國立臺灣大學理學院數學系
碩士論文
Department of Mathematics
College of Science
National Taiwan University
Master Thesis

蜘蛛圖的反魔方標號
Antimagic Labeling on Spiders

黃梓彥
Tzu-Yen Huang

指導教授：張鎮華 教授
Advisor: Professor Gerard Jennhwa Chang

中華民國 104 年 7 月
July 2015

致謝

時間像疾行的列車一樣，每當我們想端詳路上的風景時，它就開走了。短短兩年的碩士生涯，感覺做了比大學還要多的事，這段時間也過得很充實，感謝一路上鼎力相助的老師、朋友及家人，讓我順利完成學業。

首先，我要感謝台大的師長們。謝謝夏俊雄、張海潮老師，在研究所入學考試時拉我一把，讓我能順利就讀，入學後也相當關照我的生活。感謝劉豐哲、陳君明老師，給我看到了數學家應有的態度與風範。感謝朱樺老師，給我機會當微積分助教，除了扎實地複習了以前沒學好的微積分外，也藉機練了膽量和板書。另外，特別感謝從大三以來教我外文的林倩君和張雯媛老師，妳們平衡了我枯燥乏味的生活。還有兩位口委葉鴻國和顏經和老師真是辛苦了，工作雖然繁忙但仍花時間看論文以及提供修改上的方向。最後要感謝指導教授張鎮華老師，在家庭和事業兩頭燒之下還要照顧我這個常常不認真的研究生；老師除了指導寫論文、給我接助教、教我各種知識外，還有數次在研究走到瓶頸時，給了突破的方向，十分感激。

然後，我想寫我的朋友們。感謝同一師門的聖華師兄、鈺傑學長、廷豐、冠慧在論文撰寫和研究方向上的建議和指點。同間研究室的有慶、冠廷、幼鈞、居謙，謝謝你們不嫌棄我安靜木訥、依然跟我聊天打鬧；雖然我常覺得吵，可是卻還是想待著！感謝B99的承志、建鑫、啟樺，不僅讓我去你們研究室聊天，還經常一起吃飯。感謝棒球隊的學長學弟，研究所還能跟大家一起打球很開心。感謝遠在美國的冠宇、祐維，讓碩一點滿學分上限的我能順利過關。另外，特別感謝中文所的好友馳萱，經常在線上跟我聊天、分享工作上的事情，大大舒緩了課業沉重壓力。最後，我要謝謝同窗六年的好友貽婷，經常給我鼓勵與協助、傾聽與分享日常間的大小事，伴我走過一次次低潮。

最後我要感謝我的家人，幫我打理好家鄉的事情、還有跟我講電話、放假時給我回去休息，讓我一個人可以安心地在台北唸書。

如同陳之藩所說的，要謝的人實在太多，怎麼也寫不完吧！那就感謝老天，讓我這一路走來都很順利，能在舒適的環境裡學習、認識新朋友、完成學業。今後，不管時間走的快或慢，我都會記得這份感動，去面對未來的每一天。

中文摘要

設圖 G 是一由 n 個點及 m 條邊組成的有限簡單圖，圖 G 的一個標號指的是在圖 G 的每一個邊標上一個 $\{1,2,\cdots,m\}$ 內的整數，且不同邊有不同標號。給定圖 G 一個標號，定義每個頂點的頂點和是這個點所有連出去的邊的標號總和，若圖 G 所有頂點的頂點和都不一樣，則稱此標號為反魔方標號；設 f 是圖 G 的一個反魔方標號，且對於任兩個度數不同的頂點 $u, v, \deg(u) < \deg(v)$，若 u 的頂點和嚴格小於 v 的頂點和，則稱 f 是圖 G 的一個強反魔方標號。另外，若圖 G 存在一個 (強) 反魔方標號，我們稱 G 是 (強) 反魔方的。

反魔方標號一詞最早是由 Hartsfield 和 Ringel 提出，在他們的著作裡不只證明幾個簡單的例子 (圈、路徑、輪子、完全圖等) 有反魔方標號，也同時提出所有不是 K_2 的連通圖都是反魔方的猜想。幾十年來，陸陸續續有人證明滿足某些條件的圖有反魔方標號，但距離此猜想完全解決仍有很大的空間。

在本篇論文中，我們將範圍限縮到蜘蛛圖 (有一個核心和至少三隻腳，每隻腳由數條邊組成)。由於這種圖已被證實具有反魔方標號，因此我們在這裡將證明一個更強的結果：所有的蜘蛛圖都有強反魔方標號。文章最後也會討論一些蜘蛛圖的變形是反魔方的。

關鍵詞：反魔方、強反魔方、標號、蜘蛛圖。

Abstract

Let G be a simple finite graph with n vertices and m edges. A labeling of G is a bijection from the set of edges to the set $\{1, 2, \cdots, m\}$ of integers. Given a labeling of G, for each vertex, its vertex sum is defined to be the sum of labels of all edges incident to it. If all vertices have distinct vertex sums, we call this labeling antimagic. Suppose f is an antimagic labeling of G, and for any two vertices u, v with $\deg(u) < \deg(v)$, if vertex sum of u is strictly less than vertex sum of v, then we say f is a strongly antimagic labeling of G. Furthermore, a graph G is said to be (strongly) antimagic if it has (a strongly) an antimagic labeling.

The concept of antimagic labeling was first introduced by Hartsfield and Ringel. In their book, they not only proved that some graphs such as cycles, paths, wheels, complete graphs etc are antimagic, but also conjectured that all connected graphs other than K_2 are antimagic. In the past years, graphs with some restriction were gradually poven to be antimagic, but this conjecture is still widely open.

In this thesis, we restrict our graphs to spiders, which is a graph with a core and at least three legs, each leg contains some edges. Since all spiders have already been proven to be antimagic, we will prove a stronger result here, that is, all spiders are strongly antimagic. In the last chapter, we will discuss whether some variation of spiders are antimagic or not.

Keywords: antimagic, strongly antimagic, labeling, spider.

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Chapter 1

Introduction

All graphs in this thesis are finite, undirected, and simple. A *labeling* of a graph G is a bijection f from $E(G)$ to the set $\{1, 2, ..., |E(G)|\}$. Given a graph G, the *vertex sum* of a vertex $v \in V(G)$ is the sum of all labels of edges incident to v. An *antimagic* labeling of G is a labeling f such that for any two distinct $x, y \in V(G)$, vertex sums of x and y are different. i.e.

$$\sum_{e \in E(x)} f(e) \neq \sum_{e \in E(y)} f(e)$$

for any $x \neq y$ in $V(G)$, where $E(x) := \{e \in E(G) |\ e \text{ is incident to } x\}$. Hartsfield and Ringel [7] first introduced the concept of antimagic labeling of graphs in 1990. They proved that some special families of graphs, such as *paths, cycles, complete graphs*, are antimagic, and put forth the following conjecture:

Conjecture 1.1. *Every connected graph other than K_2 is antimagic.*

This conjecture received a lot of attention but is still widely open. The most significant progress is a result by Alon et al. [2]. They proved that a graph G with minimum degree $\delta(G) \geq C \log V(G)$ (for an absolute constant C) or with maximum degree $\Delta(G) \geq |V(G)| - 2$ is antimagic. In the same paper, they also proved that complete partite graph other than K_2 is antimagic. In 1999, Alon [1] introduced an algebraic theorem to prove some combinatorial problems. The so called "*Combinatorial Nullstellensatz*" was used in [8, 9] to study antimagic labeling of graphs. Hefetz et al. [9] used this algebraic tool to prove that a graph which has p^k, p is an odd prime, vertices and admits a C_p-factor is antimagic. The problem seems to be a little easier if we reduce the case to bipartite graphs. In 2009, Cranston [4] proved that regular bipartite graphs are antimagic. Few years later, he put forth the result to general regular graphs of odd degree, it is a joint work with Liang and Zhu [5]. In Liang's doctoral dissertation [11], he introduced a concept to prove that regular graphs of even degree are antimagic. However, there are still some problems unsolved. But fortunately, his team completed the whole proof this year [3]. Hence antimagicness of regular graphs is wholly completed.

The following conjecture is just the restriction of Conjecture 1.1 to trees.

Conjecture 1.2. *Every tree other than K_2 is antimagic.*

The most significant progress of Conjecture 1.2 was obtained by Kaplan, Lev and Roditty [10]. They proved that a tree with at most one vertex of $\deg 2$ is antimagic. Their method is zero-sum partitions, which is a partition of integers into pairwise disjoint subsets such that elements in the same subset sum up to zero modulo n, for some natural number n. However, their proof contains an error. In 2014, Liang et al. [12] corrected this error and used a similar technique to find out that some classes of trees are antimagic.

In the study of antimagic labeling of graphs, Hefetz [8] also introduced the concept of (ω, k)-*antimagic labeling* of graphs, where ω is a weight function and k is a non-negative integer. A weight function $\omega : V(G) \mapsto \mathbb{N}$ is a function from $V(G)$ to a set of natural numbers. A (ω, k)-antimagic labeling is an injection from $E(G)$ to the set $\{1, 2, 3, ..., |E(G)| + k\}$ such that all vertex sums are pairwise distinct, where vertex sum is the sum of labels of all edges incident to that vertex and its initial weight assigned by ω. For any given ω, Wong and Zhu [14] proved that a graph which has a vertex adjacent to all the other ones is $(\omega, 2)$-antimagic, the tool they used is the Combinatorial Nullstellensatz, which was also used to prove that a connected graph on $n \geq 3$ vertices is $(\omega, \lceil \frac{3n}{2} \rceil - 2)$-antimagic for any weight function ω.

For more concept of graph labeling and open problems, see the survey by Gallian [6].

In this thesis, we want to prove a special family of graphs being antimagic. A *spider* is formed from the disjoint union of some paths by identifying one endpoint of each path, called the *core* of the spider. These paths are called *legs* of a spider. A spider is said to be *regular* if all legs have the same length. Note that a spider with at most 2 legs is a path, and a regular spider with each leg of length 1 is a star.

In Chapter 2, we will prove that regular spiders and some variations are antimagic by designing truly antimagic labelings. In Chapter 3, we first introduce the concept of strongly antimagicness, and use it to rewrite the proof of Shang [13]. In Chapter 4, we will discuss whether some variations of spiders are antimagic or not.

Chapter 2

Regular spiders are antimagic

Theorem 2.1. *Regular spiders except K_2 are antimagic.*

Proof. Let S be a spider which has n legs $r_1, r_2, \ldots, r_n$, each has length k. For each r_i, $i = 1, 2, \ldots, n$, label the edges from outside to the core by $i, n+i, 2n+i, \ldots, k(n-1)+i$. Then vertex sums of this spider are $1, 2, \ldots, n, n+2, n+4, \ldots, 3n, 3n+2, 3n+4, \ldots, (2k-1)n, \frac{(2kn-n+1)n}{2}$. Since all vertices have different vertex sums, the labeling is antimagic. ■

Suppose that G is a graph with m edges, $n+1$ vertices, and one of which is adjacent to all the other vertices, i.e. an universal vertex, then it is straightforward to see that G is antimagic. Let v be an universal vertex of G, $e_1, e_2, \cdots, e_n$ be the edges incident to v, and $v_1, v_2, \cdots, v_n$ be other endvertices. First we use $1, 2, \cdots, m-n$ to label the edges of G which aren't incident to v, and let f be a mapping from $V(G)$ to $\mathbb{N}$, which denotes the vertex sums of every vertex at this moment. Without loss of generality, we may assume that $f(v_i) \le f(v_j)$, for $i \le j$. Then for $e_1, e_2, \cdots, e_n$, give e_k the label $m-n+k$, $k = 1, 2, \cdots, n$. Therefore the vertex sums of G are $f(v_1)+m-n+1, f(v_2)+m-n+2, \cdots, f(v_n)+m$, and $\frac{n(2m-n+1)}{2}$, since $f(v_n)+m < (m-n+1)+(m-n+2)+\cdots+m = \frac{n(2m-n+1)}{2}$, G is antimagic.

By playing a similar trick we can prove the following corollary:

Corollary 2.2. *Suppose G is an n-vertices graph without isolated vertices. For each* *vertex $v \in V(G)$, if we attach a path of length k to v, then the resulted graph G' is also antimagic.*

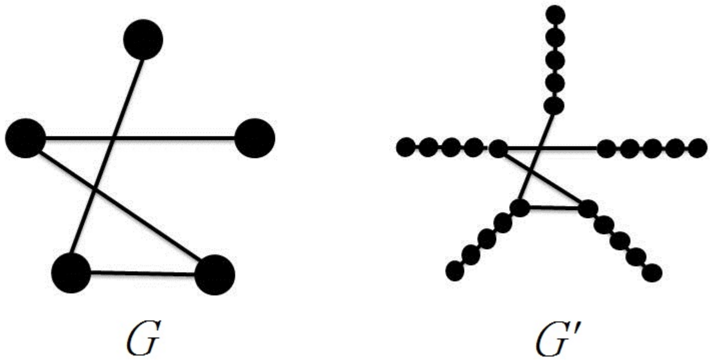


Figure 2.1: Example of G and G', $n = 5$, $k = 4$.

Proof. Suppose G' is the graph described as in the theorem. Divide G' into 2 parts: *spider part* and *core part*. Suppose furthermore that the spider part has n legs, each has length k. We have to label G' by $\{1, 2, \ldots, |E(G')|\}$. First, label the spider part with $\{1, 2, \ldots, kn\}$, by the method shown in the proof of Theorem 2.1. Then label the core part with $\{kn + 1, kn + 2, \ldots, |E(G')|\}$ arbitrarily, and denote the vertices by $v_1, v_2, \ldots, v_n$ satisfying:

For distinct $i < j$, vertex sum of v_i is less or equal to v_j.

Now attach v_i to the leg with the last edge labeled by $(k-1)n + i$. Then the vertex sums of $v_1, v_2, \ldots, v_n$ form a strictly increasing sequence. Since there's no isolated vertex in the core part, degree of $v_1, v_2, \ldots, v_n$ are all greater than 2, vertex sums of $v_1, v_2, \ldots, v_n$ are greater than those of vertices of the spider part. By Theorem 2.1, the spider part with that labeling is antimagic. Therefore, G' is antimagic. ■

Let G be a graph with no isolated vertex and S be any spider. Construct a new graph G' by attaching the core of S to each vertex of G. By a similar trick played in the proof of Corollary 2.2, we can find an antimagic labeling of G'.

Theorem 2.3. *If G, G', S are defined as above, then G' is antimagic.*

Proof. Suppose G has n vertices and m deges, and S has k legs with lengths r_1, r_2, $\cdots$, r_k, in increasing order. Divide G' into two parts: G and spider part, where the spider

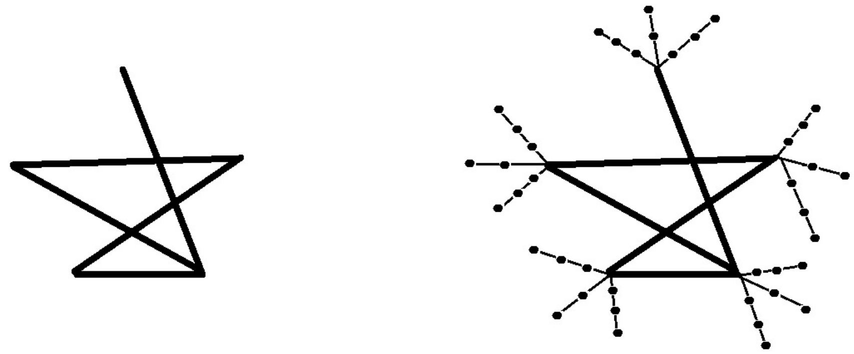


Figure 2.2: G and G'.

part contains n spiders, denoted by S_1, S_2, $\cdots$, S_n. First we label the spider part with $\{1, 2, \cdots, n|E(S)|\}$, and for each spider, label first the r_1-outermost edges of all legs with $\{1, 2, \cdots, r_1nk\}$. For S_i, label the outermost edges of all legs from the shortest to the longest by $(i-1)k+1, (i-1)k+2, \cdots, ik$, then the next edges by $nk+(i-1)k+1, nk+(i-1)k+2, \cdots, nk+2ik$, and so on. After the r_1-outermost edges are finished, ignore them and see all spiders as all their legs are cutted by r_1 edges. (i.e. Spiders with $k-1$ legs, and lengths are $r_2-r_1, r_3-r_1, \cdots, r_k-r_1$.) Repeat the same thing on the (r_2-r_1) - outermost edges with $\{r_1nk+1, r_1nk+2, \cdots, r_1nk+n(r_2-r_1)(k-1)\}$. Repeat the process until all legs are labeled.

Next label G with $\{n|E(S)|+1, n|E(S)|+2, \cdots, |E(G)|\}$ arbitrarily, and without loss of generality, we may assume that vertices of G are named by $v_1, v_2, \cdots, v_n$, where v_i has the i-th smallest vertex sum among all vertices of G. Then attach the core of S_i to v_i, a labeling of G' is constructed. We still have to check that this labeling is antimagic.

For all vertices of $V(G') - V(G)$, observe that by the above construction all vertices could be ordered so that their vertex sums are strictly monotone increasing, and have strictly smaller vertex sums than any vertex of $V(G)$, since each $v \in V(G)$ is incident to an edge which is labeled with one of the largest numbers. Finally, since S_i is attached to v_i, the all cores of $S_1, S_2, \cdots, S_n$ have mutually distinct vertex sums, hence G' is antimagic.■

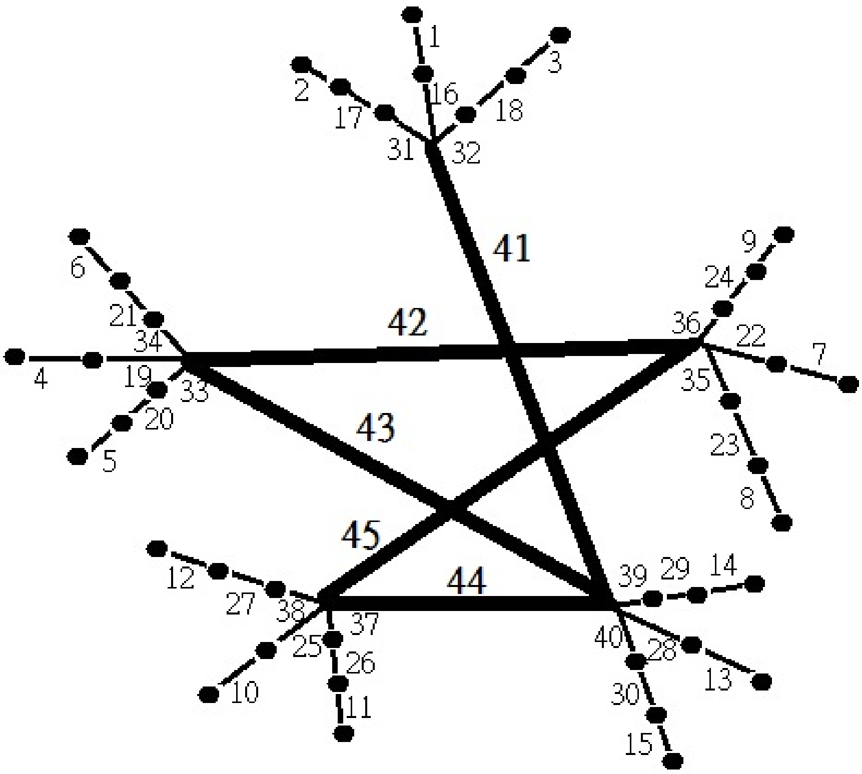


Figure 2.3: An example for theorem 2.3.

Chapter 3

General spiders are antimagic

In this chapter, we want to prove our main result, that is, all spiders except K_2 are antimagic. Actually, Shang [13] has already proven this result. Her technique is simple and direct. For a specific spider other than K_2, she assigns it a labeling, and adjusts some edge labels if necessary to make it antimagic. The labeling looks similar to the one we used for regular spiders.

Our approach is quite different. We first introduce the concept of *strongly antimagic labeling*, which is also an antimaigc labeling and for any two distinct vertices u, v with $\deg(u) < \deg(v)$, vertex sum of u is strictly less than vertex sum of v. Then we use induction hypothesis to prove that all spiders except K_2 are strongly antimagic, and hence antimagic.

With the concept of strongly antimagicness, we first rewrite the proof of theorem 2.1 as follows:

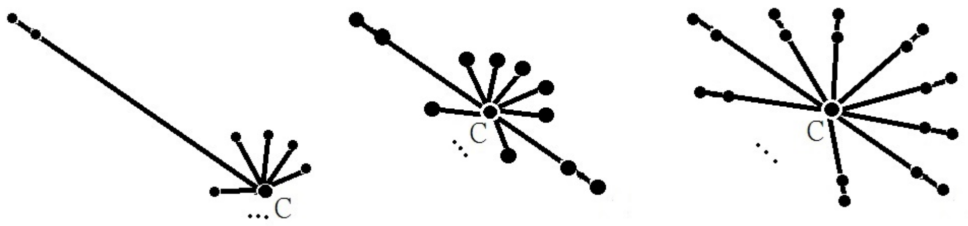
Theorem 3.1. *Regular spiders except K_2 are strongly antimagic.*

Proof. We prove the theorem by induction on the length of legs. Suppose that S is a regular spider with k legs. When all legs have length 1, the spider is a star, and it is clearly strongly antimagic. Suppose that a regular spider whose legs are all of length n has a strongly antimagic labeling. Now, for a spider G with all its legs of length $n+1$, we may first delete the outermost edges of each legs, the remaining graph is a spider which legs are all of length n. So by the induction hypothesis, it has a strongly antimagic labeling f.

Then construct a new labeling f' by adding k to the labels of each edge. By the definition of strongly antimagicness, since degrees of the outest vertices are the smallest, the vertex sums are still the smallest after adding n to all edges. Finally, since we have to recover G to a spider with length of legs $n+1$ by sticking one more edge to each legs at the outermost, we label the outermost edges by $\{1, 2, \ldots, k\}$ by the method satisfying: for $i = 1, 2, \ldots, k$, label the edge i when this edge is adjacent to a vertex whose vertex sum is the ith smallest under the labeling f'. Then the new labeling is still strongly antimagic. Hence all regular spiders are strongly antimagic. ■

Theorem 3.2. *A spider with at least two edges is strongly antimagic.*

Proof. We will prove the theorem by induction on $|E(G)|$. For all spiders except K_2, we divide all spiders into three different cases according to the number of legs with length at least 2. Case 1 consists of spiders with exactly one leg of length at least 2, Case 2 contains spiders with exactly two legs of length at least 2, and Case 3 are spiders with at least 3 legs of length greater than 2.



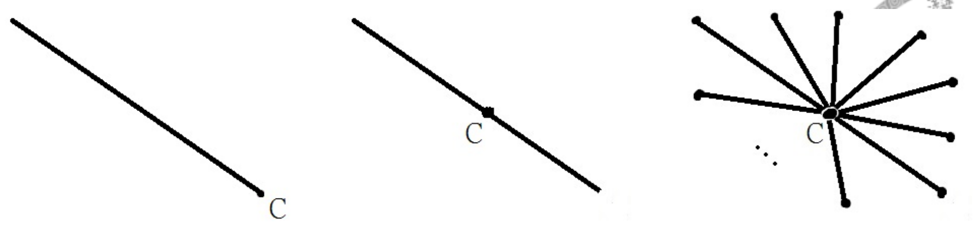
(a) Case 1 (b) Case 2 (c) Case 3

Figure 3.1: Classify all spiders due to the number of their legs of length at least 2.

This classification is based on the induction hypothesis, during which we delete each leg one edge. After deleting each leg one edge, the degrees of the core are 1, 2 ,and at least 3 respectively in the above three cases.

Now, given a spider S with k legs of length $r_1, r_2, \ldots, r_k$, $k \geq 2$. For convenience assume $r_1 \leq r_2 \leq \cdots \leq r_k$.

Construct a new spider S' by deleting each leg 1 edge, then S' has k legs with length $r'_1, r'_2, \ldots, r'_k$, where $r'_i = r_i - 1$, $i = 1, 2, \ldots, n$, note that some r'_is may be zero. There are 3 possible cases for S' as we discussed above.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 3.2: Spiders whose outermost edges are cut.

Actually, Cases 1 and 2 are primitive cases of all spiders, so they should be discussed seperately. Hence we prove Case 3 at first. By the induction hypothesis, there is a strongly antimagic Labeling L' of S' using labels $1, 2, 3, \ldots, |E(S')| = |E(S)| - k$. For each leg of S', denote the outermost edge $e'_1, e'_2, \ldots, e'_h$. Let $i_1, i_2, \ldots, i_h \in \{1, 2, \ldots, h\}$ be indices such that $L'(e'_{i_1}) < L'(e'_{i_2}) < \cdots < L'(e'_{i_h})$. Now come back to S, let $e_j, j \in \{1, 2, \ldots, h\}$ be the edges adjacent to e'_{i_j}, $j \in \{1, 2, \ldots, h\}$, and $e_{h+1}, e_{h+2}, \ldots, e_k$ be legs of S with exactly one edge. For convenience, denote the endpoints of e_i by u_i, v_i with $\deg(u_i) = 1$, $\deg v_i \geq 2$ and vertex sums of u_i, v_i by U_i, V_i. Let L be a labeling of S defined as:

$$L(e) := \begin{cases} L'(e) + k, & \text{if } e \in E(S'). \\ i, & \text{if } e = e_i \text{ , } i = 1, 2, \ldots, k. \end{cases}$$

To check that L is a strongly antimagic labeling, note that the vertex sum of the core is still the largest. For u_i, $i = 1, 2, \ldots, k$, $1 = U_1 < U_2 < \cdots < U_k = k$; for v_i, $i = 1, 2, \ldots, k$, $k+1 < V_1 < V_2 < \cdots < V_k$. For the remaining vertices, their vertex sums are also different because all vertex sums are only shifted up by $2k$ from S'. Therefore L is a strongly antimagic labeling of S. To complete the proof, we have to find strongly antimagic labelings for graphs of Cases 1 and 2.

For the Cases 1 and 2, the graphs look like a path after each leg being cutted by one edge. Although a path is obviously strongly antimagic, but we can't be sure whether the core has the largest vertex sum or not. So the above argument may fail sometimes, but we can still give them strongly antimagic labelings directly.

For Case 1, suppose the spider has k legs of length 1 and a leg of length $m, m \geq 2$. We divide m into 2 cases according to the pairity. If $m = 2n, n \in \mathbb{N}$, first label the longest leg from the core by $2n + k, n, 2n + k - 1, n - 1, \ldots, n + k + 2, 2, n + k + 1, 1$, and then label the remaining legs by $n + k, n + k - 1, \ldots, n + 1$. To check that the labeling is strongly antimagic, we compute vertex sums of all vertices and make the following table:

Degree of vertices	Corresponding vertex sums
1	$1, n+1, n+2, \ldots, n+k$
2	$n+k+2, n+k+3, \ldots, 3n+k$
$k+1$	$2n+k+\frac{k}{2}(2n+k+1)$

Table 3.1: Vertex sums of Case 1 when the longest leg is of even length.

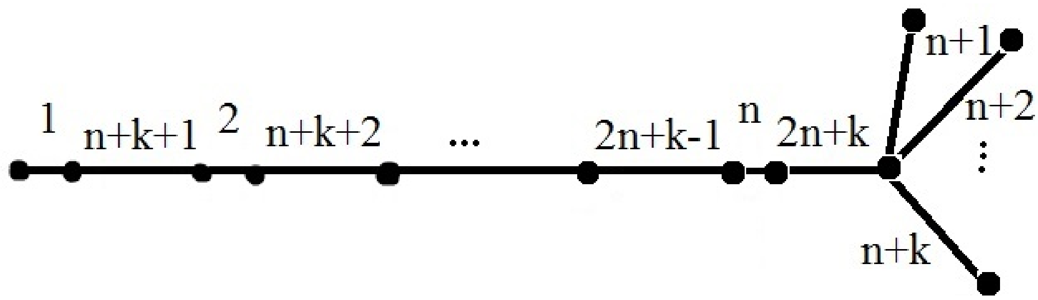


Figure 3.3: Edge-labeling of Case 1 when the lnogest leg has even length.

Observe that the labeling is antimagic and vertex with larger degree has larger vertex sums, hence it is strongly antimagic. Next if $m = 2n + 1, n \in \mathbb{N}$, we label the spider in a similar way with just a little difference. The legs of length 1 are labeled by the same numbers , but the longest leg is labeled from the core by $2n + k + 1, n, 2n + k, n - 1, \ldots, n + k + 3, 2, n + k + 2, 1, n + k + 1$. One can easily see that the labeling is also strongly antimagic.

Degree of vertices	Corresponding vertex sums
1	$n+1, n+2, \ldots, n+k+1$
2	$n+k+2, n+k+3, \ldots, 3n+k+1$
$k+1$	$2n+k+1+\frac{k}{2}(2n+k+1)$

Table 3.2: Vertex sums of Case 1 when the longest leg is of odd length.

For Case 2, suppose that a spider has k legs of length 1, and the 2 legs of length p, q, where $p, q \geq 2$.Divide p, q into 3 cases due to their pairities: both p and q are even; or they are both odd; or one of them is even and the other is odd.

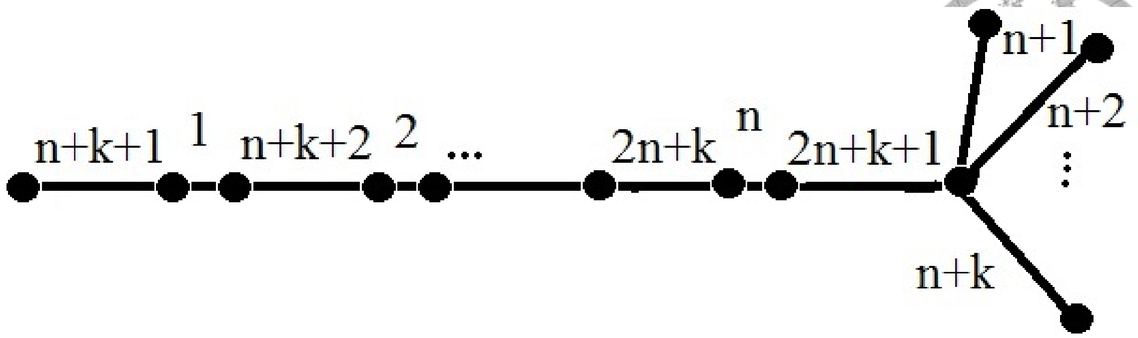


Figure 3.4: Edge-labeling of Case 1 when the lnogest leg has odd length.

First suppose $p = 2n, q = 2m$, label the leg of length $2m$ from the core by $2n + 2m + k, n + m, 2n + 2m + k - 1, n + m - 1, \ldots, 2n + m + k + 1, n + 1$, the leg of length $2n$ by $2n + m + k, n, 2n + m + k - 1, n - 1, \ldots, n + m + k + 1, 1$, and the legs of length 1 by $n + m + k, n + m + k - 1, \ldots, n + m + 1$. Then the corresponding vertex sums are shown in the following table:

Degree of vertices	Corresponding vertex sums
1	$1, n + 1, n + m + 1, n + m + 2, \ldots, n + m + k$
2	$n + m + k + 2, n + m + k + 3, \ldots, 3n + m + k, 3n + m + k + 2, 3n + m + k + 3, \ldots, 3n + 3m + k - 1, 3n + 3m + k$
$k + 2$	$3n + 4m + 2k + 2 + \frac{k}{2}(2n + 2m + k + 1)$

Table 3.3: Vertex sums of Case 2 when the two longest legs are both of even length.

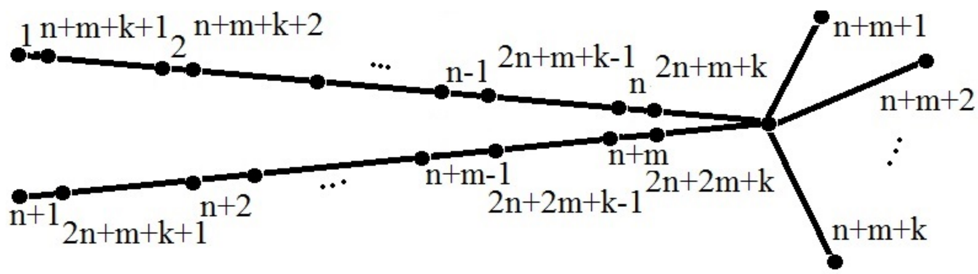


Figure 3.5: Edge-labeling of Case 2 when the two longest legs are both of even length.

Next suppose $p = 2n, q = 2m + 1$, label in a similar way but start with the leg of even length. Give it label from the core by $2n + 2m + k + 1, n + m, 2n + 2m + k, n + m - 1, \ldots, n + 2m + k + 2, m + 1$. For the leg with odd length larger than 2, label it from the core by $n + 2m + k + 1, m, n + 2m + k, m - -1, \ldots, n + m + k + 2, 1, n + m + k + 1$. The ramaing legs are all of length 1 and we assign them $n + m + k, n + m + k - 1, \ldots, n + m + 1$. It is straightforward to see that the labeling is strongly antimagic.

Degree of vertices	Corresponding vertex sums
1	$m+1, n+m+1, n+m+2, \ldots, n+m+k+1$
2	$n+m+k+2, n+m+k+3, \ldots, n+3m+k+1, n+3m+k+3, n+3m+k+4, \ldots, 3n+3m+k, 3n+3m+k+1$
$k+2$	$4n+3m+2k+\frac{k}{2}(2n+2m+k+1)$

Table 3.4: Vertex sums of Case 2 when the two longest legs have lengths of different pairities.

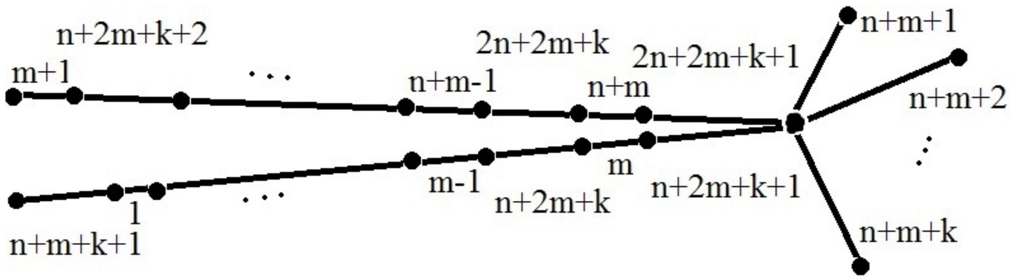


Figure 3.6: Edge-labeling of Case 2 when the two longest legs have lengths of different pairities.

Finally, suppose $p = 2n+1, q = 2m+1$, label first the leg of length $2n+1$. We label the edges of this leg from the core by $2n+2m+k+2, 2n+2m+k, n+m, 2n+2, +k-1, n+m-1, \ldots, n+2m+k+1, m+1$. For another leg, label it from the core by $2n+2m+k+1, n+2m+k, m, n+2m+k-1, m-1, \ldots, n+m+k+1, 1$. The remaining edges are still labeled by $n+m+k, n+m+k-1, \ldots, n+m+1$. Compute all vertex sums and one can find that the labeling is strongly antimagic, too. ■

Degree of vertices	Corresponding vertex sums
1	$1, m+1, n+m+1, n+m+2, \ldots, n+m+k$
2	$n+m+k+2, n+m+k+3, \ldots, n+3m+k, n+3m+k+2, n+3m+k+3, \ldots, 3n+3m+k, 3n+4m+2k+1, 4n+4m+2k+2$
$k+2$	$4n+4m+2k+3+\frac{k}{2}(2n+2m+k+1)$

Table 3.5: Vertex sums of Case 2 when the two longest legs are both of odd length.

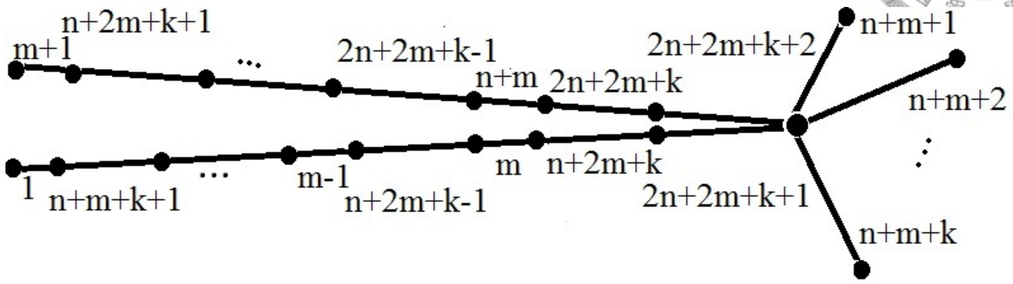


Figure 3.7: Edge-labeling of Case 2 when the two longest legs are both of odd length.

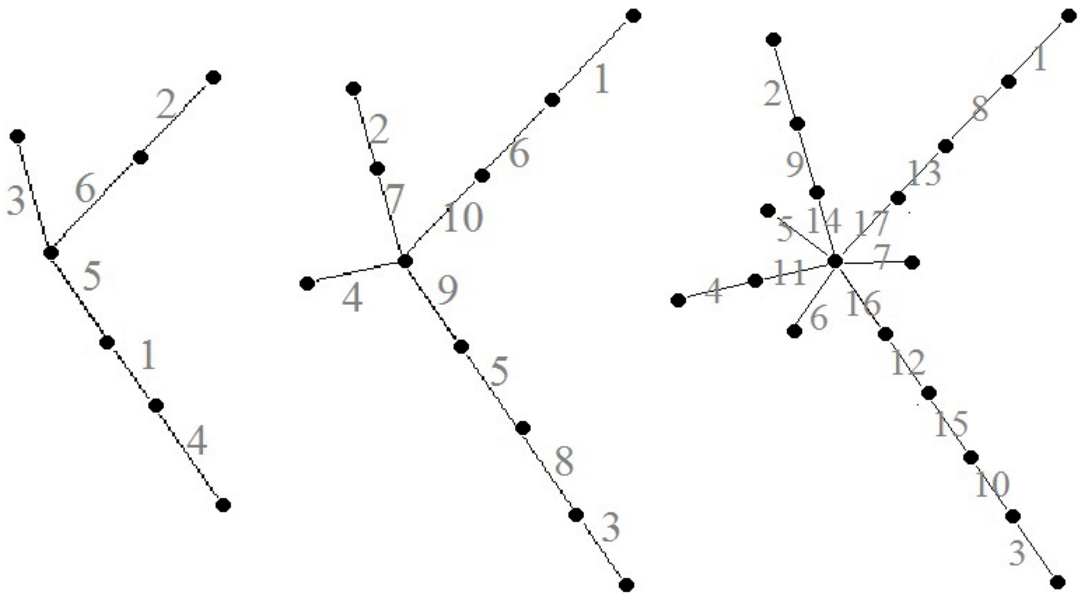


Figure 3.8: An example of strongly antimagic labeling.

Chapter 4

Some variation of spiders

In this chapter, we want to find antimagic labeling of some graphs which are similar to general spiders. The first example is a kind of graph constructed by attaching two stars to the endvertices of a path. Denote a graph by $SPS(m, n, k)$, which means this graph consists of two stars $K_{1,n}$, $K_{1,m}$ and a path P_{k+1} of k edges.

Example 4.1. *$SPS(m, n, 1)$, $m \geq n$, is antimagic.*

The proof is straightforward. First label the smaller stars (which has less edges than another) with the smallest n numbers. Then label the other star with the next m numbers, and leave the largest number to the edge that connects two stars. It is easy to see that this labeling is antimagic, because the two cores have the biggest vertex sums, and by our labeling, the cores of the two spiders also have different vertex sums.

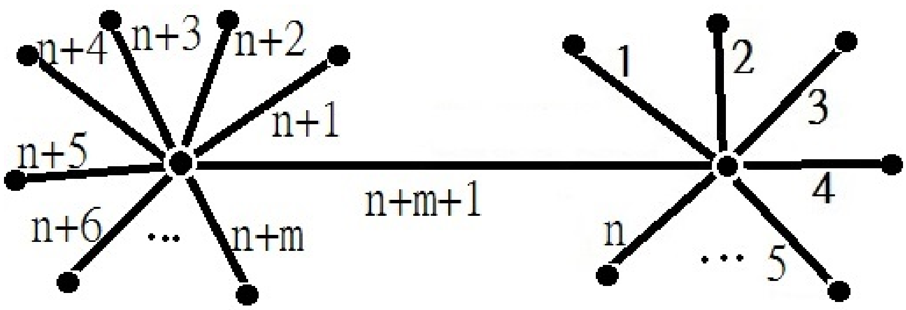


Figure 4.1: An antimagic labeling of $SPS(m, n, 1)$,with $m \geq n$.

Theorem 4.2. *Any $SPS(m, n, k)$, $m \geq n \geq 1$ and $k \geq 1$, is antimagic.*

Proof. For convenience, denote the vertices of the path, cores of the two stars by $v_0, v_1, \cdots, v_k$, with $\deg v_0 \leq \deg v_k$. First label the two stars as we did in Example 4.1 by using numbers from $\{1, 2, \cdots, m+n\}$. Next use the remaning k numbers to label the path. For the edge $v_{i-1}v_i$, $i = 1, 2, \cdots, k$, label it with $m+n+i$. The vertex sums of this labeling are: $1, 2, \cdots, n+m, 2n+2m+3, 2n+2m+5, \cdots, 2n+2m+2k-1, \frac{n(n+1)}{2}+n+m+1, \frac{m(m+1)}{2}+nm+n+m+k$. If all vertex sums are distinct, then we are done. The problem may occur only if any of the two cores has the same vertex sum as some vertex of the path. Therefore we divide all situations into the following three cases: there exists x, y with $1 < x \neq y < k-1$ such that either vertex sum of v_y = vertex sum of v_k or vertex sum of v_x = vertex sum of v_0; either vertex sum of v_{k-1} = vertex sum of v_k or vertex sum of v_1 = vertex sum of v_0; vertex sum of v_0 =vertex sum of v_1, when the path is P_3.

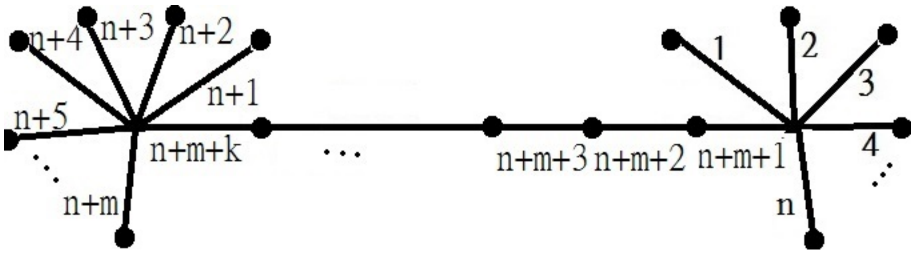


Figure 4.2: An labeling of $SPS(m,n,k)$,with $m \geq n$.

For Case 1, change the labels of v_kv_{k-1} and $v_{k-1}v_{k-2}$, then vertex sums of $v_{k-3}, v_{k-2}, v_{k-1}, v_k$ becomes $2m+2n+2k-5, 2m+2n+2k-2, 2m+2n+2k-1, 2m+2n+2y$. any other vertices still have the same vertex sums. Since the vertex sum of v_k becomes even, it is different from the one of v_y, which is $2m+2n+2y+1$, an odd number. Furthermore, if vertex sum of v_0= v_x , change the labels of $v_2v_1$and v_1v_0 ,then among all vertices, only the vertex sums of v_0, v_1, v_2, v_3 are changed, and they become $2m+2n+2x+2, 2m+2n+3, 2m+2n+4, 2m+2n+7$. Again by the same argument of v_k, we find that the new labeling is also antimagic.

For Case 2, first suppose vertex sums of v_0 and v_1 are equal (i.e. $m+n+2 = \frac{n(n+1)}{2}$). The original labels of v_1v_2, v_2v_3, are $m+n+2, m+n+3$, if we exchange the labels of the two edges mutually, then vertex sums of v_1, v_2, v_3 become $2m+2n+4, 2m+2n+5, 2m+2n+6$, and the remaining vertex sums are still the same. Since vertex sum of v_0 becomes

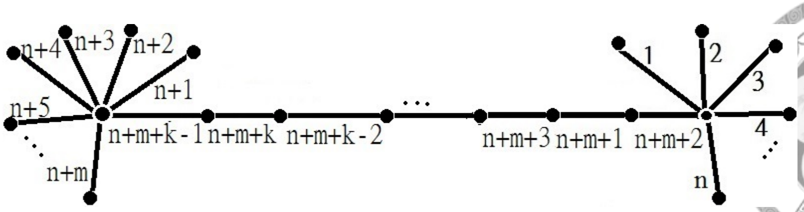


Figure 4.3: Adjustment about Case 1.

$2m+2n+3$, which has the smallest vertex sum among all vertices of the path. Furthermore, suppose vertex sums of v_{k-1} and v_k are equal (i.e. $m+n+k-1 = \frac{m(m+1)}{2} + nm$). Then, similarily, exchange mutually the labels of $v_{k-3}v_{k-2}$ and $v_{k-2}v_{k-1}$. The vertex sums of $v_{k-3}, v_{k-2}, v_{k-1}$ become $2m+2n+2k-4, 2m+2n+2k-3, 2m+2n+2k-2$, since v_k still has the largest vertex sum, so all vertices have distinct vertex sums.

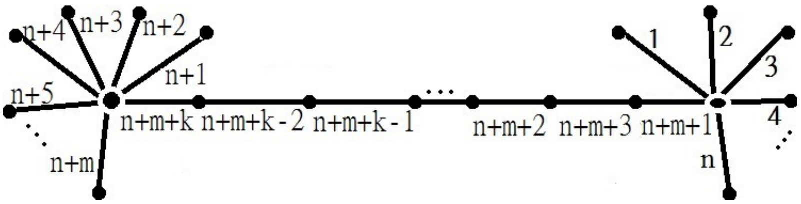


Figure 4.4: Adjustment about Case 2.

For Case 3, denote the vertices of the path by v_0, v_1, v_2, where v_0, v_2 are also the cores of the spiders and $\deg v_2 \geq \deg v_0$. If $m \geq 2$, observe that the vertex sum of v_2 is $mn + \frac{m(m+1)}{2} + m + n + 2$, which is strictly larger than the one of v_0 and $2m+2n+3$. If $m = n = 1$, it is just a path, and by our labeling it is antimagic. Therefore the problem may occur when v_0 and v_1 have the same vertex sum, that is, $m+n+2 = \frac{m(m+1)}{2}$ We could just exchange the labes of the two edges of the path, then vertex sums of this path become $\frac{n(n+1)}{2} + m + n + 2 = 2m+2n+4, 2m+2n+3, nm + \frac{m(m+1)}{2} + m + n + 1$, which are all distinct and the largest of this graph. Hence there exists an antimagic labeling. ■

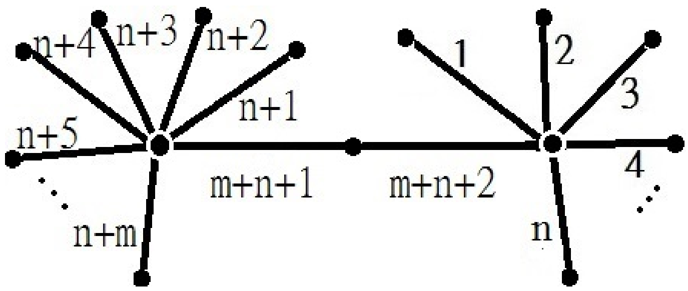


Figure 4.5: Adjustment about Case 3.

The next step we want to do is to extend the restriction of Theorem 4.2 to general spiders. In other words, the two "S" of a SPS can be replaced by general spiders. So by a $SPS(l_1, l_2, \ldots, l_{k_1}, l_{k_1+1}, \ldots, l_{k_1+k_2}; k)$, where $l_1 \leq l_2 \leq \cdots \leq l_{k_1+k_2}$, we define a graph constructed by two spiders S_1, S_2 and a path P_{k+1}, where two spiders are connected by attaching each endpoint of P_{k+1} to one core of S_1, S_2. Furthermore, $l_1, l_2, \ldots, l_{k_1}, l_{k_1+1}, \ldots, l_{k_1+k_2}$ means S_1 and S_2 have k_1+k_2 legs with length $l_1, l_2, \ldots, l_{k_1}, l_{k_1+1}, \ldots, l_{k_1+k_2}$. Without loss of generality, we suppose S_2 contains the leg with length $l_{k_1+k_2}$. And for convenience, we will follow the notation of Shang [13].

Given a graph $G := SPS(l_1, l_2, \ldots, l_{k_1}, l_{k_1+1}, \ldots, l_{k_1+k_2}; k)$, we denote the cores of S_1, S_2 by a, b. And for a leg of length l_i, $i \in \{1, 2, \ldots, k_1 + k_2\}$, let $e_{i,j}, v_{i,j}, j \in \{1, 2, \ldots, l_i\}$ denote the j-th edge and vertex of this leg from the outermost, and for the path P_{k+1}, let $e_r, r \in \{1, 2, \ldots, k\}$, denote the r-th edge from a to b, and $v_1, v_2, \ldots, v_{k-1}$ denote the vertices of this path from a to b except a, b.

Suppose G has m edges, we now design a partial labeling of G as follows. For the edges of S_1, S_2, define an order among them as follows. We say $e_{i,j} \prec e_{i',j'}$ if and only if $j < j'$, or $j = j'$ and $i < i'$. It is easy to see that $\prec$ is a linear order among these edges. In fact, $e_{1,1} \prec e_{2,1} \prec \cdots \prec e_{k_1+k_2,1} \prec e_{1,2} \prec e_{2,2} \prec \cdots \prec e_{k_1+k_2,l_{k_1+k_2}}$. Then label $e_{i,j}$ with n if $e_{i,j}$ is at the n-th position under this linear order. With this partial labeling, we have the following observation:

Observation 4.3. *Except the vertices of the path P_{k+1}, vertex sums of $V(G)$ form a strictly monotone increasing sequence, and the difference between any two consecutive vertex sums of degree-2 vertices is at least 2.*

With this observation, we can start to prove the main result of this chapter. Here we define a new verb. Given a graph G and a labeling of it, for $u, v \in V(G)$, we say u conflicts with v if their vertex sums are identical. The following theorem is true when the path is K_2. And when the path is K_3 or longer, the following argument only assures that the graph is 2-$antimagic$.

Theorem 4.4. *$SPS(l_1, l_2, \ldots, l_{k_1+k_2}; 1)$, $k_1, k_2 \geq 2$, $l_i \geq 1, i \in \{1, 2, \ldots, k_1 + k_2\}$, is antimagic.*

Proof. First label the edge of the path by m. And let the vertex sums of a, b denoted by A, B. According to the relation of A, B, we have the following 3 cases: $A > B, A = B, A < B$.

$\underline{A > B}$: Since b is incident to two edges with the largest labels, vertex sum of b is larger than all the other vertices with degree less than or equal to 2 in $V(G)$. Furthermore, since a has vertex sum larger than that of b, the labeling is antimagic.

$\underline{A = B}$: Exchange the labels of e_1 and $e_{k_1+k_2,l_{k_1+k_2}}$, then vertex sums of a, b become $A-1, B$. For any other vertex, its vertex sum is at most $2m-2$, which is less than $A-1$ because $A-1 = B-1 \geq 2m-1$. Therefore, the labeling is antimagic.

$\underline{A < B}$: At this time, b has the largest vertex sums. If vertex sum of a conflicts with some other vertex, then exchange the labels of e_1 and $e_{k_1+k_2,l_{k_1+k_2}}$. It is easy to check the new labeling is antimagic. ■

Theorem 4.5. *$SPS(l_1, l_2, \ldots, l_{k_1+k_2}; 2)$, $k_1, k_2 \geq 2$, $l_i \geq 1, i \in \{1, 2, \ldots, k_1+k_2\}$, is 1-antimagic.*

Proof. First label e_1, e_2 with $m-2, m$ and replace the label of $e_{k_1+k_2,l_{k_1+k_2}}$ by $m-1$. Now let A, B denote the vertex sum of a, b. As above, divide all situations into 3 cases according to the relation of A, B.

$\underline{A > B\ :}$ Note that the largest vertex sum of degree-2 vertex are $2m-2$. Since $A > B \geq 2m$, the vertex sum of b is larger than any other vertex of degree 2. So by observation 4.3, the labeling is antimagic.

$\underline{A = B\ :}$ Replace the labels of $e_1, e_2, e_{k_1+k_2,l_{k_1+k_2}}$ by $m-1, m+1, m$. Then vertex sums of a, b become $A+1, B+2$. Since $B+2 > A+1 > 2m$, and the largest vertex sum of all the other vertices is $2m$. Hence by observation 4.3, the labling is antimagic.

$\underline{A < B\ :}$ If $A = 2m-1$, then it is straightforward to see that the labeling is antimagic. Furthermore, suppose $A \neq 2m-1$ and A conflicts with vertex sum of some vertex, then replace the labels of $e_1, e_2, e_{k_1+k_2,l_{k_1+k_2}}$ by $m-1, m+1, m$. The vertex sum of a become $A+1$, since $A+1 \neq 2m$, a can't conflict with v_1. If $A+1 = 2m-3$ (i.e. a conflicts with $v_{k_1+k_2,l_{k_1+k_2}}$), then exchange the labels of e_2 and $e_{k_1+k_2,l_{k_1+k_2}}$. The vertex sums of $a, v_1, v_{k_1+k_2,l_{k_1+k_2}}$ become $2m-3, 2m-1, 2m-2$, since b has vertex sum larger than

$2m+2$, hence by observation 4.3 the labeling is 1-antimagic. If a still conflicts with any other vertices other than $v_1, v_{k_1+k_2,l_{k_1+k_2}}$, this means that there exist two vertices of degree 2 which have consecutive vertex sums, but it contradicts to the observation 4.3. Thus, this graph is 1-antimagic. ■

Finally, we want to discuss the case when the path is K_3 or longer. Let $G := SPS(l_1, l_2, \ldots, l_{k_1+k_2}; k)$, $k_1, k_2 \geq 2, k \geq 3, l_i \geq 1, i \in \{1, 2, \ldots, k_1+k_2\}$. For the two spiders of G, assign them a labeling as the previous paragraph. And for the path of G, label $e_1, e_2, \ldots, e_k$ with $m-k, m-k+1, \ldots, m-1$, and replace the label of $e_{k_1+k_2,l_{k_1+k_2}}$ by m. For convenience, denote the vertex sums of $a, b, v_{k_1+k_2,l_{k_1+k_2}}$ by A, B and V.

Theorem 4.6. *Let G, A, B be defined as above, according to the relation of A, B, we have the following results:*
(a) if $A > B$, G is antimagic.
(b) if $A = B$, G is 1-antimagic.
(c) if $A < B$, G is 2-antimagic.

Proof of (a). Note that v_{k-1} and $v_{k_1+k_2,l_{k_1+k_2}}$ can't conflict. So the only problem may occur if $v_{k_1+k_2,l_{k_1+k_2}}$ conflicts with some vertex on the path. When the conflict happens, exchange the labels of e_k and $e_{k_1+k_2,l_{k_1+k_2}}$. Then similar to the last 2 theorems, it is straightforward to check that the labeling is antimagic.

Proof of (b). Note that a, b have the largest vertex sums of all vertices. Here we have two subcases due to the conflict of $v_{k_1+k_2,l_{k_1+k_2}}$. First, if $v_{k_1+k_2,l_{k_1+k_2}}$ conflicts with some vertex on the path, then replace the label of $e_{k_1+k_2,l_{k_1+k_2}}$ by $m+1$. Observe that then $v_{k_1+k_2,l_{k_1+k_2}}$ don't conflict with any vertices on the path anymore, and vertex sums of a, b become A and $B+1$. This shows that the graph is 1-antimagic.

For the case that $v_{k_1+k_2,l_{k_1+k_2}}$ doesn't conflict with any other vertex in this graph, change the label of e_k into $m+1$. It is also straightforward to show that all vertices have distinct vertex sums. This means that the graph is 1-antimagic.

Proof of (c). Since $v_{k_1+k_2,l_{k_1+k_2}}$ may conflict with other vertex, we divide all situations into two cases.

(i) First suppose $v_{k_1+k_2,l_{k_1+k_2}}$ conflicts with a vertex on the path. And for convenience divide all situations into three subcases according to the relation of V and A.

$\underline{A > V:}$ If a doesn't conflict with any other vertex and $A \neq 2m-2$, then exchange the labels of $e_k, e_{k_1+k_2,l_{k_1+k_2}}$. One can easily find out that the labeling is antimagic. If a conflicts with some vertex of the graph, then for the edges labeled by $m-k, m-k+1, \ldots, m$, replace the labels by adding 1 to each label. Then vertex sums of a and $v_{k_1+k_2,l_{k_1+k_2}}$ become $A+1$ and $V+1$. Since vertex sums of vertices on the path are shifted by 2, and b still remains the largest vertex sum, so a and v will not conflict with vertices on the path. Therefore, the labeling is 1-antimagic.

$\underline{A = V:}$ If $A = V = 2m-2k+1$, replace the labels of $e_1, e_2, \ldots, e_k$, by $m-k+1, m-k+2, \ldots, m-1, m+1$. Then the vertex sums of $a, v_1, v_2, \ldots, v_{k-1}, b, v_{k_1+k_2,l_{k_1+k_2}}$ become $2m-2k+2, 2m-2k+3, 2m-2k+5, \ldots, 2m-3, 2m, B+2, 2m-2k+1$. They are the largest vertex sums and are all different, so the labeling is 1-antimagic. If $A = V = 2m-2k+2\alpha+1, \alpha \in \{1, 2, \ldots, k-2\}$, replace the labels of $e_{\alpha+1}, e_{\alpha+2}, \ldots, e_k, e_{k_1+k_2,l_{k_1+k_2}}$ by $m-k+\alpha+1, m-k+\alpha+2, \ldots, m, m+1$. Then the vertex sums of $a, v_\alpha, v_{\alpha+1}, \ldots, v_{k-1}, b, v_{k_1+k_2,l_{k_1+k_2}}$ become $2m-2k+2\alpha+1, 2m-2k+2\alpha, 2m-2k+2\alpha+3, 2m-2k+2\alpha+5, \ldots, 2m-1, B+2, 2m-2k+2\alpha+2$. Again, they are the largest vertex sums and are all different, so the labeling is 1-antimagic.

$\underline{A < V:}$ If a doesn't conflict with any other vertex, then replace the label of $e_{k_1+k_2,l_{k_1+k_2}}$ by $m+1$. If a also conflicts with some vertex in this graph, by observation 4.3, we can avoid conflict by adding 1 to the labels of a and $v_{k_1+k_2,l_{k_1+k_2}}$. By replacing the labels of $e_1, e_2, \ldots, e_k, e_{k_1+k_2,l_{k_1+k_2}}$ by $m-k+1, m-k+2, \ldots, m+1$, since all vertex sums of vertices on the path except a are shifted by 2, and b still has the largest vertex sum, a and $v_{k_1+k_2,l_{k_1+k_2}}$ won't conflict with any other vertex in this graph anymore. Hence the graph is 1-antimagic.

(ii) Next we suppose that under this labeling, $v_{k_1+k_2,l_{k_1+k_2}}$ doesn't conflict with any other vertex in this graph. Since V is larger than vertex sum of any other vertices except those on the path, there exist $\beta \in \{1, 2, \ldots, k-2\}$ such that vertex sums of $v_\beta, v_{k_1+k_2,l_{k_1+k_2}}, v_{\beta+1}$ are three consecutive integers. As usual, we divide all situations into the following three subcases.

$\underline{A > V}$: Suppose a conflicts with some vertex of this graph, then replace the labels of $e_1, e_2, \ldots, e_k$, by $m-k+1, m-k+2, \ldots, m-1, m+1$. Since vertex sum of a is shifted by 1 but vertex sums of all the other vertices on the path except b are shifted by 2. And b still has the largest vertex sum of this graph. So by observation 4.3, a doesn't conflict in this graph under the new labeling.

$\underline{A = V}$: Replace the label of $e_{k_1+k_2, l_{k_1+k_2}}$ by $m+2$. It is straightforward to check that there's no conflict under this labeling.

$\underline{A < V}$: Suppose a conflicts with some other vertex of this graph. Then we can avoid this conflict by shifting the vertex sum of a by 1. But it may produce new conflict since V could be $A+1$. So, we may replace the labels of $e_1, e_2, \ldots, e_k, e_{k_1+k_2, l_{k_1+k_2}}$ by $m-k+1, m-k+2, \ldots, m-1, m, m+2$. Since $A+1 < V+2$, a and $v_{k_1+k_2, l_{k_1+k_2}}$ don't conflict. By observation 4.3, this new labeling contains no conflict. Since the edge set we used is $\{1, 2, \ldots, m+2\}$, the graph is 2-antimagic. ■

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