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封閉曲面上拉普拉斯算子第一特徵值的最優上界
Sharp Upper Bounds of the First Eigenvalues of the Laplacian Operators on Closed Surfaces

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本論文係趙凱衛君（r01221005）在國立臺灣大學數學系、所完成之碩士學位論文，於民國一零三年七月十一日承下列考試委員審查通過及口試及格，特此證明

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（是否須簽章依各院系所規定）

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中文摘要

本論文將統整一些在球面、投影實平面以及輪胎面上求得以面積表示的拉普拉斯算子第一特徵值最優上界之方法。

Abstract

In this thesis, we will summarize some approaches to obtain sharp upper bounds of the first nonzero eigenvalues of the Laplacian operators on closed surfaces, including sphere $\mathbb{S}^2$, real projective plane $\mathbb{RP}^2$ and torus $\mathbb{T}^2$, in terms of their areas.

Contents

Chapter 1

Introduction

In 1912, H. Weyl [W] proved that given a bounded domain $\Omega \subset \mathbb{R}^n$ with volume V, the k-th Dirichlet eigenvalue λ_k and the k-th Neumann eigenvalue μ_k have asymptotic formulas as $k \to \infty$,

$$\lambda_k, \mu_k \sim C_n \Big(\frac{k}{V}\Big)^{\frac{2}{n}}$$

where $C_n = (2\pi)^2 \omega_n^{-2/n}$ is a universal constant and ω_n is the volume of the unit ball in $\mathbb{R}^n$. In 1961, Polya [Po] gave a famous conjecture that for any finite volume $\Omega \subset \mathbb{R}^n$ and for any positive integer k,

$$\lambda_k \geq C_n \Big(\frac{k}{V}\Big)^{\frac{2}{n}},$$
$$\mu_k \leq C_n \Big(\frac{k}{V}\Big)^{\frac{2}{n}}.$$

Moreover, P. Yang and S.-T. Yau [YY] proved that for any compact orientable Riemmanian surface Σ_γ of genus γ with area A, we have

$$\lambda_1 \leq \frac{8\pi(\gamma + 1)}{A}.$$

Also, P. Li and Yau [LY] proved that in the nonorientable case,

$$\lambda_1 \leq \frac{24\pi(\gamma+1)}{A}.$$

We would like to find geometrical upper bounds for λ_k of two-dimensional [1] (orientable) Riemannian surfaces Σ_γ of genus γ of the form

$$\lambda_k \leq \frac{C(\gamma)k}{\mathrm{Area}(\Sigma_\gamma)}$$

where $C(\gamma)$ is a constant depending only on γ. This question was raised by Yau in his survey article [Y1], and then N. Korevaar [K] gave an affirmative result that in the orientable case,

$$\lambda_k \leq C\frac{(\gamma+1)k}{\mathrm{Area}(\Sigma_\gamma)}$$

where C is a universal constant.

Another interesting question is: given γ and k, what is the optimal constant? Does that optimal surface exist? What are the optimal surfaces (metrics)? In the case $k = 1$, more precisely, we ask for two quantities as follows. If we fixe an orientable surface Σ of genus γ, we define

$$\lambda^*(\gamma) = \sup\{\lambda_1(g)A(g) : g \text{ smooth metric on } \Sigma\},$$

and in the case of nonorientable surface, we also define

$$\lambda^\#(\gamma) = \sup\{\lambda_1(g)A(g) : g \text{ smooth metric on } \Sigma\}.$$

[1] The analogous upper bound for λ_1 of Riemannian manifold of dimension more than two is always false. Every compact smooth manifold admits a Riemmanian metric of volume one and arbitrary large λ_1, see [CD].

There are few surfaces for which the optimal metric are known to exist.

- For $\mathbb{S}^2$, Hersch proved that $\lambda^*(0) = 8\pi$ and the round metric is the unique [2] optimal metric from 1970.
- For $\mathbb{RP}^2$, Li and Yau proved that $\lambda^\#(0) = 12\pi$ and the metric induced by Veronese minimal embedding of $\mathbb{RP}^2$ into $\mathbb{S}^4$ is a optimal metric from 1982. Uniqueness is proven by Montiel and Ros [MR] from 1986.
- For $\mathbb{T}^2$, Nadirashvili proved that $\lambda^*(1) = 8\pi^2/\sqrt{3}$ and the flat metric on equilateral torus is the unique optimal metric from 1996.
- For Klein bottle $\mathbb{K}$, we have $\lambda^\#(1) = 12\pi E(2\sqrt{2}/3)$ where $E(\cdot)$ is the complete elliptic integral of the second kind and the optimal metric induced by a minimal immersion of the Klein bottle into $\mathbb{S}^4$ is smooth and unique from work of several authors, see [N1],[JNP],[EGJ].

We also mention a parallel spectral problem for compact Riemannian manifolds with nonempty boundary here; however, it won't be discussed in detail in this thesis. Let (M, g) be a compact n-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$ and Laplacian Δ_g. Given a function $u \in \mathcal{C}^\infty(\partial M)$, then let $\tilde{u}$ be the harmonic extension of u, i.e.

$$\begin{cases} \Delta_g \tilde{u} = 0 & \text{on } M \\ \tilde{u} = u & \text{on } \partial M \end{cases}$$

Let ν be the outward unit conormal along ∂M. Define the **Dirichlet-to-Neumann map** to be the map $\mathbf{T} : \mathcal{C}^\infty(\partial M) \to \mathcal{C}^\infty(\partial M)$ given by

$$\mathbf{T}u = \frac{\partial \tilde{u}}{\partial \nu}.$$

$\mathbf{T}$ is a nonnegative self-adjoint operator with discrete spectrum $\{0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to \infty\}$. The eigenvalues in this problem were first discussed in 1902 by Steklov and

[2] In this question, it always means uniqueness up to homothety. Two surfaces (M_1, g_1) and (M_2, g_2) are homothety if there is a diffeomorphism $F : M_1 \to M_2$ so that $F^* g_2 = c g_1$ for for some positive constant c.

are often called **Steklov eigenvalues**. In [FS1], A. Fraser and R. Schoen proved a coarse upper bound for a compact surface (Σ, g) of genus γ with l boundary components:

$$\sigma_1 L_g(\partial\Sigma) \leq 2\pi(\gamma + l)$$

where $L_g(\partial\Sigma)$ is the length of $\partial\Sigma$ with respect to g. We can also ask for the quantities

$$\sigma^*(\gamma, k) = \sup\{\sigma_1(g)L_g(\partial\Sigma) : g \text{ smooth metric on } \Sigma\}$$

if Σ is orientable, and

$$\sigma^{\#}(\gamma, k) = \sup\{\sigma_1(g)L_g(\partial\Sigma) : g \text{ smooth metric on } \Sigma\}$$

if Σ is nonorientable.

The optimal cases are summarized as the following:

- For 2-disk, Weinstock [We] proved that $\sigma^*(0, 1) = 2\pi$ and is achieved uniquely (up to σ-homothety [3]) by Euclidean disk from 1954.

- For annulus, Fraser and Schoen [FS3] proved that $\sigma^*(0, 2) \approx 2\pi/1.2$ and is achieved uniquely (up to σ-homothety) by the critical catenoid.

- For Möbius band, Fraser and Schoen [FS3] proved that $\sigma^{\#}(0, 1) = 2\pi\sqrt{3}$ and is achieved uniquely (up to σ-homothety) by the critical Möbius band.

Notations

Let $x_0 \in \mathbb{R}^{n+1}$. We denote $(n+1)$-ball by $\mathbf{B}^{n+1}_r(x_0) = \{x \in \mathbb{R}^{n+1} : |x - x_0|^2 < r^2\}$ and n-sphere by $\mathbb{S}^n_r(x_0) = \partial\mathbf{B}^{n+1}_r(x_0)$. In particular, $x_0 = \mathbf{0}$ and $r = 1$, we just denote $\mathbf{B}^{n+1}$ and $\mathbb{S}^n$, respectively.

[3] Two surfaces (M_1, g_1) and (M_2, g_2) are σ-homothety if there is a conformal diffeomorphism $F : M_1 \to M_2$ which is a homothety on the boundary, i.e. $F^*g_2 = \rho g_1$ for some positive $\rho \in \mathcal{C}^\infty(M_1)$ and $\rho \equiv c$ on ∂M_1 for some positive constant c.

Let (M^m, g) be an m-dimensional compact Riemannian manifold without boundary. In local coordinate $(x^1, \dots, x^m)$, the metric can be written as $g = g_{ij}dx^i dx^j$, here we use Einstein summation convention. Denote $g^{ij} = (g_{ij})^{-1}$ and $|g| := \det(g_{ij})$. The volume of M with respect to g is $V(g) = V_g(M) = \int_M d\mu_g$ (if $m = 2$, we denote $A(g)$), where $d\mu_g = \sqrt{|g|}dx^1 \dots dx^m$ is the measure induced by g. Denote the sets $\mathcal{M}(M) = \{\text{smooth metrics on } M\}$ and $\mathcal{M}_1(M) = \{g \in \mathcal{M}(M) : V_g(M) = 1\}$. The Laplacian associated with g on M is

$$\Delta_g = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}).$$

Since M is compact without boundary, by Green's identity, Δ_g is a self-adjoint operator on Hilbert space $\mathcal{H}_1^2(M) = \overline{\mathcal{C}^\infty(M)}$ endowed with the norm $|u|_g^2 = \int_M u^2 d\mu_g + \int_M |\nabla u|^2 d\mu_g$ where $u \in \mathcal{C}^\infty(M)$. By spectrum theory, there are a unbounded increasing sequence of nonnegative numbers (counted with multiplicity)

$$\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty\}$$

satisfying the equation $\Delta_g u_k + \lambda_k u_k = 0$ for some nontrivial functions $u_k \in \mathcal{H}_1^2(M)$, which are called **eigenvalues** of Δ_g. Moreover, the set of **eigenfunctions** $\{u_k\}_{k=1}^\infty$ forms an orthogonal basis of $\mathcal{H}_1^2(M)$. Let $u \in \mathcal{H}_1^2(M)$. Denote the **Dirichlet integral** of u by

$$D(u) = D_g(u) = \int_M |\nabla_g u|^2 d\mu_g$$

where ∇_g denotes the gradient operator with respect to g and $|\nabla_g u|^2 = g(\nabla_g u, \nabla_g u)$. Also, denote the **Rayleigh quotient** of u by

$$R(u) = R_g(u) = D_g(u) / \int_M u^2 d\mu_g.$$

We can also realize eigenvalues through **Courant max-min principle**:

$$\lambda_k = \sup_{V \in \mathcal{G}_k} \inf_{u \in \mathcal{H}_1^2(M)} \{R(u) \Big| \int_M uv \, d\mu_g = 0, \forall v \in V\}$$

where $\mathcal{G}_k$ is the set of k-dimensional subspaces of $\mathcal{H}_1^2(M)$ containing constant functions. In particular,

$$\lambda_1 = \inf_{u \in \mathcal{H}_1^2(M)} \{R(u) \Big| u \in \mathcal{C}^\infty(M), \int_M u \, d\mu_g = 0\}.$$

(One can refer to [C] and [SY] for more details.)

Chapter 2

Sharp upper bound on $\mathbb{S}^2$

In section 2.1, we try to understand a type of conformal automorphisms on $\mathbb{S}^2$. In section 2.2, we prove balancing proposition that one can use these automorphisms to balance coordinate functions on $\mathbb{S}^2$ so that their integrations on $\mathbb{S}^2$ are all zero. Then, we use balancing proposition and max-min principle to prove Hersch's theorem.

2.1 Conformal automorphisms on sphere

Firstly, we summarize several formulas involving conformal transformations. Secondly, we will try to understand a type of conformal automorphisms on sphere, which is a key to prove Hersch's theorem, and is also used in proving theorems in [LY] and [MR]. Thirdly, we summarize several formulations of these conformal automorphisms and their behaviors.

Definition 2.1.1. *Let (M, g) and (N, h) be Riemannian manifolds. A differentiable map $f : M \longrightarrow N$ is called **conformal (isometric**, resp.) if $f^*h = \rho g$ for some positive function $\rho \in \mathcal{C}^\infty(M)$ ($f^*h = g$, resp.). The set of conformal diffeomorphisms (isometric diffeomorphism) of M onto N will be denoted by $\mathrm{C}\mathbf{onf}(\mathbf{M}, \mathbf{N})$ ($\mathrm{I}\mathbf{som}(\mathbf{M}, \mathbf{n})$, resp.). If $N = M$, the **group of conformal (isometric**, resp.) **automorphisms** of M is denoted by $\mathrm{C}\mathbf{onf}(\mathbf{M})$ ($\mathrm{I}\mathbf{som}(\mathbf{M})$, resp.). A **conformal class** (or **conformal structure**) on M can be represented by a metric g on M, and we denote the class by $[g] := \{g' = \rho g : \rho \in \mathcal{C}^\infty(M), \rho > 0\}$.*

When conformal transformation of metric is concerned, the formulas in the followings are useful. Let g be a smooth metric on M^m and let $\rho \in \mathcal{C}^\infty(M)$ and $\rho > 0$ on M. Then

$$\Delta_{\rho g} = \rho^{-1}\Delta_g + (\frac{m}{2} - 1)\rho^{-2}\nabla_g(\rho).$$

In particular, M is of dimension two, i.e. $m = 2$, then

$$\Delta_{\rho g} = \rho^{-1}\Delta_g. \tag{2.1}$$

If $\rho \equiv c$ for some constant $c > 0$ (without dimensional assumption on M),

$$\lambda_k(cg) = \frac{1}{c}\lambda_k(g).$$

We also have the relation

$$d\mu_{\rho g} = \rho^{m/2} d\mu_g \tag{2.2}$$

and for every $u \in \mathcal{H}_1^2(M)$

$$|\nabla_{\rho g} u|^2 = \rho^{-1}|\nabla_g u|^2. \tag{2.3}$$

Remark 1. In the case that Σ is a compact surface, Dirichlet integral is a conformal invariant. Although $\lambda_1(g)$ is not invariant under homothety, $\lambda_1(g)A(g)$ is invariant under homothety. We may normalized g so that $A(g) = 1$, then our main question becomes asking for $\sup\{\lambda_1(g)\Big| g \in \mathcal{M}_1(\Sigma)\}$.

Lemma 2.1.2. *Let $\phi : (M^m, g) \to (N, g_0) \subset \mathbb{R}^n$ be a conformal immersion, where g_0 is induced from Euclidean space. Set $\phi^* g_0 = \rho g$ for some positive function $\rho \in \mathcal{C}^\infty(M)$. Then*

$$|\nabla\phi|^2 := \sum_{j=1}^{n} |\nabla(X_j \circ \phi)|^2 = m\rho$$

where $X = (X_1, \cdots, X_n) : N \hookrightarrow \mathbb{R}^n$ is the inclusion map. In particular, if ϕ is an isometric immersion, then

$$|\nabla\phi|^2 = m.$$

Proof. Let $p \in M$ and $\{E_1, \cdots, E_m\}$ be an orthonormal basis of T_pM.

$$\begin{aligned}
|\nabla\phi|^2(p) &= \sum_{j=1}^{n} |\nabla(X_j \circ \phi)|^2(p) = \sum_{j=1}^{n} g\Big(\nabla(X_j \circ \phi), \nabla(X_j \circ \phi)\Big)(p) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m} \Big(E_i(X_j \circ \phi)\Big)^2(p) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \Big((\phi_* E_i)X_j\Big)^2(p) \\
&= \sum_{i}^{m} g_0\Big(\phi_* E_i, \phi_* E_i\Big)(p) \\
&= \sum_{i}^{m} (\phi^* g_0)(E_i, E_i)(p) \\
&= \rho(p) \sum_{i}^{m} g(E_i, E_i)(p) \\
&= m\rho(p).
\end{aligned}$$

$\square$

From now on, $\langle x, y\rangle$ denotes the standard inner product of $\mathbb{R}^n$ and $|x|^2 = \langle x, x\rangle$ for every $x, y \in \mathbb{R}^n$. Recall that given any $A \in \mathbb{S}^n$, we can define the stereographic projection with respect to A

$$\pi_A : \mathbb{S}^n \backslash \{A\} \longrightarrow A^N := \{y \in \mathbb{R}^{n+1} : \langle A, y\rangle = 0\} \cong \mathbb{R}^n$$

given by

$$\pi_A(x) = \frac{1}{1 - \langle A, x\rangle}\Big(x - \Big\langle A, x\Big\rangle A\Big), \tag{2.4}$$

and its differential map at $x \in \mathbb{S}^n \backslash \{A\}$ is

$$\mathrm{d}\pi_A|_x(v) = \frac{1}{1 - \langle A, x\rangle}\Big(v + \langle A, x\rangle \pi_A(x) - \big\langle A, x\big\rangle A\Big) \tag{2.5}$$

where $v \in \mathrm{T}_x\mathbb{S}^n$. The inverse map

$$\pi_A^{-1} : A^N \longrightarrow \mathbb{S}^n \backslash \{A\}$$

is given by

$$\pi_A^{-1}(y) = \frac{2y + (|y|^2 - 1)A}{1 + |y|^2}. \tag{2.6}$$

It is well-known that stereographic projections are conformal maps. Through stereographic maps, we can regard $\mathbb{S}^n$ as the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ endowed with the same conformal structure.

At first, we quote a classification theorem of conformal automorphisms, which can be by proven by complex analysis for $n = 2$ and is due to Liouville for the case $n \geq 3$.

Definition 2.1.3. *Let $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$. We call the map*

$$i_{x_0,r}(x) = r^2 \frac{x - x_0}{|x - x_0|^2} + x_0$$

*an **inversion** with respect to the sphere $\mathbb{S}^n_r(x_0)$.*

Proposition 2.1.4 ([BP] theorem A.3.4 & A.3.7)**.** *Let $n \geq 2$. Every conformal diffeomorphism φ between two domains of $\mathbb{R}^n$ has the form*

$$\varphi(x) = \alpha A i(x) + b$$

where $\alpha > 0$, $A \in O(n)$, i is either the identity or an inversion and $b \in \mathbb{R}^n$.

Corollary 2.1.5 ([BP] corollary A.3.8 & A.3.9)**.** *(1) By thinking of $\mathbb{S}^n$ as $\mathbb{R}^n \cup \{\infty\}$,*

$\mathrm{C}on f(\mathbb{S}^n)$ *consists of all and only the mappings of the form*

$$\varphi(x) = \alpha A i(x) + b$$

where $\alpha > 0$*,* $A \in O(n)$*,* i *is either the identity or an inversion and* $b \in \mathbb{R}^n$*.*

(2) $\mathrm{C}on f(\mathbf{B}^n)$ *consists of all and only the mappings of the form*

$$\varphi(x) = A i(x)$$

where $A \in O(n)$*,* i *is either the identity or an inversion with respect to a sphere orthogonal to* $\mathbb{S}^n$*.*

We have another interpretation of $\mathrm{C}on f(\mathbf{B}^n)$.

Proposition 2.1.6 ([BP] theorem A.4.1)**.** $\mathrm{I}som(\mathbb{B}^n) = \mathrm{C}on f(\mathbf{B}^n)$*, where* $\mathbb{B}^n$ *is endowed with hyperbolic metric* $4(1-|x|^2)^{-2}\delta_{ij}dx^i dx^j$*. In particular, the group operates transitively on* $\mathbb{B}^n$*.*

Now we list four formulations of certain conformal automorphisms of $\mathbb{S}^n$ in literatures.

(1) ([LY]) Let a be a point in $\mathbf{B}^{n+1}$ and $A = a/|a|$. At each point on $\mathbb{S}^n$, we assign a vector $V_A(x)$ to be the projection of A onto tangent plane $\mathrm{T}_x\mathbb{S}^n$, more precisely, $V_A(x) = A - \langle A, x\rangle$. The vector field can be extended on $\overline{\mathbf{B}}^{n+1}$ by solving

$$\mathcal{L}_{V_A}(\delta_{ij}dx^i dx^j) = \rho\, \delta_{ij}dx^i dx^j$$

for some positive smooth function ρ on M, where $\mathcal{L}_{V_A}$ is the Lie derivative with respect to V_A. Remark that by taking trace or using proposition 2.1.6, it is equivalent to solve

$$\mathcal{L}_{V_A}\left(\frac{4\delta_{ij}}{(1-|x|^2)^2}dx^i dx^j\right) = 0.$$

The conformal vector field generates a one-parameter group of conformal diffeomorphisms $\{\varphi_a^{(t)}\} \subset \mathrm{C}on f(\overline{\mathbf{B}}^{n+1})$ of $\overline{\mathbf{B}}^{n+1}$. There is a unique $T \geq 0$ so that

$\varphi_a^{(T)}(0) = a$. Then we denote $\varphi_a^{(T)}$ simply by φ_a.

(2) ([LY]) A homothety of $\mathbb{R}^n = \mathbb{S}^n \backslash \{A\}$, more precisely, $\pi_A^{-1} \circ \alpha \circ \pi_A(x)$ for some $\alpha > 0$.

(3) ([MR]) Define $\Psi_a \in Conf(\mathbb{S}^n)$ given by

$$\Psi_a(x) = \frac{x + (\mu\big\langle a, x\big\rangle + \lambda)a}{\lambda(\big\langle a, x\big\rangle + 1)}$$

where $\lambda = (1 - |a|^2)^{-\frac{1}{2}}$, $\mu = (\lambda - 1)|a|^{-2}$.

(4) ([SY]) Define $\Phi_a \in Conf(\overline{\mathbf{B}}^{n+1})$ given by

$$\Phi_a(x) = \frac{(1 - |a|^2)x + (1 + 2\big\langle a, x\big\rangle + |x|^2)a}{1 + 2\big\langle a, x\big\rangle + |a|^2|x|^2}.$$

We verify below that these formulations are equivalent.

(1) $\Leftrightarrow$ (2)

The pushforward of $V_A(x)$ by $\mathrm{d}\pi_A$ is

$$\begin{aligned}
\mathrm{d}\pi_A\Big(V_A(x)\Big) &= \frac{1}{1 - \langle A, x\rangle}\{\Big(A - \Big\langle A, x\Big\rangle x\Big) + \Big(1 - \langle A, x\rangle^2\Big)\pi_A(x) - \Big(1 - \langle A, x\rangle^2\Big)A\} \\
&= \Big(1 + \langle A, x\rangle\Big)\pi_A(x) - \frac{\langle A, x\rangle}{1 - \langle A, x\rangle}\{x - \Big\langle A, x\Big\rangle A\} \\
&= \Big(1 + \Big\langle A, x\Big\rangle\Big)\pi_A(x) - \langle A, x\rangle\pi_A(x) \\
&= \pi_A(x)
\end{aligned}$$

according to (2.4) and (2.5). Hence, $\varphi_a^{(t)}(x)\Big|_{\mathbb{S}^n} = \pi_A^{-1} \circ e^t \circ \pi_A(x)$.

(2) $\Leftrightarrow$ (3)

By (2.4) and (2.6),

$$\begin{aligned}
\pi_A^{-1} \circ \alpha \circ \pi_A(x) &= \pi_A^{-1}\Big(\frac{\alpha}{1 - \big\langle A, x\big\rangle}(x - \big\langle A, x\big\rangle A)\Big) \\
&= \frac{2\alpha x + \{\alpha^2(1 + \big\langle A, x\big\rangle) - 2\alpha\big\langle A, x\big\rangle + (\big\langle A, x\big\rangle - 1)\}A}{\alpha^2(1 + \big\langle A, x\big\rangle) + (1 - \big\langle A, x\big\rangle)}.
\end{aligned}$$

Choose $\alpha = \sqrt{\frac{1+|a|}{1-|a|}} = \frac{1+|a|}{\sqrt{1-|a|^2}} = \lambda(1+|a|)$ such that $\pi_A^{-1} \circ \alpha \circ \pi_A(\mathbf{0}) = a$. Then the denominator of $\pi_A^{-1} \circ \alpha \circ \pi_A(x)$ is

$$\begin{aligned}
&\frac{1+|a|}{1-|a|}(1+\frac{\langle a,x\rangle}{|a|}) + (1-\frac{\langle a,x\rangle}{|a|}) \\
&= \frac{1}{1-|a|}\{(1+\frac{\langle a,x\rangle}{|a|} + |a| + \langle a,x\rangle) + (1-\frac{\langle a,x\rangle}{|a|} - |a| + \langle a,x\rangle)\} \\
&= \frac{2}{1-|a|}(1+\langle a,x\rangle) \\
&= \frac{2}{1-|a|^2}(1+|a|)(1+\langle a,x\rangle) \\
&= 2\lambda^2(1+|a|)(1+\langle a,x\rangle)
\end{aligned}$$

and the numerator is

$$\begin{aligned}
&2\lambda(1+|a|)x + \{\frac{1+|a|}{1-|a|}(1+\frac{\langle a,x\rangle}{|a|}) - 2\lambda(1+|a|)\frac{\langle a,x\rangle}{|a|} + (\frac{\langle a,x\rangle}{|a|} - 1)\}\frac{a}{|a|} \\
&= 2\lambda(1+|a|)x \\
&\quad + \{\frac{1}{1-|a|}\Big((1+\frac{\langle a,x\rangle}{|a|} + |a| + \langle a,x\rangle) - (1-\frac{\langle a,x\rangle}{|a|} - |a| + \langle a,x\rangle)\Big) - 2\lambda(1+|a|)\frac{\langle a,x\rangle}{|a|}\}\frac{a}{|a|} \\
&= 2\lambda(1+|a|)x + \{\frac{1}{1-|a|}(1+\frac{\langle a,x\rangle}{|a|^2}) - 2\lambda(1+|a|)\frac{\langle a,x\rangle}{|a|^2}\}a \\
&= 2\lambda(1+|a|)\Big\{x + \Big(\lambda(1+\frac{\langle a,x\rangle}{|a|^2}) - \frac{\langle a,x\rangle}{|a|^2}\Big)a\Big\}.
\end{aligned}$$

Thus, we have

$$\pi_A^{-1} \circ \sqrt{\frac{1+|a|}{1-|a|}} \circ \pi_A(x) = \Psi_a(x).$$

(3) $\Leftrightarrow$ (4)

By direction calculation, one can show that $\Psi_{\beta a}(x) = \Phi_a(x)$ where $\beta = \frac{2}{1+|a|^2}$ for any

$a \in \mathbf{B}^{n+1}$. Use the relation between (2) and (3), we obtain

$$
\begin{aligned}
\Phi_a(x) &= \Psi_{\beta a}(x) \\
&= \pi_A^{-1} \circ \sqrt{\frac{1+|\beta a|}{1-|\beta a|}} \circ \pi_A(x) \\
&= \pi_A^{-1} \circ \sqrt{\frac{1+|a|^2+2|a|}{1+|a|^2-2|a|}} \circ \pi_A(x) \\
&= \pi_A^{-1} \circ \frac{1+|a|}{1-|a|} \circ \pi_A(x) \\
&= \Psi_a^2(x)
\end{aligned}
$$

Since the map in (2) is obvious conformal on $\mathbb{S}^n$, it follows that the maps in other formulations are all conformal.

From the map in (3) or (4), one can assign continuously a conformal map for every point $a \in \mathbf{B}^{n+1}$. In fact, one can show that

$$
\mathrm{C}onf(\mathbb{S}^n)/\mathbf{SO}(n+1) \cong \mathbf{B}^{n+1}, \tag{2.7}
$$

where $\mathbf{SO}(n+1) = \mathrm{I}som(\mathbb{S}^n)$.

We also verify below that the explicit form of the map in (1) is that in (4). Notice that the extended conformal automorphism on $\overline{\mathbf{B}}^{n+1}$ in (1) is NOT the map in (3). Although $\Psi_a(\mathbf{0}) = a$, it is not conformal on $\mathbf{B}^{n+1}$. The PDE in (1) is not easy to solved. We try another way to construct the map in (4) so that $\Phi_a(x) \in \mathrm{C}onf(\overline{\mathbf{B}}^{n+1})$ and $\Phi_a(\mathbf{0}) = a$. Observe that Φ_a in formulation (2) is a homothety of $\mathbb{R}^n = \mathbb{S}^n \backslash \{A\}$ and Φ_{-a} is a homothety of $\mathbb{R}^n = \mathbb{S}^n \backslash \{-A\}$ with the same dilation factor, it follows that $\Phi_a = (\Phi_{-a})^{-1}$. Hence the condition $\Phi_a(\mathbf{0}) = a$ is equivalent to $\Phi_a(-a) = (\Phi_{-a})^{-1}(-a) = \mathbf{0}$.

Now, our goal is to construct a conformal automorphism of $\mathbb{R}^{n+1} \cup \{\infty\}$ which maps $\overline{\mathbf{B}}^{n+1}$ onto itself and maps $-a$ to $\mathbf{0}$. At first, we know that the inversions have the following properties.

Proposition 2.1.7 ([BP] proposition A.3.1). *(1)* $i_{x_0,r}|_{\mathbb{S}^n_r(x_0)} = id$.

(2) *Let $i = i_{x_0,r}$. H denotes a hyperplane and S, S' denote spheres.*

(i) $x_0 \in H \overset{i}{\longleftrightarrow} H(\textit{itself}) \ni x_0$.

(ii) $x_0 \notin H \overset{i}{\longleftrightarrow} S \ni x_0$.

(iii) $x_0 \notin S \overset{i}{\longleftrightarrow} S' \not\ni x_0$.

Let $v \in \mathbb{R}^{n+1}$ and $r > 0$ to be determined. Consider the map

$$i_{0,1} \circ T_v \circ i_{-a,r}$$

which map $-a$ to $\mathbf{0}$, where $T_v(x) = x + v$ is the translation map. By proposition 2.1.7 (2) (iii), $i_{-a,r}(\mathbb{S}^n)$ is a sphere away from $-a$. We want $T_v \circ i_{-a,r}(\mathbb{S}^n) = \mathbb{S}^n$ so that $i_{0,1} \circ T_v \circ i_{-a,r}$ maps $\overline{\mathbf{B}}^{n+1}$ onto itself. It requires that $i_{-a,r}(\mathbb{S}^n)$ is a sphere of radius 1 and we can solve $r = \sqrt{1-|a|^2}$. Since $i_{-a,\sqrt{1-|a|^2}}(\mathbb{S}^n) = \mathbb{S}^n_1(-2a)$, we take $v = 2a$. Finally, we write explicitly $\Phi_a(x) = i_{0,1} \circ T_{2a} \circ i_{-a,\sqrt{1-|a|^2}}(x)$.

The map $\Phi_a(x)$ in (4) satisfies the following property:

Lemma 2.1.8. *Let $a \in \mathbf{B}^{n+1}$. If $x \in \overline{\mathbf{B}}^{n+1} \backslash \mathbf{B}^{n+1}_{\sqrt{1-|a|^2}}(-a)$. Then*

$$|\Phi_a(x) - x|^2 < 1 - |a|^2.$$

Proof. By definition,

$$\begin{aligned} |\Phi_a(x) - x|^2 &= \frac{(1-|a|^2)^2 \left| x + |x|^2 a \right|^2}{(1 + 2\left\langle a, x \right\rangle + |a|^2|x|^2)^2} \\ &= \frac{(1-|a|^2)^2 |x|^2}{(1 + 2\left\langle a, x \right\rangle + |a|^2|x|^2)} \end{aligned} \tag{2.8}$$

Since $(1 + 2\left\langle a, x \right\rangle + |a|^2|x|^2) - |x + a|^2 = (1-|a|^2)(1-|x|^2) \geq 0$, we have

$$1 + 2\left\langle a, x \right\rangle + |a|^2|x|^2 \geq |x + a|^2 > 1 - |a|^2. \tag{2.9}$$

Hence, by (2.8), (2.9) and $|x|^2 \leq 1$,

$$|\Phi_a(x) - x|^2 < 1 - |a|^2.$$

$\square$

2.2 Hersch's theorem

In the following, we will prove balancing proposition, which is an important technique to find an immersion whose coordinate functions consisting of test functions for the first eigenvalue.

Proposition 2.2.1 ([S]). *Let μ be a probability measure on $\overline{\mathbf{B}}^n$, i.e. $\mu(\overline{\mathbf{B}}^n) = 1$. Assume μ has no point mass on $\partial \mathbf{B}^n$. Then there exists $a \in \mathbf{B}^n$ such that*

$$\int_{\overline{\mathbf{B}}^n} \Phi_a(x)\, d\mu(x) = \mathbf{0}.$$

Proof. Define $F : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ by $F(a) = \int_{\overline{\mathbf{B}}^n} \Phi_a(x)\, d\mu(x)$. At first, we claim that $\forall \varepsilon > 0$, $\exists \delta > 0$ so that $|F(a) - a| < \varepsilon$ if $r(a) := \sqrt{1 - |a|^2} < \delta$.

Given $\varepsilon > 0$. Since there is no point mass on $\partial \mathbf{B}^n$, there exists $\delta_1 > 0$ such that for every $z \in \partial \mathbf{B}^n$, $\mu(\overline{\mathbf{B}}^n \cap \mathbf{B}^n_{\delta_1}(z)) < \varepsilon/3$. For every $a \in \mathbf{B}^n$ with $r = r(a) < \delta := \max\{\delta_1/2, \varepsilon/3\}$, since for every $x \in \mathbf{B}^n_r(-a)$,

$$\Big|x + a/|a|\Big|^2 \leq \Big|x + a\Big|^2 + \Big| - a + a/|a|\Big|^2 \leq r + 1 - |a| \leq 2r \leq \delta_1,$$

then we have $\mathbf{B}^n_r(-a) \subset \mathbf{B}^n_{\delta_1}(-a/|a|)$. By lemma 2.1.8 and the above observation,

$$\begin{aligned}
|F(a)-a| &= \Big| \int_{\overline{\mathbf{B}}^n} \Phi_a(x)d\mu(x) - a \Big| \\
&\leq \int_{\overline{\mathbf{B}}^n} |\Phi_a(x)-a|\, d\mu(x) \\
&\leq \int_{\overline{\mathbf{B}}^n \setminus \mathbf{B}^n_r(-a)} |\Phi_a(x)-a|\, d\mu(x) + \int_{\overline{\mathbf{B}}^n \cap \mathbf{B}^n_r(-a)} |\Phi_a(x)-a|\, d\mu(x) \\
&< \int_{\overline{\mathbf{B}}^n \setminus \mathbf{B}^n_r(-a)} r\, d\mu(x) + \int_{\overline{\mathbf{B}}^n \cap \mathbf{B}^n_{\delta_1}(-a/|a|)} (|\Phi_a(x)| + |a|) d\mu(x) \\
&< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} < \varepsilon.
\end{aligned}$$

This implies that F can be extended continuously onto $\overline{\mathbf{B}}^n$ with $F\Big|_{\partial \mathbf{B}^n} = id_{\partial \mathbf{B}^n}$. By degree theory argument, F is surjective and hence $F^{-1}(\mathbf{0})$ is not empty. □

We quote here a well-known theorem on Riemann surfaces.

Proposition 2.2.2 (Uniformization theorem)**.** *Let Σ be a oriented surface and g be a metric on Σ, then there is exact (except $\mathbb{S}^2$) one $\tilde{g} \in [g]$ so that*

$$K(\tilde{g}) = \begin{cases} 1, & \chi(\Sigma) > 0 \\ 0, & \chi(\Sigma) = 0 \\ -1, & \chi(\Sigma) < 0 \end{cases}$$

where K is Gaussian curvature and χ is Euler characteristic.

Now, we can start the proof of Hersch's theorem.

Theorem 2.2.3 ([S])**.** *For (S^2, g), we have*

$$\lambda_1(g)A(g) \leq 8\pi.$$

Equality holds if and only if g is the standard metric, up to homothety.

Proof. By uniformization theorem, there is a conformal diffeomorphism $\phi : (\mathbb{S}^2, g) \to (\mathbb{S}^2, g_0) \subset \mathbb{R}^3$, where g_0 is the standard Euclidean metric. Define a probability measure μ

on $\overline{\mathbf{B}}^3$ by

$$\mu = \phi_*(A_g)/A_g(\mathbb{S}^2)$$

where $\phi_*(A_g)$ is the pushforward measure defined by $\big(\phi_*(A_g)\big)(V) = A_g\big(\phi^{-1}(V)\big)$ for any measurable subset $V \subset \overline{\mathbf{B}}^3$. By balancing proposition 2.2.1, there is $a \in \mathbf{B}^3$ such that

$$\mathbf{0} = \int_{\overline{\mathbf{B}}^3} \Phi_a(x)\, d\mu(x) = \frac{1}{A_g(\mathbb{S}^2)} \int_{\mathbb{S}^2} \Phi_a \circ \phi \, d\mu_g.$$

Set $\psi = \Phi_a \circ \phi$ which satisfies

$$\int_\Sigma X_j \circ \psi \, d\mu_g = 0$$

for $j = 1, 2, 3$, where X_j's are coordinate functions. Take $X_j \circ \psi$'s as test functions in Courant max-min principle and then sum over j, we have

$$\sum_{j=1}^3 \lambda_1(g) \int_{\mathbb{S}^2} (X_j \circ \psi)^2 \, d\mu_g \leq \sum_{j=1}^3 \int_{\mathbb{S}^2} |\nabla(X_j \circ \psi)|^2 \, d\mu_g = \int_{\mathbb{S}^2} |\nabla\psi|^2 \, d\mu_g.$$

Set $\psi^* g_0 = \rho g$ where $\rho \in \mathcal{C}^\infty(\mathbb{S}^2)$ is positive. Since the image of ψ lies in $\mathbb{S}^2$, by (2.2) and lemma 2.1.2,

$$\lambda_1(g)A(g) = \lambda_1(g) \int_{\mathbb{S}^2} d\mu_g \leq \int_{\mathbb{S}^2} |\nabla\psi|^2 \, d\mu_g = \int_{\mathbb{S}^2} 2\rho \left(\rho^{-1}\psi^* d\mu_0\right) = 8\pi.$$

When the equality holds, $X_j \circ \psi$'s are all eigenfunctions. Namely, $\Delta\psi = -\lambda_1\psi$. It follows that

$$0 = \frac{1}{2}\Delta|\psi|^2 = \psi \cdot \Delta\psi + |\nabla\psi|^2 = -\lambda_1\psi^2 + |\nabla\psi|^2 = -\lambda_1 + |\nabla\psi|^2.$$

Hence, $\psi^* g_0 = \rho g = \frac{1}{2}|\nabla\psi|^2 g = \frac{1}{2}\lambda_1 g$. $\square$

Chapter 3

Sharp upper bound on $\mathbb{RP}^2$

In section 3.1, we recall the definition of minimal immersions in manifolds and find a connection between eigenfunction vectors and minimal immersions into spheres. In section 3.2, we define conformal area and then we prove Li-Yau's theorem that conformal area provides a sharp inequality of λ_1 by using Hersch's trick. Also, we prove that when the isometric immersion of a given surface is minimal, its conformal area is exactly area of the surface. In section 3.3, we define λ_1-minimal surfaces, which is found when the equality in Li-Yau's sharp inequality holds. Then we prove a rigidity theorem of λ_1-minimal surfaces due to Montiel and Ros. Finally, we use Li-Yau's sharp inequality, computation of conformal area to find sharp upper bound of λ_1 on $\mathbb{RP}^2$ in terms of its area. Also, we use Montiel-Ros' rigidity theorem to prove uniqueness of the optimal metric.

3.1 Minimal immersions into sphere

In this section, we summarize some results in Chap.1 §2, §3, §4 of [L] and give several important examples of minimal immersion into sphere.

Let $\phi : M^m \hookrightarrow \overline{M}^n$ be an immersed m-dimensional manifold in an ambient n-dimensional manifold. Let g be a metric on $\overline{M}$. For any $p \in M \subset \overline{M}$, we have orthogonal splitting

$$T_p\overline{M} = T_pM \oplus N_pM$$

into the tangent and normal space of M at p, respectively. For any $X \in T_p\overline{M}$, X^T denotes the tangent part and X^N denotes the normal part of X. Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. For any vector X_p and vector field Y around p, the Levi-Civita connection on M is given by

$$\nabla_X Y(p) = (\overline{\nabla}_X Y)^T(p).$$

We can also define the normal part

$$B(X,Y)(p) = (\overline{\nabla}_X Y)^N(p).$$

Note that

$$B(X,Y) = (\overline{\nabla}_X Y)^N = (\overline{\nabla}_Y X + [X,Y])^N = (\overline{\nabla}_Y X)^N = B(Y,X).$$

Hence, $B(X,Y)$ depends only on X_p and Y_p and B is a $\mathcal{C}^\infty$-section of the bundle $T^*(M) \otimes T^*(M) \otimes N(M)$ called the **second fundamental form** of M in $\overline{M}$. At every point $p \in M$, B is a symmetric bilinear map of T_pM into N_pM. We can define the **mean curvature vector field** by

$$H_p = \frac{1}{m}\mathrm{Tr}_g(B_p).$$

Locally, if $E_1, \ldots, E_m$ are an orthonormal basis of T_pM, then

$$H_p = \frac{1}{m}\sum_{i=1}^{m}(\overline{\nabla}_{E_i} E_i)^N(p).$$

If $H \equiv 0$ on M, then we call ϕ to be a **minimal immersion** and M to be a **minimal submanifold**.

Proposition 3.1.1 ([L] proposition 8). *Let* $\phi : M^m \longrightarrow \mathbb{R}^n$ *be an isometric immersion* *and let H be the mean curvature vector field of ϕ. Then*

$$\Delta\phi = mH$$

where denote $\Delta\phi := (\Delta\phi_1, \cdots, \Delta\phi_n)$.

Proof. Let $p \in M$ and choose an orthonormal basis $\{E_i\}_{i=1}^m$ of T_pM. Then for each i, $E_i\phi = E_i$ (actually = $\phi_* E_i$) and $E_iE_i\phi = \overline{\nabla}_{E_i}E_i$ where $\overline{\nabla}$ denotes the Euclidean connection. Hence,

$$\begin{aligned}\Delta\phi &= \sum_i \{E_iE_i\phi - (\nabla_{E_i}E_i)\phi\} = \sum_i \{\overline{\nabla}_{E_i}E_i - \nabla_{E_i}E_i\} \\ &= \sum_i \{\overline{\nabla}_{E_i}E_i\}^N = mH.\end{aligned}$$

□

Lemma 3.1.2 ([L], p.15)**.** *Let $\overline{M} \subset \mathbb{R}^n$ be an embedded submanifold. If $\phi : M^m \longrightarrow \overline{M} \subset \mathbb{R}^n$ is an immersion with mean curvature vector fields $\overline{H}$ in $\overline{M}$ and H in $\mathbb{R}^n$. Then*

$$m\overline{H} = (mH)^T = (\Delta\phi)^T.$$

Proof. Since

$$\overline{H} = \sum_i (\overline{\nabla}_{E_i}E_i)^N = \sum_i ((\nabla_{E_i}E_i)^T)^N = (\sum_i (\nabla_{E_i}E_i)^N)^T = (H)^T$$

where $\overline{\nabla}$, ∇ are the connections on $\overline{M}$ and $\mathbb{R}^n$, respectively. Then, the result follows the previous proposition. □

Now, we set $\overline{M} = S^n$.

Proposition 3.1.3 ([L] proposition 12)**.** *Let M^m be a Riemannian manifold with dimension m and let $\phi : M \longrightarrow \mathbb{S}^n$ be an isometric immersion. Then ϕ is minimal if and only if*

$$\Delta\phi = -m\phi.$$

Proof. By lemma 3.1.2 ϕ is minimal if and only if $\Delta\phi(p)$ is parallel to the normal direction of sphere at $\phi(p)$ for every $p \in M$, i.e. $\Delta\phi(p) = \lambda(p)\phi(p)$, for every $p \in M$. Since $|\phi|^2 = 1$, we have

$$0 = \frac{1}{2}\Delta|\phi|^2 = \langle \phi, \Delta\phi \rangle + |\nabla\phi|^2 = \lambda|\phi|^2 + |\nabla\phi|^2 = \lambda + |\nabla\phi|^2.$$

Hence, by lemma 2.3

$$\lambda = -|\nabla\phi|^2 = -m.$$

□

As an immediate consequence, we have a more general proposition.

Proposition 3.1.4 (by Takahashi, [L] proposition 13)**.** *Let M^m be a Riemannian manifold with dimension m and let $\phi : M \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion such that*

$$\Delta\phi = -\lambda\phi$$

for some constant $\lambda \neq 0$. Then we have the following:

1. *$\lambda > 0$.*

2. *$\phi(M) \subset \mathbb{S}^n(r)$ where $r^2 = \frac{m}{\lambda}$.*

3. *The immersion $\phi : M \longrightarrow \mathbb{S}^n(r)$ is minimal.*

Remark 2. The proposition above plays an important role as a connection between minimal immersions into spheres and eigenfunction vectors of Laplacian.

Example 1. *(Veronese surface) The immersion*

$$\phi_v : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$$

given by

$$\phi_v(x,y,z) = \Big(\sqrt{3}xy, \sqrt{3}xz, \sqrt{3}yz, \frac{\sqrt{3}}{2}(x^2-y^2), \frac{1}{2}(x^2+y^2-2z^2)\Big) \tag{3.1}$$

provides a minimal embedding of $\mathbb{RP}^2$ *with curvature* 1 *into* $\mathbb{S}^4$*, and it is called* ***Veronese embedding****. The area of* $\phi_{ve}(\mathbb{RP}^2)$ *is* 6π.

Example 2. *(Clifford torus) The immersion*

$$\phi_{cl} : \mathbb{R}^2 \longrightarrow \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$$

given by

$$\phi_{cl}(x,y) = \frac{1}{\sqrt{2}}\Big(e^{2\pi i x}, e^{2\pi i y}\Big) \tag{3.2}$$

provides a minimal embedding of ***Clifford torus*** $\mathbb{T}^2_{sq} := \mathbb{R}^2\big/\Gamma_{sq}$ *with normalized flat metric, whose* $\lambda_1 = 2$*, into* $\mathbb{S}^3$*, where the lattice* $\Gamma_{sq} := \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$*. The area of* $\phi_{cl}(\mathbb{T}^2_{sq})$ *is* $2\pi^2$.

Example 3. *(Equilateral torus) The immersion*

$$\phi_{eq} : \mathbb{R}^2 \longrightarrow \mathbb{S}^1(\frac{1}{\sqrt{3}}) \times \mathbb{S}^1(\frac{1}{\sqrt{3}}) \times \mathbb{S}^1(\frac{1}{\sqrt{3}}) \subset \mathbb{S}^5$$

given by

$$\phi_{eq}(x,y) = \frac{1}{\sqrt{3}}\Big(e^{\frac{4\pi i y}{\sqrt{3}}}, e^{2\pi i (x - \frac{y}{\sqrt{3}})}, e^{2\pi i (x + \frac{y}{\sqrt{3}})}\Big) \tag{3.3}$$

provides a minimal embedding of ***equilateral torus*** $\mathbb{T}^2_{eq} := \mathbb{R}^2\big/\Gamma_{eq}$ *with normalized flat metric, whose* $\lambda_1 = 2$*, into* $\mathbb{S}^5$*, where the lattice* $\Gamma_{eq} := \mathbb{Z}(1,0) \oplus \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{3}}{2})$*. The area of* $\phi_{eq}(\mathbb{T}^2_{eq})$ *is* $4\pi^2/\sqrt{3}$ *(greater than the area of Clifford torus).*

3.2 Conformal area

Definition 3.2.1. *Let Σ be a compact surface which admits a conformal map ϕ into n-dimensional sphere $\mathbb{S}^n$. Let g be the metric on Σ and g_0 be the standard metric on $\mathbb{S}^n$. Define the* **n*-dimensional area of*** ϕ *by*

$$\begin{aligned} A_c(n,\phi) &:= \sup_{\sigma\in Conf(\mathbb{S}^n)} \int_\Sigma (\sigma\circ\phi)^* d\mu_{g_0} \\ &= \sup_{\sigma\in Conf(\mathbb{S}^n)} \frac{1}{2}\int_\Sigma |\nabla(\sigma\circ\phi)|^2 d\mu_g. \end{aligned}$$

The **n*-conformal area of*** Σ *is then defined to be*

$$A_c(n,\Sigma) := \inf_\phi A_c(n,\phi)$$

where ϕ run through all conformal maps into sphere.

Theorem 3.2.2 ([LY] theorem 1)**.** *Let Σ be a compact surface with metric g. Then*

$$\lambda_1(g)A(g) \leq 2A_c(n,\Sigma) \tag{3.4}$$

for all n where $A_c(n,\Sigma)$ is defined. Equality holds if and only if Σ is a minimal surface of $\mathbb{S}^n$ given by an isometric immersion whose coordinate functions consist of first eigenfunctions.

Proof. Given $\varepsilon > 0$. Let $\phi : \Sigma \to \mathbb{S}^n$ be a conformal map so that

$$A_c(n,\phi) \leq A_c(n,\Sigma) + \varepsilon.$$

By Hersch's trick, there is $a \in \mathbf{B}^{n+1}$ such that

$$\int_\Sigma \Phi_a \circ \phi \, d\mu_g = \mathbf{0}.$$

Apply max-min principle on $X_j \circ \Phi_a \circ \phi$, for $j = 1, \ldots, n+1$, then we have

$$\begin{aligned}\lambda_1(g)A(g) = \lambda_1(g) \int_\Sigma |\Phi_a \circ \phi|^2 \, d\mu_g = \lambda_1(g) \sum_{j=1}^{n+1} \int_\Sigma (X_j \circ \Phi_a \circ \phi)^2 \, d\mu_g \\ \leq \sum_{j=1}^{n+1} \int_\Sigma |\nabla(X_j \circ \Phi_a \circ \phi)|^2 \, d\mu_g = \int_\Sigma |\nabla(\Phi_a \circ \phi)|^2 \, d\mu_g.\end{aligned}$$

By definitions of conformal area,

$$\int_\Sigma |\nabla(\Phi_a \circ \phi)|^2 \, d\mu_g \leq 2A_c(n, \phi) \leq 2(A_c(n, \Sigma) + \varepsilon).$$

Therefore,

$$\lambda_1(g)A(g) \leq 2(A_c(n, \Sigma) + \varepsilon).$$

Inequality (3.4) follows by letting $\varepsilon \to 0$.

Now, we assume the equality of (3.4) holds. Since both sides of (3.4) is invariant under homothety, we may assume

$$\lambda_1(g) = 2, \tag{3.5}$$

hence

$$A_c(n, \Sigma) = A(g). \tag{3.6}$$

Let $\phi_k : \Sigma \to \mathbb{S}^n$ be a sequence of conformal maps such that

$$\lim_{k \to \infty} A_c(n, \phi_k) = A_c(n, \Sigma). \tag{3.7}$$

Since for any k and $\varphi \in Conf(\mathbb{S}^n)$

$$\begin{aligned} A_c(n, \varphi \circ \phi_k) &= \sup_{\sigma \in Conf(\mathbb{S}^n)} \int_\Sigma (\sigma \circ \varphi \circ \phi_k)^* d\mu_{g_0} \\ &= \sup_{\tau \in Conf(\mathbb{S}^n)} \int_\Sigma (\tau \circ \phi_k)^* d\mu_{g_0} \\ &= A_c(n, \phi_k), \end{aligned}$$

we may also assume, for every k,

$$\int_\Sigma \phi_k \, d\mu_g = \mathbf{0}.$$

Therefore, for each k,

$$2A(g) = 2\int_\Sigma |\phi_k|^2 \, d\mu_g \leq \int_\Sigma |\nabla \phi_k|^2 \, d\mu_g \leq 2A_c(n, \phi_k). \tag{3.8}$$

Letting $k \to \infty$, (3.6) and (3.7) implies that

$$\lim_{k \to \infty} A_c(n, \phi_k) = A_c(n, \Sigma) = A(g), \tag{3.9}$$

then $\{\phi_k\}_{k=1}^\infty \subset \mathcal{H}_1^2(\Sigma, \mathbb{S}^n)$ is a bounded sequence. It's well-known that the embedding $\mathcal{H}_1^2(\Sigma, \mathbb{S}^n) \subset L^2(\Sigma, \mathbb{S}^n)$ is compact. Hence, up to subsequence, there is a function $\psi \in L^2(\Sigma, \mathbb{S}^n)$ so that as $k \to \infty$

$$\begin{cases} \phi_k \rightharpoonup \psi & \text{in } \mathcal{H}_1^2(\Sigma, \mathbb{S}^n), \\ \phi_k \to \psi & \text{in } L^2(\Sigma, \mathbb{S}^n). \end{cases}$$

It's clear that

$$|\psi|^2 \equiv 1, \tag{3.10}$$

almost everywhere, and by (3.8) and (3.9)

$$\int_\Sigma |\nabla\psi|^2\, d\mu_g = \lim_{k\to\infty}\int_\Sigma |\nabla\phi_k|^2\, d\mu_g = 2A(g). \tag{3.11}$$

It follows that $\phi_k \to \psi$ strongly in $\mathcal{H}_1^2(\Sigma, \mathbb{S}^n)$. From (3.10) and (3.11),

$$\int_\Sigma |\nabla\psi|^2\, d\mu_g = \lambda_1(g)\int_\Sigma |\psi|^2\, d\mu_g,$$

hence $X_j \circ \psi$'s are eigenfunction associated with $\lambda_1(g)$. In particular, ψ is a smooth conformal map from Σ to $\mathbb{S}^n$. Taking Laplacian on $|\psi|^2$, we have

$$|\nabla\psi|^2 = \lambda_1(g) = 2.$$

By lemma 2.1.2, ψ is an isometry and then it follows proposition 3.1.3 that ψ is minimal.

$\square$

Let M^m be a smooth manifold. Let T, S be tensors of (p, q)-type. Let $\left\langle T, S\right\rangle_g$ denote the pointwise inner product induced by g on (p, q)-tensor and $|T|_g^2 := \left\langle T, T\right\rangle_g$ denote the tensor norm.

Lemma 3.2.3. *Let $\Sigma \hookrightarrow M^m$ be a compact immersed surface with metric g. Then*

$$\mathring{B},\, |\mathring{B}|_g^2 d\mu_g$$

are invariant under conformal transformation of M, where $\mathring{B} := B - Hg$ is the ***trace-free second fundamental form****.*

Proof. Let $\tilde{g} = \rho g$ for some positive smooth function ρ on Σ. In local coordinate, we require $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ tangent to Σ and $\frac{\partial}{\partial x^3}, \ldots, \frac{\partial}{\partial x^m}$ normal to Σ. We shall use the following ranges of indices:

$$1 \le i, j, k, \ldots \le 2; \quad 3 \le \alpha, \beta, \gamma, \ldots \le m.$$

For metric $\tilde{g}$, by definition,

$$\tilde{B} = \tilde{\Gamma}_{ij}^{\alpha} dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^\alpha}$$

where $\tilde{\Gamma}_{ij}^{\alpha}$ is the Christoffel symbols. We have

$$\begin{aligned}
\tilde{\Gamma}_{ij}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha k}\Big(\frac{\partial}{\partial x^i}\tilde{g}_{kj} + \frac{\partial}{\partial x^j}\tilde{g}_{ik} - \frac{\partial}{\partial x^k}\tilde{g}_{ij}\Big) \\
&= \frac{1}{2}g^{\alpha k}\Big(\frac{\partial}{\partial x^i}g_{kj} + \frac{\partial}{\partial x^j}g_{ik} - \frac{\partial}{\partial x^k}g_{ij}\Big) \\
&\quad + \frac{1}{2}\rho^{-1}g^{\alpha k}\Big(\frac{\partial \rho}{\partial x^i}g_{kj} + \frac{\partial \rho}{\partial x^j}g_{ik} - \frac{\partial \rho}{\partial x^k}g_{ij}\Big) \\
&= \Gamma_{ij}^{\alpha} - \frac{1}{2}g^{\alpha k}\frac{\partial \log \rho}{\partial x^k}g_{ij}.
\end{aligned}$$

Then

$$\tilde{B} = B - \frac{1}{2}(\nabla \log \rho)^N g.$$

It follows that

$$\begin{aligned}
\tilde{H} &= \frac{1}{2}\mathrm{Tr}_{\tilde{g}}\tilde{B} \\
&= \frac{1}{2}\rho^{-1}\mathrm{Tr}_g(B - \frac{1}{2}(\nabla \log \rho)^N g) \\
&= \rho^{-1}(H - \frac{1}{2}(\nabla \log \rho)^N).
\end{aligned}$$

Therefore,

$$\tilde{B} - \tilde{H}\tilde{g} = (B - \frac{1}{2}(\nabla \log \rho)^N g) - \rho^{-1}(H - \frac{1}{2}(\nabla \log \rho)^N)\rho g = B - Hg.$$

Together with (2.2), we get

$$|\tilde{B} - \tilde{H}\tilde{g}|_{\tilde{g}}^2 d\mu_{\tilde{g}} = |B - Hg|_g^2 d\mu_g.$$

□

Proposition 3.2.4 ([LY] proposition 1)**.** *Let Σ be a compact minimal surface with metric g of $\mathbb{S}^n$ given by the isometric immersion $\phi : \Sigma \longrightarrow \mathbb{S}^n$. Then*

$$A_c(n, \phi) = A(\Sigma).$$

Proof. Let π be a stereographic projection into $\mathbb{R}^n$. Then composition $\pi \circ \phi$ is a conformal mapping of Σ into $\mathbb{R}^n$. $\hat{B}$, $\hat{H}$ and $\hat{g}$ denote the second fundamental form, mean curvature vector and metric of $\pi \circ \phi(\Sigma)$ in $\mathbb{R}^n$. On the other hand, given any $\sigma \in Conf(\mathbb{S}^n)$, composition $\sigma \circ \phi$ is a conformal mapping of Σ into $\mathbb{S}^n$. $\overline{B}$, $\overline{H}$ and $\overline{g}$ denote the second fundamental form, mean curvature vector and metric of $\sigma \circ \phi(\Sigma)$ in $\mathbb{S}^n$. By lemma 3.2.3,

$$\int_\Sigma |\hat{B} - \hat{H}\hat{g}|^2 d\mu_{\hat{g}} = \int_\Sigma |\overline{B} - \overline{H}\overline{g}| d\mu_{\overline{g}}.$$

Since $|\hat{B} - \hat{H}\hat{g}|^2 = |\hat{B}|^2 - 2|\hat{H}|^2$ and $\sigma \circ \phi(\Sigma)$ is similar,

$$\int_\Sigma (|\hat{B}|^2 - 2|\hat{H}|^2) d\mu_{\hat{g}} = \int_\Sigma (|\overline{B}|^2 - 2|\overline{H}|^2) d\mu_{\overline{g}}. \tag{3.12}$$

By Gauss equation, for the Gauss curvature $\hat{K}$ of $\pi \circ \phi(\Sigma)$ we have

$$\hat{K} = \frac{1}{2}(4|\hat{H}|^2 - |\hat{B}|^2).$$

Also, for the Gauss curvature $\overline{K}$ of $\sigma \circ \phi(\Sigma)$ we have

$$\overline{K} = 1 + \frac{1}{2}(4|\overline{H}|^2 - |\overline{B}|^2).$$

Then (3.12) becomes

$$\int_\Sigma (2|\hat{H}|^2 - 2\hat{K}) d\mu_{\hat{g}} = \int_\Sigma (2|\overline{H}|^2 - 2\overline{K} + 2) d\mu_{\overline{g}}.$$

It follows Gauss-Bonnet theorem that

$$\int_{\Sigma} |\hat{H}|^2 d\mu_{\hat{g}} = \int_{\Sigma} |\overline{H}|^2 d\mu_{\overline{g}} + A(\sigma \circ \phi(\Sigma)). \tag{3.13}$$

Note that the left hand side is independent of σ, hence it is invariant under $\mathrm{C}onf(\mathbb{S}^n)$. From the assumption that ϕ is minimal,

$$\begin{aligned} A(\phi(\Sigma)) &= \int_{\Sigma} |\hat{H}|^2 d\mu_{\hat{g}} \\ &= \int_{\Sigma} |\overline{H}|^2 d\mu_{\overline{g}} + A(\sigma \circ \phi(\Sigma)) \\ &\geq A(\sigma \circ \phi(\Sigma)). \end{aligned}$$

This implies

$$A(\Sigma) = A(\phi(\Sigma)) = A_c(n, \phi).$$

□

Corollary 3.2.5. *Let $\phi : (\Sigma, g) \to \mathbb{S}^n$ be an isometric immersion with mean curvature vectors $\overline{H}$ in $\mathbb{S}^n$ and H in $\mathbb{R}^{n+1}$. Then the* ***Willmore energy***

$$\int_{\Sigma} |H|^2 d\mu_g = \int_{\Sigma} |\overline{H}|^2 d\mu_{\overline{g}} + A(\Sigma) \tag{3.14}$$

is invariant under $\mathrm{C}onf(\mathbb{S}^n)$.

Proof. Let $E_1, E_2, \nu_3, \ldots, \nu_n, \nu_{n+1}$ be an orthonormal basis of $T_p\mathbb{R}^{n+1}$ and extended to vector fields around p so that $E_1, E_2 \in T\Sigma$ and $\nu_{n+1} \in N\mathbb{S}^n$. Denote $h_{ij}^{\alpha} := \langle B(E_i, E_j), \nu_\alpha \rangle$, $1 \leq i, j \leq 2$, $3 \leq \alpha \leq n+1$. The mean curvature is

$$H(p) = \sum_{\alpha=3}^{n+1} H^{\alpha}(p)\nu_\alpha(p) := \sum_{\alpha=3}^{n+1} \frac{1}{2} \sum_{i=1}^{2} h_{ii}^{\alpha}(p)\nu_\alpha(p).$$

Since $\nu_{n+1}(p) = \mathbf{x}(p)$, for any i = 1,2,

$$h_{ii}^{n+1}(p) = \langle \nabla_{E_i} E_i, \nu_{n+1} \rangle (p) = -\langle E_i, \nabla_{E_i} \nu_{n+1} \rangle (p) = -\langle E_i, E_i \rangle (p) = -1.$$

Then

$$H(p) = \overline{H}(p) - \nu_{n+1}(p). \tag{3.15}$$

Hence,

$$|H|^2 = \sum_{\alpha=3}^{n} (H^\alpha)^2 \nu_\alpha + 1 = |\overline{H}|^2 + 1.$$

From (3.13), we know that (3.14) is invariant under $Conf(\mathbb{S}^n)$. $\square$

3.3 λ_1-minimal surfaces

When the equality in Li-Yau's theorem holds, we find a specific class of surfaces .

Definition 3.3.1. *Let $\phi : (\Sigma, g) \longrightarrow \mathbb{R}^n$ be an isometric immersion with coordinate functions consisting of eigenfunctions associated with $\lambda_1(g)$, then g is said to be λ_1-**minimal** and ϕ is said to be λ_1-**minimal with respect to** g. We may also define λ_k-**minimal** in a similar way.*

Remark 3. Takahashi's theorem 3.1.4 asserts that if ϕ is λ_1-minimal, then $\phi : \Sigma \to \mathbb{S}^n(\sqrt{\frac{2}{\lambda_1}})$. Besides, if we assume $\phi : \Sigma \to \mathbb{S}^n$, then $\lambda_1 = 2$.

Example 4. *The minimal immersions (3.1), (3.2) and (3.3) are all λ_1-minimal.*

This class of surfaces has a rigidity theorem as follows:

Theorem 3.3.2 ([MR] Theorem 1)**.** *For any metric g on a compact surface Σ, there exists at most one metric $g' \in [g]$ which admits a λ_1-minimal isometric immersion into the unit sphere, i.e. g is λ_1-minimal with $\lambda_1(g) = 2$. In particular, any conformal automorphism of a non-spherical λ_1-minimal surface is a homothety.*

At first, we need to compute the differential map of $\Psi_a|_{\mathbb{S}^n}$. For every $x \in \mathbb{S}^n$ and $v \in T_x\mathbb{S}^n$,

$$\mathrm{d}\Psi_a|_x(v) = \frac{v + (\mu\big\langle a, v\big\rangle)a}{\lambda(\big\langle a, x\big\rangle + 1)} - \frac{\big\langle a, v\big\rangle}{\big\langle a, x\big\rangle + 1}\Psi_a(x).$$

It follows that

$$\Psi_a^*(\delta_{ij}dx^i dx^j) = \frac{1 - |a|^2}{(\big\langle a, x\big\rangle + 1)^2}\delta_{ij}dx^i dx^j.$$

If Σ is a compact surface with metric g, for any fixed branched conformal immersion $\phi : \Sigma \to \mathbb{S}^n$, we can consider the area function $A_\phi : Conf(\mathbb{S}^n) \to \mathbb{R}$ which maps a conformal automorphism σ of $\mathbb{S}^n$ to the area induced by the immersion $\sigma \circ \phi$. For any $a \in B^{n+1}$, we obtain

$$A_\phi(\Psi_a) = \frac{1}{2}\int_\Sigma \frac{1 - |a|^2}{(\big\langle a, x\big\rangle + 1)^2}|\nabla_g \phi|^2 d\mu_g. \tag{3.16}$$

In particular, if ϕ is an isometry, then

$$A_\phi(\Psi_a) = \int_\Sigma \frac{1 - |a|^2}{(\big\langle a, x\big\rangle + 1)^2} d\mu_g. \tag{3.17}$$

Let $\phi : (\Sigma, g) \to \mathbb{S}^n$ be an isometric immersion with mean curvature vectors $\overline{H}$ in $\mathbb{S}^n$ and H in $\mathbb{R}^{n+1}$. By proposition 3.1.1 and (3.15), we have $\Delta\phi = 2H = -2\phi + 2\overline{H}$. Given any $a \in \mathbf{B}^{n+1}$, we define $f : \Sigma \to \mathbb{R}$ by $f = \big\langle a, \phi\big\rangle + 1$. By direct computation,

$$\Delta \log f = f^{-2}\{-2\langle a, \phi\rangle^2 - 2\big\langle a, \phi\big\rangle + 2f\langle a, \overline{H}\rangle - |a^T|^2\}$$

where we decompose $T_\phi\mathbb{R}^{n+1} = T_\phi\phi(\Sigma) \oplus N_\phi(\phi(\Sigma), \mathbb{S}^n) \oplus N_\phi(\mathbb{S}^n, \mathbb{R}^{n+1})$ with $a = a^T + a^N + \big\langle a, \phi\big\rangle\phi$. Then $|a^T|^2 = |a|^2 - |a^N|^2 - \langle a, \phi\rangle^2$. Hence,

$$\Delta \log f = -1 + \frac{1 - |a|^2}{(\big\langle a, \phi\big\rangle + 1)^2} + \frac{2(\big\langle a, \phi\big\rangle + 1)\langle a, \overline{H}\rangle + |a^N|^2}{(\big\langle a, \phi\big\rangle + 1)^2}.$$

Integrating over Σ and applying (3.17), we get

$$A(\Sigma) = A_\phi(\Psi_a) + \int_\Sigma \frac{2(\langle a, \phi\rangle + 1)\langle a, \overline{H}\rangle + |a^N|^2}{(\langle a, \phi\rangle + 1)^2} d\mu_g. \tag{3.18}$$

We quote a rigidity theorem due to Obata here.

Theorem 3.3.3 ([O] theorem A). *For a connected complete Riemannian manifold (M^m, g), $m \geq 2$, it is isometric to the standard unit sphere if and only if there is a non-constant function f such that*

$$\nabla^2 f = -fg$$

where $\nabla^2 f$ denotes the Hessian of f.

Let Σ be a compact surface. If g_j is a metric on Σ, then Δ_j and $d\mu_j$ denote the Laplacian and the volume form corresponding to g_j.

Proof theorem 3.3.2. Suppose that there exist two λ_1-minimal metrics $g_1, g_2 \in [g]$ with $\lambda_1(g_1) = \lambda_1(g_2) = 2$. Let $\phi_j : \Sigma \to \mathbb{S}^{n_j}$, $j = 1, 2$, be the corresponding λ_1-minimal immersions. We also assume that these immersions are **full**, i.e. $\phi_j(\Sigma)$ does not lie on any hyperplane of $\mathbb{R}^{n_j+1}$ for each j. Set $g_2 = \rho_2 g_1$ for some positive function ρ.

By Hersch's trick, there is $a \in \mathbf{B}^{n_2+1}$ such that the conformal immersion $\phi_3 = \Psi_a \circ \phi_2 : \Sigma \to \mathbb{S}^{n_2}$ satisfies

$$\int_M \phi_3 \, d\mu_1 = \mathbf{0}.$$

We denote by g_3 the induced metric on Σ from the immersion ϕ_3 and set $g_3 = \rho_3 g_1$.

Applying max-min principle on $X_i \circ \phi_3$, for $i = 1, \ldots, n_2 + 1$, and from conformal invariance of the Dirichlet integrals, we get

$$\begin{aligned} \lambda_1(g_1)A(g_1) &= \lambda_1(g_1) \int_\Sigma |\phi_3|^2 d\mu_1 \\ &\leq \int_\Sigma |\nabla_{g_1}\phi_3|^2 d\mu_1 = \int_\Sigma |\nabla_{g_3}\phi_3|^2 d\mu_3. \end{aligned} \tag{3.19}$$

By lemma 2.1.2, $|\nabla_{g_3}\phi_3|^2 = 2$ in the last integral. Since Willmore energy (3.14) is invariant under conformal automorphisms of the sphere, we have

$$\int_\Sigma |\nabla_{g_3}\phi_3|^2 d\mu_3 = 2A(g_3) \le 2\int_\Sigma |H_3|^2 d\mu_3 = 2\int_\Sigma |H_2|^2 d\mu_2. \tag{3.20}$$

where H_j is the mean curvature vector in $\mathbb{R}^{n_2+1}$ associated with ϕ_j. By proposition 3.1.1, $2H_2 = \Delta\phi_2 = -\lambda_1(g_2)\phi_2 = -2\phi_2$. Hence,

$$2\int_\Sigma |H_2|^2 d\mu_2 = 2\int_\Sigma |\phi_2|^2 d\mu_2 = \lambda_1(g_2)A(g_2). \tag{3.21}$$

From (3.19), (3.20) and (3.21), we have $\lambda_1(g_1)A(g_1) \le \lambda_1(g_2)A(g_2)$. We can exchange the role of g_1 and g_2, then we obtain the equality by the same argument. Therefore, the inequalities in (3.19) and (3.20) become equality. The equality in (3.19) implies that $X_i \circ \phi_3$'s are eigenfunctions of Δ_{g_1} associated with the eigenvalue $\lambda_1(g_1)$, that is,

$$\Delta_1\phi_3 = -\lambda_1(g_1)\phi_3. \tag{3.22}$$

On the other hand, the equality in (3.20) implies that ϕ_3 is a minimal immersion into $\mathbb{S}^{n_2}$, by proposition 3.1.3,

$$\Delta_3\phi_3 = -2\phi_3. \tag{3.23}$$

It follows (2.1), (3.22) and (3.23) that

$$\rho_3^{-1}\Delta_1\phi_3 = \Delta_3\phi_3 = -2\phi_3 = \Delta_1\phi_3.$$

Then $\rho_3 \equiv 1$ and hence $g_3 = g_1$. Moreover, $\phi_3 : \Sigma \to \mathbb{S}^{n_2}$ is also λ_1-minimal with respect to g_1.

Now we have two λ_1-minimal immersions ϕ_2 and ϕ_3 such that $\phi_3 = \Psi_a \circ \phi_2$ for some

$a \in \mathbf{B}^{n+1}$. Since $A(g_3) = A(g_1) = A(g_2)$, from (3.18) we conclude that

$$0 = \int_\Sigma \frac{|a^{N_2}|^2}{(\langle a, \phi_2 \rangle + 1)^2} d\mu_{g_2}$$

where $a^{N_2} \in N_{\phi_2}(\Sigma, \mathbb{S}^{n_2})$. Therefore, $a^{N_2} \equiv \mathbf{0}$. Let the function $f : \Sigma \to \mathbb{R}$ defined by $f = \langle a, \phi_2 \rangle$. In local coordinate with ranges of indices as before, the Hessian of f with respect to g_2 is

$$\begin{aligned}
\nabla^2 f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}) &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\langle a, \phi_2 \rangle - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\langle a, \phi_2 \rangle \\
&= \langle a, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \rangle - \langle a, \Gamma_{ij}^k \frac{\partial}{\partial x^k} \rangle \\
&= \langle a, \Gamma_{ij}^{n_2+1} \phi_2 \rangle,
\end{aligned}$$

since $a^{N_2} \equiv \mathbf{0}$. Observe that

$$\begin{aligned}
\Gamma_{ij}^{n_2+1} &= \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \phi_2 \rangle \\
&= \frac{\partial}{\partial x^i}\langle \frac{\partial}{\partial x^j}, \phi_2 \rangle - \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \\
&= -(g_2)_{ij}.
\end{aligned}$$

Hence, $\nabla^2 f = -f g_2$.

There are only two cases as follows.

(i) $f \equiv 0$. Because ϕ_2 are supposed full, $a = \mathbf{0}$. Then $\phi_2 = \phi_3$ and hence $g_2 = g_1$.

(ii) $f \not\equiv 0$. By Obata's theorem 3.3.3, (Σ, g_2) is the unit 2-sphere with standard metric and $\phi_2 : \Sigma \to \mathbb{S}^2$ is the identity map. Since the metric $g_1 = g_3$ is induced by the diffeomorphism $\phi_3 = \Psi_a \circ \phi_2 : \Sigma \to \mathbb{S}^2$, (Σ, g_1) is also the standard 2-sphere. However, in this case, it may occur that $\rho_2 \not\equiv 1$.

$\square$

Theorem 3.3.4 ([LY] corollary 5 and [MR] Theorem 2)**.** *For any metric g on $\mathbb{RP}^2$, we* *have*

$$\lambda_1(g)A(g) \leq 12\pi.$$

Equality holds if and only if g is the induced metric given by Veronese embedding (3.1), up to homothety.

Proof. By uniformization theorem, $\mathbb{RP}^2$ has only one conformal class. Observe that Veronese immersion (3.1) is λ_1-minimal. From theorem 3.2.2 and proposition 3.2.4, we have the inequality above. Finally, Montiel-Ros' theorem 3.3.2 ensures the sufficient direction. $\square$

Chapter 4

Sharp upper bound on $\mathbb{T}^2$

In previous chapters, uniformization theorem ensures that $\mathbb{S}^2$ and $\mathbb{RP}^2$ both have only one conformal class, so it is easier to deal with these surfaces. However, $\mathbb{T}^2$ has infinitely many conformal classes. We need to develop a theory to deal with torus and an intuitive way is *variational method*. In section 4.1, we will introduce a general variational method of eigenvalue functionals λ_k from work of El Soufi and Ilias [EI2]. In section 4.2, we define critical points of eigenvalue functionals and study the structure of critical points on $\mathbb{T}^2$. In section 4.3, we assume the existence and regularity theorem of maximum metric on $\mathbb{T}^2$ holds, then prove the sharp inequality of first eigenvalues on $\mathbb{T}^2$.

4.1 Variational method on eigenvalue problem

Let M^m be a closed smooth manifold of dimension $m \geq 2$ with Riemannian metric g. For any $k \in \mathbb{N}$, we denote by $E_k(g) := Ker(\Delta_g + \lambda_k(g)I)$ the **eigenspace** corresponding to $\lambda_k(g)$ and by $\Pi_k : L^2(M, g) \to E_k(g)$ the **orthogonal projection** on $E_k(g)$. We will denote by δ the **variational derivative** at $t = 0$ and by δ^-, δ^+ **left** and **right** variational variational derivatives at $t = 0$, respectively.

Let k be any positive integer. It is natural to find the critical points for functional λ_k. Although the functional λ_k is continuous ([BU] theorem 2.2), it is not differentiable in general due to multiplicity. We will explore this phenomenon and investigate a proper way to define critical points of λ_k.

Proposition 4.1.1 (Basic formulas)**.** *Let $g \in \mathcal{M}(M)$. Let $\{g_t\} \subset \mathcal{M}(M)$ be a differentiable deformation of g for $t \in (-\varepsilon, \varepsilon)$ such that $g_0 = g$. Let $h := \delta g$.*

(1) $\delta \log(\det A_{ij}(t)) = A^{ij}(0)\delta A_{ij}(t)$, where $A(t)$ is a differentiable family of invertible matrices.

(2) $\delta d\mu_{g_t} = \frac{1}{2}\mathrm{Tr}_g(h)d\mu_g = \left\langle h, \frac{1}{2}g\right\rangle_g$.

(3) $\delta g_t^{ij} = -h^{ij}$.

(4) $\delta V(g_t) = \int_M \frac{1}{2}\mathrm{Tr}_g(h)\, d\mu_g = \int_M \left\langle h, \frac{1}{2}g\right\rangle_g d\mu_g$

(5) Denote $\Delta' := \delta\Delta_{g_t}$. Then, in local coordinate,

$$\begin{aligned}\Delta' &= -\frac{1}{2}\mathrm{Tr}_g(h)\Delta_g - \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}(h^{ij} - \frac{1}{2}\mathrm{Tr}_g(h)g^{ij})\frac{\partial}{\partial x^j}\right)\\ &= -h^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \mathrm{div}(h) + \frac{1}{2}\nabla\mathrm{Tr}_g(h)\end{aligned}$$

Theorem 4.1.2 ([EI2] theorem 2.1)**.** *Let $g \in \mathcal{M}(M)$ and let $\{g_t\}_{t\in(-\varepsilon,\varepsilon)}$ be an analytic deformation of g with $g_0 = g$. Then*

(1) The derivatives $\delta^-\lambda_k(g_t)$ and $\delta^+\lambda_k(g_t)$ exist and are eigenvalues of the operator $P_{k,h} := \Pi_k \circ \Delta' : E_k(g) \to E_k(g)$.

(2) If $\lambda_k(g) > \lambda_{k-1}(g)$, then $\delta^-\lambda_k(g_t)$ and $\delta^+\lambda_k(g_t)$ are the greatest and the least eigenvalues of $P_{k,h}$ on $E_k(g)$, respectively.

(3) If $\lambda_k(g) < \lambda_{k-1}(g)$, then $\delta^-\lambda_k(g_t)$ and $\delta^+\lambda_k(g_t)$ are the least and the greatest eigenvalues of $P_{k,h}$ on $E_k(g)$, respectively.

Proof. (1) Denote $\Delta_t := \Delta_{g_t}$ and $d\mu_t := d\mu_{g_t}$. Observe that, for any t, Δ_t is self-adjoint with respect to the L^2 inner product induced by g_t but not necessarily to that induced by g. Consider the unitary isomorphism $U_t : L^2(M, g) \to L^2(M, g_t)$ defined by $U_t u = (\frac{|g|}{|g_t|})^{1/4}u$ for any $u \in L^2(M, g)$. After conjugation by U_t, we obtain an analytic family $P_t := U_t^{-1} \circ \Delta_t \circ U_t$ of operators. It is easy to see that, for all $t \in (-\varepsilon, \varepsilon)$, P_t is self-adjoint with respect to the L^2 inner product induced by g. Let $n = \mathrm{dim}E_k(g)$.

Applying Rellich-Kato perturbation theory of unbounded operator (see [Ka] chapter 7) to P_t, for any $t \in (-\varepsilon, \varepsilon)$, there exist n eigenvalues $\Lambda_1(t), \ldots, \Lambda_n(t)$ of P_t associated with an $L^2(M, g)$-orthonormal family of eigenfunctions $v_1(t), \ldots, v_n(t)$ of P_t so that $\Lambda_1(0) = \cdots = \Lambda_n(0) = \lambda_k(g)$. Moreover, for all $1 \leq i \leq n$, both $\Lambda_i(t)$ and $v_i(t)$ depend analytically on t. Let $u_i(t) := U_t v_i(t)$ for each $t \in (-\varepsilon, \varepsilon)$ and $1 \leq i \leq n$. We have

$$\Delta_t u_i(t) + \Lambda_i(t) u_i(t) = 0, \tag{4.1}$$

and the family $\{u_1(t), \ldots, u_n(t)\}$ is orthonormal in $L^2(M, g_t)$. Since $\Lambda_i(t)$ is analytic with $\Lambda_i(0) = \lambda_k(g)$ and $\lambda_k(g_t)$ is continuous, by shrinking ε, there are two integers $1 \leq p, q \leq n$ such that

$$\lambda_k(g_t) = \begin{cases} \Lambda_p(t) & \text{for } t \in (-\varepsilon, 0) \\ \Lambda_q(t) & \text{for } t \in (0, \varepsilon). \end{cases}$$

Therefore,

$$\delta^- \lambda_k(g_t) = \Lambda_p'(0)$$

and

$$\delta^+ \lambda_k(g_t) = \Lambda_q'(0).$$

Differentiating both sides of (4.1) at $t = 0$, we get

$$\Delta' u_i + \Delta u_i' + \Lambda_i'(0) u_i + \lambda_k(g) u_i' = 0$$

where $u_i' := \delta u_i(t)$ and $u_i := u_i(0)$. Multiplying by u_j and integrating with $d\mu_g$ over M, we have

$$
\begin{aligned}
0 &= \int_M u_j \Delta' u_i d\mu_g + \int_M u_j \Delta u_i' d\mu_g + \Lambda_i'(0) \int_M u_j u_i d\mu_g + \int_M \lambda_k(g) u_j u_i' d\mu_g \\
&= \int_M u_j \Delta' u_i d\mu_g + \Lambda_i'(0)\delta_{ij} + \int_M (\Delta u_j + \lambda_k(g) u_j) u_i' d\mu_g
\end{aligned}
$$

From (4.1),

$$\int_M u_j \Delta' u_i d\mu_g = -\Lambda_i'(0)\delta_{ij}.$$

Since $\{u_1, \ldots, u_n\}$ forms an orthonormal basis of $E_k(g)$ with respect to the L^2 inner product induced by g, we have

$$P_{k,h} u_i = -\Lambda_i'(0) u_i.$$

In particular, $\Lambda_p'(0)$ and $\Lambda_q'(0)$ are eigenvalues of $P_{k,h}$.
(2) Assume $\lambda_k(g) > \lambda_{k-1}(g)$, then $\Lambda_i(0) > \lambda_{k-1}(g)$ for all $1 \leq i \leq n$. By continuity and shrinking ε again, we have $\Lambda_i(t) > \lambda_{k-1}(g)$ for all $1 \leq i \leq n$ and $t \in (-\varepsilon, \varepsilon)$. It follows that $\Lambda_i(t) \geq \lambda_k(g_t)$ and hence $\lambda_k(g_t) = \min\{\Lambda_1(t), \ldots, \Lambda_n(t)\}$. It implies that

$$\delta^- \lambda_k(g_t) = \max\{\Lambda_1'(0), \ldots, \Lambda_n'(0)\}$$

and

$$\delta^+ \lambda_k(g_t) = \min\{\Lambda_1'(0), \ldots, \Lambda_n'(0)\}.$$

(3) The proof is similar to that of (2). $\square$

In the proof above, one can easily deduce that $P_{k,h}$ is symmetric with respect to the L^2 inner product induced by g. Now, we write down the corresponding quadratic form explicitly as follows:

Lemma 4.1.3 ([EI2] lemma 2.1)**.** *Let $\{g_t\}$ be an analytic deformation of the smooth metric*

g and let $h := \delta g_t$. Then the operator $-P_{k,h}$ is symmetric with respect to the L^2 inner product induced by g and the corresponding quadratic form on $E_k(g)$ is given by

$$Q_h(u) := -\int_M uP_{k,h}u\, d\mu_g = -\int_M \left\langle du \otimes du - \frac{1}{4}\Delta_g u^2 g, h\right\rangle_g d\mu_g, \tag{4.2}$$

for all $u \in E_k(g)$.

Proof. In the previous proof, since $\{u_1, \ldots, u_n\}$ forms an orthonormal basis of $E_k(g)$ with respect to the L^2 inner product induced by g, it suffices to prove that (4.2) holds for all $u = u_i$. Observe that

$$Q_h(u_i) = -\int_M u_i P_{k,h} u_i\, d\mu_g = -\Lambda_i'(0)\int_M u_i^2\, d\mu_g = -\Lambda_i'(0).$$

For simplicity of indices, we omit the index i in $u_i(t)$, $u_i' := \delta u_i(t)$ and $\Lambda_i(t)$. By max-min principle,

$$\int_M |\nabla_{g_t} u(t)|^2 d\mu_g - \Lambda(t)\int_M u^2(t) d\mu_g = 0.$$

Differentiating both sides at $t = 0$, in local coordinate, we have

$$\begin{aligned} 0 = \int_M (\delta g^{ij}(t))\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\, d\mu_g + 2\int_M g^{ij}\frac{\partial u'}{\partial x^i}\frac{\partial u}{\partial x^j}\, d\mu_g + \int_M g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}(\delta d\mu_t) \\ - (\delta\Lambda(t))\int_M u^2 d\mu_g - 2\Lambda(0)\int_M uu'\, d\mu_g - \Lambda(0)\int_M u^2(\delta d\mu_t). \end{aligned} \tag{4.3}$$

Collecting the terms involving u', we obtain

$$2\int_M \left\langle \nabla_g u, \nabla_g u'\right\rangle_g d\mu_g - 2\int_M \Lambda(0)uu' d\mu_g = -2\int_M (\Delta_g u + \Lambda(0)u)u' d\mu_g = 0.$$

With proposition 4.1.1, (4.3) gives

$$0 = -\int_M \left\langle du \otimes du, h\right\rangle_g d\mu_g + \int_M \left\langle \frac{1}{2}|\nabla_g u|^2 g, h\right\rangle_g d\mu_g - \Lambda'(0) - \int_M \left\langle \frac{1}{2}\Lambda(0)u^2 g, h\right\rangle_g.$$

It follows that

$$\begin{aligned}\Lambda'(0) &= -\int_M \left\langle du \otimes du - \frac{1}{2}(|\nabla_g u|^2 + u\Delta_g u)g, h\right\rangle_g d\mu_g \\ &= -\int_M \left\langle du \otimes du - \frac{1}{4}\Delta_g u^2 g, h\right\rangle_g d\mu_g.\end{aligned}$$

□

Remark 4. In [FS2], Fraser and Schoen derived the results in this section with smooth deformations of metrics.

4.2 The structure of extremal metrics of eigenvalue functionals

In this section, we will explore the relation between the analytic notion of λ_k-extremal metric and the geometric notion of λ_k-minimal metric.

As remark 1, we regard λ_k as a functional on $\mathcal{M}_1(\Sigma)$ in our maximum problem. From now on, we will restrict λ_k to $\mathcal{M}_1(\Sigma)$.

Definition 4.2.1. *$g \in \mathcal{M}_1(M)$ is called λ_k**-critical** (or λ_k**-extremal**) if for any volume-preserving analytic deformation $\{g_t\} \subset \mathcal{M}_1(\Sigma)$ of g, one has*

$$\delta^+\lambda_k(g_t) \times \delta^-\lambda_k(g_t) \leq 0;$$

this means that either

$$\lambda_k(g_t) \leq \lambda_k(g) + o(t)$$

or

$$\lambda_k(g_t) \geq \lambda_k(g) + o(t)$$

as $t \to 0$. Namely, g is a locally maximizing or locally minimizing metric of λ_k.

Theorem 4.2.2 ([EI2] theorem 3.1). *Let M be a closed smooth manifold and $g \in \mathcal{M}_1(M)$. If g is λ_k-extremal, then g is λ_k-minimal.*

Let $S^2(M,g)$ denote the space of L^2 symmetric $(0,2)$-tensor fields on M with respect to metric g.

Lemma 4.2.3 ([EI2] proposition 3.1). *Let $g \in \mathcal{M}_1(M)$. If g is λ_k-extremal, then, for all $h \in S^2(M,g)$ satisfying $\int_M \mathrm{Tr}_g h \, d\mu_g = 0$, there exists $u \in E_k(g)\backslash\{0\}$ such that $Q_h(u) = 0$.*

Proof. Let $h \in S^2(M,g)$ satisfying $\int_M \mathrm{Tr}_g h \, d\mu_g = 0$. Consider a volume-preserving analytic deformation of g defined by $g_t = (\frac{V(g)}{V(g+th)})^{2/m}(g+th)$ for small t. From definition of λ_k-extremal and theorem 4.1.2, the operator $P_{k,h}$ admits non-negative and non-positive eigenvalues on $E_k(g)$. Namely, the quadratic form Q_h admits at least one isotropic direction. $\square$

Proof of theorem 4.2.2. Let $K \subset S^2(M,g)$ be the convex hull of

$$\{du \otimes du - \frac{1}{4}\Delta_g u^2 g : u \in E_k(g)\}.$$

We claim that $g \in K$. Suppose $g \notin K$, since K is a convex cone contained in a finite-dimensional subspace, Hahn-Banach theorem guarantees the existence of $h \in S^2(M,g)$ such that

$$\int_M \left\langle g, h \right\rangle_g d\mu_g > 0$$

and

$$\int_M \left\langle du \otimes du - \frac{1}{4}\Delta_g u^2 g, h \right\rangle_g d\mu_g < 0$$

for all $u \in E_k(g)\backslash\{0\}$. Let

$$h_0 = h - \Big(\frac{1}{mV(g)} \int_M \left\langle g, h \right\rangle_g d\mu_g\Big) g.$$

Then, $\int_M \left\langle g, h_0 \right\rangle_g d\mu_g = 0$ and, for any $u \in E_k(g)\backslash\{0\}$,

$$
\begin{aligned}
Q_{h_0}(u) &= -\int_M \left\langle du \otimes du - \frac{1}{4}\Delta_g u^2 g, h_0 \right\rangle_g d\mu_g \\
&= -\int_M \left\langle du \otimes du - \frac{1}{4}\Delta_g u^2 g, h \right\rangle_g d\mu_g + \frac{1}{mV(g)}(\int_M \left\langle g, h \right\rangle_g d\mu_g)(\int_M |\nabla u|^2 d\mu_g) \\
&\quad - \frac{1}{4V(g)}(\int_M \left\langle g, h \right\rangle_g d\mu_g)(\int_M \Delta u^2 d\mu_g) \\
&> 0.
\end{aligned}
$$

This contradicts lemma 4.2.3. Therefore, $g \in K$ and there exists independent eigenfunctions $u_1, \ldots, u_n \in E_k(g)$ such that

$$
g = \sum_{i=1}^{n}(du_i \otimes du_i - \frac{1}{4}\Delta_g u_i^2 g).
$$

The trace-free part of left hand side is zero. Hence,

$$
\sum_{i=1}^{n}\Big(du_i \otimes du_i - \frac{|\nabla u_i|^2}{m} g\Big) = 0.
$$

The remaining coefficients of g is

$$
\begin{aligned}
1 &= \sum_{i=1}^{n}\frac{|\nabla u_i|^2}{m} - \frac{1}{2}\sum_{i=1}^{n}(|\nabla u_i|^2 - \lambda_k(g)u_i^2) \\
&= -\sum_{i=1}^{n}\frac{m-2}{2m}|\nabla u_i|^2 + \frac{\lambda_k(g)}{2}\sum_{i=1}^{n}u_i^2.
\end{aligned}
$$

Then

$$
\sum_{i=1}^{n}\frac{m-2}{2m}|\nabla u_i|^2 = \frac{\lambda_k(g)}{2}\sum_{i=1}^{n}u_i^2 - 1. \tag{4.4}
$$

Set $f = \sum_{i=1}^n u_i^2 - \frac{m}{\lambda_k(g)}$. From (4.4), we get

$$
\begin{aligned}
(m-2)\Delta_g f &= 2(m-2)\Big(\Big(\sum_{i=1}^n |\nabla u_i|^2\Big) - \lambda_k(g)\Big(\sum_{i=1}^n u_i^2\Big)\Big) \\
&= 4m\Big(\frac{\lambda_k(g)}{2}\Big(\sum_{i=1}^n u_i^2\Big) - 1\Big) - 2(m-2)\lambda_k(g)\Big(\sum_{i=1}^n u_i^2\Big) \\
&= 2m\lambda_k(g)\Big(\sum_{i=1}^n u_i^2\Big) - 2(m-2)\lambda_k(g)\Big(\sum_{i=1}^n u_i^2\Big) - 4m \\
&= 4\lambda_k(g) f.
\end{aligned}
$$

If $m = 2$, then $f = 0$. If $m > 2$, since the Laplacian has no negative eigenvalues, this also implies that $f = 0$. Namely, $\sum_{i=1}^n u_i^2 = \frac{m}{\lambda_k(g)}$. Put in (4.4), then we get $\sum_{i=1}^n |\nabla u_i|^2 = m$, it follows that $\phi := (u_1, \ldots, u_n) : M \to \mathbb{S}^{n-1}$ is an isometry immersion into sphere and hence λ_k-minimal. $\square$

Remark 5. (i) In [EI2], they even prove that when $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}$, λ_k-extremal and λ_k-minimal are equivalent.

(ii) Although Nadirashvili ([N1] theorem 5) prove this theorem for 2-dimensional case earlier, we use the more transparent approach by El Soufi and Ilias.

Now, our discussion goes back to surface $\mathbb{T}^2$ and eigenvalue functional λ_1. We want to classify all λ_1-extremal metrics on $\mathbb{T}^2$. At first, we recall eigenvalue problems on flat tori. (One can refer to [C] chapter 2 §2.) Let Γ be a lattice of rank n, that is, there exists n linearly independent vectors $\gamma_1, \ldots, \gamma_n$ in $\mathbb{R}^n$ such that

$$
\Gamma = \{\sum_{j=1}^n \alpha_j \gamma_j | \alpha_j \in \mathbb{Z}, j = 1, \ldots, n\}.
$$

Then $\mathbb{R}^n/\Gamma$ determines a flat torus. We usually consider the functions on flat torus to be complex-valued. The Laplacian will act on complex-valued functions by acting on their real and imaginary parts separately. One may easily check that the same eigenvalues are obtained with the same multiplicity as that of real-valued functions. It is well-known that eigenfunctions on $\mathbb{R}^n/\Gamma$ associates with the **dual lattice** Γ^* as follows, where

$$\Gamma^* := \{\zeta \in \mathbb{R}^n | \left\langle \zeta, \gamma \right\rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}.$$

Γ^* is also a lattice of rank n. For each $\zeta \in \Gamma^*$, we define a complex-valued function u_ζ on $\mathbb{R}^n$ given by

$$u_\zeta(x) = \exp(2\pi i \left\langle \zeta, x \right\rangle).$$

Then u_ζ is invariant under the action of Γ and hence well-defined on $\mathbb{R}^n/\Gamma$. Moreover, u_ζ determines an eigenfunction satisfying

$$\Delta u_\zeta + 4\pi^2 |\zeta|^2 u_\zeta = 0. \tag{4.5}$$

Theorem 4.2.4 ([EI1] theorem 2.1)**.** *Let g be a metric on $\mathbb{T}^2$ admitting a full λ_1-minimal immersion $\phi : \Sigma \to \mathbb{S}^n$. Then either:*

(i) $(\mathbb{T}^2, g)$ is isometric to the normalized Clifford torus $(\mathbb{T}^2_{sq}, 2\pi^2 g_{sq})$, $n = 3$ and ϕ equals ϕ_{cl} up to isometry of $\mathbb{S}^3$, or

(ii) $(\mathbb{T}^2, g)$ is isometric to the normalized equilateral torus $(\mathbb{T}^2_{eq}, \frac{8\pi^2}{\sqrt{3}} g_{eq})$, $n = 5$ and ϕ equals ϕ_{eq} up to isometry of $\mathbb{S}^5$.

Lemma 4.2.5 ([EI1] proposition 2.2)**.** *Let $\eta_1, \dots, \eta_N$ be N continuous functions on a domain Ω of $\mathbb{R}^m$ and assume that the N^2 functions: $2\eta_j (1 \le j \le N)$, $\eta_j + \eta_k$ and $\eta_j - \eta_k (1 \le j < k \le N)$ are non-constant and mutually distinct modulo 2π. If $\phi := (\phi_1, \dots, \phi_{n+1}) : \Omega \to \mathbb{S}^n$ such that all its components ϕ_i are in the vector space generated by $\{\cos \eta_j, \sin \eta_j\}_{j=1}^N$, then there is an isometry R of $\mathbb{S}^n$ such that*

$$R \circ \phi = (\alpha_1 e^{i\eta_{j_1}}, \dots, \alpha_r e^{i\eta_{j_r}}, 0, \dots, 0)$$

where $r \le (n+1)/2$, $j_1, \dots, j_r \in \{1, \dots, N\}$ and $\alpha_1, \dots, \alpha_r$ are positive constants satisfying $\sum_{j=1}^r \alpha_j^2 = 1$. In particular, $R(\phi(\Omega)) \subset \mathbb{S}^1(\alpha_1) \times \cdots \times \mathbb{S}^1(\alpha_r) \times \{\mathbf{0}\}$.

We skip the elementary proof of lemma.

Proof of theorem 4.2.4. It is well-known that for any smooth metric g on $\mathbb{T}^2$ there exists $(a,b) \in \mathbb{R}^2$ satisfying $0 \leq a \leq \frac{1}{2}$ and $\sqrt{1-a^2} \leq b$ such that $(\mathbb{T}^2, g)$ is homothetic to flat torus $(\mathbb{T}^2_{a,b} := \mathbb{R}^2/\Gamma(a,b), g_{ab})$ with $\Gamma(a,b) = \mathbb{Z}(1,0) \oplus \mathbb{Z}(a,b)$ ([BGM]). Montiel-Ros' theorem 3.3.2 enables us to restrict our case that metric g is flat. The assumption that g admits a λ_1-minimal isometric immersion into the unit sphere implies that $\lambda_1(g) = 2$. Since $\lambda_1(g_{ab}) = \frac{4\pi^2}{b^2}$ via calculating dual lattice and using (4.5), $(\mathbb{T}^2, g)$ is isometric to $(\mathbb{T}^2_{a,b}, \frac{2\pi^2}{b^2} g_{ab})$. Let $E_{a,b} := E_1(g_{ab})$ and $\phi : (\mathbb{T}^2_{a,b}, \frac{2\pi^2}{b^2} g_{ab}) \to \mathbb{S}^n$ be a full λ_1-minimal isometric immersion.

- If $a^2 + b^2 > 1$, then $\mathrm{dim} E_{a,b} = 2$ and there is no such ϕ.

- If $a^2 + b^2 = 1$ and $(a,b) \neq (1/2, \sqrt{3}/2)$, then $E_{a,b}$ is generated by $\cos \eta_j, \sin \eta_j$, $1 \leq j \leq 2$, with $\eta_1(x,y) = \frac{2\pi y}{b}$ and $\eta_2(x,y) = 2\pi(x - \frac{ay}{b})$. From lemma 4.2.5, we have $n = 3$ and, up to isometry of $\mathbb{S}^3$, ϕ has the form $\phi = (\alpha_1 e^{i\eta_1}, \alpha_2 e^{i\eta_2})$ with $\alpha_1, \alpha_2 > 0$ and $\alpha_1^2 + \alpha_2^2 = 1$. Since ϕ is isometric, we deduce that $a = 0$, $b = 1$ and $\alpha_1 = \alpha_2 = \sqrt{2}/2$. Hence, ϕ equals ϕ_{cl}, up to isometry of $\mathbb{S}^3$.

- If $(a,b) = (1/2, \sqrt{3}/2)$, then $E_{a,b}$ is generated by $\cos \eta_j, \sin \eta_j$, $1 \leq j \leq 3$, with $\eta_1(x,y) = \frac{4\pi y}{\sqrt{3}}$, $\eta_2(x,y) = 2\pi(x - \frac{y}{\sqrt{3}})$ and $\eta_3(x,y) = 2\pi(x + \frac{y}{\sqrt{3}})$. From lemma 4.2.5, we have $n = 5$ and, up to isometry of $\mathbb{S}^5$, ϕ has the form $\phi = (\alpha_1 e^{i\eta_1}, \alpha_2 e^{i\eta_2}, \alpha_3 e^{i\eta_3})$ with $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Since ϕ is isometric, we deduce that $\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{3}/3$. Hence, ϕ equals ϕ_{eq}, up to isometry of $\mathbb{S}^5$.

$\square$

Corollary 4.2.6 ([EI1] corollary 2.1). $(\mathbb{T}^2, g)$ *is* λ_1*-minimal if and only if it is homothetic to* $(\mathbb{T}^2_{sq}, g_{sq})$ *or* $(\mathbb{T}^2_{eq}, g_{eq})$.

4.3 Sharp upper bound of the first eigenvalue on torus

We will skip the long and difficult proof of existence and regularity theorem.

Theorem 4.3.1 (Existence and regularity theorem of maximal metric, [N1])**.** *There exists a metric g_0 on $\mathbb{T}^2$ so that $\lambda_1(g_0)A(g_0) = \sup_{g\in\mathcal{M}(\mathbb{T}^2)} \lambda_1(g)A(g)$ and g_0 is smooth.*

Remark 6. In [N1], Nadirashvili's argument relies on coordinates of torus. In a recent preprint paper [Pe], Petrides proved that on any Riemannian surface, in any given conformal class, there always exists a maximizing metric which is smooth except at a finite set of conical singularities by a more general regularity argument introduced by Fraser and Schoen [FS3] concerning the parallel theorem in Steklov eigenvalue problem. With several modifications, one can get a new proof of the theorem above.

Theorem 4.3.2 ([N1] theorem 1)**.** *For any metric g on $\mathbb{T}^2$, we have*

$$\lambda_1(g)A(g) \leq 8\pi^2/\sqrt{3}.$$

Equality holds if and only if g is the flat metric given by the equilateral torus (3.3), up to homothety.

Proof. From existence theorem 4.3.1, there is a maximizing metric g on $\mathbb{T}^2$ and it is smooth. We may normalize g such that $g \in \mathcal{M}_1(M)$. Since maximizing metric must be λ_1-extremal, theorem 4.2.2 implies that g is λ_1-minimal. The corollary 4.2.6 of classification theorem indicates that only g_{sq} and $\frac{2}{\sqrt{3}}g_{eq}$ are candidates. Note that $\lambda_1(g_{sq}) = 4\pi^2 < \frac{8\pi^2}{\sqrt{3}} = \lambda_1(\frac{2}{\sqrt{3}}g_{eq})$, it follows that $g = \frac{2}{\sqrt{3}}g_{eq}$. $\square$

Chapter 5

Open problems

- Find the sharp upper bound for λ_1 on surface with genus $\gamma \geq 2$. On surface of genus two, in [JLNP], they conjectured that $\lambda_1^*(2) = 16\pi$. Moreover, there are some numerical evidences which show that the maximizing metric may occur finitely many conical singularities.

- Find the sharp upper bound for λ_k with $k \geq 2$ on surface with genus $\gamma \geq 1$. In [N2], Nairashvili proved an inequality of λ_2 on $\mathbb{S}^2$.

- Yau [Y2] conjectured that a minimal embedded hypersurface of a Euclidean sphere endowed with the induced metric must be λ_1-minimal.

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