$ *and any vector filed* $V = V_\alpha$ *of type* $(1,0)$ *on* M*. Here* $P_{\alpha\bar{\beta}}$ *is the purely holomorphic Hessian operator (Definition 2.3). In particular, the CR matrix Li-Yau-Hamilton inequality (2.16) holds in a closed pseudohermitian* $(2n+1)$*-manifold of nonnegative bisectional curvature and vanishing torsion. If we choose the optimal* $V = -\nabla u/u$ *and take the trace of (2.16), we recapture the CR Li-Yau gradient estimate (2.10).*

When the manifold is complete noncompact, we will need to use the CR Li-Yau Harnack inequality (2.12) and Li-Tam mean value inequality (2.14) in the proof of the CR matrix Li-Yau-Hamilton inequality (2.16). However, both estimates hold in a standard Heisenberg group $\mathbf{H}^n$ which is flat and vanishing torsion. Then as a consequence of Theorem 2.7, we are able to derive the following CR matrix Li-Yau-Hamilton inequality on $\mathbf{H}^n$.

Theorem 2.8. *(*[CFTW]*) Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (2.4) on $\boldsymbol{H}^n \times [0,\infty)$. Then the CR matrix Li-Yau-Hamilton inequality (2.16) holds.*

By applying Theorem 2.8 to the heat kernel $H(x,y,t)$ with $V = -\frac{\nabla H}{H}$, we have the following complex Hessian comparison theorem for r on $\mathbf{H}^n$. Such a Hessian comparison property seems to be new in the standard $(2n+1)$-dimensional Heisenberg group $\boldsymbol{H}^n$.

Corollary 2.9. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. Then in the sense of distribution, we have*

$$[(r^2(x))_{\alpha\overline{\beta}} + (r^2(x))_{\overline{\beta}\alpha}] \leq (16 + C_0) h_{\alpha\overline{\beta}}(x)$$

for some constant C_0. In particular, we recapture the sub-Laplacian comparison property

$$\Delta_b r^2(x) \leq (16 + C_0) n$$

in the Heisenberg group.

2.3. **CR Linear Trace Li-Yau-Hamilton Inequality and Gap Theorem.** We now consider the CR Hodge-Laplacian

$$\Delta_H = -\frac{1}{2}(\Box_b + \overline{\Box}_b)$$

for Kohn-Rossi Laplacian $\Box_b$. For any $(1,1)$-form $\eta(x,t) = \eta_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$, we study the CR Hodge-Laplacian heat equation on $M \times [0,T)$

$$\frac{\partial}{\partial t}\eta(x,t) = 4\Delta_H \eta(x,t)$$

in which connects to the existence problem of pseudo-Einstein CR $(2n+1)$-manifolds with $n \geq 2$. It follows from the CR Bochner-Weitzenbock Formula that the CR parabolic equation above is equivalent to the CR analogue of Lichnerowicz-Laplacian heat equation (2.5).

Define the Harnack quadratic by

(2.17)
$$Z(x,t)(V) := k_1 \left[\frac{1}{2}\left((\operatorname{div}\eta)_{\alpha,\bar{\alpha}} + (\operatorname{div}\eta)_{\bar{\alpha},\alpha}\right) + (\operatorname{div}\eta)_\alpha V_{\bar{\alpha}} + (\operatorname{div}\eta)_\alpha V_{\bar{\alpha}} + V_{\bar{\alpha}} V_\beta \eta_{\alpha\bar{\beta}}\right] + \frac{H}{t}$$

for any vector field $V \in T^{1,0}(M)$, $H = h^{\alpha\overline{\beta}}\eta_{\alpha\overline{\beta}}$ and k_1 to be determined later. Moreover, $\eta_{\alpha\bar{\beta},0}$ is denoted the component of covariant derivative of the tensor η with Reeb vector field T.

The following is linear trace Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation.

Theorem 2.10. *([CCF]) Let (M, J, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a nonnegative symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (2.5) on $M \times [0,T)$ and $\eta_{\alpha\bar{\beta},0}(x,0) = 0$ at $t = 0$. In additional if M is complete noncompact, we assume that there exists a constant $a > 0$ such that*

$$\int_{\delta}^{T} \int_{M} e^{-ar^2} \|\eta(x,t)\|^2 \, d\mu dt < \infty$$

and

$$\int_{\delta}^{T} \int_{M} e^{-ar^2} \|\nabla_T \eta(x,t)\|^2 \, d\mu dt < \infty,$$

where $r(x)$ is the Carnot-Carathéodory distance from a fixed point o and any $\delta > 0$. Then

$$Z(x,t) \geq 0,$$

for $0 < k_1 \leq 8$.

Then, based on the linear trace Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation. Lichnerowicz-Laplacian heat equation and then CR monotonicity of heat equation deformation of positive $(1,1)$-forms, we have the following CR gap Theorem.

Theorem 2.11. *([CF]) Let M be a complete noncompact strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. Then M is flat if*

$$\frac{1}{V_o(r)} \int_{B_o(r)} R(y) \, d\mu(y) = o\left(r^{-2}\right), \tag{2.18}$$

for some point $o \in M$. Here $R(y)$ is the Tanaka-Webster scalar curvature and $V_o(r)$ is the volume of the ball $B_o(r)$ with respect to the Carnot-Carathéodory distance. As a consequence if M is not flat, then

$$\liminf_{r \longrightarrow \infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} R(y) \, d\mu(y) > 0$$

for any $o \in M$.

2.4. **The Coupled CR Yamabe Flow.** We first study the following time-dependent CR heat equations with potentials

$$\frac{\partial u}{\partial t} = 4\Delta_b u - cRu \tag{2.19}$$

evolving by the CR Yamabe flow on $M \times [0,T)$. Here Δ_b is the time-depending sublaplacian and $R(t)$ is the Tanaka-Webster scalar curvature with respect to the contact form $\theta(t)$. We will derive differential Harnack estimates for positive solutions of (2.19) for $c = -2$.

We also present its application of Theorem 2.7 to obtain the nonlinear version of Harnack inequality for CR Lichnerowicz-Laplacian heat equation (2.5) coupled with the CR Yamabe flow (5.15).

We expect our Harnack estimate will play an important role in the study of the CR Yamabe flow. There are geometric quantities (for example the Tanaka-Webster scalar curvature) which satisfy equation (2.20) under the CR Yamabe flow in a closed CR 3-manifold. Indeed, these estimates can be used for understanding the singular models of positive Tanaka-Webster curvature under the CR Yamabe flow. In particular, this estimate should be useful in understanding the Yamabe solitons which one expects to model finite time singularities of the CR Yamabe flow.

Now we deal with $c=-2$ in (2.19). In particular, it follows that for $u=R$

$$\frac{\partial R}{\partial t}=4\Delta_t R+2R^2. \tag{2.20}$$

Then we have the CR Li-Yau-Hamilton inequality of the Yamabe flow (5.15). That is

Theorem 2.12. *([CCK]) Let $(M,J,\mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then under the CR Yamabe flow (5.15),*

$$4\frac{|\nabla_b R|^2}{R^2}-\frac{R_t}{R}-\frac{1}{t}\leq 0. \tag{2.21}$$

Furthermore, we get a subgradient estimate of logarithm of the positive Tanaka-Webster curvature

$$\frac{|\nabla_b R|^2}{R^2}\leq\frac{1}{4t}$$

for all $t\in(0,T)$.

Remark 2.1. *1. Let (M,J,θ) be a closed pseudohermitian 3-manifold. We call a CR structure J spherical if Cartan curvature tensor Q_{11} vanishes identically. Here*

$$Q_{11}=\frac{1}{6}W_{11}+\frac{i}{2}WA_{11}-A_{11,0}-\frac{2i}{3}A_{11,\bar{1}1}.$$

Note that (M,J,θ) is called a closed spherical pseudohermitian 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical pseudohermitian 3-manifold (M,J,θ) is locally CR equivalent to $(\mathbf{S}^3,\widehat{J},\widehat{\theta})$.

2. If $(M,J,\mathring{\theta})$ is a closed pseudohermitian 3-manifold with $\mathring{A}_{11}=0$, then $R_0(x,0)=0$ by the CR Bianchi identity. In additional if (M,J) is spherical, then under the CR Yamabe flow (5.15), $R_0(x,t)=0$ for all t.

By Chow connectivity theorem, there always exists a Legendrian curve joining any two points p and q, so the distance is finite. Now integrating (2.21) over $(\gamma(t),t)$ of a Legendrian path $\gamma:[t_1,t_2]\to M$ joining points x_1, x_2 in M, we obtain the following CR Harnack inequality for the positive Tanaka-Webster curvature under the CR Yamabe flow.

Corollary 2.13. *Let $(M,J,\mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then under the CR Yamabe flow (5.15), we have for all points x_1, x_2 in M and times $t_1<t_2$,*

$$R(x_1,t_1)\leq\left(\frac{t_2}{t_1}\right)^{\frac{64}{63}}R(x_2,t_2)\exp\left(\frac{1}{2}L\right),$$

where

$$L=\inf_{\gamma}\int_{t_1}^{t_2}\left(R+\frac{1}{8}|\dot{\gamma}|^2_{J,\theta_{(t)}}\right)dt$$

and the infimum is taken over all Legendrian paths γ with $\gamma(t_1)=x_1$ and $\gamma(t_2)=x_2$.

Finally, in the papers of B. Chow and R. Hamilton [CH], L. Ni and L.-F. Tam [NT1] proved the nonlinear trace Li-Yau-Hamilton inequality for the coupled the Ricci flow and Kaehler Ricci flow, respectively. Here we present its application of Theorem 2.7 to obtain the nonlinear version of Li-Yau-Hamilton inequality for CR Lichnerowicz-Laplacian heat equation (2.5) coupled with the CR Yamabe flow (5.15).

Theorem 2.14. *([CCF]) Let $(M, J, \overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Let $\eta_{1\bar{1}}(x,t)$ be a positive symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (2.5) coupled with the CR Yamabe flow (5.15) on $M \times [0,T)$ and $\eta_{1\bar{1},0} = 0$ for all t. Then*

$$Z_R := Z + RH \geq 0$$

on $M \times [0,T)$ for $k_1 = 4$. In particular, taking $V = 0$

$$2\Delta_b \eta_{1\bar{1}} + (R + \frac{1}{t})H \geq 0$$

and

$$\frac{\partial}{\partial t}\eta_{1\bar{1}} + 2(R + \frac{1}{t})H \geq 0 \tag{2.22}$$

with $H = h^{1\bar{1}}\eta_{1\bar{1}}$.

As a consequence of Theorem 2.14 with $\eta_{1\bar{1}} = R_{1\bar{1}} = Rh_{1\bar{1}}$, we have the following trace Harnack inequality for the CR Yamabe flow (5.15) which turns out to be a special case of the linear Harnack inequality for the CR Lichnerowicz-Laplacian heat equation (2.5) coupled with the CR Yamabe flow (5.15) on a closed strictly pseudoconvex spherical CR 3-manifold. This is the same as (2.21).

Corollary 2.15. *Let $(M, J, \overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then we have the following trace Harnack inequality for the CR Yamabe flow (5.15)*

$$\frac{\partial}{\partial t}(t^2 R) \geq 0.$$

Finally, we point out that, by applying Hamilton's general method, one can obtain the Harnack inequalities ([H1], [C]) to the CR Yamabe flow.

Theorem 2.16. *([CCF]) Let $(M, J, \overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then under the CR Yamabe flow*

$$\frac{\partial R}{\partial t} + \frac{2R}{t} + 2\left\langle \nabla_b R, V \right\rangle_{J,\theta} + \frac{3}{40} R \left\| V \right\|_{J,\theta}^2 \geq 0 \tag{2.23}$$

for any $V \in T^{1,0}(M)$.

It is our hope that the similar nonlinear trace Li-Yau-Hamilton (2.22) holds as well for the torsion flow (2.6) in a closed pseudohermitian 3-manifold.

3. Preliminary

First we introduce some basic materials in a pseudohermitian $(2n+1)$-manifold (see [L1], [L2] for more details). Let (M,ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with ξ is an endomorphism $J:\xi\to\xi$ such that $J^2=-1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX,Y]+[X,JY]$ and $J([JX,Y]+[X,JY])=[JX,JY]-[X,Y]$.

Let $\{T,Z_\alpha,Z_{\bar\alpha}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha}=\overline{Z_\alpha}\in T_{0,1}$ and T is the characteristic vector field. Then $\left\{\theta,\theta^\alpha,\theta^{\bar\alpha}\right\}$, which is the coframe dual to $\{T,Z_\alpha,Z_{\bar\alpha}\}$, satisfies

$$d\theta=ih_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}} \tag{3.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$, if we have this contact structure, we also call such M a strictly pseudoconvex CR $(2n+1)$-manifold.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z,W\rangle_{L_\theta}=-i\left\langle d\theta,Z\wedge\overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z},\overline{W}\right\rangle_{L_\theta}=\overline{\langle Z,W\rangle}_{L_\theta}$ for all $Z,W\in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu=\theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha\in T_{1,0}$ by

$$\nabla Z_\alpha=\theta_\alpha{}^\beta\otimes Z_\beta,\quad \nabla Z_{\bar\alpha}=\theta_{\bar\alpha}{}^{\bar\beta}\otimes Z_{\bar\beta},\quad \nabla T=0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta&=\theta^\alpha\wedge\theta_\alpha{}^\beta+\theta\wedge\tau^\beta,\\ 0&=\tau_\alpha\wedge\theta^\alpha,\\ 0&=\theta_\alpha{}^\beta+\theta_{\bar\beta}{}^{\bar\alpha},\end{aligned}$$

We can write (by Cartan lemma) $\tau_\alpha=A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma}=A_{\gamma\alpha}$. The curvature of Webster-Stanton connection, expressed in terms of the coframe $\{\theta=\theta^0,\theta^\alpha,\theta^{\bar\alpha}\}$, is

$$\begin{aligned}\Pi_\beta{}^\alpha&=\overline{\Pi_{\bar\beta}{}^{\bar\alpha}}=d\omega_\beta{}^\alpha-\omega_\beta{}^\gamma\wedge\omega_\gamma{}^\alpha,\\ \Pi_0{}^\alpha&=\Pi_\alpha{}^0=\Pi_0{}^{\bar\beta}=\Pi_{\bar\beta}{}^0=\Pi_0{}^0=0.\end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha=R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma}+W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta-W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho}\wedge\theta+i\theta_\beta\wedge\tau^\alpha-i\tau_\beta\wedge\theta^\alpha$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma}=\overline{R_{\alpha\bar\beta\sigma\bar\rho}}=R_{\bar\alpha\beta\bar\sigma\rho}=R_{\rho\bar\alpha\beta\bar\sigma},\quad W_{\beta\bar\alpha\gamma}=W_{\gamma\bar\alpha\beta}.$$

Here $R_\gamma{}^\gamma{}_{\alpha\bar\beta}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar\beta} = R_\gamma{}^\delta{}_{\alpha\bar\beta}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor. Furthermore, we define the bi-sectional curvature

$$R_{\alpha\bar\alpha\beta\bar\beta}(X,Y) = R_{\alpha\bar\alpha\beta\bar\beta}X_\alpha X_{\bar\alpha}Y_\beta Y_{\bar\beta}$$

and the bi-torsion tensor

$$T_{\alpha\bar\beta}(X,Y) := i(A_{\bar\beta\bar\rho}X_\rho Y_\alpha - A_{\alpha\rho}X_{\bar\rho}Y_{\bar\beta})$$

and the torsion tensor

$$Tor(X,Y) := h^{\alpha\bar\beta}T_{\alpha\bar\beta}(X,Y) = i(A_{\bar\alpha\bar\rho}X_\rho Y_\alpha - A_{\alpha\rho}X_{\bar\rho}Y_{\bar\alpha})$$

for any $X = X_{\bar\alpha}Z_\alpha,\ Y = Y_{\bar\alpha}Z_\alpha$ in $T_{1,0}$.

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u,\ u_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha u - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma u$.

For a smooth real-valued function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_\alpha u_{\bar\alpha}Z_\alpha + u_\alpha Z_{\bar\alpha}$. We also denote $u_0 = \mathbf{Tu}$.

We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$$

by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\bar\alpha}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta}u_{\bar\alpha\bar\beta} + u_{\alpha\bar\beta}u_{\bar\alpha\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \textstyle\sum_\alpha (u_{\alpha\bar\alpha} + u_{\bar\alpha\alpha}).$$

The Kohn-Rossi Laplacian $\Box_b$ on functions is defined by

$$\Box_b\varphi = 2\bar\partial_b^*\bar\partial_b\varphi = (\Delta_b + inT)\varphi = -2\varphi_{\bar\alpha}{}^{\bar\alpha}$$

and on (p,q)-forms is defined by

$$\Box_b = 2(\bar\partial_b^*\bar\partial_b + \bar\partial_b\bar\partial_b^*).$$

Next we recall the following commutation relations ([L1]). Let φ be a scalar function and $\sigma = \sigma_\alpha\theta^\alpha$ be a $(1,0)$ form, then we have

$$\begin{array}{rcl}
\varphi_{\alpha\beta} & = & \varphi_{\beta\alpha}, \\
\varphi_{\alpha\bar\beta} - \varphi_{\bar\beta\alpha} & = & ih_{\alpha\bar\beta}\varphi_0, \\
\varphi_{0\alpha} - \varphi_{\alpha 0} & = & A_{\alpha\beta}\varphi_{\bar\beta}, \\
\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} & = & \sigma_{\alpha,\bar\gamma}A_{\gamma\beta} - \sigma_\gamma A_{\alpha\beta,\bar\gamma}, \\
\sigma_{\alpha,0\bar\beta} - \sigma_{\alpha,\bar\beta 0} & = & \sigma_{\alpha,\gamma}A_{\bar\gamma\bar\beta} + \sigma_\gamma A_{\bar\gamma\bar\beta,\alpha},
\end{array}$$

and

$$\begin{aligned}
\sigma_{\alpha,\beta\gamma}-\sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma}\sigma_{\beta}-iA_{\alpha\beta}\sigma_{\gamma},\\
\sigma_{\alpha,\bar{\beta}\bar{\gamma}}-\sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\bar{\beta}}A_{\bar{\gamma}\bar{\rho}}\sigma_{\rho}-ih_{\alpha\bar{\gamma}}A_{\bar{\beta}\bar{\rho}}\sigma_{\rho},\\
\sigma_{\alpha,\beta\bar{\gamma}}-\sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\bar{\gamma}}\sigma_{\alpha,0}+R_{\alpha\bar{\rho}\beta\bar{\gamma}}\sigma_{\rho}.
\end{aligned}$$

Moreover for multi-index $I=(\alpha_1,...,\alpha_p)$, $\bar{J}=\left(\bar{\beta}_1,...,\bar{\beta}_q\right)$, we denote $I(\alpha_k=\mu)=(\alpha_1,...,\alpha_{k-1},\mu,\alpha_{k+1},...,\alpha_p)$. Then

$$\begin{aligned}
\eta_{I\bar{J},\mu\lambda}-\eta_{I\bar{J},\lambda\mu} &= i\sum_{k=1}^{p}\left(\eta_{I(\alpha_k=\mu)\bar{J}}A_{\alpha_k\lambda}-\eta_{I(\alpha_k=\lambda)\bar{J}}A_{\alpha_k\mu}\right)\\
&\quad -i\sum_{k=1}^{q}\left(\eta_{I\bar{J}\left(\bar{\beta}_k=\bar{\gamma}\right)}h_{\bar{\beta}_k\mu}A_{\lambda}^{\bar{\gamma}}-\eta_{I\bar{J}\left(\bar{\beta}_k=\bar{\gamma}\right)}h_{\bar{\beta}_k\lambda}A_{\mu}^{\bar{\gamma}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\eta_{I\bar{J},\lambda\bar{\mu}}-\eta_{I\bar{J},\bar{\mu}\lambda} &= ih_{\lambda\bar{\mu}}\eta_{I\bar{J},0}+\sum_{k=1}^{p}\eta_{I(\alpha_k=\gamma)\bar{J}}R_{\alpha_k}{}^{\gamma}{}_{\lambda\bar{\mu}}+\sum_{k=1}^{q}\eta_{I\bar{J}\left(\bar{\beta}_k=\bar{\gamma}\right)}R_{\bar{\beta}_k}{}^{\bar{\gamma}}{}_{\lambda\bar{\mu}}\\
\eta_{I\bar{J},0\mu}-\eta_{I\bar{J},\mu 0} &= A_{\mu}^{\bar{\rho}}\eta_{I\bar{J},\bar{\rho}}-\sum_{k=1}^{p}A_{\alpha_k\mu,\bar{\rho}}\eta_{I(\alpha_k=\rho)\bar{J}}+\sum_{k=1}^{q}A_{\mu\rho,\bar{\beta}_k}\eta_{I\bar{J}\left(\bar{\beta}_k=\bar{\rho}\right)}.
\end{aligned}$$

4. **CR Matrix Li-Yau-Hamilton Harnack Inequality**

In the seminal paper [LY], P. Li and S.-T. Yau established the parabolic Li-Yau Harnack estimate for the positive solution $u(x,t)$ of the time-independent heat equation

$$\frac{\partial u\left(x,t\right)}{\partial t}=\Delta u\left(x,t\right)$$

in a complete Riemannian l-manifold with nonnegative Ricci curvature. Here Δ is the Laplace-Beltrami operator. Later in [H2], Richard Hamilton extended the Li-Yau estimate to the full matrix version of the Hessian estimate of u under the stronger assumptions that M is Ricci parallel and of nonnegative sectional curvature. Furthermore, Hamilton [H1] proved the matrix Harnack inequality for solutions to the Ricci flow when the curvature operator is nonnegative. This inequality is called the "Li-Yau-Hamilton" type estimates. Since then, there are many additional works in this direction which cover various different geometric evolution equations such as the mean curvature flow [H2], the Kähler-Ricci flow [Ca], the Yamabe flow [C], etc.

Along this line with method of Li-Yau gradient estimate, H.-D. Cao and S.-T. Yau ([CY]) studied the heat equation

$$\frac{\partial u\left(x,t\right)}{\partial t}=Lu(x,t) \tag{4.1}$$

in a closed l-manifold with a positive measure and a subelliptic operator with respect to the sum of squares of vector fields $L=\sum_{i=1}^{h}X_i^2-Y$, $h\leq l$, with $Y=\sum_{i=1}^{h}c_iX_i$ where $X_1,\ X_2,\ ...,X_h$ are smooth vector fields satisfying Hörmander's condition : the vector fields together with their commutators up to finite order span the tangent space at every point of M. Suppose that $[X_i,[X_j,X_k]]$ can be expressed as linear combinations of $X\ _1,\ X_2,\ ...,X_h$ and their brackets $[X_1,X_2],...,[X_{l-1},X_h]$. They showed that the gradient estimate for the positive solution $u(x,t)$ of (4.1) on $M\times[0,\infty)$.

Recently in the paper of [CKL1], we obtained the CR Cao-Yau type gradient estimate for the positive solution $u(x,t)$ of the CR heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta_b u(x,t) \tag{4.2}$$

in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ) of nonnegative Tanaka-Webster curvature and vanishing torsion. Here Δ_b is the time-independent sub-Laplacian operator.

In this part, we will derive the following CR analogue of matrix Li-Yau-Hamilton inequality for any positive solution u to (4.2).

Theorem 4.1. *Let M be a closed pseudohermitian $(2n+1)$-manifold with nonnegative bi-sectional curvature and nonnegative bi-torsional tensor. Let u be the positive solution of the CR heat equation (4.2). In addition if the positive solution u satisfies the purely holomorphic Hessian operator $P_{\alpha\bar{\beta}}u = 0$. Then*

$$(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) + \frac{1}{2}[(u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}} V_\alpha) + u V_\alpha V_{\bar{\beta}}] - \frac{t}{12}\frac{|u_0|^2}{u} h_{\alpha\bar{\beta}} + \frac{4}{t} u h_{\alpha\bar{\beta}} \geq 0 \tag{4.3}$$

for $t > 0$ and any vector filed $V = V_\alpha$ of type $(1,0)$ on M. Here $P_{\alpha\bar{\beta}}$ is the purely holomorphic Hessian operator (Definition 4.1).

Corollary 4.2. *The CR matrix Li-Yau-Hamilton inequality (4.3) holds in a closed pseudo-hermitian $(2n+1)$-manifold of nonnegative bisectional curvature and vanishing torsion.*

Remark 4.1. *If we choose the optimal $V = -\nabla u/u$ and take the trace of (4.3), we recapture the following CR Li-Yau gradient estimate which was derived by Chang-Kuo-Lai in* [CKL1] *and* [CKL2] *:*

$$\frac{\partial}{\partial t}u - \frac{1}{4}\frac{\|\nabla u\|^2}{u} - \frac{nt}{12}\frac{|u_0|^2}{u} + \frac{4n}{t}u \geq 0. \tag{4.4}$$

When the manifold is complete noncompact, we will need to use the CR Li-Yau Harnack inequality (4.29) and Li-Tam mean value inequality (4.32) in the proof of the CR matrix Li-Yau-Hamilton inequality (4.3). However, the proof of both inequalities rely on CR Li-Yau gradient estimate (4.5). We refer to [CN] for some details.

As shown in section 4.2, the proof of CR Li-Yau gradient estimate (4.5) relies on the CR sub-Laplacian comparison property (4.27) and the extra u_0-growth property (see appendix in [CFTW]) with $|u_0| \leq \frac{C}{t}u$ that has no analogue in the Riemannian case. In particular, both properties holds in a standard Heisenberg group $\mathbf{H}^n$ which is flat and vanishing torsion. However, both properties are wild open in a general complete noncompact pseudohermitian $(2n+1)$-manifold.

Then we are able to derive the following CR Li-Yau gradient estimate on $\mathbf{H}^n$.

Theorem 4.3. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n \times [0,\infty)$. Let $\varphi = \ln u$, for any $\alpha < -1$, then there exists a positive constant C depending on α such that*

$$|\nabla_b \varphi|^2 + \alpha\varphi_t + t\varphi_0^2 \leq \tfrac{C}{t}. \tag{4.5}$$

By applying Theorem 4.3, we have the following CR Liouville-type theorem for a positive pseudoharmonic function u on $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$ which recaptured the Liouville theorem due to Chang-Kuo-Tie ([CKT]) and Koranyi and Stanton ([KS]) by a different method.

Corollary 4.4. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive smooth function with $\Delta_b u = 0$, then $u(x,t)$ is constant. That is, there does not exist any positive nonconstant pseudoharmonic function in $\boldsymbol{H}^n$.*

From the previous discuss and Theorem 4.1, we have the CR matrix Li-Yau-Hamilton inequality in $(\boldsymbol{H}^n, J, \theta)$ as in section 4.2.

Theorem 4.5. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n \times [0,\infty)$. Then the CR matrix Li-Yau-Hamilton inequality (4.3) holds.*

Remark 4.2. *We observe that from the proof of Theorem 4.5 that the CR matrix Li-Yau-Hamilton inequality (4.3) still holds in a complete noncompact pseudohermitian manifold whenever both the CR sub-Laplacian comparison property (4.27) and the u_0-growth property hold. We should point out that the extra u_0-growth property is equivalent to (4.35) that has no analogue in Kähler manifolds.*

By applying Theorem 4.5 to the heat kernel $H(x,y,t)$ with $V = -\frac{\nabla H}{H}$ and observe that the well-known asymptotic of $H(x,o,t)$ ([V], [Ga], [B], [Le], [T], [BBN], etc)

$$-t \log H(x,o,t) \to \frac{1}{4} r^2(x)$$

as $t \to 0$. Here $r(x)$ be the Carnot-Carathéodory distance function to the origin $o \in \boldsymbol{H}^n$. We have the following complex Hessian comparison theorem for r on $\mathbf{H}^n$. Such a Hessian comparison property seems to be new in the standard $(2n+1)$-dimensional Heisenberg group $\boldsymbol{H}^n$.

Corollary 4.6. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. Then in the sense of distribution, we have*

$$[(r^2(x))_{\alpha\overline{\beta}} + (r^2(x))_{\overline{\beta}\alpha}] \leq (16 + C_0) h_{\alpha\overline{\beta}}(x)$$

for some constant C_0. In particular, we recapture the sub-Laplacian comparison property

$$\Delta_b r^2(x) \leq (16 + C_0) n$$

in the Heisenberg group.

In the following, in section 4.1, we prove the CR matrix Li-Yau-Hamilton inequality for the CR heat equation via methods developed as in [LY], [CKL1] and [CN]. In section 4.2, we prove a CR Li-Yau gradient estimate in the standard $(2n+1)$-dimensional Heisenberg group. Combining this with Theorem 4.1, we have the CR matrix Li-Yau-Hamilton inequality and Hessian comparison property in the standard $(2n+1)$-dimensional Heisenberg group $\boldsymbol{H}^n$.

4.1. **CR Matrix Li-Yau-Hamilton Inequality.** Let $u(x,t)$ be the positive solution of the CR heat equation (4.2). For the CR Li-Yau gradient estimate as in the paper [CKL1], we observe that one of difficulties is to deal with CR Bochner formula (4.16) which involving a term $\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\rangle$ that has no analogue in the Riemannian case. In order to overcome this difficulty, we introduce a new scalar Harnack quantity $G = t[|\nabla_b\varphi|^2 + \alpha\varphi_t + t\varphi_0^2]$ with $\varphi = \ln u$ by adding an extra term $t\varphi_0^2$ to $|\nabla_b\varphi|^2 + \alpha\varphi_t$ which was appeared in Li-Yau estimate ([LY]). See section 4.2 for more details.

Now we want to find the right quantity for the CR matrix Li-Yau-Hamilton inequality. By comparing the Harnack quantity in [CN] in case of Kähler manifolds, we define the matrix Harnack quantity

$$N_{\alpha\bar{\beta}} = \frac{1}{2}(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) + 2\frac{u}{t}h_{\alpha\bar{\beta}} - b\frac{u_\alpha u_{\bar{\beta}}}{u} - at\frac{|u_0|^2}{u}h_{\alpha\bar{\beta}} \tag{4.6}$$

by adding an extra term $F := -at\frac{|u_0|^2}{u}h_{\alpha\bar{\beta}}$ in which positive constants a and b to be determined later (say $a = \frac{1}{24}$and $b = \frac{1}{4}$).

Definition 4.1. *(i) ([GL]) Define the purely holomorphic Hessian operator $P_{\alpha\bar{\beta}}$:*

$$P_{\alpha\bar{\beta}}\varphi := -2i(A_{\alpha\gamma}\varphi_{\bar{\gamma}})_{\bar{\beta}}$$

and the purely holomorphic Poisson operator Q :

$$Q\varphi := h^{\alpha\bar{\beta}}(P_{\alpha\bar{\beta}}\varphi) = -2i(A_{\alpha\gamma}\varphi_{\bar{\gamma}})_{\bar{\alpha}}$$

for any smooth function φ. Note that $P_{\alpha\bar{\beta}}\varphi = 0 = Q\varphi$ for any smooth function φ if $A_{\alpha\beta} = 0$ on M.

Lemma 4.1. *Let $u(x,t)$ be the positive solution of the CR heat equation (4.2). Then $\frac{1}{2}(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha})$ satisfies the following :*

$$\frac{1}{2}\left(\frac{\partial}{\partial t} - \Delta_b\right)(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) = 2R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_{\gamma\bar{\delta}} - R_{\alpha\bar{\delta}}u_{\delta\bar{\beta}} - R_{\delta\bar{\beta}}u_{\alpha\bar{\delta}} + C_{\alpha\bar{\beta}},$$

where

$$\begin{aligned} C_{\alpha\bar{\beta}} &:= i\left(A_{\gamma\delta,\bar{\delta}}u_{\bar{\gamma}} - A_{\bar{\gamma}\bar{\delta},\delta}u_{\gamma}\right)h_{\alpha\bar{\beta}} + i\left(A_{\gamma\delta}u_{\bar{\gamma}\bar{\delta}} - A_{\bar{\gamma}\bar{\delta}}u_{\delta\gamma}\right)h_{\alpha\bar{\beta}} \\ &\quad + in\left(A_{\bar{\gamma}\bar{\beta}}u_{\alpha\gamma} - A_{\alpha\gamma}u_{\bar{\gamma}\bar{\beta}}\right) + in\left(A_{\bar{\gamma}\bar{\beta},\alpha}u_{\gamma} - A_{\gamma\alpha,\bar{\beta}}u_{\bar{\gamma}}\right) \\ &:= -(\mathrm{Re}\,Qu)h_{\alpha\bar{\beta}} + n(\mathrm{Re}\,P_{\alpha\bar{\beta}}u). \end{aligned}$$

Note that $trC_{\alpha\bar{\beta}} = h^{\alpha\bar{\beta}}C_{\alpha\bar{\beta}} = 0$. In particular we have $C_{1\bar{1}} = 0$ for $n = 1$. In addition if the positive solution u satisfies $P_{\alpha\bar{\beta}}u = 0$ which is the case when the torsion is vanishing, then $u_{\alpha\bar{\beta}}$ satisfies the following CR Lichnerowicz-Laplacian heat equation ([CCF]) :

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)u_{\alpha\bar{\beta}} = 2R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_{\gamma\bar{\delta}} - R_{\alpha\bar{\delta}}u_{\delta\bar{\beta}} - R_{\delta\bar{\beta}}u_{\alpha\bar{\delta}}.$$

Proof. Note that

$$\begin{aligned} &\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\left(u_{\mu\bar{\lambda}} + u_{\bar{\lambda}\mu}\right) \\ &= \tfrac{\partial}{\partial t}\left(u_{\mu\bar{\lambda}} + u_{\bar{\lambda}\mu}\right) - \Delta_b\left(u_{\mu\bar{\lambda}} + u_{\bar{\lambda}\mu}\right) \\ &= [(\Delta_b u)_{\mu\bar{\lambda}} - \Delta_b u_{\mu\bar{\lambda}}] + [(\Delta_b u)_{\bar{\lambda}\mu} - \Delta_b u_{\bar{\lambda}\mu}]. \end{aligned}$$

(i) We first compute $[(\Delta_b u)_{\mu\bar{\lambda}} - \Delta_b u_{\mu\bar{\lambda}}]$: By definition, we have

$$(\Delta_b u)_{\mu\bar{\lambda}} = (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha})_{\mu\bar{\lambda}} = u_{\alpha\bar{\alpha}\mu\bar{\lambda}} + u_{\bar{\alpha}\alpha\mu\bar{\lambda}}. \tag{4.7}$$

Compute

$$\begin{aligned}
u_{\alpha\bar{\alpha}\mu\bar{\lambda}} &= (u_{\alpha\mu\bar{\alpha}} - ih_{\mu\bar{\alpha}}u_{\alpha 0} - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_\rho)_{\bar{\lambda}} \\
&= u_{\alpha\mu\bar{\alpha}\bar{\lambda}} - ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}u_\rho - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_{\rho\bar{\lambda}} \\
&= u_{\mu\alpha\bar{\alpha}\bar{\lambda}} - ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}u_\rho - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_{\rho\bar{\lambda}} \\
&= u_{\mu\alpha\bar{\lambda}\bar{\alpha}} + i\left(u_{\sigma\alpha}h_{\mu_{\bar{\alpha}}}A_{\bar{\sigma}\bar{\lambda}} - u_{\sigma\alpha}h_{\mu\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad +i\left(nu_{\mu\sigma}A_{\bar{\sigma}\bar{\lambda}} - u_{\mu\sigma}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad -ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}u_\rho - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_{\rho\bar{\lambda}} \\
&= \left(u_{\mu\bar{\lambda}\alpha} + ih_{\alpha\bar{\lambda}}u_{\mu 0} + R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_\rho\right)_{\bar{\alpha}} \\
&\quad +i\left(u_{\sigma\alpha}h_{\mu_{\bar{\alpha}}}A_{\bar{\sigma}\bar{\lambda}} - u_{\sigma\alpha}h_{\mu\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad +i\left(nu_{\mu\sigma}A_{\bar{\sigma}\bar{\lambda}} - u_{\mu\sigma}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad -ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}u_\rho - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_{\rho\bar{\lambda}} \\
&= u_{\mu\bar{\lambda}\alpha\bar{\alpha}} + ih_{\alpha\bar{\lambda}}u_{\mu 0\bar{\alpha}} - ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} \\
&\quad -R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}u_\rho - R_{\alpha\bar{\rho}\mu\bar{\alpha}}u_{\rho\bar{\lambda}} + R_{\mu\bar{\rho}\alpha\bar{\lambda},\bar{\alpha}}u_\rho + R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_{\rho\bar{\alpha}} \\
&\quad +i\left(u_{\sigma\alpha}h_{\mu_{\bar{\alpha}}}A_{\bar{\sigma}\bar{\lambda}} - u_{\sigma\alpha}h_{\mu\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) + i\left(nu_{\mu\sigma}A_{\bar{\sigma}\bar{\lambda}} - u_{\mu\sigma}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right).
\end{aligned} \tag{4.8}$$

Here we have use commutation relations

$$\begin{aligned}
u_{\mu\alpha\bar{\alpha}\bar{\lambda}} &= u_{\mu\alpha\bar{\lambda}\bar{\alpha}} + i\left(u_{\sigma\alpha}h_{\mu_{\bar{\alpha}}}A_{\bar{\sigma}\bar{\lambda}} - u_{\sigma\alpha}h_{\mu\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad +i\left(nu_{\mu\sigma}A_{\bar{\sigma}\bar{\lambda}} - u_{\mu\sigma}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right)
\end{aligned}$$

and

$$\begin{aligned}
u_{\mu\alpha\bar{\lambda}\bar{\alpha}} &= \left(u_{\mu\bar{\lambda}\alpha} + ih_{\alpha\bar{\lambda}}u_{\mu 0} + R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_\rho\right)_{\bar{\alpha}} \\
&= u_{\mu\bar{\lambda}\alpha\bar{\alpha}} + ih_{\alpha\bar{\lambda}}u_{\mu 0\bar{\alpha}} + R_{\mu\bar{\rho}\alpha\bar{\lambda},\bar{\alpha}}u_\rho + R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_{\rho\bar{\alpha}}.
\end{aligned}$$

Similiar, we have

$$\begin{aligned}
u_{\bar{\alpha}\alpha\mu\bar{\lambda}} &= u_{\mu\bar{\lambda}\bar{\alpha}\alpha} + ih_{\mu\bar{\alpha}}A_{\bar{\lambda}\bar{\rho},\alpha}u_\rho + ih_{\mu\bar{\alpha}}A_{\bar{\lambda}\bar{\rho},\alpha}u_{\rho\alpha} \\
&\quad -ih_{\mu\bar{\lambda}}A_{\bar{\alpha}\bar{\rho},\alpha}u_\rho - ih_{\mu\bar{\lambda}}A_{\bar{\alpha}\bar{\rho}}u_{\rho\alpha} \\
&\quad -i\left(nA_{\mu\rho}u_{\bar{\rho}}\right)_{\bar{\lambda}} + i\left(h_{\bar{\alpha}\mu}A_{\alpha\rho}u_{\bar{\rho}}\right)_{\bar{\lambda}} \\
&\quad +R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_{\rho\bar{\alpha}} + R_{\bar{\alpha}\rho\alpha\bar{\lambda}}u_{\mu\bar{\rho}} \\
&\quad -ih_{\mu\bar{\alpha}}u_{0\alpha\bar{\lambda}} + ih_{\alpha\bar{\lambda}}u_{\mu\bar{\alpha}0}
\end{aligned} \tag{4.9}$$

It follow from (4.7), (4.8) and (4.9) that

$$\begin{aligned}
(\Delta_b u)_{\mu\bar{\lambda}} - \Delta_b u_{\mu\bar{\lambda}} &= +2R_{\mu\bar{\rho}\alpha\bar{\lambda}}u_{\rho\bar{\alpha}} - R_{\rho\bar{\lambda}}u_{\mu\bar{\rho}} - R_{\bar{\rho}\mu}u_{\rho\bar{\lambda}} \\
&\quad +\left(R_{\mu\bar{\rho}\alpha\bar{\lambda},\bar{\alpha}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}}\right)u_\rho \\
&\quad +ih_{\alpha\bar{\lambda}}u_{\mu 0\bar{\alpha}} - ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - ih_{\mu\bar{\alpha}}u_{0\alpha\bar{\lambda}} + ih_{\alpha\bar{\lambda}}u_{\mu\bar{\alpha}0} \\
&\quad +i\left(u_{\sigma\alpha}h_{\mu_{\bar{\alpha}}}A_{\bar{\sigma}\bar{\lambda}} - u_{\sigma\alpha}h_{\mu\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) + i\left(nu_{\mu\sigma}A_{\bar{\sigma}\bar{\lambda}} - u_{\mu\sigma}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}\right) \\
&\quad +ih_{\mu\bar{\alpha}}A_{\bar{\lambda}\bar{\rho},\alpha}u_\rho + ih_{\mu\bar{\alpha}}A_{\bar{\lambda}\bar{\rho},\alpha}u_{\rho\alpha} - ih_{\mu\bar{\lambda}}A_{\bar{\alpha}\bar{\rho},\alpha}u_\rho \\
&\quad -ih_{\mu\bar{\lambda}}A_{\bar{\alpha}\bar{\rho}}u_{\rho\alpha} - i\left(nA_{\mu\rho}u_{\bar{\rho}}\right)_{\bar{\lambda}} + i\left(h_{\bar{\alpha}\mu}A_{\alpha\rho}u_{\bar{\rho}}\right)_{\bar{\lambda}}
\end{aligned} \tag{4.10}$$

By CR Bianchi identity ([L1]) and commutation relation, the third line of RHS in (4.10) becomes

$$\begin{aligned}
& R_{\mu\bar{\rho}\alpha\bar{\lambda},\bar{\alpha}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}} \\
&= R_{\alpha\bar{\rho}\mu\bar{\lambda},\bar{\alpha}} - R_{\alpha\bar{\rho}\mu\bar{\alpha},\bar{\lambda}} \\
&= R_{\bar{\rho}\alpha\bar{\lambda}\mu,\bar{\alpha}} - R_{\bar{\rho}\alpha\bar{\alpha}\mu,\bar{\lambda}} \\
&= -iA_{\bar{\rho}\bar{\alpha},\alpha}h_{\mu\bar{\lambda}} - iA_{\bar{\rho}\bar{\alpha},\mu}h_{\alpha\bar{\lambda}} + iA_{\bar{\rho}\bar{\lambda},\alpha}h_{\mu\bar{\alpha}} + iA_{\bar{\rho}\bar{\lambda},\mu}h_{\alpha\bar{\alpha}} \\
&= -iA_{\bar{\rho}\bar{\alpha},\alpha}h_{\mu\bar{\lambda}} - iA_{\bar{\rho}\bar{\alpha},\mu}h_{\alpha\bar{\lambda}} + iA_{\bar{\rho}\bar{\lambda},\alpha}h_{\mu\bar{\alpha}} + inA_{\bar{\rho}\bar{\lambda},\mu}
\end{aligned}$$

and the fourth line becomes

$$\begin{aligned}
& ih_{\alpha\bar{\lambda}}u_{\mu 0\bar{\alpha}} - ih_{\mu\bar{\alpha}}u_{\alpha 0\bar{\lambda}} - ih_{\mu\bar{\alpha}}u_{0\alpha\bar{\lambda}} + ih_{\alpha\bar{\lambda}}u_{\mu\bar{\alpha}0} \\
&= iu_{\mu 0\bar{\lambda}} - iu_{\mu 0\bar{\lambda}} - iu_{0\mu\bar{\lambda}} + iu_{\mu\bar{\lambda}0} \\
&= iu_{\mu\bar{\lambda}0} - iu_{0\mu\bar{\lambda}} \\
&= -iA_{\mu\rho,\bar{\lambda}}u_{\bar{\rho}} - iA_{\mu\rho}u_{\bar{\rho}\bar{\lambda}} - iA_{\bar{\rho}\bar{\lambda}}u_{\mu\rho} - iA_{\bar{\rho}\bar{\lambda},\mu}u_{\rho}
\end{aligned}$$

(ii) We compute $[(\Delta_b u)_{\bar{\lambda}\mu} - \Delta_b u_{\bar{\lambda}\mu}]$ by take the conjugate of $[(\Delta_b u)_{\mu\bar{\lambda}} - \Delta_b u_{\mu\bar{\lambda}}]$ and then switch index λ and μ.

Now we arrange all the torsion terms together in (i) and (ii), then we are done. □

Note that it follows from commutation relation ([CKL1]) that

$$\Delta_b u_0 = (\Delta_b u)_0 + 2\left[(A_{\alpha\beta}u_{\overline{\alpha}})_{\overline{\beta}} + (A_{\overline{\alpha}\overline{\beta}}u_{\alpha})_{\beta}\right].$$

Hence

$$[\Delta_b,\ \mathbf{T}]\, u = -2\operatorname{Im} Qu.$$

Proof of Theorem 4.1 :

Proof. As in [H2], it suffices to prove that the Hermitian symmetric $(1,1)$-tensor

$$N_{\alpha\bar{\beta}} = \frac{1}{2}(u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}) + 2\frac{u}{t}h_{\alpha\bar{\beta}} - b\frac{u_\alpha u_{\bar{\beta}}}{u} - Fh_{\alpha\bar{\beta}} \geq 0$$

for $t > 0$ and some constant a and b to be determined. Here

$$F := at\frac{|u_0|^2}{u}.$$

Now we first compute

$$\begin{aligned}
\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\tfrac{u_\alpha u_{\bar{\beta}}}{u} &= \tfrac{\partial}{\partial t}\left(\tfrac{u_\alpha u_{\bar{\beta}}}{u}\right) - \Delta_b\left(\tfrac{u_\alpha u_{\bar{\beta}}}{u}\right) \\
&= \tfrac{-1}{u^2}\Delta_b u \cdot u_\alpha u_{\bar{\beta}} + \tfrac{1}{u}\left(\Delta_b u\right)_\alpha u_{\bar{\beta}} \\
&\quad + \tfrac{1}{u}u_\alpha\left(\Delta_b u\right)_{\bar{\beta}} - \Delta_b\left(\tfrac{1}{u}u_\alpha u_{\bar{\beta}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_b\left(\tfrac{1}{u}u_\alpha u_{\bar{\beta}}\right) &= \left(\tfrac{-\Delta_b u}{u^2} + \tfrac{4}{u^3}u_{\bar{\gamma}}u_\gamma\right)u_\alpha u_{\bar{\beta}} + \tfrac{1}{u}\Delta_b\left(u_\alpha u_{\bar{\beta}}\right) \\
&\quad - \tfrac{2}{u^2}u_\gamma\left(u_\alpha u_{\bar{\beta}}\right)_{\bar{\gamma}} - \tfrac{2}{u^2}u_{\bar{\gamma}}\left(u_\alpha u_{\bar{\beta}}\right)_\gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\tfrac{u_\alpha u_{\bar{\beta}}}{u} &= -\tfrac{2}{u}u_{\alpha\gamma}u_{\bar{\beta}\bar{\gamma}} - \tfrac{2}{u}u_{\alpha\bar{\gamma}}u_{\bar{\beta}\gamma} - \tfrac{2}{u^3}\left|\nabla u\right|^2 u_\alpha u_{\bar{\beta}} \\
&\quad + \tfrac{2}{u^2}u_\gamma\left(u_\alpha u_{\bar{\beta}}\right)_{\bar{\gamma}} + \tfrac{2}{u^2}u_{\bar{\gamma}}\left(u_\alpha u_{\bar{\beta}}\right)_\gamma \\
&\quad + \tfrac{1}{u}\left((\Delta_b u)_\alpha - \Delta_b\left(u_\alpha\right)\right)u_{\bar{\beta}} + \tfrac{1}{u}\Big((\Delta_b u)_{\bar{\beta}} - \Delta_b\left(u_{\bar{\beta}}\right)\Big)u_\alpha.
\end{aligned}$$

Therefore by using Lemma 4.1, we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_b\right) N_{\alpha\bar\beta}\\
&= 2R_{\alpha\bar\gamma\delta\bar\beta}u_{\gamma\bar\delta}-R_{\alpha\bar\sigma}u_{\sigma\bar\beta}-R_{\sigma\bar\beta}u_{\alpha\bar\sigma}+C_{\alpha\bar\beta}\\
&\quad +b\left(\tfrac{2}{u}u_{\alpha\gamma}u_{\bar\beta\bar\gamma}+\tfrac{2}{u}u_{\alpha\bar\gamma}u_{\bar\beta\gamma}+\tfrac{2}{u^3}\left|\nabla u\right|^2u_\alpha u_{\bar\beta}-\tfrac{2}{u^2}u_\gamma\left(u_{\alpha\bar\gamma}u_{\bar\beta}+u_\alpha u_{\bar\beta\bar\gamma}\right)\right)\\
&\quad -b\tfrac{2}{u^2}u_{\bar\gamma}\left(u_{\alpha\gamma}u_{\bar\beta}+u_\alpha u_{\bar\beta\gamma}\right)-2\tfrac{u}{t^2}h_{\alpha\bar\beta}\\
&\quad -b\tfrac{1}{u}\left(\left(\Delta_b u\right)_\alpha-\Delta_b\left(u_\alpha\right)\right)u_{\bar\beta}-b\tfrac{1}{u}\left(\left(\Delta_b u\right)_{\bar\beta}-\Delta_b\left(u_{\bar\beta}\right)\right)u_\alpha\\
&\quad -\left(\tfrac{\partial}{\partial t}-\Delta_b\right)Fh_{\alpha\bar\beta}.
\end{aligned}
$$

Observe that

(4.11)

$$
\begin{aligned}
&\tfrac{1}{u}\left(\left(\Delta_b u\right)_\alpha-\Delta_b\left(u_\alpha\right)\right)u_{\bar\beta}\\
&=\tfrac{1}{u}\left(u_{\gamma\bar\gamma\alpha}+u_{\bar\gamma\gamma\alpha}-u_{\alpha\gamma\bar\gamma}-u_{\alpha\bar\gamma\gamma}\right)u_{\bar\beta}\\
&=\tfrac{1}{u}\left(u_{\gamma\alpha\bar\gamma}-ih_{\alpha\bar\gamma}u_{\gamma 0}-R_{\bar\gamma\alpha}u_\gamma+u_{\alpha\bar\gamma\gamma}-ih_{\alpha\bar\gamma}u_{0\gamma}-inA_{\alpha\gamma}u_{\bar\gamma}+ih_{\bar\gamma\alpha}A_{\gamma\sigma}u_{\bar\sigma}-u_{\alpha\gamma\bar\gamma}-u_{\alpha\bar\gamma\gamma}\right)u_{\bar\beta}\\
&=\tfrac{1}{u}\left(-iu_{\alpha 0}u_{\bar\beta}-R_{\bar\gamma\alpha}u_\gamma u_{\bar\beta}-iu_{0\alpha}u_{\bar\beta}-inA_{\alpha\gamma}u_{\bar\gamma}u_{\bar\beta}+iA_{\alpha\sigma}u_{\bar\sigma}u_{\bar\beta}\right)\\
&=-\tfrac{1}{u}R_{\bar\gamma\alpha}u_\gamma u_{\bar\beta}-2i\tfrac{u_{0\alpha}u_{\bar\beta}}{u}-\left(n-2\right)i\tfrac{1}{u}A_{\alpha\rho}u_{\bar\rho}u_{\bar\beta}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-\Delta_b\right)N_{\alpha\bar\beta}\\
&=2R_{\alpha\bar\gamma\delta\bar\beta}N_{\gamma\bar\delta}-R_{\alpha\bar\sigma}N_{\sigma\bar\beta}-R_{\sigma\bar\beta}N_{\alpha\bar\sigma}+C_{\alpha\bar\beta}\\
&\quad +2bR_{\alpha\bar\gamma\delta\bar\beta}\tfrac{u_\gamma u_{\bar\delta}}{u}+b\left(n-2\right)i\tfrac{1}{u}A_{\alpha\rho}u_{\bar\rho}u_{\bar\beta}-b\left(n-2\right)i\tfrac{1}{u}A_{\bar\beta\bar\rho}u_\rho u_\alpha\\
&\quad +\tfrac{2b}{u}\left(u_{\alpha\gamma}-\tfrac{u_\alpha u_\gamma}{u}\right)\left(u_{\bar\beta\bar\gamma}-\tfrac{u_{\bar\gamma}u_{\bar\beta}}{u}\right)\\
&\quad +2b\left(\tfrac{1}{u}u_{\alpha\bar\gamma}u_{\gamma\bar\beta}-\tfrac{1}{u^2}u_\gamma u_{\alpha\bar\gamma}u_{\bar\beta}-\tfrac{1}{u^2}u_{\bar\gamma}u_\alpha u_{\gamma\bar\beta}\right)\\
&\quad +2bi\tfrac{u_0}{u^2}u_\alpha u_{\bar\beta}-2bi\tfrac{u_0}{u}u_{\alpha\bar\beta}-\left(\tfrac{\partial}{\partial t}-\Delta_b\right)Fh_{\alpha\bar\beta}\\
&\quad +b\left|\nabla u\right|^2\tfrac{u_\alpha u_{\bar\beta}}{u^3}-\tfrac{2u}{t^2}h_{\alpha\bar\beta}\\
&\quad +2bi\tfrac{u_{0\alpha}u_{\bar\beta}}{u}-2bi\tfrac{u_{0\bar\beta}u_\alpha}{u}.
\end{aligned}
$$

Note that we can rewrite $N_{\alpha\bar\beta}$ as following :

$$
N_{\alpha\bar\beta}=u_{\alpha\bar\beta}-\frac{1}{2}iu_0h_{\alpha\bar\beta}+2\frac{u}{t}h_{\alpha\bar\beta}-b\frac{u_\alpha u_{\bar\beta}}{u}-Fh_{\alpha\bar\beta}.
$$

Then we replace $u_{\alpha\bar\gamma}=N_{\alpha\bar\gamma}+\frac{iu_0h_{\alpha\bar\gamma}}{2}-2\frac{u}{t}h_{\alpha\bar\gamma}+b\frac{u_\alpha u_{\bar\gamma}}{u}+Fh_{\alpha\bar\beta}$ into third and forth line of RHS as above, we have

$$
\begin{aligned}
&2b\left(\tfrac{1}{u}u_{\alpha\bar\gamma}u_{\gamma\bar\beta}-\tfrac{1}{u^2}u_\gamma u_{\alpha\bar\gamma}u_{\bar\beta}-\tfrac{1}{u^2}u_{\bar\gamma}u_\alpha u_{\gamma\bar\beta}\right)\\
&+ib\tfrac{2u_0}{u^2}u_\alpha u_{\bar\beta}-ib\tfrac{2u_0}{u}u_{\alpha\bar\beta}-\left(\tfrac{\partial}{\partial t}-\Delta_b\right)Fh_{\alpha\bar\beta}\\
&=\tfrac{2b}{u}N_{\alpha\bar\gamma}N_{\beta\bar\gamma}-\tfrac{8b}{t}N_{\alpha\bar\beta}+8b\tfrac{u}{t^2}h_{\alpha\bar\beta}+\left(b^3-2b^2\right)\left|\nabla u\right|^2\tfrac{u_\alpha u_\beta}{u^3}\\
&\quad +\left(8b-8b^2\right)\tfrac{1}{t}\tfrac{u_\alpha u_{\bar\beta}}{u}+b\tfrac{u_0^2}{2u}h_{\alpha\bar\beta}\\
&\quad +b^2\tfrac{2u_\beta u_{\bar\gamma}}{u^2}N_{\alpha\bar\gamma}+b^2\tfrac{2u_\alpha u_{\bar\gamma}}{u^2}N_{\gamma\bar\beta}-b\tfrac{2}{u^2}u_\gamma u_{\bar\beta}N_{\alpha\bar\gamma}-b\tfrac{2}{u^2}u_{\bar\gamma}u_\alpha N_{\gamma\bar\beta}\\
&\quad +\tfrac{4b}{u}FN_{\alpha\bar\beta}+\tfrac{2b}{u}F^2h_{\alpha\bar\beta}+4\left(b^2-b\right)\tfrac{Fu_\alpha u_{\bar\beta}}{u^2}\\
&\quad -2b\tfrac{4}{t}Fh_{\alpha\bar\beta}-\left(\tfrac{\partial}{\partial t}-\Delta_b\right)Fh_{\alpha\bar\beta}.
\end{aligned}
$$

Finally one obtains

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-\Delta_b\right)N_{\alpha\bar\beta}\\
&=2R_{\alpha\bar\gamma\delta\bar\beta}N_{\gamma\bar\delta}-R_{\alpha\bar\sigma}N_{\sigma\bar\beta}-R_{\sigma\bar\beta}N_{\alpha\bar\sigma}+C_{\alpha\bar\beta}\\
&\quad+2bR_{\alpha\bar\gamma\delta\bar\beta}\tfrac{u_\gamma u_{\bar\delta}}{u}+b\left(n-2\right)i\tfrac{1}{u}A_{\alpha\rho}u_{\bar\rho}u_{\bar\beta}-b\left(n-2\right)i\tfrac{1}{u}A_{\bar\beta\bar\rho}u_\rho u_\alpha\\
&\quad+\tfrac{2b}{u}\left(u_{\alpha\gamma}-\tfrac{u_\alpha u_\gamma}{u}\right)\left(u_{\bar\beta\bar\gamma}-\tfrac{u_{\bar\gamma}u_{\bar\beta}}{u}\right)+\tfrac{2b}{u}N_{\alpha\bar\gamma}N_{\beta\bar\gamma}-\tfrac{8b}{t}N_{\alpha\bar\beta}\\
&\quad+b^2\tfrac{2u_{\bar\beta}u_\gamma}{u^2}N_{\alpha\bar\gamma}+b^2\tfrac{2u_\alpha u_{\bar\gamma}}{u^2}N_{\gamma\bar\beta}-\tfrac{2b}{u^2}u_\gamma u_{\bar\beta}N_{\alpha\bar\gamma}-\tfrac{2b}{u^2}u_{\bar\gamma}u_\alpha N_{\gamma\bar\beta}+\tfrac{4b}{u}FN_{\alpha\bar\beta}\\
&\quad-\left(\tfrac{\partial}{\partial t}-\Delta_b\right)Fh_{\alpha\bar\beta}+\tfrac{2b}{u}F^2h_{\alpha\bar\beta}+4\left(b^2-b\right)\tfrac{Fu_\alpha u_{\bar\beta}}{u^2}\\
&\quad+\left(8b-2\right)\tfrac{u}{t^2}h_{\alpha\bar\beta}+b\tfrac{u_0^2}{2u}h_{\alpha\bar\beta}\\
&\quad+\left(8b-8b^2\right)\tfrac{1}{t}\tfrac{u_\alpha u_{\bar\beta}}{u}+\left(b^3-2b^2+b\right)\left|\nabla u\right|^2\tfrac{u_\alpha u_{\bar\beta}}{u^3}\\
&\quad-\tfrac{8b}{t}Fh_{\alpha\bar\beta}+2bi\tfrac{u_{0\alpha}u_{\bar\beta}}{u}-2bi\tfrac{u_{0\bar\beta}u_\alpha}{u}
\end{aligned}
\tag{4.12}
$$

Note the first and second line of RHS are positive by curvature assumption. The third and fourth line are nonnegative while we apply on null vector of $N_{\alpha\bar\beta}$.

In the following we determined F to make the rest terms nonnegative. First observe that

$$
\begin{aligned}
\left(\tfrac{\partial}{\partial t}-\Delta_b\right)\tfrac{u_0^2}{u} &= \tfrac{2u_0}{u}[T,\Delta]u-\tfrac{2\|\nabla u_0\|^2}{u}+\tfrac{4u_0\langle\nabla u_0,\nabla u\rangle}{u^2}-2u_0^2\tfrac{\|\nabla u\|^2}{u^3}\\
&= -2\left\|\tfrac{\nabla u_0}{u^{\frac12}}-\tfrac{u_0\nabla u}{u^{\frac32}}\right\|^2,
\end{aligned}
$$

where we use the fact that $[T,\Delta]u=2\operatorname{Im}Qu=0$ which is always true if $P_{\alpha\bar\beta}u=0$. The last four lines of (4.12) become

$$
\begin{aligned}
&\left(\tfrac{b}{2}-a\left(1+8b\right)\right)\tfrac{u_0^2}{u}h_{\alpha\bar\beta}+2at\left\|\tfrac{\nabla u_0}{u^{\frac12}}-\tfrac{u_0\nabla u}{u^{\frac32}}\right\|^2h_{\alpha\bar\beta}\\
&+2\left(a^2t^2b\tfrac{u_0^4}{u^3}h_{\alpha\bar\beta}+2\tfrac{\left(b^2-b\right)}{\sqrt b}at\sqrt b\tfrac{u_0^2u_\alpha u_{\bar\beta}}{u^3}+\tfrac{\left(b^2-b\right)^2}{b}\tfrac{|\nabla u|^2}{2}\tfrac{u_\alpha u_{\bar\beta}}{u^3}\right)\\
&+\left(8b-2\right)\tfrac{u}{t^2}h_{\alpha\bar\beta}\\
&+2bi\tfrac{u_{0\alpha}u_{\bar\beta}}{u}-2bi\tfrac{u_{0\bar\beta}u_\alpha}{u}+8b\left(1-b\right)\tfrac{u_\alpha u_{\bar\beta}}{tu}.
\end{aligned}
\tag{4.13}
$$

Note that the second line above is a complete square. To handle the last term, we have following

$$
\begin{aligned}
&2bi\tfrac{u_{0\alpha}u_{\bar\beta}}{u}-2bi\tfrac{u_{0\bar\beta}u_\alpha}{u}+8b\left(1-b\right)\tfrac{u_\alpha u_{\bar\beta}}{tu}\\
&=2bi\tfrac{u_{0\alpha}-\frac{u_0u_\alpha}{u}}{\sqrt u}\tfrac{u_{\bar\beta}}{\sqrt u}-2bi\tfrac{u_{0\bar\beta}-\frac{u_0u_{\bar\beta}}{u}}{\sqrt u}\tfrac{u_\alpha}{\sqrt u}+8b\left(1-b\right)\tfrac{u_\alpha u_{\bar\beta}}{tu}\\
&=b\left(\varepsilon\tfrac{u_{0\alpha}-\frac{u_0u_\alpha}{u}}{\sqrt u}-\tfrac{2}{\varepsilon}i\tfrac{u_\alpha}{\sqrt u}\right)\left(\varepsilon\tfrac{u_{0\bar\beta}-\frac{u_0u_{\bar\beta}}{u}}{\sqrt u}+\tfrac{2}{\varepsilon}i\tfrac{u_{\bar\beta}}{\sqrt u}\right)\\
&\quad-b\varepsilon^2\tfrac{u_{0\alpha}-\frac{u_0u_\alpha}{u}}{\sqrt u}\tfrac{u_{0\bar\beta}-\frac{u_0u_{\bar\beta}}{u}}{\sqrt u}-\tfrac{4b}{\varepsilon^2}\tfrac{u_\alpha u_{\bar\beta}}{u}+8b\left(1-b\right)\tfrac{u_\alpha u_{\bar\beta}}{tu}.
\end{aligned}
\tag{4.14}
$$

By taking $\varepsilon^2=\frac{4at}{b}$, we have

$$
\begin{aligned}
&2at\left\|\tfrac{\nabla u_0}{u^{\frac12}}-\tfrac{u_0\nabla u}{u^{\frac32}}\right\|^2h_{\alpha\bar\beta}-b\varepsilon^2\tfrac{u_{0\alpha}-\frac{u_0u_\alpha}{u}}{\sqrt u}\tfrac{u_{0\bar\beta}-\frac{u_0u_{\bar\beta}}{u}}{\sqrt u}\\
&=2at\left\|\tfrac{\nabla u_0}{u^{\frac12}}-\tfrac{u_0\nabla u}{u^{\frac32}}\right\|^2h_{\alpha\bar\beta}-4at\tfrac{u_{0\alpha}-\frac{u_0u_\alpha}{u}}{\sqrt u}\tfrac{u_{0\bar\beta}-\frac{u_0u_{\bar\beta}}{u}}{\sqrt u}\\
&\geq 0.
\end{aligned}
$$

Then by applying (4.13) and (4.14)

$$\begin{aligned}
&\left(\tfrac{b}{2}-a\left(1+8b\right)\right)\tfrac{u_0^2}{u}h_{\alpha\bar{\beta}}+2at\left\|\tfrac{\nabla u_0}{u^{\frac{1}{2}}}-\tfrac{u_0\nabla u}{u^{\frac{3}{2}}}\right\|^2 h_{\alpha\bar{\beta}}\\
&+2\left(a^2t^2b\tfrac{u_0^4}{u^3}h_{\alpha\bar{\beta}}+2\tfrac{\left(b^2-b\right)}{\sqrt{b}}at\sqrt{b}\tfrac{u_0^2u_\alpha u_{\bar{\beta}}}{u^3}+\tfrac{\left(b^2-b\right)^2}{b}\tfrac{|\nabla u|^2}{2}\tfrac{u_\alpha u_{\bar{\beta}}}{u^3}\right)\\
&+\left(8b-2\right)\tfrac{u}{t^2}h_{\alpha\bar{\beta}}\\
&+2bi\tfrac{u_{0\alpha}u_{\bar{\beta}}}{u}-2bi\tfrac{u_{0\bar{\beta}}u_\alpha}{u}+8b\left(1-b\right)\tfrac{u_\alpha u_{\bar{\beta}}}{tu}\\
&\geq\left(\tfrac{b}{2}-a\left(1+8b\right)\right)\tfrac{u_0^2}{u}h_{\alpha\bar{\beta}}+\left(8b-2\right)\tfrac{u}{t^2}h_{\alpha\bar{\beta}}+\left(8b\left(1-b\right)-\tfrac{b^2}{a}\right)\tfrac{u_\alpha u_{\bar{\beta}}}{tu}\\
&=0
\end{aligned}$$

when we choose a, b such that

$$\begin{aligned}
\tfrac{b}{2}-a\left(1+8b\right)&=0,\\
8b-2&=0,\\
8b\left(1-b\right)-\tfrac{b^2}{a}&=0.
\end{aligned}$$

That is

$$a=\frac{1}{24}\quad\text{and}\quad b=\frac{1}{4}.$$

Hence from (4.12)

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-\Delta_b\right)N_{\alpha\bar{\beta}}\\
&\geq 2R_{\alpha\bar{\gamma}\delta\bar{\beta}}N_{\gamma\bar{\delta}}-R_{\alpha\bar{\sigma}}N_{\sigma\bar{\beta}}-R_{\sigma\bar{\beta}}N_{\alpha\bar{\sigma}}+\tfrac{1}{2u}R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_\gamma u_{\bar{\delta}}\\
&+\tfrac{1}{2u}\left(u_{\alpha\gamma}-\tfrac{u_\alpha u_\gamma}{u}\right)\left(u_{\bar{\beta}\bar{\gamma}}-\tfrac{u_{\bar{\gamma}}u_{\bar{\beta}}}{u}\right)+\tfrac{1}{2u}N_{\alpha\bar{\gamma}}N_{\beta\bar{\gamma}}-\tfrac{2}{t}N_{\alpha\bar{\beta}}\\
&+\tfrac{1}{8}\tfrac{u_{\bar{\beta}}u_\gamma}{u^2}N_{\alpha\bar{\gamma}}+\tfrac{1}{8}\tfrac{u_\alpha u_{\bar{\gamma}}}{u^2}N_{\gamma\bar{\beta}}-\tfrac{1}{2u^2}u_\gamma u_{\bar{\beta}}N_{\alpha\bar{\gamma}}-\tfrac{1}{2u^2}u_{\bar{\gamma}}u_\alpha N_{\gamma\bar{\beta}}+\tfrac{1}{u}FN_{\alpha\bar{\beta}}\\
&+C_{\alpha\bar{\beta}}+\tfrac{1}{4u}\left(n-2\right)i[A_{\alpha\rho}u_{\bar{\rho}}u_{\bar{\beta}}-A_{\bar{\beta}\bar{\rho}}u_\rho u_\alpha].
\end{aligned}\tag{4.15}$$

which is nonnegative while we apply on null vector of $N_{\alpha\bar{\beta}}$ we assume nonnegative bisectional curvature, nonnegative bi-torsion tensor and nonnegative $C_{\alpha\bar{\beta}}$ as well. □

4.2. **The CR Gradient Estimate and Harnack inequality in Heisenberg Groups.** In this section, by using the method of CR Li-Yau gradient estimate ([LY], [CKL1]) and CR Bochner formula (4.16), we derive a CR gradient estimate and CR Harnack inequality for the positive solution of the CR heat equation (4.2) in $(2n+1)$-dimensional Heisenberg group.

We first recall the following CR version of Bochner formula in a complete pseudohermitian $(2n+1)$-manifold.

Lemma 4.2. *([Gr]) For a smooth real-valued function φ,*

$$\begin{aligned}
\tfrac{1}{2}\Delta_b\left|\nabla_b\varphi\right|^2 &= |(\nabla^H)^2\varphi|^2+\left\langle\nabla_b\varphi,\nabla_b\Delta_b\varphi\right\rangle+2\left\langle J\nabla_b\varphi,\nabla_b\varphi_0\right\rangle\\
&\quad+[2Ric-(n-2)Tor]\left(\left(\nabla_b\varphi\right)_{\mathbb{C}},\left(\nabla_b\varphi\right)_{\mathbb{C}}\right).
\end{aligned}\tag{4.16}$$

Here $\left(\nabla_b\varphi\right)_{\mathbb{C}}=\varphi^\alpha Z_\alpha$ is the corresponding complex $(1,0)$-vector of $\nabla_b\varphi$.

Since

$$|(\nabla^H)^2\varphi|^2=2\textstyle\sum_{\alpha,\beta}(|\varphi_{\alpha\beta}|^2+|\varphi_{\alpha\bar{\beta}}|^2)\geq 2\sum_\alpha|\varphi_{\alpha\bar{\alpha}}|^2\geq\frac{1}{2n}\left(\Delta_b\varphi\right)^2+\frac{n}{2}\varphi_0^2$$

and for any $v>0$,

$$2\left\langle J\nabla_b\varphi,\nabla_b\varphi_0\right\rangle \leq 2\left|\nabla_b\varphi\right|\left|\nabla_b\varphi_0\right| \leq v^{-1}\left|\nabla_b\varphi\right|^2+v\left|\nabla_b\varphi_0\right|^2.$$

Therefore, for a real-valued function φ and any $v>0$, we have the following Bochner inequality

$$\begin{array}{rcl}\frac{1}{2}\Delta_b\left|\nabla_b\varphi\right|^2 & \geq & \frac{1}{2n}\left(\Delta_b\varphi\right)^2+\frac{n}{2}\varphi_0^2+\left\langle\nabla_b\varphi,\nabla_b\Delta_b\varphi\right\rangle-v\left|\nabla_b\varphi_0\right|^2 \\ & & +[2Ric-(n-2)Tor-2v^{-1}]\left((\nabla_b\varphi)_{\mathbb{C}},(\nabla_b\varphi)_{\mathbb{C}}\right).\end{array} \tag{4.17}$$

Now let $u(x,t)$ be a positive solution of the CR heat equation (4.2) and denote

$$\varphi\left(x,t\right)=\ln u\left(x,t\right).$$

Then $\varphi\left(x,t\right)$ satisfies

$$(\Delta_b-\tfrac{\partial}{\partial t})\varphi=-\left|\nabla_b\varphi\right|^2 \tag{4.18}$$

and from Lemma 3.5 in [CKL1]

$$(\Delta_b-\tfrac{\partial}{\partial t})\varphi_0=-2\left\langle\nabla_b\varphi,\nabla_b\varphi_0\right\rangle+2V\left(\varphi\right), \tag{4.19}$$

where the operator V is defined by

$$V\left(\varphi\right)=(A_{\alpha\beta}\varphi^{\beta}),^{\alpha}+(A_{\overline{\alpha}\overline{\beta}}\varphi^{\overline{\beta}}),^{\overline{\alpha}}+A_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}+A_{\overline{\alpha}\overline{\beta}}\varphi^{\overline{\alpha}}\varphi^{\overline{\beta}}.$$

Therefore, if $A_{\alpha\beta}=0$ then one obtains $V\left(\varphi\right)=0$.

Lemma 4.3. *Let $(\boldsymbol{H}^n,J,\theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is a positive solution of (4.2) on $\boldsymbol{H}^n\times[0,\infty)$. Let $\varphi\left(x,t\right)=\ln u\left(x,t\right)$, then for any given $\alpha\leq-1$, the function*

$$G\left(x,t\right):=t[\left|\nabla_b\varphi\right|^2\left(x,t\right)+\alpha\varphi_t\left(x,t\right)+t\varphi_0^2\left(x,t\right)]$$

satisfies the inequality

$$\begin{array}{rl} & (\Delta_b-\frac{\partial}{\partial t})G \\ \geq & -2\left\langle\nabla_b\varphi,\nabla_b G\right\rangle-t^{-1}G+\alpha^{-2}n^{-1}t^{-1}G^2+\alpha^{-2}n^{-1}(\alpha+1)^2t\left|\nabla_b\varphi\right|^4 \\ & -2n^{-1}\alpha^{-2}[(\alpha+1)\left|\nabla_b\varphi\right|^2+t\varphi_0^2]G+2[\alpha^{-2}n^{-1}(\alpha+1)t^2\varphi_0^2-1]\left|\nabla_b\varphi\right|^2.\end{array} \tag{4.20}$$

Proof. Note that

$$G=t[\left|\nabla_b\varphi\right|^2+\alpha\varphi_t+t\varphi_0^2]=t[(\alpha+1)\left|\nabla_b\varphi\right|^2+\alpha\Delta_b\varphi+t\varphi_0^2].$$

By taking $v=t$ into the inequality (4.17), we compute

$$\begin{array}{rcl}\Delta_b G & = & t[\Delta_b\left|\nabla_b\varphi\right|^2+\alpha\Delta_b\varphi_t+t\Delta_b\varphi_0^2] \\ & \geq & t[\frac{1}{n}\left(\Delta_b\varphi\right)^2+n\varphi_0^2+2\left\langle\nabla_b\varphi,\nabla_b\Delta_b\varphi\right\rangle \\ & & \quad+\alpha\Delta_b\varphi_t+2t\varphi_0\Delta_b\varphi_0-2t^{-1}\left|\nabla_b\varphi\right|^2],\end{array}$$

and it follows from (4.19) that

$$\begin{array}{rcl}\frac{\partial}{\partial t}G & = & t^{-1}G+t[2\left(\alpha+1\right)\left\langle\nabla_b\varphi,\nabla_b\varphi_t\right\rangle+\alpha\Delta_b\varphi_t+\varphi_0^2+2t\varphi_0\varphi_{0t}] \\ & = & t^{-1}G+t[2\left(\alpha+1\right)\left\langle\nabla_b\varphi,\nabla_b\varphi_t\right\rangle+\alpha\Delta_b\varphi_t+\varphi_0^2 \\ & & \quad+2t\varphi_0\Delta_b\varphi_0+2t\left\langle\nabla_b\varphi,\nabla_b\varphi_0^2\right\rangle],\end{array}$$

Thus, we have

$$(4.21)\qquad (\Delta_b - \tfrac{\partial}{\partial t})G \geq -2\langle\nabla_b\varphi, \nabla_b G\rangle - t^{-1}G + t[n^{-1}(\Delta_b\varphi)^2 - 2t^{-1}|\nabla_b\varphi|^2],$$

where we used

$$\langle\nabla_b\varphi, \nabla_b G\rangle = t[(\alpha+1)\langle\nabla_b\varphi, \nabla_b\varphi_t\rangle + t\langle\nabla_b\varphi, \nabla_b\varphi_0^2\rangle - \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle].$$

However, since

$$\Delta_b\varphi = -|\nabla_b\varphi|^2 + \varphi_t = \alpha^{-1}[t^{-1}G - (\alpha+1)|\nabla_b\varphi|^2 - t\varphi_0^2],$$

thus

$$\begin{array}{rcl}(\Delta_b\varphi)^2 & \geq & \alpha^{-2}t^{-2}G^2 - 2\alpha^{-2}t^{-1}G[(\alpha+1)|\nabla_b\varphi|^2 + t\varphi_0^2] \\ & & +\alpha^{-2}[(\alpha+1)^2|\nabla_b\varphi|^4 + 2(\alpha+1)t\varphi_0^2|\nabla_b\varphi|^2].\end{array}$$

The Lemma follows from substituting this inequality into (4.21). □

Proof of Theorem 4.3:

Proof. Let B_{2R} be a ball of radius $2R$ center at $O \in \boldsymbol{H}^n$ with $R > 1$. Let $\psi \in C_0^\infty(R)$ be a cut-off function such that $0 \leq \psi \leq 1$, $\psi(t) \equiv 1$ for $t \in [0,1]$, $\psi(t) \equiv 0$ for $t \geq 2$. We also require

$$(4.22)\qquad \psi' \leq 0, \quad \psi'' \geq -C_1, \quad \text{and} \quad \tfrac{|\psi'|^2}{\psi} \leq C_2,$$

where C_1 and C_2 are positive constants. Denote by $d_c(x)$ be the Carnot-Carathéodory distance from O to x in $\boldsymbol{H}^n$. Then we define $\eta(x) = \psi\left(\frac{d_c(x)}{R}\right)$. It is clear that supp$\eta \subset B_{2R}$ and $\eta|_{B_R} \equiv 1$. For

$$G = t[|\nabla_b\varphi|^2 + \alpha\varphi_t + t\varphi_0^2]$$

we consider the function ηG with support on $B_{2R} \times (0, \mathrm{T}]$. Let $(x_0, t_0) \in B_{2R} \times (0, \mathrm{T}]$ be the maximum point of ηG. Note that at (x_0, t_0) we have the following properties

$$(4.23)\qquad \nabla_b(\eta G) = G\nabla_b\eta + \eta\nabla_b G = 0,$$

$$(4.24)\qquad \Delta_b(\eta G) \leq 0,$$

and

$$(4.25)\qquad \tfrac{\partial}{\partial t}(\eta G) = \eta G_t \geq 0.$$

In the sequel, all computations will be at the point (x_0, t_0) and we may assume that

$$(\eta G)(x_0, t_0) > 0,$$

otherwise $(\eta G)(x_0, t_0) \leq 0$, and the Theorem is true. By (4.23), $\nabla_b G = -G\nabla_b\eta/\eta$, and from (4.24)

$$(4.26)\qquad \begin{array}{rcl}0 & \geq & \Delta_b(\eta G) = G\Delta_b\eta + \eta\Delta_b G + 2\langle\nabla_b\eta, \nabla_b G\rangle \\ & = & G\Delta_b\eta + \eta\Delta_b G - 2\eta^{-1}G|\nabla_b\eta|^2.\end{array}$$

By (4.22), we have

$$\tfrac{|\nabla_b\eta|^2}{\eta} = \tfrac{|\psi'|^2|\nabla_b d_c|^2}{\psi R^2} = \tfrac{|\psi'|^2}{\psi R^2} \leq \tfrac{C_2}{R^2},$$

and

$$\Delta_b\eta = \tfrac{\psi''|\nabla_b d_c|^2}{R^2} + \tfrac{\psi'\Delta_b d_c}{R} = \tfrac{\psi''}{R^2} + \tfrac{\psi'}{R}\Delta_b d_c \geq -\tfrac{C_1}{R^2} - \tfrac{\sqrt{C_2}}{R}\Delta_b d_c.$$

By the CR sublaplacian comparison property in [CTW]

$$\Delta_b d_c \le \tfrac{C}{d_c}, \tag{4.27}$$

for some constant C, then

$$\Delta_b \eta \ge -\tfrac{C_3}{R}.$$

Substituting these into (4.26), applying the inequality (4.20) and it follows from (4.33) that we have the estimate

$$t_0^2 \varphi_0^2 \le C_5,$$

for some constant $C_5 > 0$. All these imply

$$\begin{aligned} 0 &\ge \Delta_b(\eta G) \ge -C_3 R^{-1} G - 2C_2 R^{-1} G + \eta \Delta_b G \\ &\ge -C_4 R^{-1} G + \eta[G_t - 2\langle \nabla_b \varphi, \nabla_b G\rangle - t_0^{-1} G + n^{-1}\alpha^{-2} t_0^{-1} G^2] \\ &\quad -2\eta n^{-1}\alpha^{-2}[(\alpha+1)\left|\nabla_b\varphi\right|^2 + t_0\varphi_0^2]G + n^{-1}\alpha^{-2}(\alpha+1)^2 \eta t_0 \left|\nabla_b \varphi\right|^4 \\ &\quad +2\eta[n^{-1}\alpha^{-2}(\alpha+1)C_5 - 1]\left|\nabla_b\varphi\right|^2, \end{aligned}$$

where $C_4 = C_3 + 2C_2$.

Since $\eta G_t = (\eta G)_t \ge 0$, $\eta\langle \nabla_b\varphi, \nabla_b G\rangle = G\langle \nabla_b \varphi, \nabla_b \eta\rangle$, then by the following inequality

$$\begin{aligned} &n^{-1}\alpha^{-2}(\alpha+1)^2 t_0 \left|\nabla_b\varphi\right|^4 + 2[n^{-1}\alpha^{-2}(\alpha+1)C_5 - 1]\left|\nabla_b\varphi\right|^2 \\ \ge\ & -2t_0^{-1}[n^{-1}\alpha^{-2}C_5^2 + n\alpha^2(\alpha+1)^{-2}], \end{aligned}$$

the above inequality can be reduced as

$$\begin{aligned} 0 \ \ge\ & n^{-1}\alpha^{-2} t_0^{-1}\eta G^2 - (C_4 R^{-1} + t_0^{-1}\eta)G - 2G\langle\nabla_b\varphi, \nabla_b\eta\rangle \\ & -2n^{-1}\alpha^{-2}[(\alpha+1)\left|\nabla_b\varphi\right|^2 + t_0\varphi_0^2]\eta G \\ & -2\eta t_0^{-1}[n^{-1}\alpha^{-2}C_5^2 + n\alpha^2(\alpha+1)^{-2}]. \end{aligned}$$

Then multiplying by ηt_0, since $0 \le \eta \le 1$ and $\langle \nabla_b\varphi, \nabla_b\eta\rangle \le \left|\nabla_b\varphi\right|\left|\nabla_b\eta\right|$, we get

$$\begin{aligned} 0 \ \ge\ & n^{-1}\alpha^{-2}(\eta G)^2 - (C_4 R^{-1} t_0 + 1)\eta G - 2t_0\left|\nabla_b\varphi\right|\left|\nabla_b\eta\right|\eta G \\ & -2n^{-1}\alpha^{-2}\eta t_0[(\alpha+1)\left|\nabla_b\varphi\right|^2 + t_0\varphi_0^2]\eta G \\ & -2[n^{-1}\alpha^{-2}C_5^2 + n\alpha^2(\alpha+1)^{-2}]. \end{aligned} \tag{4.28}$$

Observe that there exists a constant $C_6 > 0$ such that

$$-2n^{-1}\alpha^{-2}(\alpha+1)\eta\left|\nabla_b\varphi\right|^2 - 2\sqrt{C_2}R^{-1}\eta^{1/2}\left|\nabla_b\varphi\right| \ge C_6\alpha^2(\alpha+1)^{-1}R^{-2}.$$

Hence combining this with (4.28) and using $t_0^2\varphi_0^2 \le C_5$ again, we conclude that

$$\begin{aligned} 0 \ \ge\ & n^{-1}\alpha^{-2}(\eta G)^2 + [C_7 t_0 \alpha^2(\alpha+1)^{-1}R^{-1} - 1 - 2n^{-1}\alpha^{-2}C_5]\eta G \\ & -2[n^{-1}\alpha^{-2}C_5^2 + n\alpha^2(\alpha+1)^{-2}] \end{aligned}$$

for some constant $C_7 > 0$. This implies that at the maximum point (x_0, t_0)

$$\eta G \le C_8\alpha^2[C_5 - (\alpha+1)^{-1}(1 + \alpha^2 t_0 R^{-1})]$$

for some constant $C_8 > 0$. In particular since $t_0 \le \mathrm{T}$, when restricted on $B_{2R} \times \{\mathrm{T}\}$ we have

$$\left|\nabla_b\varphi\right|^2 + \alpha\varphi_t + \mathrm{T}\varphi_0^2 \le C_8\alpha^2[(C_5 - (\alpha+1)^{-1})\mathrm{T}^{-1} - \alpha^2(\alpha+1)^{-1}R^{-1}].$$

Theorem 4.3 follows by letting $t = \mathrm{T}$ and then taking $R \to \infty$. □

Corollary 4.7. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n \times [0,\infty)$, we have the Harnack inequality*

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \left(\frac{t_2}{t_1}\right)^C \exp\left(\frac{d_c(x_1,x_2)^2}{2(t_2-t_1)}\right) \tag{4.29}$$

for any x_1, x_2 in $\boldsymbol{H}^n$ and $0 < t_1 < t_2 < \infty$, where $d_c(x_1,x_2)$ is the Carnot-Carathéodory distance between x_1 and x_2.

Proof. Let γ be a horizontal curve with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. We define $\eta : [t_1,t_2] \to \boldsymbol{H}^n \times [t_1,t_2]$ by

$$\eta(t) = (\gamma(t), t).$$

Clearly, $\eta(t_1) = (x_1,t_1)$ and $\eta(t_2) = (x_2,t_2)$. Integrating along η, we get

$$\begin{array}{rcl} \ln u(x_1,t_1) - \ln u(x_2,t_2) & = & -\int_{t_1}^{t_2} \frac{d}{dt} \ln u dt \\ & = & \int_{t_1}^{t_2} [-\langle \dot{\gamma}, \nabla_b (\ln u)\rangle - (\ln u)_t] dt. \end{array} \tag{4.30}$$

On the other hand, Theorem 4.3 implies that

$$-(\ln u)_t \leq At^{-1} + \alpha^{-1} \left|\nabla_b(\ln u)\right|^2$$

where $A = -C_1\alpha[C_1 - (\alpha+1)^{-1}]$ for some constant C_1 depending only on n. Hence (4.30) becomes

$$\ln \frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \int_{t_1}^{t_2} [|\dot{\gamma}| \left|\nabla_b(\ln u)\right| + \alpha^{-1}\left|\nabla_b(\ln u)\right|^2 + At^{-1}] dt.$$

Using the inequality

$$\alpha^{-1}\left|\nabla_b(\ln u)\right|^2 + |\dot{\gamma}| \left|\nabla_b(\ln u)\right| \leq -\frac{\alpha}{4}|\dot{\gamma}|^2$$

and choosing

$$|\dot{\gamma}| = \frac{d_c(x_1,x_2)}{t_2-t_1},$$

we conclude that

$$\ln \frac{u(x_1,t_1)}{u(x_2,t_2)} \leq -\frac{\alpha}{4}\frac{d_c(x_1,x_2)^2}{t_2-t_1} + A \ln \frac{t_2}{t_1}.$$

By taking exponential of both sides, we have

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \left(\frac{t_2}{t_1}\right)^{-C_1\alpha[C_1-(\alpha+1)^{-1}]} \exp\left(-\frac{\alpha d_c(x_1,x_2)^2}{4(t_2-t_1)}\right).$$

The result follows by choosing $\alpha = -2$. □

As a consequence of Corollary 4.7 and [CY], we have the following upper bound estimate for the heat kernel of (4.2).

Corollary 4.8. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group and $H(x,y,t)$ be the heat kernel of (4.2) on $M \times [0,\infty)$. Then for some constant $\delta > 1$ and $0 < \epsilon < 1$, $H(x,y,t)$ satisfies the estimate*

$$H(x,y,t) \leq C(\epsilon)^\delta V^{-\frac{1}{2}}(B_x(\sqrt{t}))V^{-\frac{1}{2}}(B_y(\sqrt{t})) \exp\left(-\frac{d_{cc}^2(x,y)}{(4+\epsilon)t}\right) \tag{4.31}$$

with $C(\epsilon) \to \infty$ as $\epsilon \to 0$.

Once we have the upper bound estimate for the heat kernel and the sub-laplacian comparison property (4.27). Then by applying the arguments of Li-Tam as in [LT] or [Li], we have the following mean value inequality.

Corollary 4.9. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group and g be subsolution of the CR heat equation such that*

$$\left(\frac{\partial}{\partial t} - \Delta_b\right) g\left(x,t\right) \leq 0.$$

Then for some constant C depend on δ, τ, η, such that $0 < \delta < 1$, $0 < \tau < T$, $0 < \eta < \frac{1}{2}$, the following inequality holds for any $\rho > 2\sqrt{T}$,

$$\sup_{B_p((1-\delta)\rho)\times[\tau,T]} g \leq C \int_{(1-\eta)\tau}^{T} \int_{B_p(\rho)} g\left(y,s\right) dyds. \tag{4.32}$$

4.3. **Complete noncompact case.** In [CN], Cao and Ni derived matrix Harnack estimates for the positive solution of the heat equation on complete noncompact Kähler manifolds of nonnegative bisectional curvature by using the key estimate (4.34) which is obtained from the result of Li-Yau heat kernel estimate ([LY]). For a general complete noncompact pseudohermitian manifold, we do not have Li-Yau type heat kernel estimates. However, we do have the CR corresponding result of Li-Yau heat kernel on Heisenberg groups as in Corollary 4.8. Comparing the method of Cao-Ni, we should point out that we also need the extra u_0-growth property (4.35) that has no analogue in Kähler manifolds. First we need a lemma [CFTW].

Lemma 4.4. *Let $h_s(z,t)$ be the heat kernel on $\boldsymbol{H}^n \times [0,\infty)$, and $M = \frac{C_3}{C_2 s^m}$ where C_2 ,C_3 constant depend on n, we have*

$$|\left(\tfrac{\partial}{\partial t}\right)^m h_s(z,t)| \leq M h_s(z,t)\ . \tag{4.33}$$

Note that $\frac{\partial}{\partial t}$ is the derivative along the T direction of $\boldsymbol{H}^n$ and s is a parameter of time.

Lemma 4.5. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n \times [0,\infty)$.. We have for $0 < \delta \leq t \leq 2-\delta$, there exists a constant $b > 0$ (might depends on δ) such that*

$$u(x,t) \leq \exp(b\left(r^2\left(x\right)+1\right)) \tag{4.34}$$

and

$$\left|u_0\right|(x,t) \leq \exp(b\left(r^2\left(x\right)+1\right)). \tag{4.35}$$

Proof. Let $o \in M$ be a fixed point. Since our focus here is to obtain an upper bound on u for positive time, we may assume that $u(x,t)$ is defined on $M \times [0,2]$. By Harnack inequality in Corollary 4.7, we have, for $0 < t < 2$

$$u(x,t) \leq \frac{C}{t^{C_3}} u(o,2) \exp(ar^2\left(x\right)).$$

Here a is a constant and $r^2(x)$ is the Carnot-Carathéodory distance $d_c(o,x)$. In particular, for $0<\delta\leq t\leq 2-\delta$, there exists a constant $b>0$ such that

$$u(x,t)\leq \exp(b(r^2(x)+1)).$$

But from (4.33) as in next section, we have

$$|u_0(x,t)|\leq \frac{C}{t}u(x,t).$$

Hence we also have

$$|u_0|(x,t)\leq \exp(b\left(r^2(x)+1\right)).$$

$\square$

Lemma 4.6. *Let M be a complete pseudohermitian $(2n+1)$-manifold with nonnegative bisectional curvature and nonnegative bi-torsion tensor. Let u be the positive solution of the CR heat equation (4.2). Then*

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)\|\nabla_b u\|^2\leq -2\|u_{\alpha\beta}\|^2-\frac{1}{2}\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2+4\|\nabla_b(u_0)\|\|\nabla_b u\|.$$

In addition if the positive solution u satisfies the purely holomorphic Hessian operator $P_{\alpha\bar{\beta}}u=0$. we have

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2\leq 0.$$

Proof. We compute

$$\begin{array}{rcl}
\left(\frac{\partial}{\partial t}-\Delta_b\right)\|\nabla_b u\|^2 & = & \left(\frac{\partial}{\partial t}-\Delta_b\right)2u_\alpha u_{\bar{\alpha}}\\
& = & 2\left(\Delta_b u\right)_\alpha u_{\bar{\alpha}}+2\left(\Delta_b u\right)_{\bar{\alpha}}u_\alpha-\left(2u_\alpha u_{\bar{\alpha}}\right)_{\beta\bar{\beta}}-\left(2u_\alpha u_{\bar{\alpha}}\right)_{\bar{\beta}\beta}\\
& = & 2\left(2u_{\beta\bar{\beta}\alpha}-inu_{0\alpha}\right)u_{\bar{\alpha}}+conj.-\left(2u_\alpha u_{\bar{\alpha}}\right)_{\beta\bar{\beta}}-conj\\
& = & 4u_{\alpha\beta\bar{\beta}}u_{\bar{\alpha}}-4ih_{\alpha\bar{\beta}}u_{\beta0}u_{\bar{\alpha}}-4R_{\bar{\rho}\alpha}u_\rho u_{\bar{\alpha}}-2inu_{0\alpha}u_{\bar{\alpha}}\\
& & +4u_{\bar{\alpha}\bar{\beta}\beta}u_\alpha+4ih_{\bar{\alpha}\beta}u_{\bar{\beta}0}u_\alpha-4R_{\rho\bar{\alpha}}u_{\bar{\rho}}u_\alpha+2inu_{0\bar{\alpha}}u_\alpha\\
& & -\left(2u_{\alpha\beta\bar{\beta}}u_{\bar{\alpha}}+2u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}+2u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}+2u_\alpha u_{\bar{\alpha}\beta\bar{\beta}}\right)\\
& & -\left(2u_{\bar{\alpha}\bar{\beta}\beta}u_\alpha+2u_{\bar{\alpha}\bar{\beta}}u_{\alpha\beta}+2u_{\bar{\alpha}\beta}u_{\alpha\bar{\beta}}+2u_{\bar{\alpha}}u_{\alpha\bar{\beta}\beta}\right)\\
& = & -4ih_{\alpha\bar{\beta}}u_{\beta0}u_{\bar{\alpha}}-4R_{\bar{\rho}\alpha}u_\rho u_{\bar{\alpha}}-2inu_{0\alpha}u_{\bar{\alpha}}\\
& & +4ih_{\bar{\alpha}\beta}u_{\bar{\beta}0}u_\alpha-4R_{\rho\bar{\alpha}}u_{\bar{\rho}}u_\alpha+2inu_{0\bar{\alpha}}u_\alpha-4u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}-4u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}\\
& & +2u_{\bar{\alpha}}\left(inu_{\alpha0}+R_{\alpha\bar{\rho}}u_\rho\right)-2u_\alpha\left(inu_{\bar{\alpha}0}-R_{\bar{\alpha}\rho}u_{\bar{\rho}}\right)\\
& = & -4u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}-4u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}-4R_{\bar{\rho}\alpha}u_\rho u_{\bar{\alpha}}\\
& & -4iu_{0\alpha}u_{\bar{\alpha}}+4iu_{0\bar{\alpha}}u_\alpha+2i\left(n-2\right)A_{\bar{\alpha}\bar{\beta}}u_\beta u_\alpha-2i\left(n-2\right)A_{\alpha\beta}u_{\bar{\beta}}u_{\bar{\alpha}}.
\end{array}$$

By curvature assumptions, we have

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)\|\nabla_b u\|^2\leq -2\|u_{\alpha\beta}\|^2-\frac{1}{2}\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2+4||\nabla_b(u_0)||\,\|\nabla_b u\|$$

and

$$
\begin{aligned}
& \left(\tfrac{\partial}{\partial t}-\Delta_b\right)\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2 \\
& =2\left(\left(\tfrac{\partial}{\partial t}-\Delta_b\right)\left(u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right)\right)\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right)+conj \\
& \quad -2\left(\left(u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right)_\gamma\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right)_{\bar{\gamma}}\right)+conj \\
& \leq 2\left(\left(\tfrac{\partial}{\partial t}-\Delta_b\right)\left(u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right)\right)\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right)+conj \\
& =2\left(2R_{\alpha\bar{\gamma}\delta\bar{\beta}}u_{\gamma\bar{\delta}}-R_{\alpha\bar{\delta}}u_{\delta\bar{\beta}}-R_{\delta\bar{\beta}}u_{\alpha\bar{\delta}}+C_{\alpha\bar{\beta}}\right)\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right)+conj \\
& =\left(2R_{\alpha\bar{\gamma}\delta\bar{\beta}}\left(u_{\gamma\bar{\delta}}+u_{\bar{\delta}\gamma}\right)-R_{\alpha\bar{\delta}}\left(u_{\delta\bar{\beta}}+u_{\bar{\beta}\delta}\right)-R_{\delta\bar{\beta}}\left(u_{\alpha\bar{\delta}}+u_{\bar{\delta}\alpha}\right)+C_{\alpha\bar{\beta}}\right)\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right)+conj \\
& \leq 0.
\end{aligned}
$$

Note that we have used $[\Delta, T]u=0$, $C_{\alpha\bar{\beta}}=0$ and the following inequality :

$$
\begin{aligned}
& \left(2R_{\alpha\bar{\gamma}\delta\bar{\beta}}\left(u_{\gamma\bar{\delta}}+u_{\bar{\delta}\gamma}\right)-R_{\alpha\bar{\delta}}\left(u_{\delta\bar{\beta}}+u_{\bar{\beta}\delta}\right)-R_{\delta\bar{\beta}}\left(u_{\alpha\bar{\delta}}+u_{\bar{\delta}\alpha}\right)\right)\left(u_{\bar{\alpha}\beta}+u_{\beta\bar{\alpha}}\right) \\
& =2R_{\alpha\bar{\beta}\beta\bar{\alpha}}\lambda_\alpha\lambda_\beta-2R_{\gamma\bar{\gamma}}\left(\lambda_\gamma\right)^2 \\
& =-R_{\alpha\bar{\beta}\beta\bar{\alpha}}(\lambda_\alpha-\lambda_\beta)^2 \\
& \leq 0.
\end{aligned}
$$

Here we denote $u_{\gamma\bar{\gamma}}+u_{\bar{\gamma}\gamma}=\lambda_\gamma$ (since $u_{\gamma\bar{\delta}}+u_{\bar{\delta}\gamma}$ is symmetric and then diagonalized). □

Combining Lemma 4.5 and Lemma 4.6, we are able to obtain the following.

Lemma 4.7. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n\times[0,\infty)$. There exists a constant $\hat{b}>0$, depending only on b such that*

$$
\int_{2\delta}^{T}\int_M \exp\left(-\hat{b}r^2\right)\left(\left\|\nabla_b u_0\right\|^2+\left\|\nabla_b u\right\|^2+\left\|u_{\alpha\beta}\right\|^2+\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2\right)d\mu dt<\infty.
$$

Proof. We multiply ϕ^2 on both sides of the following equation

$$
\left(\frac{\partial}{\partial t}-\Delta_b\right)u^2\leq -2\left\|\nabla u\right\|^2,
$$

where ϕ be a cut-off function such that $\phi=0$ for $d_c(x,p)>2R$, $t<\delta$, and $\phi=1$ as $d_c(x,p)<R$, $t>2\delta$ and $|\nabla\phi|\leq\frac{C}{R}$. Then

$$
\begin{aligned}
& \textstyle\int_0^T\int_M\left\|\nabla u\right\|^2\phi^2 d\mu dt \\
& \leq\textstyle\int_0^T\int_M\left(\left(\Delta-\frac{\partial}{\partial t}\right)u^2\right)\phi^2 d\mu dt \\
& =\textstyle\int_0^T\int_M\left(\Delta u^2\right)\phi^2 d\mu dt+\int_M u^2(x,0)\,\phi^2(x,0)\,d\mu \\
& \quad -\textstyle\int_M u^2(x,T)\,\phi^2(x,T)\,d\mu+\int_0^T\int_M u^2(x,t)\left(\phi^2\right)_t d\mu dt \\
& \leq\textstyle\int_0^T\int_M u^2(x,t)\left(\phi^2\right)_t d\mu dt-\int_0^T\int_M 2\phi\left\langle\nabla u^2,\nabla\phi\right\rangle_{J,\theta}d\mu dt.
\end{aligned}
\tag{4.36}
$$

By Young's inequality we have

$$
\int_0^T\int_M 2\phi\left\langle\nabla u^2,\nabla\phi\right\rangle_{J,\theta}d\mu dt\leq\frac{1}{2}\int_0^T\int_M\left\|\nabla u\right\|^2\phi^2 d\mu dt+8\int_0^T\int_M u^2\left\|\nabla\phi\right\|^2 d\mu dt.
$$

Then (4.36) becomes

$$\int_{2\delta}^{T}\int_{M}\|\nabla u\|^{2}\phi^{2}d\mu dt\leq 2\int_{\delta}^{T}\int_{M}u^{2}\left(8\|\nabla\phi\|^{2}+\left(\phi^{2}\right)_{t}\right)d\mu dt.$$

That is, there exist a positive constant C independent of R such that

$$\int_{2\delta}^{T}\int_{B_{p}(R)}\|\nabla u\|^{2}d\mu dt\leq C\int_{\delta}^{T}\int_{B_{p}(2R)}u^{2}d\mu dt.$$

By choosing $R=2^{n}$ and $b_{1}>4b$, thus

$$\begin{aligned}&\textstyle\int_{2\delta}^{T}\int_{M}e^{-b_{1}r^{2}}\|\nabla u\|^{2}d\mu dt\\&\leq\textstyle\sum_{n=1}^{\infty}e^{-b_{1}(2^{n})^{2}}\int_{\delta}^{T}\int_{B_{p}(2^{n+1})\setminus B_{p}(2^{n})}\|\nabla u\|^{2}d\mu dt\\&\leq C\textstyle\sum_{n=1}^{\infty}e^{-b_{1}(2^{n})^{2}}\int_{\delta}^{T}\int_{B_{p}(2^{n+1})}u^{2}d\mu dt\\&\leq C\textstyle\sum_{n=1}^{\infty}e^{-b_{1}(2^{n})^{2}}e^{b2^{2n+2}}\int_{\delta}^{T}\int_{B_{p}(2^{n+1})}e^{-br^{2}}u^{2}d\mu dt\\&\leq C\textstyle\int_{\delta}^{T}\int_{M}e^{-br^{2}}u^{2}d\mu dt\cdot\sum_{n=1}^{\infty}\left(\frac{e^{4b}}{e^{b_{1}}}\right)^{4^{n}}\\&<\infty,\end{aligned}\tag{4.37}$$

where the last inequality we use the growth rate of u as in Lemma 4.5. That is

$$\int_{2\delta}^{T}\int_{M}e^{-b_{1}r^{2}}\|\nabla u\|^{2}d\mu dt<\infty.\tag{4.38}$$

Again by $[\Delta,T]u=0$, we have

$$\left(\frac{\partial}{\partial t}-\Delta_{b}\right)u_{0}^{2}\leq-2\|\nabla u_{0}\|^{2}.$$

By Lemma 4.5 and follow the proof above, for some positive constant $b_{2}>0$, the following holds

$$\int_{\delta}^{T}\int_{M}e^{-b_{2}r^{2}}\|\nabla u_{0}\|^{2}d\mu dt<\infty.\tag{4.39}$$

From Lemma 4.6 and $[\Delta,T]u=0$, we have

$$\left(\frac{\partial}{\partial t}-\Delta_{b}\right)(\|\nabla_{b}u\|^{2}+u_{0}^{2}+u^{2})\leq-2\|u_{\alpha\beta}\|^{2}-\frac{1}{2}\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^{2}.\tag{4.40}$$

We multiply a test function ϕ^{2} and integrate as in (4.36), we have

$$\begin{aligned}&\textstyle\int_{0}^{T}\int_{M}(2\|u_{\alpha\beta}\|^{2}+\frac{1}{2}\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^{2})\phi^{2}d\mu dt\\&\leq\textstyle\int_{0}^{T}\int_{M}\left(\left(\Delta-\frac{\partial}{\partial t}\right)\left(\|\nabla_{b}u\|^{2}+u_{0}^{2}+u^{2}\right)\right)\phi^{2}d\mu dt\\&\leq\textstyle\int_{0}^{T}\int_{M}\left(\|\nabla_{b}u\|^{2}+u_{0}^{2}+u^{2}\right)(\phi^{2})_{t}d\mu\\&\quad-\textstyle\int_{0}^{T}\int_{M}2\phi\left\langle\nabla\left(\|\nabla_{b}u\|^{2}+u_{0}^{2}+u^{2}\right),\nabla\phi\right\rangle_{J,\theta}d\mu dt.\end{aligned}$$

By Young's inequality again, we obtain

$$\begin{aligned}&\textstyle\int_{0}^{T}\int_{M}\|u_{\alpha\beta}\|^{2}+\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^{2}\phi^{2}d\mu dt\\&\leq C\textstyle\int_{M}\left(\|\nabla u_{0}\|^{2}+\|\nabla u\|^{2}+u_{0}^{2}+u^{2}\right)\left(\|\nabla\phi\|^{2}+\left(\phi^{2}\right)_{t}\right)d\mu.\end{aligned}$$

Now the same argument as in (4.37), for some positive constant $b_3 > 0$, we have

$$\int_{2\delta}^{T}\int_{M}\exp\left(-b_3r^2\right)\left(\left\|u_{\alpha\beta}\right\|^2+\left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2\right)d\mu dt<\infty. \tag{4.41}$$

Choose $\hat{b}=\max\{b_1,b_2,b_3\}$, and combine (4.39), (4.38) and (4.41), we are done. ☐

The result of Lemma 4.7 can be improved to the following pointwise estimates by the mean valued inequality.

Lemma 4.8. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group. If $u(x,t)$ is the positive solution of the CR heat equation (4.2) on $\boldsymbol{H}^n\times[0,\infty)$. For $t>\delta$, there exists $\tilde{b}>0$ such that*

$$\begin{array}{rcl}\left\|\nabla_b u\right\|^2(x,t) & \leq & \exp\left(\tilde{b}\left(r^2+1\right)\right)\\ \left\|u_{\alpha\bar{\beta}}+u_{\bar{\beta}\alpha}\right\|^2(x,t) & \leq & \exp\left(\tilde{b}\left(r^2+1\right)\right)\end{array}. \tag{4.42}$$

Proof. We denote $\Phi=\left\|\nabla_b u\right\|^2+u_0^2+u^2$. It follows from (4.40) that Φ is a subsolution of the CR heat equation. We multiple factor $e^{-b\left(\rho^2+1\right)}$ on both sides of the mean value inequality (4.32), we have

$$\begin{array}{l} e^{-b\left(\rho^2+1\right)}\sup_{B_p((1-\delta)\rho)\times[\tau,T]}\Phi(x,t)\\ \leq Ce^{-b\left(\rho^2+1\right)}\int_{(1-\eta)\tau}^{T}\int_{B_p(\rho)}\Phi(y,s)\,dyds\\ \leq C\int_{(1-\eta)\tau}^{T}\int_{B_p(\rho)}e^{-b\left(r^2(y)+1\right)}\Phi(y,s)\,dyds\\ <\infty, \end{array}$$

where $r(y)$ is the Carnot-Carathéodory distance between p and y. The last inequality is followed from Lemma 4.5 and Lemma 4.7. Now we substitute $\rho=\frac{1}{1-\delta}r(x)$, we have for any $x\in B_p\left(\frac{1}{1-\delta}r(x)\right)$, $\tau<t<T$,

$$\Phi(x,t)\leq C'e^{b\left(\frac{1}{1-\delta}\right)^2\left(r^2(x)+1\right)}.$$

The other inequality in (4.42) can be obtained similarly. ☐

Lemma 4.9. *Let $(\boldsymbol{H}^n, J, \theta)$ be the standard $(2n+1)$-dimensional Heisenberg group and g be a smooth function on $\boldsymbol{H}^n$ such that*

$$\exp\left(k_1\left(r^2+1\right)\right)\leq\varphi\leq\exp\left(k_2\left(r^2+1\right)\right)$$

for some constant $k_2>k_1>0$, then there exists $T_m>0$ depending only on k_2 such that the Cauchy problem

$$\begin{cases}\left(\frac{\partial}{\partial t}-\Delta_b\right)g=0\\ u(x,0)=\varphi\end{cases}$$

has a solution g on $\boldsymbol{H}^n\times[0,T]$. Moreover, there exist constants $C_1,\ C_2>0$ such that

$$C_1\exp\left(\frac{k_1}{4}r^2\right)\leq g(x,t)\leq C_2\exp\left(3k_2r^2\right)$$

on $\boldsymbol{H}^n\times[0,T_m]$.

Proof. Similar argument as in Lemma 1.1 in [NT4], where the proof only using the heat kernel estimate (4.31) and the sub-laplacian comparison property (4.27). □

Proof of Theorem 4.5 :

Proof. It follows from Lemma 4.9 with $\phi = e^{\frac{\kappa}{\delta}t}g$ for $t > \delta$, we have

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)\phi = \frac{\kappa}{\delta}\phi$$

and

$$\phi(x,t) \geq C_1 \exp\left(2\tilde{b}\left(r^2+1\right)\right)$$

for a positive constant C_1 and a positive constant κ which to be determined later.

Let $N_{\alpha\bar{\beta}}$ be the matrix Harnack quantity in (4.6) We consider the following $(1,1)$-tensor

$$\hat{N}_{\alpha\bar{\beta}} = t^2 N_{\alpha\bar{\beta}} + \varepsilon\phi h_{\alpha\bar{\beta}}. \tag{4.43}$$

We only need to prove that $\hat{N}_{\alpha\bar{\beta}} > 0$ for any $\varepsilon > 0$. We shall prove this by contradiction. Suppose it is not true, then by the growth rate of ϕ and the fact that $N_{\alpha\bar{\beta}} > 0$ at $t = 0$, there exists a first time t_0 and by Lemma 4.8, a point $x_0 \in \boldsymbol{H}^n$ and a unit vector v at x_0 such that $\hat{N}_{\alpha\bar{\beta}}(x_0,t_0)\,v^\alpha v^{\bar{\beta}} = 0$. Now we choose choose a normal coordinate around x_0 and extend v to a local unit vector field near x_0. Then at x_0

$$\Delta_b\left(\hat{N}_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}}\right) = \Delta_b\left(\hat{N}_{\alpha\bar{\beta}}\right)v^\alpha v^{\bar{\beta}}.$$

Since $\hat{N}_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}} \geq 0$ for all (x,t) with $t \leq t_0$ and x close to x_0, we see that at (x_0,t_0),

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta_b\right)(\hat{N}_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}}). \tag{4.44}$$

On the other hand, it follows from (4.15) we have, at (x_0,t_0)

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\hat{N}_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}} \\
&= \left(\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\hat{N}_{\alpha\bar{\beta}}\right)v^\alpha v^{\bar{\beta}} \\
&\geq t^2\left(2R_{\alpha\bar{\gamma}\delta\bar{\beta}}N_{\gamma\bar{\delta}} - R_{\alpha\bar{\sigma}}N_{\sigma\bar{\beta}} - R_{\sigma\bar{\beta}}N_{\alpha\bar{\sigma}} + C_{\alpha\bar{\beta}}\right)v^\alpha v^{\bar{\beta}} \\
&+t^2\left(Rm - Tor\right)\left(\tfrac{\nabla u}{\sqrt{u}}, v\right) \\
&+t^2\tfrac{1}{2u}N_{\alpha\bar{\gamma}}N_{\beta\bar{\gamma}}v^\alpha v^{\beta} \\
&+t^2\left(\tfrac{u_{\bar{\beta}}u_\gamma}{8u^2}\hat{N}_{\alpha\bar{\gamma}} + \tfrac{u_\alpha u_{\bar{\gamma}}}{8u^2}\hat{N}_{\gamma\bar{\beta}} - \tfrac{1}{2u^2}u_\gamma u_{\bar{\beta}}\hat{N}_{\alpha\bar{\gamma}} - \tfrac{1}{2u^2}u_{\bar{\gamma}}u_\alpha\hat{N}_{\gamma\bar{\beta}}\right)v^\alpha v^{\bar{\beta}} \\
&+\tfrac{3}{4}\tfrac{\kappa}{\delta}t^2\varepsilon\phi\tfrac{u_{\bar{\beta}}u_\gamma v^\gamma v^\beta}{u^2} \\
&+t^2\tfrac{F}{u}N_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}} + \varepsilon\tfrac{\kappa}{\delta}\phi\left|v\right|^2.
\end{aligned} \tag{4.45}$$

Since $\hat{N}_{\alpha\bar{\beta}}(x_0,t_0)\,v^\alpha v^{\bar{\beta}} = 0$, it follows from (4.43) that at (x_0,t_0)

$$t^2\frac{1}{u}FN_{\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}} = -\frac{F}{u}\varepsilon\phi\left|v\right|^2.$$

Now $\frac{F}{u}=\frac{t}{24}\frac{(u_0)^2}{u^2}$, by using (4.33), we have

$$t^2\frac{1}{u}FN_{\alpha\bar{\beta}}v^{\alpha}v^{\bar{\beta}}\geq-\frac{C}{t}\varepsilon\phi\left|v\right|^2\geq-\frac{C}{\delta}\varepsilon\phi\left|v\right|^2$$

for some constant C. Hence

$$t^2\frac{1}{u}FN_{\alpha\bar{\beta}}v^{\alpha}v^{\bar{\beta}}+\varepsilon\frac{\kappa}{\delta}\phi\left|v\right|^2\geq(\kappa-\frac{C}{\delta})\varepsilon\phi\left|v\right|^2>0$$

if we choose

$$\kappa>\frac{C}{\delta}.$$

That is $\left(\frac{\partial}{\partial t}-\Delta_b\right)\hat{N}_{\alpha\bar{\beta}}v^{\alpha}v^{\bar{\beta}}>0$. This contradicts to (4.44).

This shows that $\hat{N}_{\alpha\bar{\beta}}\geq0$ for all $0<\delta\leq t\leq2-\delta$. Taking $\varepsilon\to0$ and $\delta\to0$ and repeating the argument to the later time. Then we are done. $\square$

Proof of Corollary 4.6 :

Proof. By applying Theorem 4.5 to the heat kernel $H(x,y,t)$ with $V=-\frac{\nabla_bH}{H}$, we have

$$-t[(\log H(x,y,t))_{\alpha\bar{\beta}}+(\log H(x,y,t))_{\bar{\beta}\alpha}]-\frac{3}{2}t[(\log H(x,y,t))_{\alpha}(\log H(x,y,t))_{\bar{\beta}}]\leq4h_{\alpha\bar{\beta}}.$$

But $-t\log H(x,o,t)\to\frac{1}{4}r^2(x)$ as $t\to0$. Therefore

$$-t[(\log H(x,o,t))_{\alpha\bar{\beta}}+(\log H(x,o,t))_{\bar{\beta}\alpha}]\to\frac{1}{4}[(r^2(x))_{\alpha\bar{\beta}}+(r^2(x))_{\bar{\beta}\alpha}]$$

in the sense of distribution. On the other hand, one can have

$$\frac{3}{2}t|\nabla_b(\log H(x,o,t)|^2\leq C_0$$

for some constant C_0 in a Heisenberg group $\boldsymbol{H}^n$ due to the dilation δ_r in $\boldsymbol{H}^n$ as in [JS, Theorem 1.]. Therefore

$$[(r^2(x))_{\alpha\bar{\beta}}+(r^2(x))_{\bar{\beta}\alpha}]\leq(16+C_0)h_{\alpha\bar{\beta}}(x).$$

$\square$

5. **Linear Trace Li-Yau-Hamilton inequality**

In this part, one of our main goals is to prove the Li-Yau-Hamilton type estimate for positive solutions of the CR Lichnerowicz-Laplacian heat equation. In the seminal paper [LY], P. Li and S.-T. Yau established the parabolic Li-Yau gradient estimate and Harnack inequality for positive solutions of the heat equation on Riemannian manifolds with nonnegative Ricci curvature. Later, Hamilton ([H1]) proved the matrix Harnack inequality for solutions to the Ricci flow when the curvature operator is nonnegative which is called the "Li-Yau-Hamilton" type estimates. Since then, there are many additional works in this direction which cover various different geometric evolution equations such as the mean curvature flow ([H2]), the Kaehler-Ricci flow ([Ca]), the Yamabe flow ([C]), etc.

In the case of Kähler geometry, it is well-known that Kähler-Ricci curvature $(1,1)$-tensor of a Kähler-Ricci flow solution satisfies a Lichnerowicz-Laplacian heat equation. In general, the Hodge-Laplacian heat equation on symmetric (p,p)-tensors is a geometrically interesting system and has been extensively studied since the original works of Hodge and Kodaira ([Mo] and references therein). For instances, we refer to the Lichnerowicz-Laplacian heat equation on $(1,1)$-tensors as in [NT1] and the Hodge-Laplacian heat equation on (p,p)-tensors as in [NN]. In the following we discuss the case of CR geometry.

Let (M,J,θ) be a strictly pseudoconvex CR $(2n+1)$-manifold. In our recent paper ([CCT]), we consider the CR Hodge-Laplacian

$$\Delta_H = -\frac{1}{2}(\Box_b + \overline{\Box}_b)$$

for Kohn-Rossi Laplacian $\Box_b$. For any $(1,1)$-form $\eta(x,t) = \eta_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$, we study the CR Hodge-Laplacian heat equation on $M \times [0,T)$

$$\frac{\partial}{\partial t}\eta(x,t) = 4\Delta_H \eta(x,t) \tag{5.1}$$

which connects to the existence problem of pseudo-Einstein CR $(2n+1)$-manifolds with $n \geq 2$. It follows from the CR Bochner-Weitzenbock Formula (5.12) that the CR parabolic equation (5.1) is equivalent to the CR analogue of Lichnerowicz-Laplacian heat equation :

$$\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta}} = 4\left[\Delta_b \eta_{\alpha\bar{\beta}} + 2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}} - (R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}})\right]. \tag{5.2}$$

In this chapter, one of the main results is to prove such LYH type estimates for this system (5.2). From now on, we assume that $\eta_{\alpha\bar{\beta}}(x,t)$ is a symmetric $(1,1)$-tensor on a strictly pseudoconvex CR $(2n+1)$-manifold satisfying the CR Lichnerowicz-Laplacian heat equation (5.2).

Define the Harnack quadratic by

(5.3)
$$Z(x,t)(V) := k_1\left[\frac{1}{2}\left((div\eta)_{\alpha,\bar{\alpha}} + (div\eta)_{\bar{\alpha},\alpha}\right) + (div\eta)_\alpha V_{\bar{\alpha}} + (div\eta)_\alpha V_{\bar{\alpha}} + V_{\bar{\alpha}}V_\beta \eta_{\alpha\bar{\beta}}\right] + \frac{H}{t}$$

for any vector field $V \in T^{1,0}(M)$, $H = h^{\alpha\overline{\beta}}\eta_{\alpha\bar{\beta}}$ and k_1 to be determined later. Moreover, $\eta_{\alpha\bar{\beta},0}$ is denoted the component of covariant derivative of the tensor η with Reeb vector field T.

With the notation above, the following is the CR analogue of the linear trace Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation.

Theorem 5.1. *Let (M,J,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a nonnegative symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M \times [0,T)$ and $\eta_{\alpha\bar{\beta},0}(x,0) = 0$ at $t = 0$. In additional if M is complete noncompact, we assume that there exists a constant $a > 0$ such that*

$$\int_\delta^T \int_M e^{-ar^2}\|\eta(x,t)\|^2 \, d\mu dt < \infty \tag{5.4}$$

and

$$\int_{\delta}^{T}\int_{M}e^{-ar^{2}}\left\Vert \nabla_{T}\eta\left(x,t\right)\right\Vert ^{2}d\mu dt<\infty, \tag{5.5}$$

where $r\left(x\right)$ is the Carnot-Carathéodory distance from a fixed point o and any $\delta>0$. Then

$$Z\left(x,t\right)\geq0,$$

for $0<k_{1}\leq8$.

Remark 5.1. *1. The assumption for the initial condition $\eta_{\alpha\bar{\beta},0}\left(x,0\right)=0$ is valid when we apply Theorem 5.1 to the CR Lichnerowicz-Laplacian heat equation coupled with the CR Yamabe flow as in Corollary 5.4. We refer to Remark 5.3 for more details.*

2. If M is complete noncompact, the extra requirement of (5.5) is needed to preserve $\nabla_{T}\eta\left(x,t\right)=0$ and the extra requirement of (5.4) is needed in order to apply the maximum principle to the Harnack quantity $t^{2}\widehat{Z}$ as in (5.37).

3. As in the paper of [N1] *and* [N2]*, a monotonicity derived from this sharp differential estimate of Li-Yau-Hamilton type can be applied to $\eta(x,t)$ to obtain dimension estimate for the space of holomorphic functions of polynomial growth and an optimal gap theorem. Then it is natural to ask whether or not the CR analogue of corresponding estimates still hold. We shall study its applications of this Li-Yau-Hamilton type estimate in this direction in a forth coming paper.*

In our recent paper ([CKW]), we study the torsion flow in a closed pseudohermitian 3-manifold which is the CR analogue of the Hamilton Ricci flow. More precisely, let $\theta(t)$ be a family of smooth contact forms and $J(t)$ be a family of CR structures on $(M,J_{0},\mathring{\theta})$ with $J(0)=J_{0}$ and $\theta(0)=\mathring{\theta}$. We consider the following torsion flow :

$$\begin{cases}\frac{\partial}{\partial t}J=2A_{J,\theta},\\ \frac{\partial}{\partial t}\theta=-2R\theta.\end{cases} \tag{5.6}$$

on $M\times[0,T)$ with $J(t)=i\theta^{1}\otimes Z_{1}-i\theta^{\bar{1}}\otimes Z_{\bar{1}}$ and $A_{J,\theta}(t)=A_{11}\theta^{1}\otimes Z_{\bar{1}}+A_{\bar{1}\bar{1}}\theta^{\bar{1}}\otimes Z_{1}$. In particular if the initial torsion is vanishing, it follows from Lemma 5.9 that the torsion flow (5.6) is equivalent to the CR Yamabe flow (5.7) in a closed spherical CR 3-manifold.

In this section, we present its application of Theorem 5.1 to obtain the nonlinear version of Harnack inequality for CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the following CR Yamabe flow :

$$\begin{cases}\frac{\partial}{\partial t}\theta\left(t\right)=-2R\left(t\right)\theta\left(t\right),\\ \theta\left(0\right)=\overset{0}{\theta},\quad\theta=e^{2f}\hat{\theta}\end{cases} \tag{5.7}$$

on $(M,\mathring{\theta})\times[0,T)$ with $e^{2f(x,0)}\hat{\theta}=\overset{0}{\theta}$. Here $R(t)$ is the Tanaka-Webster scalar curvature with respect to the contact form $\theta\left(t\right)$.

In order to prove Theorem 5.3, we need one more key fact. By applying Hamilton's general method for obtaining Harnack inequalities ([H1], [C]) to the CR Yamabe flow, we have

Theorem 5.2. *Let $(M,\overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then under the CR Yamabe flow (5.7)*

$$\frac{\partial R}{\partial t}+\frac{2R}{t}+2\left\langle \nabla_b R,V\right\rangle_{J,\theta}+\frac{3}{40}R\left\|V\right\|_{J,\theta}^2\geq 0 \tag{5.8}$$

for any $V\in T^{1,0}(M)$.

Now we are ready to state a Harnack type inequality for CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow.

Theorem 5.3. *Let $(M,J,\overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Let $\eta_{1\bar{1}}(x,t)$ be a positive symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7) on $M\times[0,T)$ and $\eta_{1\bar{1},0}=0$ for all t. Then*

$$Z_R:=Z+RH\geq 0$$

on $M\times[0,T)$ for $k_1=4$. In particular, taking $V=0$

$$2\Delta_b\eta_{1\bar{1}}+(R+\frac{1}{t})H\geq 0$$

and

$$\frac{\partial}{\partial t}\eta_{1\bar{1}}+2(R+\frac{1}{t})H\geq 0 \tag{5.9}$$

with $H=h^{1\bar{1}}\eta_{1\bar{1}}$.

As a consequence of Theorem 5.3 with $\eta_{1\bar{1}}=R_{1\bar{1}}=Rh_{1\bar{1}}$, we have the following trace Harnack inequality for the CR Yamabe flow (5.7) which turns out to be a special case of the linear Harnack inequality for the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7) on a closed strictly pseudoconvex spherical CR 3-manifold.

Corollary 5.4. *Let $(M,J,\overset{0}{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then we have the following trace Harnack inequality for the CR Yamabe flow (5.7)*

$$\frac{\partial}{\partial t}(t^2R)\geq 0 \tag{5.10}$$

which is Theorem 5.2 by taking $V\equiv 0$.

Note that B. Chow and R. Hamilton ([CH]), L. Ni and L.-F. Tam ([NT1]) proved the similar nonlinear trace Li-Yau-Hamilton inequality for the Ricci flow and Kaehler Ricci flow, respectively. However, we conjecture that the similar nonlinear trace Li-Yau-Hamilton (5.9) holds as well for the torsion flow (5.6) in a closed pseudohermitian 3-manifold.

The rest of the thesis is organized as follows. In section 5.1, we derive the CR Bochner-Weitzenbock type formula for $(1,1)$-tensors. Then it is natural to consider the CR Hodge-Laplacian heat equation which is equivalent to the CR analogue of Lichnerowicz-Laplacian heat equation as in the Ricci flow ([CH]) and the Kähler-Ricci flow ([NT1]). In section 5.2, we prove the linear trace Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation. In section 5.3, we prove a Harnack inequality for the CR Yamabe flow. Combining this with Theorem 5.1, we have the nonlinear version of Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation coupled with the CR Yamabe flow in a closed strictly pseudoconvex spherical CR 3-manifold.

5.1. **The CR Bochner-Weitzenbock Formula.** In this section, we will derive the CR analogue of Bochner-Weitzenbock Formula. Let ϕ be an (p,q)-form and denote by

$$\phi=\phi_{\alpha_1\alpha_2...\alpha_p\bar{\beta}_1\bar{\beta}_2...\bar{\beta}_q}\theta^{\alpha_1}\wedge...\wedge\theta^{\alpha_p}\wedge\theta^{\bar{\beta}_1}\wedge\theta^{\bar{\beta}_2}\wedge...\wedge\theta^{\bar{\beta}_q}.$$

For abbreviation, we denote as $\phi=\phi_{A\bar{B}}\theta^A\wedge\theta^{\bar{B}}$, where A and $\overline{B}$ are multiple index $A=(\alpha_1,\alpha_2,...,\alpha_p)$ and $\overline{B}=\left(\bar{\beta}_1,\bar{\beta}_2,...,\bar{\beta}_q\right)$ respectively. We define

$$\begin{aligned}\langle\phi,\varphi\rangle\quad &:=\frac{1}{p!q!}\phi_{\alpha_1...\alpha_p\bar{\beta}_1...\bar{\beta}_q}\overline{\varphi_{\rho_1...\rho_p\bar{\delta}_1....\bar{\delta}_q}}h^{\alpha_1\bar{\rho}_1}...h^{\alpha_p\bar{\rho}_p}\overline{h^{\beta_1\bar{\delta}_1}...h^{\beta_q\bar{\delta}_q}}\\ \text{(or for simplicity)}\quad &:=\frac{1}{p!q!}\phi_{A(\alpha)\bar{\beta}_1...\bar{\beta}_q}\overline{\varphi_{A(\rho)\bar{\delta}_1....\bar{\delta}_q}}\overline{h^{\beta_1\bar{\delta}_1}...h^{\beta_q\bar{\delta}_q}}.\end{aligned}$$

Theorem 5.5. *Let $\psi=\psi_{\alpha_1\alpha_2...\alpha_p\bar{\beta}_1\bar{\beta}_2...\bar{\beta}_{q+1}}\theta^{\alpha_1}\wedge...\wedge\theta^{\alpha_p}\wedge\theta^{\bar{\beta}_1}\wedge\theta^{\bar{\beta}_2}\wedge...\wedge\theta^{\bar{\beta}_{q+1}}$, we have*

$$\begin{aligned}&\tfrac{1}{2}\left(\Box_b\psi\right)_{A\bar{B}}\\ &=-\psi_{A,\bar{B};\bar{\mu};\mu}-2i\left(q+1\right)\nabla_T\psi_{A,\bar{B}}-\textstyle\sum_{k=1}^{q+1}\sum_{l=1}^{p}R_{\alpha_l}{}^{\gamma\bar{\mu}}{}_{\bar{\beta}_k}\psi_{A(l=\gamma),\bar{B}(k=\bar{\mu})}\\ &\quad-\textstyle\sum_{k,l=1,k\neq l}^{q+1}R_{\bar{\beta}_k}{}^{\bar{\gamma}\bar{\mu}}{}_{\bar{\beta}_l}\psi_{A,\bar{B}(k=\bar{\gamma};l=\bar{\mu})}+\sum_{k=1}^{q+1}R^{\bar{\gamma}}{}_{\bar{\beta}_k}\psi_{A,\bar{B}(k=\bar{\gamma})}.\end{aligned}\tag{5.11}$$

Here

$$\begin{aligned}A&=\alpha_1\alpha_2...\alpha_p\\ A\left(l=\gamma\right)&=\alpha_1\alpha_2...\alpha_{l-1}\gamma\alpha_{l+1}...\alpha_p\\ \bar{B}&=\bar{\beta}_1\bar{\beta}_2...\bar{\beta}_{q+1}\\ \bar{B}\left(k=\bar{\mu}\right)&=\bar{\beta}_1\bar{\beta}_2...\bar{\beta}_{k-1}\bar{\mu}\bar{\beta}_{k+1}...\bar{\beta}_{q+1}.\end{aligned}$$

In particular, taking $p=1=q+1$ in (5.11), we have

$$\left(\frac{1}{2}\Box_b\psi\right)_{\alpha\bar{\beta}}=-\psi_{\alpha\bar{\beta},\bar{\mu},\mu}-2i\nabla_0\psi_{\alpha\bar{\beta}}-R_{\alpha}{}^{\gamma\bar{\mu}}{}_{\bar{\beta}}\psi_{\gamma\bar{\mu}}+R^{\bar{\gamma}}{}_{\bar{\beta}}\psi_{\alpha\bar{\gamma}}.$$

Now we have the following special CR Bochner-Weitzenbock formula for an $(1,1)$-form ψ.

Corollary 5.6. *For an $(1,1)$-form $\psi=\psi_{\alpha\bar{\beta}}\theta^{\alpha}\wedge\theta^{\bar{\beta}}$, we have*

$$\left(\Delta_H\psi\right)_{\alpha\bar{\beta}}=-\frac{1}{2}\left(\left(\bar{\Box}_b+\Box_b\right)\psi\right)_{\alpha\bar{\beta}}=\Delta_b\psi_{\alpha\bar{\beta}}+2R_{\alpha}{}^{\gamma\bar{\mu}}{}_{\bar{\beta}}\psi_{\gamma\bar{\mu}}-R^{\bar{\gamma}}{}_{\bar{\beta}}\psi_{\alpha\bar{\gamma}}-R^{\gamma}{}_{\alpha}\psi_{\gamma\bar{\beta}}.\tag{5.12}$$

We first derive the following Lemma.

Lemma 5.1. *Let*

$$\phi=\phi_{\alpha_1\alpha_2...\alpha_p\bar{\beta}_1\bar{\beta}_2...\bar{\beta}_q}\theta^{\alpha_1}\wedge...\wedge\theta^{\alpha_p}\wedge\theta^{\bar{\beta}_1}\wedge\theta^{\bar{\beta}_2}\wedge...\wedge\theta^{\bar{\beta}_q}$$

and

$$\psi=\psi_{\alpha_1\alpha_2\ldots\alpha_p\bar\beta_1\bar\beta_2\ldots\bar\beta_{q+1}}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_p}\wedge\theta^{\bar\beta_1}\wedge\theta^{\bar\beta_2}\wedge\ldots\wedge\theta^{\bar\beta_{q+1}},$$

we have

(5.13)

$$\begin{aligned}&\bar\partial_b\phi\\&=(-1)^p\sum_{i=1}^{q+1}(-1)^{i-1}\nabla_{\bar\beta_i}\phi_{\alpha_1\ldots\alpha_p\bar\beta_1\ldots\bar\beta_{i-1}\bar\beta_{i+1}\ldots\bar\beta_{q+1}}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_p}\wedge\theta^{\bar\beta_1}\wedge\theta^{\bar\beta_2}\wedge\ldots\wedge\theta^{\bar\beta_{q+1}}\end{aligned}$$

and

$$\begin{aligned}&\bar\partial_b^*\psi\\&=(-1)^p\tfrac{1}{q+1}\sum_{i=1}^{q+1}(-1)^i\nabla_\mu\psi_{\alpha_1\ldots\alpha_p\bar\beta_1\ldots\bar\beta_{i-1}\bar\mu\bar\beta_i\ldots\bar\beta_q}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_p}\wedge\theta^{\bar\beta_1}\wedge\theta^{\bar\beta_2}\wedge\ldots\wedge\theta^{\bar\beta_q}.\end{aligned}\tag{5.14}$$

Proof. (i) By taking exterior derivative, we obtain

$$d\phi=d_b\phi+T\phi:=\partial_b\phi+\bar\partial_b\phi+T\phi$$

where $\partial_b\phi$ is the $(p+1,q)$-form part of $d_b\phi$ and $\bar\partial_b\phi$ is the $(p,q+1)$-form part of $d_b\phi$, $T\phi$ is the form spanned by basis $\theta\wedge\theta^A\wedge\theta^{\bar B}$. In fact we have following

$$\left(\bar\partial_b\phi\right)_{\alpha_1\ldots\alpha_p\bar\beta_1\ldots\bar\beta_{q+1}}=\left(d_b\phi\right)_{\alpha_1\ldots\alpha_p\bar\beta_1\ldots\bar\beta_{q+1}}=d_b\phi\left(X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}}\right).$$

Here

$$\begin{aligned}&d_b\phi(X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}})\\=\;&\sum_{i=1}^p(-1)^{i+1}X_{\alpha_i}\phi\left(X_{\alpha_1},\ldots,X_{\alpha_{i-1}},X_{\alpha_{i+1}},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}}\right)\\&+\sum_{j=1}^{q+1}(-1)^{p+j+1}X_{\bar\beta_j}\phi(X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{j-1}},X_{\bar\beta_{j+1}},\ldots,X_{\bar\beta_{q+1}})\\&+\sum_{i<j}(-1)^{i+j}\phi\left(\left[X_{\alpha_i},X_{\alpha_j}\right],X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}}\right)\\&+\sum_{i<j}(-1)^{i+j}\phi\left(\left[X_{\bar\beta_i},X_{\bar\beta_j}\right],X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}}\right)\\&+\sum_{i=1..p,j=1..q+1}(-1)^{p+i+j}\phi\left(\left[X_{\alpha_i},X_{\bar\beta_j}\right],X_{\alpha_1},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{q+1}}\right).\end{aligned}$$

Notice that the first term and third term is zero since ϕ evaluate on $(q+1)$ conjugate tangent vector fields is zero. Moreover, since

$$\left[X_{\alpha_i},X_{\bar\beta_j}\right]=\Gamma^{\bar\gamma}_{\alpha_i\bar\beta_j}X_{\bar\gamma}-\Gamma^{\gamma}_{\bar\beta_j\alpha_i}X_\gamma-ih_{\alpha_i\bar\beta_j}T,$$

the last term will be

$$\phi\left(-\Gamma^{\gamma}_{\bar\beta_j\alpha_i}X_\gamma,X_{\alpha_1},\ldots,X_{\alpha_{i-1}},X_{\alpha_{i+1}},\ldots,X_{\alpha_p},X_{\bar\beta_1},\ldots,X_{\bar\beta_{j-1}},X_{\bar\beta_{j+1}},\ldots,X_{\bar\beta_{q+1}}\right).$$

Hence

$$
\begin{aligned}
& \bar{\partial}_b \phi \left(X_{\alpha_1}, ..., X_{\alpha_p}, X_{\bar{\beta}_1}, ..., X_{\bar{\beta}_{q+1}} \right) \\
= \; & \sum_{j=1}^{q+1} (-1)^{p+j+1} X_{\bar{\beta}_j} \phi(X_{\alpha_1}, ..., X_{\alpha_p}, X_{\bar{\beta}_1}, ..., X_{\bar{\beta}_{j-1}}, X_{\bar{\beta}_{j+1}}, ..., X_{\bar{\beta}_{q+1}}) \\
& + \sum_{i<j} (-1)^{p+i+j} \phi \left(X_{\alpha_1}, ..., X_{\alpha_p}, \left[X_{\bar{\beta}_i}, X_{\bar{\beta}_j} \right], X_{\bar{\beta}_1}, ..., X_{\bar{\beta}_{i-1}}, X_{\bar{\beta}_{i+1}}, ..., X_{\bar{\beta}_{j-1}}, X_{\bar{\beta}_{j+1}}, ..., X_{\bar{\beta}_{q+1}} \right) \\
& + \sum_{\substack{i=1..p, \\ j=1..q+1}} (-1)^{p+i+j} \phi \left(-\Gamma^{\gamma}_{\bar{\beta}_j \alpha_i} X_{\gamma}, X_{\alpha_1}, ..., X_{\alpha_{i-1}}, X_{\alpha_{i+1}}, ..., X_{\alpha_p}, X_{\bar{\beta}_1}, ..., X_{\bar{\beta}_{j-1}}, X_{\bar{\beta}_{j+1}}, ..., X_{\bar{\beta}_{q+1}} \right) \\
= \; & (-1)^p \sum_{j=1}^{q+1} (-1)^{j+1} \nabla_{\bar{\beta}_j} \phi(X_{\alpha_1}, ..., X_{\alpha_p}, X_{\bar{\beta}_1}, ..., X_{\bar{\beta}_{j-1}}, X_{\bar{\beta}_{j+1}}, ..., X_{\bar{\beta}_{q+1}}).
\end{aligned}
$$

(ii) The second formula (5.14) follows from the following computation. For $d\mu := \theta \wedge (d\theta)^n$ and $c_{p,q} = \frac{1}{p!(q+1)!}$

$$
\begin{aligned}
& \left(\left(\bar{\partial}_b \phi\right)_{A\bar{B}}, \psi_{AB} \right) \\
& = \int_M \left\langle \bar{\partial}_b \phi, \psi \right\rangle d\mu \\
& = c_{p,q} \int_M \left(\bar{\partial}_b \phi \right)_{A\bar{\beta}_1 \bar{\beta}_2 ... \bar{\beta}_{q+1}} \overline{\psi_{A\bar{\rho}_1 \bar{\rho}_2 ... \bar{\rho}_{q+1}}} \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_{q+1} \bar{\beta}_{q+1}}} d\mu \\
& = c_{p,q} \int (-1)^p \sum_{i=1}^{q+1} (-1)^{i-1} \nabla_{\bar{\beta}_i} \phi_{A\bar{\beta}_1 ... \bar{\beta}_{i-1} \bar{\beta}_{i+1} ... \bar{\beta}_{q+1}} \left(\overline{\psi_{A\bar{\rho}_1 \bar{\rho}_2 ... \bar{\rho}_{q+1}}} \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_{q+1} \bar{\beta}_{q+1}}} \right) d\mu \\
& = c_{p,q} \int (-1)^p \sum_{i=1}^{q+1} (-1)^{i} \phi_{A\bar{\beta}_1 ... \bar{\beta}_{i-1} \bar{\beta}_{i+1} ... \bar{\beta}_{q+1}} \nabla_{\bar{\beta}_i} \left(\overline{\psi_{A\bar{\rho}_1 \bar{\rho}_2 ... \bar{\rho}_{q+1}}} \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_{q+1} \bar{\beta}_{q+1}}} \right) d\mu \\
& = c_{p,q} \sum_{i=1}^{q+1} (-1)^{i+p} \int \phi_{A\bar{\beta}_1 ... \bar{\beta}_{i-1} \bar{\beta}_{i+1} ... \bar{\beta}_{q+1}} \left(\overline{h^{\rho_i \bar{\beta}_i}} \nabla_{\bar{\beta}_i} \overline{\psi_{A\bar{\rho}_1 \bar{\rho}_2 ... \bar{\rho}_{q+1}}} \right) \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_{i-1} \bar{\beta}_{i-1}} h^{\rho_{i+1} \bar{\beta}_{i+1}} ... h^{\rho_{q+1} \bar{\beta}_{q+1}}} d\mu \\
& = c_{p,q} \sum_{i=1}^{q+1} (-1)^{i+p} \int \phi_{AB} \overline{h^{\mu\bar{\eta}}} \nabla_{\bar{\eta}} \overline{\psi_{A\bar{\rho}_1 ... \bar{\rho}_{i-1} \bar{\mu} \bar{\rho}_i ... \bar{\rho}_q}} \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_q \bar{\beta}_q}} d\mu \\
& = c_{p,q} (-1)^p \int \phi_{A\overline{B}} \sum_{i=1}^{q+1} (-1)^i \, \overline{h^{\mu\bar{\eta}}} \nabla_{\bar{\eta}} \overline{\psi_{A\bar{\rho}_1 ... \bar{\rho}_{i-1} \bar{\mu} \bar{\rho}_i ... \bar{\rho}_q}} \overline{h^{\rho_1 \bar{\beta}_1} ... h^{\rho_q \bar{\beta}_q}} d\mu.
\end{aligned}
$$

$\square$

Proof of Theorem 5.5 :

Proof. It follows from (5.14) that

$$
\begin{aligned}
& \left(\bar{\partial}_b(\bar{\partial}_b^*\psi)\right)_{A\bar{\beta}_1\bar{\beta}_2\ldots\bar{\beta}_{q+1}} \\
= \;& (-1)^p \sum_{i=1}^{q+1} (-1)^{i-1}\, \nabla_{\bar{\beta}_i} \left(\bar{\partial}_b^*\psi\right)_{A\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}} \\
= \;& (-1)^{p+i-1} \sum_{i=1}^{q+1} \nabla_{\bar{\beta}_i} \left(\begin{array}{l} \frac{1}{q+1}(-1)^p \sum_{j\le i}(-1)^j\, \nabla_\mu \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_j\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}} \\ +\frac{1}{q+1}(-1)^p \sum_{j\ge i+2}^{q+2}(-1)^{j-1}\, \nabla_\mu \psi_{A\bar{\beta}_1\ldots.\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_j\ldots\bar{\beta}_{q+1}} \end{array} \right) \\
= \;& \frac{(-1)^{p+i-1}}{q+1} \sum_{i=1}^{q+1} \nabla_{\bar{\beta}_i} \left(\begin{array}{l} (-1)^p \sum_{j\le i}(-1)^j\,(-1)^{q+1-j}\, \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\beta}_j\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\mu} \\ +(-1)^p \sum_{j\ge i+2}^{q+2}(-1)^{j-1}\,(-1)^{q+1-j+1}\, \psi_{A\bar{\beta}_1\ldots.\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{j-1}\bar{\beta}_j\ldots\bar{\beta}_{q+1}\bar{\mu};\mu} \end{array} \right) \\
= \;& \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i-1}\, \nabla_{\bar{\beta}_i} \left(\sum_{j=1, j\neq i+1}^{q+2} (-1)^{q+1}\, \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\mu} \right) \\
= \;& \frac{1}{q+1} \sum_{i=1}^{q+1} \sum_{j=1,j\neq i}^{q+2} (-1)^{i+q}\, \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\mu;\bar{\beta}_i} \\
= \;& \sum_{i=1}^{q+1} (-1)^{i+q}\, \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\mu;\bar{\beta}_i},
\end{aligned}
$$

and

$$
\begin{aligned}
& \left(\bar{\partial}_b^*(\bar{\partial}_b\psi)\right)_{A\bar{\beta}_1\bar{\beta}_2\ldots\bar{\beta}_{q+1}} \\
= \;& (-1)^p \frac{1}{q+2} \sum_{i=1}^{q+2} (-1)^i\, \nabla_\mu \left(\bar{\partial}_b\psi\right)_{A\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\mu}\bar{\beta}_i\ldots\bar{\beta}_{q+1}} \\
= \;& (-1)^p \frac{1}{q+2} \sum_{i=1}^{q+2} (-1)^q\, \nabla_\mu \left(\bar{\partial}_b\psi\right)_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1}\bar{\mu}} \\
= \;& (-1)^{p+q}\, \nabla_\mu \left(\bar{\partial}_b\psi\right)_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1}\bar{\mu}} \\
= \;& (-1)^{p+q}\, \nabla_\mu \left((-1)^p \sum_{j=1}^{q+1} (-1)^{i-1}\, \nabla_{\bar{\beta}_j} \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}\bar{\mu}} + (-1)^{p+q+1}\, \nabla_{\bar{\mu}} \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1}} \right) \\
= \;& -\nabla_\mu \nabla_{\bar{\mu}} \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1}} - \sum_{j=1}^{q+1} (-1)^{q+j}\, \nabla_\mu \nabla_{\bar{\beta}_j} \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}\bar{\mu}}.
\end{aligned}
$$

Adding these together we have

$$\begin{aligned}
&\tfrac{1}{2}\Box_b\psi \\
&= \left(\bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b\right)\psi \\
&= -\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1};\bar{\mu};\mu} + \textstyle\sum_{j=1}^{q+1}(-1)^{j+q}\left(\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\mu;\bar{\beta}_j} - \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}\bar{\mu};\bar{\beta}_j;\mu}\right) \\
&= -\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1};\bar{\mu};\mu} + \textstyle\sum_{j=1}^{q+1}\left(\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1};\bar{\beta}_j;\mu} - \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1};\mu;\bar{\beta}_j}\right).
\end{aligned}$$

By commutation relation, we have

$$\begin{aligned}
&\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1};\bar{\beta}_j;\mu} - \psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1};\mu;\bar{\beta}_j} \\
&= -2i(q+1)\nabla_0\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{q+1}} - R_{\alpha_i}{}^{\gamma\bar{\mu}}{}_{\bar{\beta}_j}\psi_{\alpha_1\ldots\alpha_{i-1}\gamma\alpha_{i+1}\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}} \\
&\quad -R_{\bar{\beta}_k}{}^{\bar{\gamma}}{}_{\mu\bar{\beta}_j}\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{k-1}\bar{\gamma}\bar{\beta}_{k+1}\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}} + R^{\bar{\gamma}}{}_{\bar{\beta}_j}\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\gamma}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\tfrac{1}{2}\Box_b\psi &= -\psi_{A\overline{B};\bar{\mu};\mu} - 2i(q+1)\nabla_0\psi_{A\overline{B}} \\
&\quad -\textstyle\sum_{j=1}^{q+1}\sum_{i=1}^{p}R_{\alpha_i}{}^{\gamma\bar{\mu}}{}_{\bar{\beta}_j}\psi_{\alpha_1\ldots\alpha_{i-1}\gamma\alpha_{i+1}\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}} \\
&\quad -\textstyle\sum_{j=1}^{q+1}R_{\bar{\beta}_k}{}^{\bar{\gamma}\bar{\mu}}{}_{\bar{\beta}_j}\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{k-1}\bar{\gamma}\bar{\beta}_{k+1}\ldots\bar{\beta}_{j-1}\bar{\mu}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}} \\
&\quad +\textstyle\sum_{j=1}^{q+1}R^{\bar{\gamma}}{}_{\bar{\beta}_j}\psi_{A\bar{\beta}_1\ldots\bar{\beta}_{j-1}\bar{\gamma}\bar{\beta}_{j+1}\ldots\bar{\beta}_{q+1}}.
\end{aligned}$$

$\square$

5.2. **Linear Trace Li-Yau-Hamilton Inequality.** In this section, we will derive the LYH type estimate for the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$. We refer to [CH] and [NT1] for related estimates.

Lemma 5.2. *Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$. Then*

$$\left(\tfrac{\partial}{\partial t} - 4\Delta_b\right)(div\eta)_\alpha = -8i\eta_{\alpha\bar{\lambda},0\lambda} - 4R^{\gamma}{}_{\alpha}(div\eta)_\gamma + (Tor\ I)_\alpha.$$

Here we denote

$$(Tor\ I)_\alpha := 4\left(-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}} - 2iA_{\alpha\lambda}H_{\bar{\lambda}} + in(divA)_\gamma\eta_{\alpha\bar{\gamma}} + inA_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}} + 2iA_{\alpha\delta}(div\eta)_{\bar{\delta}}\right).$$

Proof. We first compute

$$\begin{aligned}
\tfrac{1}{4}\tfrac{\partial}{\partial t}\left(\eta_{\alpha\bar{\delta},\delta}\right) &= \tfrac{1}{4}\left(\tfrac{\partial}{\partial t}\eta\right)_{\alpha\bar{\delta},\delta} \\
&= \left(\eta_{\alpha\bar{\delta},\lambda\bar{\lambda}} + \eta_{\alpha\bar{\delta},\bar{\lambda}\lambda} - 2R_{\alpha\bar{\gamma}\bar{\delta}\mu}\eta_{\gamma\bar{\mu}} - R^{\bar{\gamma}}{}_{\bar{\delta}}\eta_{\alpha\bar{\gamma}} - R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_\delta \\
&= \eta_{\alpha\bar{\delta},\lambda\bar{\lambda}\delta} + \eta_{\alpha\bar{\delta},\bar{\lambda}\lambda\delta} - \left(2R_{\alpha\bar{\gamma}\bar{\delta}\mu}\eta_{\gamma\bar{\mu}} + R^{\bar{\gamma}}{}_{\bar{\delta}}\eta_{\alpha\bar{\gamma}} + R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_\delta.
\end{aligned}\tag{5.15}$$

(i) Now we deal with first term of RHS in (5.15) : By commutation relation

$$\begin{aligned}
\eta_{\alpha\bar{\delta},\lambda\bar{\lambda}\delta} &= \eta_{\alpha\bar{\delta},\lambda\delta\bar{\lambda}} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda0} - R_{\alpha\bar{\gamma}\delta\bar{\lambda}}\eta_{\gamma\bar{\delta},\lambda} - R_{\lambda\bar{\gamma}\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\gamma} - R_{\bar{\delta}\gamma\delta\bar{\lambda}}\eta_{\alpha\bar{\gamma},\lambda} \\
&= \eta_{\alpha\bar{\delta},\lambda\delta\bar{\lambda}} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda0} - R_{\alpha\bar{\gamma}\delta\bar{\lambda}}\eta_{\gamma\bar{\delta},\lambda} - R^{\gamma}{}_{\delta}\eta_{\alpha\bar{\delta},\gamma} + R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma},\lambda}
\end{aligned}$$

and

$$\begin{aligned}
\left(\eta_{\alpha\bar{\delta},\lambda\delta} - \eta_{\alpha\bar{\delta},\delta\lambda}\right)_{\bar{\lambda}} &= i\left(A_{\alpha\delta}\eta_{\lambda\bar{\delta}} - A_{\alpha\lambda}\eta_{\delta\bar{\delta}}\right)_{\bar{\lambda}} - i\left(\eta_{\alpha\bar{\gamma}}h_{\bar{\delta}\lambda}A^{\bar{\gamma}}_{\bar{\delta}} - \eta_{\alpha\bar{\gamma}}h_{\bar{\delta}\delta}A^{\bar{\gamma}}_{\bar{\lambda}}\right)_{\bar{\lambda}} \\
&= i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}} - A_{\alpha\lambda}H\right)_{\bar{\lambda}} - i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}} - n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}}.
\end{aligned}$$

Hence

$$\begin{aligned}\eta_{\alpha\bar{\delta},\lambda\bar{\lambda}\delta} = \eta_{\alpha\bar{\delta},\delta\lambda\bar{\lambda}} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda 0} - R_{\alpha\bar{\gamma}\delta\bar{\lambda}}\eta_{\gamma\bar{\delta},\lambda} - R^{\gamma}{}_{\delta}\eta_{\alpha\bar{\delta},\gamma} + R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma},\lambda} \\ +i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}} - A_{\alpha\lambda}H\right)_{\bar{\lambda}} - i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}} - n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}}.\end{aligned}$$

(ii) For the second term of RHS in (5.15) : Again by commutation relation

$$\eta_{\alpha\bar{\delta},\bar{\lambda}\lambda\delta} = \eta_{\alpha\bar{\delta},\bar{\lambda}\delta\lambda} + i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta} - \eta_{\delta\bar{\delta},\bar{\lambda}}A_{\alpha\lambda}\right)$$

and

$$\left(\eta_{\alpha\bar{\delta},\bar{\lambda}\delta} - \eta_{\alpha\bar{\delta},\delta\bar{\lambda}}\right)_{\lambda} = \left(-i\eta_{\alpha\bar{\delta},0}h_{\bar{\lambda}\delta} - \eta_{\gamma\bar{\delta}}R_{\alpha\bar{\gamma}\delta\bar{\lambda}} + R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma}}\right)_{\lambda}.$$

One obtains

$$\eta_{\alpha\bar{\delta},\bar{\lambda}\delta\lambda} = \eta_{\alpha\bar{\delta},\delta\bar{\lambda}\lambda} - i\eta_{\alpha\bar{\delta},0\lambda}h_{\bar{\lambda}\delta} - \left(\eta_{\gamma\bar{\delta}}R_{\alpha\bar{\gamma}\delta\bar{\lambda}} - R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma}}\right)_{\lambda}.$$

Combining all three equalities, we have

$$\begin{aligned}\eta_{\alpha\bar{\delta},\bar{\lambda}\lambda\delta} = \eta_{\alpha\bar{\delta},\delta\bar{\lambda}\lambda} - i\eta_{\alpha\bar{\delta},0\lambda}h_{\bar{\lambda}\delta} - \left(\eta_{\gamma\bar{\delta}}R_{\alpha\bar{\gamma}\delta\bar{\lambda}} - R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma}}\right)_{\lambda} \\ +i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta} - \eta_{\delta\bar{\delta},\bar{\lambda}}A_{\alpha\lambda}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\eta_{\alpha\bar{\delta},\lambda\bar{\lambda}\delta} + \eta_{\alpha\bar{\delta},\bar{\lambda}\lambda\delta} - \left(2R_{\alpha\bar{\gamma}\bar{\delta}\mu}\eta_{\gamma\bar{\mu}} + R^{\bar{\gamma}}{}_{\bar{\delta}}\eta_{\alpha\bar{\gamma}} + R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_{\delta} \\ &= \Delta_b\left(div\eta\right)_{\alpha} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda 0} - i\eta_{\alpha\bar{\delta},0\lambda}h_{\bar{\lambda}\delta} \\ &\quad + \left(-2R_{\alpha\bar{\gamma}\bar{\delta}\mu}\eta_{\gamma\bar{\mu}} - R^{\bar{\gamma}}{}_{\bar{\delta}}\eta_{\alpha\bar{\gamma}} - R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_{\delta} \\ &\quad -R_{\alpha\bar{\gamma}\delta\bar{\lambda}}\eta_{\gamma\bar{\delta},\lambda} - R^{\gamma}{}_{\delta}\eta_{\alpha\bar{\delta},\gamma} + R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma},\lambda} + \left(-\eta_{\gamma\bar{\delta}}R_{\alpha\bar{\gamma}\delta\bar{\lambda}} + R^{\bar{\gamma}}{}_{\bar{\lambda}}\eta_{\alpha\bar{\gamma}}\right)_{\lambda} \\ &\quad +i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}} - A_{\alpha\lambda}H\right)_{\bar{\lambda}} - i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}} - n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}} \\ &\quad +i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta} - \eta_{\delta\bar{\delta},\bar{\lambda}}A_{\alpha\lambda}\right) \\ &= \Delta_b\left(div\eta\right)_{\alpha} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda 0} - i\eta_{\alpha\bar{\delta},0\lambda}h_{\bar{\lambda}\delta} \\ &\quad + R_{\alpha\bar{\gamma}\mu\bar{\delta},\delta}\eta_{\gamma\bar{\mu}} - \left(R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_{\delta} \\ &\quad +i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}} - A_{\alpha\lambda}H\right)_{\bar{\lambda}} - i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}} - n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}} \\ &\quad +i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta} - \eta_{\delta\bar{\delta},\bar{\lambda}}A_{\alpha\lambda}\right).\end{aligned} \tag{5.16}$$

On the other hand, it follows from the CR Bianchi identity

$$R_{\alpha\bar{\beta}\rho\bar{\sigma},\gamma} - R_{\alpha\bar{\beta}\gamma\bar{\sigma},\rho} = iA_{\alpha\gamma,\bar{\beta}}h_{\rho\bar{\sigma}} + iA_{\alpha\gamma,\bar{\sigma}}h_{\rho\bar{\beta}} - iA_{\alpha\rho,\bar{\beta}}h_{\gamma\bar{\sigma}} - iA_{\alpha\rho,\bar{\sigma}}h_{\gamma\bar{\beta}},$$

we have

$$\begin{aligned}&R_{\alpha\bar{\gamma}\mu\bar{\delta},\delta}\eta_{\gamma\bar{\mu}} - R_{\alpha\bar{\gamma}\delta\bar{\delta},\mu}\eta_{\gamma\bar{\mu}} \\ &= \left(iA_{\alpha\delta,\bar{\gamma}}h_{\mu\bar{\delta}} + iA_{\alpha\delta,\bar{\delta}}h_{\mu\bar{\gamma}} - iA_{\alpha\mu,\bar{\gamma}}h_{\delta\bar{\delta}} - iA_{\alpha\mu,\bar{\delta}}h_{\delta\bar{\gamma}}\right)\eta_{\gamma\bar{\mu}} \\ &= \left(iA_{\alpha\delta,\bar{\delta}}h_{\mu\bar{\gamma}} - inA_{\alpha\mu,\bar{\gamma}}\right)\eta_{\gamma\bar{\mu}} \\ &= iA_{\alpha\delta,\bar{\delta}}H - inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}\end{aligned}$$

and

$$R_{\alpha\bar{\gamma}\mu\bar{\delta},\delta}\eta_{\gamma\bar{\mu}} - \left(R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_{\delta} = -R^{\gamma}{}_{\alpha}\left(div\eta\right)_{\gamma} + iA_{\alpha\delta,\bar{\delta}}H - inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}.$$

Thus (5.16) becomes

$$\begin{aligned}&\eta_{\alpha\bar{\delta},\lambda\bar{\lambda}\delta} + \eta_{\alpha\bar{\delta},\bar{\lambda}\lambda\delta} - \left(2R_{\alpha\bar{\gamma}\bar{\delta}\mu}\eta_{\gamma\bar{\mu}} + R^{\bar{\gamma}}{}_{\bar{\delta}}\eta_{\alpha\bar{\gamma}} + R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\delta}}\right)_{\delta} \\ &= \Delta_b\left(div\eta\right)_{\alpha} - ih_{\delta\bar{\lambda}}\eta_{\alpha\bar{\delta},\lambda 0} - i\eta_{\alpha\bar{\delta},0\lambda}h_{\bar{\lambda}\delta} - R^{\gamma}{}_{\alpha}\left(div\eta\right)_{\gamma} \\ &\quad +iA_{\alpha\delta,\bar{\delta}}H - inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}} + i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta} - \eta_{\delta\bar{\delta},\bar{\lambda}}A_{\alpha\lambda}\right) \\ &\quad +i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}} - A_{\alpha\lambda}H\right)_{\bar{\lambda}} - i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}} - n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}}.\end{aligned}$$

The torsion part will be

$$\begin{aligned}
&iA_{\alpha\delta,\bar{\delta}}H-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}+i\left(A_{\alpha\gamma}\eta_{\lambda\bar{\gamma}}-A_{\alpha\lambda}H\right)_{\bar{\lambda}}\\
&-i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}}-n\eta_{\alpha\bar{\gamma}}A_{\gamma\lambda}\right)_{\bar{\lambda}}+i\left(\eta_{\lambda\bar{\delta},\bar{\lambda}}A_{\alpha\delta}-H_{\bar{\lambda}}A_{\alpha\lambda}\right)\\
&=-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}+iA_{\alpha\gamma,\bar{\lambda}}\eta_{\lambda\bar{\gamma}}+iA_{\alpha\gamma}\eta_{\lambda\bar{\gamma},\bar{\lambda}}-2iA_{\alpha\lambda}H_{\bar{\lambda}}-i\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}}\right)_{\bar{\lambda}}\\
&\quad+in\left(A_{\gamma\lambda}\eta_{\alpha\bar{\gamma}}\right)_{\bar{\lambda}}+iA_{\alpha\delta}\left(div\eta\right)_{\bar{\delta}}\\
&=-i\left(n-1\right)A_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}-2iA_{\alpha\lambda}H_{\bar{\lambda}}+i\left(n-1\right)\left(divA\right)_{\gamma}\eta_{\alpha\bar{\gamma}}\\
&\quad+i\left(n-1\right)A_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}+2iA_{\alpha\delta}\left(div\eta\right)_{\bar{\delta}}.
\end{aligned}$$

To sum up, we have

$$\begin{aligned}
\left(\tfrac{1}{4}\tfrac{\partial}{\partial t}-\Delta_b\right)\left(div\eta\right)_{\alpha} &= -i\eta_{\alpha\bar{\lambda},\lambda 0}-i\eta_{\alpha\bar{\lambda},0\lambda}-R^{\gamma}{}_{\alpha}\left(div\eta\right)_{\gamma}\\
&\quad+i\left(n-1\right)\left(divA\right)_{\gamma}\eta_{\alpha\bar{\gamma}}+i\left(n-1\right)A_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}\\
&\quad+2iA_{\alpha\delta}\left(div\eta\right)_{\bar{\delta}}-i\left(n-1\right)A_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}-2iA_{\alpha\lambda}H_{\bar{\lambda}}
\end{aligned}$$

and

$$\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)\left(div\eta\right)_{\alpha} = -8i\eta_{\alpha\bar{\lambda},0\lambda}-4R^{\gamma}{}_{\alpha}\left(div\eta\right)_{\gamma}+\left(rest.I\right)_{\alpha},$$

where we have used the following commutation relation

$$-i\eta_{\alpha\bar{\lambda},\lambda 0}=-i\eta_{\alpha\bar{\lambda},0\lambda}+iA_{\gamma\lambda}\eta_{\alpha\bar{\lambda},\bar{\gamma}}-iA_{\alpha\lambda,\bar{\gamma}}\eta_{\gamma\bar{\lambda}}+iA_{\lambda\gamma,\bar{\lambda}}\eta_{\alpha\bar{\gamma}}.$$

□

Lemma 5.3. *Let $\eta_{\alpha\bar{\beta}}\left(x,t\right)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$. Then*

$$\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)\left(\left(div\eta\right)_{\alpha,\bar{\alpha}}+\left(div\eta\right)_{\bar{\alpha},\alpha}\right)=Tor\ II.$$

Here we denote

$$Tor\ II:=-16nIm(A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}+\left(divA\right)_{\gamma}\left(div\eta\right)_{\bar{\gamma}})+16Im\left(\left(divA\right)_{\lambda}H_{\bar{\lambda}}+A_{\alpha\lambda}H_{\bar{\lambda},\bar{\alpha}}\right).$$

Proof. From previous lemma

$$\begin{aligned}
\tfrac{1}{4}\tfrac{\partial}{\partial t}\left(div\eta\right)_{\alpha,\bar{\alpha}} &= \left(\Delta_b\left(div\eta\right)_{\alpha}\right)_{\bar{\alpha}}+\left(-2i\eta_{\alpha\bar{\lambda},0\lambda}-R^{\gamma}{}_{\alpha}\left(div\eta\right)_{\gamma}\right)_{\bar{\alpha}}\\
&\quad+\left(-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}-2iA_{\alpha\lambda}H_{\bar{\lambda}}\right)_{\bar{\alpha}}+\left(2iA_{\alpha\delta}\left(div\eta\right)_{\bar{\delta}}\right)_{\bar{\alpha}}\\
&\quad+\left(in\left(divA\right)_{\gamma}\eta_{\alpha\bar{\gamma}}+inA_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}\right)_{\bar{\alpha}}.
\end{aligned}\tag{5.17}$$

Note

$$\left(\Delta_b\left(div\eta\right)_{\alpha}\right)_{\bar{\alpha}}=\left(div\eta\right)_{\alpha,\lambda\bar{\lambda}\bar{\alpha}}+\left(div\eta\right)_{\alpha,\bar{\lambda}\lambda\bar{\alpha}}.$$

From the commutation relation

$$\begin{aligned}
&\left(div\eta\right)_{\alpha,\lambda\bar{\lambda}\bar{\alpha}}\\
&=\left(div\eta\right)_{\alpha,\lambda\bar{\alpha}\bar{\lambda}}+i\left(\left(div\eta\right)_{\sigma,\lambda}h_{\alpha\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}-\left(div\eta\right)_{\sigma,\lambda}h_{\alpha\bar{\alpha}}A_{\bar{\sigma}\bar{\lambda}}\right)\\
&\quad+i\left(\left(div\eta\right)_{\alpha,\sigma}h_{\lambda\bar{\lambda}}A_{\bar{\sigma}\bar{\alpha}}-\left(div\eta\right)_{\alpha,\sigma}h_{\lambda\bar{\alpha}}A_{\bar{\sigma}\bar{\lambda}}\right)\\
&=\left(div\eta\right)_{\alpha,\lambda\bar{\alpha}\bar{\lambda}}\\
&=[\left(div\eta\right)_{\alpha,\bar{\alpha}\lambda}+i\left(div\eta\right)_{\alpha,0}h_{\lambda\bar{\alpha}}+R^{\sigma}{}_{\lambda}\left(div\eta\right)_{\sigma}]_{\bar{\lambda}}\\
&=\left(div\eta\right)_{\alpha,\bar{\alpha}\lambda\bar{\lambda}}+i\left(div\eta\right)_{\alpha,0\bar{\lambda}}h_{\lambda\bar{\alpha}}+\left(R^{\sigma}{}_{\lambda}\left(div\eta\right)_{\sigma}\right)_{\bar{\lambda}}
\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}
&(div\eta)_{\alpha,\bar{\lambda}\lambda\bar{\alpha}} \\
&= (div\eta)_{\alpha,\bar{\lambda}\bar{\alpha}\lambda} + i\,(div\eta)_{\alpha,\bar{\lambda}0}\,h_{\lambda\bar{\alpha}} \\
&\quad + (div\eta)_{\sigma,\bar{\lambda}}\,R_{\alpha}{}^{\sigma}{}_{\lambda\bar{\alpha}} + (div\eta)_{\alpha,\bar{\sigma}}\,R_{\bar{\lambda}}{}^{\bar{\sigma}}{}_{\lambda\bar{\alpha}} \\
&= (div\eta)_{\alpha,\bar{\lambda}\bar{\alpha}\lambda} + i\,(div\eta)_{\alpha,\bar{\alpha}0} \\
&= [(div\eta)_{\alpha,\bar{\alpha}\bar{\lambda}} + (1-n)\,i\,(div\eta)_{\sigma}\,A_{\bar{\sigma}\bar{\lambda}}]_{\lambda} + i\,(div\eta)_{\alpha,\bar{\alpha}0} \\
&= (div\eta)_{\alpha,\bar{\alpha}\bar{\lambda}\lambda} + (1-n)\,i\,((div\eta)_{\sigma}\,A_{\bar{\sigma}\bar{\lambda}})_{\lambda} + i\,(div\eta)_{\alpha,\bar{\alpha}0}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&(\Delta_b\,(div\eta)_{\alpha})_{\bar{\alpha}} \\
&= \Delta_b\left((div\eta)_{\alpha,\bar{\alpha}}\right) + i\,(div\eta)_{\alpha,0\bar{\alpha}} + i\,(div\eta)_{\alpha,\bar{\alpha}0} \\
&\quad + (1-n)\,i\,((div\eta)_{\sigma}\,A_{\bar{\sigma}\bar{\lambda}})_{\lambda} + (R^{\sigma}{}_{\lambda}\,(div\eta)_{\sigma})_{\bar{\lambda}}
\end{aligned} \tag{5.19}$$

and from (5.17)

$$\begin{aligned}
&\left(\tfrac{1}{4}\tfrac{\partial}{\partial t} - \Delta_b\right)(div\eta)_{\alpha,\bar{\alpha}} \\
&= i\,(div\eta)_{\alpha,0\bar{\alpha}} + i\,(div\eta)_{\alpha,\bar{\alpha}0} - 2i\eta_{\alpha\bar{\lambda},0\lambda\bar{\alpha}} \\
&\quad + (1-n)\,i\,((div\eta)_{\sigma}\,A_{\bar{\sigma}\bar{\lambda}})_{\lambda} + \left(-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}} - 2iA_{\alpha\lambda}H_{\bar{\lambda}}\right)_{\bar{\alpha}} \\
&\quad + \left(in\,(divA)_{\gamma}\,\eta_{\alpha\bar{\gamma}} + inA_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}\right)_{\bar{\alpha}} + (2iA_{\alpha\delta}\,(div\eta)_{\bar{\delta}})_{\bar{\alpha}}\,.
\end{aligned} \tag{5.20}$$

By commutation relation again

$$\eta_{\alpha\bar{\lambda},0\lambda} = (div\eta)_{\alpha,0} + A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}} - A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}} + (divA)_{\sigma}\,\eta_{\alpha\bar{\sigma}}$$

and

$$(div\eta)_{\alpha,0\bar{\alpha}} = (div\eta)_{\alpha,\bar{\alpha}0} + (div\eta)_{\alpha,\sigma}\,A_{\bar{\sigma}\bar{\alpha}} + (div\eta)_{\sigma}\,(divA)_{\bar{\sigma}}\,.$$

Taking covariant derivative

$$\eta_{\alpha\bar{\lambda},0\lambda\bar{\alpha}} = (div\eta)_{\alpha,0\bar{\alpha}} + \left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}} - (A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}})_{\bar{\alpha}} + ((divA)_{\sigma}\,\eta_{\alpha\bar{\sigma}})_{\bar{\alpha}}\,.$$

Hence

$$\begin{aligned}
&i\,(div\eta)_{\alpha,0\bar{\alpha}} + i\,(div\eta)_{\alpha,\bar{\alpha}0} - 2i\eta_{\alpha\bar{\lambda},0\lambda\bar{\alpha}} \\
&= 2i\,(div\eta)_{\alpha,0\bar{\alpha}} - i\,(div\eta)_{\alpha,\sigma}\,A_{\bar{\sigma}\bar{\alpha}} - i\,(div\eta)_{\sigma}\,(divA)_{\bar{\sigma}} \\
&\quad -2i\,(div\eta)_{\alpha,0\bar{\alpha}} - 2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}} \\
&\quad +2i\,(A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}})_{\bar{\alpha}} - 2i\,((divA)_{\sigma}\,\eta_{\alpha\bar{\sigma}})_{\bar{\alpha}} \\
&= -i\,(div\eta)_{\alpha,\sigma}\,A_{\bar{\sigma}\bar{\alpha}} - i\,(div\eta)_{\sigma}\,(divA)_{\bar{\sigma}} - 2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}} \\
&\quad +2i\,(A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}})_{\bar{\alpha}} - 2i\,((divA)_{\sigma}\,\eta_{\alpha\bar{\sigma}})_{\bar{\alpha}}\,.
\end{aligned}$$

So (5.20) becomes

$$\begin{aligned}
&\left(\tfrac{1}{4}\tfrac{\partial}{\partial t} - \Delta_b\right)(div\eta)_{\alpha,\bar{\alpha}} \\
&= -i\,(div\eta)_{\alpha,\sigma}\,A_{\bar{\sigma}\bar{\alpha}} - i\,(div\eta)_{\sigma}\,(divA)_{\bar{\sigma}} \\
&\quad + \left[-2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}} + 2i\,(A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}})_{\bar{\alpha}}\right] \\
&\quad + (1-n)\,i\,((div\eta)_{\sigma}\,A_{\bar{\sigma}\bar{\lambda}})_{\lambda} + \left(-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}} - 2iA_{\alpha\lambda}H_{\bar{\lambda}}\right)_{\bar{\alpha}} \\
&\quad + \left(in\,(divA)_{\gamma}\,\eta_{\alpha\bar{\gamma}} + inA_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}\right)_{\bar{\alpha}} \\
&\quad +2iA_{\alpha\delta}\,(div\eta)_{\bar{\delta},\bar{\alpha}} - 2i\,(divA)_{\sigma,\bar{\alpha}}\,\eta_{\alpha\bar{\sigma}}.
\end{aligned} \tag{5.21}$$

Now we deal with the term $\left[-2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}} + 2i\,(A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}})_{\bar{\alpha}}\right]$.

$$
\begin{aligned}
&-2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}}\right)_{\bar{\alpha}}+2i\left(A_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda}}\right)_{\bar{\alpha}}\\
&=-2iA_{\sigma\lambda,\bar{\alpha}}\eta_{\alpha\bar{\lambda},\bar{\sigma}}-2iA_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}\bar{\alpha}}+2iA_{\alpha\lambda,\bar{\sigma}\bar{\alpha}}\eta_{\sigma\bar{\lambda}}+2iA_{\alpha\lambda,\bar{\sigma}}\eta_{\sigma\bar{\lambda},\bar{\alpha}}\\
&=-2i\left(A_{\sigma\lambda}\eta_{\alpha\bar{\lambda},\bar{\sigma}\bar{\alpha}}-A_{\alpha\lambda,\bar{\sigma}\bar{\alpha}}\eta_{\sigma\bar{\lambda}}\right)\\
&=-2iA_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}+2A_{\sigma\lambda}h_{\alpha\bar{\sigma}}A_{\bar{\alpha}\bar{\rho}}\eta_{\rho\bar{\lambda}}\\
&\quad-2nA_{\sigma\lambda}A_{\bar{\sigma}\bar{\rho}}\eta_{\rho\bar{\lambda}}-2A_{\sigma\lambda}A_{\bar{\lambda}\bar{\alpha}}\eta_{\alpha\bar{\sigma}}\\
&\quad+2\left\|A\right\|^{2}H+2i\eta_{\sigma\bar{\lambda}}\left(divA\right)_{\lambda,\bar{\sigma}}+2n\eta_{\sigma\bar{\lambda}}A_{\bar{\sigma}\bar{\rho}}A_{\rho\lambda}-2\eta_{\sigma\bar{\lambda}}h_{\lambda\bar{\sigma}}\left\|A\right\|^{2}\\
&=-2i\left(A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}-\eta_{\sigma\bar{\lambda}}\left(divA\right)_{\lambda,\bar{\sigma}}\right),
\end{aligned}
$$

where we have used the following commutation relations in third equality

$$
\begin{aligned}
&\eta_{\alpha\bar{\lambda},\bar{\sigma}\bar{\alpha}}\\
&=\eta_{\alpha\bar{\lambda},\bar{\alpha}\bar{\sigma}}+ih_{\alpha\bar{\sigma}}A_{\bar{\alpha}\bar{\rho}}\eta_{\rho\bar{\lambda}}-ih_{\alpha\bar{\alpha}}A_{\bar{\sigma}\bar{\rho}}\eta_{\rho\bar{\lambda}}-iA_{\bar{\lambda}\bar{\alpha}}\eta_{\alpha\bar{\sigma}}+iA_{\bar{\lambda}\bar{\sigma}}\eta_{\alpha\bar{\alpha}}\\
&=\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}+ih_{\alpha\bar{\sigma}}A_{\bar{\alpha}\bar{\rho}}\eta_{\rho\bar{\lambda}}-niA_{\bar{\sigma}\bar{\rho}}\eta_{\rho\bar{\lambda}}-iA_{\bar{\lambda}\bar{\alpha}}\eta_{\alpha\bar{\sigma}}+iA_{\bar{\lambda}\bar{\sigma}}H
\end{aligned}
$$

and

$$
\begin{aligned}
&A_{\alpha\lambda,\bar{\sigma}\bar{\alpha}}\\
&=A_{\alpha\lambda,\bar{\alpha}\bar{\sigma}}+ih_{\alpha\bar{\sigma}}A_{\bar{\alpha}\bar{\rho}}A_{\rho\lambda}-ih_{\alpha\bar{\alpha}}A_{\bar{\sigma}\bar{\rho}}A_{\rho\lambda}\\
&\quad+ih_{\lambda\bar{\sigma}}A_{\bar{\alpha}\bar{\rho}}A_{\alpha\rho}-ih_{\lambda\bar{\alpha}}A_{\bar{\sigma}\bar{\rho}}A_{\alpha\rho}\\
&=\left(divA\right)_{\lambda,\bar{\sigma}}-niA_{\bar{\sigma}\bar{\rho}}A_{\rho\lambda}+ih_{\lambda\bar{\sigma}}\left\|A\right\|^{2}.
\end{aligned}
$$

Next

$$
\begin{aligned}
&(1-n)\,i\left(\left(div\eta\right)_{\sigma}A_{\bar{\sigma}\bar{\lambda}}\right)_{\lambda}+\left(-inA_{\alpha\mu,\bar{\gamma}}\eta_{\gamma\bar{\mu}}-2iA_{\alpha\lambda}H_{\bar{\lambda}}\right)_{\bar{\alpha}}\\
&+\left(in\left(divA\right)_{\gamma}\eta_{\alpha\bar{\gamma}}+inA_{\gamma\lambda}\eta_{\alpha\bar{\gamma},\bar{\lambda}}\right)_{\bar{\alpha}}\\
&=(1-n)\,i\left(div\eta\right)_{\sigma,\lambda}A_{\bar{\sigma}\bar{\lambda}}+(1-n)\,i\left(div\eta\right)_{\sigma}\left(divA\right)_{\bar{\sigma}}\\
&\quad-2i\left(divA\right)_{\lambda}H_{\bar{\lambda}}-2iA_{\alpha\lambda}H_{\bar{\lambda},\bar{\alpha}}+in\left(divA\right)_{\gamma,\bar{\alpha}}\eta_{\alpha\bar{\gamma}}+in\left(divA\right)_{\gamma}\left(div\eta\right)_{\bar{\gamma}}\\
&\quad+in\left(A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}-\eta_{\sigma\bar{\lambda}}\left(divA\right)_{\lambda,\bar{\sigma}}\right).
\end{aligned}
$$

All these imply

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-4\Delta_{b}\right)\left(div\eta\right)_{\alpha,\bar{\alpha}}\\
&=4ni\left(-A_{\bar{\sigma}\bar{\lambda}}\left(div\eta\right)_{\sigma,\lambda}+A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}\right)\\
&\quad+4ni\left(-\left(div\eta\right)_{\sigma}\left(divA\right)_{\bar{\sigma}}+\left(divA\right)_{\gamma}\left(div\eta\right)_{\bar{\gamma}}\right)\\
&\quad-8i\left(divA\right)_{\lambda}H_{\bar{\lambda}}-8iA_{\alpha\lambda}H_{\bar{\lambda},\bar{\alpha}}\\
&=-8nIm(A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}+\left(divA\right)_{\gamma}\left(div\eta\right)_{\bar{\gamma}})\\
&\quad-8i\left(divA\right)_{\lambda}H_{\bar{\lambda}}-8iA_{\alpha\lambda}H_{\bar{\lambda},\bar{\alpha}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-4\Delta_{b}\right)\left(\left(div\eta\right)_{\alpha,\bar{\alpha}}+\left(div\eta\right)_{\bar{\alpha},\alpha}\right)\\
&=-16nIm(A_{\sigma\lambda}\left(div\eta\right)_{\bar{\lambda},\bar{\sigma}}+\left(divA\right)_{\gamma}\left(div\eta\right)_{\bar{\gamma}})\\
&\quad+16Im\left(\left(divA\right)_{\lambda}H_{\bar{\lambda}}+A_{\alpha\lambda}H_{\bar{\lambda},\bar{\alpha}}\right).
\end{aligned}
$$

□

Combining Lemma 5.2 and Lemma 5.3, we have

Lemma 5.4. *Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$. Then*

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)Z\\
&=k_1\left[\tfrac{1}{2}Tor\ II+(Tor\ I)_\alpha V_{\bar{\alpha}}+(Tor\ I)_{\bar{\alpha}}V_\alpha\right]\\
&\quad+8k_1\left(-i\eta_{\alpha\bar{\lambda},0\lambda}V_{\bar{\alpha}}+i\eta_{\bar{\alpha}\lambda,0\bar{\lambda}}V_\alpha\right)\\
&\quad-4k_1R^{\gamma}{}_{\alpha}(div\eta)_r V_{\bar{\alpha}}-4k_1R^{\bar{\gamma}}{}_{\bar{\alpha}}(div\eta)_{\bar{\gamma}}V_\alpha\\
&\quad+k_1(div\eta)_\alpha\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)V_{\bar{\alpha}}+k_1(div\eta)_{\bar{\alpha}}\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)V_\alpha\\
&\quad-8k_1\left((div\eta)_{\alpha,\gamma}V_{\bar{\alpha},\bar{\gamma}}+(div\eta)_{\alpha,\bar{\gamma}}V_{\bar{\alpha},\gamma}+(div\eta)_{\bar{\alpha},\bar{\gamma}}V_{\alpha,\gamma}+(div\eta)_{\bar{\alpha},\gamma}V_{\alpha,\bar{\gamma}}\right)\\
&\quad+k_1\left[\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)V_{\bar{\alpha}}\right]V_\beta\eta_{\alpha\bar{\beta}}+k_1\left[\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)V_\beta\right]V_{\bar{\alpha}}\eta_{\alpha\bar{\beta}}\\
&\quad+4k_1V_{\bar{\alpha}}V_\beta\left(2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}-R^{\bar{\gamma}}{}_{\bar{\beta}}\eta_{\alpha\bar{\gamma}}-R^{\gamma}{}_{\alpha}\eta_{\gamma\bar{\beta}}\right)\\
&\quad-8k_1\eta_{a\bar{\beta},\gamma}\left(V_{\bar{\alpha}}V_\beta\right)_{\bar{\gamma}}-8k_1\eta_{a\bar{\beta},\bar{\gamma}}\left(V_{\bar{\alpha}}V_\beta\right)_\gamma\\
&\quad-8k_1\eta_{\alpha\bar{\beta}}\left(V_{\bar{\alpha},\gamma}V_{\beta,\bar{\gamma}}+V_{\bar{\alpha},\bar{\gamma}}V_{\beta,\gamma}\right)-\tfrac{H}{t^2}.
\end{aligned}$$

Now we are going to show that the Harnack quadratic Z is nonnegative for all time. First we modify Z with small perturbation by $\tilde{\eta}$ where $\widetilde{\eta}_{\alpha\bar{\beta}}=\eta_{\alpha\bar{\beta}}+\varepsilon h_{\alpha\bar{\beta}}$ and define

$$\begin{aligned}
\widehat{Z}:=\ &\tfrac{k_1}{2}h^{\alpha\bar{\beta}}\left(\nabla_{\bar{\beta}}(div\eta)_\alpha+\nabla_\alpha(div\eta)_{\bar{\beta}}\right)\\
&+k_1\left[h^{\alpha\bar{\beta}}(div\eta)_\alpha V_{\bar{\beta}}+h^{\alpha\bar{\beta}}(div\eta)_{\bar{\beta}}V_\alpha\right]\\
&+k_1h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\left(\eta_{\alpha\bar{\delta}}+\varepsilon h_{\alpha\bar{\delta}}\right)V_{\bar{\beta}}V_\gamma+\tfrac{H+\varepsilon n}{t}.
\end{aligned}\tag{5.22}$$

Let V be the vector field which minimizes $\widehat{Z}$. Then the following lemma holds

Lemma 5.5. *Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$. Then*

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-4\Delta_b\right)\widehat{Z}\\
&=k_1\left[\tfrac{1}{2}Tor\ II+(Tor\ I)_\alpha V_{\bar{\alpha}}+(Tor\ I)_{\bar{\alpha}}V_\alpha\right]\\
&\quad-8k_1i\eta_{\alpha\bar{\lambda},0\lambda}V_{\bar{\alpha}}+8k_1i\eta_{\lambda\bar{\alpha},0\bar{\lambda}}V_\alpha\\
&\quad+8k_1\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_\gamma V_\sigma\nabla_{\bar{\gamma}}V_{\bar{\alpha}}+8k_1V_{\bar{\alpha}}V_\beta R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}\\
&\quad+\widetilde{\eta}_{\alpha\bar{\sigma}}\left[\sqrt{8k_1}\nabla_{\bar{\gamma}}V_\sigma-\tfrac{\sqrt{k_1}}{8t}h_{\sigma\bar{\gamma}}\right]\left[\sqrt{8k_1}\nabla_\gamma V_{\bar{\alpha}}-\tfrac{\sqrt{k_1}}{\sqrt{8}t}h_{\gamma\bar{\alpha}}\right]\\
&\quad-\tfrac{2\widehat{Z}}{t}+\tfrac{8-k_1}{8t^2}(H+\varepsilon n).
\end{aligned}\tag{5.23}$$

Proof. The first variation formula gives

$$(div\eta)_\alpha+\widetilde{\eta}_{\alpha\bar{\beta}}V_\beta=0,\ \text{and}\quad (div\eta)_{\bar{\alpha}}+\widetilde{\eta}_{\gamma\bar{\alpha}}V_{\bar{\gamma}}=0.\tag{5.24}$$

Differentiating it we have that

$$\begin{aligned}
\nabla_s(div\eta)_\alpha+\left(\nabla_s\eta_{\alpha\bar{\sigma}}\right)V_\sigma+\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_sV_\sigma=0,\\
\nabla_s(div\eta)_{\bar{\alpha}}+\left(\nabla_s\eta_{\sigma\bar{\alpha}}\right)V_{\bar{\sigma}}+\widetilde{\eta}_{\sigma\bar{\alpha}}\nabla_sV_{\bar{\sigma}}=0,\\
\nabla_{\bar{s}}(div\eta)_{\bar{\alpha}}+\left(\nabla_{\bar{s}}\eta_{\sigma\bar{\alpha}}\right)V_{\bar{\sigma}}+\widetilde{\eta}_{\sigma\bar{\alpha}}\nabla_{\bar{s}}V_{\bar{\sigma}}=0,\\
\nabla_{\bar{s}}(div\eta)_\alpha+\left(\nabla_{\bar{s}}\eta_{\alpha\bar{\sigma}}\right)V_\sigma+\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\bar{s}}V_\sigma=0.
\end{aligned}\tag{5.25}$$

Using above equations we can rewrite formula of Lemma 5.4 into following

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-4\Delta_b\right)Z\\
=\;& k_1\left[\frac{1}{2}Tor\ II+\left(Tor\ I\right)_{\alpha}V_{\bar{\alpha}}+\left(Tor\ I\right)_{\bar{\alpha}}V_{\alpha}\right]\\
&+8k_1\left(-i\eta_{\alpha\bar{\lambda},0\lambda}V_{\bar{\alpha}}+i\eta_{\bar{\alpha}\lambda,0\bar{\lambda}}V_{\alpha}\right)\\
&-8k_1\left[\left(div\eta\right)_{\alpha,\gamma}V_{\bar{\alpha},\bar{\gamma}}+\left(div\eta\right)_{\alpha,\bar{\gamma}}V_{\bar{\alpha},\gamma}+\left(div\eta\right)_{\bar{\alpha},\bar{\gamma}}V_{\alpha,\gamma}+\left(div\eta\right)_{\bar{\alpha},\gamma}V_{\alpha,\bar{\gamma}}\right]\\
&+8k_1V_{\bar{\alpha}}V_{\beta}R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}-8k_1\eta_{a\bar{\beta},\gamma}\left(V_{\bar{\alpha}}V_{\beta}\right)_{\bar{\gamma}}-8k_1\eta_{a\bar{\beta},\bar{\gamma}}\left(V_{\bar{\alpha}}V_{\beta}\right)_{\gamma}\\
&+8k_1\left(\left(\nabla_{\gamma}\left(div\eta\right)_{\alpha}+\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\sigma}\right)V_{\bar{\alpha},\bar{\gamma}}-k_1\left(\nabla_{\gamma}\left(div\eta\right)_{\bar{\alpha}}+\widetilde{\eta}_{\sigma\bar{\alpha}}\nabla_{\gamma}V_{\bar{\sigma}}\right)V_{\alpha,\bar{\gamma}}\right)\\
&+8k_1\left(\left(\nabla_{\bar{\gamma}}\left(div\eta\right)_{\alpha}+\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\bar{\gamma}}V_{\sigma}\right)V_{\bar{\alpha},\gamma}+\left(\nabla_{\bar{\gamma}}\left(div\eta\right)_{\bar{\beta}}+\widetilde{\eta}_{\sigma\bar{\beta}}\nabla_{\bar{\gamma}}V_{\bar{\sigma}}\right)V_{\beta,\gamma}\right)\\
&-8k_1\eta_{\alpha\bar{\beta}}\left(V_{\bar{\alpha},\gamma}V_{\beta,\bar{\gamma}}+V_{\bar{\alpha},\bar{\gamma}}V_{\beta,\gamma}\right)-\frac{H}{t^2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_b\right)\widehat{Z}\\
=\;& k_1[\frac{1}{2}Tor\ II+\left(Tor\ I\right)_{\alpha}V_{\bar{\alpha}}+\left(Tor\ I\right)_{\bar{\alpha}}V_{\alpha}]\\
&-k_1\cdot 2i\eta_{\alpha\bar{\lambda},0\lambda}V_{\bar{\alpha}}+k_1\cdot 2i\eta_{\lambda\bar{\alpha},0\bar{\lambda}}V_{\alpha}+k_1\cdot 8\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\bar{\alpha}}\nabla_{\bar{\gamma}}V_{\sigma}\\
&+k_1\cdot 8\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\sigma}\nabla_{\bar{\gamma}}V_{\bar{\alpha}}+k_1\cdot 8V_{\bar{\alpha}}V_{\beta}R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}-\frac{H+\varepsilon n}{t^2}.
\end{aligned}
\tag{5.26}
$$

It follows from (5.24) again that we can rewrite $\widehat{Z}$ as following

$$
\widehat{Z}=\frac{k_1}{2}\left(-\widetilde{\eta}_{\alpha\bar{\gamma}}\nabla_{\bar{\alpha}}V_{\gamma}-\widetilde{\eta}_{\gamma\bar{\alpha}}\nabla_{\alpha}V_{\bar{\gamma}}\right)+\frac{H+\varepsilon n}{t}
\tag{5.27}
$$

which is

$$
\frac{H+\varepsilon n}{t^2}=\frac{\widehat{Z}}{t}+\frac{k_1}{2t}\left(\widetilde{\eta}_{\alpha\bar{\gamma}}\nabla_{\bar{\alpha}}V_{\gamma}+\widetilde{\eta}_{\gamma\bar{\alpha}}\nabla_{\alpha}V_{\bar{\gamma}}\right).
\tag{5.28}
$$

Thus

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t}-\Delta_b\right)\widehat{Z}\\
&=k_1[\tfrac{1}{2}Tor\ II+\left(Tor\ I\right)_{\alpha}V_{\bar{\alpha}}+\left(Tor\ I\right)_{\bar{\alpha}}V_{\alpha}]\\
&\quad-8k_1i\eta_{\alpha\bar{\lambda},0\lambda}V_{\bar{\alpha}}+8k_1i\eta_{\lambda\bar{\alpha},0\bar{\lambda}}V_{\alpha}+8k_1V_{\bar{\alpha}}V_{\beta}R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}\\
&\quad+8k_1\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\bar{\alpha}}\nabla_{\bar{\gamma}}V_{\sigma}+8k_1\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\sigma}\nabla_{\bar{\gamma}}V_{\bar{\alpha}}\\
&\quad-\tfrac{k_1}{2t}\left(\widetilde{\eta}_{\alpha\bar{\gamma}}\nabla_{\bar{\alpha}}V_{\gamma}+\widetilde{\eta}_{\gamma\bar{\alpha}}\nabla_{\alpha}V_{\bar{\gamma}}\right)-\tfrac{\widehat{Z}}{t}.
\end{aligned}
$$

Therefore (5.23) follows from the following :

$$\begin{aligned}
&8k_1\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\bar{\gamma}}V_\sigma\nabla_\gamma V_{\bar{\alpha}} - \tfrac{k_1}{2t}\left(\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\bar{\alpha}}V_\sigma + \widetilde{\eta}_{\sigma\bar{\alpha}}\nabla_\alpha V_{\bar{\sigma}}\right)\\
&= \widetilde{\eta}_{\alpha\bar{\sigma}}\left[\sqrt{8k_1}\nabla_{\bar{\gamma}}V_\sigma - \tfrac{\sqrt{k_1}}{\sqrt{8}t}h_{\sigma\bar{\gamma}}\right]\left[\sqrt{8k_1}\nabla_\gamma V_{\bar{\alpha}} - \tfrac{\sqrt{k_1}}{\sqrt{8}t}h_{\gamma\bar{\alpha}}\right]\\
&\quad -\tfrac{1}{t}\left[\tfrac{k_1}{8t}\left(H+\varepsilon n\right) - \tfrac{k_1}{2}\left(\widetilde{\eta}_{\alpha\bar{\sigma}}\nabla_{\bar{\alpha}}V_\sigma + \widetilde{\eta}_{\sigma\bar{\alpha}}\nabla_\alpha V_{\bar{\sigma}}\right)\right]\\
&= \widetilde{\eta}_{\alpha\bar{\sigma}}\left[\sqrt{8k_1}\nabla_{\bar{\gamma}}V_\sigma - \tfrac{\sqrt{k_1}}{\sqrt{8}t}h_{\sigma\bar{\gamma}}\right]\left[\sqrt{8k_1}\nabla_\gamma V_{\bar{\alpha}} - \tfrac{\sqrt{k_1}}{\sqrt{8}t}h_{\gamma\bar{\alpha}}\right]\\
&\quad -\tfrac{\widehat{Z}}{t} + \tfrac{8-k_1}{8t^2}(H+\varepsilon n).
\end{aligned}$$

□

Lemma 5.6. *Let M be a complete strictly pseudoconvex CR $(2n+1)$-manifold. Let $f(x,t)$ be the subsolution of the heat equation satisfying*

$$\left(\frac{\partial}{\partial t} - \Delta_b\right) f(x,t) \le 0 \ \text{on } M \times [0,T)$$

with $f(x,0) \le 0$ on M. If

$$\int_0^T \int_M f^2(x,t)\, e^{-ar^2} d\mu(x)\, dt < \infty,$$

then $f(x,t) \le 0$.

Proof. The proof can be easily modified into CR case, please refer to theorem 1.2 of [NT3].
□

Lemma 5.7. *Let M be a closed strictly pseudoconvex CR $(2n+1)$-manifold with vanishing torsion. Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$ with $\eta_{\alpha\bar{\beta},0}(x,0) = 0$ at $t=0$. In additional if M is complete noncompact, we assume that*

$$\int_0^T \int_M e^{-ar^2} \left\|\nabla_T \eta(x,t)\right\|^2 d\mu dt < \infty. \tag{5.29}$$

Then $\eta_{\alpha\bar{\beta},0}(x,t) = 0$ for $t > 0$.

Proof. Since the torsion is vanishing, by CR Bianchi identity ([L1])

$$R_{\alpha\bar{\gamma}\mu\bar{\beta},0} = 0 = R_{\gamma\bar{\beta},0}$$

and ([CKL1])

$$[\Delta_b, T] = 0.$$

Thus

$$\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta},0} = 4\left[\Delta_b\eta_{\alpha\bar{\beta},0} + 2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu},0} - \left(R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma},0} + R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta},0}\right)\right].$$

(i) If M is closed, by the maximum principle, we obtain $\eta_{\alpha\bar{\beta},0}(x,t) = 0$ for all $t > 0$.

(ii) If M is complete noncompact, since $\|T\eta\|^2$ is a subsolution of heat equation as following

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_b\right)\|\nabla_T\eta\|^2\\
&=\left(\frac{\partial}{\partial t}-\Delta_b\right)\eta_{\alpha\bar{\beta},0}\eta_{\zeta\bar{\xi},0}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}+\eta_{\alpha\bar{\beta},0}\left(\frac{\partial}{\partial t}-\Delta_b\right)\eta_{\zeta\bar{\xi},0}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}\\
&\quad-2\eta_{\alpha\bar{\beta},0\gamma}\eta_{\zeta\bar{\xi},0\bar{\gamma}}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}\\
&=2\left(2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu},0}-R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma},0}-R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta},0}\right)\eta_{\zeta\bar{\xi},0}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}\\
&\quad-2\eta_{\alpha\bar{\beta},0\gamma}\eta_{\zeta\bar{\xi},0\bar{\gamma}}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}\\
&\leq 0,
\end{aligned}
$$

where we use the condition of nonnegativity of CR bisectional curvature and symmetry of η tensor in the third inequality as following

$$
\begin{aligned}
&\left(2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu},0}-R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma},0}-R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta},0}\right)\eta_{\zeta\bar{\xi},0}h_{\beta\bar{\zeta}}h_{\xi\bar{\alpha}}\\
=\;&2R_{\xi\bar{\gamma}\mu\bar{\zeta}}\eta_{\gamma\bar{\mu},0}\eta_{\zeta\bar{\xi},0}-R_{\gamma\bar{\zeta}}\eta_{\xi\bar{\gamma},0}\eta_{\zeta\bar{\xi},0}-R_{\xi\bar{\gamma}}\eta_{\gamma\bar{\zeta},0}\eta_{\zeta\bar{\xi},0}\\
=\;&2R_{\alpha\bar{\beta}\beta\bar{\alpha}}\lambda_\alpha\lambda_\beta-2R_{\gamma\bar{\gamma}}\left(\lambda_\gamma\right)^2\\
=\;&-R_{\alpha\bar{\beta}\beta\bar{\alpha}}(\lambda_\alpha-\lambda_\beta)^2\\
\leq\;&0.
\end{aligned}
\tag{5.30}
$$

Here we diagonalize $\eta_{\gamma\bar{\mu},0}$ (since symmetric) and denote $\eta_{\mu\bar{\mu},0}=\lambda_\mu$.

By assumption (5.5), we can apply the maximum principle as in Theorem 5.6 to obtain $T\eta\left(x,t\right)=0$ for all $t<T$. □

Theorem 5.7. *Let M be a closed strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative CR bisectional curvature. Let $\eta_{\alpha\bar{\beta}}\left(x,t\right)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) on $M\times[0,T)$ with $\eta\left(x,0\right)\geq 0$ at $t=0$. In additional if M is complete noncompact, we assume that $\eta\left(x,t\right)$ satisfies extra conditions (5.4) and*

$$
\int_M e^{-ar^2}\left\|\eta\left(x,0\right)\right\|d\mu<\infty,
\tag{5.31}
$$

Then $\eta\left(x,t\right)\geq 0$ for $t>0$.

Proof. (i) For M is closed, by Hamilton's tensor maximum principle, we only need to check for $v\in T^{1,0}\left(M\right)$ such that $\eta_{\alpha\bar{\beta}}v_\beta=0$, $(v_{\bar{\beta}}=\overline{v_\beta})$, we have

$$
\begin{aligned}
&2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}\left(t\right)v_{\bar{\alpha}}v_\beta-R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma}}\left(t\right)v_{\bar{\alpha}}v_\beta-R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}}\left(t\right)v_{\bar{\alpha}}v_\beta\\
&=2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}\left(t\right)v_{\bar{\alpha}}v_\beta\\
&\geq 0
\end{aligned}
$$

by curvature assumption.

(ii) For M is complete, The positivity of $\eta\left(x,t\right)$ follows from Theorem 11.6 of [NN] by assuming extra conditions (5.4) and (5.31). □

Lemma 5.8. *Let M be a complete strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative CR bisectional curvature and vanishing torsion. Moreover we assume $\nabla_T\eta\left(x,t\right)=0$ for all $t\in[0,T)$ and*

$$
\int_\delta^T\int_M e^{-ar^2}\left\|\eta\left(x,t\right)\right\|^2 d\mu dt<\infty,
$$

for some $a>0$ and any $\delta>0$, then

$$\int_{\delta}^{T}\int_{M} e^{-ar^{2}}\left\|div\eta\left(x,t\right)\right\|^{2}d\mu dt<\infty \tag{5.32}$$

and

$$\int_{\delta}^{T}\int_{M} e^{-ar^{2}}\left\|\nabla div\eta\left(x,t\right)\right\|^{2}d\mu dt<\infty. \tag{5.33}$$

Proof. Since bisectional curvature is nonnegative, we have

$$\begin{aligned}&\left(\tfrac{\partial}{\partial t}-\Delta\right)\|\eta\|^{2}\\ &=2\left(2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}}\left(t\right)\eta\left(t\right)_{\alpha\bar{\beta}}-R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma}}\left(t\right)\eta\left(t\right)_{\alpha\bar{\beta}}-R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}}\left(t\right)\eta\left(t\right)_{\alpha\bar{\beta}}\right)-2\left\langle\nabla\eta,\nabla\eta\right\rangle\\ &\leq-\left\|div\eta\right\|^{2},\end{aligned} \tag{5.34}$$

where we have used (5.30) for the last inequality. Let p be an reference point on M and ϕ be a cut-off function such that $\phi=0$ for $d(x,p)>2R$ or $t\leq\frac{\delta}{2}$, and $\phi=1$ as $d\left(x,p\right)<R$ for $t\geq\delta$. Then

$$\begin{aligned}&\int_{\delta}^{T}\int_{M}\left\|div\eta\right\|^{2}\phi^{2}d\mu dt\\ \leq\;&\int_{\delta}^{T}\int_{M}\left(\Delta-\frac{\partial}{\partial t}\right)\|\eta\|^{2}\phi^{2}d\mu dt\\ =\;&\int_{\delta}^{T}\int_{M}\Delta\|\eta\|^{2}\phi^{2}d\mu dt+\int_{M}\|\eta\|^{2}\left(x,0\right)\phi^{2}d\mu\\ &-\int_{M}\|\eta\|^{2}\left(x,T\right)\phi^{2}d\mu+\int_{0}^{T}\int_{M}\|\eta\|^{2}\left(x,T\right)\left(\phi^{2}\right)_{t}d\mu dt\\ \leq\;&\int_{\delta}^{T}\int_{M}\|\eta\|^{2}\left(\phi^{2}\right)_{t}-2\phi\left\langle\nabla\|\eta\|^{2},\nabla\phi\right\rangle_{J,\theta}d\mu dt.\end{aligned} \tag{5.35}$$

On the other hand, as before in normal coordinate, we have

$$\begin{aligned}\left\|\nabla\|\eta\|^{2}\right\|^{2}&\leq 2\sum_{\gamma}\|\eta\|_{\gamma}^{2}\|\eta\|_{\bar{\gamma}}^{2}\\ &\leq 2\sum_{\gamma}\sum_{\alpha,\beta}\left(\eta_{\alpha\bar{\beta},\gamma}\eta_{\beta\bar{\alpha}}+\eta_{\alpha\bar{\beta}}\eta_{\beta\bar{\alpha},\gamma}\right)\sum_{\zeta,\xi}\left(\eta_{\zeta\bar{\xi},\bar{\gamma}}\eta_{\xi\bar{\zeta}}+\eta_{\zeta\bar{\xi}}\eta_{\xi\bar{\zeta},\bar{\gamma}}\right)\\ &\leq 8\sum_{\gamma}\sum_{\alpha,\beta}\eta_{\beta\bar{\beta},\gamma}\eta_{\beta\bar{\beta}}\eta_{\alpha\bar{\alpha},\bar{\gamma}}\eta_{\alpha\bar{\alpha}}\\ &\leq 8\|\eta\|^{2}\sum_{\alpha,\beta,\gamma}\eta_{\beta\bar{\beta},\gamma}\eta_{\alpha\bar{\alpha},\bar{\gamma}}\\ &\leq 8\|\eta\|^{2}\sum_{\alpha,\beta,\gamma}\frac{\left|\eta_{\beta\bar{\beta},\gamma}\right|^{2}+\left|\eta_{\alpha\bar{\alpha},\bar{\gamma}}\right|^{2}}{2}\\ &\leq 8n\|\eta\|^{2}\|div\eta\|^{2}.\end{aligned}$$

Then (5.35) becomes

$$\begin{aligned}(5.36)\qquad &\int_\delta^T \int_M \|div\eta\|^2 \phi^2 d\mu dt \\ &\leq \int_\delta^T \int_M \|\eta\|^2 \left(\phi^2\right)_t + 2\sqrt{8n}\phi \|\eta\| \|div\eta\| |\nabla\phi| \, d\mu dt.\end{aligned}$$

By $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$,

$$\begin{aligned}&\int_\delta^T \int_M 2\sqrt{8n}\phi \|\eta\| \|div\eta\| |\nabla\phi| \, d\mu dt \\ &\leq 32n\varepsilon \int_\delta^T \int_M \|\eta\|^2 |\nabla\phi|^2 d\mu dt + \tfrac{1}{4\varepsilon} \int_\delta^T \int_M \phi^2 \|div\eta\|^2 d\mu dt.\end{aligned}$$

Let $\varepsilon = \frac{1}{2}$ and combine with (5.36)

$$\int_\delta^T \int_M \|div\eta\|^2 \phi^2 d\mu dt \leq 32n \int_\delta^T \int_M \|\eta\|^2 \left(|\nabla\phi|^2 + \left(\phi^2\right)_t\right) d\mu dt$$

and then

$$\int_\delta^T \int_{B_p(R)} \|div\eta\|^2 d\mu dt \leq C \int_\delta^T \int_{B_p(2R)} \|\eta\|^2 d\mu dt,$$

where C depend on n and $|\nabla\phi|$. This implies (5.32).

Next following the same process as above with the fact that $\nabla_T \eta(x,t) = 0$ for all t and the torsion is vanishing, we have the similar estimate as in (5.34)

$$\left(\frac{\partial}{\partial t} - \Delta\right) \|div\eta\|^2 \leq -\|\nabla div\eta(x,t)\|^2$$

and

$$\int_\delta^T \int_M e^{-ar^2} \|\nabla div\eta(x,t)\|^2 d\mu dt \leq C \int_\delta^T \int_M e^{-ar^2} \|div\eta(x,t)\|^2 d\mu dt.$$

This completes the proof. □

Proof of Theorem 5.1 :

Proof. Since the torsion is vanishing, it follow from Lemma 5.5 and Lemma 5.7

$$\left(\frac{\partial}{\partial t} - 4\Delta_b\right) t^2 \widehat{Z} \geq 0$$

for the vector field which minimizes $\widehat{Z}$ and $0 < k_1 \leq 8$. Since $\tilde{\eta}_{\alpha\bar{\beta}} \geq \varepsilon h_{\alpha\bar{\beta}}$ on $M \times [0,T)$, by first and second variation formula of Z (5.24,5.25), we have

$$\|V\| \leq \|\widetilde{\eta} * (div\eta)_\alpha\| \leq C(\varepsilon) \|div\eta\|$$

and

$$\begin{aligned}\|\nabla_s V_\sigma\| &\leq \left\|\widetilde{\eta}^{-1} * \nabla_s (div\eta)_\alpha + \widetilde{\eta}^{-1} * (\nabla_s \eta_{\alpha\bar{\sigma}}) V_\sigma\right\| \\ &\leq C(\varepsilon) \|\nabla div\eta\| + C(\varepsilon) \|div\eta\|^2 .\end{aligned}$$

Hence

$$\begin{aligned}
\left\| t^2 \widehat{Z} \right\|^2 &\leq t^2 \left\| \widetilde{\eta}_{\alpha\bar{\gamma}} \nabla_{\bar{\alpha}} V_\gamma \right\| + tH + t\varepsilon n \\
&\leq t^2 \left\| -\widetilde{\eta}_{\alpha\bar{\gamma}} \widetilde{\eta}^{-1}_{\gamma\bar{\beta}} \nabla_{\bar{\alpha}} (div\eta)_\beta - \widetilde{\eta}_{\alpha\bar{\gamma}} \widetilde{\eta}^{-1}_{\gamma\bar{\beta}} \left(\nabla_{\bar{\alpha}} \eta_{\beta\bar{\sigma}} \right) V_\sigma \right\| + t \|\eta\| + t\varepsilon n \\
&\leq t^2 \left\| \nabla_{\bar{\alpha}} (div\eta)_\alpha \right\| + t^2 \left\| (div\eta)_{\bar{\sigma}} V_\sigma \right\| + t \|\eta\| + t\varepsilon n \\
&\leq t^2 \|\nabla div\eta\|^2 + t^2 \|div\eta\|^2 + t^2 \|V\|^2 + t \|\eta\| + t\varepsilon n \\
&\leq t^2 \|\nabla div\eta\|^2 + t^2 \|div\eta\|^2 + C(\varepsilon)^2 \|div\eta\|^2 + t \|\eta\| + t\varepsilon n.
\end{aligned}$$

By lemma 5.8

$$\int_0^T \int_M e^{-ar^2} \left\| t^2 \widehat{Z} \right\|^2 d\mu dt < \infty. \tag{5.37}$$

But $t^2 \widehat{Z} = 0$ at $t = 0$. By the maximum principle as in Lemma 5.6, we have $\widehat{Z} \geq 0$ for any $(1,0)$ vector field V. Let $\varepsilon \to 0$, we have $Z \geq 0$. □

5.3. **Nonlinear Version for Li-Yau-Hamilton Inequality.** As an application of Theorem 5.1, we derive the positivity of Harnack quantity of the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7). For simplicity, we work the CR Harnack inequality on a closed strictly pseudoconvex spherical CR 3-manifold.

Definition 5.1. *Let (M, J, θ) be a closed pseudohermitian 3-manifold. We call a CR structure J spherical if Cartan curvature tensor Q_{11}*

$$Q_{11} = \frac{1}{6} R_{,11} + \frac{i}{2} R A_{11} - A_{11,0} - \frac{2i}{3} A_{11,\bar{1}1} \tag{5.38}$$

vanishes identically. Here R is the Tanaka-Webster scalar curvature and A_{11} is the pseudohermitian torsion. Note that (M, J, θ) is called a closed spherical pseudohermitian 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical pseudohermitian 3-manifold (M, J, θ) is locally CR equivalent to $(\mathbf{S}^3, J_{can}, \theta_{can})$.

Lemma 5.9. *([CC]) 1. Let $\left(M^3,\ J,\ \overset{0}{\theta} \right)$ be a closed pseudohermitian 3-manifold with positive initial Tanaka-Webster curvature $\overset{0}{R}(x) > 0$. Then*

$$R(x,t) > 0$$

is preserved under the CR Yamabe flow (5.7) on $M \times [0, T)$. Here $R(x, 0) = \overset{0}{R}(x) > 0$.

2. Let $\left(M^3,\ J,\ \overset{0}{\theta} \right)$ be a closed spherical pseudohermitian 3-manifold with vanishing initial torsion $\overset{0}{A}_{11}(x) = 0$. Then

$$A_{11}(x,t) = 0$$

is preserved under the CR Yamabe flow (5.7) on $M \times [0, T)$. Here $A_{11}(x, 0) = \overset{0}{A}_{11}(x) = 0$.

Lemma 5.10. *Under the CR Yamabe flow (5.7), we have*

$$\frac{\partial}{\partial t}h_{\alpha\bar{\beta}}\left(t\right)=-2Rh_{\alpha\bar{\beta}}\quad \text{and}\quad \frac{\partial}{\partial t}h^{\alpha\bar{\gamma}}\left(t\right)=2Rh^{\alpha\bar{\gamma}} \tag{5.39}$$

and

$$\frac{\partial}{\partial t}\Gamma_{\beta\alpha}^{\sigma}\left(t\right)=-2R_{,\beta}\delta_{\alpha}^{\sigma}-2R_{,\alpha}\delta_{\beta}^{\sigma}\quad \text{and}\quad \frac{\partial}{\partial t}\Gamma_{\bar{\beta}\alpha}^{\sigma}\left(t\right)=2R^{,\sigma}h_{\bar{\beta}\alpha}. \tag{5.40}$$

Proof. For $\theta=e^{2f}\hat{\theta}$ with $\theta(x,0)=e^{2f(x,0)}\hat{\theta}=\overset{0}{\theta}$, the CR Yamabe flow is equivalent to

$$\frac{\partial}{\partial t}f\left(t\right)=-R\left(t\right). \tag{5.41}$$

Moreover, as in ([L1]) we choose admissible coframe $\left\{\theta^{\alpha}=\hat{\theta}^{\alpha}+2if^{\alpha}\hat{\theta}\right\}$ and the Levi form is given by the matrix $h_{\alpha\bar{\beta}}=e^{2f}\hat{h}_{\alpha\bar{\beta}}$, then

$$\frac{\partial}{\partial t}h_{\alpha\bar{\beta}}\left(t\right)=-2Rh_{\alpha\bar{\beta}}.$$

On the other hand, since

$$\Gamma_{\beta\alpha}^{\sigma}=\widehat{\Gamma}_{\beta\alpha}^{\sigma}+2\delta_{\alpha}^{\sigma}f_{\beta}+2f_{\alpha}\delta_{\beta}^{\sigma},$$

by (5.41) we have

$$\frac{\partial}{\partial t}\Gamma_{\beta\alpha}^{\sigma}=-2R_{,\beta}\delta_{\alpha}^{\sigma}-2R_{,\alpha}\delta_{\beta}^{\sigma}$$

and the other formula can be proved analogously. □

Lemma 5.11. *Let $\eta_{\alpha\bar{\beta}}\left(x,t\right)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7) on $M\times[0,T)$, we have*

$$\tfrac{1}{2}\tfrac{\partial}{\partial t}\left(div\eta\right)_{\alpha,\bar{\alpha}}=I+\tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(\nabla_{\bar{\tau}}\nabla_{\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right) \tag{5.42}$$

and

$$\tfrac{\partial}{\partial t}\left(h^{\alpha\bar{\beta}}\left(div\eta\right)_{\alpha}V_{\bar{\beta}}\right)=II+h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\nabla_{\delta}\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right)V_{\bar{\beta}}+h^{\alpha\bar{\beta}}\left(div\eta\right)_{\alpha}\tfrac{\partial}{\partial t}V_{\bar{\beta}}. \tag{5.43}$$

Here we denote

$$\begin{aligned}I=&\,2R\left(div\eta\right)_{\alpha,\bar{\alpha}}+\left(\left(1-n\right)R_{,\gamma\bar{\tau}}\eta_{\tau\bar{\gamma}}+\left(1-n\right)R_{,\gamma}\left(div\eta\right)_{\bar{\gamma}}+R_{,\tau\bar{\tau}}H+R_{,\tau}H_{,\bar{\tau}}\right)\\&+\left(1-n\right)R_{,\bar{\alpha}}\left(div\eta\right)_{\alpha}+\tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(\nabla_{\bar{\tau}}\nabla_{\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right)\end{aligned}$$

and

$$\begin{aligned}II=&\,4h^{\alpha\bar{\beta}}R\left(div\eta\right)_{\alpha}V_{\bar{\beta}}+2\left(1-n\right)h^{\alpha\bar{\beta}}R_{,\gamma}\eta_{\alpha\bar{\gamma}}V_{\bar{\beta}}+2h^{\alpha\bar{\beta}}R_{,\alpha}V_{\bar{\beta}}H\\&+h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\nabla_{\delta}\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right)V_{\bar{\beta}}+h^{\alpha\bar{\beta}}\left(div\eta\right)_{\alpha}\tfrac{\partial}{\partial t}V_{\bar{\beta}}.\end{aligned}$$

Proof. We justify the identities by a normal coordinate at one point p with $\Gamma(p) = 0$. By definition of covariant derivative

$$
\begin{aligned}
\frac{\partial}{\partial t}\eta_{\alpha\bar{\gamma},\delta} &= \frac{\partial}{\partial t}\left(\partial_\delta \eta_{\alpha\bar{\gamma}} - \Gamma^{\sigma}_{\delta\alpha}\eta_{\sigma\bar{\gamma}} - \Gamma^{\bar{\sigma}}_{\delta\bar{\gamma}}\eta_{\alpha\bar{\sigma}}\right) \\
&= \partial_\delta\left(\frac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right) - \left(-2R_{,\delta}\delta^{\sigma}_{\alpha} - 2R_{,\alpha}\delta^{\sigma}_{\delta}\right)\eta_{\sigma\bar{\gamma}} - 2R^{,\bar{\sigma}}h_{\delta\bar{\gamma}}\eta_{\alpha\bar{\sigma}} \\
&= \nabla_\delta\left(\frac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right) - \left(-2R_{,\delta}\delta^{\sigma}_{\alpha} - 2R_{,\alpha}\delta^{\sigma}_{\delta}\right)\eta_{\sigma\bar{\gamma}} - 2R^{,\bar{\sigma}}h_{\delta\bar{\gamma}}\eta_{\alpha\bar{\sigma}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}\left(h^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha V_{\bar{\beta}}\right) \\
&= h^{\alpha\bar{\beta}}\tfrac{\partial}{\partial t}\left(h^{\delta\bar{\gamma}}\eta_{\alpha\bar{\gamma},\delta}\right)V_{\bar{\beta}} + 2Rh^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha V_{\bar{\beta}} + h^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha \tfrac{\partial}{\partial t}V_{\bar{\beta}} \\
&= h^{\alpha\bar{\beta}}\left(2Rh^{\delta\bar{\gamma}}\right)\eta_{\alpha\bar{\gamma},\delta}V_{\bar{\beta}} + h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\left(2R_{,\delta}\delta^{\sigma}_{\alpha} + 2R_{,\alpha}\delta^{\sigma}_{\delta}\right)\eta_{\sigma\bar{\gamma}}V_{\bar{\beta}} \\
&\quad - h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\left(2R^{,\bar{\sigma}}h_{\delta\bar{\gamma}}\right)\eta_{\alpha\bar{\sigma}}V_{\bar{\beta}} + h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\nabla_\delta\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right)V_{\bar{\beta}} \\
&\quad + 2Rh^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha V_{\bar{\beta}} + h^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha \tfrac{\partial}{\partial t}V_{\bar{\beta}} \\
&= 4h^{\alpha\bar{\beta}}R\left(div\eta\right)_\alpha V_{\bar{\beta}} + 2\left(1-n\right)h^{\alpha\bar{\beta}}R_{,\gamma}\eta_{\alpha\bar{\gamma}}V_{\bar{\beta}} + h^{\alpha\bar{\beta}}2R_{,\alpha}V_{\bar{\beta}}H \\
&\quad + h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\nabla_\delta\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right)V_{\bar{\beta}} + h^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha \tfrac{\partial}{\partial t}V_{\bar{\beta}} \\
&:= II + h^{\alpha\bar{\beta}}h^{\delta\bar{\gamma}}\nabla_\delta\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\gamma}}\right)V_{\bar{\beta}} + h^{\alpha\bar{\beta}}\left(div\eta\right)_\alpha \tfrac{\partial}{\partial t}V_{\bar{\beta}}.
\end{aligned}
$$

Next we compute

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta},\gamma\bar{\tau}} \\
&= \tfrac{\partial}{\partial t}\left(\partial_{\bar{\tau}}\left(\eta_{\alpha\bar{\beta},\gamma}\right) - \Gamma^{\delta}_{\bar{\tau}\alpha}\eta_{\delta\bar{\beta},\gamma} - \Gamma^{\bar{\delta}}_{\bar{\tau}\beta}\eta_{\alpha\bar{\delta},\gamma} - \Gamma^{\delta}_{\bar{\tau}\gamma}\eta_{\alpha\bar{\beta},\delta}\right) \\
&= \partial_{\bar{\tau}}\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta},\gamma}\right) - 2R^{,\delta}h_{\bar{\tau}\alpha}\eta_{\delta\bar{\beta},\gamma} - \Gamma^{\delta}_{\bar{\tau}\alpha}\tfrac{\partial}{\partial t}\eta_{\delta\bar{\beta},\gamma} \\
&\quad - \left(-2R_{,\bar{\tau}}\delta^{\bar{\delta}}_{\bar{\beta}} - 2R_{,\bar{\beta}}\delta^{\bar{\delta}}_{\bar{\tau}}\right)\eta_{\alpha\bar{\delta},\gamma} - \Gamma^{\bar{\delta}}_{\bar{\tau}\bar{\beta}}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\delta},\gamma} \\
&\quad - 2R^{,\delta}h_{\bar{\tau}\gamma}\eta_{\alpha\bar{\beta},\delta} - \Gamma^{\delta}_{\bar{\tau}\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta},\delta} \\
&= \partial_{\bar{\tau}}\left(\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta},\gamma}\right) - 2R^{,\delta}h_{\bar{\tau}\alpha}\eta_{\delta\bar{\beta},\gamma} + 2R_{,\bar{\tau}}\eta_{\alpha\bar{\beta},\gamma} \\
&\quad + 2R_{,\bar{\beta}}\eta_{\alpha\bar{\tau},\gamma} - 2R^{,\delta}h_{\bar{\tau}\gamma}\eta_{\alpha\bar{\beta},\delta} \\
&= \nabla_{\bar{\tau}}\nabla_\gamma\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}} + 2R_{,\gamma\bar{\tau}}\eta_{\alpha\bar{\beta}} + 2R_{,\gamma}\eta_{\alpha\bar{\beta},\bar{\tau}} \\
&\quad + 2R_{,\alpha\bar{\tau}}\eta_{\gamma\bar{\beta}} + 2R_{,\alpha}\eta_{\gamma\bar{\beta},\bar{\tau}} - 2R_{,\sigma\bar{\tau}}\eta_{\alpha\bar{\sigma}}h_{\gamma\bar{\beta}} \\
&\quad - 2R_{,\sigma}\eta_{\alpha\bar{\sigma},\bar{\tau}}h_{\gamma\bar{\beta}} - 2R^{,\delta}h_{\bar{\tau}\alpha}\eta_{\delta\bar{\beta},\gamma} + 2R_{,\bar{\tau}}\eta_{\alpha\bar{\beta},\gamma} \\
&\quad + 2R_{,\bar{\beta}}\eta_{\alpha\bar{\tau},\gamma} - 2R^{,\delta}h_{\bar{\tau}\gamma}\eta_{\alpha\bar{\beta},\delta},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\partial}{\partial t}\left(div\eta\right)_{\alpha,\bar{\alpha}}\\
&= \tfrac{1}{2}\tfrac{\partial}{\partial t}\left(h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\eta_{\alpha\bar{\beta},\gamma\bar{\tau}}\right)\\
&= 2R\left(div\eta\right)_{\alpha,\bar{\alpha}} + \tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta},\gamma\bar{\tau}}\\
&= 2R\left(div\eta\right)_{\alpha,\bar{\alpha}} + \tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(\nabla_{\bar{\tau}}\nabla_{\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right)\\
&\quad +h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(R_{,\gamma\bar{\tau}}\eta_{\alpha\bar{\beta}} + R_{,\gamma}\eta_{\alpha\bar{\beta},\bar{\tau}} + R_{,\alpha\bar{\tau}}\eta_{\gamma\bar{\beta}} + R_{,\alpha}\eta_{\gamma\bar{\beta},\bar{\tau}} - R_{,\sigma\bar{\tau}}\eta_{\alpha\bar{\sigma}}h_{\gamma\bar{\beta}} - R_{,\sigma}\eta_{\alpha\bar{\sigma},\bar{\tau}}h_{\gamma\bar{\beta}}\right)\\
&\quad +h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(-R^{,\delta}h_{\bar{\tau}\alpha}\eta_{\delta\bar{\beta},\gamma} + R_{,\bar{\tau}}\eta_{\alpha\bar{\beta},\gamma} + R_{,\bar{\beta}}\eta_{\alpha\bar{\tau},\gamma} - R^{,\delta}h_{\bar{\tau}\gamma}\eta_{\alpha\bar{\beta},\delta}\right)\\
&= 2R\left(div\eta\right)_{\alpha,\bar{\alpha}} + \left(\left(1-n\right)R_{,\gamma\bar{\tau}}\eta_{\tau\bar{\gamma}} + \left(1-n\right)R_{,\gamma}\left(div\eta\right)_{\bar{\gamma}} + R_{,\tau\bar{\tau}}H + R_{,\tau}H_{,\bar{\tau}}\right)\\
&\quad + \left(1-n\right)R_{,\bar{\alpha}}\left(div\eta\right)_{\alpha} + \tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(\nabla_{\bar{\tau}}\nabla_{\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right)\\
&:= I + \tfrac{1}{2}h^{\alpha\bar{\tau}}h^{\gamma\bar{\beta}}\left(\nabla_{\bar{\tau}}\nabla_{\gamma}\tfrac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right).
\end{aligned}
$$

□

Lemma 5.12. *Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7) on $M\times[0,T)$, we have*

(5.44)

$$
\begin{aligned}
\tfrac{\partial}{\partial t}\left[h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\left(\eta_{\alpha\bar{\delta}} + \varepsilon h_{\alpha\bar{\beta}}\right)V_{\bar{\beta}}V_{\gamma}\right] &= III + 2\varepsilon Rh_{\beta\bar{\gamma}}V_{\bar{\beta}}V_{\gamma} + h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\tfrac{\partial}{\partial t}\left(\eta_{\alpha\bar{\delta}}V_{\bar{\beta}}V_{\gamma}\right),\\
\tfrac{\partial}{\partial t}\tfrac{H+2\epsilon n}{t} &= IV + h^{\alpha\bar{\beta}}\tfrac{\partial}{\partial t}\left(\tfrac{\eta_{\alpha\bar{\beta}}}{t}\right),\\
\tfrac{\partial}{\partial t}R &= \left(2n+2\right)\Delta R + 2R^{2},\\
\left(\tfrac{\partial}{\partial t} - 4\Delta_{b}\right)(RH) &= \left(2n-2\right)(\Delta_{b}R)H + 4R^{2}H - 8\left(R_{,\tau}H_{,\bar{\tau}} + R_{,\bar{\tau}}H_{,\tau}\right)\\
&:= V
\end{aligned}
$$

with

$$III := 4Rh^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\eta_{\alpha\bar{\delta}}V_{\bar{\beta}}V_{\gamma} \quad and \quad IV := 2R\frac{H}{t}.$$

Proof. It is by straightforward computation as in the previous Lemma. □

Now we define

$$Z_R = Z + k_2 RH$$

with k_2 to be determined and Z is

$$Z := \frac{k_1}{2}\left[h^{\alpha\bar{\beta}}\left(div\eta\right)_{\alpha,\bar{\beta}} + conj\right] + k_1\left[h^{\alpha\bar{\beta}}\left(div\eta\right)_{\alpha}V_{\bar{\beta}} + conj\right] + k_1 h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\eta_{\alpha\bar{\delta}}V_{\bar{\beta}}V_{\alpha} + \frac{H}{t}$$

as before.

Lemma 5.13. *Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$-tensor satisfying the CR Lichnerowicz-Laplacian heat equation (5.2) coupled with the CR Yamabe flow (5.7) on $M\times[0,T)$, we have*

$$
\begin{aligned}
&\left(\tfrac{\partial}{\partial t} - 4\Delta_{b}\right)Z_{R}\\
&= Y_{1} + 8k_{1}\eta_{\alpha\bar{\sigma}}\nabla_{\gamma}V_{\sigma}\nabla_{\bar{\gamma}}V_{\bar{\alpha}} - \tfrac{2Z_{R}}{t} + \tfrac{8-k_{1}}{8t^{2}}H\\
&\quad +\eta_{\alpha\bar{\sigma}}\left[\sqrt{8k_{1}}\nabla_{\bar{\gamma}}V_{\sigma} - \tfrac{\sqrt{k_{1}}}{\sqrt{8t}}h_{\sigma\bar{\gamma}} - \tfrac{\sqrt{k_{1}}}{\sqrt{2}}Rh_{\sigma\bar{\alpha}}\right]\left[\sqrt{8k_{1}}\nabla_{\gamma}V_{\bar{\alpha}} - \tfrac{\sqrt{k_{1}}}{\sqrt{8t}}h_{\gamma\bar{\alpha}} - \tfrac{\sqrt{k_{1}}}{\sqrt{2}}Rh_{\gamma\bar{\alpha}}\right],
\end{aligned}
$$

where

$$Y_1 = k_1(\Delta_b R)H + k_1\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) - 8k_2\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) \\ +k_1(1-n)\left(R_{,\gamma\bar\tau} + R_{,\bar\tau\gamma}\right)\eta_{\tau\bar\gamma} + k_2(2n-2)\Delta_b RH \\ +\left(4k_2 - \tfrac{k_1}{2}\right)R^2H + 2k_1\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma} \\ +8k_1V_{\bar\alpha}V_\beta R_{\alpha\bar\beta\mu\bar\gamma}\eta_{\gamma\bar\mu} + \left(2k_2 + 2 - \tfrac{k_1}{2}\right)\tfrac{RH}{t}. \tag{5.45}$$

In particular if $n=1$, by taking $\overline{V} = \frac{2(8k_2+k_1)}{k_1}V$, (5.45) becomes

$$Y_1 = \tfrac{k_1}{2}[2\Delta_b R + \left\langle \nabla_b R, \overline{V}\right\rangle + \left(\tfrac{8k_2}{k_1} - 1\right)R^2 \\ + \tfrac{2k_1^2}{(8k_2+k_1)^2}R\left\|\overline{V}\right\|^2 + \tfrac{2}{k_1}\left(2k_2 + 2 - \tfrac{k_1}{2}\right)\tfrac{R}{t}]H. \tag{5.46}$$

Proof. As before we compute

$$\left(\tfrac{\partial}{\partial t} - 4\Delta_b\right)Z_R \\ = E + 8\eta_{\alpha\bar\sigma}\nabla_\gamma V_\sigma \nabla_{\bar\gamma}V_{\bar\alpha} \\ +\eta_{\alpha\bar\sigma}\left[\sqrt{8k_1}\nabla_{\bar\gamma}V_\sigma - \tfrac{\sqrt{k_1}}{\sqrt{8t}}h_{\sigma\bar\gamma}\right]\left[\sqrt{8k_1}\nabla_\gamma V_{\bar\alpha} - \tfrac{\sqrt{k_1}}{\sqrt{8t}}h_{\gamma\bar\alpha}\right] \\ -\tfrac{2Z_R}{t} + \tfrac{8-k_1}{8t^2}H,$$

where the extra term E comes from covariant derivative of the Levi metric $h_{\alpha\bar\beta}$ under the CR Yamabe flow and lemma 5.5

$$E = k_1I + k_1II + k_1III + k_1IV + k_2V + 8k_1V_{\bar\alpha}V_\beta R_{\alpha\bar\beta\mu\bar\gamma}\eta_{\gamma\bar\mu} \\ = -2k_1R\eta_{\alpha\bar\sigma}\nabla_{\bar\alpha}V_\sigma - 2k_1R\eta_{\sigma\bar\alpha}\nabla_\alpha V_{\bar\sigma} + k_1\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) \\ -8k_2\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) + k_1(\Delta_b R)H \\ +k_1(1-n)\left(R_{,\gamma\bar\tau} + R_{,\bar\tau\gamma}\right)\eta_{\tau\bar\gamma} + k_2(2n-2)(\Delta_b R)H \\ +4k_2R^2H + 2k_1\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma} \\ +8k_1V_{\bar\alpha}V_\beta R_{\alpha\bar\beta\mu\bar\gamma}\eta_{\gamma\bar\mu} + 2\tfrac{RH}{t}.$$

Rearranging these terms we obtain

$$\left(\tfrac{\partial}{\partial t} - 4\Delta_b\right)Z_R \\ = Y_1 + 8\eta_{\alpha\bar\sigma}\nabla_\gamma V_\sigma \nabla_{\bar\gamma}V_{\bar\alpha} \\ +\eta_{\alpha\bar\sigma}\left[\sqrt{8k_1}\nabla_{\bar\gamma}V_\sigma - \tfrac{\sqrt{k_1}}{\sqrt{8t}}h_{\sigma\bar\gamma} - \tfrac{\sqrt{k_1}}{\sqrt{2}}Rh_{\sigma\bar\gamma}\right]\left[\sqrt{8k_1}\nabla_\gamma V_{\bar\alpha} - \tfrac{\sqrt{k_1}}{\sqrt{8t}}h_{\gamma\bar\alpha} - \tfrac{\sqrt{k_1}}{\sqrt{2}}Rh_{\gamma\bar\alpha}\right] \\ +\tfrac{8-k_1}{8t^2}H - \tfrac{2Z_R}{t},$$

where

$$Y_1 = k_1\Delta_b RH + k_1\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) - 8k_2\left(R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau}\right) \\ +k_1(1-n)\left(R_{,\gamma\bar\tau} + R_{,\bar\tau\gamma}\right)\eta_{\tau\bar\gamma} + k_2(2n-2)\Delta_b RH \\ +\left(4k_2 - \tfrac{k_1}{2}\right)R^2H + 2k_1\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma} \\ +8k_1V_{\bar\alpha}V_\beta R_{\alpha\bar\beta\mu\bar\gamma}\eta_{\gamma\bar\mu} + \left(2k_2 + 2 - \tfrac{k_1}{2}\right)\tfrac{RH}{t}.$$

Since $\eta_{\alpha\bar\beta}$ is positive definite, by (5.24) we have $R_{,\tau}H_{,\bar\tau} + R_{,\bar\tau}H_{,\tau} = -\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma}$ so that

$$Y_1 = k_1(\Delta_b R)H + (8k_2 + k_1)\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma} \\ +k_1(1-n)\left(R_{,\gamma\bar\tau} + R_{,\bar\tau\gamma}\right)\eta_{\tau\bar\gamma} + k_2(2n-2)\Delta_b RH \\ +\left(4k_2 - \tfrac{k_1}{2}\right)R^2H + 2k_1\left(R_{,\alpha}V_{\bar\alpha} + R_{,\bar\alpha}V_\alpha\right)h_{\gamma\bar\tau}\eta_{\tau\bar\gamma} \\ +8k_1V_{\bar\alpha}V_\beta R_{\alpha\bar\beta\mu\bar\gamma}\eta_{\gamma\bar\mu} + \left(2k_2 + 2 - \tfrac{k_1}{2}\right)\tfrac{RH}{t}.$$

For $n=1$

$$\begin{aligned}Y_1 = \tfrac{k_1}{2}[2\Delta_b R &+ \tfrac{2}{k_1}\left(8k_2+k_1\right)\left(R_{,\tau}V_{,\bar{\tau}}+R_{,\bar{\tau}}V_{,\tau}\right) \\ &+\tfrac{2}{k_1}\left(4k_2-\tfrac{k_1}{2}\right)R^2+8R\left\|V\right\|^2+\tfrac{2}{k_1}\left(2k_2+2-\tfrac{k_1}{2}\right)\tfrac{R}{t}]H.\end{aligned}$$

Taking $\overline{V}=\frac{2(8k_2+k_1)}{k_1}V$,

$$\begin{aligned}Y_1 = \tfrac{k_1}{2}[2\Delta_b R &+ \left\langle \nabla_b R, \overline{V}\right\rangle + \left(\tfrac{8k_2}{k_1}-1\right)R^2 \\ &+\tfrac{2k_1^2}{(8k_2+k_1)^2}R\left\|\overline{V}\right\|^2+\tfrac{2}{k_1}\left(2k_2+2-\tfrac{k_1}{2}\right)\tfrac{R}{t}]H.\end{aligned} \tag{5.47}$$

□

Proof of Theorem 5.2 :

Proof. It is known that

$$2\Delta_b R + R^2 + \frac{R}{t} + \left\langle \nabla_b R, V\right\rangle_{J,\theta} = 0$$

is the equation for a CR Yamabe expanding soliton ([CC]) with the CR vector field V. The Hamilton-type Harnack quantity (we refer to [H1] for some details) for the CR Yamabe flow is defined as

$$Z_\Theta\left(\theta,V\right) = 2\Delta_b R + R^2 + \frac{R}{t} + \left\langle \nabla_b R, V\right\rangle_{J,\theta} + \frac{\Theta}{8}R\left\|V\right\|_{J,\theta}^2$$

where $\Theta\le 1$ is to be determined (see Remark 5.2). A straightforward computation for $\square=\partial_t-4\Delta_b$ yields

$$\begin{aligned}&\square Z_\Theta\left(\theta,V\right) \\ &= 12R\Delta_b R - 4\left\|\nabla_b R\right\|_{J,\theta}^2 + 4R^3 + 2R^2/t - R/t^2 \\ &\quad + R\left\langle \nabla_b R, V\right\rangle_{J,\theta} + \tfrac{\Theta}{4}R^2\left\|V\right\|_{J,\theta}^2 + \left\langle \nabla_b R + \tfrac{\Theta}{4}RV, \square V\right\rangle_{J,\theta} \\ &\quad -8\left\langle \nabla_b\left(\nabla_b R\right), \nabla_b V\right\rangle_{J,\theta} - \Theta R\left\|\nabla_b V\right\|_{J,\theta}^2 - \Theta\left\langle \nabla_b R, \nabla_b\left(\left\|V\right\|_{J,\theta}^2\right)\right\rangle_{J,\theta}.\end{aligned} \tag{5.48}$$

We prove the theorem by contradiction. Suppose that $Z_\Theta\left(\theta,V\right)\le 0$ at some space-time point for some V. Then there exists a first time t_0, a point p_0 such that at (p_0,t_0),

$$Z_\Theta\left(\theta,V\right) = 0. \tag{5.49}$$

We can extend V so that at (p_0,t_0)

$$V_{\bar{1},1} = -\frac{4R_{,\bar{1}1}}{\Theta R} - \frac{R_{,1}V_{\bar{1}}}{R} \quad \text{and} \quad V_{1,1} = -\frac{R_{,1}V_1}{R}.$$

Then the last three terms of (5.48) become

$$\begin{aligned}&-8\left\langle \nabla_b\left(\nabla_b R\right), \nabla_b V\right\rangle_{J,\theta} - \Theta R\left\|\nabla_b V\right\|_{J,\theta}^2 - \Theta\left\langle \nabla_b R, \nabla_b\left(\left\|V\right\|_{J,\theta}^2\right)\right\rangle_{J,\theta} \\ &= 2\Theta R\left|\tfrac{4R_{,\bar{1}1}}{\Theta R} + \tfrac{R_{,1}V_{\bar{1}}}{R}\right|^2 + 2\Theta R\left|\tfrac{R_{,1}V_1}{R}\right|^2,\end{aligned}$$

where we have used $R_{,11}=0$ due to Lemma 5.9 and (5.38).

Now if $\nabla_b R + \frac{\Theta}{4}RV \ne 0$ at (p_0,t_0), we extend V by choosing the value of $\square V$ at (p_0,t_0) to kill all terms on the right-hand side of (5.48) except, say $2R^2/t$. Then it follows that

$$0 \ge \partial_t Z = 4\Delta_b Z + 2R^2/\tau \ge 2R^2/\tau$$

at (p_0,t_0). This is a contradiction. So we may assume

$$\nabla_b R + \frac{\Theta}{4} R V = 0 \tag{5.50}$$

at (p_0,t_0). By (5.49) and (5.50), we have

$$2\Delta_b R = \ -R^2 - \tfrac{R}{t} + \tfrac{\Theta}{8} R \left\| V \right\|_{J,\theta}^2 \tag{5.51}$$

at (p_0,t_0). Now combining (5.48), (5.50) and (5.51)

$$\begin{aligned} &\Box Z_\Theta(\theta, V) \\ &= \left(-2+\tfrac{2}{\Theta}\right) R^3 + \left(-4+\tfrac{4}{\Theta}\right)\tfrac{R^2}{t} \\ &\quad + \left(\tfrac{3-\Theta^2}{4} - \tfrac{(1-\Theta)}{2}\right) R^2 \left\| V \right\|_{J,\theta}^2 + \left(-1+\tfrac{2}{\Theta}\right)\tfrac{R}{t^2} \\ &\quad + \left(\tfrac{\Theta(1-\Theta)^2}{32} + \tfrac{\Theta^3}{32}\right) R \left\| V \right\|_{J,\theta}^4 - \tfrac{(1-\Theta)}{2}\tfrac{R\left\| V \right\|_{J,\theta}^2}{t}. \end{aligned} \tag{5.52}$$

We apply the Young's inequality for the last term of (5.52) and obtain

$$\begin{aligned} &\left(-1+\tfrac{2}{\Theta}\right)\tfrac{R}{t^2} + \left(\tfrac{\Theta(1-\Theta)^2}{32} + \tfrac{\Theta^3}{32}\right) R \left\| V \right\|_{J,\theta}^4 - \tfrac{(1-\Theta)}{2}\tfrac{R\left\| V \right\|^2}{t} \\ \geq &\left(-1+\tfrac{2}{\Theta}-\epsilon^2\right)\tfrac{R}{t^2} + \left(\tfrac{\Theta(1-\Theta)^2}{32} + \tfrac{\Theta^3}{32} - \tfrac{(1-\Theta)^2}{16\epsilon^2}\right) R \left\| V \right\|_{J,\theta}^4. \end{aligned} \tag{5.53}$$

This will lead a contradiction again if we can choose ϵ and Θ to make the RHS of (5.53) to be positive which is possible by taking $\epsilon^2 = \frac{2-\Theta}{\Theta}$ and

$$\frac{\Theta\left(1-\Theta\right)^2}{32} + \frac{\Theta^3}{32} > \frac{\left(1-\Theta\right)^2}{16\epsilon^2}.$$

That is

$$\Theta^2 - 2\Theta + \frac{1}{2} < 0$$

which is true if $1-\frac{\sqrt{2}}{2} < \Theta < 1+\frac{\sqrt{2}}{2}$. Hence we may choose $\Theta = \frac{3}{10}$. Then we are done. $\Box$

Remark 5.2. *In the paper of* [CC], *we have the following Harnack inequality*

$$2\Delta_b R + R^2 + \frac{R}{t} + \left\langle \nabla_b R, V \right\rangle_{J,\theta} + \frac{\Theta}{8} R \left\| V \right\|_{J,\theta}^2 \geq 0$$

for $\Theta = 1$. However, it is not enough to obtain $Y_1 \geq 0$ unless, say $\Theta = \frac{3}{10}$ so that there exist k_1, k_2 satisfying (5.57).

Proof of Theorem 5.3 :

Proof. From Lemma 5.9, the vanishing torsion and positive Tanaka-Webster curvature are preserved under the CR Yamabe flow. By imitating the argument as in theorem 5.1, Theorem 5.3 if Y_1 is nonnegative. To determined Y_1 is nonnegative, by Theorem 5.2, we require the coefficients in (5.46) satisfy

$$\left(\frac{8k_2}{k_1}-1\right) \geq 1 \tag{5.54}$$

$$\frac{2k_1^2}{(8k_2+k_1)^2} \geq \frac{3}{80} \tag{5.55}$$

$$\frac{2}{k_1}\left(2k_2+2-\frac{k_1}{2}\right) \geq 1 \tag{5.56}$$

and also we require $0 \leq k_1 \leq 8$. These are equivalent to

$$\frac{8}{\sqrt{\frac{160}{3}}-1}k_2 \leq k_1 \leq \min\{8, 4k_2, 2k_2+2\} \tag{5.57}$$

By choosing $k_2 = 1$ and $k_1 = 4$. We are done.

Note that for $n = 1$, the CR Lichnerowicz-Laplacian heat equation (5.2) will be the special form

$$\frac{\partial}{\partial t}\eta_{1\bar{1}} = 4\Delta_b\eta_{1\bar{1}} + 2R_{1\bar{1}}\eta_{1\bar{1}} - (R_{1\bar{1}}\eta_{1\bar{1}} + R_{1\bar{1}}\eta_{1\bar{1}}) = 4\Delta_b\eta_{1\bar{1}}.$$

Hence (5.9) follows. □

Remark 5.3. *1. Let $(M, J, \overset{0}{\theta})$ be a closed pseudohermitian 3-manifold with vanishing initial torsion $\overset{0}{A}_{11}(x) = 0$, then $\overset{0}{R}_{,0}(x) = 0$. In additional if (M, J) is spherical, then*

$$R_{,0}(x,t) = 0$$

under the CR Yamabe flow (5.7) on $M \times [0, T)$. 2. Let $\left(M^3,\ J,\ \overset{0}{\theta}\right)$ be a closed strictly pseudoconvex spherical CR 3-manifold. Since $R_{1\bar{1}} = Rh_{1\bar{1}}$, then $\eta_{1\bar{1}} := R_{1\bar{1}}$ satisfies the CR Lichnerowicz-Laplacian heat equation (5.2)

$$\frac{\partial}{\partial t}R_{1\bar{1}} = 4\Delta_b R_{1\bar{1}}$$

coupled with the CR Yamabe flow (5.7) on $M \times [0, T)$ satisfying

$$A_{11}(x,t) = 0$$

and

$$\eta_{1\bar{1},0}(x,t) = R_{1\bar{1},0}(x,t) = 0$$

for all t.

Proof of Corollary 5.4 :

Proof. It follows from (5.39) that $R_{1\bar{1}} = Rh_{1\bar{1}}$ satisfies

$$\begin{aligned}
\tfrac{\partial}{\partial t}R_{1\bar{1}} &= (\tfrac{\partial}{\partial t}R)h_{1\bar{1}} + R(\tfrac{\partial}{\partial t}h_{1\bar{1}}) \\
&= (4\Delta_b R + 2R^2)h_{1\bar{1}} + R(-2Rh_{1\bar{1}}) \\
&= 4(\Delta_b R)h_{1\bar{1}} = 4\Delta_b(Rh_{1\bar{1}}) \\
&= 4\Delta_b R_{1\bar{1}}.
\end{aligned}$$

Hence we apply Theorem 5.3 by taking $\eta_{1\bar{1}} := R_{1\bar{1}}$. We obtain

$$\begin{aligned} Z_R &= 2\Delta_b R + R^2 + \tfrac{R}{t} + 4\langle \nabla_b R, V\rangle + 2R\|V\|^2 \\ &= \tfrac{1}{2}(\partial_t R + \tfrac{2R}{t} + 8\langle \nabla_b R, V\rangle + 4R\|V\|^2) \\ &\geq 0. \end{aligned}$$

In particular, taking $V = 0$, we have

$$\frac{\partial}{\partial t}(t^2 R) \geq 0$$

on $M \times [0, T)$. □

6. CR Gap Theorem

In [GW1], [S] and [Y], it is conjectured that a complete noncompact Kähler manifold of positive holomorphic bisectional curvature of complex dimension m is biholomorphic to $\mathbf{C}^m$. The first result concerning this conjecture was obtained by Mok-Siu-Yau ([MSY]) and Mok ([Mok2]). Let M be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension $m \geq 2$. They proved that M is isometrically biholomorphic to $\mathbf{C}^m$ with the standard flat metric under the assumptions of the maximum volume growth condition

$$V_o(r) \geq \delta r^{2m}$$

for some point $o \in M$, $\delta > 0$, $r(x) = d(o, x)$ and the scalar curvature R decays as

$$R(x) \leq \frac{C}{1 + r^{2+\varepsilon}}, \quad x \in M$$

for $C > 0$ and any arbitrarily small positive constant ε. Since then there are several further works aiming to prove the optimal result and reader is referred to [Mok1], [CTZ], [CZ2], [N4] and [NT2]. A key common ingredient used in the previous works such as [MSY], [N4] and [NT2] is to solve the so-called Poincare Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$, for a given d-closed real $(1, 1)$- form ρ and then show that $trace(\rho) = 0$ by using (6.1). In particular in [NT2], Ni and Tam showed that the solution $u(x)$ of $\sqrt{-1}\partial\bar{\partial}u = Ric$ is of $o(logr(x))$ growth with the extra condition $\liminf_{r\to\infty} \exp(-ar^2)\int_{B_o(r)} R^2(y)\,d\mu(y) < \infty$ for some $a > 0$. Then the result follows from the Liouville theorem for plurisubharmonic functions which asserts that any continuous plurisubharmonic function with upper growth bound of $o(logr(x))$ must be a constant.

In 2012, L. Ni finally obtained an optimal gap theorem ([N2]) on M with nonnegative bisectional curvature without the maximum volume growth condition, provided the following scalar decays

$$\frac{1}{V_o(r)}\int_{B_o(r)} R(y)\,d\mu(y) = o\left(r^{-2}\right). \tag{6.1}$$

In the paper of [N2], L. Ni adapted a different method which has also succeeded in the recent resolution of the fundamental gap conjecture in [AC]. The key step is , using a sharp differential estimate and monotonicity of heat equation deformation of positive $(1, 1)$-forms as in [N1], it provided an alternate argument of proving the above mentioned Liouville theorem.

A Riemannian version of [MSY] was proved in [GW2] shortly afterwards. This part is concerned with an analogue of CR gap theorem on a complete noncompact strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature. Recently, enlightened by the work of [N1] as above, we obtained the linear trace version of Li-Yau-Hamilton inequality for positive solutions of the CR Lichnerowicz-Laplacian heat equation and then CR monotonicity of heat equation deformation of positive $(1,1)$-forms is available in order to prove the following CR gap Theorem :

Theorem 6.1. *Let M be a complete noncompact strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. Then M is flat if*

$$\frac{1}{V_o(r)}\int_{B_o(r)} S(y)\,d\mu(y) = o\left(r^{-2}\right), \tag{6.2}$$

for some point $o \in M$. Here $S(y)$ is the Tanaka-Webster scalar curvature and $V_o(r)$ is the volume of the ball $B_o(r)$ with respect to the Carnot-Carathéodory distance. As a consequence if M is not flat, then

$$\liminf_{r\longrightarrow\infty} \frac{r^2}{V_o(r)}\int_{B_o(r)} S(y)\,d\mu(y) > 0$$

for any $o \in M$.

Here we adapt the method as in [N2]. Below is the main idea in our proof. We first work on degenerated parabolic systems in CR manifolds which is different to Kähler manifolds :

$$\begin{cases} \frac{\partial}{\partial t}\phi(x,t) & = & \Delta_H \phi(x,t), \\ \phi(x,0) & = & Ric(x) \geq 0. \end{cases}$$

Here Δ_H is the CR Hodge-Laplacian operator, $Ric(x) = iR_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$ is the pseudohermitian Ricci form of a strictly pseudoconvex CR $(2n+1)$-manifold.

Let M be a complete noncompact strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. It follows from Proposition 6.1 that there exists a long time solution $\phi(x,t)$ with $\phi(x,t) \geq 0$ on $M \times [0,\infty)$. Now let $u(x,t) = \Lambda(\phi)$ which is nonnegative and satisfies the CR heat equation with $u(x,0) = S(x)$. Li-Yau-Hamilton Harnack quantity (6.21) and monotonicity property (6.35) with vanishing mixed-term implies that $tu(x,t)$ is nondecreasing in t for any x. Finally, the assumption (6.2) and CR moment type estimate (6.12) imply $\lim_{t\to\infty} tu(x_0,t) = 0$. Hence the monotonicity and maximum principle imply $tu(x,t) \equiv 0$ for all $t > 0$ and any $x \in M$. The flatness then follows from $u(x,0) = 0$ which is clear by continuity.

This chapter is organized as follows. In section 6.1, we obtain the CR moment type estimate which is the first key estimate for the proof of main theorem. In section 6.2, we relate the linear trace Li-Yau-Hamilton type inequality of the CR Lichnerowicz-Laplacian heat equation to a monotonicity formula of the heat solution. In section 6.3, we prove the CR optimal gap Theorem.

6.1. **CR Moment-Type Estimates.** Let (M,J,θ) be a strictly pseudoconvex CR $(2n+1)$-manifold. In our recent paper ([CCT] and [CCF]), we consider the CR Hodge-Laplacian

$$\Delta_H = -\frac{1}{2}(\Box_b + \overline{\Box}_b)$$

for Kohn-Rossi Laplacian $\Box_b$. For any $(1,1)$-form $\phi(x,t)=\phi_{\alpha\overline{\beta}}\theta^{\alpha}\wedge\theta^{\overline{\beta}}$, we study the CR Hodge-Laplacian heat equation on $M\times[0,T)$

$$\frac{\partial}{\partial t}\phi(x,t)=\Delta_H\phi(x,t). \tag{6.3}$$

It follows from the CR Bochner-Weitzenbock Formula ([CCF]) that the CR parabolic equation (6.3) is equivalent to the CR analogue of Lichnerowicz-Laplacian heat equation :

$$\frac{\partial}{\partial t}\phi_{\alpha\bar{\beta}}=\Delta_b\phi_{\alpha\bar{\beta}}+2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\phi_{\gamma\bar{\mu}}-(R_{\gamma\bar{\beta}}\phi_{\alpha\bar{\gamma}}+R_{\alpha\bar{\gamma}}\phi_{\gamma\bar{\beta}}). \tag{6.4}$$

In this section, we consider the following Dirichlet problem of degenerate parabolic systems :

$$\left\{\begin{array}{lcll}\left(\frac{\partial}{\partial t}-\Delta_H\right)\phi & = & 0, & \text{on } \Omega\times[0,\infty),\\ \phi\left(x,t\right) & = & 0, & \text{on } \partial\Omega\times[0,\infty),\\ \phi\left(x,0\right) & = & \phi_{ini}\left(x\right) & \text{on } \Omega.\end{array}\right. \tag{6.5}$$

In contrast to Kähler case, the regularity of a solution for Δ_H up to $\partial\Omega$ may depend on geometry around the characteristic point at the boundary ([J1] and [J2]) in the CR setting. In fact,

Proposition 6.1. *There exists "sweetsop" exhaustion domains Ω_μ such that the solutions ϕ_μ of (6.5) are $C\left(C^{2,\alpha}\left(\bar{\Omega}_\mu,\Lambda^{1,1}\right),[0,T)\right)$.*

We will give a detail proof of Proposition 6.1 in Appendix **A.** After the construction of the "sweetsop" exhaustion domain Ω_μ for Δ_H as in Proposition 6.1, one is able to apply semigroup method ([P]) to obtain better regularity of the solution of the CR Lichnerowitz-Laplacian heat equation (6.5) which depends on regularity of the initial condition. One more tensor maximum principle below is needed in the proof of main theorem in order to have nonnegativity of the constructed solution ϕ_μ if the initial data is nonnegative.

Proposition 6.2. *Let (M,J,θ) be a strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature. Let Ω be bounded domain in M. Assume that $\phi(x,t)$ is a $(1,1)$-form satisfies*

$$\left\{\begin{array}{lcll}\left(\frac{\partial}{\partial t}-\Delta_H\right)\phi & = & 0, & \text{on } \Omega\times[0,\infty),\\ \phi\left(x,t\right) & \geq & 0, & \text{on } \partial\Omega\times[0,\infty),\\ \phi\left(x,0\right) & \geq & 0 & \text{on } \Omega.\end{array}\right.$$

Then $\phi(x,t)\geq 0$ on $\Omega\times[0,\infty)$.

Proof. Similar to proposition 11.1 in [NN]. □

The first key estimate for the proof of main theorem is the moment type estimate. This estimate is first introduced by L. Ni [N3]. By using Li-Yau type heat kernel estimate, he proved that a nonnegative solution $u(x,t)$ of the heat equation are $t^{d/2}$ growth if and only if the average function $k(x,r):=\frac{1}{V(r)}\int_{B_x(r)}f(y)\,dy$ of the initial data $f(y)$ grows as r^d in a certain complete Kaehler manifold. In our CR setting, we only has the CR moment type estimate for a nonnegative heat solution which can be express as P_tf for a smooth bounded function f on M. In contrast to the Kähler case, in general, we do not know if any nonnegative heat solution could hold.

To introduce our version, we will follow from semigroup method as in [M] (also [BBGM]). It is known that the heat semigroup $(P_t)_{t\geq 0}$ is given by

$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda$$

for the spectral decomposition of $\Delta_b = -\int_0^\infty \lambda dE_\lambda$ in $L^2(M)$. It is a one-parameter family of bounded operators on $L^2(M)$. We denote

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y),$$

for $f \in C_0^\infty(M)$. Here $p(x, y, t) > 0$ is the so-called symmetric heat kernel associated to P_t. Due to hypoellipticity of Δ_b, the function $(x, t) \to P_t f(x)$ is smooth on $M \times (0, \infty)$.

In the following we use $V(r)$ and $B_x(r)$ denote the volume of a unit ball with respect to the Carnot-Carathéodory distance and measure $d\mu = \theta \wedge (d\theta)^n$. We recall some facts from [M] (also [BG] and [BBGM]). For $f, g, h \in C^\infty(M)$, we define

(i)
$$\Gamma(f, g) = \tfrac{1}{2}\Delta_b(fg) - f\Delta_b g - g\Delta_b f.$$

(ii)
$$\Gamma_2(f, g) = \tfrac{1}{2}\left[\Delta_b \Gamma(f, g) - \Gamma(f, \Delta_b g) - \Gamma(g, \Delta_b f)\right].$$

(iii)
$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h).$$

(iv)
$$\Gamma_2^Z(f, g) = \tfrac{1}{2}\left[\Delta_b \Gamma^Z(f, g) - \Gamma^Z(f, \Delta_b g) - \Gamma^Z(g, \Delta_b f)\right].$$

Here we denote $\Gamma(f) = \Gamma(f, f)$, $\Gamma_2(f) = \Gamma_2(f, f)$, $\Gamma^Z(f) = \Gamma^Z(f, f)$ and $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$. Note that in a complete strictly pseudoconvex CR $(2n+1)$-manifold with vanishing torsion. One can have $\Gamma(f, f) = (\nabla_b f, \nabla_b f)$ and $\Gamma_2(f) = \|\nabla_b^2 f\|^2 + Ric(\nabla_b f, \nabla_b f) + \frac{n}{2}\|\nabla_T \nabla_b f\|^2$ and $\Gamma^Z(f, g) = (\nabla_T f, \nabla_T g)$.

Definition 6.1. *We say that (M, J, θ) satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with respect to Δ_b if there exist constants ρ_1 a real number, $\rho_2 > 0$, $\kappa \geq 0$, and $d \geq 2$ such that the inequality*

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(\Delta_b f)^2 + (\rho_1 - \frac{\kappa}{\nu})\Gamma(f) + \rho_2\Gamma^Z(f)$$

holds for every $f \in C^\infty(M)$ and every $\nu > 0$.

We define

$$D := d\left(1 + \frac{3\kappa}{2\rho_2}\right) \tag{6.6}$$

and

$$\rho_1^- = \max(-\rho_1, 0).$$

Lemma 6.1. *(i) (*[M, Theorem 4]*) Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of vanishing torsion with*

$$Ric \geq \rho_1.$$

Then M satisfies the generalized curvature-dimension inequality $CD\left(\rho_1, \frac{n}{2}, 1, 2n\right)$ with $\rho_2 = \frac{n}{2}, \kappa = 1$ and $d = 2n$. Moreover for any given $R_0 > 0$, there exists a constant $C\left(d, \kappa, \rho_2\right) > 0$ such that

$$\mu\left(B\left(x, R\right)\right) \leq C\left(d, \kappa, \rho_2\right) \frac{\exp\left(2d\rho_1^- R_0^2\right)}{R_0^D p\left(x, x, R_0^2\right)} R^D \exp\left(2d\rho_1^- R^2\right)$$

for every $x \in M$ and $R \geq R_0$. In particular if M is a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative Ricci curvature and vanishing torsion, then there exists a constant $C_1 > 0$ such that

$$\mu\left(B\left(x, R\right)\right) \leq \frac{C_1}{R_0^D p\left(x, x, R_0^2\right)} R^D \tag{6.7}$$

for $R \geq R_0$.

(ii) ([BG]*) Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative Ricci curvature and vanishing torsion. Then, for any $\varepsilon > 0$, there exists a constant $C_3(d, \rho_2, \kappa, \varepsilon) > 0$ such that*

$$p\left(x, y, t\right) \leq \frac{C\left(d, \rho_2, \kappa, \varepsilon\right)}{\mu\left(B\left(x, \sqrt{t}\right)\right)^{\frac{1}{2}} \mu\left(B\left(y, \sqrt{t}\right)\right)^{\frac{1}{2}}} \exp\left(-\frac{d^2\left(x, y\right)}{\left(4+\varepsilon\right) t}\right). \tag{6.8}$$

(iii) ([BBGM]*) Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative Ricci curvature and vanishing torsion. Then there exists a constant $C_2 > 0$ such that*

$$p\left(x, x, 2R^2\right) \geq \frac{C_2}{\mu\left(B\left(x, R\right)\right)}. \tag{6.9}$$

Remark 6.1. *Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative Ricci curvature and vanishing torsion. (6.7) and (6.9) together imply the doubling property. That is*

$$\mu\left(B\left(x, R\right)\right) \leq \tfrac{C_1}{R_0^D p\left(x, x, R_0^2\right)} R^D \leq C\left(\tfrac{R}{R_0}\right)^D \mu\left(B\left(x, \tfrac{R_0}{\sqrt{2}}\right)\right). \tag{6.10}$$

By taking $R_0 = \frac{R}{\sqrt{2}}$, then there exists a constant $C_4 > 0$ such that

$$\mu\left(B\left(x, R\right)\right) \leq C_4\left(n, D\right) \mu\left(B\left(x, \tfrac{R}{2}\right)\right). \tag{6.11}$$

Applying above Lemma 6.1, we are able to prove the following moment type estimate for those solution of form $P_t f$.

Theorem 6.2. *Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative Ricci curvature and vanishing torsion. Assume that u is a solution of CR heat equation*

$$\frac{\partial}{\partial t} u = \Delta_b u$$

such that

$$u(x,t) = P_t f$$

for a nonnegative bounded function f. Assume that for any $a > -D-2$ (where $D = 2n+6$ is defined in 6.6), we have

$$\frac{1}{V(r)} \int_{B_x(r)} f(y)\, d\mu(y) \leq A r^a$$

for a constant $A > 0$ and $r \geq R \geq 1$. Then there exists a constant $C(n,d)$ such that

$$u(x,t) \leq C(n,d)\, A t^{\frac{a}{2}} \tag{6.12}$$

for all $t \geq R^2$.

Proof. Let $\delta = \frac{d(x,y)}{\sqrt{t}}$. Thus

$$B_x\left(\sqrt{t}\right) \subseteq B_y\left((\delta+1)\sqrt{t}\right). \tag{6.13}$$

It follow from (6.11) and (6.13) that

$$V_x\left(\sqrt{t}\right) \leq V_y\left((\delta+1)\sqrt{t}\right) \leq C(d,\kappa,\rho_2)(\delta+1)^D V_y\left(\sqrt{t}\right).$$

That is,

$$\frac{V_x\left(\sqrt{t}\right)}{V_y\left(\sqrt{t}\right)} \leq C(d,\kappa,\rho_2)(\delta+1)^D. \tag{6.14}$$

We can rewrite (6.8) as

$$\begin{aligned} p(x,y,t) &\leq \frac{C(d,\rho_2,\kappa,\varepsilon)}{\mu\left(B\left(x,\sqrt{t}\right)\right)^{\frac{1}{2}} \mu\left(B\left(y,\sqrt{t}\right)\right)^{\frac{1}{2}}} \exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t}\right) \\ &\leq \frac{C(d,\rho_2,\kappa,\varepsilon)}{\mu\left(B\left(x,\sqrt{t}\right)\right)} \left(\frac{\mu\left(B\left(x,\sqrt{t}\right)\right)}{\mu\left(B\left(y,\sqrt{t}\right)\right)}\right)^{\frac{1}{2}} \exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t}\right) \\ &\leq \frac{C(d,\rho_2,\kappa,\varepsilon)}{\mu\left(B\left(x,\sqrt{t}\right)\right)} \exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t}\right). \end{aligned} \tag{6.15}$$

Then, based on (6.14) and (6.15), Theorem 6.2 follows from the proof of Theorem 3.1 in [N3] in case of $u(x,t) = P_t f$ for a nonnegative bounded function f. The use of volume comparison can be replace by (6.10). □

6.2. **CR Lichnerowicz-Laplacian heat equation.** In this section, we first relate the linear trace Li-Yau-Hamilton type inequality of the CR Lichnerowicz-Laplacian heat equation to a monotonicity formula of the heat solution. More precisely, let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$ tensor satisfying the CR Lichnerowicz-Laplacian heat equation

$$\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta}} = \Delta_b \eta_{\alpha\bar{\beta}} + 2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{\gamma\bar{\mu}} - \left(R_{\gamma\bar{\beta}}\eta_{\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}}\eta_{\gamma\bar{\beta}}\right) \tag{6.16}$$

on $M \times [0,T)$. As in the paper of [CCF], we define following Harnack quantity

$$Z(x,t)(V) := k_1\left(\frac{1}{2}\left((\mathrm{div}\eta)_{\alpha,\bar{\alpha}} + (\mathrm{div}\eta)_{\bar{\alpha},\alpha}\right) + (\mathrm{div}\eta)_{\alpha} V_{\bar{\alpha}} + (\mathrm{div}\eta)_{\alpha} V_{\bar{\alpha}} + V_{\bar{\alpha}} V_{\beta}\eta_{\alpha\bar{\beta}}\right) + \frac{H}{t}$$

for any vector field $V \in T^{1,0}(M)$, $H = h^{\alpha\overline{\beta}}\eta_{\alpha\bar{\beta}}$ and $0 < k_1 \leq 8$. We proved

Theorem 6.3. *([CCF]) Let (M, J, θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold of nonnegative bisectional curvature and vanishing torsion. Let $\eta_{\alpha\bar{\beta}}(x,t)$ be a symmetric $(1,1)$ tensor satisfying the CR Lichnerowicz-Laplacian heat equation (6.16) on $M \times (0,T)$ with*

$$\eta_{\alpha\bar{\beta}}(x,0) \geq 0,$$

and

$$\nabla_T \eta(x,0) = 0.$$

Then

$$Z(x,t) \geq 0$$

on $M \times (0,T)$ for any $(1,0)$ vector field V and $0 < k_1 \leq 8$ if there exists constant $a > 0$ such that

$$\int_0^T \int_M e^{-ar^2} \left\|\eta(x,t)\right\|^2 d\mu dt < \infty, \tag{6.17}$$

$$\int_0^T \int_M e^{-ar^2} \left\|\nabla_T \eta(x,t)\right\|^2 d\mu dt < \infty, \tag{6.18}$$

$$\int_M e^{-ar^2} \left\|\eta(x,0)\right\| d\mu < \infty. \tag{6.19}$$

Let ϕ be a (p,q)-form. Define contraction operator $\Lambda : \Lambda^{p,q} \to \Lambda^{p-1,q-1}$ as follow

$$(\Lambda\phi)_{\alpha_1\ldots\alpha_{p-1}\bar{\beta}_1\ldots\bar{\beta}_{q-1}} = \frac{1}{\sqrt{-1}}(-1)^{p-1} h^{\alpha\bar{\beta}} \phi_{\alpha\alpha_1\ldots\alpha_{p-1}\bar{\beta}\bar{\beta}_1\ldots\bar{\beta}_{q-1}}.$$

Then it is a straightforward computation, we have

Lemma 6.2. *([CCT]) Let (M, J, θ) be a strictly pseudoconvex CR $(2n+1)$-manifold. We have the Kähler type identities*
(i)

$$[\partial_b, \Lambda] = -\sqrt{-1}\bar{\partial}_b^* \quad \textit{and} \quad \left[\bar{\partial}_b, \Lambda\right] = \sqrt{-1}\partial_b^*.$$

(ii)

$$[\bar{\partial}_b, \Box_b] = 2iT\bar{\partial}_b \quad \textit{and} \quad [\partial_b, \Box_b] = 0.$$

(iii)

$$\left[\bar{\partial}_b, \Delta_H\right] = -iT\bar{\partial}_b \quad \textit{and} \quad [\Lambda, \Delta_H] = 0.$$

Lemma 6.3. *Let ϕ be a nonnegative $(1,1)$-form. Define $Q(\phi, V)$ as*
(6.20)

$$Q(\phi, V, k_2) = k_2\left(\frac{1}{2\sqrt{-1}}\left(\bar{\partial}_b^*\partial_b^* - \partial_b^*\bar{\partial}_b^*\right)\phi + \frac{1}{\sqrt{-1}}\left(\bar{\partial}_b^*\phi\right)_V - \frac{1}{\sqrt{-1}}\left(\partial_b^*\phi\right)_V + \phi_{V,\bar{V}}\right) + \frac{\Lambda\phi}{t}.$$

Then this is equivalent to

$$Q(\eta, V, k_2) = k_2\left(\frac{1}{2}\left((div\eta)_{\alpha,\bar{\alpha}} + (div\eta)_{\bar{\alpha},\alpha}\right) + (div\eta)_{\alpha} V_{\bar{\alpha}} + (div\eta)_{\alpha} V_{\bar{\alpha}} + \eta_{\alpha\bar{\beta}}V_{\alpha}V_{\bar{\beta}}\right) + \frac{H}{t}$$

for a symmetric $(1,1)$-tensor $\eta_{\alpha\bar{\beta}} := \frac{1}{\sqrt{-1}}\phi_{\alpha\bar{\beta}}$. In particular, by taking $V = 0$, and $k_2 = 2$, we have

$$Q(\phi, V) = -\Delta_H\Lambda\phi + \left(\bar{\partial}_b^*\Lambda\bar{\partial}_b + conj\right)\phi + \frac{u}{t} \tag{6.21}$$

for $u = \Lambda\phi$.

Proof. As in [CCF], we have the formula for a $(p,q+1)$-form ψ

$$\left(\bar{\partial}_b^*\psi\right)_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}=(-1)^p\,\frac{1}{q+1}\sum_{i=1}^{q+1}(-1)^i\,\nabla_\mu\psi_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\mu}\bar{\beta}_i\ldots\bar{\beta}_q}$$

and a $(p+1,q)$-form φ

$$\left(\partial_b^*\varphi\right)_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}=(-1)\,\frac{1}{p+1}\nabla_{\bar{\mu}}\varphi_{\mu\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}.$$

Thus for a $(1,1)$-form ϕ, we have

$$\left(\partial_b^*\phi\right)_{\bar{\gamma}}=-\nabla_{\bar{\mu}}\phi_{\mu\bar{\gamma}}$$

and

$$\bar{\partial}_b^*\partial_b^*\phi=\nabla_\gamma\left(\nabla_{\bar{\mu}}\phi_{\mu\bar{\gamma}}\right).$$

Then the first term of (6.20) become

$$\begin{array}{rcl}\frac{1}{2\sqrt{-1}}\left(\bar{\partial}_b^*\partial_b^*-\partial_b^*\bar{\partial}_b^*\right)\phi & = & \frac{1}{2\sqrt{-1}}\bar{\partial}_b^*\partial_b^*\phi+conj.\\ & = & \frac{1}{2\sqrt{-1}}\nabla_\gamma\left(\nabla_{\bar{\mu}}\phi_{\mu\bar{\gamma}}\right)+conj.\\ & = & \frac{1}{2}\Big((\mathrm{div}\eta)_{\alpha,\bar{\alpha}}+conj.\Big)\end{array}$$

We are done. On the other hand, taking $V=0$ and $k_2=2$, by lemma 6.2 we have

$$\begin{array}{rcl}\frac{2}{2\sqrt{-1}}\left(\bar{\partial}_b^*\partial_b^*-\partial_b^*\bar{\partial}_b^*\right)\phi & = & \bar{\partial}_b^*[\bar{\partial}_b,\Lambda]\phi-\partial_b^*\left[\partial_b,\Lambda\right]\phi\\ & = & -\bar{\partial}_b^*\bar{\partial}_b\Lambda\phi-\partial_b^*\partial_b\Lambda\phi+\bar{\partial}_b^*\Lambda\bar{\partial}_b\phi+\partial_b^*\Lambda\partial_b\phi\\ & = & -\Delta_H\Lambda\phi+\bar{\partial}_b^*\Lambda\bar{\partial}_b\phi+\partial_b^*\Lambda\partial_b\phi.\end{array}$$

Here we use the fact that $\bar{\partial}_b^*f=\partial_b^*f=0$ for any scalar function. Then formula (6.21) follows. □

Remark 6.2. *The regularity of the heat solution in Proposition 6.1 and the following Lemma is used to prove the "mix-term" $\left(\bar{\partial}_b^*\Lambda\bar{\partial}_b+conj\right)\phi$ in (6.21) vanishing as in (6.35) and (6.36) which is the key step in the proof of our main theorem.*

Lemma 6.4. *Let (M,J,θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold with nonnegative bisectional curvature and vanishing torsion. Let ϕ be a solution of the CR Hodge-Laplace heat equation (6.4). Then $\left\|\Lambda\bar{\partial}_b\phi\right\|$ satisfies*

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)\left\|\Lambda\bar{\partial}_b\phi\right\|\ \leq||\Lambda T\bar{\partial}_b\phi||.$$

Proof. We have the formula for a (p,q)-form ψ

$$\left(\bar{\partial}_b\psi\right)_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_{q+1}}=(-1)^p\sum_{i=1}^{q+1}(-1)^{i-1}\,\nabla_{\bar{\beta}_i}\psi_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_{i-1}\bar{\beta}_{i+1}\ldots\bar{\beta}_{q+1}}.$$

So that

$$\left(\bar{\partial}_b\phi\right)_{\alpha\bar{\beta}\bar{\gamma}}=-\nabla_{\bar{\beta}}\phi_{\alpha\bar{\gamma}}+\nabla_{\bar{\gamma}}\phi_{\alpha\bar{\beta}}$$

and

$$\begin{array}{rcl}\left(\Lambda\bar{\partial}_b\phi\right)_{\bar{\gamma}} & = & ih^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\phi_{\alpha\bar{\gamma}}-ih^{\alpha\bar{\beta}}\nabla_{\bar{\gamma}}\phi_{\alpha\bar{\beta}}\\ & = & h^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\eta_{\alpha\bar{\gamma}}-h^{\alpha\bar{\beta}}\nabla_{\bar{\gamma}}\eta_{\alpha\bar{\beta}}\\ & = & (\mathrm{div}\eta)_{\bar{\gamma}}-\nabla_{\bar{\gamma}}u.\end{array}$$

Note that $\Lambda\bar{\partial}_b\phi$ satisfies the CR Hodge Laplace heat equation, i.e.,

$$\begin{aligned}\left(\tfrac{\partial}{\partial t}+\Delta_H\right)\Lambda\bar{\partial}_b\phi &= -\Lambda\bar{\partial}_b\Delta_H\phi+\Delta_H\Lambda\bar{\partial}_b\phi \\ &= -\Lambda\Delta_H\bar{\partial}_b\phi+\Delta_H\Lambda\bar{\partial}_b\phi+i\Lambda T\bar{\partial}_b\phi \\ &= [\Delta_H,\Lambda]\bar{\partial}_b\phi+i\Lambda T\bar{\partial}_b\phi \\ &= i\Lambda T\bar{\partial}_b\phi.\end{aligned}$$

Hence we have

$$\begin{aligned}\left(\tfrac{\partial}{\partial t}-\Delta_b\right)\sqrt{\left\|\Lambda\bar{\partial}_b\phi\right\|^2} &= \tfrac{\Lambda\bar{\partial}_b\phi}{\left\|\Lambda\bar{\partial}_b\phi\right\|}\cdot\left(-\Delta_H\Lambda\bar{\partial}_b\phi-\Delta_b\Lambda\bar{\partial}_b\phi\right)+\tfrac{\Lambda\bar{\partial}_b\phi}{\left\|\Lambda\bar{\partial}_b\phi\right\|}\cdot i\Lambda T\bar{\partial}_b\phi \\ &= -\tfrac{1}{\left\|\Lambda\bar{\partial}_b\phi\right\|}R_{\alpha\bar{\beta}}\left(\Lambda\bar{\partial}_b\phi\right)_\alpha\overline{\left(\Lambda\bar{\partial}_b\phi\right)}_{\bar{\beta}}+\tfrac{\Lambda\bar{\partial}_b\phi}{\left\|\Lambda\bar{\partial}_b\phi\right\|}\cdot i\Lambda T\bar{\partial}_b\phi\end{aligned}$$

where in second line we use formula (3.1) of [CCF] for $(1,0)$-form $\Lambda\bar{\partial}_b\phi$. □

Before going any further for the proof of our main theorem, we need two more lemmas.

Lemma 6.5. *([NT2]) Let $f\geq 0$ be a function on a complete noncompact Riemannian manifold M^m with*

$$R_{ij}\geq-(m-1)K$$

for some $K\geq 0$. Let

$$u(x,t):=\int_M H(x,y,t)\,f(y)\,dy.$$

Assume that u is defined on $M\times[0,T]$ for some $T>0$ and that for $0<t\leq T$,

$$\lim_{r\longrightarrow\infty}\exp\left(-\frac{r^2}{20t}\right)\int_{B_o(r)}f=0. \tag{6.22}$$

and $p\geq 1$,

$$\begin{aligned}&\tfrac{1}{V_o(r)}\textstyle\int_{B_o(r)}u^p dx \\ \leq\ & C_{m,p}\left[\tfrac{1}{V_o(4r)}\textstyle\int_{B_o(4r)}f^p dx+\left(C_2(K,t)\int_{4r}^{\infty}\left(\tfrac{s}{\sqrt{t}}+\tfrac{s^2}{t}\right)\exp\left(-\tfrac{s^2}{40t}\right)\tfrac{1}{V_o(s)}\int_{B_o(s)}fd\left(\tfrac{s^2}{t}\right)\right)^p\right]\end{aligned}$$

where $C_2(K,t)=C_mte^{C_mKt}$ and C_m is constant only depend on dimension M.

Lemma 6.6. *([Li]) Let (M,J,θ) be a complete strictly pseudoconvex CR $(2n+1)$-manifold and $f(x,t)$ be the subsolution of the heat equation satisfying*

$$\left(\tfrac{\partial}{\partial t}-\Delta_b\right)f(x,t)\leq 0\ \text{ on } M\times[0,T)$$

with $f(x,0)\leq 0$ on M. Then $f(x,t)\leq 0$ for all $t<T$ if there exists $a>0$ such that

$$\textstyle\int_0^T\int_M f^2(x,t)\,e^{-ar^2}d\mu(x)\,dt<\infty.$$

6.3. **Proof of CR Optimal Gap Theorem.** In this section, by using the CR moment type estimate (Theorem 6.2) and the linear trace LYH inequality (Theorem 6.3), we are able to prove the CR optimal gap theorem.

Proof of the main theorem:

Proof. Here is the main idea : In the following we first use proposition 6.1 to construct η_μ on exhaustion domain Ω_μ. Schauder estimates provide the convergence of η_μ (**Step 1**) to a unique solution η. Define $u := tr_h\eta$ and u is a solution of sublaplacian heat equation with initial condition $S(y)$. By uniqueness theorem ([D]) of the nonnegative heat solution we have $u^{(i)} \to u$. Now this allows us in one hand using trace linear Harnack estimate on $tr_h\eta$ to obtain monotonicity formula

$$(tu)_t \geq 0 \tag{6.23}$$

which apply to every nonnegative heat solution and on the other hand using moment type estimate (which only apply to heat solution with $P_t f$ type and f is bounded) on $u^{(i)} := P_t\rho^{(i)}S$ to obtain that

$$u^{(i)} = o\left(t^{-1}\right).$$

Hence as well as u. Combing these results, the initial condition are forced to be zero and the gap theorem holds.

Note that we derived the monotonicity property (6.23) by lemma 6.3, 6.4, and the vanishing of mixed term in LYH quantity (6.21). The condition (6.2) is applied while we use Theorem 6.2 for $a = -2$ to obtain

$$u = o\left(t^{-1}\right).$$

Now we split the detail proof into two steps :

(i) **Step 1 : Convergence of $\eta_\mu^{(i)}$:** Let Ω_μ be an sweetsop exhaustion domains, $\rho^{(i)}$ be a cut-off function support in $B(2R_i)$ such that $0 \leq \rho^{(i)} \leq 1$, $\rho^{(i)} = 1$ in $B(R_i)$, $\left\|\nabla_b^{m_1}\nabla_T^{m_2}\rho^{(i)}\right\| \leq \frac{C}{R_i}$ for $m_1, m_2 = 0, 1, 2,\ m_1 + m_2 \geq 1$ and some constant C. Note that for each i, there exists N_i such that for $\mu \geq N_i$, $B(2R_i) \subset \Omega_\mu$. Let $\eta_\mu^{(i)}$ be the solution as in Proposition 6.1 on Ω_μ for any $\mu \geq N_i$ with initial condition $\rho^{(i)}Ric$. Now we define

$$u^{(i)}(x,t) := \int_M p(x,y,t)\,\rho^{(i)}S(y)\,d\mu(y).$$

Then $u^{(i)}(x,t)$ satisfies

$$\frac{\partial}{\partial t}u^{(i)}(x,t) - \Delta_\varepsilon u^{(i)}(x,t) = -\varepsilon^2 u_{00}^{(i)}(x,t), \tag{6.24}$$

where $\Delta_\varepsilon = \Delta_b + \varepsilon^2 T^2$ is Riemannian Laplacian with respect to the adapted metric $h_\varepsilon := h + \varepsilon^{-2}\theta^2$. Moreover, proposition 6.2 imply $\eta_\mu^{(i)}(x,t)$ is nonnegative and

$$\left\|\eta_\mu^{(i)}(x,t)\right\| \leq tr_h\eta_\mu^{(i)}(x,t) \leq u^{(i)}(x,t), \tag{6.25}$$

for all $\mu \geq N_i$. Now we estimate $u_{00}^{(i)}(x,t)$ first. Since $u^{(i)}(x,t)$ is a solution of sub-Laplacian heat equation, we have

$$\frac{\partial}{\partial t}u_{00}^{(i)}(x,t) - \Delta_b u_{00}^{(i)}(x,t) = 0$$

due to vanishing torsion. We define $l^{(i)}(x,t) = \left|u_{00}^{(i)}(x,t)\right|$, and observe that it is a subsolution of heat equation with initial condition satisfying the followings

$$\begin{array}{rl} & \left|l^{(i)}(x,t)(x,0)\right| \\ = & \left|\nabla_T \nabla_T \rho^{(i)} S(y)\right| \\ \leq & \frac{C}{R_i} \chi_{B_{2R_i \setminus R_i}} S(y), \end{array}$$

where $\chi_{B_{2R_i \setminus R_i}}(y)$ is a function with 1 in annulus $B(2R_i) \backslash B(R_i)$ and zero elsewhere. By maximum principle $l^{(i)}(x,t)$ is controlled by a sub-Laplacian heat solution.

Next we define

$$g(x,t) := \int_M p(x,y,t) \frac{C}{R_i} \chi_{B_{2R_i \setminus R_i}} S(y)\, dy.$$

By moment type estimate

$$g(x,t) = \tfrac{1}{R_i} o\left(t^{-1}\right), \tag{6.26}$$

where the particular coefficient in $o\left(t^{-1}\right)$ does not depend on i. To summarize, we have

$$\left|u_{00}^{(i)}(x,t)\right| = l^{(i)}(x,t) \leq g(x,t) = \tfrac{1}{R_i^2} o\left(t^{-1}\right). \tag{6.27}$$

We return to equation (6.24). Now we restricted on $B(r) \times [\epsilon, T]$ and try to obtain estimate not depend on index i. Now we define

$$L^{(i)}(x,t) = u^{(i)}(x,t) + \varepsilon^2 e^{T-t} \sup_{B(r) \times [\epsilon,T]} g(x,t)$$

so that $L^{(i)}(x,t)$ satisfy

$$\tfrac{\partial}{\partial t} L^{(i)}(x,t) - \Delta_\varepsilon L^{(i)}(x,t) \leq 0. \tag{6.28}$$

Applying mean value theorem (Theorem 1.2 in [LT]) to function $L^{(i)}(x,t)$, we have

$$\begin{aligned} \sup_{B_\varepsilon((1-\delta)r) \times [\epsilon,T]} L^{(i)} \leq\ & C_{16} \left\{ \tfrac{1}{(\delta r)^{2n+3}} \tfrac{V_\varepsilon\left(\frac{2}{\varepsilon^2}, 2r\right)}{V_\varepsilon(r)} \left(r \tfrac{\sqrt{2}}{\varepsilon} \coth\left(r \tfrac{\sqrt{2}}{\varepsilon} \right) + 1 \right) \exp\left(C_{17} \tfrac{2}{\varepsilon^2} T \right) \right\} \\ & \times \textstyle\int_\epsilon^T ds \int_{B_\varepsilon(r)} L^{(i)}(y,s)\, d\mu_\varepsilon(y) + (1+\varepsilon_1) \sup_{B_\varepsilon(r)} L^{(i)}(\cdot, \epsilon). \end{aligned}$$

Let $B_\varepsilon(r), d\mu_\varepsilon(y)$ denote the ball with radius r and volume element which is respected to metric h_ε. The above inequality also means

$$\begin{aligned} \sup_{B_\varepsilon((1-\delta)r) \times [\epsilon,T]} u^{(i)} \leq\ & C_{16} \left\{ \tfrac{1}{(\delta r)^{2n+3}} \tfrac{V_\varepsilon\left(\frac{2}{\varepsilon^2}, 2r\right)}{V_\varepsilon(r)} \left(r \tfrac{\sqrt{2}}{\varepsilon} \coth\left(r \tfrac{\sqrt{2}}{\varepsilon} \right) + 1 \right) \exp\left(C_{17} \tfrac{2}{\varepsilon^2} T \right) \right\} \\ & \times \textstyle\int_0^T ds \int_{B_\varepsilon(r)} L^{(i)}(y,s)\, d\mu_\varepsilon(y) \\ & + (1+\varepsilon_1) \sup_{B_\varepsilon(r)} u^{(i)}(\cdot, \epsilon) + (1+\varepsilon_1) \varepsilon^2 e^{T-\epsilon} \sup_{B_\varepsilon(r)} g(x, \epsilon). \end{aligned} \tag{6.29}$$

We only need to estimate the first term of (6.29) below, since the other terms are bounded. We define

$$L_\varepsilon^{(i)}(y,s) := \textstyle\int_M H_\varepsilon(x,y,t) \left\|\rho^{(i)} S\right\|(y)\, d\mu_\varepsilon(y)$$

and again we have

$$L^{(i)}(y,s) \le L_{\varepsilon}^{(i)}(y,s) + \varepsilon^2 e^T \sup_{B(r)\times[\epsilon,T]} g(x,t) \le L_{\varepsilon}^{(i)}(y,s) + \varepsilon^2 e^T \frac{o\left(t^{-1}\right)}{R_i^2}. \tag{6.30}$$

Now the first term of (6.29) is estimated by using (6.30) and Lemma 6.5 as following

$$\begin{array}{rl} & \frac{1}{V_{o,\varepsilon}(r)} \int_{B_o(r)} L_{\varepsilon}^{(i)}(y,t)\, d\mu_{\varepsilon}(y) \\ \le & C_{m,1} \frac{1}{V_{o,\varepsilon}(4r)} \int_{B_{o,\varepsilon}(4r)} \left\|\rho^{(i)} S\right\|(y)\, d\mu_{\varepsilon}(y) \\ + & C_m e^{C_m \frac{1}{\varepsilon^2} t} \int_{4r}^{\infty} \left(\frac{s}{\sqrt{t}} + \frac{s^2}{t}\right) \exp\left(-\frac{s^2}{40t}\right) \frac{1}{V_{o,\varepsilon}(s)} \int_{B_{o,\varepsilon}(s)} \left\|\rho^{(i)} S\right\|(y)\, d\mu_{\varepsilon}(y)\, d\left(\frac{s^2}{t}\right). \end{array} \tag{6.31}$$

The integral $\int_{B_{o,\varepsilon}(4r)} \left\|\rho^{(i)} S\right\|(y)\, d\mu_{\varepsilon}(y)$ inside both terms in (6.31) are estimated by assumption (6.2) and is controlled by quantity that not depend on i. Hence (6.29) and (6.31) imply

$$\sup_{B_{\varepsilon}((1-\delta)r)\times[\epsilon,T]} u^{(i)} \le C\left(\varepsilon, r, T, n, \rho^{(i)} S\right) \tag{6.32}$$

and

$$\max_{B_{\varepsilon}(r)\times[\epsilon,T]} tr_h \eta_{\mu}^{(i)}(x,t) \le C\left(\varepsilon, r, T, n, \rho^{(i)} S\right). \tag{6.33}$$

Now the interior Schauder estimate can be applied to extract a convergent subsequence $\eta_{\mu_k}^{(i)} \to \eta^{(i)}$ that satisfies the CR Lichnerowiz-subLaplacian heat equation on $[0,T]$. Note that $tr_h \eta^{(i)}(x,0) = u^{(i)}(x,0)$, and by uniqueness of bounded sub-Laplacian heat solution (from lemma 6.6) we actually have

$$tr_h \eta^{(i)}(x,t) = u^{(i)}(x,t).$$

By (6.24), (6.27), (6.32) and Schauder estimates, there is a subsequence $u^{(i_j)} \to u$ and $\eta^{(i_j)} \to \eta$ in any fixed compact subset with an arbitrary chosen Hölder norm (by choosing β_0 large for sweetsop domain, see appendix). Note in (6.27) as i goes to infinity we can conclude $\nabla_T \nabla_T u(x,t) = 0$ and similarly $\nabla_T u(x,t) = 0$ and $\nabla_T \eta(x,t) = 0$ by using that $\left\|\eta_0^{(i)}\right\|$ is a subsolution of sub-Laplacian heat equation as follows

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)\left\|\eta_0^{(i)}\right\| = \frac{1}{\left\|\eta_0^{(i)}\right\|}\left(2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\eta_{0\gamma\bar{\mu}} - (R_{\gamma\bar{\beta}}\eta_{0\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}}\eta_{0\gamma\bar{\beta}})\eta_{0\zeta\bar{\xi}} h_{\beta\bar{\zeta}} h_{\xi\bar{\alpha}}\right) \le 0.$$

Here we use the facts that bisectional curvature is nonnegative and vanishing torsion. Moreover, requirement for applying maximum principle is garanteed by similar argument as (6.25), we have

$$\left\|\eta_0^{(i)}\right\|(x,t) \le C \int p(x,y,t)\, |\nabla\rho|\, S(y)\, d\mu(y) \le \frac{C}{R_i} o\left(t^{-1}\right). \tag{6.34}$$

As i goes to infinity, $\eta_0 = 0$. However, by now we do not know yet through the subsequence the two functions $tr_h \eta(x,t)$ and $u(x,t)$ are the same even they have the same initial condition. One regards both $u(x,t)$ and $tr_h \eta(x,t)$ as solutions of Laplacian heat equations associated to adapted metric (due to $\nabla_T u(x,t) = \nabla_T tr_h \eta(x,t) = 0$), and the manifold are seen as Riemannian manifold with Riemannian curvature bounded below by $-\frac{1}{\varepsilon^2}$ (Theorem 4.9 in [CC1]). Now by the uniqueness of nonnegative Laplacian heat solution ([D]) on complete manifold with Riemannian Ricci curvature bounded below, we can conclude that

$$u(x,t) = tr_h \eta(x,t).$$

Note that u is the unique sub-Laplacian heat solution with $\nabla_T u(x,t) = 0$, and since any such u we can find a sequence of $u^{(i)}$ that satisfy moment type estimates converge to u. Hence u satisfy the moment type estimate.

(ii) **Step 2 : Monotonicity of** tu **:** By our assumptions on Ric, and the upper bound of $\eta(x,t)$ by $u(x,t) = o(t^{-1})$, (6.17), (6.18) and (6.19) in Theorem 6.3 are satisfied. Hence by Lemma 6.3 and (6.21), $tr_h\eta$ satisfy

$$u_t + \left(\bar{\partial}_b^* \Lambda \bar{\partial}_b + conj\right)\phi + \tfrac{u}{t} \geq 0. \tag{6.35}$$

In the following we are going to prove the mixed terms $\left(\bar{\partial}_b^* \Lambda \bar{\partial}_b + conj\right)\phi$ of (6.35) vanishing so the monotonicity

$$(tu)_t \geq 0 \tag{6.36}$$

follows. Hence

$$tu(x,t) \equiv 0,$$

for any x and $t > 0$. The flatness then follows from $u(x,0) \equiv 0$.

In fact, we first define $\sigma^{(i)} := \Lambda\bar{\partial}_b \eta^{(i)}$ (note $\eta^{(i)} = \frac{1}{\sqrt{-1}}\phi^{(i)}$). Then direct calculation shows that

$$\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\left\|\eta^{(i)}_{\mu_k}\right\|^2 \leq -2\left\|\nabla \eta^{(i)}_{\mu_k}\right\|^2.$$

We integrate on both sides over Ω_{μ_k} and apply Dirichlet condition (using boundary regularity in Proposition 6.1). After taking $\mu_k \to \infty$, we have

$$2\int_0^t \int_M \left\|\nabla_b \eta^{(i)}\right\|^2 (x,s)\, d\mu ds \leq \int_M \left\|\eta^{(i)}(x,0)\right\|^2 d\mu = \int_M \left\|\rho^{(i)} Ric\right\|^2 d\mu. \tag{6.37}$$

Due to $\left\|\sigma^{(i)}\right\|(x,t) \leq \left\|\nabla_b \eta^{(i)}\right\|(x,t)$, (6.37) and assumption (6.2) we have for some $a' > 0$,

$$\int_0^t \int_M e^{-a' r^2}\left\|\sigma^{(i)}\right\|(x,s)\, d\mu ds < \infty. \tag{6.38}$$

By Lemma 6.4, and direct calculation shows that

$$\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\left\|\sigma^{(i)}\right\|(x,t) \quad \leq \quad \left\|\sigma^{(i)}_0\right\|.$$

Since $\left\|\sigma^{(i)}_0\right\| \leq \left\|\nabla \eta^{(i)}_0\right\|$ and $\left\|\eta^{(i)}_0\right\|(x,t)$ satisfy 6.34, by Schauder estimates [Si] we have for any $\tilde{\varepsilon} > 0$, there exists $n_{\tilde{\varepsilon}} > 0$ such that $\left\|\sigma^{(i)}_0\right\| \leq \left\|\nabla \eta^{(i)}_0\right\| \leq \tilde{\varepsilon}$ for any $i \geq n_{\tilde{\varepsilon}}$. This shows that for any $(x,t) \in [0,T)$

$$\left(\tfrac{\partial}{\partial t} - \Delta_b\right)\left\|\sigma^{(i)}\right\|(x,t) + e^{T-t}\tilde{\varepsilon} \quad \leq \quad 0.$$

We define $v^{(i)}(x,t)$ as follow

$$v^{(i)}(x,t) = \int_M p(x,y,t)\left\|\Lambda\bar{\partial}_b\left(\rho^{(i)} Ric\right)\right\|(y)\, d\mu(y).$$

Duo to (6.38) and maximum principle we have

$$\left\|\sigma^{(i)}\right\|(x,t) + e^{T-t}\tilde{\varepsilon} \leq v^{(i)}(x,t) + e^{T-t}\tilde{\varepsilon}.$$

Since torsion is vanishing, it implies $\bar{\partial}_b Ric = 0$ and by nonnegativity of Ricci curvature it follows that

$$\left\|\Lambda\bar{\partial}_b\left(\rho^{(i)}Ric\right)\right\|(y) \quad \leq \quad \tfrac{C}{R_i}\chi_{B_{2R_i}\setminus R_i}S(y)\,. \tag{6.39}$$

Similarly as (6.26), (6.39) gives that $v^{(i)} \to 0$ uniformly on any compact subset as $i \to \infty$. Since $\tilde{\varepsilon}$ is arbitrary, we have

$$\|\sigma\|(x,t) = 0.$$

Finally, as a result we have $(tu)_t \geq 0$ and then $u = o\left(t^{-1}\right)$. This completes the proof. $\square$

Appendix A.

In this appendix, we construct "nice" domains to avoid the possibility of the bad regularity for heat solutions in the case of degenerated parabolic systems. In fact, we will give a proof on existence and regularity result for $(1,1)$-form ϕ of the Lichnerowicz-subLaplacian heat equation. In the proof of main theorem, one required some regularity of the heat solution in order to prove the mixed terms $\left(\bar{\partial}_b^*\Lambda\bar{\partial}_b + conj\right)\phi$ of (6.35) vanishing (then the monotonicity follows). While we construct heat solution on complete manifolds with exhaustion domains, we need the interior regularity at least $C^{2,\alpha}\left(\Omega_\mu\right)$ and boundary regularity as continuous function in $C\left(\bar{\Omega}_\mu\right)$. This requirement are needed for Arzela Ascoli theorem and integration by part in (6.37). In semigroup method, better regularity of evolution equation comes from the regularity of infinitesimal generator.

We denote $C^{2,\alpha}\left(\Omega,\Lambda^{1,1}\right)$ as $C^{2,\alpha}$ sections of $\Lambda^{1,1}$ on bounded domain Ω. In our case, it is Δ_H on Banach space $C^{2,\alpha}\left(\Omega,\Lambda^{1,1}\right) \cap C\left(\bar{\Omega},\Lambda^{1,1}\right)$. Note $\Delta_H = -\frac{1}{2}\left(\Box_b + \bar{\Box}_b\right)$. Here we denote u as solution of following Dirichlet problem

$$\Box_b\phi = g \tag{A.1}$$

for $g \in C^\infty\left(\Omega,\Lambda^{1,1}\right)$. First we state some results :

Theorem A.1 (Kohn)**.** *Let M be a strictly pseudoconvex CR $(2n+1)$-manifold. If $1 \leq q \leq n-1$, then $\|\phi\|_{\frac{1}{2}}^2 \leq C\left[(\Box_b\phi,\phi) + \|\phi\|_0^2\right]$ for $\phi \in C^\infty\left(\Lambda^{0,q}\right)$. $\|.\|_s$ stands for the L^2 Sobolev norm of order s.*

Remark A.1. *1. From the hypothesis in above theorem it requires $n \geq 2$. When $n = 1$, one refers to* [J1].

2. Even though the operator $\Box_b$ is not Δ_H, in [J2] *(see p.146) they actually prove the case for $\alpha = 0$. Moreover, we have $\Delta_H = \mathcal{L}_\alpha$ with $\alpha = 0$ up to lower order terms. Here $\mathcal{L}_\alpha = -\Delta_b + i\alpha T$.*

The following is the interior and boundary regularity result by Jerison [J1].

Theorem A.2. *Let U be the open subset of M containing no characteristic points of $\partial\Omega$. If $\psi,\varphi \in C_0^\infty(U)$, $\psi = 1$ in the neighborhood of the support of φ, and u satisfies (A.1) with $\psi g \in \Gamma_\beta\left(\bar{\Omega},\Lambda^{0,q}\right)$, then $\varphi\phi \in \Gamma_{\beta+2}\left(\bar{\Omega},\Lambda^{0,q}\right)$ and*

$$\|\varphi\phi\|_{\Gamma_{\beta+2}} \leq c\left(\|\psi g\|_{\Gamma_\beta} + \|\psi\phi\|_{L^2}\right).$$

When an isolated characteristic boundary point occurs, Jerison proved the regularity result when the neighborhood have strictly convexity property. The convexity is defined by Folland-Stein local coordinates $\Theta(p,-) : U \to \mathrm{H}^n$, and the boundary near point p is corresponding to graph $\tilde{t} = \sum \alpha_i \tilde{x}_i^2 + \beta_j \tilde{y}_j^2 + e(\tilde{x},\tilde{y})$, where $e(\tilde{x},\tilde{y}) = O\left(|\tilde{x}|^3 + |\tilde{y}|^3\right)$. Strictly convex means $\alpha_i, \beta_j > 0$ (see eq. (7.4) and A.3 in [J2]). In the following we state the theorem in the form we want. Reader who is confused can refer to theorem 7.6, Proposition 7.11, and Corollary 10.2 in [J2].

Theorem A.3. *Let p be an isolated characteristic point on $\partial\Omega$ and in some neighborhood U_p of p the geometry $U_p \cap \Omega$ is like the domain $\left\{(x,y,t) : M_c\left(|x|^2 + |y|^2\right) < t\right\}$ in the Heisenberg group, where M_c a positive number . Then $\varphi\phi \in \Gamma_{\beta+2}\left(\bar{\Omega}, \Lambda^{0,q}\right)$, where the best β depends on M_c. Moreover, as $M_c \nearrow \infty$, one can choose $\beta \nearrow \infty$.*

Remark A.2. *In Theorem A.3, one required $g \in \Gamma_\beta\left(\bar{\Omega}, \Lambda^{0,q}\right)$ for $\beta > 2$. Moreover, β has upper bound $\beta_0 - 2$, where β_0 is an index related to the geometry of the boundary. In* [J2]*, they proved $M_c \nearrow \infty$, then $\beta_0 \nearrow \infty$.*

In order to construct a $C^{2,\alpha}$ Lichnerowitz-subLaplacian heat solution, we need the exhaustion domain which satisfy the property above. In the following we prove that it is possible by perturbing the boundary of exhaustion domain.

Theorem A.4. *For any given positive number M_c, there exists exhaustion domains Ω_μ such that $\partial\Omega_\mu$ consist only isolated characteristic points with property as in Theorem A.3 with given M_c.*

Proof. We construct the exhaustion domain with smooth boundary arbitrarily. Since $\partial\Omega_\mu$ is compact, we define Ξ_μ the set consisting all the characteristic points. Then the closure of Ξ_μ is compact. At each point there exist coordinate V_p such that we can express the boundary as $r(z,t) = t - q(z) + e(x,y)$ in $B_p(\varepsilon_p)$ for some ε_p depend on p, where $q(z) = \alpha_i x_i^2 + \beta_j y_j^2$ for some real numbers α_i, β_j. Since injective radius (with respect to some adapted metric) is uniformly bounded below on $\partial\Omega_\mu$, ε_p can be chosen to not depend on p but μ only. These Folland-Stein coordinate neighborhoods form an open covering for $\bar{\Xi}_\mu$.

Now we claim there is a small modification to boundary so that $\bar{\Xi}_\mu$ contains only isolated characteristic points.

Assume $B_{p_i}(\varepsilon)$ are the covering of $\bar{\Xi}_\mu$, we can choose $\varepsilon_1 < \varepsilon_2 < \varepsilon$ such that $B_{p_i}(\varepsilon_1)$ are still a covering of $\bar{\Xi}_\mu$. We start at point p_1. First we deform the graph in the coordinate of $B_{p_1}(\varepsilon_1)$ to plane $t = 0$ and smoothly attached to graph on $\partial B_{p_i}(\varepsilon_2)$. Under the deformation we keep point p_1 as the only characteristic point. This is possible by noticing that we only need to take $q(z)$ into consideration (because this term dominate all the other inside small ball.) and we only need to consider the case in the Heisenberg group with graph $t = q(z)$ in $B_{p_i}(\varepsilon)$. We modify $q(z)$ into new one $\tilde{q}(z)$ by define $\tilde{q}(z) = -\max_{|z|=\varepsilon_2} q(z)$ in $B_{p_i}(\varepsilon_1)$ and $\varphi(|z|,\theta)$ in $B_{p_i}(\varepsilon_2) \backslash B_{p_i}(\varepsilon_1)$ where $\varphi(|z|,\theta)$ is a smooth monotone function in $|z|$ for each θ such that the function smoothly attached to the value $q(z)$ on $\partial B_{p_i}(\varepsilon_2)$ and $\tilde{q}(z) = q(z)$ on $B_{p_i}(\varepsilon) \cap B_{p_i}^c(\varepsilon_2)$. This modification clearly imply the origin is the only characteristic point in $B_{p_i}(\varepsilon_1)$. Moreover, we can choose $\varphi(|z|,\theta)$ very steep so that all the point $(z, q(z))$ for

$z \in B_{p_1}(\varepsilon_2) \setminus (0,0)$ are noncharacteristic. We define the new domain as $\Omega_{\mu,1}$. Specifically,

$$\Omega_{\mu,1} = \{\Omega_\mu \backslash B_{p_1}(\varepsilon_2)\} \cup (M \cap \{(z,t) : t > \tilde{q}(z) - R(z,t) \text{ for } z \in B_{p_1}(\varepsilon_2)\}).$$

Then we continue the same process on p_2, and the new domain is $\Omega_{\mu,2}$. Observe that the process do not create new characteristic points but eliminate all the characteristic point inside $B_{p_i}(\varepsilon_2)$ except p_i. Continuing this process we are able to deform domain Ω_μ into new one that only consist isolated characteristic points on the boundary with $M_c = 0$.

To modify M_c into any value we want is easier. One can do the same process by deforming the graph into parabolic. □

For convenience, we call the domain in above theorem as sweetsop domain.

Remark A.3. *The above theorem can be simplified if we can construct strictly convex domain in M. But the existence to this kind of exhaustion domain isn't known yet.*

We recall theorems from semigroup method. For the definition of analytic semigroup, one can refer to definition 12.30 in [R] ([P]). We cited the characterization of infinitesimal generator of analytic semigroups. Notation here X is Banach space and A is operator defined on X. Note A can be unbounded operator $A : D(A) \to X$, where $D(A)$ is a subset in X such that Ax can be defined. As before, we denote Γ_β the Lipschitz classes associated to nonisotropic distance (refered [J2]) and $\Gamma_\beta\left(\bar{\Omega}, \Lambda^{1,1}\right)$ the restriction to $\bar{\Omega}$ of sections of $\Lambda^{1,1}$ with coefficients in $\Gamma_\beta\left(\bar{\Omega}\right)$. We denote $\|\|_{\Gamma_\beta}$ the norm of Banach space $\Gamma_\beta\left(\bar{\Omega}, \Lambda^{1,1}\right)$, and $R_\lambda(A)$ as the inverse operator of $A_\lambda := A - \lambda I$ as A_λ is one-to-one. The resolvent set of the operator A is the subset of $\mathbb{C}$ that $R_\lambda(A)$ exists, bounded, and the domain is dense in X. When we apply, we let $X = \Gamma_\beta\left(\bar{\Omega}, \Lambda^{1,1}\right)$ and $A = \Delta_H$. Here we state general theorems for following evolution systems

$$\dot{u} = Au + f$$

where $f \in X$.

Theorem A.5. *(*[R, Theorem 12.31]*) A closed, densely defined operator A in X is the generator of an analytic semigroup if and only if there exists ω a real number such that the half-plane $Re\lambda > \omega$ is contained in the resolvent set of A and, moreover, there is a constant C such that*

$$\|R_\lambda(A)\| \leq \frac{C}{|\lambda - \omega|} \tag{A.2}$$

for $Re\lambda > \omega$ and $\|.\|$ is the norm of X.

Theorem A.6. *(*[R, Theorem 12.33]*) Let A be the infinitesimal generator of an analytic semigroup and assume that the spectrum of A is entirely to the left of the line $Re\lambda = \omega$. Then there exists a constant M such that*

$$\left\|e^{At}\right\| \leq Me^{\omega t},$$

where $\|.\|$ is the norm of X.

One can refer to section 7.1 in [P] or page 421 in [E] for the application of semigroup theory for A is a strong elliptic operator. In our case, the missing boundary regularity is replaced by theorem A.3 (refer to [GV], [GV2]). Then one can follow Stewart [SH] and consider Hölder spaces as interpolation space [LA] to obtain the resolvent estimates A.2. As a result, the regularity of the parabolic systems follows by theorem A.6.

In conclusion, we are able to choose exhaustion domain with β_0 large enough, then follow theorem above, we can choose β large enough to make sure the function space X is contained in $C^{2,\alpha}$. This is possible by relation $C^{\beta} \subset \Gamma_{\beta} \subset C^{\beta/2}$ as in 20.5, 20.6 of [FS]. This completes the proof of Proposition 6.1.

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