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在 N=7 超重力理論中優化漸進趨勢以及 BCFW

Bonus Scaling and BCFW in N=7 Supergravity

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本論文係 柳君諭 君（R02245006）在國立臺灣大學物理學系、所完成之碩士學位論文，於民國 104 年 7 月 28 日承下列考試委員審查通過及口試及格，特此證明

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摘要

本篇文章中將簡短的回顧散射振幅。回顧內容包含兩部分。第一部分裡，我們回顧散射振幅的定義以及旋量螺度，並且使用旋量螺度的方式來表示楊-米爾斯理論的散射振幅。在第二部分中，我們將簡短回顧超對稱。這部分的回顧僅止于使用在建構超對稱散射振幅的基本程度。在這之後我們還會介紹超重力理論。

在概覽完散射振幅的簡介以後，第三部分我們將開始尋找俱有自然性質的建構散射振幅元件。我們將會給出一套系統方式去建構散射振幅，這套模式中每一個建構元件在高能量時都俱有更好的漸進行為z^{-2}，就好比散射振幅一樣。我們將在 N=7 超重力理論中使用布里托、卡查索、馮以及維滕的遞迴關係，並且使用特定的動量形變以展現更好的漸進行為。並且我們將會解釋這個更好的行為是因為使用了 N=8 超重力理論中的附加關係式。

關鍵詞：散射振幅、旋量螺度、遞迴關係、超對稱、超重力

Abstract

We review some ideas of scattering amplitudes. The review consists of two parts. In Part I, we review the definition of scattering amplitudes and spinor helicity. We use the technology of spinor helicity to represent scattering amplitudes in Yang-Mills theory. In part II, we review supersymmetry. The review will be on a basic level to introduce superamplitudes. We then introduce supergravity amplitudes. After introducing amplitudes, we search for natural building blocks for supergravity amplitudes in part III. We want to show a systematic way to find the building blocks which are term-by-term bonus z^{-2} large momentum scaling just like amplitudes. For a given choice of deformation legs, we present such an expansion in the form of the Britto, Cachazo, Feng and Witten recursion relation in N=7 supergravity based on a special shift. We will show that this improved scaling behavior, with respect to the fully N=8 representation, is due to its automatic incorporation of the so called bonus relations.

Keywords: Scattering amplitudes, Spinor helicity, Recursion relations, Supersymmetry, Supergravity

Contents

1 Introduction

Scattering amplitudes are the main physical observables in high energy physics. Recently, there have been many great successes in studying this region. One of the result is finding recursion relations. This powerful result can help us to compute Yang-Mills theory, N=4 Super-Yang-Mills theory and N=8 supergravity scattering amplitudes easily.

In amplitudes language we only use physical observables to construct amplitudes and the reason why we focus on using this language is that there are no gauge redundancy. We will construct amplitudes by using momentum, helicity$\cdots$ of outgoing particles and amazing result is that using Lorentz symmetry and above informations are enough for us to determine three point tree level amplitudes. And how about higher points tree level amplitudes? There is a systematic method to construct higher points on-shell amplitudes called BCFW recursion relation. As we know three point amplitudes and recursion relations, we can construct n point tree level amplitudes. We will briefly introduce the starting point of recursion relations here.

In field theory, we try to write down the Lagrangian for our theory and quantize them using path integral or canonical quantization. Feynman tell us we can use Feynman rules to simplify our calculation.

Feynman rules greatly help us to compute scattering processes but they become very complicated when we consider higher points amplitudes and loop amplitudes. Fortunately, Feynman rules give us a very good experience in studying scattering amplitudes. We can see tree level scattering amplitudes have singular points when their propagators go on-shell. This means we know where the poles are and the residue on each pole. We will naturally want to use complex analysis to study scattering amplitudes and this is what recursion relations do. We introduce a complex number z into our amplitudes by shifting the real momentum to complex. The key point in recursion relation is knowing the singularities and discontinuities of the functional form of the shifting amplitudes on z-plane. In four dimension spacetime, we achieve recursion relations by shifting two legs of outgoing particles. This is so called BCFW recursion relations. It is convenience to use $[ij\rangle$ to denote which legs we shift. $[ij\rangle$ means we shift particle i and particle j as

$$|i] \to |i] + z|j\rangle, \quad |j\rangle \to |j\rangle - z|i\rangle. \tag{1.1}$$

Shifting as $[i^+j^+\rangle$, $[i^-j^+\rangle$ and $[i^-j^-\rangle$ are called good shifts and shifting as $[i^+j^-\rangle$ are called bad shifts. Plus sign and minus sign are used to denote positive helicity and negative helicity of the particles. The reason why we call good shifts and bad shifts is that we can use good shifts to shift the amplitudes and get better large z behavior than bad shifts. More concrete process of this powerful tool will be introduced in Part I of this thesis.

Recursion relation can be generalized to constructing superamplitudes by just modifying the shifting. The process of constructing on-shell superamplitudes is called super BCFW. We can use super BCFW to construct higher point superamplitudes.

To ensure recursion relations work, we can use eq.(1.1) to test the amplitudes behavior in high energy limits. To do this we shift our amplitudes by using eq.(1.1) and set $z \to \infty$. The shifting results are: (1). N=8 supergravity amplitudes have high energy behavior $1/z^2$ under shifting. We will not use plus sign and minus sign to denote the shifting $[ij\rangle$ in N=8 theory, because all of the superfields $i, j \cdots$ in N=8 contain positive helicity graviton and negative helicity graviton simultaneously. (2). N=7$\cdots$0 supergravity amplitudes behavior are depended on shifted legs. If we are using good shift, then N=7$\cdots$0 supergravity amplitudes behavior are $1/z^2$. If we are using bad shift, then N=7$\cdots$0 supergravity amplitudes behavior are z^{8-N}/z^2.

Validity of recursion relations relies on vanishing of boundary contribution, which means we can not use bad shift recursion relations in N=6$\cdots$0 supergravity theory. Bad shift recursion relations only work in N=7 supergravity theory. Validity of N=7 bad shift relies on using $1/z^2$ fall off of the full amplitude. We can see this from reducing supersymmetry from N=8 to N=7. In N=8 theory amplitudes have $1/z^2$ fall off. If we choose the shifting superfields $[ij\rangle$ in N=8 and reduce to N=7 by choosing i to be a plus superfield which has positive helicity graviton and j to be negative one which has negative helicity graviton. Then we will get an extra z factor. So the N=7 amplitude in bad shifts will look like z/z^2, which vanishes in high energy by appearance of z^2 on denominator. More details are in the Section.5.1. Although N=8 supergravity amplitudes also vanishes in high energy limits under $[ij\rangle$ shift, they don't have extra z factor as N=7. In other words, N=8 theory blind to the improved fall off $1/z^2$. So that we believe there are something interesting in constructing N=7 amplitudes by using bad shift.

As we know that N=8 amplitudes $A^{N=8}$ have $1/z^2$ behavior under high energy limits. We can multiply a linear function of z which we call $f(z)$ on our amplitudes and claim that $f(z)A^{N=8}$ vanishes while $z \to \infty$. We can compute $f(z)A^{N=8}$ by using recursion relation and we will find its building blocks are all contained in the building block for constructing $A^{N=8}$. So they will give us new relations between the building blocks. We call these relations bonus relations. We will give a concrete example in MHV amplitudes in Section 5.4.

In part III of this thesis, we will present a proof that the building blocks of bad shift BCFW in N=7 term by term manifest $1/z^2$ behavior under special shifting leg. Note that the generic BCFW building blocks can behave as $1/z$ individually and only behave as $1/z^2$ in sum. We call it bonus scaling if each building block manifests $1/z^2$ behavior. We will show that this bonus scaling comes from the bonus relation in N=8 theory in MHV amplitudes.

The reason why we are interested in each building block manifesting $1/z^2$ in high energy limits is that gravity theories don't have Yangian invariance which arises in the building block of N=4 super Yang-Mills theory and N=6 super Chern-Simons matter theory. We will naturally want to ask, if there are natural building blocks for gravity amplitudes, what would be a desirable property similar to Yangian invariance. One special property is the asymptotic behaviour.

In the end of part III, we will apply bad shift BCFW in superstring amplitudes. We show bad shift BCFW building blocks have better large z fall of than the whole string amplitudes under special kinematics region. In superstring gluon amplitudes, large z scaling under $[ij\rangle$ shift is improved by $z^{-\alpha' s_{ij}}$ where α' is string tension and s_{ij} is the Mandestam variables. Combing this improvement with bonus scaling results in Section 5.2, bad shift building blocks have better large z fall of than the whole amplitudes in the region $3-N < Re[\alpha' s_{ij}] < 4-N$ in superstring gluon amplitudes where N is the number supersymmetry. We will use superstring four point gluon amplitudes for example to show this fact. Similar results can be found in the closed superstring. The improvement is $z^{-2\alpha' s_{ij}}$ in superstring graviton amplitudes and the special region that building block have better behavior than whole amplitudes is $6 - N < Re[2\alpha' s_{ij}] < 7 - N$.

Part I

2 Spinor Formalism

In this chapter, we are going to introduce spinor helicity formalism. We will briefly review the representation of the Lorentz group and we will study some algebra of spinor fields. At the end of this section, we will use the technology of spinor helicity to represent scattering amplitudes in Yang-Mills theory.

2.1 Representation of Lorentz group

Lorentz symmetry is a spacetime symmetry which describes difference reference frames observation. A Lorentz transformation is a linear, homogeneous change of coordinates

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{2.1}$$

It will preserve the interval x^2 and so that we know that the matrix $\Lambda^{\mu}{}_{\nu}$ obey

$$g_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} g_{\rho\sigma}, \tag{2.2}$$

where $g_{\mu\nu}$ is the metric of Minkowski space

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{2.3}$$

In quantum theory, symmetries are represented by unitary operators on Hilbert space. So we use $U(\Lambda)$ to represent Lorentz transformation on Hilbert space. The definition of group tells us these operators $U(\Lambda)$ must obey the composition rule

$$U(\Lambda'\Lambda) = U(\Lambda)U(\Lambda'). \tag{2.4}$$

For an infinitesimal Lorentz transformation, we can expand $\Lambda^{\mu}{}_{\nu}$ like

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}. \tag{2.5}$$

And expand $U(\Lambda)$ to first order

$$U(1+\delta\omega) = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu} + \mathcal{O}\{\delta\omega^2\} \tag{2.6}$$

We can define $M^{\mu\nu}$ as an antisymmetry matrix so that we can use $M^{\mu\nu}$ to represent the generators of Lorentz group. We will use composition rule and eq.(2.5), eq.(2.6) to show the commutation relations of generator $M^{\mu\nu}$ later. Now we should define what scalar, vector and tensor are by using Lorentz transformation.
The definition of scalar, vector and tensor transformation:
Scalar transformation

$$U(\Lambda)^{-1} C U(\Lambda) = C. \tag{2.7}$$

Vector transformation

$$U(\Lambda)^{-1}V^{\mu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}V^{\rho}. \tag{2.8}$$

Two rank tensor transformation

$$U(\Lambda)^{-1}T^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}. \tag{2.9}$$

This is a general result: any operator carrying space-time indices will transform similarly. So that we can easily define n-rank tensor transformation

$$U(\Lambda)^{-1}T^{\mu\nu\cdots\eta}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\cdots\Lambda^{\eta}{}_{\xi}T^{\rho\sigma\cdots\xi} \tag{2.10}$$

There is another way to find two rank anti-symmetry tensor transformation. Starting from $U(\Lambda^{-1}\Lambda'\Lambda) = U(\Lambda^{-1})U(\Lambda')U(\Lambda)$ we expand Λ' to first order by using eq.(2.5) and expand $U(\Lambda')$ to first order by using eq.(2.6). The result is

$$U(\Lambda^{-1}(\delta^{\mu}{}_{\nu} + \delta\omega'^{\mu}{}_{\nu})\Lambda) = U(\Lambda^{-1})(I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu})U(\Lambda). \tag{2.11}$$

As we know $\Lambda^{-1} = \Lambda^{T}$, the first order term of $\delta\omega'^{\mu}{}_{\nu}$ give us

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{2.12}$$

which is exactly two ranks tensor transformation.

Now we can show the commutation relations of $M^{\mu\nu}$. We expand Λ and $U(\Lambda)$ in eq.(2.12) to first order of $\delta\omega^{\mu}{}_{\nu}$, we get

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\{(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}) - (g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho})\} \tag{2.13}$$

The commutation relation specify the Lie algebra of the Lorentz group. We can identify the components of angular momentum operator $\vec{J}$ as

$$J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk} \tag{2.14}$$

and components of boost operator $\vec{K}$ as

$$K_i = M^{i0}. \tag{2.15}$$

We find eq.(2.13) can be rewritten as

$$[J_i, J_j] = +\varepsilon_{ijk}J_k \tag{2.16}$$

$$[J_i, K_j] = -\varepsilon_{ijk}K_k \tag{2.17}$$

$$[K_i, K_J] = -\varepsilon_{ijk}J_k \tag{2.18}$$

Here we start to describe how scalar field, vector field and tensor field transformation in Lorentz group. In quantum mechanics, the Heisenberg picture which describes time dependent operator $\phi_{Heisnberg}(t)$ as

$$e^{iHt}\phi_{Schrodinger}e^{-iHt} = \phi_{Heisnberg}(t)$$

where $\phi_{Schrodinger}$ is time independent operator. Eq.(2.1) gives us a hint that field theory should transform like this formula.
So that we define scalar field transformation

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x), \tag{2.19}$$

vector fields transformation

$$U(\Lambda)^{-1}V^{\rho}U(\Lambda) = \Lambda^{\mu}{}_{\rho}V^{\rho}(\Lambda^{-1}x) \tag{2.20}$$

and tensor field transformation

$$U(\Lambda)^{-1}B^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}B^{\rho\sigma}(\Lambda^{-1}x). \tag{2.21}$$

In general, we can consider a field which carries a generic Lorentz index, ψ_A (Index A is $\mu\nu\rho\sigma\cdots$.) which transform as

$$U(\Lambda)^{-1}\psi_A U(\Lambda) = L_A{}^{B}(\Lambda)\psi_B(\Lambda^{-1}x) \tag{2.22}$$

where $\Lambda_A{}^B(\Lambda) = \Lambda_{\mu}{}^{\eta}\Lambda_{\nu}{}^{\xi}\cdots$. For an infinitesimal transformation, we expand $L_A{}^B(\Lambda)$ as

$$L_A{}^B(1+\delta\omega) = \delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A{}^B. \tag{2.23}$$

Now we use eq.(2.6) and eq.(2.23) to expand eq.(2.22) to first order

$$[\psi_A(x), M^{\mu\nu}] = -i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\psi_A(x) + (S^{\mu\nu})_A{}^B\psi_B(x). \tag{2.24}$$

We want to find the irreducible representations of Lorentz group. We start from the algebra of Lorentz group generator J_i and K_i. Studying commutation relations more carefully, we can rewrite the relations as follows

$$[N_i, N_j] = i\varepsilon_{ijk}N_k, \tag{2.25}$$
$$[N_i^{\dagger}, N_j^{\dagger}] = i\varepsilon_{ijk}N_k^{\dagger}, \tag{2.26}$$
$$[N_i, N_j^{\dagger}] = 0. \tag{2.27}$$

N_i and $N_i^{\dagger}$ are defined as

$$N_i \equiv \frac{1}{2}(J_i - iK_i), \tag{2.28}$$
$$N_i^{\dagger} \equiv \frac{1}{2}(J_i + iK_i). \tag{2.29}$$

From the results of studying SO(3) group, the commutations relation of angular momentum are $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ which just like eq.(2.25) and eq.(2.26). So that we can construct the spectrum of the Lorentz group as we have done in quantization of angular momentum. The spectrum of SO(3) can be denoted by two quantum numbers (j,m) where j and m are quantum number of J^2 and J_z respectively. Quantum number j is integer or half integer and quantum number m run from -j, -j+1, $\cdots$, j-1, j+1. In Lorentz group, we have two set of commutation relations eq.(2.25) and eq.(2.26) which means we can use SU(2)⊗SU(2) to represent SO(1,3). As we know the spectrum of N_i constitute all of the inequivalent, irreducible representations of SU(2) and so does $N_i^\dagger$, we can construct the irreducible representations of the Lorentz group. We use two numbers (m,n) to denote the representations where m is the number of states which transform under N_i and n is the number of states which transform under $N_i^\dagger$. Some representations have special names:

$$\begin{aligned}(1,1) =& \ scalar,\\ (2,1) =& \ left-handed \ \ spinor,\\ (1,2) =& \ right-handed \ \ spinor,\\ (2,2) =& \ vector.\end{aligned}$$

Left-handed spinor field is also named left-handed Weyl field and right-handed spinor field is also named right-handed Weyl field.

In this section we briefly review Lorentz group and demonstrate irreducible representations from Lorentz algebra. Then spinor representation was naturally found. In next section we will start from studying free electron Lagrangian and solving Dirac equation. The solutions of Dirac equation which are called Dirac spinor will naturally split to two parts in Weyl representation. One part is left-handed Weyl spinor and the other is right-handed Weyl spinor.

2.2 Spinor Fields

The Lagrangian for a free electron Ψ is

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi. \tag{2.30}$$

where $\bar{\Psi}$ is the Dirac conjugate of Ψ

$$\bar{\Psi} = \Psi^\dagger \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{b}} \\ \delta_a{}^b & 0 \end{pmatrix}. \tag{2.31}$$

And the equation of motion for $\bar{\Psi}$ give us the Dirac equation

$$(-i\partial\!\!\!/ + m)\Psi = 0 \tag{2.32}$$

We can multiply the Dirac equation by $(i\partial\!\!\!/ + m)$ and get the Klein-Gordon equation

$$(i\partial\!\!\!/ + m)(-i\partial\!\!\!/ + m)\Psi = (-\partial^2 + m^2)\Psi = 0 \tag{2.33}$$

It is easy to find the solution to the Klein-Gordon equation. We expand field $\Psi(x)$ in momentum space

$$\Psi(x) \sim u(p)e^{ipx} + v(p)e^{-ipx}. \tag{2.34}$$

And Klein-Gordon equation only give us one constraint $p^2 = p^\mu p_\mu = -m^2$. If we want to find the solution to the Dirac equation, then we will find another constraints

$$(p\!\!\!/ + m)u(p) = 0, \tag{2.35}$$

$$(-p\!\!\!/ + m)v(p) = 0. \tag{2.36}$$

There are two independent solution for u(p) and v(p) respectively which we use $\pm$ to denote. For massive particles, we can choose our basis to be in the rest-frame of the particle. Then $u_{\pm}$ and $v_{\pm}$ are eigenstates of the z- components of the spin matrix. So that "+" denotes spin up and "-" denotes spin down. For massless fermions, $\pm$ denotes helicity, which is the projection of the spin along the momentum of the particle. We rewrite the solution as

$$\Psi(x) = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} [b_s(p)u_s(p)e^{ipx} + d_s^\dagger(p)v_s(p)e^{-ipx}]. \tag{2.37}$$

Write down $\not{p}$ in matrix formalism

$$\not{p} = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}, \tag{2.38}$$

where

$$p_{a\dot{b}} \equiv p_\mu(\sigma^\mu)_{a\dot{b}} = \begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \tag{2.39}$$

and

$$p^{\dot{a}b} \equiv p_\mu(\bar{\sigma^\mu})^{\dot{a}b} = \begin{pmatrix} -p^0-p^3 & -p^1+ip^2 \\ -p^1-ip^2 & -p^0+p^3 \end{pmatrix}. \tag{2.40}$$

The determinant of $p_{a\dot{b}}$ and $p^{\dot{a}b}$ are Lorentz-invariant,

$$det\ p = -p^\mu p_\mu = m^2. \tag{2.41}$$

Because we are studying high-energy scattering amplitudes, neglecting the mass term would be a good approximation. When m = 0, the Dirac equation becomes

$$\not{p}u(p) = 0, \tag{2.42}$$
$$-\not{p}v(p) = 0. \tag{2.43}$$

But actually we are more interested in the solution of $v_\pm(p)$ and $\bar{u}_\pm(p)$ which come from ψ and $\bar{\psi}$ respectively. The reason is that $v_\pm(p)$ and $\bar{u}_\pm(p)$ is associated with an outgoing anti-fermion and fermion respectively. Don't worry about $u_\pm(p)$ and $\bar{v}_\pm(p)$! We have crossing symmetry which relate $u_\pm = v_\mp$ and $\bar{v}_\pm = \bar{u}_\mp$. We write down the solution of $\not{p}v_\pm(p) = 0$ and $\bar{u}_\pm(p)\not{p} = 0$

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}, \tag{2.44}$$

$$\bar{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix}, \bar{u}_+(p) = \begin{pmatrix} [p|^a \\ 0 \end{pmatrix}. \tag{2.45}$$

We define $|p]_a$ and $\langle p|_{\dot{b}}$ using invariant symbol ϵ^{ab} and $\epsilon^{\dot{a}\dot{b}}$

$$|p]_a = \epsilon_{ab}[p|^b, \quad \langle p|_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}|p\rangle^{\dot{b}}. \tag{2.46}$$

From the spinor completeness relations with m = 0, we have

$$p_{a\dot{b}} = -|p]_a\langle p|_{\dot{b}}, \tag{2.47}$$
$$p^{\dot{a}b} = -|p\rangle^{\dot{a}}[p|^b. \tag{2.48}$$

We can do a Dirac conjugate to $\Psi(x)$, it seems that $[p|^a$ is related to $|p\rangle^{\dot{a}}$

$$[p|^a = (|p\rangle^{\dot{a}})^\star. \tag{2.49}$$

So do $\langle p|_{\dot{a}}$ and $|p]_a$

$$\langle p|_{\dot{a}} = (|p]_a)^\star. \tag{2.50}$$

But this relation rely on that the momentum p^μ is real valued. And here we are using momentum which is complex, so that $[p|^a$, $|p\rangle^{\dot{a}}$, $[p|^a$ and $|p\rangle^{\dot{a}}$ are all independent.

In this section, we know how to use $|p\rangle$ and $|p]$ to represent Weyl spinor. We can start to use the spinor formalism to express amplitudes which are calculated by Feynman rule. In the next section, we will try to express amplitudes of Yang-Mills theory in spinor formalism. As we claim before, spinor formalism is directly related to physical observables so that we can simplify amplitudes by using spinor formalism.

2.3 Yang-Mills Theory

The Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}), \tag{2.51}$$

where $F_{\mu\nu}$ is field strength defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]. \tag{2.52}$$

A_μ is defined as $A_\mu = A^a_\mu T^a$. We consider general cases Yang-Mills theory whose gauge group is SU(N). This means we consider that gluon have N colors. For convenience, we choose the Gervais-Neveu gauge and normalize the generators T^a as

$$Tr(T^aT^b) = \delta^{ab}, \tag{2.53}$$

$$[T^a, T^b] = i\sqrt{2}f^{abc}T^c. \tag{2.54}$$

In the Gervais-Neveu gauge, we rewrite our Lagrangian

$$\mathcal{L} = Tr(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \sqrt{2}g\partial^\mu A^\nu A_\nu A_\mu + \frac{g^2}{4}A^\mu A^\nu A_\nu A_\mu). \tag{2.55}$$

Using Feynman rules, we can construct our amplitude. For example, we consider 4 point gluon tree amplitude. The color factors which depend on structure constants are c_s, c_t and c_u.

$$c_s \equiv 2f^{a_1a_2b}f^{ba_3a_4}, \tag{2.56}$$

$$c_t \equiv 2f^{a_1a_3b}f^{ba_4a_2}, \tag{2.57}$$

$$c_s \equiv 2f^{a_1a_4b}f^{ba_2a_3}. \tag{2.58}$$

We use the relation

$$i\sqrt{2}f^{abc} = Tr(T^aT^bT^c) - Tr(T^bT^aT^c), \tag{2.59}$$

and the completeness relation

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \frac{1}{N} \delta_i{}^j \delta_k{}^l \tag{2.60}$$

to rewrite $f^{a_1a_2b}f^{ba_3a_4}$

$$\begin{aligned} f^{a_1a_2b}f^{ba_3a_4} \propto\, & Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - Tr(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ & + Tr(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) - Tr(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) \end{aligned} \tag{2.61}$$

Using eq.(2.61), we can write the 4 points gluon amplitude as

$$\begin{aligned} A_4 = g^2(& A_4[1234]Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + A_4[1243]Tr(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ & A_4[1342]Tr(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + A_4[1432]Tr(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) \end{aligned} \tag{2.62}$$

where $A_4[1234], A_4[1243], A_4[1342]$ and $A_4[1432]$ are called color-order amplitudes.

Now, we try to use spinor formalism to express color-order amplitudes. The simplest example is the 3 point gluon amplitude

$$A_3[123] = -\sqrt{2}[(\epsilon_1\epsilon_2)(\epsilon_3p_1) + (\epsilon_2\epsilon_3)(\epsilon_1p_2) + (\epsilon_3\epsilon_1)(\epsilon_2p_3). \tag{2.63}$$

We choose gluon 1 and gluon 2 to have negative helicity while gluon 3 has positive helicity. Using Fierz identity:

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24] \tag{2.64}$$

and 3-particle kinematics, we will get

$$A_3[1^-2^-3^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{2.65}$$

Another choice is that gluon 1 and gluon 2 have positive helicity while gluon 3 has negative helicity. The result is

$$A_3[1^+2^+3^-] = \frac{[12]^4}{[12][23][31]}. \tag{2.66}$$

We can use recursion relations to construct higher points amplitudes and the famous result is the Parke-Taylor n gluon tree amplitude

$$A_n[1^+, \cdots, i^-, \cdots, j^-, \cdots, n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{2.67}$$

where only gluon i and j carry negative helicity and other gluons have positive helicity.

2.4 Little Group

Little group is a special case of the Lorentz group. This special transformation will leave the momentum of an particle invariant. To see the little group more concretely, we describe the one particle case as an example. For a massive particle, we can choose the rest frame where the particle momentum $p^\mu = (E, 0, 0, 0)$. When we rotate x-y plane, y-z plane and x-z plane of the frame, p^μ is invariant, so the little group is SO(3). For a massless particle, we can choose the reference frame where the particle momentum

$p^\mu = (E,0,0,E)$. When we rotate x-y plane of the frame, p^μ is invariant, so the little group is SO(2).

Little group scaling is a transformation which leave the momentum of on-shell particles invariant. As we know, for a massless particle momentum $p_{a\dot{b}} = -|p]_a\langle p|_{\dot{b}}$ (eq.(2.47)). This relation is invariant under the scaling

$$|p\rangle \to t|p\rangle, \quad |p] \to t^{-1}|p]. \tag{2.68}$$

Equation(2.68) is just U(1) transformation. U(1) and SO(2) are isomorphic groups which means what little group scaling do is exactly rotate $p^\mu = (E,0,0,E)$ on x-y plane.

A crucial result for little group scaling: For an amplitude of massless particles, scaling the particle of this amplitude will transform the amplitude homogeneously like

$$A_n(\psi_1,\psi_2,\cdots,\psi_i(t_i|p_i\rangle, t_i^{-1}|p_i], h_i),\cdots,\psi_n) = t_i^{-2h_i} A_n(\psi_1,\psi_2,\cdots,\psi_i(|p_i\rangle,|p_i],h_i),\cdots,\psi_n). \tag{2.69}$$

h_i is the helicity of particle i. We can see eq.(2.69) from Feynman rules: (1). For scalar field theories, there are constant factor 1 on each fields and scalar field has zero helicity. Eq.(2.69) is obviously true. (2). For spinor fields, there are one angle spinor for an left-handed Weyl field and one square spinor for an right-handed Weyl field. Left-handed and right-handed Weyl field have helicity $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively. Eq.(2.69) works again! (3). For spin-1 boson, the polarization vectors $\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}$ and $\epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}$ have helicity -1 and +1 respectively. Polarization vectors $\epsilon_\pm$ scale as $t^{\pm 2}$.

This powerful result fix the massless three particles amplitude. For example, we consider the color order amplitude $A_3(1^-2^-3^+)$ which is scattering of three gluons. From LSZ reduction, we know the tree point scattering amplitudes in Yang-Mills theory has mass dimension 1. Three particle special kinematics show a non-vanishing on-shell 3-particle amplitude can only depend on either angle brackets or square brackets. We choose the angle brackets to represent the amplitude, so that

$$A_3(1^-2^-3^+) \propto |1\rangle^a|2\rangle^b|3\rangle^c.$$

The power a,b and c are fixed by little group

$$A_3(1^-2^-3^+) \propto |1\rangle^2|2\rangle^2|3\rangle^{-2}.$$

Non-vanishing $A_3(1^-2^-3^+)$ which has correct mass-dimension must be

$$A_3(1^-2^-3^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

which we have shown in eq.(2.65).

3 BCFW Recursion Relation

Recursion relations are a method for building higher point amplitudes from lower point amplitudes. Considering an on-shell amplitude, the key idea is to use complex analysis. The most famous on-shell recursion relation is Britto, Cachazo, Feng and Witten (BCFW) recursion relation. We describe the general BCFW recursion relation in Section 3.1 and Section 3.2 shows how to do BCFW when we have boundary contribution in the complex plane.

3.1 BCFW

An n-point on shell amplitude A_n is a function of momenta. In general, we can write it as $A_n(p_1, p_2, \cdots, p_n)$. Here we focus on massless particles so $p_i^2 = 0$ for i=1, 2, ..., n. However we can consider a more general functional form of amplitudes that A_n is a function not only of momenta but also a complex number z. To do this, we introduce n complex vectors v_i^μ which have the property that

$$\sum_{i=1}^{n} v_i^\mu = 0$$
$$v_i^\mu \cdot v_{j\mu} = 0$$
$$p_i^\mu \cdot v_{i\mu} = 0.$$

The last condition just contracts the index μ. (Index i no sum.) We introduce complex number z by shifting momenta like

$$\hat{p}_i^\mu \equiv p_i^\mu + z v_i^\mu.$$

Instead all of momenta of A_n to the shifted momenta. Now our amplitude depends on momenta p_i and a complex number z. Obviously the amplitude $A_n(p_1, p_2, \cdots, p_n, z)$ is equal to the unshifted amplitude when z is zero. So that we integrate the shifted amplitude

$$\oint_0 \frac{\hat{A}_n(z)}{z} \tag{3.1}$$

around z=0. Using Cauchy's theorem we find

$$\hat{A}_n(z=0) = -\sum_{z_i} Res_{z=z_i} \frac{\hat{A}_n(z)}{z} + B. \tag{3.2}$$

B is the residue of the pole at $z = \infty$. When we are doing good shift of momenta, we can drop the boundary term B out and find the contribution on finite z-plane. The problem is where are the poles of the amplitudes? How can we systematically find out all of the finite poles in the complex plane? To answer these questions, we need to use properties of vectors v_i and p_i. We note that the shifted momenta preserve momentum conservation $\sum_{i=1}^{n} \hat{p}_i^\mu = 0$ and the shifted momenta are also on-shell.

$$\begin{aligned} \hat{p}_i^2 &= (p_i^\mu + z v_i^\mu)^2 \\ &= p_i^\mu p_{i\mu} + 2z p_i^\mu v_{i\mu} + z^2 r_i^\mu r_{i\mu} = 0. \end{aligned}$$

The Feynman diagram tell us that amplitudes diverge while propagator approaches to zero. If we can find out all kind of shifted propagators, then we have already found all the poles on the complex plane. Here we focus on the discussion on tree amplitudes, so that we only need to consider single pole contributions.

By using momentum conservation, the generic shifted propagator looks like

$$\frac{1}{\hat{P}_I^2}$$

where $\hat{P}_I^\mu = \sum_{i\in I} p_i^\mu$. To find $\hat{P}_I^2 = 0$, we can expand out $\hat{P}_I^2$

$$\hat{P}_I^2 = (\sum_{i\in I} p_i^\mu)^2 = P_I^2 + 2zP_I \cdot V_I$$

with $V_I = \sum_{i\in I} v_i$. We can rewrite

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \;\; with \;\; z_I = -\frac{P_I^2}{2P_I \cdot R_I}. \tag{3.3}$$

Insert the result of eq.(3.3) into eq.(3.2). Because of the propagator go on-shell, the shifted amplitude factorize into two on-shell amplitudes which we call $\hat{A}_L$ and $\hat{A}_R$.

$$-\sum_{z_i} Res_{z=z_i} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_i)\frac{1}{P_i^2}\hat{A}_R(z_i).$$

$\hat{A}_L$ and $\hat{A}_R$ are lower point amplitude than $\hat{A}_n(z)$. This is so called recursion relations.

In D=4 spacetime, we choose two particles and shift their momenta. Using spinor representation to demonstrate the shifted momentum:

Particle i:

$$|\hat{i}] = |i] + z|j], \tag{3.4}$$

$$|\hat{i}\rangle = |i\rangle. \tag{3.5}$$

Particle j:

$$|\hat{j}] = |j], \tag{3.6}$$

$$|\hat{j}\rangle = |j\rangle - z|i\rangle. \tag{3.7}$$

We call this $[i, j\rangle$-shift, and this is BCFW recursion relation. Note that $\langle\hat{i}\hat{j}\rangle = \langle ij\rangle$ and $[\hat{i}\hat{j}] = [ij]$ remain unshifted because $\langle ii\rangle$ and $[jj]$ equal to zero.

Although we can use this powerful method to construct higher points tree amplitudes efficiently, but we may want to ask a question when does this method work? Can we use BCFW recursion relation on any theory we know? To answer this question, we should step back and study the Lagrangian more carefully. Then we would find the recursion relation workability rely on symmetry preservation on the Lagrangian. We will try to explain what this mean more concretely by using scalar-QED Lagrangian and ϕ^4 theory Lagrangian as examples. But before we investigate Lagrangians, we can answer this workability question in a simple way.

From the above derivation, BCFW recursion relation relies on vanish of boundary contribution after we do contour integration. This statement is discussed on eq.(3.2) before but we can show good shift and bad shift more concretely here. In pure Yang-Mills theory, [10] show the color-ordered gluon tree amplitudes under BCFW shift will have large z behavior like

$$\begin{aligned}
[--\rangle &: \lim_{z\to\infty} \hat{A(z)} \sim \frac{1}{z}, \\
[-+\rangle &: \lim_{z\to\infty} \hat{A(z)} \sim \frac{1}{z}, \\
[++\rangle &: \lim_{z\to\infty} \hat{A(z)} \sim \frac{1}{z}, \\
[+-\rangle &: \lim_{z\to\infty} \hat{A(z)} \sim z^3.
\end{aligned} \tag{3.8}$$

Minus and plus sign means -1 helicity gluon and +1 helicity gluon respectively. Here we choose two adjacent particles to shift. If we choose two shifted particles non-adjacent then we will get extra power 1/z in each case. Shifting like $[--\rangle$, $[-+\rangle$ and $[++\rangle$ are called good shift and shifting like $[+-\rangle$ which would have boundary contribution is called bad shift. So a condition for BCFW recursion relation works is that the theory must have good shift. But this condition is so strong that we can only use BCFW recursion relation on some theories. This problem push us to extend recursion relation to theories which don't have good shift. A systematic method called Multi-step BCFW [20] is one way to use recursion relation when we have boundary contribution. We will briefly introduce Multi-step BCFW in the next section.

Now we try to study the scalar-QED Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}^2 + (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{4}\lambda|\phi|^4 \tag{3.9}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariance derivative. The interaction between scalar field ϕ and photon A_μ is encoded in $(D_\mu\phi)^*(D^\mu\phi)$. We can expand out this term

$$(D_\mu\phi)^*(D^\mu\phi) = |\partial\phi|^2 + ieA^\mu[(\partial_\mu\phi)^* - \phi^*\partial_\mu\phi] - e^2A^\mu A_\mu\phi^*\phi. \tag{3.10}$$

We find that $-e^2A^\mu A_\mu\phi^*\phi$ which will give us four-point vertex is not gauge invariant. But as we know four-point on-shell scattering amplitudes is a physical quantity which must be invariant under gauge transformation. We have same results on three-point vertex and three points on-shell scattering amplitudes. If we want to preserve gauge symmetry in Lagrangian, then we will have explicit relation between three points and four points amplitude. Actually when we write down the covariant derivative like eq.(3.10) the formulation between three-point vertex and four-point vertex are fixed. In this sense, we expect recursion relation work in scalar-QED. But we shouldn't forget whether boundary contribution exist or not. We compute 4-point scalar amplitude for example. Using BCFW recursion relation, we will find

$$A_{BCFW}(\phi\phi^*\phi\phi^*) = \tilde{e}^2\frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.11}$$

We can compare this result with 4-point amplitude which we compute using Feynman rule

$$A_{Feynman}(\phi\phi^*\phi\phi^*) = -\lambda + \tilde{e}^2 + \tilde{e}^2\frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.12}$$

We meet a problem! First two terms on RHS ($-\lambda$ and $\tilde{e}^2$) are the boundary terms when we are doing BCFW. As we claim BCFW recursion relation works when there are no boundary contributions, if $-\lambda + \tilde{e}^2 = 0$ then A_{BCFW} should be the answer. This statement agrees with eq.(3.11) and eq.(3.12).

Now we try to study ϕ^4 theory Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{3.13}$$

We may think all of the information of n points amplitude($n > 4$) as encoded in ϕ^4. So that we can use recursion relation construct higher points amplitude. From the above experience, we must be careful about the boundary contribution. On 6 points amplitudes we find that there is no way to shift the amplitude without boundary term. This means we can not use recursion relations.

Above two examples show when we can use BCFW recursion relation. We can do the same analysis to Yang-Mills theory color-ordered amplitudes, and this time we will find BCFW work in some good shifts.

3.2 Multi-step BCFW

From eq.(3.2), we know recursion relations rely on boundary contributions to be missing. When we are doing good shifts, the boundary term will always vanish, so there is no problem to calculate the amplitude using good shifts. But for a general theory we do not know whether or not a good shift exists, and in fact we do know that there are a lot of theories where there is no good shift. So we need to ask a question how to use recursion relations with bad shifts. When we are doing bad shifts, the boundary term arises and recursion relations can not work as before. In this section we will introduce the systematic algorithm called multi-step BCFW to determine the boundary contribution.

Our purpose is that we want to construct amplitudes, and the amplitude should cover all physical poles which have correct residues. Obviously we are not sure the residual of some poles when we are doing BCFW by choosing arbitrary leg $[i_1 i_2\rangle$. But complex analysis tell us recursion relation can give us the correct amplitude. If we still believe complex analysis, then we can start from eq.(3.2) to find the physical amplitude. From complex analysis, we can expand the function $A(z_{\underline{0}})$ into single pole parts and polynomial parts

$$\hat{A}_n^0(z_{\underline{0}}) = -\sum_{z_{i,\underline{0}}} \hat{A}_L(z_{i,\underline{0}})\frac{1}{P_i^2}\hat{A}_R(z_{i,\underline{0}}) + C_0^{\underline{0}} + \sum C_i^{\underline{0}} z_{\underline{0}}^i. \tag{3.14}$$

For convenience, we use underline zero to denote the step in which we use BCFW to construct amplitude. Later we will use underline number($\underline{1}, \underline{2}, \cdots$) to denote which step we are doing. We call the first part of eq(3.14) the recursive part

$$R^{\underline{0}}(z_{\underline{0}}) = -\sum_{z_{i,\underline{0}}} \hat{A}_L(z_{i,\underline{0}})\frac{1}{P_i^2}\hat{A}_R(z_{i,\underline{0}}),$$

which means we can factorize the amplitude into two parts. The other part is called the boundary part

$$B^{\underline{0}}(z_{\underline{0}}) = C_0^{\underline{0}} + \sum C_i^{\underline{0}} z_{\underline{0}}^i.$$

Boundary contributions only come from $C_0^{\underline{0}}$ term if the coefficients $C_0^{\underline{0}}$ and $C_i^{\underline{0}}$ are all independent of z_0.

We believe that the tree amplitude diverges when propagators go on-shell. So we claim that if we don't have boundary contribution, then the recursive part is our physical amplitude. Now if the boundary part doesn't vanish, then we will intuitively say there are some physical pole on boundary. And tracing these physical poles is what multi-step BCFW is doing.

When we are doing BCFW recursion relation, we find that some physical poles depended on shifted parameter z and others do not. Physical poles which depend on z are called detectable propagators. The reason why is that we can find these poles on finite z plane. Physical poles that do not depend on z are undetectable poles. In addition to physical poles, there are some spurious poles. Actually, when we study complex z plane we will find that infinite poles (boundary parts) are undetectable propagators and spurious poles. Recursive parts are combined with detectable propagators and spurious poles. Multi-step BCFW is trying to pull back the undetectable poles by using another shift which can detect those physical poles. For convenience, we use $D^{\underline{0}}$ to denote the set of detectable propagators, $U^{\underline{0}}$ denote the set of undetectable propagators and $S^{\underline{0}}$ denote the set of spurious poles.

The systematic method construct amplitudes:

(1). Choosing one kind of shifting $[ij\rangle$

(2). Finding the recursive parts $R^{\underline{0}}$ of $[ij\rangle$ shifting

(3). Choosing another shifting $[kl\rangle$ which contain physical pole that $[ij\rangle$ do not detect

(4). Finding the recursive parts $R^{\underline{1}}$ of $[kl\rangle$ shifting

(5). Shift $R^{\underline{0}}$ by $[kl\rangle$ and expand it as a series of $z^{\underline{1}}$.

(6). Factorize $R^{\underline{0}}$ into recursive part $RR^{\underline{01}}$ and boundary part $RB^{\underline{01}}$.

(7). Construct the recursive part $R^{\underline{01}}$ which include all detectable physical pole in $[ij\rangle$ and $[kl\rangle$

$$R^{\underline{01}} = R^{\underline{0}} + R^{\underline{1}} - RR^{\underline{01}} \tag{3.15}$$

(8). Repeat steps 1 through 7 until the recursive part include all physical poles and no spurious pole.

Intuitively our detectable propagators after union will larger than before and undetectable propagators will smaller than before. But we should note that spurious poles sometimes may be larger than before. After finite step shifting, we can construct the amplitude. And we expect the boundary part will become vanish under n step BCFW. In fact this is the condition for multi-step BCFW workability.

Part II

4 Supersymmetry

In this section, we will briefly introduce supersymmetry. Starting from N=1 supersymmetry, we then introduce supersymmetry Ward identity and superspace. Lastly we will cover N=4 super Yang-Mills theory and super-BCFW recursion relations.

4.1 N=1 Supersymmetry

Considering the free Lagrangian for a scalar field ϕ and Weyl fermion field ψ

$$\mathcal{L}_0 = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \partial_\mu \bar{\phi} \partial^\mu \phi. \tag{4.1}$$

Obviously the Lagrangian has Poincare symmetry, and you may notice that the Lagrangian is invariant (up to a total derivative) under

$$\begin{aligned} \delta_\epsilon \phi &= \epsilon\psi, & \delta_\epsilon \bar{\phi} &= \epsilon^\dagger \psi^\dagger \\ \delta_\epsilon \psi_a &= i\sigma^\mu_{a\dot{b}} \epsilon^{\dagger\dot{b}} \partial_\mu \phi, & \delta_\epsilon \psi^\dagger_{\dot{a}} &= i\partial_\mu \bar{\phi} \epsilon^b \sigma^\mu_{b\dot{a}}, \end{aligned} \tag{4.2}$$

where ϵ and $\epsilon^\dagger$ are anti-commuting constant spinors. The $\epsilon\psi$ and $\epsilon^\dagger\psi^\dagger$ mean $\epsilon^a\psi_a$ and $\epsilon^\dagger_{\dot{a}}\psi^{\dagger\dot{a}}$. This is supersymmetry transformation. We act the operator $[\delta_{\epsilon_1}, \delta_{\epsilon_1}]$ on the fields and the result is shifting the fields on the spacetime. This means combination of two supersymmetry transformation is a spacetime translation

$$\{Q_a, \bar{Q}_{\dot{a}}\} = p_{a\dot{a}} \tag{4.3}$$

Now we start to study supersymmetry algebra by choosing a reference frame. For convenience, we choose a frame so that the momentum of the massless particle is like

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.4}$$

The momentum $p_{a\dot{a}}$ in eq.(4.3) is

$$\{Q_a, \bar{Q}_{\dot{a}}\} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

We have the anti-commutations relation for supersymmetry generator

$$\begin{aligned} \{Q_1, \bar{Q}_{\dot{1}}\} = 1, \quad \{Q_1, \bar{Q}_{\dot{2}}\} = 0, \\ \{Q_2, \bar{Q}_{\dot{1}}\} = 0, \quad \{Q_2, \bar{Q}_{\dot{2}}\} = 0. \end{aligned}$$

The non-zero term just look like the creation and annihilation operator in quantum mechanics. They have similar algebra means we can construct the supermultiplet just as the harmonic oscillator. Because there is only one set of creation and annihilation operator, we use N=1 to label this theory. We can define a state $|0\rangle$ which satisfy

$$Q_1|0\rangle = 0. \tag{4.6}$$

We are using Q_1 to represent annihilate operator and $\bar{Q}_{\dot{1}}$ represent creation operator. Note that operator Q and $Q^\dagger$ are fermion operators which means $Q^2 = Q^{\dagger 2} = 0$. So that we have two difference states

$$\begin{aligned} 0 \quad &state: \ |0\rangle \\ \frac{1}{2} \quad &state: \ |\frac{1}{2}\rangle = \bar{Q}|0\rangle. \end{aligned}$$

From the CPT symmetry, we must have another state with $-\frac{1}{2}$ helicity

$$
\begin{aligned}
-\frac{1}{2} \quad state: & \quad |-\frac{1}{2}\rangle \\
0 \quad state: & \quad |0\rangle = \bar{Q}|-\frac{1}{2}\rangle.
\end{aligned}
$$

This is the spectrum of massless N=1 theory.

If we consider a massive particle, then we can choose a reference frame where p^μ looks like

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.7}$$

Again, we insert p^μ to eq.(4.3)

$$\{Q_a, \bar{Q}_{\dot{a}}\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

This time we have different commutation relations

$$
\begin{aligned}
\{Q_1, \bar{Q}_{\dot{1}}\} = 1, \quad \{Q_1, \bar{Q}_{\dot{2}}\} = 0, \\
\{Q_2, \bar{Q}_{\dot{1}}\} = 0, \quad \{Q_2, \bar{Q}_{\dot{2}}\} = 1.
\end{aligned}
$$

There are two sets of creation and annihilation operators which means the massive spectrum are double that of massless spectrum.

Because of we are focusing on spinor representation, we will try to use spinors to represent supersymmetry operator here. We start from rewriting the spinor field ψ_a using Majorana field

$$\Psi_M = \begin{pmatrix} \psi_a \\ \psi^{\dagger \dot{a}} \end{pmatrix}. \tag{4.9}$$

Define the projection operators P_L and P_R as

$$P_L = \frac{1-\gamma_5}{2} \tag{4.10}$$

$$P_R = \frac{1+\gamma_5}{2}. \tag{4.11}$$

We have the relation

$$\psi_a = P_L \Psi_M. \tag{4.12}$$

Expand $\phi(x)$ and $\psi(x)$ as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}[a_-(p)e^{ipx} + a_+^\dagger(p)e^{-ipx}], \tag{4.13}$$

$$\psi_a(x) = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p}[b_s(p)P_L u_s(p)e^{ipx} + b_s^\dagger(p)P_L v_s(p)e^{-ipx}]. \tag{4.14}$$

We use $s = \pm$ to describe fermions with helicity $\pm\frac{1}{2}$, but the meaning of $\pm$ in scalar field is different. In scalar field, $\pm$ just label two different solution without meaning of helicity. Upon canonical quantization, they satisfy the algebra of bosonic/fermionic creation-annihilation operators:

$$[a_\pm(p), a_\pm^\dagger(p')] =(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p'}), \tag{4.15}$$
$$[b_\pm(p), b_\pm^\dagger(p')] =(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p'}). \tag{4.16}$$

Using eq.(4.13) and eq.(4.14) to expand eq.(4.2), we get the results

$$\begin{aligned}
\delta_\epsilon a_-(p) =&[\epsilon p] b_-(p),\\
\delta_\epsilon a_+(p) =&\langle \epsilon p\rangle b_+(p),\\
\delta_\epsilon b_-(p) =&\langle \epsilon p\rangle a_-(p),\\
\delta_\epsilon b_+(p) =&[\epsilon p] a_+(p).
\end{aligned} \tag{4.17}$$

The generator $Q_M = \begin{pmatrix} Q_a \\ Q^{\dagger\dot{a}} \end{pmatrix}$ can be found from $\delta_\epsilon = [\bar{\delta}_M Q_M, \mathcal{O}] = [[\epsilon Q] + \langle \epsilon Q\rangle, \mathcal{O}]$. $\mathcal{O}$ are the creation/annihilation operators. We find that

$$|Q]_a = \int \frac{d^3p}{(2\pi)^3 2E_p} |p]_a (a_+(p) b_+^\dagger(p) - b_-(p) a_-^\dagger(p)), \tag{4.18}$$
$$|Q^\dagger\rangle^{\dot{a}} = \int \frac{d^3p}{(2\pi)^3 2E_p} |p\rangle^{\dot{a}} (a_-(p) b_-^\dagger(p) - b_+(p) a_+^\dagger(p)), \tag{4.19}$$

reproduce eq.(4.17). Rewriting eq.(4.17) using operator Q and $Q^\dagger$

$$\begin{aligned}
&[Q, a_-(p)] = |p] b_-(p), && \{Q^\dagger, b_-(p)\} = |p\rangle a_-(p)\\
&\{Q, b_-(p)\} = 0, && [Q^\dagger, a_-(p)] = 0\\
&\{Q, b_+(p)\} = |p] a_+(p), && [Q^\dagger, a_+(p)] = |p\rangle b_+(p)\\
&[Q, a_+(p)] = 0, && \{Q^\dagger, b_+(p)\} = 0.
\end{aligned} \tag{4.20}$$

You may notice that the above descriptions are only for free particle on spacetime. But we are calculating scattering amplitudes, we need some interactions! In chiral model, we introduce a superpotential interaction of the form

$$\mathcal{L}_\mathcal{I} = \frac{1}{2} g\phi\psi\psi + \frac{1}{2} g^* \bar{\phi} \psi^\dagger \psi^\dagger - \frac{1}{4}|g|^2 |\phi|^4. \tag{4.21}$$

This Lagrangian($\mathcal{L}_0 + \mathcal{L}_\mathcal{I}$) is invariant under transformation like

$$\begin{aligned}
&\delta_\epsilon \phi = \epsilon\psi, && \delta_\epsilon \bar{\phi} = \epsilon^\dagger \psi^\dagger\\
&\delta_\epsilon \psi_a = i\sigma^\mu_{a\dot{b}} \epsilon^{\dagger\dot{b}} \partial_\mu \phi + \frac{1}{2} g^* \bar{\phi}^2 \epsilon_a, && \delta_\epsilon \psi^\dagger_{\dot{a}} = i\partial_\mu \bar{\phi} \epsilon^b \sigma^\mu_{b\dot{a}} + \frac{1}{2} g\phi^2 \epsilon^\dagger_{\dot{a}}.
\end{aligned} \tag{4.22}$$

We find that eq.(4.22) is just doing a small modification of eq.(4.2). We can construct operators Q and $Q^\dagger$ by using same method we just discussed.

Extending supersymmetry to $N > 1$, there are 2^N states in the massless supermultiplets. For example, for $N = 2$ one supermultiplet consists of a helicity -1 photon, two photinos with $h = -\frac{1}{2}$ and a scalar $h = 0$.

4.2 Supersymmetry Ward Identities

From studying of previous section, we have a Lagrangian which contain both bosonic field and fermionic field

$$\begin{aligned}\mathcal{L} =&\mathcal{L}_0 + \mathcal{L}_\mathcal{I}\\ =&i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \partial_\mu\bar{\phi}\partial^\mu\phi + \frac{1}{2}g\phi\psi\psi + \frac{1}{2}g^*\bar{\phi}\psi^\dagger\psi^\dagger - \frac{1}{4}|g|^2|\phi|^4.\end{aligned} \tag{4.23}$$

This Lagrangian is invariant under supersymmetry transformation between scalar field and fermion field. In this section we will study the effect of supersymmetry on the scattering amplitudes.

In Yang-Mills theory, we can use Feynman rules to compute scattering amplitudes under different scattering processes. But you may find there are some relations between different scattering processes. For example, the 4-point tree amplitudes are presented as below

$$\begin{aligned}A_4(\phi\phi\bar{\phi}\bar{\phi}) =& -|g|^2,\\ A_4(\phi f^- f^+ \bar{\phi}) =& -|g|^2\frac{\langle 24\rangle}{\langle 34\rangle},\\ A_4(f^- f^- f^+ f^+) =&|g|^2\frac{\langle 12\rangle}{\langle 34\rangle}.\end{aligned} \tag{4.24}$$

These three amplitudes have linear relations as

$$A_4(\phi f^- f^+ \bar{\phi}) =\frac{\langle 24\rangle}{\langle 34\rangle}A_4(\phi\phi\bar{\phi}\bar{\phi}), \tag{4.25}$$

$$A_4(f^- f^- f^+ f^+) = -\frac{\langle 12\rangle}{\langle 24\rangle}A_4(\phi f^- f^+ \bar{\phi}). \tag{4.26}$$

These relations seems arbitrary in Yang-Mills theory but actually they come from supersymmetry prescription. Note that we are using 4-point tree amplitudes for example here but these relations hold not only for the tree-level amplitudes. Supersymmetry will ensure these relations hold at all orders in perturbation expansion. We will explain these linear relations starting from the supersymmetry algebra. Suppose we have a supersymmetry vacuum $|0\rangle$ which is defined as

$$Q|0\rangle = Q^\dagger|0\rangle = 0. \tag{4.27}$$

Obviously, we have relations

$$\langle 0|[Q^\dagger, \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\cdots\mathcal{O}_n(p_n)]|0\rangle = 0, \tag{4.28}$$

where $\mathcal{O}_i(p_i)$ is the creation or annihilation operator of bosonic field or fermionic field. We can change the order of operator $Q^\dagger$ and $\mathcal{O}_i$ on LHS. When two operators are all fermionic we use anti-commutation relation. For one fermionic operator and one bosonic operator we use commutation relation. This statement can be easily understood by using an example. We consider the relation

$$\langle 0|[Q^\dagger, a^+(p_1)b^-(p_2)b^-(p_3)]|0\rangle = 0$$

We can rewrite this equation as

$$
\begin{aligned}
&\langle 0|[Q^\dagger, a^+(p_1)b^-(p_2)b^-(p_3)]|0\rangle \\
=&\langle 0|[Q^\dagger, a^+(p_1)]b^-(p_2)b^-(p_3) - a^+(p_1)\{Q^\dagger, b^-(p_2)\}b^-(p_3) + a^+(p_1)b^-(p_2)\{Q^\dagger, b^-(p_3)\}|0\rangle \\
=&|p_1\rangle\langle 0|b^+(p_1)b^-(p_2)b^-(p_3)|0\rangle - |p_2\rangle\langle 0|a^+(p_1)a^-(p_2)b^-(p_3)|0\rangle + |p_3\rangle\langle 0|a^+(p_1)b^-(p_2)a^-(p_3)|0\rangle
\end{aligned}
$$

where a is bosonic operator, b is fermionic operator and Q is obviously fermionic operator. The last line we used supersymmetry algebra to rewrite commutation and anti-commutation relations. There is misleading in the notation $|\cdot\rangle$ here. $|0\rangle$ is the vacuum state and $|p\rangle$ is the angle-spinor.

We can generalize the above example and find the general formulation as

$$\sum_{i=1}^{n}(-1)^{\sum_{j<i}|\mathcal{O}_i|}\langle 0|\mathcal{O}_1(p_1)\cdots[Q^\dagger, \mathcal{O}_i(p_i)]\cdots\mathcal{O}_n(p_n)]|0\rangle, \tag{4.29}$$

where $|\mathcal{O}_i|$ is 0 when the operator $Q^\dagger$ pass through the bosonic operator and 1 if it pass through fermionic operator. Taking supersymmetry algebra eq.(4.20) into eq.(4.29), we will get the linear relation of n point scattering amplitudes. Such relations are called supersymmetry Ward identities. From above example, considering four point amplitudes

$$\langle 0|[Q^\dagger, a_-(p_1)b_-(p_2)a_+(p_3)a_+(p_4)]|0\rangle = 0. \tag{4.30}$$

Supersymmetry algebra tell us

$$|2\rangle A_4(\phi\phi\bar{\phi}\bar{\phi}) - |3\rangle A_4(\phi f^- f^+\bar{\phi}) - |4\rangle A_4(\phi f^-\bar{\phi} f^+) = 0. \tag{4.31}$$

This relation is a spinor relation which means it contains two linear independent relations in one equation. We can contract this relation with $\langle 4|$

$$\langle 42\rangle A_4(\phi\phi\bar{\phi}\bar{\phi}) - \langle 43\rangle A_4(\phi f^- f^+\bar{\phi}) = 0, \tag{4.32}$$

which is precisely one of the relation in eq.(4.24) that we found to be true at tree-level. Other relations can be found by contracting with $\langle 2|$ and supersymmetry Ward identities for operator Q

$$\sum_{i=1}^{n}(-1)^{\sum_{j<i}|\mathcal{O}_i|}\langle 0|\mathcal{O}_1(p_1),\cdots,[Q, \mathcal{O}_i(p_i)],\cdots,\mathcal{O}_n(p_n)]|0\rangle = 0. \tag{4.33}$$

All of the relations in eq.(4.24) can be found by supersymmetry Ward identity.

4.3 N=4 Super Yang-Mills Theory

In this section we will generalize N=1 supersymmetry theory to N=4 super Yang-Mills theory. We write down N=4 super Yang-Mills action and discuss it's spectrum.

$$\mathcal{S} = \int dx^4 Tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D\phi_I)^2 - \frac{i}{2}\bar{\psi}\not{D}\psi + \frac{g}{2}\bar{\psi}\Gamma^I[\phi_I, \psi] + \frac{g^2}{4}[\phi_I, \phi_J]^2) \tag{4.34}$$

where ϕ_I are scalar fields and I run from $1, \cdots, 6$. This means we have six kinds of scalar fields. ψ is ten-dimension Majorana-Weyl fields and Γ^I are ten dimension gamma matrices. D is covariant derivative. All fields are in the adjoint of the gauge group SU(N). The supersymmetry algebra contains global SO(6) R-symmetry. We can also use the isomorphic group SU(4) to represent.

Scalar potential V in this action is $[\phi_I, \phi_j]^2$. If the scalar potential $V \neq 0$ then supersymmetry is spontaneous broken. Other words if our theory has supersymmetry then scalar potential must be equal to zero. We can demonstrate why in a example. As we know $\{Q_\alpha, Q^\dagger_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu$, we can see the vacuum expectation value

$$\langle 0|\{Q_\alpha, Q^\dagger_{\dot\beta}\}|0\rangle = 2(\sigma^\mu)_{\alpha\dot\beta}\langle 0|P_\mu|0\rangle. \tag{4.35}$$

If the theory has supersymmetry, then $\langle 0|\{Q_\alpha, Q^\dagger_{\dot\beta}\}|0\rangle = 0$. But as we know $V \geq 0$, the RHS of eq.(4.35) is greater than or equal to zero, we must break supersymmetry algebra when $V \neq 0$. If we want to preserve supersymmetry in our theory, we need to find out the solutions of $V = 0$. There is a moduli space of N=4 supersymmetric vacua with $[\phi_I, \phi_j]^2 = 0$. At the origin of moduli space, all of the scalar vevs vanish which means all states are massless.
The spectrum of N=4 SYM:

$$\begin{aligned} 1\ gluon &\quad g^+ \\ 4\ gluinos &\quad \lambda^A \\ 6\ scalars &\quad S^{AB} \\ 4\ gluinos &\quad \lambda^{ABC} \\ 1\ gluon &\quad g^-. \end{aligned}$$

A, B and C run from 1, 2, 3, 4 are the SU(4) R-symmetry index. We can use on-shell superspace to keep track the states which means either bosonic state or fermionic state are living on superspace. The difference between spuperspace and usual Minkowski space are the Grassmann parameter η_A. We can easily think of superspace as our usual bosonic coordinates x^μ plus fermionic coordinate η_A. For convenience, we can use on-shell chiral superfield which group 16 states above in a compact formulation

$$\Omega = g^+ + \eta_A - \frac{1}{2!}\eta_A\eta_B S^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C\lambda^{ABC} + \eta_1\eta_2\eta_3\eta_4 g^- \tag{4.36}$$

to denote and studying supersymmetry Ward identities will give us the relation between components amplitudes in superamplitude $A_n(\Omega_1, \Omega_2, \cdots, \Omega_n)$. To see supersymmetry Ward identities, we start from the invariance of superamplitudes under supersymmetry

$$Q^A A = 0 \quad and \quad \tilde{Q}^A A = 0 \tag{4.37}$$

where $\tilde{Q}^A \equiv Q^\dagger$ are supercharges. Using η to classify and their coefficients are supersymmetry Ward identities.

Although we have not construct recursion relation in N=4 super Yang-Mills theory, but we know that as a starting point of recursion relation 3 point superamplitudes must be known. These three points superamplitudes can easily fix down the formula by using

eq.(4.37), little group weighting and mass dimension analysis.
Three point MHV superamplitude:

$$A_3^{MHV}[1,2,3] = \frac{\delta^8(\tilde{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{4.38}$$

Three points anti-MHV superamplitude:

$$A_3^{\overline{MHV}}[1,2,3] = \frac{\delta^4([12]\eta_3 + [23]\eta_1 + [31]\eta_2)}{[12][23][31]}. \tag{4.39}$$

We can use derivative of Grassmann variables or setting Grassmann variable to zero to pick up the desired external states. For example, we can try to pick up three gluon scattering amplitudes $A(g_1^-, g_2^-, g_3^+)$ by

$$(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{\hat{P}A}})(\prod_{B=1}^{4} \frac{\partial}{\partial \eta_{\hat{P}B}})\frac{\delta^8(\tilde{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\Bigg|_{\eta_{\hat{p}1},\eta_{\hat{p}2},\eta_{\hat{p}3},\eta_{\hat{p}4}=0} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{4.40}$$

In the next section, we will introduce super BCFW which help us construct higher points superamplitudes. Then we can write down n points tree amplitudes in N=4 super Yang-Mills theory.

4.4 Super BCFW

According to Section 3.1, we should not expect constructing ϕ^4 theory higher points amplitudes by using recursion relation. But we expect recursion relation work in N=4 super Yang-Mills theory which contains pure scalar scattering process. The reason why is that N=4 super Yang-Mills Lagrangian should preserve not only Poincare symmetry but also supersymmetry. Preserving supersymmetry will fix the relation between lower point vertex and higher point vertex. In this section, we will introduce recursion relation in the supersymmetry theory.

We want to construct higher points superamplitudes by using lower points superamplitudes. In Section 3.1, we shift the particle momenta and preserve the on-shell conditions $p_i^2 = 0$ and momentum conservation $\sum\limits_{i=1}^{n} p_i = 0$. However this shifting doesn't preserve supermomentum conservation $\sum\limits_{i=1}^{n} |i\rangle\eta_{iA} = 0$ that means we will break supersymmetry. So that we modify BCFW shift as:

Field i:

$$|\hat{i}] = |i] + z|j], \tag{4.41}$$

$$|\hat{i}\rangle = |i\rangle, \tag{4.42}$$

$$\hat{\eta}_{1A} = \eta_{iA} + z\eta_{jA}. \tag{4.43}$$

Field j:

$$|\hat{j}] = |j], \tag{4.44}$$

$$|\hat{j}\rangle = |j\rangle - z|i\rangle. \tag{4.45}$$

This is $[i,j\rangle$-supershift which preserve supermomentum conservation as well. We can construct higher points superamplitudes by using super BCFW not only in super Yang-Mills theory but also super gravity theory. In part III, we will discuss searching natural block while we are using super BCFW in N=7 supergravity theory.

4.5 N=8 Supergravity Amplitudes

In this section we will try to expand our discussion from N=4 super Yang-Mills theory to N=8 supergravity which has SU(8) R-symmetry. First we write down the spectrum of N=8 supergravity theory as

$$\begin{gathered}
1 \;\; graviton \;\; h^+, \quad 1 \;\; graviton \;\; h^-, \\
8 \;\; gravitinos \; \psi^A, \quad 8 \;\; gravitinos \; \psi^{ABCDEFG}, \\
28 \;\; gravi-photons \; \nu^{AB}, \quad 28 \;\; gravi-photons \; \nu^{ABCDEF}, \\
56 \;\; gravi-photinos \; \chi^{ABC}, \quad 56 \;\; gravi-photinos \; \chi^{ABCDE}, \\
70 \;\; scalar \;\; S^{ABCD}.
\end{gathered}$$

where $A, B, \cdots, H = 1, 2, \cdots, 8$ are the SU(8) R-symmetry indices. The number in the spectrum is easily to compute. We can start from graviton h^- and using $\tilde{Q}^A$ raise the helicity by $+\frac{1}{2}$ step by step. Acting $\tilde{Q}^A$ once, there are eight gravitnos with helicity $-\frac{3}{2}$. Acting $\tilde{Q}^A$ twice, we should note that Grassmann nature make R-symmetry indices totally anti-symmetric so that there are twenty-eight gravi-photons.

As N=4 SYM, we glue the component fields into a superfield

$$\Phi_i = h^+ + \eta_{iA}\psi^A - \frac{1}{2}\eta_{iA}\eta_{iB}\nu^{AB} + \cdots + \eta_{iA}\eta_{iB}\eta_{iC}\eta_{iD}\eta_{iE}\eta_{iF}\eta_{iG}\eta_{iH}h^- \tag{4.46}$$

and we will compute superamplitudes in N=8 supergravity. We start from studying pure graviton amplitudes. The Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}R + S_{matter} \tag{4.47}$$

and we consider pure gravity which means we set $S_{matter} = 0$ here. We can compute graviton amplitudes just like what we done in quantum field theory. In D=4 dimensions, the three point graviton amplitudes are

$$M_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^8}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}, \tag{4.48}$$

$$M_3(1^+, 2^+, 3^-) = \frac{[12]^8}{[12]^2[23]^2[31]^2}. \tag{4.49}$$

The on-shell three point supergravity amplitudes are just square of Yang-Mills amplitudes

$$\begin{aligned}
M_3(1^-, 2^-, 3^+) &= A_3(1^-, 2^-, 3^+)^2, \\
M_3(1^+, 2^+, 3^-) &= A_3(1^+, 2^+, 3^-)^2
\end{aligned}$$

where A means the amplitudes forms in Yang-Mills theory.

The general results of n points gravity amplitudes can be constructed by using BCFW but an important difference is that gravity amplitudes are not color-ordered. The famous results for the n points MHV amplitudes are presented by Berends, Giele and Kuijf (BGK)

$$M_n^{tree}(1^-,2^-,3^+,\cdots,n^+) = \sum_{P(3,4,\cdots,n-1)} \frac{\langle 12\rangle^8 \prod_{l=3}^{n-1}\langle n|2+3+\cdots+(l-1)|l]}{(\prod_{i=1}^{n-2}\langle i,i+1\rangle)\langle 1,n-1\rangle\langle 1n\rangle^2\langle 2n\rangle^2(\prod_{l=3}^{n-1}\langle ln\rangle)}. \tag{4.50}$$

$P(3,4,\cdots,n-1)$ means permute the label $(3,4,\cdots,n-1)$. And the relationships between gravity and Yang-Mills theory in MHV is

$$M_n^{tree}(1^-,2^-,3^+,\cdots,n^+) = \sum_{P(i_3,i_4,\cdots,i_n)} S_{1i_n}(\prod_{k=4}^{n-1}\beta_k)A_n(1^-,2^-,i_3^+,\cdots,i_n^+)^2 \tag{4.51}$$

where $n \geq 4$ and

$$\beta_k = -\frac{\langle i_k, i_{k+1}\rangle}{\langle 2, i_{k+1}\rangle}\langle 2|i_3+i_4+\cdots+i_{k-1}|i_k]. \tag{4.52}$$

There are more general results beyond MHV amplitudes. Kawai, Lewellen, and Tye (KLT) derived KLT relation in string theory. Under string tension limits $\alpha' \to 0$, gravity amplitudes M_n^{tree} and color-order amplitudes A_n^{tree} have KLT relations:

$$M_4^{tree}(1,2,3,4) = -\, s_{12}A_4^{tree}(1,2,3,4)A_4^{tree}(1,2,4,3), \tag{4.53}$$

$$M_5^{tree}(1,2,3,4,5) = -\, s_{23}s_{45}A_5^{tree}(1,2,3,4,5)A_5^{tree}(1,3,2,5,4) + (3 \leftrightarrow 4), \tag{4.54}$$

$$\begin{aligned} M_6^{tree}(1,2,3,4,5,6) = &-\, s_{12}s_{45}A_6^{tree}(1,2,3,4,5,6)[s_{35}A_6^{tree}(1,5,3,4,6,2) \\ &+ (s_{34}+s_{35})A_6^{tree}(1,5,4,3,6,2)] + P(2,3,4). \end{aligned} \tag{4.55}$$

The last line $P(2,3,4)$ means permutation of legs $(2,3,4)$. We can use graviton amplitudes to construct other components amplitudes in superamplitudes by using supersymmetry Ward identity. For example, the MHV amplitudes of graviton and gravitino are restricted

$$M_n(1^-,\psi^-,\psi^+,4^+,\cdots,n^+) = \frac{\langle 13\rangle}{\langle 12\rangle}M_n(1^-,2^-,3^+,4^+,\cdots,n^+). \tag{4.56}$$

As we know graviton amplitudes, we can start to construct superamplitudes. Superamplitudes preserve momentum conservation and supermomentum conservation which means superamplitudes must contain two delta functions

$$M_n = \delta^4(\sum p)\delta^{16}(\tilde{Q})F \tag{4.57}$$

where F is a kinematics invariant factor. As we know n point MHV amplitudes, M_n must project to pure graviton MHV amplitudes correctly. So that the only way is

$$M_n^{MHV} = \delta^4(\sum p)\delta^{16}(\tilde{Q})\frac{M_n^{graviton}(1^-,2^-,3^+,\cdots,n^+)}{\langle 12\rangle^8}. \tag{4.58}$$

It is easily to derive 3 point $\overline{MHV}$ superamplitudes using same arguments, and the formula just like MHV amplitudes

$$M_3^{\overline{MHV}} = \delta^4(\sum p)\delta^8([12]\eta_3 + [23]\eta_1 + [31]\eta_2)\frac{1}{[12]^2[23]^2[31]^2}. \tag{4.59}$$

Having 3 point superamplitudes, we can use super BCFW to construct all kinds n points superamplitude in tree level. In the next section, we will start to search the natural building block when we try to use BCFW recursion relations. They will rely on understanding of superamplitudes formula in this section.

Part III

5 Bonus scaling and BCFW in $\mathcal{N} = 7$ supergravity

One of the fascinating themes in the study of planar $\mathcal{N} = 4$ SYM, is that the amplitude is often a solution to a geometric question. The now famous example is the realization that the building blocks for the n-point $\mathcal{N} = 4$ SYM amplitude with k- negative helicity gluons, constructed via the Britto, Cachazo, Feng and Witten (BCFW) recursion relation [1], are associated with positive cells of a Grassmannian $G(k, n)$ [2, 3], the moduli space of k-planes in n-dimensional space.

A natural question is whether such structure exists outside of $\mathcal{N} = 4$ SYM. Certain progress has been made for $\mathcal{N} = 6$ super-Chern-Simons matter theory (CSM) [4, 5], in the context of an orthogonal Grassmannian [6, 7, 8]. The common property between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 6$ CSM theory is that both allow for color decomposition such that color ordered amplitudes can be defined, and the theories enjoy an infinite dimensional Yangian symmetry [9]. In fact the building blocks that arise from the recursion are individually Yangian invariant.

Both of the above properties are absent in gravity, and thus it may be unclear how to proceed. However we may ask, if there are natural building blocks for gravity amplitudes, what would be a desirable property similar to Yangian invariance for the gauge theories. One special property of gravity amplitudes is the asymptotic behavior in the large momentum limit. Indeed it was known that in the BCFW recursion, if one shifts $|i\rangle$ and $|j]$, where i and j are a positive and negative helicity graviton respectively, as the deformation parameter z is taken to infinity, the amplitude behaves as $1/z^2$ [10].[1] This is to be compared with $1/z$ of Yang-Mills.

Thus we propose that a criteria for a "good" building block is good large-z scaling under any pair of shifted momenta. Note that in a generic BCFW representation, individual terms can behave as $1/z$ and only cancel in the sum. To begin, we will relax our criteria and ask: if one chooses two particular legs to deform, is there a representation such that individual terms scale as $1/z^2$ under large deformation? We will show that indeed such a representation exists, in the form of a BCFW recursion in $\mathcal{N} = 7$ supergravity, constructed out of a "bad-shift". $\mathcal{N} = 7$ supergravity has the same on-shell degrees of freedom as with $\mathcal{N} = 8$ supergravity, only with a reduced set of supersymmetry being

[1] Recently, it has been shown that this asymptotic behavior can be attributed to the permutation invariance of gravity amplitudes [11].

manifest. However the reduced symmetry allows us to exploit the $1/z^2$ fall off of the full $\mathcal{N} = 8$ amplitude. More precisely we claim that if one constructs the $\mathcal{N} = 7$ amplitude under the following $[j^+, i^-\rangle$ bad shift:

$$|j^+] \to |j^+] + w\,|i^-], \quad |i^-\rangle \to |i^-\rangle - w\,|j^+\rangle, \quad \eta_{j^+} \to \eta_{j^+} + w\,\eta_{i^-}. \tag{5.1}$$

Then the individual terms in the BCFW expansion scale at large z as $1/z^2$ under the following $[i^-, j^+\rangle$ shift of the same primary shifted legs:

$$|i^-] \to |i^-] + z\,|j^+], \quad |j^+\rangle \to |j^+\rangle - z\,|i^-\rangle, \quad \eta_{i^-} \to \eta_{i^-} + z\,\eta_{j^+}. \tag{5.2}$$

Note that the bad-shift in $\mathcal{N}$-supergravity behaves as $z^{8-\mathcal{N}}/z^2 = z^{6-\mathcal{N}}$, and thus it has sufficient fall off for a valid recursion relation for $\mathcal{N} = 7, 8$. As we will argue, the reason why $\mathcal{N} = 7$ bad-shift recursion allows for term by term $1/z^2$ fall off is because it secretly uses the $1/z^2$ fall off of the full amplitude. For a valid BCFW representation, all one needs is that the amplitude vanish as $z \to \infty$, thus even though gravity amplitudes behave as $1/z^2$, the usual BCFW recursion is blind to such improved fall off. On the other hand, for the $\mathcal{N} = 7$ bad-shift, the large-z fall off behaves as $1/z$ precisely because of the $1/z^2$ of the full $\mathcal{N} = 8$ amplitude. Thus, the $1/z$ fall off is crucial for the validity of the $\mathcal{N} = 7$ bad shift. The presence of $1/z^2$ fall off implies extra "bonus relations" for individual BCFW terms [12]. As we will show, for MHV amplitudes, it is precisely due to these bonus relations that the $\mathcal{N} = 7$ bad shift exhibit improved fall off relative to $\mathcal{N} = 8$.

Note that representations with term by term $1/z^2$ fall off are already known for MHV amplitudes [13]. However, no known expression with such properties exist beyond the MHV sector. The $\mathcal{N} = 7$ bad shift allows for such a representation beyond MHV level. This special property of the $\mathcal{N} = 7$ bad-shift has already been noted at the six-point level in Hodges work [14]. In this chapter we present a proof extending to general tree-level amplitudes.

This chapter is organized as follows: first we introduce BCFW recursion in the formalism of $\mathcal{N} = 7$ supergravity, and examine its validity under different scenarios, leading us to investigate the large z behavior of the $[+, -\rangle$ "bad shift" representation. We then present a proof for term-by-term $\mathcal{O}(z^{-2})$ scaling of the "bad shift" representation under a correspondingly chosen test shift. Furthermore, we discover the improved scaling in $\mathcal{N} = 7$ is related to bonus relations in $\mathcal{N} = 8$.

5.1 $\mathcal{N} = 7$ superamplitudes

Here we review the derivation of $\mathcal{N} = 7$ supergravity amplitudes from its $\mathcal{N} = 8$ counterpart, as well as its large z behavior. This discussion follows [15].

5.1.1 From $\mathcal{N} = 8$ to $\mathcal{N} = 7$

We formulate $\mathcal{N} = 8$ supergravity using an on-shell superspace by introducing eight Grassmann variables η^A, labeled by the SU(8) index $A = 1...8$. This allows us to associate the states of various helicities in the $\mathcal{N} = 8$ theory with components of different orders of η in an on-shell chiral superfield, which we write as

$$\Omega = h^+ + \psi_A\eta^A + \frac{1}{2!}v_{AB}\eta^A\eta^B + \frac{1}{3!}\chi_{ABC}\eta^A\eta^B\eta^C + \frac{1}{4!}S_{ABCD}\eta^A\eta^B\eta^C\eta^D + \frac{1}{3!}\chi^{ABC}\eta^5_{ABC} + \frac{1}{2!}v^{AB}\eta^6_{AB} + \psi^A\eta^7_A + h_-\eta^8, \tag{5.3}$$

where $\eta^5_{ABC} \equiv \frac{1}{5!}\epsilon_{ABCDEFGH}\eta^D\eta^E\eta^F\eta^G\eta^H$, and other η polynomials are similarly defined.

When we reduce the manifest supersymmetry from $\mathcal{N} = 8$ to $\mathcal{N} = 7$, the on-shell states separate into two superfields, which are obtained respectively from two different ways of reducing supersymmetry: setting η^8 to zero or integrating away η^8.

$$\Phi^+ \equiv \Omega|_{\eta^8\to 0} = \int \mathrm{d}\eta^8\ \eta^8\,\Omega\,, \tag{5.4a}$$

$$\Phi^- \equiv \int \mathrm{d}\eta^8\ \Omega\,. \tag{5.4b}$$

The explicit forms of the superfields are:

$$\Phi^+ = h^+ + \psi_A\eta^A + \frac{1}{2!}v_{AB}\eta^A\eta^B + \frac{1}{3!}\chi_{ABC}\eta^A\eta^B\eta^C + \frac{1}{3!}S^{8ABC}\eta^4_{ABC} + \frac{1}{2!}\chi^{8AB}\eta^5_{AB} + v^{8A}\eta^6_A + \psi^8\eta^7, \tag{5.5a}$$

$$\Phi^- = \psi_8 + v_{8A}\eta^A + \frac{1}{2!}\chi_{8AB}\eta^A\eta^B + \frac{1}{3!}S_{8ABC}\eta^A\eta^B\eta^C + \frac{1}{3!}\chi^{ABC}\eta^4_{ABC} + \frac{1}{2!}v^{AB}\eta^5_{AB} + \psi^A\eta^6_A + h^-\eta^7. \tag{5.5b}$$

The indices are now summed from 1 to 7, and $\eta^4_{ABC} \equiv \frac{1}{4!}\epsilon_{ABCDEFG}\eta^D\eta^E\eta^F\eta^G$. Note that setting η^8 to zero can be represented by an integration over η^8 after multiplying by η^8. The Φ^+ multiplet has helicity $+2$, and contains the positive helicity graviton h^+, while Φ^- has helicity $+3/2$, and contains the negative helicity graviton h^-. We will use a $+$ sign to mark quantities associated with the Φ^+ multiplet, while quantities associated with the Φ^- multiplet will be marked with a $-$ sign.

Using the same operations, $\mathcal{N} = 7$ amplitudes can be derived from the corresponding $\mathcal{N} = 8$ amplitudes. As an example, the $\mathcal{N} = 7$ MHV 3-point graviton scattering amplitude is obtained from the $\mathcal{N} = 8$ MHV 3-point amplitude as follows:

$$\mathcal{M}_3(1^-2^-3^+) = \int \mathrm{d}\eta_1^8\mathrm{d}\eta_2^8\mathrm{d}\eta_3^8\ \eta_3^8\,\mathcal{M}_3^{\mathrm{MHV}}(123). \tag{5.6}$$

Here the first subscript of η refers to the associated particle number, while the superscript refers to the SU(8) index.

For a general $\mathrm{N^kMHV}$ amplitude, there will be $k+2$ external legs in the Φ^- multiplet, which we denote by the set $\{x\}$, and $n-k-2$ external legs in the Φ^+ multiplet, which we denote by the set $\{y\}$. Then we have the following map between $\mathcal{N} = 7$ and $\mathcal{N} = 8$ amplitudes:

$$\mathcal{M}^{\mathcal{N}=7}(\{x\},\{y\}) = \int \left[\prod_{a=1}^{n}\mathrm{d}\eta_a^8\right]\left[\prod_{b\in\{y\}}\eta_b^8\right]\mathcal{M}^{\mathcal{N}=8}. \tag{5.7}$$

Or more explicitly,

$$\mathcal{M}_n^{\mathcal{N}=7}(1^-,\cdots,(k+2)^-,(k+3)^+,\cdots,n^+) = \int \mathrm{d}\eta_1^8\cdots\mathrm{d}\eta_n^8\;\eta_{k+3}^8\cdots\eta_n^8\,\mathcal{M}_n^{\mathcal{N}=8}(1,\cdots,n). \tag{5.8}$$

5.1.2 BCFW in the $\mathcal{N}=7$ formalism

Validity of a BCFW representation requires the amplitude vanish as the deformation parameter z goes to infinity:

$$\begin{gathered}|\hat{i}] = |i] + z|j], \quad |\hat{j}\rangle = |j\rangle + z|i\rangle, \quad \hat{\eta}_i = \eta_i + z\,\eta_j, \\ \mathcal{M}(z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty\,.\end{gathered} \tag{5.9}$$

$\mathcal{N}=8$ amplitudes scale as $\mathcal{O}\left(z^{-2}\right)$ for large z. In the case of $\mathcal{N}=7$, we can deduce the large z behavior by relating the $\mathcal{N}=7$ amplitude to the parent $\mathcal{N}=8$ using (5.7). Unlike in the case of $\mathcal{N}=8$, amplitudes in $\mathcal{N}=7$ specialize into different supermultiplet configurations for lines i,j which may show different large z behavior.

Note that in order to deduce the large z behavior of $\mathcal{N}=7$ from $\mathcal{N}=8$ using (5.7), we need to take into the subtlety that for $\mathcal{N}=8$, we shift $\hat{\eta}_i^A$ for $A=1...8$, while for $\mathcal{N}=7$, we only shift for $A=1...7$. Thus we need to somehow unshift $\hat{\eta}_i^8$. This can easily be done by a change of variables. We define

$$\eta_i^8 = \overline{\eta}_i^8 - z\overline{\eta}_j^8, \quad \eta_a^8 = \overline{\eta}_a^8 \quad \text{for } a \neq i. \tag{5.10}$$

The Jacobian is simply 1. Now we can promote (5.7) into a relation for the shifted variables:

$$\mathcal{M}^{\mathcal{N}=7}(z) = \int \left[\prod_{a=1}^{n}\mathrm{d}\overline{\eta}_a^8\right]\left[\prod_{b\in\{y\}}\eta_b^8(\overline{\eta}_c^8)\right]\mathcal{M}^{\mathcal{N}=8}(z)\,, \tag{5.11}$$

where η_b^8 is a function of $\overline{\eta}_c^8$, as defined by (5.10).

We can now analyze different scenarios for which multiplet the lines i,j in our $[i,j\rangle$ shift sits in:

- For $[i^-,j^+\rangle$ and $[i^-,j^-\rangle$: Since i is not in the Φ^+ multiplet, η_b^8 does not contain any z dependence, and hence the $\mathcal{N}=7$ amplitude behaves as $\mathcal{O}\left(z^{-2}\right)$ at large z exactly like $\mathcal{N}=8$.

- For $[i^+,j^+\rangle$: Now i belongs to the Φ^+ multiplet, so naively applying a change of variable, one would pick up a z factor. However the z will be proportional to $\overline{\eta}_j$ which is already present in η_b^8 and thus this term drops out, i.e. $(\overline{\eta}_i - z\overline{\eta}_j)\overline{\eta}_j = \overline{\eta}_i\overline{\eta}_j$. Thus we see for this shift, the $\mathcal{N}=7$ amplitude again behaves as $\mathcal{O}\left(z^{-2}\right)$ at large z exactly like $\mathcal{N}=8$.

- For $[i^+,j^-\rangle$: Now i belongs to the Φ^+ multiplet, while j does not, so η_b^8 obtains an overall factor of z. Thus the large z behavior for $\mathcal{N}=7$ amplitude behaves as $\mathcal{O}\left(z\right)\times\mathcal{O}\left(z^{-2}\right) = \mathcal{O}\left(z^{-1}\right)$.

From the above we conclude that for the "good" shifts $[i^-, j^+\rangle$, $[i^-, j^-\rangle$, $[i^+, j^+\rangle$, the $\mathcal{N}=7$ amplitude behaves as $1/z^2$ just as the $\mathcal{N}=8$ parent. The BCFW built for $\mathcal{N}=7$ from the good shifts will be using the same $1/z$ pole as the $\mathcal{N}=8$ parent. Thus the BCFW built from the $[+,-\rangle$ "bad" shift in $\mathcal{N}=7$ is secretly using information of the $1/z^2$ behavior of the $\mathcal{N}=8$ amplitude. In the following section, we will demonstrate that the $\mathcal{N}=7$ BCFW expansion built from the $[j^+, i^-\rangle$ "bad shift" indeed has bonus behavior in the form of term-by-term $\mathcal{O}(z^{-2})$ large-z scaling under the $[i^-, j^+\rangle$ test shift.

5.2 Bonus z scaling of $\mathcal{N}=7$ "bad shift" BCFW terms

5.2.1 A particular $[-,+\rangle$ test shift: N^{k}MHV amplitudes

We would like to prove that the $\mathcal{N}=7$ $[j^+, i^-\rangle$ "bad shift" BCFW terms have $\mathcal{O}(z^{-2})$ large z fall off under the secondary $[i^-, j^+\rangle$ test shift. Note our analysis can be easily applied to other helicity configurations as well, where the $\mathcal{O}(z^{-2})$ fall off is no longer present. Therefore, we start without fixing which superfields particles i and j belong to and construct the $[j,i\rangle$ BCFW representation of the amplitude (see Fig. 1):

$$\mathcal{M}_n(1,\cdots,i,\cdots,j,\cdots,n) = \sum \int d^7\eta_{\hat{P}}\ \mathcal{M}_L(-\hat{P},\hat{j},\cdots)\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{P},\hat{i},\cdots)\ |_{\hat{P}^2=0}\,, \tag{5.12}$$

$$|\hat{j}] = |j] + w|i], \quad |\hat{i}\rangle = |i\rangle - w|j\rangle, \quad \hat{\eta}_j = \eta_j + w\,\eta_i\,. \tag{5.13}$$

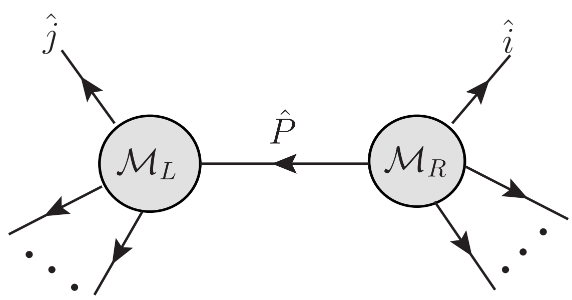

Figure 1: Diagram of a BCFW term.

For the on-shell condition $\hat{P}^2 = (P + w\,|i]\langle j|)^2 = 0$, we can solve for w and $\hat{P}$ in terms of i, j and P. Leaving details of derivation to Appendix A, the result is[2]

$$w = -\frac{P^2}{\langle j|P|i]}\,, \tag{5.14}$$

$$\hat{P} = \frac{P|j\rangle[i|P}{\langle j|P|i]}\,. \tag{5.15}$$

Let us now deform (5.12) by an $[i,j\rangle$ test shift:

$$|i](z) = |i] + z|j], \quad |j\rangle(z) = |j\rangle - z|i\rangle, \quad \eta_i(z) = \eta_i + z\,\eta_j. \tag{5.16}$$

[2] We adopt the "mostly minus" metric convention, such that $p_k = |k]\langle k|$ and $s_{ij} = (p_i + p_j)^2 = [ij]\langle ji\rangle$ for massless particles.

Under the test shift, the amplitude is deformed into

$$\mathcal{M}_n(z) = \sum \int d^7\eta_{\hat{P}}\, \mathcal{M}_L(-\hat{P}(z), \hat{j}(z), \cdots)\, \frac{1}{P^2(z)}\, \mathcal{M}_R(\hat{P}(z), \hat{i}(z), \cdots) \tag{5.17}$$

Now $|i], |j\rangle, \eta_i, P^2, |\hat{j}], |\hat{i}\rangle, \hat{\eta}_j, |\hat{P}], |\hat{P}\rangle$ have become functions of z. Since the BCFW terms must have zero little group weight in $\hat{P}$, the z dependence of the BCFW terms only comes from $|i], |j\rangle, \eta_i, P^2, |\hat{j}], |\hat{i}\rangle, \hat{\eta}_j, \hat{P}$. By analyzing their large z behavior individually, we can deduce the large z behavior of the BCFW term as a whole. We thus proceed to do so.

From the $[i, j\rangle$ test shift (5.16), deriving the large-z behavior of $|i], |j\rangle, \eta_i, P^2$ is straightforward:

$$|i](z) \longrightarrow \mathcal{O}(z)\,, \quad |j\rangle(z) \longrightarrow \mathcal{O}(z)\,, \quad \eta_i(z) \longrightarrow \mathcal{O}(z)\,. \tag{5.18}$$

$$P^2(z) = P^2 - z\langle i|P|j] \longrightarrow \mathcal{O}(z)\,. \tag{5.19}$$

The primary deformed quantities $|\hat{j}], |\hat{i}\rangle, \hat{\eta}_j, \hat{P}$ transform under the test shift as

$$|\hat{j}](z) = |j] + w(z)|i](z), \quad |\hat{i}\rangle(z) = |i\rangle - w(z)|j\rangle(z), \quad \hat{\eta}_j(z) = \eta_j + w(z)\eta_i(z), \tag{5.20}$$

$$\hat{P}(z) = \frac{-(P - p_j)|i\rangle[j|(P + p_i)}{\langle i|P|j]} + \mathcal{O}\left(z^{-1}\right) \longrightarrow \mathcal{O}\left(z^0\right)\,. \tag{5.21}$$

To determine the large-z behavior of $|\hat{j}], |\hat{i}\rangle, \hat{\eta}_j$, we solve for the z-deformed primary shift parameter $w(z)$, and expand it in powers of z:

$$w(z) = -\frac{1}{z} + \frac{-P^2 - \langle j|P|j] + \langle i|(P - p_j)|i]}{\langle i|P|j]}\frac{1}{z^2} + \mathcal{O}\left(z^{-3}\right). \tag{5.22}$$

We expand to $\mathcal{O}\left(z^{-2}\right)$ since the leading term gets canceled when we plug in expressions (5.16) and (5.22) into (5.20). We get:

$$\begin{aligned}
|\hat{j}](z) &= \left(-|i] + \frac{-P^2 - \langle j|P|j] + \langle i|(P - p_j)|i]}{\langle i|P|j]}|j]\right)\frac{1}{z} + \mathcal{O}\left(z^{-2}\right), \\
|\hat{i}\rangle(z) &= \left(|j\rangle + \frac{-P^2 - \langle j|P|j] + \langle i|(P - p_j)|i]}{\langle i|P|j]}|i\rangle\right)\frac{1}{z} + \mathcal{O}\left(z^{-2}\right), \\
\hat{\eta}_j(z) &= \left(-\eta_i + \frac{-P^2 - \langle j|P|j] + \langle i|(P - p_j)|i]}{\langle i|P|j]}\eta_j\right)\frac{1}{z} + \mathcal{O}\left(z^{-2}\right)\,.
\end{aligned} \tag{5.23}$$

Now we can read off their large-z behavior. The results are organized below:

$$\boxed{\begin{gathered}
|i](z) \longrightarrow \mathcal{O}(z)\,, \quad |j\rangle(z) \longrightarrow \mathcal{O}(z)\,, \quad \eta_i(z) \longrightarrow \mathcal{O}(z)\,, \\
|\hat{j}](z) \longrightarrow \mathcal{O}\left(z^{-1}\right), \quad |\hat{i}\rangle(z) \longrightarrow \mathcal{O}\left(z^{-1}\right), \quad \hat{\eta}_j(z) \longrightarrow \mathcal{O}\left(z^{-1}\right)\,, \\
\hat{P}(z) \longrightarrow \mathcal{O}\left(z^0\right)\,, \\
P^2(z) \longrightarrow \mathcal{O}(z)\,.
\end{gathered}} \tag{5.24}$$

With the large-z scaling of $|i], |j\rangle, \eta_i, P^2, |\hat{j}], |\hat{i}\rangle, \hat{\eta}_j, \hat{P}$ in hand, we can know how the BCFW term behaves at large z by counting the orders of these contributing components.

From (5.24) we see that $|i], \eta_i$, which have helicity $1/2$, behave as $\mathcal{O}(z)$. On the other hand, $|\hat{i}\rangle$, which has helicity $-1/2$, scales oppositely as $\mathcal{O}(z^{-1})$. We can write a general Ansatz that if particle i contributes to the amplitude in the form of $|i]^a \eta_i^b |\hat{i}\rangle^c$, then it scales as $\mathcal{O}\left(z^{a+b-c}\right)$.

In general, determining the orders of the spinors and the Grassmann variable can be nontrivial. However, in this case little group scaling of external leg i trivializes the counting by fixing $a+b-c=2h_i$, where h_i and h_j are the helicities of the superfield corresponding to legs i and j. Therefore, particle i contributes $\mathcal{O}\left(z^{2h_i}\right)$ at large z. A similar analysis shows that particle j contributes $\mathcal{O}\left(z^{-2h_j}\right)$ at large z. Since $\hat{P}$ approaches a constant at $z \to \infty$, the large z scaling of each BCFW term is of:

$$\boxed{\mathcal{O}\left(z^{2(h_i-h_j)-1}\right).} \tag{5.25}$$

Crucial to this result is the choice of the $[j, i\rangle$ primary shift followed by $[i, j\rangle$ test shift, which enjoys the cancellation of order z^0 terms while obtaining (5.23) and thus ensures that the square spinors and the Grassmann variable scale oppositely to the angle spinors. Other choices would not have allowed us to determine the large z scaling from the helicities alone. For example, if we chose a $[j, i\rangle$ primary shift followed by a $[k, j\rangle$ test shift, where $i \neq k$, then $|k]$ and η_k would scale as $\mathcal{O}(z)$ while $|k\rangle$ scale as $\mathcal{O}(z^0)$. If particle k contributes to the amplitude in the form of $|k]^a \eta_k^b |k\rangle^c$, then it would scale as $\mathcal{O}\left(z^{a+b}\right)$, so $a+b-c=2h_k$ would not be sufficient to determine the large z scaling contributed by particle k.

Note that up until this point we have not designated the helicities of superfields i and j. If we choose a $[j^+, i^-\rangle$ "bad" $\mathcal{N}$ supershift for supergravity, h_j and h_i would be separated by $\frac{8-\mathcal{N}}{2}$, such that the large z scaling of each BCFW term be:

$$\boxed{\mathcal{O}\left(z^{\mathcal{N}-9}\right).} \tag{5.26}$$

We now specialize to the $\mathcal{N}=7$ $[j^+, i^-\rangle$ "bad shift" BCFW expansion under the secondary $[i^-, j^+\rangle$ test shift. From the expressions for the $\mathcal{N}=7$ superfields (5.5), superfield i has helicity $+3/2$ and therefore contributes $\mathcal{O}(z^3)$ at large z, while superfield j has helicity $+2$ and gives us $\mathcal{O}(z^{-4})$. $1/P^2$ gives $\mathcal{O}(z^{-1})$. Collectively, we find that the large z scaling for the BCFW term is of:

$$\boxed{\mathcal{O}\left(z^{-2}\right).} \tag{5.27}$$

We are lead to this result only if we specialize to the case where the $[j^+, i^-\rangle$ bad shift is the primary shift. Other choices can result in $\mathcal{O}(z^{-1})$ or worse fall off. However, note that our counting is only indicative of the worst behavior, so the terms can actually have better fall off than shown by the counting. For example, both $\mathcal{N}=7$ $[j^+, i^+\rangle$ and $[j^-, i^-\rangle$ count to $\mathcal{O}(z^{-1})$, but explicit calculations have shown that some but not all of their BCFW terms behave as $\mathcal{O}(z^{-2})$.

Finally, note that the place where $\mathcal{N}=7$ plays a crucial role is the fact that the bad shift BCFW recursion is not valid for $\mathcal{N}<7$, while $\mathcal{N}=8$ does not distinguish between different shifts.

5.2.2 General $[-,+\rangle$ test shifts: the MHV case

The above result fails for general BCFW test shifts other than the $[i^-, j^+\rangle$ shift, and an alternative analysis is required. In general, there are many combinations of test shifts that we can choose from, however we are mainly concerned with the $[-,+\rangle$ test shift, since it is the most relevant in the high energy limit. In the following, we analyze the large z scaling under general $[-,+\rangle$ test shifts in the MHV case. (See Fig. 2)

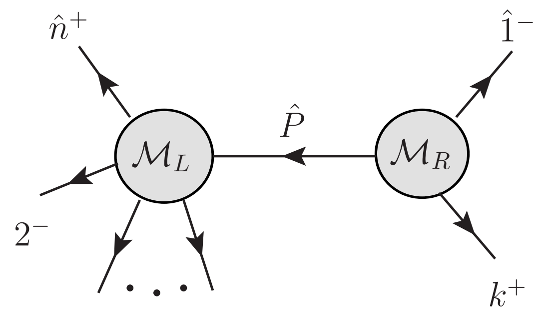

Figure 2: Diagram of a MHV "bad shift" BCFW term.

Choosing the $[n^+, 1^-\rangle$ primary shift, the amplitude factorizes into an $n-1$ point MHV subamplitude and a 3-point $\overline{\text{MHV}}$ subamplitude. Similar to our previous analysis, first we solve for w and $\hat{P}$:

$$w = \frac{\langle 1k\rangle}{\langle nk\rangle}\,, \tag{5.28}$$

$$\hat{P} = -\left(|k] + \frac{\langle n1\rangle}{\langle nk\rangle}|1]\right)\langle k|. \tag{5.29}$$

We now analyze the large z scaling under different $[-,+\rangle$ test shifts:

- For the $[1^-, n^+\rangle$ shift: The proof in the previous section applies, and there is $\mathcal{O}(z^{-2})$ term by term behavior.
- For the $[2^-, n^+\rangle$ shift: There is $\mathcal{O}(z^{-2})$ term by term behavior. The large z behavior of the deformed quantities are:

$$\begin{aligned}
\hat{P} &\longrightarrow \mathcal{O}\left(z^0\right)\\
|2](z) &= |2] + z|n]\\
|n\rangle(z) &= |n\rangle - z|2\rangle\\
|\hat{n}](z) &= |n] + w|1] \longrightarrow |n]\\
|\hat{1}\rangle(z) &= |1\rangle - w(|n\rangle - z|2\rangle) \longrightarrow \mathcal{O}\left(z^0\right).
\end{aligned} \tag{5.30}$$

  In the large z limit, dependence on z only comes from the $n-1$ point subamplitude $\mathcal{M}_L$, also we see that $|\hat{n}] \to |n]$. Therefore, the chosen test shift is precisely a BCFW shift on the subamplitude $\mathcal{M}_L$ at large z, so the BCFW term must scale as $\mathcal{O}(z^{-2})$.

- For a $[2^-, m^+\rangle$ shift (where $m \neq n$): Individual terms scale as $\mathcal{O}(z^{-2})$. The same argument as above applies if m is not on the 3 point amplitude, so terms scale as $\mathcal{O}(1/z^2)$. Moreover, the BCFW expansion is summed over all possible permutations, but there is only one diagram where m is on the 3 point amplitude, therefore this term must also scale as $\mathcal{O}(z^{-2})$, since the existence of an $\mathcal{O}(z^{-1})$ part cannot be canceled by other terms.

- For a $[1^-, m^+\rangle$ test shift: The above argument fails and there are terms which do not behave as $\mathcal{O}(z^{-2})$.

Summarizing the results above, we have demonstrated that for the MHV case, the $\mathcal{N} = 7$ $[n^+, 1^-\rangle$ bad shift BCFW representation has $\mathcal{O}(z^{-2})$ term by term large z scaling under $[1^-, n^+\rangle$, $[2^-, n^+\rangle$ and $[2^-, m^+\rangle$ test shifts.

5.2.3 Comparison to other formulas for supergravity amplitudes

The large z scaling of the "bad shift" BCFW representation can be compared with the tree formula for MHV amplitudes by Nguyen, Spradlin, Volovich, and Wen [13], which also manifest $\mathcal{O}(z^{-2})$ large z fall off term-by-term under certain test shifts. The formula chooses two legs as special, and involves a sum of terms each represented by a tree diagram. By directly counting the orders of z in the z deformed formula, we see that if at least one of test shift legs are special, then the term will scale as $\mathcal{O}(z^{-2})$. Otherwise, for an $[i, j\rangle$ test shift where neither i or j is a special leg, the term scales as $\mathcal{O}\left(z^{\deg(i)+\deg(j)-4}\right)$. The degree of a leg refers to the number of propagators that connect to the leg in the tree diagram. The best fall off occurs when both leg i and j have only one connection, where the term scales as $\mathcal{O}(z^{-2})$. The tree formula and the $\mathcal{N} = 7$ BCFW is complementary in the sense that both manifest the $\mathcal{O}(z^{-2})$ scaling term by term, but under different conditions of test shift legs.

5.3 $\mathcal{N} = 8$ bonus relations and $\mathcal{N} = 7$ bonus scaling: the MHV case

After demonstrating our proof, we would like to show that $\mathcal{N} = 7$ BCFW terms manifest the improved scaling because they are using "bonus relations", which come from the $\mathcal{O}(z^{-2})$ fall off of $\mathcal{N} = 8$ amplitudes. The bonus scaling of $\mathcal{N} = 8$ amplitudes enables us to multiply a linear function of z on our amplitude and deform z as in BCFW recursion, except that we do not have to consider the boundary integral. These extra relations are called "bonus relations". Multiplying by the s channel, we have the sum over residues at $z = z_k$,

$$s(0)\, \mathcal{M}_n^{\mathcal{N}=8} = \sum_k s(z_k) \int \mathrm{d}^8\eta_{\hat{P}}\, \mathcal{M}_L \, \frac{1}{P^2}\, \mathcal{M}_R. \tag{5.31}$$

Our purpose is to use the bonus relations to recombine $\mathcal{N} = 8$ terms and cancel out linear relations between terms, such that the remaining expression corresponds to the $\mathcal{N} = 7$ representation. The following analysis focuses on the MHV case for simplicity and parallels Appendix C of [16]. Note that the BCFW representation for the $\mathcal{N} = 8$ n-point MHV amplitude will always have one more diagram than $\mathcal{N} = 7$. We will show that we can use the bonus relation to express the additional $\mathcal{N} = 8$ term using terms appearing in $\mathcal{N} = 7$. More explicitly, we write the $\mathcal{N} = 8$ n-point MHV amplitude as $\mathcal{M}(123\cdots n)$ or $\mathcal{M}_n^{\mathcal{N}=8}$, the $\mathcal{N}=7$ amplitude as $\mathcal{M}(1^-2^-3^+\cdots n^+)$ or $\mathcal{M}_n^{\mathcal{N}=7}$, and construct the BCFW representation using the $[n^+,1^-\rangle$ shift:

$$|\hat{n}]=|n]+w|1],\quad |\hat{1}\rangle=|1\rangle-w|n\rangle,\quad \hat{\eta}_n=\eta_n+w\,\eta_1. \tag{5.32}$$

The $\mathcal{N}=8$ representation has $n-2$ diagrams while the $\mathcal{N}=7$ representation has $n-3$ diagrams. The additional term for $\mathcal{N}=8$ can be written as

$$\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}2). \tag{5.33}$$

Intuitively, we want to expand this term into the other $n-3$ terms, so we separate the additional term and multiply S_{12} on each side

$$\mathcal{M}_n^{\mathcal{N}=8}=\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}2)+\sum_{k=3}^{n-1}\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}k), \tag{5.34}$$

$$s_{12}(0)\,\mathcal{M}_n^{\mathcal{N}=8}=\sum_{k=3}^{n-1}s_{12}(z_k)\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}k). \tag{5.35}$$

After some manipulation, we successfully expand the additional term in $\mathcal{N}=8$ using others terms which have correspondence with $\mathcal{N}=7$.

$$\mathcal{M}_n^{\mathcal{N}=8}=\sum_{k=3}^{n-1}\frac{s_{12}(z_k)}{s_{12}(0)}\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}k). \tag{5.36}$$

To compare with $\mathcal{N}=7$, we need to reduce the $\mathcal{N}=8$ terms to $\mathcal{N}=7$. In the MHV case, legs 1 and 2 are in multiplet Φ^-, which have helicity $+3/2$, while the other particles are in multiplet Φ^+, which has helicity $+2$, so we integrate out η_1^8,η_2^8 and $\hat{\eta_P}$ in the integral in (5.36) as follows:

$$\begin{aligned}
&\int \mathrm{d}^8\eta_{\hat{P}}\,\mathcal{M}_L\,\frac{1}{P^2}\,\mathcal{M}_R(\hat{1}\hat{P}k)\\
&=\int \mathrm{d}\eta_1^8\mathrm{d}\eta_2^8\int \mathrm{d}\hat{\eta}_P^8\,\delta(|n\rangle\hat{\eta}_n^8+|\hat{P}\rangle\eta_{\hat{P}}^8+\cdots)\,\delta([1k]\eta_{\hat{P}}^8+[k\hat{P}]\eta_1^8+[\hat{P}1]\eta_k^8)\int \mathrm{d}^7\eta_{\hat{P}}\,\widetilde{\mathcal{M}}_L\,\frac{1}{P^2}\,\widetilde{\mathcal{M}}_R\\
&=(w\langle 2n\rangle[1k]+[k\hat{P}]\langle 2\hat{P}\rangle)\int \mathrm{d}^7\eta_{\hat{P}}\,\widetilde{\mathcal{M}}_L\,\frac{1}{P^2}\,\widetilde{\mathcal{M}}_R\\
&=\langle 12\rangle[1k]\int \mathrm{d}^7\eta_{\hat{P}}\,\widetilde{\mathcal{M}}_L\,\frac{1}{P^2}\,\widetilde{\mathcal{M}}_R,
\end{aligned} \tag{5.37}$$

where $\widetilde{\mathcal{M}}_L$ and $\widetilde{\mathcal{M}}_R$ are $\mathcal{M}_L$ and $\mathcal{M}_R$ with the supermomentum conservation delta function stripped off. Combining this result with (5.36), we obtain

$$\sum_{k=3}^{n-1}\int \mathrm{d}^7\eta_{\hat{P}}\,\langle\hat{1}2\rangle[1k]\,\widetilde{\mathcal{M}}_L\,\frac{1}{P^2}\,\widetilde{\mathcal{M}}_R, \tag{5.38}$$

which is exactly the explicit form for the corresponding $\mathcal{N}=7$ BCFW representation:

$$\sum_{k=3}^{n-1}\int \mathrm{d}^7\eta_{\hat{P}}\,\mathcal{M}_L^{\mathcal{N}=7}\,\frac{1}{P^2}\,\mathcal{M}_R^{\mathcal{N}=7}=\sum_{k=3}^{n-1}\int \mathrm{d}^7\eta_{\hat{P}}\,\langle 2\hat{P}\rangle[\hat{P}k]\widetilde{\mathcal{M}}_L\frac{1}{P^2}\widetilde{\mathcal{M}}_R. \tag{5.39}$$

What we have demonstrated is that we can use a bonus relation to relate $\mathcal{N}=8$ BCFW terms to $\mathcal{N}=7$ BCFW terms. In other words, the reason why $\mathcal{N}=7$ BCFW terms have nicer large z behavior in this example is precisely because they are implicitly using bonus relations to cancel out linear dependent terms which appear in the $\mathcal{N}=8$ representation.

The next question we can ask is whether the result applies to the general n-point $\mathrm{N^kMHV}$ case. To answer this question, we try the same analysis on the 6-point NMHV amplitude. Now we have 14 terms in $\mathcal{N}=8$ compared with 9 terms in $\mathcal{N}=7$, so we require 5 bonus relations to reduce the additional 5 terms to the other 9 terms. We cannot continue, since we only have one bonus relation and it is impossible to solve 5 parameters with one condition in general. This implies the $\mathcal{O}(z^{-2})$ large z behavior of $\mathcal{N}=7$ individual terms include not only bonus relations which cancel out linear dependence but also some unknown property in $\mathcal{N}=7$.

5.4 Bonus scaling of "bad shift" BCFW for string amplitudes

Applications of BCFW recursion to string amplitudes have demonstrated improved large z scaling compared to field theory amplitudes in certain kinematic regimes [17] [18]. This not only validates the construction of a "bad shift" recursion formula without the requirement of $\mathcal{N}=7$ supersymmetry, but also enables the application of our previous argument to pursue even better term-by-term large z bonus scaling.

Since we encounter an infinite tower of massive states in string theory, we first demonstrate the validity of our argument in the case of a massive propagator. The previous derivation is modified such that the on-shell condition becomes $\hat{P}^2=(P+w\,|i]\langle j|)^2=m^2$. The primary shift parameter w and $|\hat{j}],|\hat{i}\rangle,\hat{\eta}_j,\hat{P}$ become:

$$w_m=\frac{-P^2+m^2}{\langle j|P|i]} \tag{5.40}$$

$$|\hat{j}]_m=|j]+w_m|i],\quad |\hat{i}\rangle_m=|i\rangle-w_m|j\rangle,\quad \hat{\eta}_{j_m}=\eta_j+w_m\eta_i, \tag{5.41}$$

$$\hat{P}_m=\frac{P|j\rangle[i|P-m^2|i]\langle j|}{\langle j|P|i]}\,. \tag{5.42}$$

In the numerator of w_m, the additional m^2 term scales as z^0 while the original P^2 scales as z, so the large z scaling of w_m and hence $|\hat{j}],|\hat{i}\rangle,\hat{\eta}_j$ are not affected. The large z scaling of $\hat{P}_m$ is $\mathcal{O}(z^0)$, which is also unchanged compared to that of the massless $\hat{P}$. Hence making the propagator massive does not affect the large z behavior under the $[i,j\rangle$ test shift.

It was shown in [18] that the large z scaling under an $[i,j\rangle$ shift of superstring gluon amplitudes is improved by $z^{-\alpha' s_{ij}}$ compared to the corresponding field theory amplitude. For a $[j^+,i^-\rangle$ adjacent bad shift, the superstring amplitude scales as $z^{-\alpha' s_{ij}+3-\mathcal{N}}$ since the corresponding super-Yang-Mills amplitude scales as $z^{3-\mathcal{N}}$, thus by requiring the amplitude fall off faster than z^0, this leads to the kinematic condition $\mathrm{Re}\,[3-\mathcal{N}-\alpha' s_{ij}]<0$ for a valid representation. Following our previous result (5.25), under an $[i^-,j^+\rangle$ test shift the $\mathcal{N}$ bad shift representation has $z^{\mathcal{N}-5}$ term-by-term scaling, compared to the $z^{-\alpha' s_{ij}-1}$ large z fall off of the whole amplitude. Note the curious result that for $3-\mathcal{N}<\mathrm{Re}\,[\alpha' s_{ij}]<4-\mathcal{N}$, the term-by-term scaling is actually better than the whole amplitude. We turn to a specific amplitude for further investigation.

As an example, we look at the superstring four-point gluon component amplitude, which is given by:

$$A_4(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)}{\Gamma(1+\alpha'(s+t))} \tag{5.43}$$

Here the s and t are the usual Mandelstam variables, which in our convention read as $s = s_{12} = (p_1 + p_2)^2$, $t = s_{23} = (p_2 + p_3)^2$, and $u = s_{13} = (p_1 + p_3)^2$. The kinematic constraint for a valid recursion for this amplitude $\mathrm{Re}\,[3 - \alpha' t] < 0$ was first given in [17] by demonstrating the vanishing of the boundary term. We construct a bad shift representation by first deforming the amplitude with a $\mathcal{N} = 0$ $[3^+, 2^-\rangle$ shift,

$$A_4(w) = \frac{(\langle 12\rangle - w\langle 13\rangle)^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{\Gamma(1+\alpha' s + w\alpha'[12]\langle 13\rangle)\Gamma(1+\alpha' t)}{\Gamma(1+\alpha'(s+t) + w\alpha'[12]\langle 13\rangle)}. \tag{5.44}$$

From the asymptotic expansion of the ratio of gamma functions, which can be obtained by using Stirling's series,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}\mathcal{O}\left(z^{-2}\right)\right], \tag{5.45}$$

we can readily see that $A_4(w)$ indeed scales as $w^{-\alpha' t + 3}$.

Using the function $\frac{A_4(w)}{z}$, we can form the $[3^+, 2^-\rangle$ representation of the amplitude as the sum of the residues at $w = -\frac{k+\alpha' s}{\alpha'[12]\langle 13\rangle}, k \in \mathbb{N}$. This representation can be simplified into

$$A_4(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{-1}{\alpha'^3 s^3} \sum_{k=1}^{\infty} \binom{\alpha' t}{k} \frac{(-1)^k k^4}{k + \alpha' s}. \tag{5.46}$$

Through direct summation using Mathematica, we can observe the convergence of the bad shift representation (5.46) to the closed form of the amplitude (5.43) within the kinematic regime $\mathrm{Re}\,[3 - \alpha' t] < 0$. Another way to look at the convergence of the series is through the alternating series test. The ratio between terms of the series a_k expands at large k as

$$r = \left|\frac{a_{k+1}}{a_k}\right| = 1 + \frac{3 - \alpha' t}{k} + \mathcal{O}\left(k^{-2}\right). \tag{5.47}$$

We obtain the condition $3 - \alpha' t < 0$ by requiring $r < 1$ for sufficiently large k such that the series converges.

Under the $[2^-, 3^+\rangle$ test shift, the $[3^+, 2^-\rangle$ bad shift representation deforms into

$$A_4(z) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle(\langle 34\rangle + z\langle 24\rangle)\langle 41\rangle} \frac{-1}{\alpha'^3(s - z\langle 12\rangle[13])^3} \sum_{k=1}^{\infty} \binom{\alpha' t}{k} \frac{(-1)^k k^4}{k + \alpha'(s - z\langle 12\rangle[13])}. \tag{5.48}$$

From this form, we can observe directly that individual terms of the series fall off as z^{-5} as predicted. Also note that for $\alpha' t = n \in \mathbb{N}$, the series terminates after n terms and $A_4(z)$ has finite poles, in contrast to the case for $\alpha' t$ at generic values. This property can also be observed by shifting the closed form formula for A_4.

We now turn to the previously mentioned curiosity at $3 < \mathrm{Re}[\alpha' t] < 4$. Firstly, it is tested numerically by Mathematica that the series converges in this kinematic region and that under the $[2^-, 3^+\rangle$ test shift, individual terms scale as z^{-5} at large z, better than the $z^{-\alpha' t-1}$ scaling of the amplitude in its closed form. We observe that the series converges slower at larger z, such that the number of terms required to sum to a certain fraction of the amplitude increases with z. From this, we expect that convergence issues may arise at the large z limit, allowing the large z fall off for individual terms to be better than the closed form in this kinematic region.

Similar analysis can be applied to the closed superstring. In our previous reasoning for supergravity, we noted that our argument for bonus scaling only applies to $\mathcal{N} = 7$ since the amplitude scales as $z^{6-\mathcal{N}}$ under the bad shift, and thus only offers a valid representation for $\mathcal{N} > 6$. For gravitons in the superstring, the condition for a valid $[j^+, i^-\rangle$ "bad shift" representation is:

$$\mathrm{Re}\,[6 - \mathcal{N} - 2\alpha' s_{ij}] < 0. \tag{5.49}$$

In this kinematic regime, the $[j^+, i^-\rangle$ bad shift representation has $z^{\mathcal{N}-9}$ term-by-term large z scaling under an $[i^-, j^+\rangle$ test shift according to (5.25), compared to the $z^{-2\alpha' s_{ij}^2 - 2}$ scaling of the whole amplitude. Similarly, note that the term-by-term large z fall off is better than the whole amplitude for $6 - \mathcal{N} < \mathrm{Re}\,[2\alpha' s_{ij}] < 7 - \mathcal{N}$.

6 Conclusion and Future directions

In this thesis, we briefly review spinor formalism and use spinor to present amplitudes. We introduce the powerful recursion relation help us construct higher points amplitudes. We generalize recursion relation to superamplitudes in part II. The last part we prove that the "bad shift" BCFW representation of $\mathcal{N}=7$ supergravity gives building blocks that exhibit term by term bonus $\mathcal{O}(z^{-2})$ fall off. In particular, we prove that using the $[j^+, i^-\rangle$ BCFW representation of $\mathrm{N^kMHV}$ amplitudes, each term vanishes as $\mathcal{O}(z^{-2})$ under the $[i^-, j^+\rangle$ deformation. Focusing on the MHV case, we find that the $\mathcal{O}(z^{-2})$ behavior is also present for a large number of other $[-,+\rangle$ deformations. For example, in the $[n^+, 1^-\rangle$ representation, all $[2^-, m^+\rangle$ deformation exhibits term by term $\mathcal{O}(z^{-2})$ asymptotic behavior. The reason that the "bad shift" is a valid BCFW shift can be traced back to the $\mathcal{O}(z^{-2})$ fall off of $\mathcal{N}=8$ supergravity, which allows for the susy reduction to still have vanishing asymptotic, i.e. the shift behaves as $\mathcal{O}(z^{-2})$. Thus the "bad shift" BCFW representation of $\mathcal{N}=7$ supergravity is the only BCFW recursion that utilizes the $\mathcal{O}(z^{-2})$ fall off of the amplitude. We demonstrate this claim by showing that for the MHV case, we can use the bonus relation to recombine building blocks in $\mathcal{N}=8$ BCFW into building blocks of the $\mathcal{N}=7$ bad shift.

Our previous analysis only allows us to relate the BCFW representation of $\mathcal{N}=8$ supergravity to the $\mathcal{N}=7$ bad shift representation for the MHV amplitude. This relation is no longer straightforward for NMHV amplitude and beyond. For example the six-point NMHV contains 14 diagrams in $\mathcal{N}=8$ supergravity versus 9 diagrams for $\mathcal{N}=7$ bad-shift representation. Since there is only one bonus relation at each multiplicity, it is insufficient to convert one representation to the other, unless one incorporates the information of the bonus relations for the lower point amplitudes. This would require us to further expand the BCFW representation. Indeed it is known that using all bonus relation, one can express the supergravity amplitudes in terms of $(n-3)!$ building blocks [19]. It will be interesting to see if one can utilize these building blocks to form term by term $\mathcal{O}(z^{-2})$ fall off for all deformations.

Recent studies [11] have shown how BCFW terms of gravitational amplitudes can pair into combinations with improved permutation invariance, such that leading $\mathcal{O}(z^{-1})$ pieces cancel and $\mathcal{O}(z^{-2})$ fall off is exposed. However, it appears that to have $\mathcal{O}(z^{-2})$ fall off for all shifts, one eventually requires the combination of everything and end up with the full amplitude, which is similar to the $\mathcal{N}=7$ bad shift result. Thus it would appear that the improved fall off obtained by implementing partial permutation invariance can be similarly achieved without. It might be interesting to perform a general search of rational functions of spinor products that satisfies the correct helicity weight, mass dimension, at most simple poles and $\mathcal{O}(z^{-2})$ fall off for all shifts. These are very stringent constraints, and it is likely that the solution can serve as the true building blocks for the amplitude.

Finally, we note that the "bad shift" BCFW recursion is also valid for string amplitudes under certain kinematic conditions. Unlike the story for the $\mathcal{N}=7$ theory, whose validity of the "bad shift" BCFW is attributed to the bonus fall off of $\mathcal{N}=8$ gravity, here the validity of the string amplitude representation is tied to its improved high-energy behavior. Due to the enhanced large z scaling of string amplitudes, the restriction to the $\mathcal{N}=7$ representation is lifted and we can further reduce supersymmetry to expose better term-by-term large z fall off compared to field theory. Furthermore, just as the bonus scaling of the $\mathcal{N} = 7$ bad shift representation may be considered as the incorporation of $\mathcal{N} = 8$ bonus relations, the improved behavior of BCFW terms of string amplitudes hint at possible relations inviting deeper investigation. It would be interesting to understand further, whether or not new symmetry or new amplitude relations emerge from this picture.

A Derivation of $\hat{P}$

Consider a $[j,i\rangle$ BCFW representation:

$$\mathcal{M}_n = \sum \int \mathrm{d}^7\eta_{\hat{P}}\ \mathcal{M}_L \frac{1}{P^2}\mathcal{M}_R\ |_{\hat{P}^2=m^2}\,, \tag{A.1}$$

$$|\hat{j}] = |j] + w|i], \quad |\hat{i}\rangle = |i\rangle - w|j\rangle, \quad \hat{\eta}_j = \eta_j + w\,\eta_i\,, \tag{A.2}$$

$$\hat{P} = P + w\,|i]\langle j|. \tag{A.3}$$

We can evaluate w using the on-shell condition $\hat{P}^2 = m^2$.

$$\begin{aligned}
\hat{P}^2 &= (P + w\,|i]\langle j|)^2 \\
&= P^2 + 2P\cdot w\,|i]\langle j| \\
&= P^2 + w\,\langle j|P|i] = m^2.
\end{aligned}$$

Therefore,

$$\boxed{w = \frac{-P^2+m^2}{\langle j|P|i]}.} \tag{A.4}$$

Plugging the expression for w into $\hat{P}$,

$$\begin{aligned}
\hat{P} &= P + \frac{(-P^2+m^2)}{\langle j|P|i]}|i]\langle j| \\
&= \frac{[i|P|j\rangle P - P^2|i]\langle j| + m^2|i]\langle j|}{\langle j|P|i]}\,.
\end{aligned}$$

This can be simplified by invoking the Schouten identity as follows:

$$\begin{aligned}
\langle j|P|i]P_{a\dot{b}} &= j_{\dot{c}}P^{\dot{c}d}i_d P_{a\dot{b}} \\
&= -P_{\dot{c}}{}^{d} i_d P_a{}^{\dot{c}} j_{\dot{b}} - P_{a\dot{c}} j^{\dot{c}} P_{\dot{b}}{}^{d} i_d \\
&= P_{a\dot{c}}P^{\dot{c}d} i_d j_{\dot{b}} + P_{a\dot{c}} j^{\dot{c}} P_{\dot{b}d} i^d.
\end{aligned}$$

Using $P^a_{\dot{c}}P^{\dot{c}d} = P^2\epsilon^{ad}$, we have

$$\begin{aligned}
\langle j|P|i]P &= P^2\delta_a{}^{d} i_d j_{\dot{b}} + P_{a\dot{c}} j^{\dot{c}} P_{\dot{b}d} i^d \\
&= P^2 i_a j_{\dot{b}} + P_{a\dot{c}} j^{\dot{c}} P_{\dot{b}d} i^d \\
&= P^2|i]\langle j| + P|j\rangle[i|P.
\end{aligned}$$

We obtain for $\hat{P}$:

$$\boxed{\hat{P} = \frac{P|j\rangle[i|P + m^2|i]\langle j|}{\langle j|P|i]}.} \tag{A.5}$$

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