國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

零維二變數 Gorenstein 理想

Zero-Dimensional Gorenstein Ideals in Two Variables

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中華民國 103 年 6 月

June 2014

國立臺灣大學碩士學位論文
口試委員會審定書

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本論文係王乙珊君（R00221018）在國立臺灣大學數學系完成之碩士學位論文，於民國一零三年六月二十五日承下列考試委員審查通過及口試及格，特此證明

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（是否須簽章依各院系所規定）

Acknowledgements

To my advisor: For kindness and patience.

To my family: For encouragement and support.

中文摘要

本論文著眼於形式如 $((x^n, y^n) : F_k)$ 的零維 Gorenstein 理想，其中 F_k 是在 $K[x, y]$ 中的一個次數爲 k 的齊次多項式，K 爲代數封閉體。首先，在 $k \leq n$ 且 F_k 中 x^k 的係數 c_0 不爲 0 的情況下，我們給出一個齊次多項式屬於 $((x^n, y^n) : F_k)$ 的充要條件。接下來，我們説明在此情形下 $((x^n, y^n) : F_k)$ 可以由二個元素生成。然後將結果推廣到任意的 c_0 與 k。最後，我們介紹 Genoway，Ortiz-Albino 與 Tavares [8] 文章中的一些引理並改寫證明，再加上一個三變數的例子。

Abstract

In this thesis, we are interested in zero-dimensional Gorenstein ideals of the form $((x^n, y^n) : F_k)$ where F_k is a homogeneous polynomial of degree k in $K[x, y]$, K an algebraically closed field. Firstly, we figure out the necessary and sufficient condition for a homogenous polynomial to be in $((x^n, y^n) : F_k)$ where $k \leq n$ and the coefficient of x^k, denoted by c_0, is nonzero. Next, we declare that in this case $((x^n, y^n) : F_k)$ can be generated by two elements. Then expand the result to arbitrary c_0 and k. At last, we introduce some lemmas from the work of Genoway, Ortiz-Albino, and Tavares [8] along with revised proofs and an example in 3 variables.

Contents

§0. Introduction

Gorenstein rings, named after D. Gorenstein, have been intensively studied because of its ubiquity and the various characterizations of rings belonging to it. It is H. Bass [1, 1963] who developed the theory of Gorenstein rings including that as rings of finite injective dimension as well as an historical outline of the subject. There are two main aspects of viewing Gorenstein rings.

One is algebraic geometry approach. In algebraic geometry, the canonical bundle, the highest exterior power of the cotangent bundle, is the most easily accessible and important. If the variety is affine, then the sections of the canonical bundle form a module over the coordinate ring of the variety called the canonical module. Because it is the module of sections of a line bundle, it is locally free of rank 1. If the variety has singularities, then there is still a canonical module, but it may not be locally free. Local Gorenstein rings are those local rings for which the canonical module is free [4, Chapter 21].

The other way is ring theory approach. In a Cohen-Macaulay ring, all ideals generated by sets of parameters have irredundant decompositions of the same length, which leads to the notion of the type of a Cohen-Macaulay ring. We define Gorenstein rings to be the rings of type equal to 1. It turns out that these are precisely those rings which, when regarded as modules over themselves, have finite injective dimension.

Now we follow the route introduced by Balcerzyk and Józefiak [2, Chapter IV].

Theorem. (1) *Any two irredundant presentations of an $\mathfrak{m}$-primary ideal Q in the form of an intersection of irreducible ideals have the same length. This length is called the type of the ideal Q, and is denoted $r(Q)$.*
(2) *If $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring of dimension d, and if Q, Q' are both ideals generated by sets of parameters of R, then $r(Q) = r(Q')$. This number is equal to*

$$\dim_{R/\mathfrak{m}}\mathrm{Ext}^d_R(R\mathfrak{m}, R).$$

We call it the type $r(R)$ of the ring R.

Theorem. *The following properties are equivalent for a local ring $(R, \mathfrak{m})$:*
(i) *the ring R is Cohen-Macaulay and there exists a set of parameters of R generating an irreducible ideal, i.e. R is a Cohen-Macaulay ring of type 1,*
(ii) *every set of parameters of the ring R generates an irreducible ideal.*
A local ring is called a Gorenstein ring when it has the equivalent properties stated above.

If R is a local Gorenstein ring and $x \in R$ is not a zero-divisor of R, then $R/(x)$ is also a Gorenstein ring.

A zero-dimensional local ring is Gorenstein if and only of the zero ideal is irreducible.

Theorem. *Let $(R, \mathfrak{m})$ be a local ring. The following properties are equivalent:*
(i) *the ring R is a zero-dimensional Gorentein ring,*
(ii) *the ring R is an injective R-module,*
(iii) $\dim R = 0$ *and the mapping $I \mapsto 0 : I$ between the ideals of R, sends finite intersections of ideals to their sums (the converse is always true),*
(iv) $0 : (0 : I) = I$ *for any ideal I of R.*

Corollary. *Let R be a zero-dimensional Gorenstein ring and Q an ideal of R. The following properties are equivalent:*
(i) *Q is an irreducible ideal,*
(ii) *$0 : Q$ is a principal ideal,*
(iii) *$Q = 0 : (x)$ for some $x \in R$, $x \neq 0$,*
(iv) *$0 : R \approx R/Q$, or equivalently $\mathrm{Hom}_R(R/Q, R) \approx R/Q$.*

Corollary. *Let $(R, \mathfrak{m})$ be a local ring and Q an irreducible $\mathfrak{m}$-primary ideal. Denote by $\mathcal{P}$ the family of all ideals of R which contain Q. Then:*
(i) *the mapping $I \mapsto Q : I$ sends finite intersections of ideals of $\mathcal{P}$ into their sums.*
(ii) *$Q : (Q : I) = I$ for any $I \in \mathcal{P}$,*

(iii) *and ideal I of $\mathcal{P}$ is irreducible of and only if $I = Q : (x)$ for some $x \in R$.*

Those ideals I in a Gorenstein ring R, for which the factor ring R/I is also Gorenstein, are called Gorenstein ideals. In a ring R, we say an ideal I is zero-dimensional Gorenstein if R/I is zero-dimensional Gorenstein[3]. For more detail, please refer to [2].

Let $(R, \mathfrak{m})$ be a Gorenstein ring and let $x_1, \cdots, x_d$ be a maximal regular sequence of R contained in $\mathfrak{m}$. It can be proved that Q is an irreducible $\mathfrak{m}$-primary ideal precisely when there exists a positive integer s and an element $x \in R$ such that $Q = (x_1^s, \cdots, x_d^s) : (x)$ [2, Ex 4, p.146].

According to Eisenbud [4, Chapter 21], Macaulay's method of inverse system is principally of interest in constructing zero-dimensional Gorenstein rings. Here we give the settings established by Eisenbud: Let $S = k[x_1, \cdots, x_r]$. For each $d \geq 0$ let S_d be the vector space of forms of degree d in the x_i. Let $T = k[x_1^{-1}, \cdots, x_r^{-1}] \subset K(S) = k(x_1, \cdots, x_r)$ be the polynomial ring on the inverse of the x_i.

We make T into an S-module as follows: Let $L \subset K(S)$ be the vector space generated by the monomials in the x_i that are not in T. Notice that L is an S-submodule of $K(S)$. We identify the S-module $K(S)/L$ with T by means of the maps $T \subset K(S) \to K(S)/L$. More directly put: If $m \in S$ and $n \in T$ are monomials, then $m \cdot n$ is the monomial $mn \in K(S)$ if this happens to lie in T, and 0 otherwise.

Theorem. *With notation as above, there is a one-to-one inclusion reversing correspondence between finitely generated S-submodules $M \subset T$ and ideals $I \subset S$ such that $I \subset (x_1, \cdots, x_r)$ and S/I is a local zero-dimensional ring, given by*

$$M \mapsto (0 :_S M), \quad \text{the annihilator of } M \text{ in } S;$$

$$I \mapsto (0 :_T I), \quad \text{the submodule of } T \text{ annihilated by } I.$$

If M and I correspond then $M \simeq \omega_{S/I}$, so the ideals $I \subset (x_1, \cdots, x_r)$ such that S/I is a local zero-dimensional Gorenstein ring are precisely the ideals of the form $I = (0 :_S f)$ for some nonzero element $f \in T$. For further information, please see

[4, Chapter 21].

Next, Chen [5] had introduced another form of the ideal $I = (0 :_S f)$ for the convenience of computation as following:

Lemma.[5, Proposition 2.1] *Let $f \in T$ be a homogeneous polynomial, we write $f = \dfrac{\tilde{f}}{x_1^{l_1} x_2^{l_2} \cdots x_\gamma^{l_\gamma}}$ where $\tilde{f} \in S$ and all terms of $\tilde{f}$ do not have a common divisor. Set $l = \max_{1 \le i \le \gamma}\{l_i\}$. Then*

$$(0 :_S f) = ((x_1^{l+1}, x_2^{l+1}, \cdots, x_\gamma^{l+1}) :_S x_1^{l-l_1} x_2^{l-l_2} \cdots x_\gamma^{l-l_\gamma} \tilde{f}).$$

Proof. ($\subseteq$) Suppose $g \in (0 :_S f)$, then we have $g \in S$ and

$$g \cdot f = g \cdot \frac{\tilde{f}}{x_1^{l_1} x_2^{l_2} \cdots x_\gamma^{l_\gamma}} \in L.$$

If $m = \dfrac{x_1^{m_1} x_2^{m_2} \cdots x_\gamma^{m_\gamma}}{x_1^{l_1} x_2^{l_2} \cdots x_\gamma^{l_\gamma}}$ is a term appears in $g \cdot f$, then we have

$$\begin{aligned}
&\frac{x_1^{m_1} x_2^{m_2} \cdots x_\gamma^{m_\gamma}}{x_1^{l_1} x_2^{l_2} \cdots x_\gamma^{l_\gamma}} \in L \\
\Rightarrow &\exists i \quad \text{such that} \quad m_i > l_i \Rightarrow m_i - l_i \ge 1 \\
\Rightarrow &\frac{x_1^{m_1} x_2^{m_2} \cdots x_\gamma^{m_\gamma}}{x_1^{l_1} x_2^{l_2} \cdots x_\gamma^{l_\gamma}} \cdot x_1^l x_2^l \cdots x_\gamma^l = x_1^{l+m_1-l_1} x_2^{l+m_2-l_2} \cdots x_i^{l+m_i-l_i} \cdots x_\gamma^{l+m_\gamma-l_\gamma}.
\end{aligned}$$

Since $(l-l_j)+m_j \ge 0$ for $1 \le j \le \gamma$ and $l+(m_i-l_i) \ge l+1$, we have $m \cdot x_1^l x_2^l \cdots x_\gamma^l \in (x_i^{l+1}) \subset (x_1^{l+1}, x_2^{l+1}, \cdots, x_\gamma^{l+1})$. That is,

$$g \cdot x_1^{l-l_1} x_2^{l-l_2} \cdots x_\gamma^{l-l_\gamma} \tilde{f} \in (x_1^{l+1}, x_2^{l+1}, \cdots, x_\gamma^{l+1}).$$

($\supseteq$) Suppose $g \in ((x_1^{l+1}, x_2^{l+1}, \cdots, x_\gamma^{l+1}) :_S x_1^{l-l_1} x_2^{l-l_2} \cdots x_\gamma^{l-l_\gamma} \tilde{f})$, then $g \in S$ and

$$\begin{aligned}
&g \cdot x_1^{l-l_1} x_2^{l-l_2} \cdots x_\gamma^{l-l_\gamma} \tilde{f} = g f x_1^l x_2^l \cdots x_\gamma^l \in (x_1^{l+1}, x_2^{l+1}, \cdots, x_\gamma^{l+1}) \\
\Rightarrow \quad &g f x_1^l x_2^l \cdots x_\gamma^l = h_1 x_1^{l+1} + h_2 x_2^{l+1} + \cdots + h_\gamma x_\gamma^{l+1}
\end{aligned}$$

for some $h_1, h_2, \cdots, h_\gamma \in S$. Therefore,

$$gf = h_1 \cdot \frac{x_1}{x_2^l \cdots x_\gamma^l} + h_2 \cdot \frac{x_2}{x_1^l x_3^l \cdots x_\gamma^l} + h_\gamma \cdot \frac{x_\gamma}{x_1^l \cdots x_{\gamma-1}^l} \in L$$

and we can conclude that $g \in (0 :_S f)$. □

Eisenbud [4, Chapter 15] suggested a project concerning how many generators does $I = ((x_1^s, \cdots, x_r^s) : p)$, p is homogeneous, require. He particularly mentioned trying the case $r = 2$ with more complicated p on the machine and next try $r = 3$, various p. Generators of Gorenstein ideals in some cases where p is binomial are computed in [5], [6], and [7]. In [7], $I_s : p$ with p a trinomial of the form $x^\rho y_1^{\sigma_1} \cdots y_s^{\sigma_s} z_1^{\tau_1} \cdots z_t^{\tau_t}(x^\omega - (y_1^{u_1} \cdots y_s^{u_s} + z_1^{v_1} \cdots z_t^{v_t}))$ is also discussed. On the other hand, Genoway, Ortiz-Albino, and Tavares [8, 2001] proposed a conjecture which contends that for any polynomial $p \notin I_s = (x^s, y^s)$ in two variables, $I_s : p$ has two generators in a minimal generating set. Our main theory in this essay is to prove this conjecture for any homogeneous p. As for $r = 3$, we give an example which need more than 3 generators.

Main Theorem. *Let $F_k = c_0x^k + c_1x^{k-1}y + c_2x^{k-2}y^2 + \cdots + c_ky^k$ be a homogeneous polynomial of degree k with coefficients in an algebraically closed field K of characteristic 0, then $I_{n,k} = ((x^n, y^n) : F_k)$ can be generated by two elements.*

In section 1, for the later proofs, we construct two matrices from the coefficients of F_k and discuss some properties of these matrices. Subsequently, we consider F_k where $k \leq n$ and $c_0 \neq 0$ in section 2. In the beginning of section 2, we figure out the necessary and sufficient condition for a homogenous polynomial to be in $I_{n,k}$. Next, we declare that in this case $I_{n,k}$ can be generated by two elements. Then expand the result to arbitrary c_0 and k in section 3. At last, in section 4, we introduce some theorems from [8] along with revised proofs and an example in 3 variables.

§1. Some Lemmas

In this thesis, we are interested in zero-dimensional Gorenstein ideals of the form $((x^n, y^n) : F_k)$ where F_k is a homogeneous polynomial of two variables with coefficients in an algebraic closed field K of characteristic 0. Let

$$F_k = c_0x^k + c_1x^{k-1}y + c_2x^{k-2}y^2 + \cdots + c_ky^n. \tag{1}$$

For the proof of our theorem, we define two matrices according to the coefficients of F_k:

$$C_k = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ -c_k & -c_{k-1} & \cdots & -c_2 & -c_1 \end{bmatrix},$$

and

$$D_{k,l} = \begin{bmatrix} c_k & c_{k-1} & \cdots & \cdots & \cdots & c_1 \\ 0 & c_k & c_{k-1} & \cdots & \cdots & c_2 \\ & & \cdots & \cdots & & \\ & & \cdots & \cdots & & \\ 0 & \cdots & 0 & c_k & \cdots & c_{k-l} \end{bmatrix} C_k^{n-k} \begin{bmatrix} \mathbf{0} \\ \\ \mathbf{I}_l \end{bmatrix},$$

where $\mathbf{I}_l$ is the identity matrix of size l, we denote $D_{k,l}$ as

$$D_{k,l} = \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ \vdots & \vdots & & \vdots \\ d_{k-l,l} & d_{k-l,l-1} & \cdots & d_{k-l,1} \end{bmatrix}.$$

Now we prove some formulae relating to matrices C_k and $D_{k,l}$.

Lemma 1. *Let $\mathbf{e}_i$ be the ith element of the standard basis of the column space K^k. For $1 \le i \le k-1$, we have*

(a) $(C_k^i + c_1 C_k^{i-1} + c_2 C_k^{i-2} + \cdots + c_i \mathbf{I}_k)\mathbf{e}_k = \mathbf{e}_{k-i}$,

(b) $(c_k, c_{k-1}, \cdots, c_1)(C_k^i + c_1 C_k^{i-1} + c_2 C_k^{i-2} + \cdots + c_i \mathbf{I}_k) = (0, \cdots, 0, c_k, \cdots, c_{i+1})$.

Proof. (a) By induction on i. Assume that

$$(C_k^{i-1} + c_1 C_k^{i-2} + c_2 C_k^{i-3} + \cdots + c_{i-1} \mathbf{I}_k)\mathbf{e}_k = \mathbf{e}_{k-i+1}.$$

Then

$$\begin{aligned}
& (C_k^i + c_1 C_k^{i-1} + c_2 C_k^{i-2} + \cdots + c_i \mathbf{I}_k)\mathbf{e}_k \\
= \;& C_k(C_k^{i-1} + c_1 C_k^{i-2} + \cdots + c_{i-1})\mathbf{e}_k + c_i \mathbf{I}_k \mathbf{e}_k \\
= \;& C_k \mathbf{e}_{k-i+1} + c_i \mathbf{e}_k = \mathbf{e}_{k-i}.
\end{aligned}$$

(b) By induction on i. Assume that

$$(c_k, c_{k-1}, \cdots, c_1)(C_k^{i-1} + c_1 C_k^{i-2} + c_2 C_k^{i-3} + \cdots + c_{i-1} \mathbf{I}_k) = (0, \cdots, 0, c_k, \cdots, c_i).$$

Then

$$\begin{aligned}
& (c_k, c_{k-1}, \cdots, c_1)(C_k^i + c_1 C_k^{i-1} + c_2 C_k^{i-2} + \cdots + c_i \mathbf{I}_k) \\
= \;& (c_k, c_{k-1}, \cdots, c_1)(C_k^{i-1} + c_1 C_k^{i-2} + \cdots + c_{i-1} \mathbf{I}_k) C_k + (c_k, c_{k-1}, \cdots, c_1) c_i \mathbf{I}_k \\
= \;& (0, \cdots, 0, c_k, \cdots, c_i) C_k + (c_k, c_{k-1}, \cdots, c_1) c_i \mathbf{I}_k \\
= \;& (-c_i c_k, -c_i c_{k-1}, \cdots, c_k - c_i c_{k-i}, c_{k-1} - c_i c_{k-i-1}, \cdots, c_{i+2} - c_i c_2, c_{i+1} - c_i c_1) \\
& + (c_i c_k, c_i c_{k-1}, \cdots, c_i c_{k-i}, c_i c_{k-i-1}, \cdots, c_i c_2, c_i c_1) \\
= \;& (0, \cdots, 0, c_k, c_{k-1}, \cdots, c_{i+2}, c_{i+1}).
\end{aligned}$$

□

Definition 2. *Given a matrix A, the skew-transpose is the transpose of A about the non-main diagonal. More precisely, if*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m1} & a_{m2} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix}_{m\times n},$$

then the skew-transpose of A is

$$A^{st} = \begin{bmatrix} a_{mn} & a_{m-1,n} & \cdots & a_{2n} & a_{1n} \\ a_{m,n-1} & a_{m-1,n-1} & \cdots & a_{2,n-1} & a_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m2} & a_{m-1,2} & \cdots & a_{22} & a_{12} \\ a_{m1} & a_{m-1,1} & \cdots & a_{21} & a_{11} \end{bmatrix}_{n\times m}.$$

Depending on the definition of $D_{k,l}$ and Lemma 1, we have

$$\begin{aligned} d_{i,j} &= (0,\cdots,0,c_k,\cdots,c_i)C_k^{n-k}\mathbf{e}_{k-j+1} \\ &= (0,\cdots,0,c_k,\cdots,c_i)C_k^{n-k}(C_k^{j-1}+c_1C_k^{j-2}\cdots+c_{j-1}\mathbf{I}_k)\mathbf{e}_k \qquad \text{by (a)} \\ &= (c_k,c_{k-1},\cdots,c_1)(C_k^{i-1}+\cdots+c_{i-1}\mathbf{I}_k)C_k^{n-k}(C_k^{j-1}+\cdots+c_{j-1}\mathbf{I}_k)\mathbf{e}_k \qquad \text{by (b).} \end{aligned}$$

Thus $d_{i,j} = d_{j,i}$ and

$$D_{k,k-l} \text{ is the skew-transpose of } D_{k,l}. \tag{2}$$

In addition, $\text{rank}D_{k,l} = \text{rank}D_{k,k-l}$.

Subsequently, we can derive the main relations between the $d_{i,j}$s.

Lemma 3. *Let $c_0 = 1$, then*

$$d_{i,j} = d_{i+1,j-1} + c_{j-1} d_{i,1} - c_i d_{1,j-1}, \tag{3}$$

$$d_{i,j} = d_{i-1,j+1} - c_j d_{i-1,1} + c_{i-1} d_{1,j}. \tag{4}$$

Proof.

$$\begin{aligned}
d_{i,j} &= (c_k, c_{k-1}, \cdots, c_1)\left(\sum_{l=0}^{i-1} c_l C_k^{i-1-l}\right) C_k^{n-k} \left(\sum_{l=0}^{j-1} c_l C_k^{j-1-l}\right) \mathbf{e}_k \\
&= (c_k, c_{k-1}, \cdots, c_1)\left(\sum_{l=0}^{i-1} c_l C_k^{i-l}\right) C_k^{n-k} \left(\sum_{l=0}^{j-2} c_l C_k^{j-2-l}\right) \mathbf{e}_k \\
&\quad +(c_k, c_{k-1}, \cdots, c_1)\left(\sum_{l=0}^{i-1} c_l C_k^{i-1-l}\right) C_k^{n-k} \left(c_{j-1}\mathbf{I}_k\right) \mathbf{e}_k \\
&= (c_k, c_{k-1}, \cdots, c_1)\left(\sum_{l=0}^{l} c_l C_k^{i-l}\right) C_k^{n-k} \left(\sum_{l=0}^{j-2} c_l C_k^{j-2-l}\right) \mathbf{e}_k \\
&\quad +(c_k, c_{k-1}, \cdots, c_1)\left(\sum_{l=0}^{i-1} c_l C_k^{i-1-l}\right) C_k^{n-k} (c_{j-1}\mathbf{I}_k) \mathbf{e}_k \\
&\quad -(c_k, c_{k-1}, \cdots, c_1)(c_i\mathbf{I}_k) C_k^{n-k} \left(\sum_{l=0}^{j-2} c_l C_k^{j-2-l}\right) \mathbf{e}_k \\
&= d_{i+1,j-1} + c_{j-1} d_{i,1} - c_i d_{1,j-1}.
\end{aligned}$$

The second equation is obtained from the first by setting $i' = i+1, j' = j-1$. □

Now we prove a result which is essential to the proof of our Main Theorem.

Proposition 4. *For $1 \le l \le \lfloor \frac{k}{2} \rfloor - 1$, we have* $\mathrm{rank} D_{k,l+1} \ge \mathrm{rank} D_{k,l}$.

Proof. Let

$$D_{k,l} = \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,2} & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,2} & d_{2,1} \\ & \cdots & \cdots & \cdots & \\ d_{k-l,l} & d_{k-l,l-1} & \cdots & d_{k-l,2} & d_{k-l,1} \end{bmatrix},$$

$$D_{k,l+1} = \begin{bmatrix} d_{1,l+1} & d_{1,l} & d_{1,l-1} & \cdots & d_{1,2} & d_{1,1} \\ d_{2,l+1} & d_{2,l} & d_{2,l-1} & \cdots & d_{2,2} & d_{2,1} \\ & \cdots & \cdots & \cdots & \cdots & \\ d_{k-l-1,l+1} & d_{k-l-1,l} & d_{k-l-1,l-1} & \cdots & d_{k-l-1,2} & d_{k-l-1,1} \end{bmatrix},$$

and define

$$D'_{k,l+1} := \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,2} & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,2} & d_{2,1} \\ & \cdots & \cdots & \cdots & \\ d_{k-l-1,l} & d_{k-l-1,l-1} & \cdots & d_{k-l-1,2} & d_{k-l-1,1} \end{bmatrix}.$$

Let $\text{rank} D_{k,l} = r$.

Suppose $\text{rank} D_{k,l+1} \geq \text{rank} D_{k,l}$ is not true, we may assume $\text{rank} D_{k,l+1} = r-1$ and $\text{rank} D'_{k,l+1} = r-1$. In fact, if $\text{rank} D_{k,l+1} \leq r-2$ then $\text{rank} D'_{k,l+1} \leq r-2$ and since $D_{k,l}$ is obtained from $D'_{k,l+1}$ by adding the last row, so $\text{rank} D_{k,l} \leq \text{rank} D'_{k,l+1} + 1 \leq r-1$, a contradiction. Hence the first column of $D_{k,l+1}$ is generated by the columns of $D'_{k,l+1}$, say

$$\begin{bmatrix} d_{1,l+1} \\ d_{2,l+1} \\ \vdots \\ d_{k-l-1,l+1} \end{bmatrix} = D'_{k,l+1} \begin{bmatrix} \beta_l \\ \beta_{l-1} \\ \vdots \\ \beta_1 \end{bmatrix} \tag{5}$$

for some β_i.

Let

$$D_l := \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ \vdots & \vdots & & \vdots \\ d_{l-1,l} & d_{l-1,l-1} & \cdots & d_{l-1,1} \\ d_{l,l} & d_{l,l-1} & \cdots & d_{l,1} \end{bmatrix}$$

be the submatrix of $D_{k,l}$ obtained by taking the first l rows. By assumption,

$\text{rank}D_l \leq \text{rank}D'_{k,l+1} = r-1$ since $k-l-1 \geq l$ for $1 \leq l \leq \lfloor \frac{k}{2} \rfloor - 1$.

To complete our proof, now we want to show that all rows of $D_{k,l}$ are linear combinations of rows in D_l, hence $\text{rank}D_{k,l} = \text{rank}D_l \leq r-1$ and it leads to a contradiction.

Step 1. Consider the $(l+1)$th row of $D_{k,l}$, make use of the fact that $d_{i,j} = d_{j,i}$, we get

$$
\begin{aligned}
& (d_{l+1,l}, d_{l+1,l-1}, \cdots, d_{l+1,1}) \\
= \; & (d_{l,l+1}, d_{l-1,l+1}, \cdots, d_{1,l+1}) \\
= \; & (\beta_1, \beta_2, \cdots, \beta_l) \begin{bmatrix} d_{l,1} & d_{l-1,1} & \cdots & d_{1,1} \\ d_{l,2} & d_{l-1,2} & \cdots & d_{1,2} \\ \vdots & \vdots & & \vdots \\ d_{l,l-1} & d_{l-1,l-1} & \cdots & d_{1,l-1} \\ d_{l,l} & d_{l-1,l} & \cdots & d_{1,l} \end{bmatrix} \qquad \text{by (5)} \\
= \; & (\beta_1, \beta_2, \cdots, \beta_l) \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ \vdots & \vdots & & \vdots \\ d_{l-1,l} & d_{l-1,l-1} & \cdots & d_{l-1,1} \\ d_{l,l} & d_{l,l-1} & \cdots & d_{l,1} \end{bmatrix} \qquad \text{by } d_{i,j} = d_{j,i} \\
= \; & (\beta_1, \beta_2, \cdots, \beta_l) D_l
\end{aligned}
$$

which is a linear combination of rows in D_l.

Step 2. By induction, we assume that for some i, $l+1 \leq i \leq k-l-1$,

$$
(d_{i,l}, d_{i,l-1}, \cdots, d_{i,1}) = (\gamma_1, \gamma_2, \cdots, \gamma_l) D_l. \tag{6}
$$

For some $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \cdots, \gamma_l)$.

Now we consider the $(i+1)$th-row of $D_{k,l}$. Note that we have

$$
\begin{aligned}
d_{i,l+1} &= (d_{i,l}, d_{i,l-1}, \cdots, d_{i,1}) \begin{bmatrix} \beta_l \\ \beta_{l-1} \\ \vdots \\ \beta_1 \end{bmatrix} \\
&= (\gamma_1, \gamma_2, \cdots, \gamma_l) D_l \begin{bmatrix} \beta_l \\ \beta_{l-1} \\ \vdots \\ \beta_1 \end{bmatrix} \\
&= (\gamma_1, \gamma_2, \cdots, \gamma_l) \begin{bmatrix} d_{1,l+1} \\ d_{2,l+1} \\ \vdots \\ d_{l,l+1} \end{bmatrix} = \boldsymbol{\gamma} \begin{bmatrix} d_{1,l+1} \\ d_{2,l+1} \\ \vdots \\ d_{l,l+1} \end{bmatrix}.
\end{aligned} \tag{7}
$$

Next, apply Lemma 3,

$$
\begin{aligned}
&(d_{i+1,l}, d_{i+1,l-1}, \cdots, d_{i+1,1}) \\
&\overset{(4)}{=} (d_{i,l+1}, d_{i,l}, \cdots, d_{i,2}) - d_{i,1}(c_l, c_{l-1}, \cdots, c_1) + c_i(d_{1,l}, d_{1,l-1}, \cdots, d_{1,1}) \\
&\overset{(6)(7)}{=} \boldsymbol{\gamma} \begin{bmatrix} d_{1,l+1} & d_{1,l} & \cdots & d_{1,2} \\ d_{2,l+1} & d_{2,l} & \cdots & d_{2,2} \\ \vdots & \vdots & & \vdots \\ d_{l,l+1} & d_{l,l} & \cdots & d_{l,2} \end{bmatrix} - \boldsymbol{\gamma} \begin{bmatrix} d_{1,1} \\ d_{2,1} \\ \vdots \\ d_{l,1} \end{bmatrix} (c_l, c_{l-1}, \cdots, c_1) \\
&\quad + c_i(d_{1,l}, d_{1,l-1}, \cdots, d_{1,1}) \\
&\overset{(3)}{=} \boldsymbol{\gamma} \left(\begin{bmatrix} d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ d_{3,l} & d_{3,l-1} & \cdots & d_{3,1} \\ \vdots & \vdots & & \vdots \\ d_{l+1,l} & d_{l+1,l-1} & \cdots & d_{l+1,1} \end{bmatrix} + \begin{bmatrix} d_{1,1} \\ d_{2,1} \\ \vdots \\ d_{l,1} \end{bmatrix} (c_l, c_{l-1}, \cdots, c_1) \right.
\end{aligned}
$$

$$
- \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_l \end{bmatrix} (d_{1,l}, d_{1,l-1}, \cdots, d_{1,1}) \Big) - \boldsymbol{\gamma} \begin{bmatrix} d_{1,1} \\ d_{2,1} \\ \vdots \\ d_{l,1} \end{bmatrix} (c_l, c_{l-1}, \cdots, c_1)
$$

$$
+ c_i(d_{1,l}, d_{1,l-1}, \cdots, d_{1,1})
$$

$$
\stackrel{(5)}{=} \gamma_1(d_{2,l}, d_{2,l-1}, \cdots, d_{2,1}) + \gamma_2(d_{3,l}, d_{3,l-1}, \cdots, d_{3,1}) + \cdots + \gamma_{l-1}(d_{l,l}, d_{l,l-1}, \cdots, d_{l,1})
$$

$$
+ \gamma_l(\beta_1, \beta_2, \cdots, \beta_l) \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ \vdots & \vdots & & \vdots \\ d_{l-1,l} & d_{l-1,l-1} & \cdots & d_{l-1,1} \\ d_{l,l} & d_{l,l-1} & \cdots & d_{l,1} \end{bmatrix}
$$

$$
+ \Big(- \boldsymbol{\gamma} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_l \end{bmatrix} + c_i \Big)(d_{1,l}, d_{1,l-1}, \cdots, d_{1,1})
$$

$$
= (\gamma_l\beta_1 - (\sum_{t=1}^{l} \gamma_t c_t) + c_i, \gamma_1 + \gamma_l\beta_2, \cdots, \gamma_{l-1} + \gamma_l\beta_l) \cdot \begin{bmatrix} d_{1,l} & d_{1,l-1} & \cdots & d_{1,1} \\ d_{2,l} & d_{2,l-1} & \cdots & d_{2,1} \\ \vdots & \vdots & & \vdots \\ d_{l,l} & d_{l,l-1} & \cdots & d_{l,1} \end{bmatrix}
$$

$$
= (\gamma_l\beta_1 - \gamma_1 c_1 - \cdots - \gamma_l c_l + c_i, \gamma_1 + \gamma_l\beta_2, \cdots, \gamma_{l-1} + \gamma_l\beta_l) D_l
$$

Thus the proof is complete. □

§2. Cases with $c_0 \neq 0$ and $k \leq n$

Now we begin to prove our Main Theorem. Let F be an algebraic closed field of characteristic 0. We assume that $k \leq n$ and $c_0 \neq 0$, may assume $c_0 = 1$, and

$$F_k = x^k + c_1 x^{k-1} y + c_2 x^{k-2} y^2 + \cdots + c_k y^k \in K[x, y].$$

Denote $I_{n,k} = ((x^n, y^n) : F_k)$. Let

$$f_m = a_0 x^m + a_1 x^{m-1} y + a_2 x^{m-2} y^2 + \cdots + a_m y^m \in K[x, y].$$

In this section, we intend to give a necessary and sufficient conditions on a_i such that f_m belongs to $I_{n,k}$. Consider

$$f_m \cdot F_k = b_0 x^{m+k} + b_1 x^{m+k-1} y + b_2 x^{m+k-2} y^2 + \cdots + b_{m+k} y^{m+k}. \tag{8}$$

Suppose $m + k < n$, that is, $m < n - k$, then $f_m \notin I_{n,k}$. On the other hand, if $m > 2n - k - 2 > 0$, then $f_m \in I_{n,k}$. In this situation, for any terms $b_i x^{m+k-i} y^i$ in (8), we have $m + k - i \geq n$ or $i \geq n$, such that $b_i x^{m+k-i} y^i \in (x^n, y^n)$. Thus we may assume

$$n - k \leq m \leq 2n - k - 2. \tag{9}$$

Proposition 5. *Suppose* $f_m = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m$.
(a) *For* $n - k \leq m \leq n - 2$, *the necessary and sufficient condition for* $f_m \in I_{n,k}$ *is*

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \in \mathrm{Ker} D_{k,m+k-n+1} \text{ and } a_i = (0, \cdots, 0, 1) C_k^{i-m-k+n} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

for $m+k-n+1 \le i \le m$.

(b) *For* $m \ge n-1$,

$$f_m \in I_{n,k} \Leftrightarrow a_i = (0,\cdots,0,1)C_k^{i-m-k+n} \begin{bmatrix} a_{m-n+1} \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

for $m+k-n+1 \le i \le n-1$.

Proof. Let

$$\begin{aligned} f_m \cdot F_k &= (a_0x^m + a_1x^{m-1}y + \cdots + a_my^m)(x^k + c_1x^{k-1}y + \cdots + c_ky^k) \\ &= b_0x^{m+k} + b_1x^{m+k-1}y + b_2x^{m+k-2}y^2 + \cdots + b_{m+k}y^{m+k}. \end{aligned}$$

Observe that for $0 \le i \le m$, we can express the coefficients b_i in terms of a_i, c_i,

$$b_i = \left(\sum_{j+l=i} c_ja_l\right) + a_i \qquad \text{where} \quad \begin{cases} c_j = 0 & \text{for } j > k \text{ or } j < 1 \\ a_l = 0 & \text{for } l > m \text{ or } l < 0 \end{cases}. \tag{10}$$

For $i \ge m+1$,

$$b_i = \sum_{j+l=i} c_ja_l \qquad \text{where} \quad \begin{cases} c_j = 0 & \text{for } j > k \\ a_l = 0 & \text{for } l > m \text{ or } l < 0 \end{cases}. \tag{11}$$

We first prove (a).

Case 1. $k \le m \le n-2$.

For $k \le m \le n-2$, we can rewrite the above relations (10) and (11) in the matrix form

$$
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \\ \vdots \\ \vdots \\ \vdots \\ b_m \\ b_{m+1} \\ \vdots \\ \vdots \\ b_{m+k} \end{bmatrix}
=
\begin{bmatrix}
1 & & & & & & & & & \\
c_1 & 1 & & & & & & & & \\
c_2 & c_1 & 1 & & & & & 0 & & \\
c_3 & c_2 & c_1 & 1 & & & & & & \\
 & & & & \ddots & & & & & \\
c_k & c_{k-1} & \cdots & & c_1 & 1 & & & & \\
 & \ddots & & & & & \ddots & & & \\
 & & \ddots & & & & & \ddots & & \\
 & & & \ddots & & & & & \ddots & \\
 & & & & c_k & c_{k-1} & \cdots & & c_1 & 1 \\
 & & & & & c_k & \cdots & & & c_1 \\
 & & 0 & & & & & & & \vdots \\
 & & & & & & & \ddots & & \\
 & & & & & & & & & c_k
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_m \end{bmatrix}.
$$

Since $n-k \leq m \leq 2n-k-2$ and

$$f_m \cdot F_k \equiv b_{m+k-n+1} x^{n-1} y^{m+k-n+1} + \cdots + b_{n-1} x^{m+k-n+1} y^{n-1} \mod (x^n, y^n),$$

we have

$$f_m \in I_{n,k} \Leftrightarrow b_{m+k-n+1} = b_{m+k-n+2} = \cdots = b_{n-1} = 0. \tag{12}$$

Step 1. Suppose $b_{m+k-n+1} = b_{m+k-n+2} = \cdots = b_m = 0$, we may express $a_{m+k-n+1}, a_{m+k-n+2}, \cdots, a_m$ in terms of $a_0, a_1, \cdots, a_{m+k-n}$. Observe that for $m+k-n+1 \le i \le m$,

$$b_i = (0, \cdots, 0, c_k, c_{k-1}, \cdots, c_1, 1, 0, \cdots, 0) \begin{bmatrix} a_0 \\ \vdots \\ a_i \\ \vdots \\ a_m \end{bmatrix}$$

$$= (c_k, c_{k-1}, \cdots, c_1, 1) \begin{bmatrix} a_{i-k} \\ \vdots \\ a_i \end{bmatrix} = 0$$

$$\Rightarrow a_i = -(c_k, c_{k-1}, \cdots, c_1) \begin{bmatrix} a_{i-k} \\ \vdots \\ a_{i-1} \end{bmatrix}.$$

We can reformulate this as

$$\begin{bmatrix} a_{i-k+1} \\ \\ \vdots \\ \\ a_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ & 0 & & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ -c_k & -c_{k-1} & \cdots & -c_2 & -c_1 \end{bmatrix} \begin{bmatrix} a_{i-k} \\ \\ \vdots \\ \\ a_{i-1} \end{bmatrix}$$
$$= C_k \begin{bmatrix} a_{i-k} \\ \\ \vdots \\ \\ a_{i-1} \end{bmatrix} = C_k^{i-(m+k-n)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}, \tag{13}$$

for $m+k-n+1 \le i \le m$, where $a_{i-j} = 0$ for $i-j < 0$.

Step 2. $b_{m+1} = b_{m+2} = \cdots = b_{n-1} = 0$

$$\Leftrightarrow \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{m+1} \\ b_{m+2} \\ \vdots \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & c_k & c_{k-1} & \cdots & & c_1 \\ & & & 0 & c_k & \cdots & & c_2 \\ & & & & \cdots & & & \\ & & 0 & & \cdots & & & \\ & & & & & c_k & \cdots & c_{n-m-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{m-k+1} \\ \vdots \\ a_m \end{bmatrix}.$$

We can rewrite the above matrices as:

$$\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{m+1} \\ b_{m+2} \\ \vdots \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} c_k & c_{k-1} & \cdots & & c_1 \\ 0 & c_k & \cdots & & c_2 \\ & \cdots & & & \\ & 0 & \cdots & & \\ & & c_k & \cdots & c_{n-m-1} \end{bmatrix} \begin{bmatrix} a_{m-k+1} \\ \\ \vdots \\ \\ a_m \end{bmatrix}$$

$$= \begin{bmatrix} c_k & c_{k-1} & \cdots & & c_1 \\ 0 & c_k & \cdots & & c_2 \\ & \cdots & & & \\ & 0 & \cdots & & \\ & & c_k & \cdots & c_{n-m-1} \end{bmatrix} C_k^{n-k} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \qquad \text{by (13)}$$

$$= \begin{bmatrix} c_k & c_{k-1} & \cdots & & c_1 \\ 0 & c_k & \cdots & & c_2 \\ & \cdots & & & \\ & 0 & \cdots & & \\ & & c_k & \cdots & c_{n-m-1} \end{bmatrix} C_k^{n-k} \begin{bmatrix} \mathbf{0} \\ \\ \mathbf{I}_{m+k-n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_{m+k-n} \end{bmatrix} \qquad (14)$$

$$= D_{k,m+k-n+1} \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_{m+k-n} \end{bmatrix},$$

where $\mathbf{I}_{m+k-n+1}$ is the identity matrix of size $m+k-n+1$. That is, $\begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \in$ $\text{Ker}D_{k,m+k-n+1}$.

Case 2. $m \leq k-1$.

We have

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_m \\ \vdots \\ b_k \\ \vdots \\ b_{m+k} \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ c_1 & 1 & & & & & \\ c_2 & c_1 & 1 & & \cdots & 0 & \\ c_3 & c_2 & c_1 & 1 & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ c_m & c_{m-1} & & \cdots & & c_1 & 1 \\ & & \cdots & & \cdots & \cdots & \\ c_k & c_{k-1} & & \cdots & & & c_{k-m} \\ & & & \ddots & & & \vdots \\ & & 0 & & & & c_k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_m \end{bmatrix}.$$

We also have

$$f_m \in I_{n,k} \Leftrightarrow b_{m+k-n+1} = b_{m+k-n+2} = \cdots = b_{n-1} = 0.$$

Step 1. Suppose $b_{m+k-n+1} = b_{m+k-n+2} = \cdots = b_m = 0$, then for $m+k-n+1 \leq i \leq m$, as in Case 1, we have

$$\begin{bmatrix} a_{i-k+1} \\ \\ \vdots \\ \\ a_i \end{bmatrix} = C_k^{i-(m+k-n)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

where $a_{i-j} = 0$ if $i - j < 0$.

Step 2. For $m \leq k-1$, $b_{m+1} = b_{m+2} = \cdots = b_{n-1} = 0 \Leftrightarrow$

$$\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{m+1} \\ b_{m+2} \\ \vdots \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} c_{m+1} & c_m & c_{m-1} & \cdots & & c_1 \\ c_{m+2} & c_{m+1} & c_m & \cdots & & c_2 \\ & & & & \cdots & \\ & & 0 & & \cdots & \\ & & & c_k & \cdots & c_{n-m-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \\ \vdots \\ \\ a_m \end{bmatrix}$$

$$= \begin{bmatrix} c_k & c_{k-1} & \cdots & c_{m+1} & c_m & \cdots & c_1 \\ 0 & c_k & c_{k-1} & \cdots & c_{m+1} & \cdots & c_2 \\ & \cdots & & & & & \\ & 0 & \cdots & & & & \\ & 0 & c_k & \cdots & c_{m+1} & \cdots & c_{n-m-1} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_m \end{bmatrix}$$

$$= \begin{bmatrix} c_k & c_{k-1} & \cdots & & c_1 \\ 0 & c_k & \cdots & & c_2 \\ & \cdots & & & \\ & 0 & \cdots & & \\ & & c_k & \cdots & c_{n-m-1} \end{bmatrix} C_k^{n-k} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

$$= \begin{bmatrix} c_k & c_{k-1} & \cdots & & c_1 \\ 0 & c_k & \cdots & & c_2 \\ & \cdots & & & \\ & 0 & \cdots & & \\ & & c_k & \cdots & c_{n-m-1} \end{bmatrix} C_k^{n-k} \begin{bmatrix} \mathbf{0} \\ \\ \mathbf{I_{m+k-n+1}} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

$$= D_{k,m+k-n+1} \begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}.$$

We get the same equation as (14).

(b) $m \geq n-1$. In this case, observe that for $0 \leq i \leq n-1 \leq m$,

$$b_i = \left(\sum_{j+l=i} c_j a_l \right) + a_i \qquad \text{where} \quad \begin{cases} c_j = 0 & \text{for } j > k \text{ or } j < 1 \\ a_l = 0 & \text{for } l > m \text{ or } l < 0 \end{cases}.$$

By the same argument given in Case 1, Step 1 of (a), we also get the relation (13) for $m+k-n+1 \leq i \leq n-1$. Note that in this case $a_0, a_1, \cdots, a_{m+k-n}$ and $a_n, a_{n+1}, \cdots, a_m$ are free. □

In order to prove Proposition 7, we need a Proposition. Here we let $R = K[x,y]$, $R = R_0 \oplus R_1 \oplus \cdots$. For any homogeneous polynomial f, we write $(Rf)_m$ to be the m-degree part of the R-submodule Rf.

Proposition 6.

(a) *For each $m \geq n-k+\lfloor \frac{k}{2} \rfloor$, there must exist a homogeneous polynomial f_m of degree m belongs to $I_{n,k}$.*

(b) *Suppose there is no f_{n-k} of degree $n-k$ in $I_{n,k}$, then*

(i) *we can choose a polynomial $G_{n-M} \in I_{n,k}$, where $k - \lfloor \frac{k}{2} \rfloor \leq M \leq k-1$, such that the degree $n-M$ is the least.*

(ii) *There is* $G_{n-N} \in I_{n,k}$ *where* $1 \leq N \leq M \leq k-1$, $M+N=k$ *such that* G_{n-N} *is not a multiple of* G_{n-M}.

(iii) $(RG_{n-M})_m \cap (RG_{n-N})_m = \{0\}$ *for* $n-N \leq m \leq 2n-k-1$.

Proof.

(a) Let $f_m = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$.

Case 1. $n-k+\lfloor \frac{k}{2} \rfloor \leq m \leq n-2$. In this case, from Proposition 5(a), we have $f_m \in I_{n,k} \Leftrightarrow$

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \in \mathrm{Ker}D_{k,m+k-n+1}, \text{ and } a_i = (0,\cdots,0,1)C_k^{i-m-k+n} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix}$$

for $m+k-n+1 \leq i \leq m$.

Note that $D_{k,m+k-n+1}$ has $m+k-n+1$ columns and $n-m-1$ rows. For $n-k+\lfloor \frac{k}{2} \rfloor \leq m \leq n-2$, since $m \geq n-k+\frac{k-1}{2} > n-\frac{k}{2}-1$,

$$\Rightarrow 2m > 2n-k-2 \Rightarrow m+k-n+1 > n-m-1,$$

therefore $D_{k,m+k-n+1}$ has more columns than rows, the dimension of $\mathrm{Ker}D_{k,m+k-n+1}$ is greater than 1, we can find $\begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ such that $\begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \in \mathrm{Ker}D_{k,m+k-n+1}$. That is, $f_m \in I_{n,k}$ exists for $n-k+\lfloor \frac{k}{2} \rfloor \leq m \leq n-2$.

Case 2. $m \geq n-1$. From proposition 5(b), $f_{n-1} \in I_{n,k}$ exists.

(b)(i) By (a), for some $M \geq k-\lfloor \frac{k}{2} \rfloor$, f_{n-M} exists. Thus we can choose G_{n-M} such that the degree $n-M$ is the least.

(ii) Let $N = k - M$.

Observe that $\text{rank}D_{k,k-N+1} = \text{rank}D_{k,M+1} = \text{rank}D_{k,k-M-1} = k - M - 1$.

$\Rightarrow \dim(\text{Ker}D_{k,k-N+1}) = (k - N + 1) - (k - M - 1) = M - N + 2$.

On the other hand,

$$(RG_{n-M})_{n-N} = \sum_{i+j=M-N} kx^i y^j G_{n-M} \Rightarrow \dim(RG_{n-M})_{n-N} = M - N + 1,$$

that is, there exists an G_{n-N} which is not a multiple of G_{n-M}.

(iii) In fact, if we have proved that G_{n-M} and G_{n-N} are relatively prime, then the least common multiple of G_{n-M} and G_{n-N} is $G_{n-M} \cdot G_{n-N}$ which has degree $(n - M) + (n - N) = 2n - k$, so $(RG_{n-M} \cap RG_{n-N})_m = 0$ for $m < 2n - k$ and the result follows.

Claim: G_{n-M} and G_{n-N} are relatively prime.

Suppose $d(x, y) := \gcd(G_{n-M}, G_{n-N}) \neq 1$, where $\deg d = s \geq 1$. Let

$$g_{n-M-s} = \frac{G_{n-M}}{d} \quad \text{and} \quad g_{n-N-s} = \frac{G_{n-N}}{d}.$$

Note that $\gcd(g_{n-M-s}, g_{n-N-s}) = 1$. Since K is algebraic closed, we may write

$$g_{n-M-s} = g_1 g_{n-M-s-1}$$

where $\deg g_1 = 1$.

$$\Rightarrow \frac{G_{n-M}}{g_1} = \frac{G_{n-N}}{g_{n-N-s}} g_{n-M-s-1}.$$

Observe that

$$\frac{G_{n-M}}{g_1} \cdot F_k = Ax^n + Bx^n + H \notin (x^n, y^n)$$

where $A, B, H \in K[x, y]$ and $H \notin (x^n, y^n)$. We have

$$g_1 \cdot \frac{G_{n-M}}{g_1} \cdot F_k = g_1 Ax^n + g_1 By^n + g_1 H \in (x^n, y^n) \Rightarrow g_1 H \in (x^n, y^n).$$

Next,

$$\begin{aligned} g_{n-N-s} \cdot \frac{G_{n-M}}{g_1} \cdot F_k &= g_{n-M-s-1} G_{n-N} \cdot F_k \\ &= g_{n-N-s} A x^n + g_{n-N-s} B y^n + g_{n-N-s} H \in (x^n, y^n) \\ &\Rightarrow g_{n-N-s} H \in (x^n, y^n). \end{aligned}$$

Let $g_1 = \alpha x + \beta y$, $(\alpha, \beta) \neq (0, 0)$. Assume $\alpha \neq 0$ and consider $g_{n-N-s} = Q g_1 + R$. Note that $R \neq 0$ since $\gcd(g_{n-M-s}, g_{n-N-s}) = 1$. Therefore

$$g_{n-N-s} = Q g_1 + \gamma y^{n-N-s}$$

with $\gamma \neq 0$.

$\Rightarrow g_{n-N-s} H = Q g_1 H + \gamma y^{n-N-s} H \in (x^n, y^n) \Rightarrow \gamma y^{n-N-s} H \in (x^n, y^n)$.

Suppose $x^p y^q$ appears in H, then we must have $p \leq n-1$, $q \leq n-1$ and $p + q = n + N - 1$.

$\gamma y^{n-N-s} H \in (x^n, y^n) \Rightarrow$

$$n - N - s + q \geq n \tag{15}$$

$\Rightarrow q \geq N + s$.

Since $H = \sum_{l=0}^{n-1} h_l x^{n-N-l-1} y^l \neq 0$, there exists $j \geq N + s$ such that $h_i = 0$ for $i < j$ and $h_j \neq 0$.

$$\Rightarrow H = h_j x^{n+N-j-1} y^j + h_{j+1} x^{n+N-j-2} y^{j+1} + \cdots + h_{n-1} x^N y^{n-1}.$$

Consider

$$\begin{aligned} g_1 H &= (\alpha x + \beta y)(h_j x^{n+N-j-1} y^j + h_{j+1} x^{n+N-j-2} y^{j+1} + \cdots + h_{n-1} x^N y^{n-1}) \\ &= \alpha h_j x^{n+N-j} y^j + \cdots \end{aligned}.$$

Note

$$j \geq N + s \Rightarrow n - N - j \leq n - s \leq n - 1$$

since $s \geq 1$. Thus we must have $h_j = 0$, a contradiction. □

Proposition 7. *Let $n \geq k$ and $F_k = x^k + c_1 x^{k-1} y + c_2 x^{k-2} y^2 + \cdots + c_k y^k$ be a homogeneous polynomial of degree k, then $I_{n,k} = ((x^n, y^n) : F_k)$ can be generated by two elements. In fact, if $\text{rank} D_{k,1} = 0$, then $I_{n,k} = (G_{n-k}, y^n)$. Otherwise, $I_{n,k} = (G_{n-M}, G_{n-M})$ where G_{n-M} and G_{n-N} are stated in Proposition 6.*

Proof.

Case 1. $\text{rank} D_{k,1} = 0$

By Proposition 5(a), there exists non-zero $G_{n-k} \in I_{n,k}$ and the coefficient of x^{n-k} in G_{n-k} is nonzero.

Claim: $I_{n,k} = (G_{n-k}, y^n)$.

It is obvious that $(G_{n-k}, y^n) \subset I_{n,k}$. Now suppose $f_m \in I_{n,k}$ where $\deg f_m = m$. Note that $m \geq n - k$, and

$$f_m = H y^n + \tilde{H} G_{n-k} + R$$

where

$$R = \sum_i r_i x^{m-i} y^i$$

with $m - i \leq n - k - 1$, and $i \leq n - 1$; $r_i = 0$ for $m - i < 0$ or $i < 0$.

Suppose $R \neq 0$, then there exists j such that $r_i = 0$ for $i < j$ and $r_j \neq 0$.

$$\Rightarrow R = r_j x^{m-j} y^j + r_{j+1} x^{m-j-1} y^{j+1} + \cdots + r_{n-1} x^{m-n+1} y^{n-1} \in I_{n,k}$$

$$\Rightarrow R \cdot F_k = r_j x^{m+k-j} y^j + \cdots \in (x^n, y^n).$$

We have $j \leq n-1$ and $m-j \leq n-k-1$,

$$\Rightarrow m+k-j \leq n-1$$

$$\Rightarrow r_j = 0,$$

contradicts to the assumption $r_j \neq 0$. Hence $R=0$, that is, $f_m \in (G_{n-k}, y^n)$.

Case 2. $\text{rank}D_{k,1} = 1$

By Proposition 6(b)(i) we can find $G_{n-M}, G_{n-N} \in I_{n,k}$ such that $k - \lfloor \frac{k}{2} \rfloor \leq M \leq k-1$, $n-M$ is the least, $M+N=k$ and G_{n-N} is not a multiple of G_{n-M}. Note that from Proposition 4, when $\text{rank}D_{k,1} = 1$, we have $\text{rank}D_{k,j} = j-1$ or j for $2 \leq j \leq \lfloor \frac{k}{2} \rfloor$. If $\text{rank}D_{k,j} = j-1$, then there exists a polynomial $f_{n-(k+1-j)}$ of degree $n-(k+1-j)$ in $I_{n,k}$ by Proposition 5. On the other hand, if $\text{rank}D_{k,j} = j$, then such polynomial does not exist. In conclusion, $\text{rank}D_{k,j} = j-1$ if and only if $f_{n-(k+1-j)} \in I_{n,k}$ exists. By Proposition 6(b)(i), $n-M$ is the smallest, we see that $\text{rank}D_{k,l} = l$ for $l \leq N$ and $\text{rank}D_{k,N+1} = N$. It is clear that $(G_{n-M}, G_{n-N}) \subseteq I_{n,k}$. Now suppose $f_m \in I_{n,k}$,

$$f_m = a_0 x^m + a_1 x^{m-1} y + a_2 x^{m-2} y^2 + \cdots + a_m y^m.$$

We separate the verification for $f_m \in (G_{n-M}, G_{n-N})$ into four cases according to m.

(i) For $n-M \leq m \leq n-N-1$, let $(RG_{n-M})_{n-M+i}$ be the $(n-M+i)$-degree part of the R-submodule generated by G_{n-M} in R_{n-M+i}, where $R = K[x,y]$.

We claim $\text{rank}D_{k,l} \geq N$ for $N+1 \leq l \leq M$. For $N+2 \leq l \leq \lfloor \frac{k}{2} \rfloor$, by Proposition 4, $\text{rank}D_{k,l} \geq \text{rank}D_{k,N+1} = N$. Also for $\lfloor \frac{k}{2} \rfloor + 1 \leq l \leq M-1$, $\text{rank}D_{k,l} = \text{rank}D_{k,k-l} \geq N$ since $D_{k,l}$ is the skew-transpose of $D_{k,k-l}$ by (2). At last, $\text{rank}D_{k,M} = \text{rank}D_{k,N} = N$ since $D_{k,M}$ is the skew-transpose of $D_{k,N}$ by (2). Therefore $\dim(\text{Ker}D_{k,N+1+i}) \leq (N+1+i) - N = 1+i$ for $N+2 \leq N+1+i \leq M$.

Since $G_{n-M} \in I_{n,k} \Rightarrow (RG_{n-M})_{n-M+i} \subset (R_{n-M+i} \cap I_{n,k})$. Therefore by Proposition 5(a), $\dim(RG_{n-M})_{n-M+i} \leq \dim(R_{n-M+i} \cap I_{n,k}) = \dim(\text{Ker}D_{k,N+1+i})$. On the other hand, $\dim(RG_{n-M})_{n-m+i} = 1+i \geq \dim(\mathrm{Ker}D_{k,N+1+i})$, thus $\dim(RG_{n-M+i}) = \dim(\mathrm{Ker}D_{N+1+i})$ and $f_{n-M+i} \in (G_{n-M}) \Leftrightarrow f_{n-M+i} \in I_{n,k}$.

(ii) $n - N \leq m \leq n-2$

From Proposition 5(a), $f_m \in I_{n,k} \Leftrightarrow \begin{bmatrix} a_0 \\ \vdots \\ a_{m+k-n} \end{bmatrix} \in \mathrm{Ker}D_{k,m+k-n+1}$. For $m = n - N$, from Proposition 6(b)(ii), $f_m \in I_{n,k} \Leftrightarrow f_m \in (G_{n-M}, G_{n-N})$. Now for $n - N+1 \leq m \leq n-2$, consider $D_{k,m+k-n+1}$. By assumption, we have $\mathrm{rank}D_{k,m+k-n+1} = \mathrm{rank}D_{k,n-m-1} = n-m-1$ since $1 \leq n-m-1 \leq N-1$,

$\Rightarrow \dim(\mathrm{Ker}D_{k,m+k-n+1}) = m+k-n+1-(n-m-1) = 2m+k-2n+2.$

From 6(b)(iii), $(RG_{n-M})_m \cap (RG_{n-N})_m = 0$ for $n-N \leq m \leq n-2$, therefore

$$\begin{aligned} & \dim(RG_{n-M})_m + \dim(RG_{n-N})_m \\ = \ & (m-(n-M)+1)+(m-(n-N)+1) = 2m-2n+k+2 \\ = \ & \dim(\mathrm{Ker}D_{k,m+k-n+1}), \end{aligned}$$

that is, $f_m \in I_{n,k} \Leftrightarrow f_m \in (G_{n-M}, G_{n-N})$.

(iii) $n-1 \leq m \leq 2n-k-2$

Suppose

$$f_m = a_0x^m + a_1x^{m-1}y + a_2x^{m-2}y^2 + \cdots + a_my^m.$$

According to Proposition 5(b), $a_0, a_1, \cdots, a_{m+k-n}$ and $a_n, a_{n+1}, \cdots, a_m$ are free, therefore

$$\dim(R_m \cap I_{n,k}) = 1 + (m+k-n) + (m-(n-1)) = 2m-2n+k+2.$$

In fact, from 6(b)(iii)

$$\begin{aligned} \dim(RG_{n-M})_m + \dim(RG_{n-N})_m = \ & (m-(n-M)+1)+(m-(n-N)+1) \\ = \ & 2m-2n+k+2 \end{aligned}$$

for $n-1 \leq m \leq 2n-k-2$. Thus we have $\dim(R_m \cap I_{n,k}) = \dim(RG_{n-M})_m + \dim(RG_{n-N})_m = 2m-2n+k+2$ for $n-1 \leq m \leq 2n-k-2$, that is, $f_m \in I_{n,k} \Leftrightarrow f_m \in (G_{n-M}, G_{n-N})$.

(iv) For $m \geq 2n-k-1$.

Suppose

$$f_m = a_0x^m + a_1x^{m-1}y + a_2x^{m-2}y^2 + \cdots + a_my^m.$$

Note that $f_m \in I_{n-k}$ for all f_m with degree greater than $2n-k-1$. Therefore $\dim(R_m \cap I_{n,k}) = m+1$ for $m \geq 2n-k-1$. For $m = 2n-k-1$,

$$\dim(RG_{n-M})_{2n-k-1} + \dim(RG_{n-N})_{2n-k-1} = 2n-k = \dim(R_{2n-k-1} \cap I_{n,k}).$$

However, since the degree of the least common multiple of G_{n-M} and G_{n-N} is $(n-M)+(n-N) = 2n-k$,

$$\dim(RG_{n-M})_m + \dim(RG_{n-N})_m = 2n-k+(m-(2n-k-1)) = m+1$$

for $m \geq 2n-k$. Thus

$$\dim(R_m \cap I_{n,k}) = \dim(RG_{n-M})_m + \dim(RG_{n-N})_m = m+1,$$

that is, $f_m \in I_{n,k} \Leftrightarrow f_m \in (G_{n-M}, G_{n-N})$.

Hence we can conclude that $I_{n,k} = (G_{n-M}, G_{n-N})$. □

Corollary 8. *Let $n \geq k$ and $F_k = c_0x^k + c_1x^{k-1}y + c_2x^{k-2}y^2 + \cdots + c_ky^k$ be a homogeneous polynomial of degree k, where $c_k \neq 0$, then $I_{n,k} = ((x^n, y^n) : F_k)$ can be generated by two elements.*

Proof. Consider this case by the symmetry of x and y. For

$$F_k(x,y) = c_0x^k + c_1x^{k-1}y + c_2x^{k-2}y^2 + \cdots + c_ky^k$$

with $c_k \neq 0$, $F_k(y,x)$ has nonzero leading coefficient. Hence by Proposition 7, $((y^n, x^n) : F_k(y,x))$ can be generated by two elements and $((x^n, y^n) : F_k(x,y))$ can also be generated by two elements. □

§3. Cases with $c_0 = 0$ and $k \leq n$

Lemma 9. *Suppose* $F_k = x^k + c_1x^{k-1}y + \cdots + c_{k-1}xy^{k-1} + c_ky^k$, *then* $((x^n, y^n) : x^jy^jF_k) = ((x^{n-j}, y^{n-j}) : F_k)$.

Proof. ($\supseteq$) If $f \in ((x^{n-j}, y^{n-j}) : F_k)$, then $f \cdot F_k = Ax^{n-j} + By^{n-j}$ for some polynomials A, B. We have

$$\begin{aligned} f \cdot x^jy^jF_k &= (f \cdot F_k) \cdot x^jy^j \\ &= (Ax^{n-j} + By^{n-j})x^jy^j = Ax^ny^j + Bx^jy^n \in (x^n, y^n). \end{aligned}$$

($\subseteq$) If $f \in ((x^n, y^n) : x^jy^jF_k)$, then $f \cdot F_kx^jy^j = Cx^n + Dy^n$. Suppose

$$f \cdot F_k = h_1 + h_2 + h_3 + \cdots + h_l$$

where $h_s \neq h_t$ for $s \neq t$. Then $h_sx^jy^j = C_sx^n$ or $h_sx^jy^j = D_sy^n$, thus $h_sy^j = C_sx^{n-j}$ or $h_sx^j = D_sy^{n-j}$ and $h_s = C'_sx^{n-j}$ or $h_s = D'_sy^{n-j}$. Thus we have $f \cdot F_k = h_1 + h_2 + \cdots + h_l \in (x^{n-j}, y^{n-j})$. □

Proposition 10. *Let* $n \geq k$ *and* $F_k = c_0x^k + c_1x^{k-1}y + c_2x^{k-2}y^2 + \cdots + c_ky^k$ *be a homogeneous polynomial of degree* k*, where* $c_0 = 0$*, then* $I_{n,k} = ((x^n, y^n) : F_k)$ *can be generated by two elements.*

Proof. If $c_0 = c_1 = \cdots = c_{j-1} = 0$ and $c_{k-i+1} = c_{k-i+2} = \cdots = c_{k-1} = c_k = 0$, while $c_j \neq 0$ and $c_{k-i} \neq 0$, where $i + j < k$, then

$$F_k = c_jx^{k-j}y^j + c_{j-1}x^{k-j-1}y^{j+1} + \cdots + c_{k-i}x^iy^{k-i}.$$

If $i \geq j$, then

$$F_k = x^jy^j(c_jx^{k-2j} + c_{j+1}x^{k-2j-1}y + \cdots + c_{k-i}x^{i-j}y^{k-i-j})$$

where $c_j \neq 0$. From Proposition 7 and Lemma 9, $((x^n, y^n) : F_k)$ can be generated by two elements.

If $i < j$, then

$$F_k = x^i y^i (c_j x^{k-i-j} y^{j-i} + c_{j+1} x^{k-i-j-1} y^{j-i+1} + \cdots + c_{k-i} y^{k-2i}).$$

Let

$$g = c_j x^{k-i-j} y^{j-i} + c_{j+1} x^{k-i-j-1} y^{j-i+1} + \cdots + c_{k-i} y^{k-2i}.$$

Note that $c_{k-i} \neq 0$. Therefore by Lemma 9 and Corollary 8,

$$((x^n, y^n) : F_k) = ((x^n, y^n) : x^i y^i g) = ((x^{n-i}, y^{n-i}) : g)$$

and it can be generated by two elements. $\square$

Corollary 11. *Let* $F_k = c_0 x^k + c_1 x^{k-1} y + c_2 x^{k-2} y^2 + \cdots + c_k y^k$ *where* $k \geq n$.

(a) *If* $k \geq 2n - 1$, *then* $((x^n, y^n) : F_k) = (1)$.

(b) *If* $n \leq k \leq 2n - 2$, *let*

$$\hat{F}_k = c_{k-n+1} x^{n-1} y^{k-n+1} + c_{k-n+2} x^{n-2} y^{k-n+2} + \cdots + c_{n-1} x^{k-n+1} y^{n-1},$$

then $((x^n, y^n) : F_k) = ((x^n, y^n) : \hat{F}_k)$ *and* $((x^n, y^n) : F_k)$ *can be generated by two elements.*

Proof.

(a) If $k \geq 2n - 1$, then every term $x^i y^j$ appears in F_k must have $i \geq n$ or $j \geq n$, therefore $1 \cdot F_k \in (x^n, y^n)$.

(b) If $n \leq k \leq 2n - 2$,

(i) Claim: $((x^n, y^n) : F_k) = ((x^n, y^n) : \hat{F}_k)$.

Since $F_k = (F_k - \hat{F}_k) + \hat{F}_k$ and $F_k - \hat{F}_k \in (x^n, y^n)$,

$$
\begin{aligned}
f \in ((x^n, y^n) : F_k) &\Leftrightarrow f \cdot F_k \in ((x^n, y^n), \\
&\Leftrightarrow f \cdot [(F_k - \hat{F}_k) + \hat{F}_k] \in (x^n, y^n), \\
&\Leftrightarrow f \cdot (F_k - \hat{F}_k) + f \cdot \hat{F}_k \in (x^n, y^n), \\
&\Leftrightarrow f \cdot \hat{F}_k \in (x^n, y^n).
\end{aligned}
$$

(ii) We have

$$\hat{F}_k = y^{k-n+1}(c_{k-n+1}x^{n-1} + c_{k-n+2}x^{n-2}y + \cdots + c_{n-1}x^{k-n+1}y^{2n-k-2}).$$

By Proposition 7 and 10, the minimal generating set of $((x^n, y^n) : \hat{F}_k)$ has two generators.

From (i),(ii), we can conclude that $((x^n, y^n) : F_k) = ((x^n, y^n) : \hat{F}_k)$ and the minimal generating set of $((x^n, y^n) : F_k)$ has two generators. □

§4. Discussion

Let $p(x_1, \cdots, x_r)$ be a homogeneous polynomial in r variables, $s \geq 1$ be an integer, and $I_s = (x_1^s, \cdots, x_r^s)$. In Genoway, Ortiz-Albino and Tavares's paper [8], they discussed some conditions under which the colon ideal $I = (x_1^s, \cdots, x_r^s) : p$ can be generated by no more that r elements. They provide a partial characterization of those polynomials p for which I requires more generators than the number of variables r; this collection is called $C_{(r,s)}$. That is, to characterize

$$p \in C_{(r,s)} = \{p | (x_1^s, \cdots, x_r^s) : p \quad \text{requires more than } r \text{ generators}\}.$$

Here we state the Theorems given in the paper and revise the original proofs.

Lemma. [8, Theorem 3] *For any r and s, if p is a non-zero monomial and $p \notin I_s$. Then $p \notin C_{(r,s)}$.*

Proof. Let $p = cx_1^{t_1} x_2^{t_2} \cdots x_r^{t_r}$ where $c \neq 0$ and $p \notin I_s$. Since $p \notin I_s$, we have $t_i < s$ for all i. We prove $p \notin C_{(r,s)}$ by showing $I_s : p$ can be generated by r elements.

Claim: $I_s : p = (x_1^{s-t_1}, x_2^{s-t_2}, \cdots, x_r^{s-t_r})$.

($\supseteq$) We have $x_i^{s-t_i} \cdot p = cx_1^{t_1} \cdots x_i^{t_i+(s-t_i)} \cdots x_r^{t_r} \in I_s$ for $1 \leq i \leq r$, that is, $x_i^{s-t_i} \in I_s : p$ for $1 \leq i \leq r$.

($\subseteq$) Assume $f \in I_s : p$ and $f = h_1 + h_2 + \cdots + h_l$ where $h_\alpha \neq h_\beta$ for $\alpha \neq \beta$. Then for $1 \leq \alpha \leq l$, we must have

$$\begin{aligned} & h_\alpha \cdot cx_1^{t_1} x_2^{t_2} \cdots x_r^{t_r} \in (x_i^s) \quad \text{for some } i. \\ \Rightarrow \quad & h_\alpha \cdot x_i^{t_i} \in (x_i^s) \quad \text{for some } i. \\ \Rightarrow \quad & h_\alpha \in (x_i^{s-t_i}) \quad \text{for some } i. \end{aligned}$$

Thus we can conclude that $I_s : p = (x_1^{s-t_1}, x_2^{s-t_2}, \cdots, x_r^{s-t_r})$. □

If the homogeneous polynomial $p(x_1, ..., x_j)$, $j < r$, does not involve all the variables, then the following Lemma provides an important tool to improve the efficiency of computing quotient ideals.

Lemma. [8, Theorem 4] *Let* $I_s = (x_1^s, \cdots, x_r^s)$, $p := p(x_1, \cdots, x_j)$ *and* $(x_1^s, \cdots, x_j^s) : p = (g_1, \cdots, g_k)$. *Then* $I_s : p = (g_1, \cdots, g_k, x_{j+1}^s, \cdots, x_r^s)$.

Proof. Without loss of generality, we may assume $p = p(x_1, \cdots, x_{r-1})$ and prove $I_s : p = (g_1, \cdots, g_k, x_r^s)$.

Suppose $f \in I_s : p$. We can rewrite f as a homogeneous polynomial in terms of x_r with coefficients in $k[x_1, \cdots, x_{r-1}]$, such as the following form:

$$f(x_1, \cdots, x_r) = f_t(x_1, \cdots, x_{r-1})x_r^t + \cdots + f_s(x_1, \cdots, x_{r-1})x_r^s + \cdots + f_0(x_1, \cdots, x_{r-1}).$$

Then

$$\begin{aligned} f \cdot p &= (f_t x_r^t + \cdots + f_s x_r^s + \cdots + f_0) \cdot p \\ &= f_t x_r^t \cdot p + \cdots + f_s x_r^s \cdot p + \cdots + f_0 \cdot p \in I_s. \end{aligned}$$

Note we have $f_l x_r^l \cdot p \in I_s$ for $0 \leq l \leq t$. For $l < s$,

$$\begin{aligned} f_l x_r^l \cdot p \in I_s &\Rightarrow f_l \cdot p \in (x_1^s, \cdots, x_{r-1}^s) \\ &\Rightarrow f_l \in (g_1, \cdots, g_k) \\ &\Rightarrow f_l x_r^l \in (g_1, \cdots, g_k). \end{aligned}$$

For $l \geq s$, $f_l x_r^l \in (x_r^s)$. Hence $f \in (g_1, \cdots, g_k, x_r^s)$.

On the other hand, $g_i \cdot p \in (x_1^s, \cdots, x_{r-1}^s) \subset (x_1^s, \cdots, x_{r-1}^s, x_r^s) = I_s$ for $1 \leq i \leq k$, therefore $g_i \in I_s : p$ for $1 \leq i \leq k$. Besides, $x^s \in I_s : p$.

Thus $(g_1, \cdots, g_k, x_r^s) \subseteq I_s : p$. □

Proposition. [8, Theorem 5] *Let* t *be the total degree of the homogeneous polynomial* p. *If* $t \geq r(s-1)$, *then* $p \notin C_{(r,s)}$.

Proof. Case 1. $t > r(s-1)$. We prove $p \in I_s$ which leads to $I_s : p = (1)$.

Claim: $p \in I_s$.

If $p \notin I_s$, then there exists some monomial m of p such that $m = x_1^{v_1} x_2^{v_2} \cdots x_r^{v_r} \notin I_s$. We must have $v_i \leq s-1$ for $1 \leq i \leq r$ and $t = \sum_{i=1}^{r} v_i \leq r(s-1)$, a contradiction.

Case 2. $t = r(s-1)$.

Observe that the only monomial of p which is not in I_p is $m = a(x_1 \cdots x_r)^{s-1}$. If $a = 0$, then $p \in I_s$ and $I_s : p = (1)$. Suppose $a \neq 0$, then $m \notin I_s$ while $p - m \in I_s$.

Claim: $I_s : p = I_s : m$.

$$
\begin{aligned}
f \cdot p \in I_s &\Leftrightarrow f \cdot (p-m) + f \cdot m \in I_s \\
&\Leftrightarrow f \cdot m \in I_s \\
&\Leftrightarrow f \in I_s : m.
\end{aligned}
$$

Finally, $I_s : p = I_s : m = (x_1^{s-(s-1)}, \cdots, x_r^{s-(s-1)}) = (x_1, \cdots, x_r)$. Hence $p \notin C_{(r,s)}$. □

Next, we give an example in 3 variables in which the minimal generating set has more than 3 elements.

Example. $((x^2, y^2, z^2) : x+y+z) \in C_{(3,2)}$.

Proof. Let $I = ((x^2, y^2, z^2) : x+y+z)$ and assume $\{f_1, f_2, f_3\}$ is a minimal generating set of I. Firstly, we show that $\deg f_i \neq 1$ for $i = 1, 2, 3$. Assume there exists some $f = \alpha x + \beta y + \gamma z \in I$, then we have

$$
\begin{aligned}
&(\alpha x + \beta y + \gamma z)(x + y + z) \\
=&\alpha x^2 + \alpha xy + \alpha xz + \beta yx + \beta y^2 + \beta yz + \gamma zx + \gamma zy + \gamma z^2 \\
=&\alpha x^2 + (\alpha + \beta)xy + (\alpha + \gamma)xz + \beta y^2 + (\beta + \gamma)yz + \gamma z^2 \in (x^2, y^2, z^2)
\end{aligned}
$$

and this implies

$$\begin{cases} \alpha + \beta = 0 \\ \alpha + \gamma = 0 \\ \beta + \gamma = 0 \end{cases} \Rightarrow \alpha = \beta = \gamma = 0,$$

a contradiction. Hence we must have $\deg f_i \geq 2$ for $i = 1, 2, 3$.

Case 1. Assume $\deg f_1 = 2$, $\deg f_2, \deg f_3 \geq 3$, then x^2, y^2, z^2 must be generated by f_1, therefore $f_1 | x^2$, $f_1 | y^2$, and $f_1 | z^2$, a contradiction.

Case 2. Assume $\deg f_1 = \deg f_2 = 2$, and $\deg f_3 \geq 3$. Suppose

$$f_1 = \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + m_1$$
$$f_2 = \alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 + m_2$$

where m_1 and m_2 are composed of mixed monomials. Since x^2, y^2 and z^2 must be generated by f_1 and f_2, we have

$$x^2 = \alpha f_1 + \beta f_2$$
$$y^2 = \gamma f_1 + \delta f_2.$$

We may replace (f_1, f_2, f_3) by (x^2, y^2, f_3). However, it is impossible that z^2 could be generated by (x^2, y^2, f_3).

Case 3. Suppose

$$f_1 = \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + m_1$$
$$f_2 = \alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 + m_2$$
$$f_3 = \alpha_3 x^2 + \beta_3 y^2 + \gamma_3 z^2 + m_3.$$

Since x^2, y^2, z^2 can be generated by f_1, f_2, f_3, comparing the coefficients of x^2, y^2, z^2 leads to the fact that $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ can be generated by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$. Thus f_1, f_2, f_3 can be reduced to

$$f_1 = x^2 + m'_1$$
$$f_2 = y^2 + m'_2$$
$$f_3 = z^2 + m'_3.$$

Observe that x^2 must be generated by f_1, thus $f_1 = x^2$. Similarly, $f_2 = y^2$ and $f_3 = z^2$. Thus we have $I_2 : p = I_2$. However, since for $\mathbb{S} = K[x, y, z]/I_2$, x, y, z are nilpotent elements, so is $p = x + y + z$. Hence p is a nonzero divisor in $\mathbb{S}$ and $\exists q \neq 0$ in $\mathbb{S}$ such that $pq = 0$ in $\mathbb{S}$, that is, $I_2 : p \neq I_2$, a contradiction. $\square$

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