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逆平均曲率流之研究

A Study of the Inverse Mean Curvature Flow

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中文摘要

這篇文章中，我們討論超曲面在對曲率做合理假設下的旋轉對稱空間中沿著逆平均曲率流之行爲。我們針對逆平均曲率流的初始曲面爲封閉、星形且mean-convex之情形的長時間存在性以及近似行爲做細部分析。另外我們利用逆平均曲率流來證明歐氏空間中定義域爲星形且mean-convex之quermassintegrals的isoperimetric不等式。

Abstract

In this thesis, we study the behavior of the motion of hypersurfaces by their inverse mean curvature flow (abbreviated as IMCF) in the rotational symmetric space with reasonable condition on its curvatures. In particular, we give a detailed analysis about the long time existence and the asymptotic behavior of the IMCF when the initial surface is closed star-shaped and mean-convex. We also present an application of the IMCF to the proof of the isoperimetric inequality for quermassintegrals of mean-convex star-shaped domains in Euclidean space.

Contents

1. Introduction

A classical solution of the IMCF in an $(n+1)$-dimensional Riemannian manifold $(\mathbb{N}^{n+1}, \overline{g})$ is a smooth family $X : M^n \times [0,T) \to \mathbb{N}^{n+1}$ of closed hypersurfaces satisfying

$$\frac{\partial}{\partial t} X(p,t) = \frac{1}{H}\nu(p,t),\ p \in M^n,\ 0 \le t < T, \tag{1.1}$$

where $H(p,t)$ is the mean curvature and $\nu(p,t)$ is the outward unit normal vector of the surface $X(\cdot,t)(M^n)$ at the point $X(p,t)$.

The IMCF was first proposed by Geroch [4] and Jang and Wald [9] in the seventies as an approach to the proof of the positive mass theorem. It was first studied rigourously by Gerhardt [3] and Urbas [12] independently in Euclidean space. Gerhardt proved [3] (also [12]) that for a smooth, closed, star-shaped initial hypersurface of positive mean curvature, the IMCF has a unique smooth solution for all times, moreover the rescaled surfaces

$$\widehat{X}(t) \coloneqq e^{-\frac{t}{n}} \cdot X(t)$$

converge exponentially fast to a sphere.

In 2001 Huisken and Ilmanen [7] used a level-set approach and developed the notion of weak solutions for the IMCF to prove the Penrose inequality for asymptotically flat 3-manifolds with non-negative scalar curvature. Huisken and Ilmanen also proved [8] higher regularity properties of the IMCF in Euclidean space.

In [2], Ding has studied the IMCF when the ambient space is a rotationally symmetric space with non-positive sectional curvature and Euclidean volume growth.

Let $\mathbb{N}^{n+1}$ be a rotationally symmetric space, whose metric is

$$\overline{g} = dr^2 + \lambda^2(r)\sigma_{ij}dx^i dx^j \tag{1.2}$$

under the geodesic polar coordinates, where $\sigma \coloneqq \sigma_{ij}dx^i dx^j$ is the canonical metric of $\mathbb{S}^n, \lambda \in C^\infty, \lambda(0) = 0, \lambda'(0) = 1$ and $\lambda(r) > 0$ for any $r > 0$. The manifolds has non-positive sectional curvature if $\lambda'' \geq 0$. Furthermore, if the manifold $\mathbb{N}$ also has Euclidean volume growth, which means that λ' is uniformly bounded, Ding proved the following theorem.

Theorem 1.1. [2] *Let $\mathbb{N}^{n+1}$ be a rotationally symmetric space with non-positive sectional curvature, M_0^n be closed, mean-convex, star-shaped hypersurface of $\mathbb{N}^{n+1}$, which is given as an embedding*

$$X_0 : \mathbb{S}^n \to \mathbb{N}^{n+1}.$$

The IMCF has a unique smooth solution for all times. If λ' is uniformly bounded, then the rescaled surfaces

$$\widehat{X}(t) \coloneqq e^{-\frac{t}{n}X(t)}$$

converges to a uniquely determined sphere of radius $\frac{1}{\lambda'(\infty)}(\frac{Area(M_0)}{|\mathbb{S}^n|})^{\frac{1}{n}}$, where $\lambda'(\infty) \coloneqq \lim_{r\to\infty}\lambda'(r)$, Area($M_0$) is the area of M_0 in $\mathbb{N}^{n+1}$, and $|\mathbb{S}^n|$ is Lebesgue measure of n-sphere in Euclidean space.

In [11], Li and Wei has studied the IMCF in Schwarzschild space. The Schwarzschild space is an $(n+1)-$dimensional manifold $\mathbb{N}^{n+1} = [s_0, \infty) \times \mathbb{S}^n$ equipped with the metric

$$\overline{g} = \frac{1}{1 - 2ms^{1-n}}ds^2 + s^2\sigma_{ij}dx^i dx^j$$

where $m > 0$ is a constant, s_0 is the unique positive solution of $1 - 2ms_0^{1-n} = 0$. This metric is asymptotically flat in the sense that the sectional curvatrue of $(\mathbb{N}^{n+1}, \overline{g})$

approaches to zero near infinity. By change of variable, this metric can be written as the following warped metric

$$\overline{g} = dr^2 + \lambda^2(r)\sigma_{ij}dx^i dx^j$$

where $\lambda : [0, \infty) \to [s_0, \infty)$ satisfies $\lambda'(r) = \sqrt{1 - 2m\lambda^{1-n}(r)}$. Li and Wei proved the following result.

Theorem 1.2. *The IMCF starting from a closed, mean-convex, star-shaped hypersurface M_0^n in the Schwarzschild space $(\mathbb{N}^{n+1}, \overline{g})$ will exist for all time. The flow hypersurface M_t^n converges to infinity while preserving mean-convexity and star-shpaedness. Moreover, there exists positive constants $\overline{\lambda}$ and β' such that the induced metric on M_t^n satisfies*

$$e^{-\frac{2t}{n}} g_{ij} \to \overline{\lambda}\sigma_{ij}$$

exponentially fast and the second fundamental form h_i^j satisfies

$$|\frac{\lambda}{\lambda'} h_i^j - \delta_i^j| = O(e^{-\beta' t})$$

as $t \to \infty$, where σ_{ij} denotes the compnents of the round metric. In other words, the flows M_t^n converges to a large coordinate sphere as $t \to \infty$.

We can prove a similar result with some reasonable decay on λ'' and allow the sectional curvatures to be positive in some directions.

Theorem 1.3. *Let $\mathbb{N}^{n+1}$ be a rotationally symmetric space with the metric*

$$\overline{g} = dr^2 + \lambda^2(r)\sigma_{ij}dx^i dx^j$$

where $\lambda'' \geq 0$, $\lambda > 0$, $\lambda^{1+\varepsilon}\lambda'' \leq C$, *where* $\varepsilon > 0$, *and* $\lim_{r\to\infty} \lambda'(r) = 1$ *and the covariant derivative of the curvature of* $\mathbb{N}^{n+1}$ *have the decay* $|\overline{\nabla} R| \leq \frac{C}{\lambda^3}$. *Suppose* M_0^n *is a closed, mean-convex, star-shaped hypersurface of* $\mathbb{N}^{n+1}$, *which is given as an embedding*

$$X_0 : \mathbb{S}^n \to \mathbb{N}^{n+1}.$$

The IMCF has a unique smooth solution for all times and the rescaled surfaces

$$\widehat{X}(t) \coloneqq e^{-\frac{t}{n}} X(t)$$

converges to a uniquely determined sphere of radius $(\frac{Area(M_0^n)}{|\mathbb{S}^n|})^{\frac{1}{n}}$, *where* $Area(M_0^n)$ *is the area of* M_0^n *in* $\mathbb{N}^{n+1}$, *where* $|\mathbb{S}^n|$ *is Lebesgue measure of n-sphere in Euclidean space.*

The condition of λ which we assume implies that the curvature of $\mathbb{N}^{n+1}$ has the quadratic decay and the covariant derivative of the curvature of $\mathbb{N}^{n+1}$ have cubic decay. Also, our condition $\lambda'' \geq 0$, and $\lim_{r\to\infty} \lambda'(r) = 1$ implies that the support function grows exponentially and the star-shapedness is preserved by the inverse curvature flow. Also, the condition $\lambda''\lambda^{1+\varepsilon} \leq C$ is used to show that $\overline{Ric}(\nu, \nu) \leq \frac{C}{\lambda^3}$ which is used to prove the decay estimate of the mean curvature. In the case of an asymptotically flat manifolds, the curvature tensor is required to fall off as $O(\frac{1}{r^3})$. So our assumption of the curvature decay is more reasonable than Ding's assumption.

In [5], Guan and Li used the IMCF to prove the following isoperimetric inequality for quermassintegrals of closed, mean-convex, star-shaped hypersurfaces in Euclidean space.

Theorem 1.4. *Suppose Σ is a closed, mean-convex, star-shaped hypersurface in $\mathbb{R}^{n+1}$, then inequality*

$$\frac{\int_{\Sigma} H d\mu_g}{(\int_{\Sigma} d\mu_g)^{\frac{(n-1)}{n}}} \geq \frac{\int_{\mathbb{S}^n} H d\mu_g}{(\int_{\mathbb{S}^n} d\mu_g)^{\frac{(n-1)}{n}}}$$

is true. The equality holds if and only if Σ is a round sphere. Here H is the mean curvature of the hypersurface.

In this thesis, we will give a detailed proof of Theorem 1.3 and Theorem 1.4. The organization of this thesis is as follows: In section 2, we include the basic results about the Riemannian geometry, the geometry of submanifolds and the maximum principle for parabolic equation. In section 3, we compute the Riemannian curvatures, Ricci curvature and the scalar curvature of the rotationally symmetric space. We also derive the second fundamental forms and various geometric quantities of a star-shaped hypersurface in terms of its radial function. In section 4, we prove that the mean-convexity and star-shapedness is preserved by the IMCF and derive the estimate of λ, the decay estimates of the radial function, mean curvature, the second fundamental form that will be used to prove the convergence of the rescaled flow later. In section 5, we prove Theorem 1.3. In section 6, we prove Theorem 1.4.

2. Preliminary

In this section, we include the basic results about the Riemannian geometry and the geometry of submanifolds and the maximum principle for parabolic equation.

Let $(\mathbb{N}^{n+1}, \bar{g})$ be a smooth complete Riemannian manifold without boundary. We denote by a bar for all quantities on $\mathbb{N}^{n+1}$, for example by $\bar{g} = \{\bar{g}_{\alpha\beta}\}$, $0 \leq \alpha, \beta \leq n$, the metric, by $\bar{y} = \{\bar{y}^{\alpha}\}$ coordinates, by $\bar{\Gamma} = \{\bar{\Gamma}^{\gamma}_{\alpha\beta}\}$ the Levi-Civita connection, by

$\bar{\nabla}$ the covariant derivative and by $\overline{Riem} = \{\overline{Riem}_{\alpha\beta\gamma\delta}\}$ the Riemannian curvature tensor. Components are sometimes taken with respect to the tangent vector fields $\{\frac{\partial}{\partial \bar{y}^\alpha}\}$, $0 \leq \alpha \leq n$ associated with a local coordinate chart $\bar{y} = \{\bar{y}^\alpha\}$ and sometimes with respect to a moving orthonormal frame $\{e_\alpha\}$, $0 \leq \alpha \leq n$, where $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. We write $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$ for the inverse of the metric and use the Einstein summation convention for the sum of repeated indices. The Ricci curvature $\bar{Ric} = \bar{R}_{\alpha\beta}$ and scalar curvature $\bar{R}$ of $(\mathbb{N}^{n+1}, \bar{g})$ are then given by

$$\bar{R}_{\alpha\beta} = \bar{g}^{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta}, \bar{R} = \bar{g}^{\alpha\beta}R_{\alpha\beta}$$

and the sectional curvatures (in an orthonormal frame) are given by $\overline{\mathrm{Sec}}_{\alpha\beta} = \bar{R}_{\alpha\beta\alpha\beta}$.

Now let $X : M^n \to \mathbb{N}^{n+1}$ be a smooth hypersurface immersion. For simplicity we restrict our attention to closed surfaces, i.e., compact without boundary. The induced metric on M^n will be denoted by g, in local coordinates we have

$$\begin{aligned} g_{ij}(p) &= \langle \frac{\partial X}{\partial x^i}(p), \frac{\partial X}{\partial x^j}(p)\rangle_M \\ &= \bar{g}_{\alpha\beta}(X(p))\frac{\partial X^\alpha}{\partial x^i}(p)\frac{\partial X^\beta}{\partial x^j}(p),\ p \in M^n. \end{aligned} \tag{2.1}$$

Furthermore, $\{\Gamma^i_{jk}\}$, ∇ and $Riem = \{R_{ijkl}\}$ with Latin indices i, j, k, l ranging from 1 to n describe the intrinsic geometry of the induced metric g on the hypersurface. If ν is a local choice of unit normal for $X(M^n)$, we often work in an adapted orthonormal frame $e_0(= \nu), e_1, \cdots, e_n$ in a neighbourhood of $X(M^n)$ such that $e_1(p), \cdots, e_n(p) \in T_pM^n \subset T_pM^{n+1}$ and $g(p)(e_i(p), e_j(p)) = \delta_{ij}$ for $p \in M^n$, $1 \leq i, j \leq n$. The second fundamental form $A = \{h_{ij}\}$ as a bilinear form $A(p) : T_pM^n \times T_pM^n \to \mathbb{R}$ and the Weingarten map $W = \{h^i_j\} = \{g^{ik}h_{kj}\}$ as an operator $W : T_pM^n \to T_pM^n$ are then given by

$$h_{ij} = \langle \bar{\nabla}_{e_i} \nu, e_j \rangle = -\langle \nu, \bar{\nabla}_{e_i} e_j \rangle.$$

In local coordinates x^i, $1 \le i \le n$, near $p \in M^n$ and $\{\bar{y}^\alpha\}$, $0 \le \alpha \le n$, near $X(p) \in M^{n+1}$ these relations are equivalent to the Weingarten equations:

$$\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial X^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\delta} \frac{\partial X^\beta}{\partial x^i} \frac{\partial X^\delta}{\partial x^j} = -h_{ij} \nu^\alpha; \tag{2.2}$$

$$\frac{\partial \nu^\alpha}{\partial x^i} + \bar{\Gamma}^\alpha_{\beta\delta} \frac{\partial X^\beta}{\partial x^i} \nu^\delta = h_{ij} g^{jl} \frac{\partial X^\alpha}{\partial x^l} \tag{2.3}$$

Recall that $A(p)$ is symmetric, i.e., W is self-adjoint, and the eigenvalues $\kappa_1(p), ..., \kappa_n(p)$ are called the principal curvatures of $X(M)$ at $X(p)$. Also note that at a given point $p \in M^n$ by choosing normal coordinates and then possibly rotating them we can always arrange that at this point

$$g_{ij} = \delta_{ij}, (\bar{\nabla}_{e_i} e_j)^T = 0, \ h_{ij} = h^i_j = diag(\kappa_1, \cdots, \kappa_n).$$

Here $(\bar{\nabla}_{e_i} e_j)^T$ means the projection of the vector $\bar{\nabla}_{e_i} e_j$ from $T_M(p)$ to $T_M(p)$.

The mean curvature and the norm of the second fundamental form are given by

$$H := tr(W) = h^i_i = \kappa_1 + \cdots + \kappa_n$$

and

$$|A|^2 := \kappa_1^2 + \cdots + \kappa_n^2.$$

The commutator of second derivatives of a vector field V and a one-form ω on M^n are given by

$$\nabla_i \nabla_j V^k - \nabla_j \nabla_i V^k = R_{ijlp} g^{kl} V^p$$

and

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = R_{ijkl} g^{lp} \omega_p.$$

More generally, the commutator of second derivatives for an arbitrary tensor involves one curvature term as above for each of the indices of the tensor. The corresponding laws of course also hold for the metric $\bar{g}$.

The curvature of the hypersurface and ambient manifold are related by the equations of Gauss:

$$R_{ijkl} = \overline{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk},\ 1 \le i, j, k, l \le n, \tag{2.4}$$

the equations of Codazzi-Mainardi:

$$\nabla_i h_{jk} - \nabla_k h_{ij} = \bar{R}_{0jki}. \tag{2.5}$$

the second derivatives of the second fundamental form satisfies the identities:

$$\begin{aligned}\nabla_k \nabla_l h_{ij} &= \nabla_i \nabla_j h_{kl} + h_{kl}h_{im}h_{mj} - h_{km}h_{mj}h_{il} + h_{kj}h_{im}h_{ml} - h_{km}h_{ml}h_{ij} \\ &+ \bar{R}_{kilm}h_{mj} + \bar{R}_{kijlm}h_{ml} + \bar{R}_{mjil}h_{km} + \bar{R}_{0i0j}h_{kl} - \bar{R}_{0k0l}h_{ij} + \bar{R}_{mljk}h_{im} \\ &+ \bar{\nabla}_k R_{0jil} + \bar{\nabla}_i R_{0ljk}.\end{aligned}$$

Here we state and prove the scalar Maximum Principle for parabolic equation that will be used later for our various estimate for inverse curvature flow.

Theorem 2.1. *Suppose that $g(t)$ is a family of metrics on a closed manifold M and $u:\ M \times [0,T) \to\ \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t}u \le Lu + \langle X(t),\ \nabla u\rangle + F(u), \tag{2.6}$$

where $X(t)$ is a time-dependent vector field and F is a Lipschitz Function. If $u \le c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u(x,t) \le U(t)$ for all $x \in M^n$ and $t \ge 0$, where $U(t)$ is the *solution to the ODE*

$$\frac{dU}{dt} = F(U)$$

with U(0)=c.

Proof. By (2.6), we have

$$\frac{\partial}{\partial t}(u - U) \leq L(u - U) + \langle X(t), \nabla(u - U)\rangle + F(u) - F(U),$$

then according to the Lipschitz property of F,we have:

$$\frac{\partial}{\partial t}(u - U) \leq L(u - U) + \langle X(t), \nabla(u - U)\rangle + C|u - U|$$

This implies that $v =: e^{-Ct}(u - U)$ satisfies

$$\frac{\partial}{\partial t}v \leq L(u - U) + \langle X(t), \nabla v\rangle + C(|v| - v)$$

Hence $v_\varepsilon =: v - \varepsilon(1 + t)$ satisfies $v_\varepsilon(0) \leq -\epsilon$ and

$$\frac{\partial}{\partial t}v_\varepsilon \leq Lv_\varepsilon + \langle X(t), \nabla v_\varepsilon\rangle + C(|v| - v) - \varepsilon.$$

We claim $v_\varepsilon < 0$ for all $t \geq 0$. If not, then there exists a first time t_0 at which there is a point x_0 such that $v_\varepsilon(x_0, t_0) = 0$. Then $v = |v| = \varepsilon(1 + t_0)$ at (x_0, t_0) and

$$0 \leq \frac{\partial}{\partial t}v_\varepsilon \leq Lv_\varepsilon + \langle X(t), \nabla v_\varepsilon\rangle + C(|v| - v) - \epsilon \leq -\epsilon.$$

at (x_0, t_0), which is a contradiction. Hence $v_\varepsilon < 0$ for all $t \geq 0$ and $\epsilon > 0$. The result follows from taking $\varepsilon \to 0$. ■

3. The Geometry of Star-Shaped Hypersurfaces in a Rotationally Symmetric Space

In this section, we compute the Riemannian curvatures, Ricci curvature and the scalar curvature of the rotationally symmetric space. We also derive the second fundamental forms and various geometric quantities of a star-shaped hypersurface in terms of its radial function.

Let $\mathbb{N}^{n+1}$ be a rotationally symmetric space, whose metric is

$$\overline{g} = dr^2 + \lambda^2(r)\sigma_{ij}dx^i dx^j$$

under the geodesic polar coordinates, where $\sigma := \sigma_{ij}dx^i dx^j$ is the canonical metric of $\mathbb{S}^n, \lambda \in C^\infty, \lambda'' \geq 0,\ \lambda > 0,\ \lambda^{1+\varepsilon}\lambda'' \leq C$, where $\varepsilon > 0$, and $\lim\limits_{r\to\infty} \lambda'(r) = 1$ and the covariant derivative of the curvature of $\mathbb{N}^{n+1}$ have the decay $|\overline{\nabla} R| \leq \frac{C}{\lambda^3}$

We calculate the Riemannian curvature tensors in the next lemma. Let $x = \{x^i\}_{i=1}^n$ be a local coordinate system on $\mathbb{S}^n$ and let $\frac{\partial}{\partial x^i}$ be the corresponding coordinate vector fields in $\mathbb{N}^{n+1}$. Let D, ∇ and $\overline{\nabla}$ be the Levi-Civita connections of $\mathbb{S}^n$, M^n and $\mathbb{N}^{n+1}$, respectively. $\widetilde{\Gamma}_{ij}^k$ denote the Christoffel symbols of $\mathbb{S}^n$ with respect to the tangent basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, and $\overline{\Gamma}_{ij}^k$ denote the Christoffel symbols of $\mathbb{N}^{n+1}$ with respect to the tangent basis $\{\frac{\partial}{\partial r}\} \cup \{\frac{\partial}{\partial x^i}\}_{i=1}^n$ ($\frac{\partial}{\partial r}$ is indexed by $i = 0$). We write $\overline{g}_{ij} := \overline{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \lambda^2(r)\sigma_{ij}$.

Lemma 3.1. *Let $\overline{R}_{\alpha\beta\gamma\mu}$ denote the Riemannian curvature tensor of the rotationally symmetric metric $\overline{g} = dr^2 + \lambda^2(r)\sigma_{ij}dx^i dx^j$. Then*

$$\overline{g}(\overline{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial r})\frac{\partial}{\partial x^j}, \frac{\partial}{\partial r}) = -\frac{\lambda''}{\lambda}\overline{g}_{ij}$$

$$\overline{g}\big(\overline{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k},\frac{\partial}{\partial r}\big)=0$$

$$\overline{g}\big(\overline{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\big)=\frac{1-(\lambda')^2}{\lambda^2}(\overline{g}_{ik}\overline{g}_{jl}-\overline{g}_{jk}\overline{g}_{il})$$

$$\overline{Ric}=-n\frac{\lambda''}{\lambda}dr^2+\big[(n-1)\frac{1-\lambda'^2}{\lambda^2}-\frac{\lambda''}{\lambda}\big]\bar{g} \tag{3.1}$$

$$\overline{R}=-2n\frac{\lambda''}{\lambda}+n(n-1)\frac{1-\lambda'^2}{\lambda^2}$$

Proof. We first compute some components of the Christoffel symbols of $\mathbb{N}^{n+1}$ with respect to the tangent basis $\{\frac{\partial}{\partial r}\}\cup\{\frac{\partial}{\partial x^i}\}_{i=1}^n$. Using the fact that $\overline{g}_{i0}=0,\ \overline{g}^{00}=1,\ \overline{g}_{00}=1,\frac{\partial\overline{g}_{jl}}{\partial r}=0$ and $\frac{\partial}{\partial x^i}\overline{g}_{jl}=\lambda^2\frac{\partial}{\partial x^i}\sigma_{jl}$, we have

$$\overline{\Gamma}_{ij}^k=\frac{1}{2}\overline{g}^{kl}(\frac{\partial\overline{g}_{jl}}{\partial x^i}+\frac{\partial\overline{g}_{il}}{\partial x^j}-\frac{\partial\overline{g}_{ij}}{\partial x^l})=\frac{1}{2}\frac{1}{\lambda^2}\sigma^{kl}(\lambda^2\frac{\partial\sigma_{jl}}{\partial x^i}+\lambda^2\frac{\partial\sigma_{il}}{\partial x^j}-\lambda^2\frac{\partial\sigma_{ij}}{\partial x^l})=\widetilde{\Gamma}_{ij}^k$$

and

$$\begin{aligned}\overline{\Gamma}_{ij}^0&=\frac{1}{2}\overline{g}^{0l}(\frac{\partial\overline{g}_{jl}}{\partial x^i}+\frac{\partial\overline{g}_{il}}{\partial x^j}-\frac{\partial\overline{g}_{ij}}{\partial x^l})=\frac{1}{2}\overline{g}^{00}(\frac{\partial\overline{g}_{j0}}{\partial x^i}+\frac{\partial\overline{g}_{i0}}{\partial x^j}-\frac{\partial\overline{g}_{ij}}{\partial r})\\&=\frac{1}{2}(-\frac{\partial(\lambda^2\sigma_{ij})}{\partial r})=-\lambda\lambda'\sigma_{ij}.\end{aligned}$$

This implies that

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=-\lambda\lambda'\sigma_{ij}\frac{\partial}{\partial r}+\widetilde{\Gamma}_{ij}^k\frac{\partial}{\partial x^k} \tag{3.2}$$

where $\widetilde{\Gamma}_{ij}^k$ is the Christoffel symbols of the standard metric on the sphere.

This implies that

$$\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j},\frac{\partial}{\partial r})=-\lambda\lambda'\sigma_{ij}. \tag{3.3}$$

Since $\overline{g}(\frac{\partial}{\partial r},\frac{\partial}{\partial r})=1$, we have $\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r},\frac{\partial}{\partial r})=0$. Using $\overline{g}(\frac{\partial}{\partial r},\frac{\partial}{\partial x^j})=0$, we have $\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r},\frac{\partial}{\partial x^j})=-\overline{g}(\frac{\partial}{\partial r},\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})=\lambda\lambda'\sigma_{ij}$. This implies that

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r}=\lambda'\lambda\sigma_{ij}\overline{g}^{jk}\frac{\partial}{\partial x^k}=\frac{\lambda'}{\lambda}\frac{\partial}{\partial x^i}.$$

Now we also have

$$\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial x^i}) = -\overline{g}(\frac{\partial}{\partial r},\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^i}) = -\overline{g}(\frac{\partial}{\partial r},\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r}) = 0.$$

Thus

$$\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0.$$

Each level set of the function r is a round sphere with induced metric $\lambda(r)^2\sigma$ and the second fundamental form $h_{ij} = -\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j},\frac{\partial}{\partial r}) = \lambda\lambda'\sigma_{ij}$ from (3.3). From Gauss equation (2.4), we have

$$\begin{aligned}
&\overline{R}_{ijkl} = R_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}\\
=&\lambda^2(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}) - \lambda^2\lambda'^2(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk})\\
=&\lambda(r)^2\,(1-\lambda'(r)^2)\,(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk})\\
=&\frac{1-(\lambda')^2}{\lambda^2}(\overline{g}_{ik}\overline{g}_{jl} - \overline{g}_{jk}\overline{g}_{il})
\end{aligned}$$

Since each level set of r is umbilical with second fundamental form $h_{ij} = \lambda\lambda'\sigma_{ij}$, from the Codazzi equation (2.5), we derive

$$\overline{R}(\frac{\partial}{\partial r},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^i}) = \nabla_i h_{jk} - \nabla_k h_{ij} = \nabla_i(\lambda\lambda'\sigma_{jk}) - \nabla_k(\lambda\lambda'\sigma_{ij}) = 0$$

The remaining components of the curvature tensors are

$$\begin{aligned}
&\overline{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial r},\frac{\partial}{\partial x^j},\frac{\partial}{\partial r})\\
&= -\overline{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial r},\frac{\partial}{\partial r},\frac{\partial}{\partial x^j})\\
&= \overline{g}\big((\overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial r}} - \overline{\nabla}_{\frac{\partial}{\partial r}}\overline{\nabla}_{\frac{\partial}{\partial x^i}})\frac{\partial}{\partial r},\frac{\partial}{\partial x^j}\big)\\
&= -\overline{g}\big(\overline{\nabla}_{\frac{\partial}{\partial r}}\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r},\frac{\partial}{\partial x^j}\big) \quad (\text{use } \overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0)
\end{aligned}$$

$$
\begin{aligned}
&= -\overline{g}\big(\overline{\nabla}_{\frac{\partial}{\partial r}}(\frac{\lambda'}{\lambda}\frac{\partial}{\partial x^i}), \frac{\partial}{\partial x^j}\big) \quad (\text{use} \overline{\nabla}_{\frac{\partial}{\partial r}}\overline{\nabla}_{\frac{\partial}{\partial x^i}} = \frac{\lambda'}{\lambda}\frac{\partial}{\partial x^i}) \\
&= -(\frac{\lambda''}{\lambda} - \frac{\lambda'^2}{\lambda^2})\lambda^2\sigma_{ij} - \frac{\lambda'}{\lambda}\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\
&= (-\lambda''\lambda + \lambda'^2)\sigma_{ij} - \frac{\lambda'}{\lambda}\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j}) \\
&= (-\lambda''\lambda + \lambda'^2)\sigma_{ij} - \frac{\lambda'}{\lambda}\cdot\frac{\lambda'}{\lambda}\cdot\lambda^2\sigma_{ij} \\
&= -\lambda\,\lambda''\,\sigma_{ij} = -\frac{\lambda''}{\lambda}\overline{g}_{ij}.
\end{aligned}
$$

Now we can use the curvature formulae to compute the Ricci curvature

$$
\begin{aligned}
&\overline{Ric}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}) \\
=&\overline{g}\big(\overline{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\big)\overline{g}^{jl} + \overline{g}\big(\overline{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial r})\frac{\partial}{\partial x^k}, \frac{\partial}{\partial r}\big) \\
=&\frac{1-(\lambda')^2}{\lambda^2}(\overline{g}_{ik}\overline{g}_{jl} - \overline{g}_{jk}\overline{g}_{il})\overline{g}^{jl} - \frac{\lambda''}{\lambda}\overline{g}_{ik} \\
=&\big[(n-1)\frac{1-(\lambda')^2}{\lambda^2} - \frac{\lambda''}{\lambda}\big]\overline{g}_{ik}
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \\
=&\overline{g}\big(\overline{R}(\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial r}, \frac{\partial}{\partial x^l}\big)\overline{g}^{jl} \\
=&-\frac{\lambda''}{\lambda}\overline{g}_{jl}\overline{g}^{jl} \\
=&-\frac{n\lambda''}{\lambda}.
\end{aligned}
$$

■

Corollary 3.2. *Under our assumption $\lambda > 0$, $\lambda'' \geq 0$ and $\lim_{r\to\infty}\lambda'(r) = 1$, we have $\lambda'(r) \leq 1$, $\overline{Sec}(\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial r}) \leq 0$ and $\overline{Sec}(\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}) \geq 0$. Moreover, $|\overline{Rm}| \leq \frac{C}{\lambda^2}$ if $\lambda\lambda'' \leq C$ for some positive constant $C > 0$.*

A star-shaped hypersurface $M^n \subset \mathbb{N}^{n+1}$ can be parameterized by

$$M^n = \{(r(x), x) : \ x \in S^n\}$$

for a smooth function r on $\mathbb{S}^n$. Considering this embedding

$$X : \mathbb{S}^n \to M^n := X(\mathbb{S}^n) \hookrightarrow \mathbb{N}^{n+1}.$$

We assume that $X := (r(x), x^1, \cdots, x^n)$ in the coordinate system $(r, x^1, \cdots, x^n)$.

We next define a new function. For the convenience, we define function

$$\varphi(r) := \int_c^r \frac{1}{\lambda(s)} ds$$

on $\mathbb{S}^n$. To make the integral been meaningful, c is supposed be an arbitrary fixed positive constant. Then we have

$$\begin{aligned} D_i\varphi &= \frac{1}{\lambda} \cdot D_i r \\ Dr = D_i r \sigma^{ij} \frac{\partial}{\partial x^j} &= \lambda \cdot D_i \varphi \sigma^{ij} \frac{\partial}{\partial x^j} = \lambda D\varphi \\ \varphi_{i,j} = D_j D_i \varphi = D_j(\frac{1}{\lambda} \cdot D_i r) &= \frac{1}{\lambda} r_{i,j} - \frac{\lambda'}{\lambda^2} D_j r D_i r, \end{aligned} \tag{3.4}$$

where $\varphi_{i,j}$ and $r_{i,j}$ denote the second covariant derivative of φ and r with respect to the round metric $\sigma := \sigma_{ij} dx^i dx^j$, respectively. Moreover, let

$$v := \sqrt{1 + |D\varphi|^2_{\mathbb{S}^n}}. \tag{3.5}$$

In the next proposition, we express the exterior unit normal vector, metric and second fundamental form of M in terms of covariant derivatives of φ as in [2]:

Proposition 3.3. *Let ν be the exterior unit normal vector, g_{ij} be the induced metric on M and h_{ij} be the second fundamental form in term of the basis $\{\frac{\partial X}{\partial x^i} = D_i r \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}\}_{i=1}^n$. Then*

$$g_{ij} = \lambda^2 \left(\sigma_{ij} + D_i\varphi D_j\varphi\right) \tag{3.6}$$

$$g^{ij} = \frac{1}{\lambda^2}(\sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}) = \frac{1}{\lambda^2}\widetilde{\sigma}^{ij} \tag{3.7}$$

$$\nu = \frac{1}{v}(\frac{\partial}{\partial r} - \frac{D\varphi}{\lambda})$$

$$h_{ij} = \frac{\lambda}{v}\left(\lambda'\left(\sigma_{ij} + D_i\varphi D_j\varphi\right) - \varphi_{i,j}\right)$$

$$h^i_j = \frac{1}{v\lambda}(\lambda'\delta^i_j - \widetilde{\sigma}^{ik}\varphi_{k,j})$$

and

$$H = \frac{1}{v\lambda}(n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j})$$

where $\widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}$ and $D^i\varphi = \sigma^{ik}D_k\varphi$.

Proof. The local coordinate vector fields of M can be expressed as

$$e_i := \frac{\partial X}{\partial x^i} = D_i r\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i},\ 1 \le i \le n.$$

The induced metric of M is

$$g_{ij} = \bar{g}(e_i, e_j) = \bar{g}(D_i r\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}, D_j r\frac{\partial}{\partial r} + \frac{\partial}{\partial x^j}) = D_i r D_j r + \lambda^2\sigma_{ij} = \lambda^2\left(\sigma_{ij} + D_i\varphi D_j\varphi\right).$$

To find the inverse of g_{ij}, we use the fact that: A is an $n \times n$ invertible matrix and w is an $n \times 1$ matrix, then $(A + ww^T)^{-1} = A^{-1} - \frac{1}{1+w^TA^{-1}w}A^{-1}ww^TA^{-1}$. Now take $A = (\sigma_{ij})$ and $w = (D_i\varphi)$, then

$$(\sigma_{ij} + D_i\varphi D_j\varphi)^{-1} = (A + ww^T)^{-1} = A^{-1} - \frac{1}{1 + w^TA^{-1}w}A^{-1}ww^TA^{-1}$$

$$= \sigma^{ij} - \frac{1}{1 + D_p\varphi\sigma^{qp}D_q\varphi}\sigma^{ik}D_k\varphi D_k\varphi\sigma^{kj}$$

$$= \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{1 + |D\varphi|^2_{\mathbb{S}^n}} = \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}.$$

where $D^i\varphi = \sigma^{ij}D_j\varphi$. Therefore,

$$g^{ij} = \frac{1}{\lambda^2}(\sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}) = \frac{1}{\lambda^2}\widetilde{\sigma}^{ij}$$

where $\widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}$. Since (g^{ij}) is the inverse of (g_{ij}), (g^{ij}) is positive definite and so is $\widetilde{\sigma}^{ij}$.

Next we find the outward unit normal vector ν of M as below. Let $\vec{a}$ be a normal vector of the form $\vec{a} = \frac{\partial}{\partial r} + B^j\frac{\partial}{\partial x^j}$. Since $\vec{a}$ is normal, we have $\overline{g}(\vec{a}, e_i) = 0$, for $i = 1, 2, ...n$. This implies that

$$\overline{g}(\vec{a}, e_i) = \overline{g}(\frac{\partial}{\partial r} + B^j\frac{\partial}{\partial x^j}, D_ir\frac{\partial}{\partial r} + \frac{\partial}{\partial x_i}) = D_ir + B^j\overline{g}_{ji} = D_ir + B^j\overline{g}_{ij} = D_ir + B^j\lambda^2\sigma_{ij} = 0.$$

To solve B^j, multiply on the both side of the last equation by σ^{ik}, the inverse matrix of σ_{ik}, and sum over i, then

$$D_ir\sigma^{ik} + B^j\lambda^2\sigma_{ij}\sigma^{ik} = D_ir\sigma^{ik} + B^j\lambda^2\delta^k_j = D_ir\sigma^{ik} + B^k\lambda^2 = 0$$

So $B^k = \frac{-D_ir}{\lambda^2}\sigma^{ik}$. Hence $\vec{a} = \frac{\partial}{\partial r} - \frac{D_ir}{\lambda^2}\sigma^{ij}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial r} - \frac{D\varphi}{\lambda}$ and $\overline{g}(\vec{a}, \vec{a}) = 1 + |D\varphi|^2_{\mathbb{S}^n} =: v^2$. We have the exterior unit normal vector

$$\nu = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{v}(\frac{\partial}{\partial r} - \frac{D\varphi}{\lambda}) \tag{3.8}$$

Next we compute the second fundamental form of M^n in $\mathbb{N}^{n+1}$. To compute $h_{ij} := -\overline{g}(\nu, \overline{\nabla}_{e_j}e_i)$, we first compute $\overline{\nabla}_{e_j}e_i = \overline{\nabla}_{(\frac{\partial r}{\partial x^j}\frac{\partial}{\partial r} + \frac{\partial}{\partial x^j})}(\frac{\partial r}{\partial x^i}\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i})$. Since $\frac{\partial r}{\partial x^i}$ is a funtion of x only and $\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}=0$, we have

$$\overline{\nabla}_{(\frac{\partial r}{\partial x^j}\frac{\partial}{\partial r})}(\frac{\partial r}{\partial x^i}\frac{\partial}{\partial r})=\frac{\partial r}{\partial x^j}\Big(\frac{\partial}{\partial r}(\frac{\partial r}{\partial x^i})+\frac{\partial r}{\partial x^i}\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}\Big)=0$$

Using $\overline{\nabla}_{\frac{\partial}{\partial r}}(\frac{\partial}{\partial x^i})=\overline{\nabla}_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial r})=\frac{\lambda'}{\lambda}\frac{\partial}{\partial x^i}$, we have

$$\overline{\nabla}_{(\frac{\partial r}{\partial x^j}\frac{\partial}{\partial r})}(\frac{\partial}{\partial x^i})=\frac{\partial r}{\partial x^j}\Big(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r}\Big)=\frac{\partial r}{\partial x^j}\frac{\lambda'}{\lambda}\frac{\partial}{\partial x^i}=\lambda'\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}$$

Using (3.2) and (3.3), we can get

$$\overline{\nabla}_{\frac{\partial}{\partial x^j}}(\frac{\partial r}{\partial x^i}\frac{\partial}{\partial r}+\frac{\partial}{\partial x^i})=\frac{\partial^2 r}{\partial x^i\partial x^j}\frac{\partial}{\partial r}+\lambda'\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}-\lambda\lambda'\sigma_{ij}\frac{\partial}{\partial r}+\widetilde{\Gamma}^k_{ij}\frac{\partial}{\partial x^k}.$$

Combining previous three equations, we have

$$\overline{\nabla}_{e_j}e_i=(\frac{\partial^2 r}{\partial x^i\partial x^j}-\lambda\lambda'\sigma_{ij})\frac{\partial}{\partial r}+\lambda'\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}+\lambda'\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}+\widetilde{\Gamma}^k_{ij}\frac{\partial}{\partial x^k}$$

Thus, the second fundamental form is given by

$$\begin{aligned}
h_{ij}&=-\overline{g}\big((\frac{\partial^2 r}{\partial x^i\partial x^j}-\lambda\lambda'\sigma_{ij})\frac{\partial}{\partial r}+\lambda'\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}+\lambda'\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}+\widetilde{\Gamma}^k_{ij}\frac{\partial}{\partial x^k},\frac{1}{v}(\frac{\partial}{\partial r}-\frac{1}{\lambda}\sigma^{lm}\frac{\partial\varphi}{\partial x^l}\frac{\partial}{\partial x^m})\big)\\
&=\frac{1}{v}\,(\lambda\lambda'\,\sigma_{ij}+2\lambda'\lambda\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}-\frac{\partial^2 r}{\partial x^i\partial x^j}+\widetilde{\Gamma}^k_{ij}\frac{\partial r}{\partial x^k})\\
&=\frac{\lambda}{v}\left[\lambda'\,(\sigma_{ij}+D_i\varphi D_j\varphi)-\varphi_{ij}\right],
\end{aligned}$$

$$\begin{aligned}
h^i_j=g^{ik}h_{jk}&=\lambda^{-2}(\sigma^{ik}-\frac{D^i\varphi D^k\varphi}{v^2})\cdot\frac{\lambda}{v}\left[\lambda'(\sigma_{jk}+D_j\varphi D_k\varphi)-\varphi_{j,k}\right]\\
&=\frac{1}{v\lambda}[\lambda'(\sigma^{ik}-\frac{D^i\varphi D^k\varphi}{v^2})(\sigma_{jk}+D_j\varphi D_k\varphi)-(\sigma^{ik}-\frac{D^i\varphi D^k\varphi}{v^2})\varphi_{j,k}]. \qquad (3.9)
\end{aligned}$$

Because of (3.6) and (3.7), the first term

$$(\sigma^{ik}-\frac{D^i\varphi D^k\varphi}{v^2})(\sigma_{jk}+D_j\varphi D_k\varphi)=\delta^i_j.$$

Thus,

$$h^i_j = \frac{1}{v\lambda}(\lambda' \delta^i_j - \widetilde{\sigma}^{ik}\varphi_{k,j}),$$

and the mean curvature is

$$H = h^i_i = \frac{1}{v\lambda}(n\lambda' - \widetilde{\sigma}^{ij}\varphi_{j,i}) \tag{3.10}$$

■

Next we compute two more quantities $\overline{Ric}(\nu, \frac{\partial}{\partial r})$ and $\overline{Ric}(\nu, \nu)$ that will be used later.

Lemma 3.4.

$$\overline{Ric}(\nu, \frac{\partial}{\partial r}) = -\frac{n\lambda''}{v\lambda} \tag{3.11}$$

and

$$\overline{Ric}(\nu, \nu) = -n\frac{\lambda''}{\lambda} - |D\varphi|^2_{\mathbb{S}^n} \cdot \frac{n-1}{v^2} \cdot \frac{(\lambda')^2 - 1 - \lambda\lambda''}{\lambda^2} \tag{3.12}$$

Proof. Recall (3.1), we have

$$\overline{Ric} = -n\frac{\lambda''}{\lambda}dr^2 + \big[(n-1)\frac{1-\lambda'^2}{\lambda^2} - \frac{\lambda''}{\lambda}\big]\overline{g}.$$

Thus

$$\overline{Ric}(\nu, \frac{\partial}{\partial r}) = \overline{Ric}\big(\frac{1}{v}(\frac{\partial}{\partial r} - \frac{D^j r}{\lambda^2}\frac{\partial}{\partial x^j}), \frac{\partial}{\partial r}\big) = -\frac{n\lambda''}{v\lambda}$$

and

$$\begin{aligned}
&\overline{Ric}(\nu, \nu)\\
=&\overline{Ric}\big(\frac{1}{v}(\frac{\partial}{\partial r} - \frac{D^i\varphi}{\lambda}\frac{\partial}{\partial x^i}), \frac{1}{v}(\frac{\partial}{\partial r} - \frac{D^j\varphi}{\lambda}\frac{\partial}{\partial x^j})\big)\\
=&\frac{1}{v^2}\big[\overline{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) + \frac{D^i\varphi D^j\varphi}{\lambda^2}\overline{Ric}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j})\big]\\
=&\frac{1}{v^2}\Big\{-n\frac{\lambda''}{\lambda} + |D\varphi|^2_{\mathbb{S}^n}\big[(n-1)\frac{1-\lambda'^2}{\lambda^2} - \frac{\lambda''}{\lambda}\big]\Big\}
\end{aligned}$$

$$= -n\frac{\lambda''}{\lambda} - |D\varphi|^2_{\mathbb{S}^n} \cdot \frac{n-1}{v^2} \cdot \frac{(\lambda')^2 - 1 - \lambda\lambda''}{\lambda^2}.$$

In the last step, we use the fact that $v^2 = 1 + |D\varphi|^2_{\mathbb{S}^n}$.

We summary the notations as below

object	$\mathbb{S}^n$	$\xrightarrow{X}$	M	$\subseteq$	$\mathbb{N}^{n+1}$
coordinate system	$(x^1, \cdots, x^n)$		$(r(x^1, \cdots, x^n)$ $, x^1, \cdots, x^n)$		$(r, x^1, \cdots, x^n)$
basis of tangent bundle	$\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$		$\{e_i = D_i r \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}\}_{i=1}^n$		$\{\frac{\partial}{\partial r}\} \cup \{\frac{\partial}{\partial x^i}\}_{i=1}^n$
metric	$\sigma = \sigma_{ij} dx^i dx^j$		g		$\overline{g} = dr^2$ $+\lambda^2 \sigma_{ij} dx^i dx^j$
Levi-Civita connection	D		∇		$\overline{\nabla}$
Christoffel symbols	$\widetilde{\Gamma}^k_{ij}$				$\overline{\Gamma}^k_{ij}$
curvature tensor	$\widetilde{R}$		R		$\overline{R}$
Ricci tensor	$\widetilde{Ric}$		Ric		$\overline{Ric}$

4. The Inverse Mean Curvature Flow

In this section, we prove that the mean-convexity and star-shapedness is preserved by the IMCF and derive the estimate of λ, the decay estimates of the radial function, mean curvature, the second fundamental form that will be used to prove the convergence of the rescaled flow later.

Let M_0^n be a mean-convex star-shaped hypersurface in $\mathbb{N}^{n+1}$ which is given by an embedding

$$X_0 : \mathbb{S}^n \to \mathbb{N}^{n+1}$$

Let $X_t \coloneqq X(\cdot, t) : \mathbb{S}^n \to \mathbb{N}^{n+1}$, $t \in [0, T)$, be the solution of the IMCF with initial data given by X_0. In other words,

$$\frac{\partial X_t}{\partial t} = \frac{1}{H}\,\nu, \tag{4.1}$$

where ν is the unit outer normal vector and H is the mean curvature. We shall call (4.1) the parametric form of the IMCF.

Let $e_i = X_{t*}(\frac{\partial}{\partial x^i}) = \frac{\partial X_t}{\partial x^i}, i = 1, 2, \cdots, n$ be coordinate vector fields on $X_t(\mathbb{S}^n) \coloneqq M_t^n$. Denote by g_{ij} and h_{ij} the components of the first and second fundamental form, by $H = g^{ij}h_{ij}$ the mean curvature and $|A|^2 = h_{ik}h_{lj}g^{il}g^{jk}$ the squared norm of the second fundamental form, by $\phi = \overline{g}(\lambda(r)\frac{\partial}{\partial r}, \nu) = \frac{\lambda}{v}$ the support function, by $\psi = \frac{1}{H\phi}$ and by $d\mu_t$ the area element on M_t^n. We first collect the evolution equations for various geometric quantities under the IMCF.

Lemma 4.1 (Evolution equations)**.** *Under the IMCF* (1.1)*, we have*

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \frac{2}{H}h_{ij},\\
\frac{\partial}{\partial t}\mu_t &= d\mu_t,\\
\frac{\partial}{\partial t}\nu &= \frac{1}{H^2}\nabla H,\\
\frac{\partial}{\partial t}h_i^j &= \frac{\Delta h_i^j}{H^2} + \frac{|A|^2}{H^2}h_i^j - \frac{2}{H}h_i^k h_k^j - \frac{2}{H^3}\nabla_i H \nabla^j H - \frac{2}{H}\bar{R}_{\nu i \nu k}g^{kj}\\
&\quad + \frac{2}{H^2}g^{lj}g^{ks}h_k^m \bar{R}_{misl} + \frac{1}{H^2}g^{lj}g^{ks}h_i^m \bar{R}_{mksl} + \frac{1}{H^2}g^{lj}g^{ks}h_l^m \bar{R}_{mksi}\\
&\quad + \frac{1}{H^2}\overline{Ric}(\nu,\nu)h_i^j + \frac{1}{H^2}g^{lj}g^{ks}\left(\bar{\nabla}_k \bar{R}_{\nu isl} + \bar{\nabla}_l \bar{R}_{\nu ksi}\right)
\end{aligned} \tag{4.2}$$

$$\frac{\partial}{\partial t}H = \frac{\Delta H}{H^2} - 2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\overline{Ric}(\nu,\nu)}{H}$$

$$\frac{\partial}{\partial t}\phi \;= \frac{1}{H^2}(\triangle\phi + |A|^2\phi) + \frac{1}{H^2}(\phi\overline{Ric}(\nu,\nu) - \lambda\overline{Ric}(\nu, \frac{\partial}{\partial r}) \tag{4.3}$$

$$\frac{\partial}{\partial t}\psi \;= \phi^2\psi^2 \triangle \psi + 2\phi\psi\nabla\phi\cdot\nabla\psi - n\phi^2\psi^3\frac{\lambda''}{\lambda} \tag{4.4}$$

where ∇ *and* Δ *are gradient and Laplacian operator with respect to the induced metric on the flow hypersurface* Σ_t*.*

We could use the evolution equation (4.3) of the support function to show that under the IMCF, the evolved hypersurface M_t^n remains star-shaped and mean-convex.

Lemma 4.2. *Under the IMCF* (1.1)*, the evolved hypersurface* M_t^n *remains star-shaped and mean-convex if* M_0 *is star-shaped and mean-convex. Moreover,* $\phi \geq e^{\frac{t}{n}} \min_{M_0}\phi > 0$*.*

Proof. From the expression (3.4) of Ricci curvature and ϕ, we have

$$\phi\overline{Ric}(\nu,\nu) - \lambda\overline{Ric}(\nu, \frac{\partial}{\partial r}) \tag{4.5}$$

$$=\frac{\lambda}{v}\big[-n\frac{\lambda''}{\lambda} - |D\varphi|^2_{\mathbb{S}^n}\cdot\frac{n-1}{v^2}\cdot\frac{(\lambda')^2-1-\lambda\lambda''}{\lambda^2}\big] - \lambda(-\frac{n\lambda''}{v\lambda}) \tag{4.6}$$

$$=-|D\varphi|^2_{\mathbb{S}^n}\cdot\frac{n-1}{v^3}\cdot\frac{(\lambda')^2-1-\lambda\lambda''}{\lambda} \geq 0 \tag{4.7}$$

under our assumption $\lambda'' \geq 0$ and $\lambda' \leq 1$.

Thus we have

$$\frac{\partial\phi}{\partial t} \geq \frac{1}{H^2}(\triangle\phi + |A|^2\phi)$$

from the evolution equation (4.3) of the support function.

Since $\phi > 0$ on the initial hypersurface M_0^n, in view of the inequality $|A|^2 \geq \frac{H^2}{n}$ and using the parabolic maximum principle, we conclude that

$$\phi \geq e^{\frac{t}{n}} \min_{M_0^n} \phi > 0$$

which implies the star-shapedness of M_t^n is preserved. Using the evolution equation of $\psi = \frac{1}{H\phi}$ in (4.4) and the maximum principle, we have $\psi \leq \sup_{M_0^n} \psi$. Thus $0 < \min_{M_0^n} H\phi \leq H\phi$ and the mean-convexity is preserved by the IMCF. ■

We can write the initial hypersurface M_0^n as the graph of a function r_0 defined on the unit sphere if M_0^n is star-shaped:

$$M_0^n = \{(r_0(x), x) : x \in S^n\}.$$

If each M_t^n is star-shaped, it can be parameterized them as the graph

$$X_t : \mathbb{S}^n \to M_t^n := X_t(\mathbb{S}^n) \hookrightarrow \mathbb{N}^{n+1}, \forall t \in [0, T).$$

In this case, the IMCF can be written as a parabolic PDE for r. As long as the solution of (4.1) exists and remains star-shaped, it is equivalent to

$$\frac{\partial r}{\partial t} = \frac{v}{H}, \tag{4.8}$$

where $v = \sqrt{1 + |D\varphi|^2_{\mathbb{S}^n}}$.

The equation (4.8) will be referred as the non-parametric form of the IMCF.

We assume that $X_t := (r(x(t), t), x^1(t), \cdots, x^n(t))$ in the coordinate system $(r, x^1, \cdots, x^n)$. We can compute $\frac{\partial X}{\partial t} = \frac{dr}{dt}\frac{\partial}{\partial r} + \frac{dx^i}{dt}\frac{\partial}{\partial x^i}$. From (4.1) and (3.8), we have

$$\frac{dr}{dt}\frac{\partial}{\partial r} + \frac{dx^i}{dt}\frac{\partial}{\partial x^i} = \frac{1}{vH}(\frac{\partial}{\partial r} - \frac{D^i\varphi}{\lambda}\frac{\partial}{\partial x^i}).$$

Comparing the components of the vector, we get

$$\frac{dr}{dt} = \frac{1}{vH} \qquad \text{and} \qquad \frac{dx^i}{dt} = -\frac{D^i\varphi}{\lambda v H}. \tag{4.9}$$

Then $r = r(x(t), t)$, $\frac{dr}{dt} = \frac{\partial r}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial r}{\partial t} = D_i r \frac{dx^i}{dt} + \frac{\partial r}{\partial t}$

$$\frac{\partial r}{\partial t} = \frac{dr}{dt} - D_i r\frac{dx^i}{dt} = \frac{1}{vH} + D_i r\frac{D^i r}{\lambda^2 vH} = \frac{\lambda^2 + |Dr|^2}{\lambda^2 vH} = \frac{\lambda^2 + \lambda^2(v^2-1)}{\lambda^2 vH} = \frac{v}{H} \tag{4.10}$$

Recall that

$$h^i_j = \frac{1}{v\lambda}(\lambda'\delta^i_j - \widetilde{\sigma}^{ik}\varphi_{k,j}),$$

and the mean curvature is

$$H = h^i_i = \frac{1}{v\lambda}(n\lambda' - \widetilde{\sigma}^{ij}\varphi_{j,i}) \tag{4.11}$$

Evolution equation (4.9) becomes

$$\frac{dr}{dt} = \frac{1}{vH} = \frac{\lambda}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}, \tag{4.12}$$

and (4.10) can also be rewritten as $\lambda\frac{\partial\varphi}{\partial t} = \frac{\partial r}{\partial t} = \frac{v}{H} = \frac{v^2\lambda}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}$, hence

$$\frac{\partial\varphi}{\partial t} = \frac{v^2}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}. \tag{4.13}$$

Next we prove a sharp estimates on λ.

Proposition 4.3. *Let r_1, r_2 be constants such that*

$$r_1 \le r(x,0) \le r_2$$

holds on the initial hypersurface M^n_0. Then on M^n_t we have the estimate

$$\lambda(r_1)e^{\frac{t}{n}} \le \lambda(r(x,t)) \le \lambda(r_2)e^{\frac{t}{n}}, \qquad \forall\ t \in [0,T).$$

Moreover, there exists a constant C such that the support function has the estimate $\frac{1}{C}e^{\frac{t}{n}} \leq \phi \leq Ce^{\frac{t}{n}}$

Proof. Let x_t be the maximizer that $r(x_t, t) = \max_{x \in \mathbb{S}^n} r(x, t)$ which is dependent on t. Substitute x_t for x in (3.4) and get $\varphi_{i,j} = \frac{1}{\lambda} r_{i,j}$ which is a negative-definite matrix since r occurs a maximum at x_t. Because of that $\widetilde{\sigma}^{ij}$ is positive-definite, $\widetilde{\sigma}^{ij}\varphi_{i,j} \leq 0$. Substitute x_t for x in (4.12) and get

$$\frac{dr}{dt}(x_t, t) = \frac{\lambda}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}} \leq \frac{\lambda}{n\lambda'}$$

or

$$n\lambda'(r)\frac{dr}{dt}(x_t, t) - \lambda(r) \leq 0.$$

Multiply on both side by $e^{-\frac{t}{n}}$ then get

$$\frac{d}{dt}\left(e^{-\frac{t}{n}}\lambda(r(x_t, t))\right) \leq 0.$$

Integrate and we have

$$e^{-\frac{t}{n}}\lambda(r(x_t, t)) - \lambda(r(x_0, 0)) \leq 0.$$

Since λ is increasing w.r.t. r,

$$\lambda(r(x, t)) \leq \lambda(r(x_t, t)) \leq e^{\frac{t}{n}}\lambda(r(x_0, 0)) = e^{\frac{t}{n}}\lambda(\max_{x \in \mathbb{S}^n} r(x, 0)).$$

A similar argument can show that

$$e^{\frac{t}{n}}\lambda(\min_{x \in \mathbb{S}^n} r(x, 0)) \leq \lambda(r(x, t)) \leq e^{\frac{t}{n}}\lambda(\max_{x \in \mathbb{S}^n} r(x, 0)).$$

Since $\phi = \frac{\lambda}{v}$ and $v \geq 1$, we have $\phi \leq \lambda \leq Ce^{\frac{t}{n}}$. The lower estimate of the support function follows from Lemma 4.2. ■

With this estimate on λ, we can estimate the mean curvature.

Lemma 4.4. *There is a constant $C_1 > 0$ such that $He^{\frac{t}{n}} \le C_1$ and $H\lambda \le C_1$ if we also assume $\lambda''\lambda^{1+\varepsilon} \le C$, where $\varepsilon > 0$.*

Proof. From the evolution equation of H, we have

$$\frac{\partial}{\partial t}H^2 = \frac{\Delta H^2}{H^2} - 4\frac{|\nabla H|^2}{H^2} - 2|A|^2 - 2\overline{Ric}(\nu,\nu).$$

Since $\lambda'' \ge 0$ and $\lambda'^2 \le 1$, we have $\overline{Ric}(\nu,\nu) = -n\frac{\lambda''}{\lambda} - |D\varphi|^2 \cdot \frac{n-1}{v^2} \cdot \frac{(\lambda')^2 - 1 - \lambda\lambda''}{\lambda^2} \ge -n\frac{\lambda''}{\lambda}$. From $\lambda''\lambda^{1+\varepsilon} \le C$ and $c_1 e^{\frac{t}{n}} \le \lambda \le c_2 e^{\frac{t}{n}}$, we have $-2\overline{Ric}(\nu,\nu) \le O(\lambda^{-2-\varepsilon}) = O(e^{-\frac{(2+\varepsilon)t}{n}})$. Using $-2\overline{Ric}(\nu,\nu) \le O(e^{-\frac{(2+\varepsilon)t}{n}})$ and the inequality $|A|^2 \ge \frac{1}{n}H^2$, we obtain that

$$\frac{d}{dt}H^2_{\max} \le -\frac{2}{n}H^2_{\max} + O(e^{-\frac{(2+\varepsilon)t}{n}}).$$

From this, we get $He^{\frac{t}{n}} \le C_1$. Using Proposition 4.3, we have $H\lambda \le C_2$. ∎

Let $F := \frac{n\lambda' - \tilde{\sigma}^{ij}\varphi_{i,j}}{v^2}$. Then we can rewrite (4.13) as

$$\frac{\partial \varphi}{\partial t} = \frac{1}{F} \tag{4.14}$$

and the mean curvature can be rewritten as

$$H = \frac{vF}{\lambda}. \tag{4.15}$$

Since $v^2 = 1 + |D\varphi|^2_{\mathbb{S}^n}$ and $\widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2}$, we can regard F as a function of $r, D_i\varphi, \varphi_{i,j}$. Let $a^i := \frac{\partial F}{\partial(D_i\varphi)}$ and $a^{ij} := -\frac{\partial F}{\partial \varphi_{i,j}} = \frac{\tilde{\sigma}^{ij}}{v^2}$.

Moreover, (3.7) becomes

$$\lambda^2 g^{ij} = \sigma^{ij} - \frac{D^i\varphi D^j\varphi}{v^2} = \widetilde{\sigma}^{ij} = v^2 a^{ij}.$$

So (a^{ij}) is positive definite.

Proposition 4.5. *There is a constant $C > 0$ such that $He^{\frac{t}{n-1}} \geq C$ and $\frac{\partial \varphi}{\partial t} \leq \sup_{x\in\mathbb{S}^n} \frac{\partial \varphi}{\partial t}(x,0)$ is uniformly bounded.*

Proof. If we differentiate (4.14) with respect to t and use the notations above, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t}\frac{1}{F} &= -\frac{1}{F^2}\Big[\frac{\partial F}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial F}{\partial (D_i\varphi)}\frac{\partial}{\partial t}D_i\varphi + \frac{\partial F}{\partial \varphi_{i,j}}\frac{\partial \varphi_{i,j}}{\partial t}\Big] \\
&= -\frac{1}{F^2}\Big[\frac{n\lambda''}{v^2}\frac{\partial r}{\partial t} + a^i D_i\frac{\partial \varphi}{\partial t} - a^{ij}(\frac{\partial \varphi}{\partial t})_{i,j}\Big] \\
&= \frac{1}{F^2}(a^{ij}D_{ij}(\frac{\partial \varphi}{\partial t}) - a^i D_i\frac{\partial \varphi}{\partial t}) - \frac{1}{F^3}\frac{n\lambda\lambda''}{v^2} \\
&\leq \frac{1}{F^2}(a^{ij}D_{ij}(\frac{\partial \varphi}{\partial t}) - a^i D_i\frac{\partial \varphi}{\partial t}).
\end{aligned}$$

where we have used the fact that $(a^{ij}) > 0$, $F = \frac{H}{v\lambda} > 0$, $\lambda > 0$ and $\lambda'' > 0$. By maximum principle, we have $\frac{\partial \varphi}{\partial t}(x,t) \leq \sup_{x\in\mathbb{S}^n} \frac{\partial \varphi}{\partial t}(x,0)$. Noting that $\frac{\partial}{\partial t}\varphi = \frac{v}{\lambda H}$ and $v \geq 1$, we obtain that $\lambda H \geq C > 0$. The assertion follows from the above inequality and by using Proposition 4.3.

■

Now we have a good decay estimate on mean curvature.

Proposition 4.6. *The mean curvature $H(x,t)$ of M_t^n satisfies*

$$C_1 \leq H(x,t)e^{\frac{t}{n}} \leq C_2$$

and

$$C_1 \leq H(x,t)\lambda \leq C_2$$

, where C_1 and C_2 are positive constants.

Proof. This follows from Proposition 4.3, Lemma 4.4 and previous Proposition. ■

Next we derive the evolution equation of $\omega = \frac{1}{2}|D\varphi|^2_{\mathbb{S}^n}$.

Lemma 4.7.

$$\frac{\partial}{\partial t}\omega - \frac{a^{ij}\omega_{i,j} - a^i D_i\omega}{F^2} + \frac{1}{F^2}(\frac{2n\lambda''\omega}{v^2}\lambda + a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} - a^{ij}D_i\varphi D_j\varphi) = -\frac{a^{ij}}{F^2}\sigma^{kl}\varphi_{i,k}\varphi_{j,l}. \tag{4.16}$$

Proof. Note that

$$\frac{\partial}{\partial t}\omega = \frac{\partial}{\partial t}(\frac{1}{2}\sigma^{ij}D_i\varphi D_j\varphi) = \sigma^{ij}D_i\frac{\partial\varphi}{\partial t}D_j\varphi = D^i\varphi D_i(\frac{1}{F}).$$

Since $F = F(r, D_i\varphi, \varphi_{i,j})$ where $i, j = 1, 2...n$, by chain rule, then

$$\begin{aligned}\frac{\partial}{\partial t}\omega &= -\frac{D^k\varphi}{F^2}(\frac{\partial F}{\partial r}D_k r + \frac{\partial F}{\partial(D_i\varphi)}D_kD_i\varphi + \frac{\partial F}{\partial\varphi_{i,j}}D_k\varphi_{i,j}) \\ &= -\frac{D^k\varphi}{F^2}(\frac{n\lambda''}{v^2}D_k r + a^i D_iD_k\varphi - \frac{\tilde{\sigma}^{ij}}{v^2}D_k\varphi_{i,j}) \\ &= -\frac{1}{F^2}(\frac{nD^k\varphi\lambda''}{v^2}\lambda D_k\varphi + a^iD^k\varphi\varphi_{i,j} - D^k\varphi a^{ij}D_k\varphi_{i,j}) \\ &= -\frac{1}{F^2}(\frac{n\lambda''|D\varphi|^2_{\mathbb{S}^n}}{v^2}\lambda + \frac{1}{2}a^iD_i|D\varphi|^2_{\mathbb{S}^n} - a^{ij}D^k\varphi D_k\varphi_{i,j}) \\ &= -\frac{1}{F^2}(\frac{n\lambda''|D\varphi|^2_{\mathbb{S}^n}}{v^2}\lambda + a^iD_i\omega - a^{ij}D^k\varphi D_k\varphi_{i,j}).\end{aligned}$$

We have

$$\frac{\partial}{\partial t}\omega + \frac{1}{F^2}(\frac{2n\lambda''\omega}{v^2}\lambda + a^iD_i\omega - a^{ij}D^k\varphi D_k\varphi_{i,j}) = 0. \tag{4.17}$$

On the other hand,

$$D_i\omega = D_i(\frac{1}{2}\sigma^{kl}D_k\varphi D_l\varphi) = \frac{1}{2}\sigma^{kl}\big[(D_iD_k\varphi)D_l\varphi + D_k\varphi(D_iD_l\varphi)\big]$$

$$\begin{aligned}D_jD_i\omega = \frac{1}{2}\sigma^{kl}\big[&(D_jD_iD_k\varphi)D_l\varphi + (D_iD_k\varphi)(D_jD_l\varphi) \\ &+ (D_jD_k\varphi)(D_iD_l\varphi) + D_k\varphi(D_jD_iD_l\varphi)\big]\end{aligned}$$

$$= \sigma^{kl}\big[(D_jD_iD_k\varphi)D_l\varphi + (D_iD_k\varphi)(D_jD_l\varphi)\big]$$

$$= \sigma^{kl}\big[(\widetilde{R}_{jkib}\sigma^{ba}D_a\varphi + D_kD_jD_i\varphi)D_l\varphi + (D_iD_k\varphi)(D_jD_l\varphi)\big]$$

by Ricci identity $D_jD_iD_k\varphi - D_kD_jD_i\varphi = \widetilde{R}_{jkib}\sigma^{ba}D_b\varphi$

$$= \sigma^{kl}\big[(\sigma_{ji}\sigma_{kb} - \sigma_{jb}\sigma_{ki})\sigma^{ba}D_a\varphi D_l\varphi + (D_kD_jD_i\varphi)D_l\varphi + (D_iD_k\varphi)(D_jD_l\varphi)\big]$$

since $\widetilde{R}_{jkib} = \sigma_{ji}\sigma_{kb} - \sigma_{jb}\sigma_{ki}$

$$= \sigma^{kl}\sigma_{ji}\sigma_{kb}\sigma^{ba}D_a\varphi D_l\varphi - \sigma^{kl}\sigma_{jb}\sigma_{ki}\sigma^{ba}D_a\varphi D_l\varphi$$

$$+ \sigma^{kl}(D_kD_jD_i\varphi)D_l\varphi + \sigma^{kl}(D_iD_k\varphi)(D_jD_l\varphi)$$

$$= \sigma_{ji}|D\varphi|^2_{\mathbb{S}^n} - D_i\varphi D_j\varphi + D^k\varphi(D_kD_jD_i\varphi) + \sigma^{kl}(D_iD_k\varphi)(D_jD_l\varphi).$$

Then (4.17) becomes

$$\frac{\partial}{\partial t}\omega + \frac{1}{F^2}\Big(\frac{2n\lambda''\omega}{v^2}\lambda + a^iD_i\omega - a^{ij}\omega_{i,j} + a^{ij}\sigma_{ij}|D\varphi|^2 - a^{ij}D_i\varphi D_j\varphi\Big)$$
$$= -\frac{a^{ij}}{F^2}\sigma^{kl}\varphi_{i,k}\varphi_{j,l}.$$

■

We need the following Lemma to prove the decay estimate of ω.

Lemma 4.8. *We have*

$$\frac{1}{v^4}(\sigma^{ij})_{n\times n} \le (a^{ij})_{n\times n} \le \frac{1}{v^2}(\sigma^{ij})_{n\times n}$$

and

$$a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} - a^{ij}D_i\varphi D_j\varphi \ge \frac{n-1}{v^4}|D\varphi|^2_{\mathbb{S}^n}.$$

Proof. Let $w = w^i \frac{\partial}{\partial x^i}$ be a tangent vector on $\mathbb{S}^n$, then

$$
\begin{aligned}
a^{ij} w_i w_j = (\frac{\sigma^{ij}}{v^2} - \frac{D^i\varphi D^j\varphi}{v^4}) w_i w_j &= \frac{|w|^2_{\mathbb{S}^n}}{v^2} - \frac{(\sigma^{ik} D_k\varphi \; w_i)^2}{v^4} \\
&\geq \frac{v^2 |w|^2_{\mathbb{S}^n}}{v^4} - \frac{|D\varphi|^2_{\mathbb{S}^n} |w|^2_{\mathbb{S}^n}}{v^4} \text{ by Cauchy inequality} \\
&= \frac{1}{v^4}[(1 + |D\varphi|^2_{\mathbb{S}^n})|w|^2_{\mathbb{S}^n} - |D\varphi|^2_{\mathbb{S}^n}|w|^2_{\mathbb{S}^n}] = \frac{|w|^2_{\mathbb{S}^n}}{v^4} = \frac{\sigma^{ij} w_i w_j}{v^4}.
\end{aligned}
$$

Also,

$$
\frac{|w|^2_{\mathbb{S}^n}}{v^2} - \frac{(\sigma^{ik} D_k\varphi \; w_i)^2}{v^4} \leq \frac{|w|^2_{\mathbb{S}^n}}{v^2} = \frac{\sigma^{ij}}{v^2} w_i w_j.
$$

Recall that $a^{ij} D_i\varphi D_j\varphi = (\frac{\sigma^{ij}}{v^2} - \frac{D^i\varphi D^j\varphi}{v^4}) D_i\varphi D_j\varphi = \frac{|D\varphi|^2_{\mathbb{S}^n}}{v^2} - \frac{|D\varphi|^4_{\mathbb{S}^n}}{v^4} = \frac{|D\varphi|^2_{\mathbb{S}^n}}{v^4}$. Note that we have used $v^2 = 1 + |D\varphi|^2_{\mathbb{S}^n}$.

Using $a^{ij} \geq \frac{\sigma^{ij}}{v^4}$, we have $a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} \geq \frac{\sigma^{ij}}{v^4}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} = \frac{n|D\varphi|^2_{\mathbb{S}^n}}{v^4}$. Thus we have $a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} - a^{ij} D_i\varphi D_j\varphi \geq \frac{n-1}{v^4}|D\varphi|^2_{\mathbb{S}^n}$.

■

For the first order space derivatives of φ, we have the following estimate.

Lemma 4.9. *There are constants $\theta > 0$ and $C > 0$ such that $|D\varphi|^2_{S^n} \leq Ce^{-\theta t}$.*

Proof. Applying Lemma 4.8 to the RHS of (4.16), we have

$$
-\frac{a^{ij}}{F^2}\sigma^{kl}\varphi_{i,k}\varphi_{j,l} \leq -\frac{1}{v^4 F^2}\sigma^{ij}\sigma^{kl}\varphi_{i,k}\varphi_{j,l} \leq 0
$$

and

$$
0 \geq \frac{\partial}{\partial t}\omega + \frac{1}{F^2}\big(a^i D_i\omega - a^{ij}\omega_{i,j} + a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} + \frac{2n\lambda\lambda''}{v^2}\omega - a^{ij} D_i\varphi D_j\varphi\big)
$$

From Lemma 4.8, we have $a^{ij}\sigma_{ij}|D\varphi|^2_{\mathbb{S}^n} - a^{ij} D_i\varphi D_j\varphi \geq \frac{n-1}{v^4}|D\varphi|^2_{\mathbb{S}^n}$ and obtain

$$
0 \geq \frac{\partial}{\partial t}\omega + \frac{1}{F^2}\big(a^i D_i\omega - a^{ij}\omega_{i,j} + \frac{2n\lambda\lambda''}{v^2}\omega + \frac{n-1}{v^4}|D\varphi|^2_{\mathbb{S}^n}\big).
$$

Using $\lambda > 0$, $\lambda'' > 0$, $\omega \geq 0$ and $F = \frac{H\lambda}{v}$, we get

$$\frac{\partial}{\partial t}\omega + \frac{1}{F^2}\left(a^i D_i\omega - a^{ij}\omega_{i,j}\right) \leq -\frac{2(n-1)\omega}{H^2\lambda^2 v^2}.$$

Thus $\omega = \frac{1}{2}|D\varphi|^2_{\mathbb{S}^n}$ is uniformly bounded and so $v = 1 + |D\varphi|^2_{\mathbb{S}^n}$ is uniformly bounded. From Proposition 4.6, we know that $H^2\lambda^2$ bounded from above. Thus we can find a positive constant $\theta > 0$ such that

$$\frac{\partial}{\partial t}\omega + \frac{1}{F^2}\left(a^i D_i\omega - a^{ij}\omega_{i,j}\right) \leq -\theta\omega$$

and $\omega \leq \omega_0 e^{-\theta t}$ where $\omega_0 = \sup_{x\in\mathbb{S}^n} \omega(x,0)$.

■

Corollary 4.10. *There exists C such that $C\sigma^{ij} \leq \widetilde{\sigma}^{ij} \leq \sigma^{ij}$ is uniformly bounded.*

Proof. Due to Lemma 4.8, $\frac{1}{v^2}(\sigma^{ij})_{n\times n} \leq v^2(a^{ij})_{n\times n} = (\widetilde{\sigma}^{ij})_{n\times n} \leq (\sigma^{ij})_{n\times n}$. Since v is uniformly bounded, we have the desire result. ■

Next we can estimate the second fundamental form.

Lemma 4.11. *There is a constant C such that $|A| \leq Ce^{-\frac{t}{n}}$ if $|\overline{R}| \leq \frac{C}{\lambda^2} = O(e^{-\frac{2t}{n}})$ and $|\overline{\nabla R}| \leq \frac{C}{\lambda^3} = O(e^{-\frac{3t}{n}})$.*

Proof. Recall that the evolution equation of h_i^j is

$$\begin{aligned}\frac{\partial h_i^j}{\partial t} &= \frac{\Delta h_i^j}{H^2} - 2\,\frac{\nabla_i H\,\nabla^j H}{H^3} + \frac{|A|^2}{H^2}\,h_i^j - 2\,\frac{h_i^k h_k^j}{H} \\ &\quad + \frac{2}{H^2}\,g^{kl}\,g^{sj}\,\overline{R}_{miks}\,h_l^m - \frac{1}{H^2}\,g^{kl}\,g^{sj}\,\overline{R}_{mksl}\,h_i^m - \frac{1}{H^2}\,g^{kl}\,\overline{R}_{mkil}\,h^{mj} \\ &\quad + \frac{1}{H^2}\,\overline{\mathrm{Ric}}(\nu,\nu)\,h_i^j - \frac{2}{H}\,g^{mj}\,\overline{R}_{\nu i\nu m} \\ &\quad - \frac{1}{H^2}\,g^{kl}\,g^{mj}\,\overline{\nabla}_m\overline{R}_{\nu kil} - \frac{1}{H^2}\,g^{kl}\,g^{mj}\,\overline{\nabla}_k\overline{R}_{\nu iml}.\end{aligned}$$

Since $|\overline{R}| \leq \frac{C}{\lambda^2} = O(e^{-\frac{2t}{n}})$ and $|\overline{\nabla} \overline{R}| \leq \frac{C}{\lambda^3} = O(e^{-\frac{3t}{n}})$, we have

$$|\frac{2}{H^2}\, g^{kl}\, g^{sj}\, \overline{R}_{miks}\, h_l^m - \frac{1}{H^2}\, g^{kl}\, g^{sj}\, \overline{R}_{mksl}\, h_i^m - \frac{1}{H^2}\, g^{kl}\, \overline{R}_{mkil}\, h^{mj}| \leq \frac{|A|}{H^2} O(e^{-\frac{2t}{n}}),$$

$$|\frac{2}{H}\, g^{mj}\, \overline{R}_{\nu i \nu m}| \leq \frac{1}{H} O(e^{-\frac{2t}{n}}) \leq \frac{|A|}{H^2} O(e^{-\frac{2t}{n}})$$

and

$$|\frac{1}{H^2}\, g^{kl}\, g^{mj}\, \overline{\nabla}_m \overline{R}_{\nu kil} - \frac{1}{H^2}\, g^{kl}\, g^{mj}\, \overline{\nabla}_k \overline{R}_{\nu iml}| \leq \frac{1}{H^2} O(e^{-\frac{3t}{n}}).$$

We obtain

$$\begin{aligned} \frac{\partial h_i^j}{\partial t} &= \frac{\Delta h_i^j}{H^2} - 2\, \frac{\nabla_i H\, \nabla^j H}{H^3} + \frac{|A|^2}{H^2}\, h_i^j - 2\, \frac{h_i^k h_k^j}{H} + \frac{1}{H^2}\, \overline{\mathrm{Ric}}(\nu, \nu)\, h_i^j \\ &+ \frac{|A|}{H^2}\, O(e^{-\frac{2}{n}t}) + \frac{1}{H^2} O(e^{-\frac{3t}{n}}). \end{aligned} \tag{4.18}$$

Recall that the evolution of the mean curvature is given by

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2\, \frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\overline{Ric}(\nu, \nu)}{H}. \tag{4.19}$$

Combining (4.19) and (4.18), we obtain the following evolution equation for the tensor $M_i^j = H\, h_i^j$:

$$\begin{aligned} \frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - 2\, \frac{\nabla^k H\, \nabla_k M_i^j}{H^3} - 2\, \frac{\nabla_i H\, \nabla^j H}{H^2} - 2\, \frac{M_i^k\, M_k^j}{H^2} \\ &+ \frac{|M|}{H^2}\, O(e^{-\frac{2}{n}t}) + \frac{1}{H} O(e^{-\frac{3t}{n}}). \end{aligned}$$

Let μ denote the largest eigenvalue of the tensor M_i^j, and let $\mu_{\max}(t)$ denote the maximum of μ at a given time t. Since the trace of M is positive, we have $|M| \leq C\mu$ for some constant C. Since $c_1 e^{-\frac{2}{n}t} \leq H^2 \leq c_2 e^{-\frac{2}{n}t}$, we obtain

$$\frac{d}{dt}\mu_{\max} \leq -\frac{1}{C}\, e^{\frac{2}{n}t} \mu_{\max}^2 + C\, \mu_{\max} + C e^{-\frac{2}{n}t}$$

for some uniform constant C. Thus

$$e^{\frac{2}{n}t}\frac{d}{dt}\mu_{\max} \le -\frac{1}{C}\, e^{\frac{4}{n}t}\mu_{\max}^2 + C\, e^{\frac{2}{n}t}\mu_{\max} + C \tag{4.20}$$

Let $w = e^{\frac{2}{n}t}\mu_{\max}$. Then $w' = e^{\frac{2}{n}t}\mu'_{\max} + \frac{2}{n}e^{\frac{2}{n}t}\mu_{\max}$, $e^{\frac{2}{n}t}\mu'_{\max} = w' - \frac{2}{n}w$ and

$$w' \le -\frac{1}{C}w^2 + (C + \frac{2}{n})w + C.$$

Thus w is uniform bounded and $\mu_{\max} \le Ce^{-\frac{2t}{n}}$. Thus $H|A| \le Ce^{-\frac{2t}{n}}$ and $|A| \le Ce^{-\frac{t}{n}}$. ■

Using the decay estimate of the second fundamental form, we can obtain the C^2 estimate of φ.

Lemma 4.12. $|D^2\varphi|_{\mathbb{S}^n}$ *is uniformly bounded.*

Proof. Since $h^i_j = \frac{1}{v\lambda}(\lambda'\delta^i_j - \widetilde{\sigma}^{ik}\varphi_{k,j})$, we get $\widetilde{\sigma}^{ik}\varphi_{k,j} = \lambda'\delta^i_j - v\lambda h^i_j$. Using $|A| \le Ce^{-\frac{t}{n}}$, $v \le C$, $\lambda \le Ce^{\frac{-t}{n}}$, $\lambda' \le C$ and $C\sigma^{ij} \le \widetilde{\sigma}^{ij} \le \sigma^{ij}$, we have $|D^2\varphi|_{\mathbb{S}^n}$ is uniformly bounded. ■

Using the following result of Evans-Krylov [10], we can obtain $C^{2,\alpha}$ estimate of φ. This is from Theorem 6 of Ben Andrews's paper [1].

Theorem 4.13. *Let* $u \in C^4(\Omega \times (0, T])$ *satisfy*

$$\frac{\partial u}{\partial t} = G(D^2u, Du, u, x, t)$$

where G is C^2, $\lambda I \le [\dot{G}^{ij}] \le \Lambda I$ for some $0 < \lambda \le \Lambda$, and $\lambda I \le [\dot{G}^{ij}] \le \Lambda I$ for some $\Lambda \ge \lambda > 0$, and $\ddot{G}^{ij,kl}M_{ij}M_{kl} \le 0$ for all matrices $[M_{ij}]$ for which $\dot{G}^{ij}M_{ij} = 0$. Then *for any* $\tau > 0$ *and* $\Omega' \subset\subset \Omega$,

$$\sup_{s,t\in[\tau,T],p,q\in\Omega'} \left(\frac{|D^2u(p,t) - D^2u(q,t)|}{|p-q|^\alpha + |s-t|^{\frac{\alpha}{2}}} + \frac{|\partial_t u(p,t) - \partial_t u(q,t)|}{|p-q|^\alpha + |s-t|^{\frac{\alpha}{2}}} \right)$$

$$+ \sup_{p\in\Omega',\tau\le s,t\le T} \frac{|Du(p,t) - Du(p,s)|}{|s-t|^{\frac{(1+\alpha)}{2}}} \le C$$

where α *depends on* n, λ *and* Λ, *and* C *depends* n, λ, Λ, $\sup_{\Omega\times(0,T]}|D^2u|$, $\sup_{\Omega\times(0,T]}|\partial_t u|$, $d(\Omega',\partial\Omega)$, τ *and bounds for the first and second derivatives of* G *(other than the second derivative in the first argument).*

Lemma 4.14. $|D^k\varphi|_{\mathbb{S}^n} \le C_k$ *for any* $k \ge 3$

Proof. Recall that $\frac{\partial}{\partial t}\varphi = \frac{1}{F}$ and $F = \frac{n\lambda' - \tilde{\sigma}^{ij}\varphi_{i,j}}{v^2}$. We have $\frac{\partial}{\partial\varphi_{i,j}}(\frac{1}{F}) = \frac{\tilde{\sigma}^{ij}}{v^2F^2}$ and $(\ddot{\frac{1}{F}})^{ij,kl} = \frac{\partial^2}{\partial\varphi_{k,l}\partial\varphi_{i,j}}(\frac{1}{F}) = \frac{2\tilde{\sigma}^{ij}\tilde{\sigma}^{kl}}{v^4F^3}$. Hence $C_1\sigma^{ij} \le [(\dot{\frac{1}{F}})^{ij}] \le C_2\sigma^{ij}$ from Corollary 4.10. Moreover, $(\ddot{\frac{1}{F}})^{ij,kl}M_{ij}M_{kl} = \frac{2\tilde{\sigma}^{ij}\tilde{\sigma}^{kl}M_{ij}M_{kl}}{v^4F^3} = 0$ for all matrices $[M_{ij}]$ for which $(\dot{\frac{1}{F}})^{ij}M_{ij} = \frac{\tilde{\sigma}^{ij}M_{ij}}{v^2F^2} = 0$. Since $|\frac{\partial}{\partial t}\varphi|_{\mathbb{S}^n}$, $|D\varphi|_{\mathbb{S}^n}$ and $|D^2\varphi|_{\mathbb{S}^n}$ are uniformly bounded, we can apply Theorem 4.13 to get $|\varphi|_{C^{2,\alpha}} \le C$. The standard Schauder estimates for uniformly parabolic equation imply that $|D^k\varphi|_{\mathbb{S}^n} \le C_k$ for $k \ge 3$. ■

Here we recall a result by Hamilton ([6], Corollary 12.6) which be used to prove decay estimate of $|D^2\varphi|_{\mathbb{S}^n}$ later.

Theorem 4.15. *If* T *is any tensor and if* $1 \le i \le m-1$ *where* m *is any integer then with a constant* $C = C(n,m)$ *which is independent of the metric and connection we have the estimate*

$$\int |\partial^i T|^{\frac{2m}{i}} d\mu \le C \max_{\mathbb{N}} |T|^{2(\frac{m}{i}-1)} \int |\partial^m T|^2 d\mu.$$

Proposition 4.16. $|D^2\varphi|_{\mathbb{S}^n} \le Ce^{-\theta t}$ *for some positive constants* C *and* θ.

Proof. Choose $T = D\varphi$ and $i = 1 \leq m = n$. From Theorem 4.15, $|D\varphi|_{\mathbb{S}^n} \leq Ce^{-\theta t}$ and $|D^n\varphi|_{\mathbb{S}^n} \leq C_n$, we have

$$\int_{\mathbb{S}^n} |D^2\varphi|^{2n}_{\mathbb{S}^n} d\sigma \leq C \max_{\mathbb{S}^n} |D\varphi|^{2(n-1)}_{\mathbb{S}^n} \int_{\mathbb{S}^n} |D^n\varphi|^2_{\mathbb{S}^n} d\sigma \leq C_1 e^{-\theta_1 t}$$

for some $\theta_1 > 0$. Similarly, when $T = D\varphi$ and $i = 2 \leq m = 2n$, then we have

$$\int_{\mathbb{S}^n} |D^3\varphi|^{2n}_{\mathbb{S}^n} d\sigma \leq C \max_{\mathbb{S}^n} |D\varphi|^{2(n-1)}_{\mathbb{S}^n} \int_{\mathbb{S}^n} |D^{2n+1}\varphi|^2_{\mathbb{S}^n} d\sigma \leq C_2 e^{-\theta_2 t}.$$

for some $\theta_2 > 0$. Sobolev's inequality states that if $p > n$, then

$$\max_{\mathbb{S}^n} |f|^p \leq C \int_{\mathbb{S}^n} (|f|^p + |Df|^p) d\sigma.$$

Take $f = D^2\varphi$ and $p = 2n$, then

$$|D^2\varphi|^{2n}_{\mathbb{S}^n} \leq \max_{\mathbb{S}^n} |D^2\varphi|^{2n}_{\mathbb{S}^n} \leq C \int_{\mathbb{S}^n} (|D^2\varphi|^p_{\mathbb{S}^n} + |D^3\varphi|^p_{\mathbb{S}^n}) d\sigma \leq C_3 e^{-\theta_3 t}.$$

That is,

$$|D^2\varphi|_{\mathbb{S}^n} \leq C_4 e^{-\theta_4 t}.$$

∎

Lemma 4.17. $|\widetilde{\sigma}^{ij}\varphi_{i,j}|_{\mathbb{S}^n} \leq Ce^{-\theta t}$ *for some positive constant* C *and* θ*.*

Proof. By Corollary 4.10 and Proposition 4.16, $|\widetilde{\sigma}^{ij}\varphi_{i,j}|_{\mathbb{S}^n} \leq |\sigma^{ij}\varphi_{i,j}|_{\mathbb{S}^n} \leq C|D^2\varphi|_{\mathbb{S}^n} \leq Ce^{-\theta t}$.

∎

5. Proof of Theorem 1.3

Since the second fundamental form is bounded, the solution of (1.1) has long time existence. We consider the rescaled surfaces

$$\widehat{X}(x,t) := X(x,t)e^{-\frac{t}{n}}, \qquad \forall x \in \mathbb{S}^n,\ t \in [0,\infty).$$

Let $\widehat{r}(x,t) := r(x,t)e^{-\frac{t}{n}}$ be the radial function of $\widehat{X}(\mathbb{S}^n,t)$. Denote the first, second fundamental form by $\widehat{g}_{ij}$ and $\widehat{h}_{ij}$. Also define a function $\widehat{\varphi}(x,t)$ in analogous way. We are going to prove that $\widehat{r}(x,t)$ has a common limit κ as $t \to \infty$ for any $x \in \mathbb{S}^n$, $\widehat{g}_{ij} \to \lambda^2(\kappa)\sigma_{ij}$ and $\widehat{h}^i_j \to \frac{\lambda'(\kappa)}{\lambda(\kappa)}\delta^i_j$ as $t \to \infty$. Fortunately, we can determine the constants κ.

Lemma 5.1. *$\widehat{r}(x,t)$ is bounded.*

Proof. Combine (4.10), Proposition 4.6 and Lemma 4.9, we have

$$C_2 e^{\frac{t}{n}} \le \frac{\partial r}{\partial t}(x,t) = \frac{v}{H} \le C_1 e^{\frac{t}{n}}$$

for some positive constants C_1 and C_2. Integrate both sides and get

$$C_2(e^{\frac{t}{n}} - 1) \le r(x,t) - r(x,0) \le C_1(e^{\frac{t}{n}} - 1) \le Ce^{\frac{t}{n}}.$$

Therefore

$$\left[C_2(1 - e^{\frac{t}{n}}) + r(x,0)\right]e^{-\frac{t}{n}} \le \widehat{r}(x,t) = r(x,t)e^{-\frac{t}{n}} \le \left[C_1 e^{\frac{t}{n}} + r(x,0)\right]e^{-\frac{t}{n}} \le C_1 + \sup_{x\in\mathbb{S}^n} r(x,0).$$

∎

Notice that the lower bound $C_2 + \left[r(x,0) - C_2\right]e^{-\frac{t}{n}}$ is greater than a positive number for t large enough. This guarantees the limit is not 0 if it exists.

Proposition 5.2. *For any $x \in \mathbb{S}^n$, $\widehat{r}(x,t) \to \kappa$ uniformly as $t \to \infty$ where κ is a positive constant which is independent of x.*

Proof. Differentiate $\widehat{r}$ w.r.t. t,

$$\frac{\partial \widehat{r}}{\partial t} = \frac{\partial r}{\partial t} e^{-\frac{t}{n}} - \frac{r}{n} e^{-\frac{t}{n}} = \frac{v}{H} e^{-\frac{t}{n}} - \frac{r}{n} e^{-\frac{t}{n}} = \left(\frac{\lambda v^2}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}} - \frac{r}{n}\right) e^{-\frac{t}{n}}.$$

Define $f(x,t) := \frac{\lambda v^2}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}} - \frac{\lambda}{n\lambda'}$, it can be seen that $|f(x,t)| \le Ce^{(\frac{1}{n}-\theta)t}$ for a sufficiently small $\theta > 0$ through Proposition 4.3, Lemma 4.9, Proposition 4.16, and Lemma 4.17.

Hence

$$\frac{\partial \widehat{r}}{\partial t} = \left(f + \frac{\lambda}{n\lambda'} - \frac{r}{n}\right) e^{-\frac{t}{n}} = f e^{-\frac{t}{n}} + \frac{\lambda - r\lambda'}{n\lambda'} e^{-\frac{t}{n}}$$

where

$$\begin{aligned}
\lambda - r\lambda' &= \lambda(r) - \lambda(0) - r\lambda'(r) = \int_0^r \lambda'(s)ds - r\lambda'(r) = \int_0^r \big[\lambda'(s) - \lambda'(r)\big] ds \\
&= \int_0^r \big[\lambda'(s) - \lambda'(r)\big] ds = \int_0^r \int_r^s \lambda''(u) du ds = -\int_0^r \int_0^u \lambda''(u) ds du \\
&= -\int_0^r u\lambda''(u) du.
\end{aligned}$$

So

$$\widehat{r}(x,t) - \widehat{r}(x,0) = \int_0^t \frac{\widehat{r}}{\partial t}(x,s) ds = \int_0^t f(x,s) e^{-\frac{s}{n}} ds - \int_0^t \frac{e^{-\frac{s}{n}}}{n\lambda'} \int_0^r u\lambda''(u) du ds.$$

Observe that

$$\left|\int_0^t f(x,s) e^{-\frac{s}{n}} ds\right| \le \int_0^t |f(x,s)| e^{-\frac{s}{n}} ds \le \int_0^t e^{-\frac{\theta s}{n}} ds = \frac{n}{\theta}(1 - e^{-\frac{\theta t}{n}}) \le \frac{n}{\theta}$$

is bounded. Also the left hand side is bounded by Lemma 5.1. Therefore $\int_0^t \frac{e^{-\frac{s}{n}}}{n\lambda'} \int_0^r u\lambda''(u) du ds$ is bounded and increasing w.r.t t and so the limit exists as $t \to \infty$. Then $\lim_{t\to\infty} \widehat{r}(x,t) = \kappa(x)$ exists for all $x \in \mathbb{S}^n$. It still need to prove that the limit is independent of x.

Differentiate $\widehat{r}$ w.r.t. the operator D of $\mathbb{S}^n$ and get

$$D\widehat{r} = e^{-\frac{t}{n}} Dr = e^{-\frac{t}{n}} \lambda D\varphi.$$

Combine Proposition 4.3 and Lemma 4.9 and see

$$|D\widehat{r}| \leq Ce^{-\theta t} \to 0 \text{ as } t \to \infty$$

uniformly and so $D\kappa(x) = 0$. Hence $\widehat{r}$ converges to a positive constant κ uniformly and κ which is independent of x. ■

Let $\widehat{e}_i := \frac{\partial}{\partial x^i} \widehat{X}_t = e^{-\frac{t}{n}} D_i r \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}$, then $\widehat{g}_{ij} = g(\widehat{e}_i, \widehat{e}_j) = e^{-\frac{2t}{n}} D_i r D_j r + \lambda^2(\widehat{r})\sigma_{ij}$.

Proposition 5.3. *$\widehat{g}_{ij} \to \lambda^2(\kappa)\sigma_{ij}$ as $t \to \infty$.*

Proof. Combine Proposition 4.3 and Lemma 4.9, we have

$$|e^{-\frac{2t}{n}} D_i r D_j r| = |\lambda^2 e^{-\frac{2t}{n}} D_i \varphi D_j \varphi| \leq Ce^{-\theta t} \to 0$$

as $t \to \infty$ for some positive constants C and θ. This completes the proof. ■

Lemma 5.4. *$\widehat{\varphi}_{i,j} \to 0$ as $t \to \infty$.*

Proof.

$$\widehat{\varphi}_{i,j} = \frac{1}{\lambda(\widehat{r})} \widehat{r}_{i,j} - \frac{\lambda'(\widehat{r})}{\lambda^2(\widehat{r})} \widehat{r}_i \widehat{r}_j = \frac{1}{\lambda(\widehat{r})} \frac{\lambda(r)}{e^{\frac{t}{n}}} \frac{r_{i,j}}{\lambda(r)} - \frac{\lambda'(\widehat{r})}{\lambda^2(\widehat{r})} \left[\frac{\lambda(r)}{e^{\frac{t}{n}}}\right]^2 \varphi_i \varphi_j$$

Because of $\varphi_{i,j} \to 0$ and (3.4), $\frac{r_{i,j}}{\lambda(r)} \to 0$ as $t \to \infty$. On the other hand, $\frac{\lambda(r)}{e^{\frac{t}{n}}}$ is bounded by Proposition 4.3. Apply the Lemma 4.9 and get the result. ■

Proposition 5.5. *$\widehat{h}^i_j \to \frac{\lambda'(\kappa)}{\lambda(\kappa)} \delta^i_j$ as $t \to \infty$.*

Proof.

$$\widehat{h}^i_j = \frac{\lambda'(\widehat{r})}{\lambda(\widehat{r})\widehat{v}}\delta^i_j - \frac{1}{\lambda(\widehat{r})\widehat{v}}\widehat{\sigma}^{ik}\widehat{\varphi}_{k,j}$$

Since $\widehat{v}^2 = \frac{|D\widehat{r}|^2_{\mathbb{S}^n}}{\lambda^2(\widehat{r})} + 1 \to 1$ as $t \to \infty$ and $\widehat{\sigma^{i,k}} = \sigma^{ik} - \frac{e^{\frac{-2t}{n}} D^i\varphi D^j\varphi}{\widehat{v}^2}$ is bounded. Using the Lemma above and get the desire result. ■

Now we determine this κ. Apply the L'Hopital rule to the limit $\lim_{s\to\infty} \frac{\lambda(s)}{s} = \lim_{s\to\infty} \frac{\lambda'(s)}{1} = 1$, and see that $\frac{\lambda(r)}{e^{\frac{t}{n}}} = \frac{\lambda(\widehat{r}e^{\frac{t}{n}})}{\widehat{r}e^{\frac{t}{n}}}\widehat{r} \to \kappa$.

By (4.2),

$$\frac{\partial}{\partial t}\sqrt{\det g} = \frac{\sqrt{\det g}}{2} g^{ij}\frac{\partial}{\partial t}g^{ij}\frac{2}{H}h_{ij} = \sqrt{\det g}.$$

So

$$\frac{d}{dt}\text{Area}(M_t) = \frac{d}{dt}\int_{M_t} d\mu_t = \int_{M_t} d\mu_t = \text{Area}(M_t).$$

Then we get Area(M_t)=Area(M_0) $\cdot\, e^t$. Then

$$\text{Area}(M_0) = e^{-t}\text{Area}(M_t) = e^{-t}\int_{\mathbb{S}^n}\lambda^n(r)d\sigma = \kappa^n|\mathbb{S}^n|$$

by taking the limit $t \to \infty$, where $|\mathbb{S}^n|$ is Lebesgue measure of n-sphere in Euclidean space. Therefore we conclude that

$$\kappa = \left(\frac{\text{Area}(M_0)}{|\mathbb{S}^n|}\right)^{\frac{1}{n}}.$$

6. Proof of Theorem 1.4

We present the proof of Theorem 1.4, i.e.

$$\frac{\left(\int_{\Sigma} H d\mu\right)^{\frac{1}{n-1}}}{\left(|\Sigma|\right)^{\frac{1}{n}}} \geq \frac{\left(\int_{\mathbb{S}^n} H d\mu\right)^{\frac{1}{n-1}}}{\left(|\mathbb{S}^n|\right)^{\frac{1}{n}}}$$

Proof. We consider the IMCF of Σ. Let Σ_t denote the solution of the IMCF at time t.

Recall that $\frac{\partial}{\partial t}\mu = d\mu$ and

$$\partial_t H = div(\frac{\nabla H}{H^2}) - \frac{|A|^2}{H} .$$

So

$$\begin{aligned}
\frac{d}{dt}\int_{\Sigma_t} H d\mu &= \int_{\Sigma_t} \partial_t H d\mu + \int_{\Sigma_t} H \partial_t (d\mu) \\
&= \int_{\Sigma_t} div(\frac{\nabla H}{H^2}) - \frac{|A|^2}{H} + H d\mu \\
&= \int_{\Sigma_t} -\frac{|A|^2}{H} + H d\mu \\
&\leq \int_{\Sigma_t} \frac{(n-1)H}{n} d\mu
\end{aligned}$$

here we have used the fact that $H^2 \leq n|A|^2$.

Using $\frac{d}{dt}|\Sigma_t| = |\Sigma_t|$, we obtain

$$\frac{d}{dt}\ln(\int_{\Sigma_t} H d\mu) \leq \frac{n-1}{n} = \frac{n-1}{n}\frac{d}{dt}\ln(|\Sigma_t|)$$

and

$$\frac{d}{dt}\ln\frac{(\int_{\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\Sigma_t|^{\frac{1}{n}}} \leq 0.$$

Thus $\frac{(\int_{\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\Sigma_t|^{\frac{1}{n}}}$ is nonincreasing and

$$\frac{(\int_{\Sigma} H d\mu)^{\frac{1}{n-1}}}{|\Sigma|^{\frac{1}{n}}} \geq \lim_{t\to\infty} \frac{(\int_{\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\Sigma_t|^{\frac{1}{n}}}$$

$$= \lim_{t\to\infty} \frac{(\int_{\exp(\frac{-t}{n})\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\exp(\frac{-t}{n})\Sigma_t|^{\frac{1}{n}}}$$

$$= \frac{(\int_{\mathbb{S}^n} H d\mu)^{\frac{1}{n-1}}}{|\mathbb{S}^n|^{\frac{1}{n}}}$$

Here we have used the fact that the expression $\frac{(\int_{\Sigma} H d\mu)^{\frac{1}{n-1}}}{|\Sigma|^{\frac{1}{n}}}$ is scale invariant and and

$$\frac{(\int_{\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\Sigma_t|^{\frac{1}{n}}} = \frac{(\int_{\exp(\frac{-t}{n})\Sigma_t} H d\mu)^{\frac{1}{n-1}}}{|\exp(\frac{-t}{n})\Sigma_t|^{\frac{1}{n}}}.$$

The equality holds if $|A|^2 = nH^2$ on Σ_t for all t. This implies that Σ_t is umbilical, Σ_t is a sphere for all time and Σ is a sphere. ■

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