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Gross-Pitaevskii方程之行波解

Traveling Waves for the Gross-Pitaevskii Equation

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中文摘要

本文的目標是探討Fabrice Bethuel，Philippe Gravejat，和 Jean-Claude Saut 在Gross-Pitaevskii方程式中關於二，三維度的行波解

$$c\partial_1 u + \Delta u + u(1 - |u|^2) = 0$$

前四章節我們透過最小化能量在動量固定下來探討解之存在性，並提供一些構造這些定理的動機。

最後一章節我們討論Gross-Pitaevskii equation未來的研究方向。

Abstract

In this thesis, Fabrice Bethuel, Philippe Gravejat, and Jean-Claude Saut discussed the existence of travelling wave solutions to the Gross-Pitaevskii equation in $\mathbb{R}^N$, where $N = 2, 3$.

$$c\partial_1 u + \Delta u + u(1 - |u|^2) = 0$$

In the first four sections, we survey the theorems based on minimizing energy under momentum constraint. Also, we give some motivations about how the theorems are constructed.

In the final section, we discuss the future works of Gross-Pitaevskii equation.

目錄 Contents

1 Introduction

1.1 Background

First, take a look at the Gross-Pitaevskii equation

$$i\partial_t \Psi = \Delta\Psi + \Psi(1 - |\Psi|^2) \quad \text{on} \quad x \in \mathbb{R}^N,\, t \in \mathbb{R}. \tag{GP}$$

From physics, we know that it is a nonlinear Schrödinger equation.
In classical quantum physics, the Schrödinger equation has its own Energy and Momentum. So, in this paper, we also want to define the Energy and Momentum for the Gross-Pitaevskii equation.
It is natural to think that the Energy part comes from the variation of the PDE.
But the question is, " What is the momentum ? ".
In order to answer this question, we need to focus on the classical Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi.$$

Now for the Schrödinger equation, we want to find it's Momentum. We could formally compute it by the following strategy.
(a) First, we notice that the Lagrangian of Schrödinger.

$$\mathcal{L}(\nabla_x\Psi, \Psi_t) = -\frac{\hbar^2}{2m}\int_{\mathbb{R}^N}|\nabla_x\Psi|^2 dx + \int_{\mathbb{R}^N}\overline{\Psi}(i\hbar\Psi_t)dx.$$

(b) Second, we find the Legendre transform of $\mathcal{L}$ with respect to $\Psi_t = \Phi$.

$$\mathcal{L}^*(\nabla_x\Psi, \Lambda) = \sup_{\Phi}\{\langle\Lambda, \Phi\rangle - \mathcal{L}(\nabla_x\Psi, \Phi)\},$$

where $\langle\Lambda, \Phi\rangle = Re\int_{\mathbb{R}^N}\overline{\Lambda}\Phi dx$.

(c) Third, we find it's Hamiltonian.

$$H(\nabla_x\Psi, \Lambda) = \mathcal{L}^*(\nabla_x\Psi, \Lambda) = \frac{\hbar^2}{2m}\int_{\mathbb{R}^N}|\nabla_x\Psi|^2 dx$$

$$\mathcal{L}(\nabla_x\Psi, \Psi_t) = -H(\nabla_x\Psi, \Psi_t) - \langle\Psi, -i\hbar\partial_t\Psi\rangle.$$

(d) Replace $\frac{\partial}{\partial t} = -\boldsymbol{c}\cdot\nabla$, where $\boldsymbol{c}$ is the wave speed.

$$\mathcal{L}(\nabla_x\Psi, \Psi_t) = -H(\nabla_x\Psi, \Psi_t) + \boldsymbol{c}\cdot\langle\Psi, p(\Psi)\rangle,$$

where $p(\Psi) = -i\hbar\nabla\Psi$ is the classical momentum in Schrödinger equation.
Since we always hope that the solution is a " *Travelling wave solution* ". That is, we hope that the solution has the form : $\Psi(x,t) = \widetilde{\Psi}(x - ct)$.
Just like wave equation, travelling wave $\Psi(x,t) = \widetilde{\Psi}(x - ct)$ must be a solution of

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2}.$$

But, the traveling wave is NOT always a solution for Schrödinger equation, so we need to add more assumptions.

$$\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = -\boldsymbol{c}\cdot\nabla\widetilde{\Psi}(\boldsymbol{x} - \boldsymbol{c}t) = -\boldsymbol{c}\cdot\nabla\Psi(\boldsymbol{x},t).$$

In other words, we will assume : $\frac{\partial}{\partial t} = -\boldsymbol{c}\cdot\nabla$.
This is exactly the analogue of the classical equation :

$$\mathcal{L}(q,\dot{q}) = -H(p,q) + p\cdot\dot{q},$$

where q means the position and p means the momentum.
Thus, this suggests to us the way to define " *General Momentum Operator* ".
Now for Gross-Pitaevskii equation, we want to find it's Momentum. We could formally compute it by the similar strategy.
(a) First, we recall the Lagrangian of Gross-Pitaevskii.

$$\mathcal{L}(\Psi, \nabla_x\Psi, \Psi_t) = -[\ \frac{1}{2}\int_{\mathbb{R}^N}|\nabla\Psi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N}(1-|\Psi|^2)^2 dx\ ]$$
$$-\frac{1}{2}Re\int_{\mathbb{R}^N} i\overline{\Psi_t}(1-\Psi)dx.$$

(b) Second, we find the Legendre transform of $\mathcal{L}$ with respect to $\Psi_t = \Phi$.

$$\mathcal{L}^*(\Psi, \nabla_x\Psi, \Lambda) = \sup_{\Phi}\{\langle\Lambda,\Phi\rangle - \mathcal{L}(\Psi,\nabla_x\Psi,\Phi)\},$$

where $\langle\Lambda,\Phi\rangle = Re\int_{\mathbb{R}^N}\overline{\Lambda}\Phi dx$.
(c) Third, we find it's Hamiltonian.

$$H(\Psi,\nabla_x\Psi,\Lambda) = \mathcal{L}^*(\Psi,\nabla_x\Psi,\Lambda) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla\Psi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N}(1-|\Psi|^2)^2 dx$$

$$\mathcal{L}(\nabla_x, \Psi_t) = -H(\nabla_x \Psi, \Psi_t) - \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_t \Psi, \Psi - 1\rangle dx,$$

where $\langle i\partial_t \Psi, \Psi - 1\rangle \equiv Re(\overline{i\partial_t \Psi}(\Psi - 1))$.

(d) Replace $\frac{\partial}{\partial t} = -\boldsymbol{c} \cdot \nabla$,where $\boldsymbol{c}$ is the wave speed.

$$\mathcal{L}(\nabla_x \Psi, \Psi_t) = -H(\nabla_x \Psi, \Psi_t) + \boldsymbol{c} \cdot \frac{1}{2}\int_{\mathbb{R}^N} \langle i\nabla \Psi, \Psi - 1\rangle dx.$$

Thus, the momentum operator for Gross-Pitaevskii equation is :

$$\boldsymbol{P}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\nabla \Psi, \Psi - 1\rangle dx.$$

The Gross-Pitaevskii equation appears in several models in various areas of physics :

non-linear optics, fluid mechanics, and Bose-Einstein condensation...(see for instance [5,25,30,31,32,38]).On a formal level, the Gross-Pitaevskii equation is Hamiltonian. Also, the conserved Hamiltonian is a Ginzburg-Landau energy, namely (in previous observations)

$$E(\Psi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \Psi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |\Psi|^2)^2 dx.$$

1.2 Statement of the main results

We investigate the existence of travelling waves to the Gross-Pitaevskii equation

$$i\partial_t \Psi = \Delta \Psi + \Psi(1 - |\Psi|^2) \quad \text{on} \quad x \in \mathbb{R}^N,\, t \in \mathbb{R}, \tag{GP}$$

where $\Psi(x,t) : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{C}$ is an unknown function, x is the space variable and t is the time variable. $N = 2, 3$, and Δ is the standard Laplace operator. We are interested in the solution of the form $\Psi(x,t) = u(x_1 - ct, x_2, ..., x_N)$, called the travelling wave solution. Here u is called the profile and c is called the wave speed. The equation of profile u is given by

$$ic\partial_1 u + \Delta u + u(1 - |u|^2) = 0. \tag{TWc}$$

Here, the parameter $c \in \mathbb{R}$ corresponding to the speed of the travelling waves, may be restricted to the case $c \geq 0$, since we may consider the complex conjugation.

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2 dx = \int_{\mathbb{R}^N} e(u). \tag{Energy}$$

Since our travelling wave speed is only considered in the first dimension, so the momentum $\boldsymbol{P}$ in Gross-Pitaevskii equation in the first dimension will be denoted as p, which is hence a scalar

$$p(u) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_1 u, u-1\rangle dx, \qquad \text{(Momentum)}$$

where the notation $\langle\ ,\ \rangle$ stands for the canonical scalar product of the complex plane $\mathbb{C}$ identified to $\mathbb{R}^2$, that is

$$\langle z_1, z_2\rangle \equiv Re(z_1)Re(z_2) + Im(z_1)Im(z_2) = Re(\bar{z}_1 z_2).$$

In order to make the previous definition well-defined, we introduce the space

$$V(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{C} | \nabla u \in L^2(\mathbb{R}^N), Re(u) \in L^2(\mathbb{R}^N), Im(u) \in L^4(\mathbb{R}^N), \nabla Re(u) \in L^{\frac{4}{3}}(\mathbb{R}^N)\}$$

$$W(\mathbb{R}^N) = 1 + V(\mathbb{R}^N).$$

We check : $\langle i\partial_1 v, v-1\rangle$ is integrable for any $v \in W(\mathbb{R}^N)$, so $p(v)$ is well-defined.
First observe that :

$$\langle i\partial_1 v, v-1\rangle = \partial_1(Re(v))Im(v) - \partial_1(Im(v))(Re(v)-1).$$

Let $u = v - 1 \in V(\mathbb{R}^N)$ and apply Holder's inequality

$$\begin{aligned}\int_{\mathbb{R}^N} |\langle i\partial_1 v, v-1\rangle| dx &\leq \int_{\mathbb{R}^N} |\partial_1 Re(v)||Im(v)| + |\partial_1 Im(v)||Re(v)-1| \\ &\leq \int_{\mathbb{R}^N} |\nabla Re(v)||Im(v)| + |\nabla v||Re(v)-1| \\ &= \int_{\mathbb{R}^N} |\nabla Re(u)||Im(u)| + |\nabla u||Re(u)|.\end{aligned}$$

The following two main theorem are from Béthuel [4].

Theorem 1. *For $N = 2$, let $\mathfrak{p} > 0$. There exists a non-constant finite energy solution $u_{\mathfrak{p}} \in W(\mathbb{R}^N)$ to equation (TWc) , with $c = c(u_{\mathfrak{p}})$ s.t.*

$$p(u_{\mathfrak{p}}) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_1 u_{\mathfrak{p}}, u_{\mathfrak{p}} - 1\rangle dx = \mathfrak{p},$$

and such $u_{\mathfrak{p}}$ is the solution of the minimization problem

$$E(u_{\mathfrak{p}}) = E_{min}(\mathfrak{p}) = \inf\{E(u) | u \in W(\mathbb{R}^N), p(u) = \mathfrak{p}\}.$$

Theorem 2. *For $N = 3$, there exists some constant $\mathfrak{p}_0 > 0$ such that :*
For $\mathfrak{p} \geq \mathfrak{p}_0$, there exists a non-constant finite energy solution $u_\mathfrak{p} \in W(\mathbb{R}^N)$ to equation (TWc), with $c = c(u_\mathfrak{p})$ s.t.

$$p(u_\mathfrak{p}) = \mathfrak{p}, \qquad E(u_{\mathfrak{p}_o}) = E_{min}(\mathfrak{p}_o) = \sqrt{2}\mathfrak{p}_o,$$

and, for $\mathfrak{p} > \mathfrak{p}_0$, we have

$$E(u_\mathfrak{p}) = E_{min}(\mathfrak{p}) < \sqrt{2}\mathfrak{p}.$$

Moreover, for $0 < \mathfrak{p} < \mathfrak{p}_0$,

$$E_{min}(\mathfrak{p}) = \inf\{E(u) | u \in W(\mathbb{R}^N), p(u) = \mathfrak{p}\} = \sqrt{2}\mathfrak{p},$$

and the infimum is not achieved in $W(\mathbb{R}^3)$.

1.3 Starting point of the proofs

The starting point of the proofs was due to the analysis of the curve $\mathfrak{p} \mapsto E_{\min}(\mathfrak{p})$. First, we do some linearization of $E_{min}(\mathfrak{p})$ by using Taylor formula. For $\mathfrak{p} > \hat{\mathfrak{p}} > 0$,

$$E_{min}(\mathfrak{p}) \simeq E_{min}(\hat{\mathfrak{p}}) + \frac{d^+}{d\mathfrak{p}} E_{min}(\hat{\mathfrak{p}})(\mathfrak{p} - \hat{\mathfrak{p}}).$$

Assume E_{min} is achieved by some map $u_{\hat{\mathfrak{p}}}$, by Euler-Lagrange equation and scaling on ψ_1, we have

$$c = cDp(u_{\hat{\mathfrak{p}}})(\psi_1) = DE(u_{\hat{\mathfrak{p}}})(\psi_1).$$

Consider the curve $\gamma : \mathbb{R} \mapsto W(\mathbb{R}^N)$ defined by $\gamma(t) = u_{\hat{\mathfrak{p}}} + t\psi_1$, since the functions E and p are smooth on $W(\mathbb{R}^N)$, using Taylor formula, we have

$$p(\gamma(t)) = \hat{\mathfrak{p}} + \mathfrak{s}, \text{ where } \mathfrak{s} = t + p(\psi_1)t^2,$$

$$E(\gamma(t)) = E(u_{\hat{\mathfrak{p}}}) + DE(u_{\hat{\mathfrak{p}}})(t\psi_1) + \underset{t \longrightarrow 0}{\mathcal{O}}(t^2) = E_{\min}(\hat{\mathfrak{p}}) + ct + \underset{t \longrightarrow 0}{\mathcal{O}}(t^2).$$

Since $p(\gamma(t)) = \hat{\mathfrak{p}} + \mathfrak{s} \implies E_{\min}(\hat{\mathfrak{p}} + \mathfrak{s}) \leq E(\gamma(t))$, so we obtain

$$E_{\min}(\hat{\mathfrak{p}} + \mathfrak{s}) - E_{\min}(\hat{\mathfrak{p}}) \leq E(\gamma(t)) - E_{\min}(\hat{\mathfrak{p}}) \leq c\mathfrak{s} + \underset{\mathfrak{s} \longrightarrow 0}{\mathcal{O}}(\mathfrak{s}^2).$$

Taking $\mathfrak{s} \longrightarrow 0^+$,

$$\frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\hat{\mathfrak{p}})\big) \leq c,$$

also by Gravejat's paper [19], there is no travelling wave solution existing for $c > \sqrt{2}$, so we may hope for $c \leq \sqrt{2}$ (This is exactly the idea of the proof in Lemma 1). Taking $\hat{\mathfrak{p}} \longrightarrow 0^+$, we have the following approximate inequality

$$E_{\min}(\mathfrak{p}) \lesssim \sqrt{2}\mathfrak{p}. \tag{1.1}$$

This inequality corresponds in some sense to a linearization of the equation.
We performed some observation to obtain some feeling for its proof, first, considered a map $v \in \{1\} + C_c^\infty(\mathbb{R}^N)$ such that

$$\delta = \inf_{x \in \mathbb{R}^N} |v(x)| \geq \frac{1}{2},$$

so that we may write $v = \rho \exp i\varphi$. To obtain (1.1), we need to construct v so that $E(v) \simeq \sqrt{2}|p(v)|$, thus $E_{min}(\mathfrak{p}) \lesssim \sqrt{2}\mathfrak{p}$.
From the formula (2.2) for another type of momentum and $\delta \leq \rho$, we have

$$|p(v)| = \left|\frac{1}{2}\int_{\mathbb{R}^N}(1-\rho^2)\partial_1\varphi\right| \leq \frac{1}{2\delta}\int_{\mathbb{R}^N}\left|1-\rho^2\right|\left|\rho\partial_1\varphi\right|.$$

From the inequality $ab \leq \frac{1}{2}(a^2+b^2)$, we set $a = \sqrt[4]{2}|\rho\partial_1\varphi|$, $b = \frac{|1-\rho^2|}{\sqrt[4]{2}}$, and also viewed the formula (2.1) for another type of energy

$$|p(v)| \leq \frac{1}{\sqrt{2}\delta}\left(\frac{1}{2}\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2 + \frac{1}{4}\int_{\mathbb{R}^N}\left(1-\rho^2\right)^2\right) \leq \frac{1}{\sqrt{2}\delta}E(v)$$

i.e. $\sqrt{2}\delta|p(v)| \leq E(v)$.
In order to obtain a map, such that $E(v) \simeq \sqrt{2}|p(v)|$, we will need to make δ close to 1, and the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ close to an equality, that is $a \simeq b$ or in other words,

$$\partial_1\varphi \simeq \frac{1-\rho^2}{\sqrt{2}}.$$

This idea introduced the following (see Lemma 3.4).

Let $\mathfrak{s} > 0$, there exists $(\gamma_n)_{n\in\mathbb{N}}$ in $\{1\} + C_c^\infty(\mathbb{R}^N)$ s.t. $p(\gamma_n) = \mathfrak{s}$ and $E(\gamma_n) \longrightarrow \sqrt{2}\mathfrak{s}$.

From the density of $V(\mathbb{R}^N)$, we may suppose that E_{min} has the following form

$$E_{\min}(\mathfrak{p}) = \inf\{E(1+v)|v \in C_c^\infty(\mathbb{R}^N), p(1+v) = \mathfrak{p}\},$$

this lead us to the Lemma 3.2. Combining all observations above, we can proof Theorem 3.

Theorem 3. *Let* $N = 2, 3$. *For any* $\mathfrak{p}, \mathfrak{q} \geq 0$ *we have :*

$$|E_{min}(\mathfrak{p}) - E_{min}(\mathfrak{q})| \leq \sqrt{2}|\mathfrak{p} - \mathfrak{q}|,$$

that is, the real-valued function $\mathfrak{p} \longmapsto E_{min}(\mathfrak{p})$ *is Lipschitz and thus is concave, increasing on* $\mathbb{R}_+$ *.*
Set $\Xi(\mathfrak{p}) = \sqrt{2}\mathfrak{p} - E_{min}(\mathfrak{p})$, *then the function* $\mathfrak{p} \longmapsto \Xi(\mathfrak{p})$ *is non-negative, convex, increasing on* $\mathbb{R}_+$, *tending to* $+\infty$ *as* $\mathfrak{p} \to +\infty$.
In particular, there exists $\mathfrak{p}_0 \geq 0$ *such that* $\Xi(\mathfrak{p}) = 0$, *if* $\mathfrak{p} \leq \mathfrak{p}_0$, *and* $\Xi(\mathfrak{p}) > 0$, *as otherwise .*
Let us define $\Sigma(u) = \sqrt{2}p(u) - E(u)$ *for* $u \in W(\mathbb{R}^N)$, *then* $\Xi(\mathfrak{p}) = \sup\{\Sigma(u)|u \in W(\mathbb{R}^N), p(u) = \mathfrak{p}\}$.

An important consequence of the concavity to the function $E_{\min}(\mathfrak{p})$, is the following inequality.

Corollary 1. *The function* $E_{\min}$ *is subadditive,*
that is, for any non-negative numbers $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$, *we have the inequality,*

$$\sum_{i=1}^{\ell} E_{\min}(\mathfrak{p}_i) \geq E_{\min}\Big(\sum_{i=1}^{\ell} \mathfrak{p}_i\Big).$$

Moreover, if $\ell \geq 2$ *and the previous equation is an equality, then the function* $E_{\min}$, *will be linear on* $(0, \mathfrak{p})$, *where* $\mathfrak{p} \equiv \sum_{i=1}^{\ell} \mathfrak{p}_i$.

Proof. Notice $E_{\min}(0) = 0$ and $E_{\min}$ is concave, so its graph lies above the line segment joining $(0, 0)$ and $(\mathfrak{p}, E_{\min}(\mathfrak{p}))$.
In particular, for any $0 \leq \mathfrak{q} \leq \mathfrak{p}$, we have

$$\frac{E_{min}(\mathfrak{q}) - E_{min}(0)}{\mathfrak{q} - 0} \geq \frac{E_{min}(\mathfrak{p}) - E_{min}(0)}{\mathfrak{p} - 0} \implies E_{\min}(\mathfrak{q}) \geq \mathfrak{q}\frac{E_{\min}(\mathfrak{p})}{\mathfrak{p}}.$$

For any $0 \leq \mathfrak{p}_i \leq \mathfrak{p}$,

$$E_{\min}(\mathfrak{p}_i) \geq \mathfrak{p}_i \frac{E_{\min}(\mathfrak{p})}{\mathfrak{p}} \implies \sum_{i=1}^{\ell} E_{\min}(\mathfrak{p}_i) \geq \sum_{i=1}^{\ell} \mathfrak{p}_i \frac{E_{\min}(\mathfrak{p})}{\mathfrak{p}} = E_{min}(\mathfrak{p})$$

Now, if it is an equality, then $E_{\min}(\mathfrak{p}_i) = \mathfrak{p}_i \frac{E_{\min}(\mathfrak{p})}{\mathfrak{p}}$ and the graph needs to be linear, that is, the function $E_{\min}$ will be linear on $(0, \mathfrak{p})$. □

Since the function $E_{\min}$ is Lipschitz, non-decreasing and concave, its left and right derivatives exist for any $\mathfrak{p} \geq 0$ and will be equal on $\mathbb{R}_+$, except a countable subset $\mathcal{Q}$ (Lipschitz $\implies$ differentiable almost everywhere), are non-negative and non-increasing and will satisfy the inequality,

$$0 \leq \frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq \frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq \sqrt{2},$$

where we let

$$\frac{d^\pm}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \equiv \lim_{\Delta\mathfrak{p} \longrightarrow 0^+} \frac{E_{\min}(\mathfrak{p} \pm \Delta\mathfrak{p}) - E_{\min}(\mathfrak{p})}{\pm\Delta\mathfrak{p}}.$$

Lemma 1 (Control the speed $c(u_{\mathfrak{p}})$)**.** *Let $\mathfrak{p} > 0$ and assume that $E_{\min}(\mathfrak{p})$ is achieved by a solution $u_{\mathfrak{p}}$ of (TWc) of speed $c(u_{\mathfrak{p}})$. Then we have,*

$$\frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq c(u_{\mathfrak{p}}) \leq \frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big).$$

This is the main control for speed c, the derivatives are related to the speed $c(u_{\mathfrak{p}})$. Also, using the Lipschitz constant we have a bound for speed $c(u_{\mathfrak{p}})$,

$$0 \leq c(u_{\mathfrak{p}}) \leq \sqrt{2}.$$

Lemma 2 (The property of affine energy $E_{min}(\mathfrak{p})$)**.** *Let $0 \leq \mathfrak{p}_1 < \mathfrak{p}_2$ and assume the function $E_{\min}$ is affine on $(\mathfrak{p}_1, \mathfrak{p}_2)$. Then, for any $\mathfrak{p}_1 < \mathfrak{p} < \mathfrak{p}_2$, the infimum $E_{\min}(\mathfrak{p})$ will not be achieved in $W(\mathbb{R}^N)$.*

Lemma 3. *Let $v \in W(\mathbb{R}^N)$ and assume $p(v) > 0$. Then we have,*

$$\inf_{x \in \mathbb{R}^N} |v(x)| \leq \max\Big\{\frac{1}{2}, 1 - \frac{\Sigma(v)}{\sqrt{2}p(v)}\Big\}.$$

Proof. Now define δ as previous,

$$\delta \equiv \inf_{x \in \mathbb{R}^N} |v(x)|.$$

If $\delta \leq \frac{1}{2}$, the result holds. Otherwise, $|v| \geq \delta > \frac{1}{2}$, v has a lifting, i.e. we may write $v = \rho \exp i\varphi$, and recall in previous, we have

$$|p(v)| \leq \frac{1}{\sqrt{2}\delta}\left(\frac{1}{2}\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 + \frac{1}{4}\int_{\mathbb{R}^N}\left(1-\rho^2\right)^2\right) \leq \frac{1}{\sqrt{2}\delta}E(v)$$

$$\sqrt{2}\delta p(v) \leq \sqrt{2}\delta|p(v)| \leq E(v) = \sqrt{2}p(v) - \Sigma(v),$$

and hence,

$$1 - \delta \geq \frac{\Sigma(v)}{\sqrt{2}p(v)} \implies \delta \leq 1 - \frac{\Sigma(v)}{\sqrt{2}p(v)},$$

that is,

$$\inf_{x \in \mathbb{R}^N} |v(x)| = \delta \leq \max\left\{\frac{1}{2}, 1 - \frac{\Sigma(v)}{\sqrt{2}p(v)}\right\}.$$

□

Lemma 3 is the main tool to make the finite energy solutions of (TWc) into nontrivial solutions .

Previously, we haven't talked about the existence of the solution of (TWc). In order to overcome the difficulties of finding minimizing sequences in whole space $\mathbb{R}^N$, there are several ways to proceed. In this thesis, we consider the corresponding minimization problem on expanding tori. This choice has several advantages.
First, the torus is compact, so that, the existence of minimizers, have no difficulty (see Proposition 1).
Second, it has no boundary, so that, the elliptic theory, is essentially the local one and concentration near the boundary is avoided. The torus also captures some of the translation invariance for the problem on $\mathbb{R}^N$.
Finally, the Pohozaev's identities will give bounds for the Lagrange multipliers, which provide a uniform control on the ellipticity of (TWc). Our strategy, to obtain compactness for the sequence of approximate minimizers, is then to develop the elliptic theory for the equation on tori, derive several estimates which do not rely on

the size of the torus and then to pass to the limit, when the size of the torus tends to infinity (see Proposition 1,2).

More precisely, we introduce the flat torus, for $N = 2$ and $N = 3$, defined by

$$\mathbb{T}_n^N \simeq \Omega_n^N \equiv [-\pi n, \pi n]^N.$$

Now, introduce the Energy and Momentum on flat torus : $\mathbb{T}_n^N$, we define the energy E_n and p_n on $X_n^N = H^1(\mathbb{T}_n^N, \mathbb{C})$ as follow

$$E_n(u) = \frac{1}{2}\int_{\mathbb{T}_n^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{T}_n^N} (1-|u|^2)^2 dx,$$

as well,

$$p_n(u) = \frac{1}{2}\int_{\mathbb{T}_n^N} \langle i\partial_1 u, u\rangle dx,$$

which defines a quadratic functional on X_n^N, and the discrepancy term

$$\Sigma_n(v) = \sqrt{2}p_n(v) - E_n(v).$$

We introduce the set $\Gamma_n^N(\mathfrak{p})$, defined in dimension three by $\Gamma_n^3(\mathfrak{p}) \equiv \{u \in X_n^3, p_n(u) = \mathfrak{p}\}$, but in dimension two, its definition is a little different and is given by

$$\Gamma_n^2(\mathfrak{p}) \equiv \{u \in X_n^2, p_n(u) = \mathfrak{p}\} \cap \mathcal{S}_n^0.$$

The set $\mathcal{S}_n^0$, corresponds to a topological sector of the energy E_n, following the approach of Almeida [1]. Let us just mention that we introduce the set $\mathcal{S}_n^0$, to have appropriate lifting properties far from the possibly vorticity set. For more detail, see Béthuel [4, Section 4].

Similarly, we consider the minimization problem on torus,

$$E_{min}^n(\mathfrak{p}) = \inf\{E_n(u) | u \in \Gamma_n^N(\mathfrak{p})\}. \qquad (\mathcal{P}_n^N(\mathfrak{p}))$$

Proposition 1. *Assume $N = 2, 3$, and $n \geq \widetilde{n}(\mathfrak{p})$, where $\widetilde{n}(\mathfrak{p})$ is some integer depending on $E_{min}(\mathfrak{p})$ then there exists a minimizer $u_{\mathfrak{p}}^n \in \Gamma_n^N(\mathfrak{p})$ for $E_{min}^n(\mathfrak{p})$ and some constant $c_{\mathfrak{p}}^n \in \mathbb{R}$, such that $u_{\mathfrak{p}}^n$ satisfies ($TWc_{\mathfrak{p}}^n$) i.e.*

$$ic_{\mathfrak{p}}^n \partial_1 u_{\mathfrak{p}}^n + \Delta u_{\mathfrak{p}}^n + u_{\mathfrak{p}}^n(1 - |u_{\mathfrak{p}}^n|^2) = 0 \text{ on } \mathbb{T}_n^N.$$

In particular, $u^n_{\mathfrak{p}}$ *is smooth.*

Moreover, if $\Xi(\mathfrak{p}) > 0$*, then there exists a constant* $K(\mathfrak{p})$ *and an integer* $n(\mathfrak{p})$ *s.t.*

$$|c^n_{\mathfrak{p}}| \leq K(\mathfrak{p}), \text{ for any } n \geq n(\mathfrak{p}).$$

In particular, for any $k \in \mathbb{N}$*, there exists some constant* $K_k(\mathfrak{p})$*, such that*

$$\|u^n_{\mathfrak{p}}\|_{C^k(\mathbb{T}^N_n)} \leq K_k(\mathfrak{p}).$$

Proposition 2. *For* $N = 2, 3$*,* $\mathfrak{p} > 0$*, and assume* $\Xi(\mathfrak{p}) > 0$*, then there exists a non-trivial finite energy solution* $u_{\mathfrak{p}}$ *to (TWc), such that ,by passing to a subsequence, we have*

$$u^n_{\mathfrak{p}} \longrightarrow u_{\mathfrak{p}} \text{ in } C^k(K) \text{ as } n \to +\infty,$$

for any $k \in \mathbb{N}$ *and compact* K *in* $\mathbb{R}^N$*. Moreover, we have* $E(u_{\mathfrak{p}}) \leq E_{min}(\mathfrak{p})$ *and*

$$|u_{\mathfrak{p}}(0)| \leq \sup\{\frac{1}{2}, 1 - \frac{\Xi(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\} < 1.$$

Proposition 3. *For* $N = 2, 3$*,* $\mathfrak{p} > 0$ *and assume* $\Xi(\mathfrak{p}) > 0$*. Let* $u_{\mathfrak{p}}$ *and* $u^n_{\mathfrak{p}}$ *be in proposition 2. Then there exists* ℓ *finite energy solutions* $u_1 = u_{\mathfrak{p}}, u_2, u_3, ..., u_\ell$ *to (TWc), such that*

$$E_{min}(\mathfrak{p}) = \sum_{i=1}^{\ell} E(u_i) \qquad \mathfrak{p} = \sum_{i=1}^{\ell} p(u_i).$$

Also, u_i *are minimizers of* $E_{min}(\mathfrak{p}_i)$*, where* $\mathfrak{p}_i = p(u_i)$ *and* $0 < c(u_{\mathfrak{p}}) < \sqrt{2}$*. Moreover,*

$$\limsup_{n \longrightarrow \infty} E_n(u^n_{\mathfrak{p}}) = \limsup_{n \longrightarrow \infty} E^n_{min}(\mathfrak{p}) = E_{min}(\mathfrak{p}).$$

Theorem 4. *Assume* $N = 2, 3$*, if* $\mathfrak{p} > \mathfrak{p}_0$*, where* $\mathfrak{p}_0 \geq 0$ *is defined in Theorem 3. Then* $E_{min}(\mathfrak{p})$ *is achieved by the map,* $u_{\mathfrak{p}} \in W(\mathbb{R}^N)$*, constructed in Proposition 2.*

2 Preliminaries

2.1 Finite energy solutions for (TWc)

Lemma 2.1. *Let $n \in \mathbb{N}$ and let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. There exist some constants $K(N)$ and $K(c,k,N)$ s.t.*

$$\|1 - |v|\|_{L^\infty((\mathbb{R}^N)} \leq \max\{1, \frac{c}{2}\}$$

$$\|\nabla v\|_{L^\infty(\mathbb{R}^N)} \leq K(N)(1 + \frac{c^2}{4})^{\frac{3}{2}},$$

and more generally,

$$\|v\|_{C^k(\mathbb{R}^N)} \leq K(c,k,N), \forall k \in \mathbb{N}.$$

Proof. In paper [3,13,20,40], a finite energy solution v to (TWc) is a smooth bounded function on $R^{\mathbb{N}}$, such that,

$$|v| \longrightarrow 1, \text{ as } |x| \longrightarrow \infty,$$

thus, $\|v\|_{L^\infty(\mathbb{R}^N)} \geq 1$, (otherwise, for any $x \in \mathbb{R}^N$, $|v(x)| \leq \|v\|_{L^\infty(\mathbb{R}^N)} < 1$).
We compute for $\Delta|v|^2$,

$$\begin{aligned}\Delta|v|^2 &= 2\langle v, \Delta v\rangle + 2|\nabla v|^2 \\ &= 2|\nabla v|^2 - 2c\langle i\partial_1 v, \Delta v\rangle - 2|v|^2(1 - |v|^2) \\ &\geq 2|\nabla v|^2 - 2|\partial_1 v|^2 - \frac{c^2}{2}|v|^2 - 2|v|^2(1 - |v|^2).\end{aligned}$$

Since $|2c\langle i\partial_1 v, v\rangle| \leq 2c|\partial_1 v||v| \leq \frac{(2|\partial_1 v|)^2}{2} + \frac{(c|v|^2)}{2}$, by Cauchy and Young's inequality, we have,

$$\Delta|v|^2 + 2|v|^2(1 + \frac{c^2}{4} - |v|^2) \geq 2(|\nabla v|^2 - |\partial_1 v|^2) \geq 0.$$

Assume $\|v\|_{L^\infty(\mathbb{R}^N)} > 1$, let $u = |v|^2$ and $\|u\|_{L^\infty(\mathbb{R}^N)} > 1$.
Using maximum principle argument :
For $\epsilon = \frac{\|u\|_{L^\infty(\mathbb{R}^N)} - 1}{2} > 0$, there exists $M > 0$, such that, $u(x) < 1 + \epsilon$ for all $|x| > M$.
Thus,

$$\sup_{|x|>M} |u| \leq 1 + \epsilon < \|u\|_{L^\infty(\mathbb{R}^N)} \implies \|u\|_{L^\infty(\mathbb{R}^N)} = \sup_{|x|\leq M} |u|.$$

Say $\sup_{|x|\leq M}|u| = |u(x_0)| = u(x_0)$ and we claim that :

$$u = |v|^2 \leq (1+\frac{c^2}{4}).$$

If not, $\exists z, u(z) > 1+\frac{c^2}{4} \implies u(x_0) = \|u\|_{L^\infty(\mathbb{R}^N)} \geq |u(z)| > 1+\frac{c^2}{4}$,
by

$$\Delta u + 2u(1+\frac{c^2}{4} - u) \geq 0 \implies \Delta u(x_0) > 0,$$

contradicts to the maximum.
(The idea comes from : if $\exists z, u(z) > 1+\frac{c^2}{4} \implies \|u\|_{L^\infty(\mathbb{R}^N)} > 1+\frac{c^2}{4}$ and $|u| \longrightarrow 1$, as $|x| \longrightarrow \infty$, so $|u|$ must have max in some ball $\bar{B}_M(0)$.)
Assume $\|v\|_{L^\infty(\mathbb{R}^N)} = 1$, the above inequality also holds, so it holds in any case.
In particular,

$$\|1-|v|\|_{L^\infty(\mathbb{R}^N)} \leq \max\{1, (\sqrt{1+\frac{c^2}{4}} - 1)\} \leq \max\{1, \frac{c}{2}\}.$$

Now consider, $w(x) = v(x)\exp{(i\frac{c}{2}x_1)}$ by (TWc) w satisfies,

$$\Delta w + w(1+\frac{c^2}{4} - |w|^2) = 0.$$

Combine with previous

$$|\Delta w| \leq |w||(1+\frac{c^2}{4}) - |w|^2| \leq |w|((1+\frac{c^2}{4}) + |w|^2) \leq 2(1+\frac{c^2}{4})^{\frac{3}{2}},$$

$$\|\Delta w\|_{L^\infty(B(x_0,1))} \leq 2(1+\frac{c^2}{4})^{\frac{3}{2}}.$$

By standard elliptic theory in Gilbarg Trudinger [29], there exists constant $K(N)$, such that

$$|\nabla w(x_0)| \leq K(N)(\|\Delta w\|_{L^\infty(B(x_0,1))} + \|w\|_{L^\infty(B(x_0,1))}).$$

Moreover,

$$|\nabla w(x_0)| \leq 2K(N)(1+\frac{c^2}{4})^{\frac{3}{2}}.$$

Also, by definition of w,

$$|\nabla v(x_0)| \leq |\nabla w(x_0)| + \frac{c}{2}|v(x_0)| \leq (2K(N)+1)(1+\frac{c^2}{4})^{\frac{3}{2}}.$$

Since the estimate holds for any $x_0 \in \mathbb{R}^N$, so we have the following;

$$\|\nabla v\|_{L^\infty(\mathbb{R}^N)} \le K(N)(1+\frac{c^2}{4})^{\frac{3}{2}}$$

$$\|\nabla w\|_{L^\infty(\mathbb{R}^N)} \le K(N)(1+\frac{c^2}{4})^{\frac{3}{2}}.$$

Finally, by standard estimates for elliptic equation in Gilbarg Trudinger [29] with bootstrap argument, we obtain,

$$\|v\|_{C^k(\mathbb{R}^N)} \le K(c,k,N) \text{ for any } k \in \mathbb{N}.$$

□

Lemma 2.2. *Let $r > 0$ and let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. There exist some constants $K(N)$ s.t. for any $x_0 \in \mathbb{R}^N$,*

$$\|1-|v|\|_{L^\infty(B(x_0,\frac{r}{2}))} \le \max\{K(N)(1+\frac{c^2}{4})^2 E(v,B(x_0,r))^{\frac{1}{N+2}}, \frac{K(N)}{r^{N/2}}E(v,B(x_0,r))^{\frac{1}{2}}\},$$

where $E(v,B(x_0,r)) = \int_{B(x_0,r)} e(v)$.

Proof. Let $\eta = 1-|v|^2$, by Lemma 2.1, the function η is smooth on $\mathbb{R}^N$ and satisfies

$$\|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} \le 2\|v\nabla v\|_{L^\infty(\mathbb{R}^N)} \le K(N)(1+\frac{c^2}{2})^2.$$

Let $\bar{x} \in \bar{B}(x_0,\frac{r}{2})$, such that the sup is attached,

$$|\eta(\bar{x})| = \sup_{y\in \bar{B}(x_0,\frac{r}{2})} |\eta(y)|,$$

we have,

$$|\eta(\bar{x})| - |\eta(y)| \le |\eta(y)-\eta(\bar{x})| \le |\nabla\eta(\zeta)||y-\bar{x}| \le K(N)(1+\frac{c^2}{2})^2|y-\bar{x}|.$$

Let $\mu = \frac{|\eta(\bar{x})|}{2K(N)(1+\frac{c^2}{2})^2}$.
For any $y \in B(\bar{x},\mu)$, $|y-\bar{x}| < \mu \implies |\eta(\bar{x})| - |\eta(y)| \le K(N)(1+\frac{c^2}{2})^2\mu \le \frac{|\eta(\bar{x})|}{2}$, that is,

$$|\eta(y)| \ge \frac{|\eta(\bar{x})|}{2}$$

$$E(v,B(x_0,r)) = \frac{1}{2}\int_{B(x_0,r)} |\nabla u|^2 dx + \frac{1}{4}\int_{B(x_0,r)} (1-|u|^2)^2 dx.$$

Let $l = \min(\mu, \frac{r}{2})$,

$$\frac{1}{4}\int_{B(x_0,r)}(1-|u|^2)^2dx \geq \frac{1}{4}\int_{B(\bar{x},l)}(\eta(y))^2dy \geq \frac{1}{4}\int_{B(\bar{x},l)}(\frac{|\eta(\bar{x})|}{2})^2dy \geq \frac{1}{16}\eta(\bar{x})^2|B(\bar{x},l)|,$$

the first inequality follows on by $B(\bar{x},l) \subset B(x_0,r)$, also see Figure 1.

By definition of $l = \min(\mu, \frac{r}{2})$ and the measure of ball,

$$\frac{1}{16}\eta(\bar{x})^2|B(\bar{x},l)| = \min\{\frac{|\eta(\bar{x})|^{N+2}|B(x_0,1)|}{2^{N+4}K(N)^N(1+\frac{c^2}{4})^{2N}}, \frac{|\eta(\bar{x})|^2r^N|B(x_0,1)|}{2^{N+4}}\},$$

also, notice that, $(1-|v|)(1+|v|) = \eta \implies |1-|v|| = \frac{|\eta|}{|1+|v||} \leq |\eta|$,

we obtain,

$$\|1-|v|\|_{L^\infty(B(x_0,\frac{r}{2}))} \leq \|\eta\|_{L^\infty(B(x_0,\frac{r}{2}))} = |\eta(\bar{x})|.$$

Finally, we have the estimate,

$$\|1-|v|\|_{L^\infty(B(x_0,\frac{r}{2}))} \leq \max\{K(N)(1+\frac{c^2}{4})^2E(v,B(x_0,r))^{\frac{1}{N+2}}, \frac{K(N)}{r^{N/2}}E(v,B(x_0,r))^{\frac{1}{2}}\}.$$

□

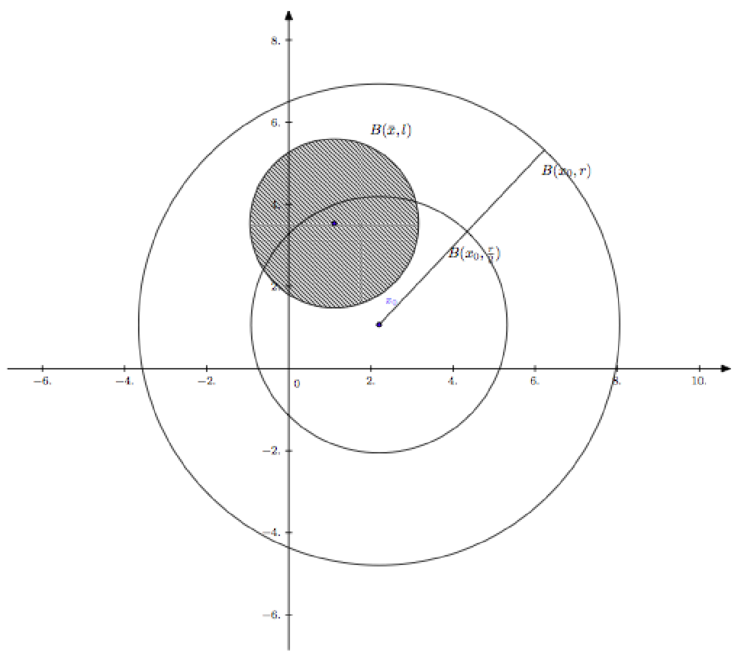

Figure 1: Energy estimate

Now, we introduce a new function space,

$$\Lambda(\mathbb{R}^N) = \{v \in C^0(\mathbb{R}^N) | E(v) < \infty \text{ and } \exists R(v) > 0 \text{ s.t. } |v(x)| \geq \frac{1}{2}, \forall |x| \geq R(v)\}.$$

Corollary 2.3. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. Then v belongs to space $\Lambda(\mathbb{R}^N)$.*

Proof. $\int_{\mathbb{R}^N} e(v) < \infty \implies \lim_{R \longrightarrow \infty} \int_{\mathbb{R}^N \setminus B(0,R-1)} e(v) = 0$ by Lebesgue's Theorem.

For $\frac{1}{(2K(N)(1+\frac{c^2}{4})^2)^{N+2}} > 0$, there exists $R = R(v) > 1$, such that

$$\int_{\mathbb{R}^N \setminus B(0,R-1)} e(v) \leq \frac{1}{(2K(N)(1+\frac{c^2}{4})^2)^{N+2}},$$

where $K(N)$ is the constant in Lemma 2.2.

For any $|x_0| \geq R$, also see Figure 2

$$E(v, B(x_0,1)) = \int_{B(x_0,1)} e(v) \leq \int_{\mathbb{R}^N \setminus B(0,R-1)} e(v)$$

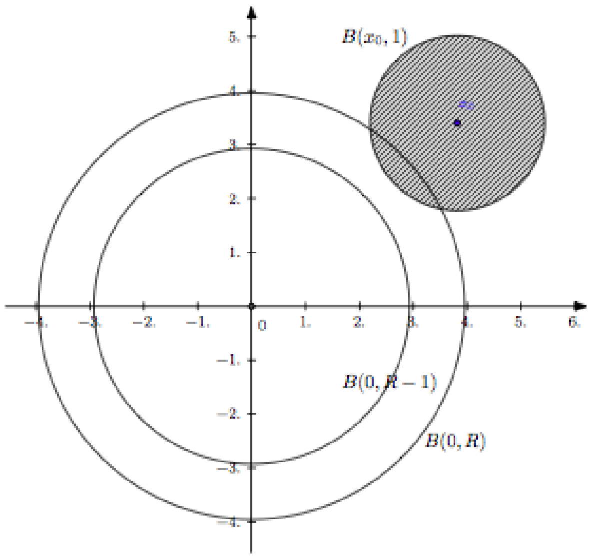

Figure 2: Energy estimate

$$E(v, B(x_0,1)) \leq \frac{1}{(2K(N)(1+\frac{c^2}{4})^2)^{N+2}} < 1 \implies (E(v, B(x_0,1)))^{\frac{1}{N+2}} \geq (E(v, B(x_0,1)))^{\frac{1}{2}},$$

thus, $K(N)(1+\frac{c^2}{4})^2(E(v, B(x_0,1)))^{\frac{1}{N+2}}$ is larger than $K(N)(E(v, B(x_0,1)))^{\frac{1}{2}}$, then by Lemma 2.2,

$$\|1-|v|\|_{L^\infty(B(x_0,\frac{1}{2}))} \leq 2K(N)(1+\frac{c^2}{4})^2(E(v, B(x_0,1)))^{\frac{1}{N+2}},$$

so, we have,

$$1-|v(x_0)| \leq K(N)(1+\frac{c^2}{4})^2(E(v, B(x_0,1)))^{\frac{1}{N+2}} \leq \frac{1}{2} \implies |v(x_0)| \geq \frac{1}{2},$$

that is, $v \in \Lambda(\mathbb{R}^N)$.

□

2.2 Alternate definitions of the momentum

If $v \in \Lambda(\mathbb{R}^N)$, we may write, for $|x| > R(v)$

$$v = \rho \exp(i\varphi),$$

where φ is a real function on $\mathbb{R}^N \backslash B(0, R(v))$, defined modulo a multiple of 2π.
Also, we have,

$$\partial_j v = (i\rho\partial_j\varphi + \partial_j\rho)\exp(i\varphi) \implies |\partial_j v|^2 = |i\rho\partial_j\varphi|^2 + |\partial_j\rho|^2 + 2Re\overline{(i\rho\partial_j\varphi)}(\partial_j\rho) = |i\rho\partial_j\varphi|^2 + |\partial_j\rho|^2.$$

Moreover,

$$\langle i\partial_1 v, v\rangle = -\rho^2\partial_1\varphi \quad \text{and} \quad e(v) = \frac{1}{2}(|\nabla\rho|^2 + \rho^2|\nabla\varphi|^2) + \frac{1}{4}(1-\rho^2)^2.$$

Lemma 2.4. *Let ρ and φ be C^1 scalar functions on a domain $\mathcal{U}$ in $\mathbb{R}^N$, such that ρ is positive. Let $v = \rho\exp(i\varphi)$. Then, we have the pointwise bound*

$$|(\rho^2-1)\partial_1\varphi| \leq \frac{\sqrt{2}}{\rho}e(v).$$

Proof.

$$e(v) = \frac{1}{2}(|\nabla\rho|^2 + \rho^2|\nabla\varphi|^2) + \frac{1}{4}(1-\rho^2)^2 \geq \frac{1}{2}(\rho^2|\partial_1\varphi|^2) + \frac{1}{2}(\frac{1}{2}(1-\rho^2)^2).$$

Let $a = \frac{1}{\sqrt{2}}(\rho^2 - 1)$, $b = \rho\partial_1\varphi$, using Young's inequality $|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$,

$$|\frac{1}{\sqrt{2}}(\rho^2 - 1)\rho\partial_1\varphi| \leq \frac{1}{2}(\rho^2|\partial_1\varphi|^2) + \frac{1}{2}(\frac{1}{2}(1-\rho^2)^2) \leq e(v).$$

□

Now, we can have an alternative definition of the momentum on the space $\Lambda(\mathbb{R}^N)$. We consider the function,

$$g(v) = \langle i\partial_1 v, v\rangle + \partial_1((1-\chi)\varphi),$$

where $v = \rho\exp(i\varphi)$ on $\mathbb{R}^N \backslash B(0, R(v))$ and χ is an arbitrary smooth function with compact support, such that $\chi = 1$ on $B(0, R(v))$ and $0 \leq \chi \leq 1$.

Lemma 2.5. *If v belongs to $\Lambda(\mathbb{R}^N)$, then $g(v)$ belongs to $L^1(\mathbb{R}^N)$. Moreover, the integral*

$$\widetilde{p}(v) \equiv \frac{1}{2}\int_{\mathbb{R}^N} g(v) = \frac{1}{2}\int_{\mathbb{R}^N}(\langle i\partial_1 v, v\rangle + \partial_1((1-\chi)\varphi)),$$

for any smooth function χ with compact support.

Proof. $v \in \Lambda(\mathbb{R}^N) \implies \rho = |v| \geq \frac{1}{2}$ on $\mathbb{R}^N \backslash B(0, R(v))$. Notice that,

$$g(v) = -\rho^2\partial_1\varphi + \partial_1((1-\chi)\varphi) = (1-\rho^2)\partial_1\varphi \text{ on } \mathbb{R}^N \backslash \operatorname{supp}(\chi),$$

so, by Lemma 2.4 :

If $x \in \mathbb{R}^N \backslash \operatorname{supp}(\chi) \subset \mathbb{R}^N \backslash B(0, R(v))$ (since $B(0, R(v)) \subset \operatorname{supp}(\chi)$)

$$|g(v)| = |(1-\rho^2)\partial_1\varphi| \leq \frac{\sqrt{2}}{\rho}e(v) \leq 2\sqrt{2}e(v),$$

thus we can obtain,

$$\int_{\mathbb{R}^N \backslash \operatorname{supp}(\chi)} |g(v)| \leq 2\sqrt{2}\int_{\mathbb{R}^N \backslash \operatorname{supp}(\chi)} e(v) \leq 2\sqrt{2}E(v) < \infty,$$

since v is smooth on $\mathbb{R}^N$, so, $g(v)$ is also smooth on compact set in $\mathbb{R}^N$

$$\int_{\operatorname{supp}(\chi)} |g(v)| < \infty,$$

that is, the function $g(v)$ is integrable on $\mathbb{R}^N$.

Now, using integration by parts,

$$\begin{aligned}\int_{\mathbb{R}^N} g(v) &= \int_{\mathbb{R}^N} (\langle i\partial_1 v, v\rangle + \partial_1((1-\chi)\varphi)) \\ &= \int_{\mathbb{R}^N} \langle i\partial_1 v, v\rangle + \partial_1\varphi - \int_{\mathbb{R}^N} \partial_1(\chi\varphi) \\ &= \int_{\mathbb{R}^N} \langle i\partial_1 v, v\rangle + \partial_1\varphi - \int_{\text{supp}(\chi)} \nabla\cdot(\chi\varphi)n_1 \\ &= \int_{\mathbb{R}^N} \langle i\partial_1 v, v\rangle + \partial_1\varphi - \int_{\partial\ \text{supp}(\chi)} (\chi\varphi)n_1\cdot n\, dS_x \\ &= \int_{\mathbb{R}^N} \langle i\partial_1 v, v\rangle + \partial_1\varphi,\end{aligned}$$

where $n_1 = (1,0,0)$ and n is the outer normal vector on $\partial(\text{supp}(\chi))$, so, the integral does not depend on the choice of χ. □

2.3 Decay properties for (TWc)

Proposition 2.6. *Let v be a finite energy solution to (TWc).*

(1) There exists a constant v_∞, such that $|v_\infty| = 1$ and

$$v(x) \longrightarrow v_\infty, \text{ as } |x| \longrightarrow \infty.$$

Without loss of generality, we may assume $v_\infty = 1$.

(2) Assume $c(v) < \sqrt{2}$. Then, there exists some constant $K > 0$, depending only on $c(v)$, $E(v)$ and the dimension N, such that the following estimates hold for any $x \in \mathbb{R}^N$,

$$\begin{aligned} |Im(v(x))| &\leq \frac{K}{1+|x|^{N-1}} & \qquad |Re(v(x)) - 1| &\leq \frac{K}{1+|x|^N} \\ |\nabla Im(v(x))| &\leq \frac{K}{1+|x|^N} & \qquad |\nabla Re(v(x))| &\leq \frac{K}{1+|x|^{N+1}}. \end{aligned}$$

(3) Assume $N = 3$ and $c(v) = \sqrt{2}$. Then, $Re(v) - 1$ and $\nabla Im(v)$ belongs to $L^p(\mathbb{R}^3)$ for any $p > \frac{5}{3}$, $\nabla Re(v)$ belongs to $L^p(\mathbb{R}^3)$ for any $p > \frac{5}{4}$, whereas, $Im(v)$ belongs to $L^p(\mathbb{R}^3)$ for any $p > \frac{15}{4}$.

Corollary 2.7. *Let v be a finite energy solution to (TWc) and assume $v_\infty = 1$. Then v belongs to $W(\mathbb{R}^N)$.*

Remark 2.8. *Since any finite energy solution v to (TWc) has a limit v_∞ at infinity, we may write $v = \rho\exp(i\varphi)$ outside some ball $B(0,R)$, for some $R > 0$, where φ is smooth function on $\mathbb{R}^N \backslash B(0,R)$, which is defined up to an integer multiple of 2π. Moreover, the function φ has a limit at infinity φ_∞, which we may take as equal to 0, if we assume that $v_\infty = 1$. The statements given in [18,20,21], are actually expressed in terms of the real function ρ and φ, as follow,*
(1) Assume $0 \le c(v) < \sqrt{2}$. Then, there exists some constant $K > 0$, depending only on $c(v)$, $E(v)$ and the dimension N, such that the following estimates hold for any $x \in \mathbb{R}^N$,

$$|\varphi(x)| \le \frac{K}{1+|x|^{N-1}} \qquad |1-\rho(x)| \le \frac{K}{1+|x|^N}$$
$$|\nabla\varphi(x)| \le \frac{K}{1+|x|^N} \qquad |\nabla\rho(x)| \le \frac{K}{1+|x|^{N+1}}.$$

(2) If $c(v) = \sqrt{2}$. Then, the function φ belongs to $L^p(\mathbb{R}^3 \backslash B(0,R))$ for any $p > \frac{5}{3}$, whereas, $\nabla\rho$ belongs to $L^p(\mathbb{R}^3 \backslash B(0,R))$ for any $p > \frac{5}{4}$.

Proof. since $v = \rho\cos\varphi + i\rho\sin\varphi$, so $Re(v) - 1 = \rho\cos\varphi - 1$, $Im(v) = \rho\sin\varphi$

$$|Re(v) - 1| \le K(|\rho - 1| + \varphi^2) \qquad |Im(v)| \le K|\varphi|$$
$$|\nabla Re(v)| \le K(|\nabla\rho| + |\rho||\nabla\varphi|) \qquad |\nabla Im(v)| \le K(|\nabla\varphi| + |\varphi||\nabla\rho|)$$

where $K > 0$ is a constant. □

Proposition 2.9. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. Then, we have*

$$\widetilde{p}(v) = p(v).$$

Proof. Since v is a finite energy solution, so $v \in \Lambda(\mathbb{R}^N)$, that is, exists $R(v) > 0$ s.t.

$$|v| \ge \frac{1}{2} \text{ on } \mathbb{R}^N \backslash B(0, R(v)).$$

By Remark 2.8, without loss of generality, we may assume, $v_\infty = 1$ and $\varphi_\infty = 0$. Observe that,

$$Re(v) + iIm(v) = v = \rho\exp i\varphi = \rho(\cos\varphi + i\sin\varphi) \implies \sin\varphi = \frac{Im(v(x))}{\rho(x)}.$$

Using Taylor expansion of sin, we have, if x is sufficiently large

$$|\frac{Im(v(x))}{\rho(x)} - \varphi(x)| = |\sin\varphi - \varphi(x)| \leq \frac{|\varphi(x)|^3}{3!},$$

we want to estimate, $|p(v) - \widetilde{p}(v)| = \frac{1}{2}|\int_{\mathbb{R}^N}(\langle i\partial_1 v, v-1\rangle - g(v))|$.

For $R > R(v)$ is sufficiently large. Using integration by parts,

$$\begin{aligned}
\int_{B(0,R)} \langle i\partial_1 v, 1\rangle = \int_{B(0,R)} Re(\overline{i\partial_1 v}) &= \int_{B(0,R)} -Im(\partial_1 v) \\
&= -\int_{B(0,R)} \nabla \cdot (Im(\partial_1 v), 0, 0.., 0) \\
&= -\int_{\partial B(0,R)} (Im(\partial_1 v), 0, 0.., 0) \cdot n dS_x \\
&= -\int_{\partial B(0,R)} Im(\partial_1 v)\frac{x_1}{R} dS_x,
\end{aligned}$$

where, the outer unit normal $n = (\frac{x_1}{R}, \frac{x_2}{R}, ...)$.

Similarly,

$$\int_{B(0,R)} \partial_1((1-\chi)\varphi) = \int_{\partial B(0,R)} \varphi(x)\frac{x_1}{R} dS_x,$$

so, we have,

$$\int_{B(0,R)} (\langle i\partial_1 v, v-1\rangle - g(v)) = \frac{1}{R}\int_{\partial B(0,R)} (Im(v(x)) - \varphi(x)) x_1 dS_x.$$

Let $Im(v(x)) - \varphi(x) = (\frac{Im(v(x))}{\rho} - \varphi(x)) + Im(v)\frac{\rho-1}{\rho}$, so that,

$$\begin{aligned}
|\int_{B(0,R)} (\langle i\partial_1 v, v-1\rangle - g(v))| &= \frac{1}{R}|\int_{\partial B(0,R)} (Im(v(x)) - \varphi(x)) x_1 dS_x| \\
&= \frac{1}{R}|\int_{\partial B(0,R)} [(\frac{Im(v(x))}{\rho} - \varphi(x)) + Im(v)\frac{\rho-1}{\rho}] x_1 dS_x| \\
&\leq \frac{1}{R}\int_{\partial B(0,R)} |(\frac{Im(v(x))}{\rho} - \varphi(x)) + Im(v)\frac{\rho-1}{\rho}||x_1| dS_x \\
&\leq \frac{1}{R}\int_{\partial B(0,R)} |(\frac{Im(v(x))}{\rho} - \varphi(x)) + Im(v)\frac{\rho-1}{\rho}| R dS_x \\
&= \int_{\partial B(0,R)} |\frac{Im(v(x))}{\rho} - \varphi(x)| + |Im(v)\frac{\rho-1}{\rho}| dS_x \\
&\leq \int_{\partial B(0,R)} \frac{|\varphi(x)|^3}{3!} + |Im(v)\frac{\rho-1}{\rho}| dS_x \\
&\leq \int_{\partial B(0,R)} \frac{|\varphi(x)|^3}{3!} + 2|Im(v)||\rho-1| dS_x,
\end{aligned}$$

the last inequality follows by, $\rho = |v| \geq \frac{1}{2} \implies \frac{1}{\rho} \leq 2$.

Now, we have two cases.

Case 1. $N = 2$ or 3 and $c(v) \leq \sqrt{2}$

by Proposition 2.6 and Remark 2.8, we have,

$$\frac{|\varphi(x)|^3}{3!} + 2|Im(v)||\rho - 1| \leq \frac{K}{1+|x|^N},$$

we conclude that,

$$|\int_{B(0,R)} (\langle i\partial_1 v, v-1\rangle - g(v))| \leq \int_{\partial B(0,R)} \frac{K}{1+|x|^N} dS_x = \frac{K|\partial B(0,R)|}{1+R^N} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

Case 2. $N = 3$ and $c(v) = \sqrt{2}$

by Remark 2.8 : $\varphi \in L^q(\mathbb{R}^3 \backslash B(0, R(v)))$ for any $q > \frac{15}{4}$ and $\rho - 1 \in L^q(\mathbb{R}^3 \backslash B(0, R(v)))$ for any $q > \frac{5}{3}$ and by Proposition 2.6, the function $f \equiv \frac{|\varphi(x)|^3}{3!} + 2|Im(v)||\rho - 1| \in L^q(\mathbb{R}^3 \backslash B(0, R(v)))$ for any $q > \frac{5}{4}$.

We claim that : $\forall R > R(v), \forall q > \frac{5}{4}$ there $\exists R', R \leq R' \leq 2R, \exists K(q) > 0$, such that

$$\int_{\partial B(0,R')} f^q dS_x \leq \frac{K(q)}{R}.$$

If not, $\exists R > R(v), \exists q > \frac{5}{4}$ s.t. $\forall R', R \leq R' \leq 2R, \forall K(q) > 0, \int_{\partial B(0,R')} f^q dS_x > \frac{K(q)}{R}$, so,

$$\int_R^{2R} (\int_{\partial B(0,R')} f^q dS_x) dR' \geq \int_R^{2R} (\frac{K(q)}{R}) dR' = K(q) \implies \int_C f^q dx \geq K(q),$$

where $C = B(0, 2R) \backslash B(0, R)$, also, see Figure 3. Since $f \in L^q(\mathbb{R}^3 \backslash B(0, R(v))) \implies f \in L^q(C)$. Taking $q \longrightarrow \infty$ and taking $K(q) = q$, we obtain a contradiction.

$$\infty > \int_C f^q dx \geq K(q) \longrightarrow \infty.$$

Now, by Holders inequality :

$$\int_{\partial B(0,R')} f dS_x \leq (\int_{\partial B(0,R')} f^q dS_x)^{\frac{1}{q}} (\int_{\partial B(0,R')} 1^p dS_x)^{\frac{1}{p}} \leq (\frac{K(q)}{R})^{\frac{1}{q}} |\partial B(0,R')|^{\frac{1}{p}} \leq \bar{K}(q) R^{2-\frac{3}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $|\partial B(0,R')|^{\frac{1}{p}} = (4\pi R'^2)^{\frac{1}{p}} \leq (4\pi)^{1-\frac{1}{q}} (2R)^{2(1-\frac{1}{q})} = \tilde{K}(q) R^{2-\frac{2}{q}}$,

that is, we take $q = \frac{4}{3}$

$$|\int_{B(0,R')} (\langle i\partial_1 v, v-1\rangle - g(v))| \leq \tilde{K}(q) R^{2-\frac{2}{q}} \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

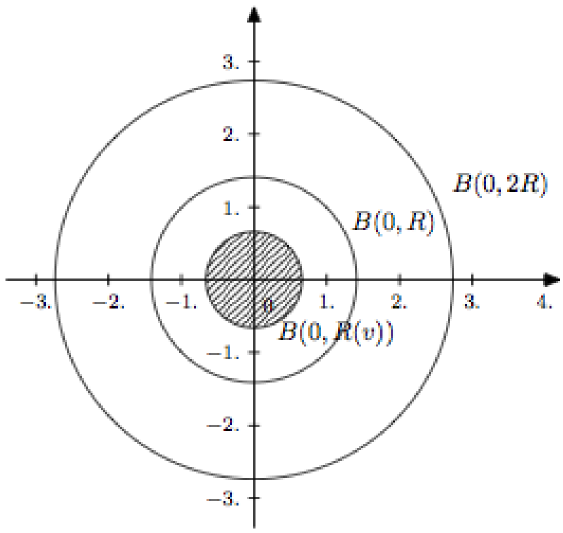

Figure 3: Energy estimate

which yields the conclusion, since the integrand is integrable by Lemma 2.5 and Corollary 2.7. □

2.4 Pohozaev's type identities

Lemma 2.10. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$, with speed $c = c(v)$. Then, we have the identities*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2,$$

and for any $2 \leq j \leq N$,

$$E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + c(v)p(v).$$

Moreover, if $c(v) > 0$ and v is not constant, then $p(v) > 0$.

Proof. The first identity was established in [19], and the second identity was proved there for any $2 \leq j \leq N$,

$$E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + c(v)\tilde{p}(v).$$

By Proposition 2.9 we have,

$$p(v) = \tilde{p}(v) \implies E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + c(v)p(v).$$

Notice, adding the identities in previous, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^2 + (N-1)c(v)p(v) \\
&= \int_{\mathbb{R}^N} |\partial_1 v|^2 + \Big(\int_{\mathbb{R}^N} |\partial_2 v|^2 + c(v)p(v) \Big) + \ldots + \Big(\int_{\mathbb{R}^N} |\partial_N v|^2 + c(v)p(v) \Big) \\
&= E(v) + (N-1)E(v) = NE(v) \\
&= \frac{N}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{4} \int_{\mathbb{R}^N} (1-|v|^2)^2,
\end{aligned}$$

combine all calculations, we get

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{4} \int_{\mathbb{R}^N} (1-|v|^2)^2 - c(v)(N-1)p(v) = 0.$$

Assume $c(v) > 0$ and $p(v) \leq 0$,

if $N = 3$, using the previous identity

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{4} \int_{\mathbb{R}^N} (1-|v|^2)^2 = c(v)(N-1)p(v) \leq 0 \implies |v| = 1 \text{ and } \nabla v = 0,$$

thus, v is constant, a contradiction.

If $N = 2$, we only have $|v| = 1$, that is $p(v) = 0$, also we may write $v = \rho \exp i\varphi$, by Lemma 2.13 (the proof of Lemma 2.13 doesn't depend on the Lemma 2.10)

$$c(v)p(v) = \int_{\mathbb{R}^N} \rho^2 |\nabla \varphi|^2 = 0 \implies \nabla \varphi = 0, \rho = 0,$$

thus, v is constant, a contradiction.

□

Now, introduce the quantities $\Sigma(v) = \sqrt{2}p(v) - E(v)$ and $\varepsilon(v) = \sqrt{2 - c(v)^2}$, combine with the second identity in Lemma 2.10, we have, for $2 \leq j \leq N$

$$\int_{\mathbb{R}^N} |\partial_j v|^2 + \Sigma(v) = \Big(\sqrt{2} - c(v)\Big)p(v) = \Big(\sqrt{2} - \sqrt{2 - \varepsilon(v)^2}\Big)p(v) = \frac{\varepsilon(v)^2}{\sqrt{2} + \sqrt{2 - \varepsilon(v)^2}} p(v).$$

Corollary 2.11. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$, with speed $c = \sqrt{2}$ and such that $\Sigma(v) \geq 0$. Then, v is a constant.*

Proof. Since $\varepsilon(v) = 0$ and $\Sigma(v) \geq 0$, by previous, for any $2 \leq j \leq N$,

$$\int_{\mathbb{R}^N} |\partial_j v|^2 = \frac{\varepsilon(v)^2}{\sqrt{2} + \sqrt{2 - \varepsilon(v)^2}} p(v) - \Sigma(v) = -\Sigma(v) \leq 0 \implies \int_{\mathbb{R}^N} |\partial_j v|^2 = 0,$$

so that v depends only on x_1 variable, say $v = f(x_1)$, but,

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2 dx = \int_{\mathbb{R}^N} |f'(x_1)|^2 dx_1 dx_2 ... dx_N = \Big(\int_{\mathbb{R}} |f'(x_1)|^2 dx_1 \Big) \Big(\int_{\mathbb{R}} 1 dx_2 \Big) ... \Big(\int_{\mathbb{R}} 1 dx_N \Big).$$

This implies that $|f'(x_1)|^2 = 0$, since v is a finite energy solution (i.e. $E(v) < \infty$), so, v is a constant. □

Moreover, if $\Sigma(v) > 0$, then identity in previous gives, for any $2 \leq j \leq N$

$$\int_{\mathbb{R}^N} |\partial_j v|^2 \leq \int_{\mathbb{R}^N} |\partial_j v|^2 + \Sigma(v) = \frac{\varepsilon(v)^2}{\sqrt{2} + \sqrt{2 - \varepsilon(v)^2}} p(v) \leq \varepsilon(v) p(v).$$

Combine with this inequality, the next result gives another version of Corollary 2.11.

Lemma 2.12. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. Then, there exists a constant $K(c) > 0$, depending on c, such that*

$$\|\eta\|_{L^\infty(\mathbb{R}^N)}^{N+1} \leq K(c) \int_{\mathbb{R}^N} \Big(\lambda |\partial_j v|^2 + \frac{\eta^2}{\lambda} \Big),$$

for any $2 \leq j \leq N$, and for any $\lambda > 0$.
In particular, we have,

$$\|\eta\|_{L^\infty(\mathbb{R}^N)}^{N+1} \leq K(c) \Big(\lambda \Big(\varepsilon(v) p(v) - \Sigma(v) \Big) + \frac{E(v)}{\lambda} \Big).$$

Proof. Let $\eta_\infty = \|\eta\|_{L^\infty(\mathbb{R}^N)}$. We may assume without loss of generality, say that $|\eta(0)| = \eta_\infty$. In view of the uniform bound for $|\nabla v|$, there exists some constant $K(N, c)$, such that

$$\eta_\infty - |\eta(x)| = |\eta(0)| - |\eta(x)| \leq |\eta(0) - \eta(x)| \leq |\nabla \eta(\zeta)||x - 0| \leq 2|v \nabla v(\zeta)||x| \leq K(N, c)|x|,$$

so, we have, if $|x| < \frac{\eta_\infty}{2K(N,c)}$,

$$\eta_\infty - |\eta(x)| \leq \frac{1}{2} \eta_\infty \implies \frac{\eta_\infty}{2} \leq |\eta(x)|.$$

In other words,

$$|\eta(x)| \geq \frac{\eta_\infty}{2}, \forall x \in B\Big(0, \frac{\eta_\infty}{2K(N,c)}\Big).$$

We next consider for any point $a = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_N)$, the line $D_j(a)$ parallel to the axis x_j, that is, $D_j(a) = \{a_j(x) \equiv (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_N), x \in \mathbb{R}\}$. We claim that,

$$|\eta_\infty|^2 \leq 4\int_{D_j(a)} \Big(\lambda(\partial_j\eta)^2 + \frac{\eta^2}{\lambda}\Big),$$

for any $a = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_N) \in B(0, \frac{\eta_\infty}{2K(c)})$.

Given $a = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_N) \in B(0, \frac{\eta_\infty}{2K(c)})$, we have $|\eta(a)| \geq \frac{\eta_\infty}{2}$, since $\eta(x) \longrightarrow 0$, as $|x| \longrightarrow +\infty$, and by Fundamental Theorem of Calculus,

$$|\eta_\infty|^2 \leq 4\eta(a)^2 = 4\int_{-\infty}^0 \partial_j(\eta^2(a_j(x)))dx = 8\int_{-\infty}^0 \partial_j\eta(a_j(x))\eta(a_j(x))dx$$

Using Young's inequality, $ab = (a\sqrt{\lambda})(\frac{b}{\sqrt{\lambda}}) \leq \frac{\lambda a^2}{2} + \frac{b^2}{2\lambda}$ for any $\lambda > 0$, where we take $a = \sqrt{2}\partial_j\eta(a_j(x))$ and $b = \sqrt{2}\eta(a_j(x))^2$,

$$\begin{aligned}
2\int_{-\infty}^0 \partial_j\eta(a_j(x))\eta(a_j(x))dx &\leq \int_{\infty}^0 \Big(\lambda\big(\partial_j\eta(a_j(x))\big)^2 + \frac{\eta(a_j(x))^2}{\lambda}\Big)dx \\
&\leq \int_{\mathbb{R}} \Big(\lambda\big(\partial_j\eta(a_j(x))\big)^2 + \frac{\eta(a_j(x))^2}{\lambda}\Big)dx. \\
&= \int_{D_j(a)} \Big(\lambda\big(\partial_j\eta(z)\big)^2 + \frac{\eta^2(z)}{\lambda}\Big)dz.
\end{aligned}$$

Now, we integrate the inequality on $a = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_N) \in B(0, \frac{\eta_\infty}{2K(N,c)})$.

$$\begin{aligned}
\int_{B(0,\frac{\eta_\infty}{2K(N,c)})} |\eta_\infty|^2 da &\leq 4\int_{B(0,\frac{\eta_\infty}{2K(N,c)})}\int_{D_j(a)} \Big(\lambda\big(\partial_j\eta(z)\big)^2 + \frac{\eta^2(z)}{\lambda}\Big)dzda \\
&\leq 4\int_{\mathbb{R}^N} \Big(\lambda\big(\partial_j\eta\big)^2 + \frac{\eta^2}{\lambda}\Big),
\end{aligned}$$

that is,

$$|\eta_\infty|^2\Big(\omega_{N-1}|\frac{\eta_\infty}{2K(N,c)}|^{N-1}\Big) \leq 4\int_{\mathbb{R}^N} \Big(\lambda\big(\partial_j\eta\big)^2 + \frac{\eta^2}{\lambda}\Big)$$

where ω_{N-1} is the volume of unit ball in $\mathbb{R}^N - 1$.

Finally,

$$\|\eta\|^{N+1}_{L^\infty(\mathbb{R}^N)} \leq K(N,c)\int_{\mathbb{R}^N} \Big(\lambda|\partial_j v|^2 + \frac{\eta^2}{\lambda}\Big),$$

for any $2 \leq j \leq N$, and for any $\lambda > 0$. □

2.5 Solutions without vortices

In this subsection, we consider solutions v to (TWc) on $\mathbb{R}^N$ which do not vanish. Moreover, we will assume that

$$|v| \geq \frac{1}{2},$$

so that, v may be written as $v = \rho \exp i\varphi$. Also, the energy can be written as variables ρ and φ,

$$E(v) = E(\rho, \varphi) \equiv \frac{1}{2} \int_{\mathbb{R}^N} \Big(|\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2 + \frac{(1-\rho^2)^2}{2} \Big), \tag{2.1}$$

whereas, for the momentum, we have $\langle i\partial_1 v, v\rangle = -\rho^2 \partial_1 \varphi$. Therefore, it follows from Proposition 2.9 that,

$$p(v) = \tilde{p}(v) = \frac{1}{2} \int_{\mathbb{R}^N} \Big(-\rho^2 \partial_1 \varphi + \partial_1\big((1-\chi)\varphi\big) \Big) = \frac{1}{2} \int_{\mathbb{R}^N} (1-\rho^2) \partial_1 \varphi. \tag{2.2}$$

The system for ρ and φ is written

$$\begin{cases} \frac{c}{2} \partial_1 \rho^2 + \nabla \cdot \Big(\rho^2 \nabla \varphi \Big) = 0, \\ c\rho \partial_1 \varphi - \Delta \rho - \rho \Big(1 - \rho^2 \Big) + \rho |\nabla \varphi|^2 = 0. \end{cases}$$

Notice that the quantity $\eta = 1 - \rho^2$ satisfies the equation,

$$\Delta^2 \eta - 2\Delta \eta + c^2 \partial_1^2 \eta = -2\Delta(|\nabla v|^2 + \eta^2 - c\eta \partial_1 \varphi) - 2c\partial_1 \nabla \cdot (\eta \nabla \varphi),$$

where the L.H.S is linear with respect to η, whereas, the R.H.S is quadratic.
A first elementary result is,

Lemma 2.13. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$, satisfying $|v| \geq \frac{1}{2}$. Then, we have the identity*

$$cp(v) = \int_{\mathbb{R}^N} \rho^2 |\nabla \varphi|^2.$$

Proof. The system for ρ and φ is written

$$\begin{cases} \frac{c}{2} \partial_1 \rho^2 + \nabla \cdot \Big(\rho^2 \nabla \varphi \Big) = 0, \\ c\rho \partial_1 \varphi - \Delta \rho - \rho \Big(1 - \rho^2 \Big) + \rho |\nabla \varphi|^2 = 0. \end{cases}$$

Multiplying the first equation by φ and integrating by parts, also, use the decay properties.

$$\frac{c}{2}\int_{\mathbb{R}^N}\varphi\partial_1\rho^2+\int_{\mathbb{R}^N}\varphi\nabla\cdot\Big(\rho^2\nabla\varphi\Big)=0,$$

$$\nabla\cdot\Big(\varphi\rho^2\nabla\varphi\Big)=\rho^2|\nabla\varphi|^2+\varphi\nabla\cdot\Big(\rho^2\nabla\varphi\Big),$$

$$-\frac{c}{2}\int_{\mathbb{R}^N}\varphi\partial_1\rho^2=\int_{\mathbb{R}^N}\varphi\nabla\cdot\Big(\rho^2\nabla\varphi\Big)=\int_{\mathbb{R}^N}\nabla\cdot\Big(\varphi\rho^2\nabla\varphi\Big)-\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2.$$

Now we claim,

$$\int_{\mathbb{R}^N}\nabla\cdot\Big(\varphi\rho^2\nabla\varphi\Big)=0.$$

We may assume $\rho^2\leq 2$, since $\rho\longrightarrow 1$ as $R=|x|\longrightarrow\infty$ and by divergent theorem,

$$\begin{aligned}|\int_{B_R}\nabla\cdot\Big(\varphi\rho^2\nabla\varphi\Big)|&=|\int_{\partial B_R}\Big(\varphi\rho^2\nabla\varphi\Big)\cdot ndS_x|=\int_{\partial B_R}|\varphi||\rho^2||\nabla\varphi|dS_x\\&\leq 2\int_{\partial B_R}|\varphi||\nabla\varphi|dS_x\leq\frac{K}{(1+R^{N-1})(1+R^N)}|\partial B_R|\longrightarrow 0,\end{aligned}$$

where n is the unit outer normal vector on ∂B_R, the last inequality follows by the decay property, that is,

$$\frac{c}{2}\int_{\mathbb{R}^N}\varphi\partial_1\rho^2=\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2,$$

and similarly,

$$\frac{c}{2}\int_{\mathbb{R}^N}\varphi\partial_1\rho^2=\frac{c}{2}\int_{\mathbb{R}^N}\partial_1(\varphi\rho^2)-\frac{c}{2}\int_{\mathbb{R}^N}(\partial_1\varphi)\rho^2,$$

again, we claim,

$$\int_{\mathbb{R}^N}\partial_1(\varphi\rho^2)=\int_{\mathbb{R}^N}\partial_1\varphi,$$

thus,

$$\frac{c}{2}\int_{\mathbb{R}^N}\varphi\partial_1\rho^2=\frac{c}{2}\int_{\mathbb{R}^N}\partial_1\varphi-\frac{c}{2}\int_{\mathbb{R}^N}(\partial_1\varphi)\rho^2=\frac{c}{2}\int_{\mathbb{R}^N}\partial_1\varphi(1-\rho^2).$$

We may say $\rho\leq 2$, since $\rho\longrightarrow 1$ as $R=|x|\longrightarrow\infty$ and by divergent theorem,

$$\begin{aligned}|\int_{B_R}\partial_1(\varphi\rho^2-\varphi)|&\leq|\int_{\partial B_R}|\varphi\rho^2-\varphi|dS_x=\int_{\partial B_R}|\varphi||\rho-1||\rho+1|dS_x\\&\leq\frac{K}{(1+R^{N-1})(1+R^N)}|\partial B_R|\longrightarrow 0,\end{aligned}$$

the last inequality also follows by the decay property.

Finally,

$$cp(v) = \frac{c}{2}\int_{\mathbb{R}^N}(1-\rho^2)\partial_1\varphi = \int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2.$$

□

Lemma 2.14. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$ satisfying $|v| \geq \frac{1}{2}$. Then*

$$E(v) \leq 7c(v)^2\int_{\mathbb{R}^N}\eta^2.$$

Proof. By Lemma 2.13 and Cauchy-Schwarz's inequality,

$$\begin{aligned}\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2 = \frac{c}{2}\int_{\mathbb{R}^N}(1-\rho^2)\partial_1\varphi &\leq c\Big(\int_{\mathbb{R}^N}\eta^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}|\nabla\varphi|^2\Big)^{\frac{1}{2}}\\ &\leq 2c\Big(\int_{\mathbb{R}^N}\eta^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2\Big)^{\frac{1}{2}},\end{aligned}$$

the last inequality follows by $\rho = |v| \geq \frac{1}{2} \implies 4\rho^2 \geq 1$.

Hence, we get,

$$\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2 \leq 4c^2\int_{\mathbb{R}^N}\eta^2.$$

It remains to bound the integral of $|\nabla\rho|^2$, recall the equations

$$\begin{cases}\frac{c}{2}\partial_1\rho^2 + \nabla\cdot\Big(\rho^2\nabla\varphi\Big) = 0,\\ c\rho\partial_1\varphi - \Delta\rho - \rho\Big(1-\rho^2\Big) + \rho|\nabla\varphi|^2 = 0.\end{cases}$$

For that purpose, we multiply the second equation by $1-\rho^2$ and integrate by parts on $\mathbb{R}^N$, using the decay properties in Remark 2.8.

$$\int_{\mathbb{R}^N}\Big((\Delta\rho)(1-\rho^2) + \rho(1-\rho^2)^2\Big) = c\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi + \int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2,$$

$$\int_{\mathbb{R}^N}(\Delta\rho)(1-\rho^2) = \int_{\mathbb{R}^N}\nabla\cdot\big((\nabla\rho)(1-\rho^2)\big) - \nabla\rho\cdot\nabla(1-\rho^2) = \int_{\mathbb{R}^N}2\rho|\nabla\rho|^2,$$

thus, we have,

$$\int_{\mathbb{R}^N}\Big(2\rho|\nabla\rho|^2 + \rho(1-\rho^2)^2\Big) = c\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi + \int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2,$$

also, by $1 \le 2\rho$ and $(1-\rho^2) \le 2\rho$ and Cauchy-Schwarz's inequality, we deduced

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(|\nabla\rho|^2 + \frac{1}{2}(1-\rho^2)^2 \right) &\le \int_{\mathbb{R}^N} \left((2\rho)|\nabla\rho|^2 + (2\rho)\frac{1}{2}(1-\rho^2)^2 \right) \\
&= c\int_{\mathbb{R}^N} \rho(1-\rho^2)\partial_1\varphi + \int_{\mathbb{R}^N} \rho(1-\rho^2)|\nabla\varphi|^2 \\
&\le c\Big(\int_{\mathbb{R}^N} \rho^2|\partial_1\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} (1-\rho^2)^2\Big)^{\frac{1}{2}} + 2\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 \\
&\le c\Big(\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} \eta^2\Big)^{\frac{1}{2}} + 8c^2\int_{\mathbb{R}^N} \eta^2 \\
&\le 2c^2\Big(\int_{\mathbb{R}^N} \eta^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} \eta^2\Big)^{\frac{1}{2}} + 8c^2\int_{\mathbb{R}^N} \eta^2 \\
&= 10c^2\int_{\mathbb{R}^N} \eta^2.
\end{aligned}$$

Finally,

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\rho^2|\nabla\varphi|^2 + |\nabla\rho|^2 + \frac{1}{2}(1-\rho^2)^2 \right) \le \frac{1}{2}(4c^2\int_{\mathbb{R}^N} \eta^2 + 10c^2\int_{\mathbb{R}^N} \eta^2).$$

□

2.6 Subsonic vortexless solutions

We next assume that the solution v satisfies the additional condition,

$$0 < c(v) < \sqrt{2}.$$

For such a solution, we let

$$\varepsilon(v) = \sqrt{2 - c(v)^2}.$$

Proposition 2.15. *Let v be a non-trivial finite energy solution to (TWc) on $\mathbb{R}^N$ satisfying $0 < c(v) < \sqrt{2}$. Then,*

$$\Big\| 1 - |v| \Big\|_{L^\infty(\mathbb{R}^N)} \ge \frac{\varepsilon(v)^2}{10}.$$

Proof. Let $\delta = \|1-|v|\|_{L^\infty(\mathbb{R}^N)}$. If $\delta \ge \frac{1}{2}$, then the proof is straightforward. Otherwise, $\delta < \frac{1}{2} \implies |v| \ge \frac{1}{2}$, we can use Lemma 2.4,

$$\delta = \|1-|v|\|_{L^\infty(\mathbb{R}^N)} \ge 1 - |v| \implies \rho \ge 1 - \delta \implies \frac{1}{\rho} \le \frac{1}{1-\delta},$$

$$cp(v) \le \frac{c}{2}\Big|\int_{\mathbb{R}^N}(1-\rho^2)\partial_1\varphi\Big| \le \frac{c}{2}\int_{\mathbb{R}^N}|1-\rho^2||\partial_1\varphi| \le \frac{c}{2}\int_{\mathbb{R}^N}\frac{\sqrt{2}}{\rho}e(v) \le \frac{c}{\sqrt{2}(1-\delta)}\int_{\mathbb{R}^N}e(v),$$

so, by Lemma 2.13,

$$\int_{\mathbb{R}^N}\rho^2|\nabla\varphi|^2 \le \frac{c}{\sqrt{2}(1-\delta)}\int_{\mathbb{R}^N}e(v).$$

Recall the identity in Lemma 2.14, now we will estimate the R.H.S.

$$\int_{\mathbb{R}^N}\Big(2\rho|\nabla\rho|^2+\rho(1-\rho^2)^2\Big) = c\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi + \int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2,$$

using Lemma 2.4 we have

$$c\Big|\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi\Big| \le \sqrt{2}c\int_{\mathbb{R}^N}e(v),$$

and using $|v|\ge\frac{1}{2}$, $|\frac{1-\rho^2}{\rho}|\le|\frac{1+\rho}{\rho}|\delta\le(1+\frac{1}{\rho})\delta\le 3\delta$

$$\Big|\int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2\Big| = \Big|\int_{\mathbb{R}^N}(\rho^2|\nabla\varphi|^2)(\frac{1-\rho^2}{\rho})\Big| \le 6\delta\int_{\mathbb{R}^N}e(v).$$

Combining previous calculation and using the fact that $\rho\ge 1-\delta>0$, we can get,

$$\begin{aligned}
\int_{\mathbb{R}^N}\Big(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4}\Big) &\le \int_{\mathbb{R}^N}\Big(\rho|\nabla\rho|^2+\rho\frac{(1-\rho^2)^2}{2}\Big)\\
&= \frac{1}{2}\Big(c\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi+\int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2\Big)\\
&\le |\frac{c}{2}\int_{\mathbb{R}^N}\rho(1-\rho^2)\partial_1\varphi| + |\frac{1}{2}\int_{\mathbb{R}^N}\rho(1-\rho^2)|\nabla\varphi|^2|\\
&\le \frac{\sqrt{2}c}{2}\int_{\mathbb{R}^N}e(v)+\frac{6\delta}{2}\int_{\mathbb{R}^N}e(v)\\
&= \frac{\sqrt{2}c+6\delta}{2}\int_{\mathbb{R}^N}e(v)\\
&\le \frac{\sqrt{2}c+6\delta}{4(1-\delta)}\int_{\mathbb{R}^N}e(v).
\end{aligned}$$

Finally, we derive,

$$\begin{aligned}
E(v) = \int_{\mathbb{R}^N}\Big((|\nabla\rho|^2+\frac{(1-\rho^2)^2}{2})+(\rho^2|\nabla\varphi|^2)\Big) &\le \Big(\frac{\sqrt{2}c+6\delta}{4(1-\delta)}+\frac{c}{2\sqrt{2}(1-\delta)}\Big)E(v)\\
= \Big(\frac{3\delta}{2(1-\delta)}+\frac{c}{\sqrt{2}(1-\delta)}\Big)E(v),
\end{aligned}$$

that is,

$$\lambda\int_{\mathbb{R}^N}e(v)\le 0,$$

where, we let $\lambda = 1 - \frac{c}{\sqrt{2}(1-\delta)} - \frac{3\delta}{2(1-\delta)}$.

Since v is non-trivial, its energy is not equal to 0, so that $\lambda \leq 0$ and

$$\delta \geq \frac{2}{5}\Big(1 - \frac{c}{\sqrt{2}}\Big) = \frac{2}{5}\Big(1 - \sqrt{1 - \frac{\varepsilon^2}{2}}\Big) \geq \frac{\varepsilon^2}{10},$$

which is our goal. □

Combining Lemma 2.10 and Lemma 2.13, we have,

Lemma 2.16. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$ satisfying $|v| \geq \frac{1}{2}$ and $0 < c(v) < \sqrt{2}$. Then,*

$$\Sigma(v) + \frac{1}{N}\int_{\mathbb{R}^N} |\nabla\rho|^2 = \frac{\varepsilon(v)^2}{\sqrt{2} + c(v)} p(v).$$

Moreover, if $N = 2$, then we have,

$$\int_{\mathbb{R}^2} |\nabla\rho|^2\Big(1 + \frac{1}{\rho^2}\Big) = \int_{\mathbb{R}^2} \eta|\nabla\varphi|^2.$$

Proof. Recall the identity in Lemma 2.10,

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{4}\int_{\mathbb{R}^N} \eta^2 = c(N-1)p(v),$$

and using the identity $|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\varphi|^2$,

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\rho|^2 + \frac{N-2}{2}\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2,$$

thus,

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\rho|^2 + \frac{N-2}{2}\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 + \frac{N}{4}\int_{\mathbb{R}^N} \eta^2 = c(N-1)p(v).$$

Now, add the following equality in Lemma 2.13 on both sides,

$$\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 = cp(v),$$

we get,

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\rho|^2 + \frac{N}{2}\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 + \frac{N}{4}\int_{\mathbb{R}^N} \eta^2 = cNp(v),$$

this yields,

$$E(v) - cp(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla\rho|^2,$$

by the definitions of $\Sigma(v)$ and $\varepsilon(v)$ we get,

$$\Sigma(v) + \frac{1}{N}\int_{\mathbb{R}^N} |\nabla\rho|^2 = \frac{\varepsilon(v)^2}{\sqrt{2} + c(v)} p(v).$$

For another equality , recall the system for ρ and φ is written

$$\begin{cases} \frac{c}{2}\partial_1\rho^2 + \nabla\cdot\Big(\rho^2\nabla\varphi\Big) = 0, \\ c\rho\partial_1\varphi - \Delta\rho - \rho\Big(1-\rho^2\Big) + \rho|\nabla\varphi|^2 = 0. \end{cases}$$

We multiply the second equation by $\frac{\eta}{\rho}$ and integrate on $\mathbb{R}^N$. This implies,

$$\int_{\mathbb{R}^N} \Big(c\eta\partial_1\varphi - \frac{\Delta\rho}{\rho}\eta - \eta^2 + \eta|\nabla\varphi|^2\Big) = 0.$$

Using integrating by parts, and $\nabla\Big(\frac{1-\rho^2}{\rho}\Big) = -(\nabla\rho)\Big(1 + \frac{1}{\rho^2}\Big)$

$$\int_{\mathbb{R}^N} \frac{\Delta\rho}{\rho}\eta = -\int_{\mathbb{R}^N} \nabla\rho\nabla\Big(\frac{\eta}{\rho}\Big) = -\int_{\mathbb{R}^N} \nabla\rho\nabla\Big(\frac{1-\rho^2}{\rho}\Big) = \int_{\mathbb{R}^N} |\nabla\rho|^2\Big(1+\frac{1}{\rho^2}\Big).$$

Now, for $N = 2$, and the following identity,

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\rho|^2 + \frac{N}{2}\int_{\mathbb{R}^N} \rho^2|\nabla\varphi|^2 + \frac{N}{4}\int_{\mathbb{R}^N} \eta^2 = cNp(v),$$

we get,

$$\int_{\mathbb{R}^2} \eta^2 = 2cp(v) = c\int_{\mathbb{R}^2} \eta\partial_1\varphi.$$

Finally,

$$\int_{\mathbb{R}^N} |\nabla\rho|^2\Big(1+\frac{1}{\rho^2}\Big) = \int_{\mathbb{R}^N} \frac{\Delta\rho}{\rho}\eta = c\int_{\mathbb{R}^2} \eta\partial_1\varphi - \int_{\mathbb{R}^N} \eta^2 + \int_{\mathbb{R}^N} \eta|\nabla\varphi|^2 = \int_{\mathbb{R}^N} \eta|\nabla\varphi|^2.$$

□

Now, we state the following Theorem, which shows that $u_{\mathfrak{p}}$ is analytic. For more detail, see Béthuel [4, Proposition 2.3].

Theorem 2.17. *Let v be a finite energy solution of (TWc) on $\mathbb{R}^N$, with speed $0 \le c < \sqrt{2}$, then each component of v is real-analytic on $\mathbb{R}^N$.*

2.7 Estimates of Fourier transform

We consider $\xi \in \mathbb{R}^N$, and a function f defined on $\mathbb{R}^N$, its Fourier transform $\widehat{f}(\xi)$, defined by the integral,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-ix\cdot\xi} dx.$$

Lemma 2.18. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^N$ satisfying $|v| \geq \frac{1}{2}$. Then, for any $\xi \in \mathbb{R}^N$ we have,*

$$\left(|\xi|^2 + 2 - c^2 \frac{\xi_1^2}{|\xi|^2}\right)\widehat{\eta}(\xi) = 2\widehat{R_0}(\xi) - 2c\sum_{j=2}^{N} \frac{\xi_j^2}{|\xi|^2}\widehat{R_1}(\xi) + 2c\sum_{j=2}^{N} \frac{\xi_1 \xi_j}{|\xi|^2}\widehat{R_j}(\xi)$$

where $R_0 = |\nabla v|^2 + \eta^2$ and $R_j = \eta\partial_j\varphi$.

Proof. Since $|v| \geq \frac{1}{2}$, we may write $v = \rho e^{i\varphi}$, which satisfies the system for ρ and φ

$$\begin{cases} \frac{c}{2}\partial_1\rho^2 + \nabla\cdot\Big(\rho^2\nabla\varphi\Big) = 0, \\ c\rho\partial_1\varphi - \Delta\rho - \rho\Big(1-\rho^2\Big) + \rho|\nabla\varphi|^2 = 0. \end{cases}$$

Also, $\eta = 1 - \rho^2$ satisfies the equation,

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_1^2\eta = -2\Delta(|\nabla v|^2 + \eta^2 - c\eta\partial_1\varphi) - 2c\partial_1\nabla\cdot(\eta\nabla\varphi),$$

it suffices to consider the Fourier transform on this equation. □

Lemma 2.19. *Let $N = 2$ and let v be a finite energy solution to (TWc) on $\mathbb{R}^2$ satisfying $|v| \geq \frac{1}{2}$ and $0 < c(v) < \sqrt{2}$. Then, there exists some universal constant $K > 0$, such that*

$$\varepsilon(v) \leq KE(v).$$

Proof. For any integrable function f on $\mathbb{R}^N$ and any $\xi \in \mathbb{R}^N$, we can see that $|\widehat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R}^N)} \implies \|\widehat{f}\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}$, and Lemma 2.4, $\rho = |v| \geq \frac{1}{2}$, we have the following,

$$\|\widehat{R_i}\|_{L^\infty(\mathbb{R}^N)} \leq \|R_i\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |R_i| = \int_{\mathbb{R}^N} |(\rho^2-1)\partial_i\varphi| \leq \int_{\mathbb{R}^N} \frac{\sqrt{2}}{\rho} e(v) \leq KE(v).$$

By integrating equation in Lemma 2.18, we have for any $1 \leq q \leq +\infty$,

$$\int_{\mathbb{R}^N} |\widehat{\eta}(\xi)|^q d\xi \leq K(1+c^q)\Big(\int_{\mathbb{R}^N} |L_\varepsilon(\xi)|^q d\xi\Big) E(v)^q,$$

where, for any $\xi \in \mathbb{R}^N$ we let

$$L_\varepsilon(\xi) = \frac{1}{|\xi|^2 + 2 - c^2\frac{\xi_1^2}{|\xi|^2}} = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}.$$

For case $q = 2$, by Plancherel's theorem $\|\widehat{\eta}\|_{L^2(\mathbb{R}^N)} = \|\eta\|_{L^2(\mathbb{R}^N)}$

$$\int_{\mathbb{R}^N} |\eta(x)|^2 dx \leq K(1+c^2)\Big(\int_{\mathbb{R}^N} |L_\varepsilon(\xi)|^2 d\xi\Big) E(v)^2.$$

We will proof the following claim after finishing the Lemma 2.19,

$$\int_{\mathbb{R}^2} |L_\varepsilon(\xi)|^2 d\xi = \frac{\pi}{\sqrt{2}\varepsilon(c)}.$$

By Lemma 2.14, $0 < c < \sqrt{2}$, and previous claim, we have,

$$E(v) \leq 7c^2 \int_{\mathbb{R}^N} |\eta|^2 \leq 7c^2 K(1+c^2)\Big(\int_{\mathbb{R}^N} |L_\varepsilon(\xi)|^2 d\xi\Big) E(v)^2 \leq \frac{K}{\varepsilon(c)} E(v)^2.$$

(Now we proof the claim)

Using polar coordinates $\xi_1 = r\cos\theta, \xi_2 = r\sin\theta$, we have,

$$\int_{\mathbb{R}^2} L_\varepsilon(\xi)^2 d\xi = \int_0^\infty \int_0^{2\pi} \frac{r dr d\theta}{(r^2 + 2 - c^2\cos^2(\theta))^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2 - c^2\cos^2(\theta)}.$$

Now, we change the variables $t = \tan(\theta)$ and $u = \sqrt{\frac{2}{2-c^2}}t$, we obtain,

$$\int_{\mathbb{R}^2} L_\varepsilon(\xi)^2 d\xi = 2\int_0^\infty \frac{dt}{2 - c^2 + 2t^2} = \sqrt{\frac{2}{2-c^2}} \int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{\sqrt{2}\varepsilon(c)}.$$

□

Lemma 2.20. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^3$ satisfying $|v| \geq \frac{1}{2}$ and $0 < c < \sqrt{2}$. Then, there exists some universal constant $K > 0$, such that*

$$E(v) \geq \mathcal{E}_0(c) \equiv \frac{K}{c(1+c^2)\arcsin\left(\frac{c}{\sqrt{2}}\right)}.$$

Proof. The argument is similar to the argument in Lemma 2.19.

We will proof the following claim after finishing the Lemma 2.20,

$$\int_{\mathbb{R}^3} |L_\varepsilon(\xi)|^2 d\xi = \frac{\pi^2}{c}\arcsin\Big(\frac{c}{\sqrt{2}}\Big).$$

By Lemma 2.14, $0 < c < \sqrt{2}$, and previous claim, there exists some constant $K > 0$ and $\tilde{K} > 0$, such that

$$E(v) \leq 7c^2 \int_{\mathbb{R}^3} |\eta|^2 \leq 7c^2 K(1+c^2)\Big(\int_{\mathbb{R}^3} |L_\varepsilon(\xi)|^2 d\xi\Big) E(v)^2 = \frac{c(1+c^2)}{\tilde{K}} \arcsin\Big(\frac{c}{\sqrt{2}}\Big) E^2(v).$$

(Now we proof the claim)

Using spherical coordinates and using the change of variables $u = \cos(\theta)$, we have,

$$\int_{\mathbb{R}^3} L_\varepsilon(\xi)^2 d\xi = 2\pi \int_0^\infty \Big(\int_0^\pi \frac{r^2 \sin(\theta) d\theta}{(r^2 + 2 - c^2\cos^2(\theta))^2}\Big) dr = 4\pi \int_0^\infty \Big(\int_0^1 \frac{r^2 du}{(r^2+2-c^2u^2)^2}\Big) dr.$$

Using integration by parts with respect to the variable r,

$$\int_0^\infty \frac{1}{(r^2+2-c^2u^2)} dr = \frac{r}{(r^2+2-c^2r^2)}\Big|_0^\infty - \int_0^\infty \frac{-2r^2}{(r^2+2-c^2r^2)^2} dr = \int_0^\infty \frac{2r^2}{(r^2+2-c^2r^2)^2} dr,$$

thus,

$$\int_{\mathbb{R}^3} L_\varepsilon(\xi)^2 d\xi = 2\pi \int_0^\infty \Big(\int_0^1 \frac{du}{r^2+2-c^2u^2}\Big) dr.$$

Using the change of variables $v = \frac{r}{\sqrt{2-c^2u^2}}$ and $w = \frac{cu}{\sqrt{2}}$, we obtain,

$$\begin{aligned}\int_{\mathbb{R}^3} L_\varepsilon(\xi)^2 d\xi =& 2\pi \int_0^\infty \Big(\int_0^1 \frac{du}{\sqrt{2-c^2u^2}}\Big) \frac{dv}{1+v^2} = \pi^2 \int_0^1 \frac{du}{\sqrt{2-c^2u^2}} = \frac{\pi^2}{c} \int_0^{\frac{c}{\sqrt{2}}} \frac{dw}{\sqrt{1-w^2}} \\ =& \frac{\pi^2}{c} \arcsin\Big(\frac{c}{\sqrt{2}}\Big).\end{aligned}$$

□

Lemma 2.21. *Let v be a non-trivial finite energy solution to (TWc) on $\mathbb{R}^3$. Then,*

$$E(v) \geq \mathcal{E}_0,$$

where $\mathcal{E}_0$ is some positive universal constant.

Proof. According to the results of Gravejat [19, Theorem 1], we know that, if $c(v) > \sqrt{2}$, then every finite energy solution to (TWc), must be constant solution. Now, since our v is nontrivial finite energy solution to (TWc), so $0 \leq c \leq \sqrt{2}$. Recall the Pohozaev's type inequality in the proof of Lemma 2.10,

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{4} \int_{\mathbb{R}^N} (1-|v|^2)^2 - c(v)(N-1)p(v) = 0,$$

if $c = 0$, then for $N = 3$, we have v is constant, contradict to nontrivial solution, thus we have,

$$0 < c \leq \sqrt{2}.$$

Moreover, in dimensions $N > 2$, recall from Lemma 2.2.
Let $r > 0$ and let v be a finite energy solution to (TWc) on $\mathbb{R}^N$. There exist some constants $K(N)$ s.t. for any $x_0 \in \mathbb{R}^N$,

$$\|1-|v|\|_{L^\infty(B(x_0,\frac{r}{2}))} \leq \max\{K(N)(1+\frac{c^2}{4})^2 E(v,B(x_0,r))^{\frac{1}{N+2}}, \frac{K(N)}{r^{N/2}}E(v,B(x_0,r))^{\frac{1}{2}}\},$$

where $E(v,B(x_0,r)) = \int_{B(x_0,r)} e(v)$.
Now, we let $x_0 = 0$ and for $r > 0$ large enough, since the energy is finite, we have,

$$\|1-|v|\|_{L^\infty(B(0,\frac{r}{2}))} \leq K(N)(1+\frac{c^2}{4})^2 E(v,B(0,r))^{\frac{1}{N+2}},$$

by Monotone convergence theorem,

$$\|1-|v|\|_{L^\infty(\mathbb{R}^N)} \leq KE(v)^{\frac{1}{N+2}}.$$

Assume $|v| < \frac{1}{2} \implies \frac{1}{2} \leq \|1-|v|\|_{L^\infty(\mathbb{R}^N)} \leq KE(v)^{\frac{1}{N+2}} \implies \mathcal{E}_0 = (\frac{1}{2K})^{N+2} \leq E(v)$.
Assume $|v| \geq \frac{1}{2}$, then v may be written as $v = \rho \exp i\varphi$.
If $0 < c(v) < \sqrt{2}$, then by Lemma 2.20, the function $c \mapsto c(1+c^2)\arcsin(\frac{c}{\sqrt{2}})$ is bounded on $(0,\sqrt{2})$, so $E(v)$ has a universal lower bound.
If $c(v) = \sqrt{2}$, we can get from the proof in Lemma 2.20 ($\varepsilon = \sqrt{2-c^2}$),

$$\int_{\mathbb{R}^3} L_0(\xi)^2 d\xi = \frac{\pi^2}{c}\arcsin\Big(\frac{c}{\sqrt{2}}\Big)\Big|_{c=\sqrt{2}} = \frac{\pi^3}{2\sqrt{2}} < +\infty,$$

then, use the same argument in Lemma 2.20, we can get the result. □

With same spirit, we may get the following Lemma in Béthuel [4, Lemma 2.15].

Lemma 2.22. *Let $\frac{5}{3} < q < +\infty$, and let v be a finite energy solution to (TWc) on $\mathbb{R}^3$, satisfying $|v| \geq \frac{1}{2}$. Then, there exists a constant $K(q)$, only depending on q, such that*

$$\|\eta\|_{L^q(\mathbb{R}^3)} \leq K(q)E(v)^{\frac{1}{q}+\frac{2}{5}}.$$

Corollary 2.23. *Let v be a non-trivial finite energy solution to (TWc) on $\mathbb{R}^3$. There exists some constants $K > 0$ and $\alpha > 0$, such that*

$$\|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{K}{E(v)^\alpha}.$$

Proof. If $\|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{1}{2}$, then by Lemma 2.21, take $K = \frac{\mathcal{E}_0^\alpha}{2}$ and any $\alpha > 0$.

$$\|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{1}{2} = \frac{K}{\mathcal{E}_0^\alpha} \geq \frac{K}{E(v)^\alpha}.$$

If $\|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{2} \implies |v| \geq \frac{1}{2}$. By Lemmas 2.14 and 2.22, and $0 \leq c \leq \sqrt{2}$, for any $\frac{5}{3} < q < 2$,

$$E(v) \leq 7c^2\|\eta\|^2_{L^2(\mathbb{R}^3)} \leq 14\|\eta\|^q_{L^q(\mathbb{R}^3)}\|\eta\|^{2-q}_{L^\infty(\mathbb{R}^3)} \leq K(q)\|\eta\|^{2-q}_{L^\infty(\mathbb{R}^3)}E(v)^{1+\frac{2q}{5}},$$

also,

$$|1-|v|| = |1-\rho| = \frac{|1-\rho^2|}{|1+\rho|} \geq \frac{|\eta|}{\|1+\rho\|_{L^\infty(\mathbb{R}^3)}} \implies \|1-|v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{\|\eta\|_{L^\infty(\mathbb{R}^3)}}{\|1+\rho\|_{L^\infty(\mathbb{R}^3)}}.$$

Hence, notice that $\|\rho\|_{L^\infty(\mathbb{R}^3)} - 1 \leq \|1-\rho\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{2}$, and using previous inequality,

$$\|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{\|\eta\|_{L^\infty(\mathbb{R}^3)}}{1 + \|\rho\|_{L^\infty(\mathbb{R}^3)}} \geq \frac{2}{5}\|\eta\|_{L^\infty(\mathbb{R}^3)} \geq \frac{K(q)}{E(v)^{\frac{2q}{5(2-q)}}}.$$

Finally, we choose $q = \frac{7}{4}$ and $\alpha = \frac{14}{5}$. □

Lemma 2.24. *Let v be a finite energy solution to (TWc) on $\mathbb{R}^3$, such that $\Sigma(v) > 0$. Then, there exists some constant $K(c)$, depending only on c, and some universal constant $\alpha > 0$, such that*

$$\varepsilon(v)p(v) \geq \frac{K(c)}{E(v)^{8\alpha+1}}.$$

Proof. By Lemma 2.12, we have,

$$\|\eta\|^{N+1}_{L^\infty(\mathbb{R}^N)} \leq K(c)\Big(\lambda\Big(\varepsilon(v)p(v) - \Sigma(v)\Big) + \frac{E(v)}{\lambda}\Big).$$

Let $N = 3$ and use Corollary 2.23,

$$\|\eta\|_{L^\infty(\mathbb{R}^3)} = \|1 - |v|^2\|_{L^\infty(\mathbb{R}^3)} \geq \|1 - |v|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{K(c)}{E(v)^\alpha},$$

thus,

$$\lambda\varepsilon(v)p(v) + \frac{E(v)}{\lambda} \geq K(c)\|\eta\|^4_{L^\infty(\mathbb{R}^3)} \geq \frac{K(c)}{E(v)^{4\alpha}}, \forall\lambda > 0.$$

Finally, we choice $\lambda = \frac{2E(v)^{4\alpha+1}}{K(c)}$ to get the result. □

3 Properties for the function $E_{min}(\mathfrak{p})$

In this section, we will introduce some properties of function $E_{min}(\mathfrak{p})$ and complete all the details of Theorem 3, Lemma 1 and Lemma 2, at the starting point of the proofs.

3.1 Proof of Theorem 3 : The Lipschitz condition for $E_{min}(\mathfrak{p})$

Define the space $\mathcal{E}(\mathbb{R}^N) = \{v : \mathbb{R}^N \longrightarrow \mathbb{C} | E(v) < +\infty\}$.

Lemma 3.1. *For $N = 2$ and $N = 3$, we have the inclusion $W(\mathbb{R}^N) \subset \mathcal{E}(\mathbb{R}^N)$. Moreover, the functions E and p are continuous on $W(\mathbb{R}^N)$.*

Proof. Concerning the momentum p, we have already seen that, in view of the proof of $p(v)$ is well-defined, we can similarly show that p is continuous on $W(\mathbb{R}^N)$.
For the energy E, we observe the identity,

$$(1 - |1 + w|^2)^2 = 4Re(w)^2 + 4Re(w)|w|^2 + |w|^4,$$

for any $w \in V(\mathbb{R}^N)$.
By Young's inequality, $Re(w)|w|^2 \leq \frac{Re(w)^2}{2} + \frac{|w|^4}{2}$ and $|w|^4 = (Re(w)^2 + Im(w)^2)^2$,

$$(1-|1+w|^2)^2 \leq 6Re(w)^2+3|w|^4 \leq 6Re(w)^2+3Re(w)^4+3Im(w)^4+6Re(w)^2Im(w)^2,$$

again, by Young's inequality,

$$Re(w)^2Im(w)^2 \leq \frac{Re(w)^4}{2}+\frac{Im(w)^4}{2} \implies (1-|1+w|^2)^2 \leq 6Re(w)^2+6Re(w)^4+6Im(w)^4.$$

We just need to claim : $Re(w)^4 \in L^1(\mathbb{R}^N)$ then the L.H.S of this identity belongs to $L^1(\mathbb{R}^N)$, whenever w belongs to $V(\mathbb{R}^N)$. Hence, $W(\mathbb{R}^N)$ is included in $\mathcal{E}(\mathbb{R}^N)$, also, we can similarly show that E is continuous on $W(\mathbb{R}^N)$.
(Claim : $Re(w)^4 \in L^1(\mathbb{R}^N)$)
For $N = 2$, let $p = \frac{4}{3}$ and $p^* = \frac{Np}{N-p} = 4$, also $v = Re(w)$ and φ is a smooth cut off with 1 on $B(0,1)$ and support in $B(0,2)$, and $\nabla v \in L^{\frac{4}{3}}(\mathbb{R}^N)$, and $v \in L^2(\mathbb{R}^N)$, we need to show $v \in L^4(\mathbb{R}^N)$.

Let ϕ_n be standard smooth mollifier, $\|\phi_n\|_{L^1(\mathbb{R}^N)} = 1$, then $\phi_n * v$ is smooth.
By Zygmund [41, Prob 7.2 and (9.6)], $\phi_n * v \longrightarrow v$ a.e. and

$$v_n \longrightarrow v \text{ in } L^2(\mathbb{R}^N) \quad \text{and} \quad \nabla v_n = \phi_n * \nabla v \longrightarrow \nabla v \text{ in } L^{\frac{4}{3}}(\mathbb{R}^N),$$

where $v_n = (\phi_n * v)$, and take $\eta^n_m = \varphi(\frac{x}{m})v_n$ and $\phi^n_m = \varphi(\frac{x}{m})\nabla v_n$ in $C^\infty_c(\mathbb{R}^N)$.
Using diagonal process,

$$\begin{array}{cccccccccc}
\eta^1_1 & \eta^1_2 & \eta^1_3 & \eta^1_4 & . & . & . & . & v_1 & \text{in } L^2(\mathbb{R}^N) \\
\\
\eta^2_1 & \eta^2_2 & \eta^2_3 & \eta^2_4 & . & . & . & . & v_2 & \text{in } L^2(\mathbb{R}^N) \\
\\
\eta^3_1 & \eta^3_2 & \eta^3_3 & \eta^3_4 & . & . & . & . & v_3 & \text{in } L^2(\mathbb{R}^N) \\
. & . & . & . & & & & & . &
\end{array}$$

let j fixed, $\{\eta^j_m\}^\infty_{m=1}$ and $\{\phi^j_m\}^\infty_{m=1}$, both Cauchy in $L^2(\mathbb{R}^N)$ and $L^{\frac{4}{3}}(\mathbb{R}^N)$.

For $\frac{1}{j} > 0$ $\exists N_j$ s.t. $\forall i \geq N_j$, $\|\eta^j_i - \eta^j_{N_j}\|_{L^2(\mathbb{R}^N)} < \frac{1}{j}$ and $\|\phi^j_i - \phi^j_{N_j}\|_{L^{\frac{4}{3}}(\mathbb{R}^N)} < \frac{1}{j}$,

claim that : $\{\eta^j_{N_j}\}^\infty_{j=1}$ converges in $L^2(\mathbb{R}^N)$ and $\{\phi^j_{N_j}\}^\infty_{j=1}$ converges in $L^{\frac{4}{3}}(\mathbb{R}^N)$.
Given $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\forall k \geq N_0$, $\frac{1}{k} < \frac{\epsilon}{2}$ and $\|v_k - v\|_{L^2(\mathbb{R}^N)} < \frac{\varepsilon}{2}$,
for any $i > N_k$,

$$\|\eta^k_{N_k} - v\|_{L^2(\mathbb{R}^N)} \leq \|\eta^k_{N_k} - \eta^k_i\|_{L^2(\mathbb{R}^N)} + \|\eta^k_i - v_k\|_{L^2(\mathbb{R}^N)} + \|v_k - v\|_{L^2(\mathbb{R}^N)} \leq \|\eta^k_i - v_k\|_{L^2(\mathbb{R}^N)} + \varepsilon,$$

taking $i \longrightarrow \infty$, then $\eta^k_{N_k}$ converge to v in $L^2(\mathbb{R}^N)$. Similar for $\{\phi^j_{N_j}\}^\infty_{j=1}$.
That is,

$$\eta^n_{N_n} \longrightarrow v \text{ in } L^2(\mathbb{R}^N) \text{ and } \phi^n_{N_n} \longrightarrow \nabla v \text{ in } L^{\frac{4}{3}}(\mathbb{R}^N).$$

Let $\eta_n = \eta^n_{N_n}$ and notice that,

$$\nabla \eta_n = \frac{1}{N_n}\Big(\nabla\varphi(\frac{x}{N_n})\Big)v_n + \varphi(\frac{x}{N_n})\nabla v_n = \frac{1}{N_n}\Big(\nabla\varphi(\frac{x}{N_n})\Big)v_n + \phi^n_{N_n},$$

also,

$$\|\nabla\eta_n - \nabla v\|_{L^{\frac{4}{3}}(\mathbb{R}^N)} \leq \|\frac{1}{N_n}\Big(\nabla\varphi(\frac{x}{N_n})\Big)v_n\|_{L^{\frac{4}{3}}(\mathbb{R}^N)} + \|\phi^n_{N_n} - \nabla v\|_{L^{\frac{4}{3}}(\mathbb{R}^N)},$$

by Young's convolution inequality,

$$\|v_n\|_{L^2(\mathbb{R}^N)} = \|\phi_n * v\|_{L^2(\mathbb{R}^N)} \leq \|\phi_n\|_{L^1(\mathbb{R}^N)}\|v\|_{L^2(\mathbb{R}^N)} = \|v\|_{L^2(\mathbb{R}^N)},$$

by Holder inequality,

$$\|\frac{1}{N_n}\Big(\nabla\varphi(\frac{x}{N_n})\Big)v_n\|_{L^{\frac{4}{3}}(\mathbb{R}^N)} \leq \frac{1}{(N_n)^{\frac{1}{2}}}\|\nabla\varphi\|_{L^4(\mathbb{R}^N)}\|v_n\|_{L^2(\mathbb{R}^N)} \leq \frac{1}{(N_n)^{\frac{1}{2}}}\|\nabla\varphi\|_{L^4(\mathbb{R}^N)}\|v\|_{L^2(\mathbb{R}^N)},$$

the last term will converge to 0 as $n \longrightarrow \infty$, that is, there exists $\eta_n \in C_c^\infty(\mathbb{R}^N)$ s.t.

$$\eta_n \longrightarrow v \text{ in } L^2(\mathbb{R}^N) \text{ and } \nabla\eta_n \longrightarrow \nabla v \text{ in } L^{\frac{4}{3}}(\mathbb{R}^N),$$

by Gagliardo-Nirenberg-Sobolev inequality,

$$\|\eta_n - \eta_m\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla\eta_n - \nabla\eta_m\|_{L^p(\mathbb{R}^N)},$$

thus, η_n is Cauchy in $L^{p^*}(\mathbb{R}^N)$, say $\eta_n \longrightarrow v^*$ in $L^{p^*}(\mathbb{R}^N)$

$$\begin{cases} \eta_n \longrightarrow v^* \text{ in } L^{p^*}(\mathbb{R}^N) \\ \eta_n \longrightarrow v \text{ in } L^2(\mathbb{R}^N). \end{cases}$$

For any K compact,

$$\|v - v^*\|_{L^2(K)} \leq \|v - \eta_n\|_{L^2(K)} + \|\eta_n - v^*\|_{L^2(K)},$$

since $p^* = 4$,

$$\|\eta_n - v^*\|^2_{L^2(K)} = \int_K |\eta_n - v^*|^2 \leq \Big(\int_{\mathbb{R}^N} |\eta_n - v^*|^4\Big)^{\frac{1}{2}}|K|^{\frac{1}{2}} \longrightarrow 0$$

thus, $v = v^*$ a.e. (i.e. $v \in L^{p^*}(\mathbb{R}^N)$).

For $N = 3$, let $p = 2$ and $p^* = 6$, also $v \in L^2(\mathbb{R}^N)$, $\nabla v \in L^2(\mathbb{R}^N)$, with the same argument, we can get $v \in L^{p^*}(\mathbb{R}^N)$, that is, v in $L^6(\mathbb{R}^N)$,

by Young's inequality,

$$\|v\|^4_{L^4(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |v|^3|v| \leq \int_{\mathbb{R}^N} \Big(\frac{|v|^6}{2} + \frac{|v|^2}{2}\Big) < \infty,$$

thus, $v \in L^4(\mathbb{R}^N)$. □

Lemma 3.2. *Assume $N = 2$ or $N = 3$ and let $v = 1 + w$ be in $W(\mathbb{R}^N)$ and $p(v) > 0$. There exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^N)$, such that $w_n \longrightarrow w$ in $V(\mathbb{R}^N)$ as $n \longrightarrow +\infty$, and*

$$p(v_n) = p(v), \text{ and } E(v_n) \longrightarrow E(v) \text{ as } n \longrightarrow +\infty.$$

where, $v_n = 1 + w_n$.
In particular, for any $\mathfrak{p} > 0$, there exists a sequence $(w_n)_{n\in N}$ in $C_c^\infty(\mathbb{R}^N)$, such that

$$p(1 + w_n) = \mathfrak{p}, \text{ and } E(1 + w_n) \longrightarrow E_{\min}(\mathfrak{p}) \text{ as } n \longrightarrow +\infty,$$

and thus, we have,

$$E_{\min}(\mathfrak{p}) = \inf\{E(1 + v) | v \in C_c^\infty(\mathbb{R}^N), p(1 + v) = \mathfrak{p}\}.$$

Moreover, for $\mathfrak{p} > 0$

$$\Gamma^N(\mathfrak{p}) = \{w \in W(\mathbb{R}^N) | p(w) = \mathfrak{p}\}$$

is not empty.

Proof. By Lemma 3.1 and the density of $C_c^\infty(\mathbb{R}^N)$ in $V(\mathbb{R}^N)$, we have the following. By density : given any $w \in V(\mathbb{R}^N)$, there exists $(\tilde{w}_n)_{n\in N}$ in $C_c^\infty(\mathbb{R}^N)$, such that

$$\tilde{w}_n \longrightarrow w \text{ in } V(\mathbb{R}^N),$$

let $\tilde{v}_n = 1 + \tilde{w}_n$, and $v = 1 + w$, implies $\tilde{v}_n, v \in W(\mathbb{R}^N) = 1 + V(\mathbb{R}^N)$ and

$$\tilde{v}_n \longrightarrow v \text{ in } W(\mathbb{R}^N).$$

By continuity of E, p on $W(\mathbb{R}^N)$:

$$p(\tilde{v}_n) \longrightarrow p(v), \text{ and } E(\tilde{v}_n) \longrightarrow E(v) \text{ as } n \longrightarrow +\infty.$$

Since $p(v) > 0$, so by continuity, we have $p(\tilde{v}_n) > 0$, for n sufficiently large,
let,

$$w_n = \Big(\sqrt{\frac{p(v)}{p(\tilde{v}_n)}}\Big)\tilde{w}_n, \text{ and } v_n = 1 + w_n,$$

by calculation we can get,

$$p(v_n) = \int_{\mathbb{R}^N} \langle i\partial_1 v_n, v_n - 1\rangle = \int_{\mathbb{R}^N} \langle i\partial_1 w_n, w_n\rangle = \Big(\frac{p(v)}{p(\tilde{v}_n)}\Big) \int_{\mathbb{R}^N} \langle i\partial_1 \tilde{w}_n, \tilde{w}_n\rangle$$
$$= \Big(\frac{p(v)}{p(\tilde{v}_n)}\Big) \int_{\mathbb{R}^N} \langle i\partial_1 \tilde{v}_n, \tilde{v}_n - 1\rangle = p(v),$$

also, notice that, $w_n \longrightarrow w$ in $V(\mathbb{R}^N)$, implies $v_n \longrightarrow v$ in $W(\mathbb{R}^N)$, again by continuity of E on $W(\mathbb{R}^N)$, we have $E(v_n) \longrightarrow E(v)$. □

Corollary 3.3. *Let* $\mathfrak{p} > 0$. *Then,*

$$\limsup_{n\longrightarrow+\infty} \big(E^n_{\min}(\mathfrak{p})\big) \leq E_{\min}(\mathfrak{p}).$$

Proof. By Lemma 3.2,

$$E_{\min}(\mathfrak{p}) = \inf\{E(1+v), v \in C_c^\infty(\mathbb{R}^N), p(1+v) = \mathfrak{p}\}.$$

For any $\varepsilon > 0$, there exists $w = 1 + v \in \{1\} + C_c^\infty(\mathbb{R}^N)$, such that

$$E_{\min}(\mathfrak{p}) \leq E(1+v) \leq E_{\min}(\mathfrak{p}) + \varepsilon, \text{ and } p(1+v) = \mathfrak{p}.$$

We may assume v has compact support in some ball $B(0, R)$, for some radius $R > 0$, if $\pi n > R$, then, the restriction of v to the set Ω_n^N, will vanish on the boundary $\partial\Omega_n^N$, and hence defines a map in $H^1(\mathbb{T}_n^N)$. Adding to v the constant function 1, we have similarly $w = 1 + v \in H^1(\mathbb{T}_n^N, \mathbb{C})$.
Moreover, in the two-dimensional case, if $n \geq \big(\frac{R}{\pi}\big)^{\frac{4}{3}}$, then $v \in \mathcal{S}_n^0$, where $\mathcal{S}_n^0$ is a suitable topological space that v has a lifting (see Béthuel [4, (4.14)]), this implies,

$$E(w) \geq E^n_{\min}(\mathfrak{p}), \ \forall n \geq \Big(\frac{R}{\pi}\Big)^{\frac{4}{3}},$$

hence,

$$E^n_{\min}(\mathfrak{p}) \leq E(w) \leq E_{\min}(\mathfrak{p}) + \varepsilon, \ \forall n \geq \Big(\frac{R}{\pi}\Big)^{\frac{4}{3}},$$

that is,

$$\limsup_{n\longrightarrow\infty} E^n_{\min}(\mathfrak{p}) \leq E_{\min}(\mathfrak{p}) + \varepsilon,$$

and the conclusion follows letting ε tends to zero. □

Next, we state the following approximation lemma, which we have mentioned in Chapter 1. For more detail, see Béthuel [4, Lemma 3.3].

Lemma 3.4. *Let $N \geq 2$ and $\mathfrak{s} > 0$ be given. There exists a sequence of non-constant maps, $(\gamma_n)_{n \in \mathbb{N}}$ in $\{1\} + C_c^\infty(\mathbb{R}^N)$, such that*

$$p(\gamma_n) = \mathfrak{s}, \|\gamma_n\|_{W(\mathbb{R}^N)} \leq K\sqrt{\mathfrak{s}}, \text{ and } E(\gamma_n) \longrightarrow \sqrt{2}\mathfrak{s}, \text{ as } n \longrightarrow +\infty,$$

where K is some universal constant. In particular, $E_{\min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p}$, for any $\mathfrak{p} \geq 0$, and the map $\mathfrak{p} \mapsto \Xi(\mathfrak{p})$ is non-negative.

Lemma 3.5. *For any $\mathfrak{p}, \mathfrak{q} \geq 0$,*

$$|E_{\min}(\mathfrak{p}) - E_{\min}(\mathfrak{q})| \leq \sqrt{2}|\mathfrak{p} - \mathfrak{q}|.$$

In particular, the function $\mathfrak{p} \mapsto E_{\min}(\mathfrak{p})$ is Lipschitz continuous on $\mathbb{R}_+$, with Lipschitz's constant $\sqrt{2}$, and the function $p \mapsto \Xi(\mathfrak{p})$ is non-negative, non-decreasing and continuous on $\mathbb{R}_+$.

Proof. We use the approximation argument.
We may assume without loss of generality, that $\mathfrak{q} > \mathfrak{p}$, and claim that,

$$E_{\min}(\mathfrak{q}) \leq E_{\min}(\mathfrak{p}) + \sqrt{2}(\mathfrak{q} - \mathfrak{p}).$$

By Lemma 3.2 and the definition of inf we have the following :
given $\varepsilon > 0$, there exists $w_\varepsilon = 1 + v_\varepsilon$, where $v_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, such that

$$p(w_\varepsilon) = \mathfrak{p}, \text{ and } E(w_\varepsilon) \leq E_{\min}(\mathfrak{p}) + \frac{\varepsilon}{2}.$$

Let $\mathfrak{s} = \mathfrak{q} - \mathfrak{p} > 0$ and by Lemma 3.4, we have the following,
there exists f_ε in $C_c^\infty(\mathbb{R}^N)$, such that $p(1 + f_\varepsilon) = \mathfrak{s}$ and $E(1 + f_\varepsilon) \leq \sqrt{2}\mathfrak{s} + \frac{\varepsilon}{2}$.
Set

$$v = 1 + v_\varepsilon + f_\varepsilon(\cdot - a_\varepsilon),$$

since v_ε and f_ε both in $C_c^\infty(\mathbb{R}^N)$, so we may choose $a_\varepsilon \in \mathbb{R}^N$, such that the support of v_ε and $f_\varepsilon(\cdot - a_\varepsilon)$ do not intersect.

Observe that :

$$
\begin{aligned}
E(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2 \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon + \nabla f_\varepsilon(\cdot - a_\varepsilon)|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|1+v_\varepsilon + f_\varepsilon(\cdot - a_\varepsilon)|^2)^2 \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + |\nabla f_\varepsilon(\cdot - a_\varepsilon)|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|1+v_\varepsilon + f_\varepsilon(\cdot - a_\varepsilon)|^2)^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big(1-|1+v_\varepsilon + f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)^2 =& \Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2 - |v_\varepsilon|^2 - 2Re(v_\varepsilon)\overline{(1+f_\varepsilon(\cdot - a_\varepsilon))}\Big)^2 \\
=& \Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)^2 + \Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\overline{(1+f_\varepsilon(\cdot - a_\varepsilon))}\Big)^2 \\
&- 2Re\Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)\overline{\Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\overline{(1+f_\varepsilon(\cdot - a_\varepsilon))}\Big)},
\end{aligned}
$$

also,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} Re\Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)\overline{\Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\overline{(1+f_\varepsilon(\cdot - a_\varepsilon))}\Big)} \\
=&\int_{\mathbb{R}^N} Re\Big(-|f_\varepsilon(\cdot - a_\varepsilon)|^2 - 2Re(f_\varepsilon(\cdot - a_\varepsilon))\Big)\overline{\Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\Big)} = 0,
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^N} \Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\overline{(1+f_\varepsilon(\cdot - a_\varepsilon))}\Big)^2 = \int_{\mathbb{R}^N} \Big(|v_\varepsilon|^2 + 2Re(v_\varepsilon)\Big)^2 = \int_{\mathbb{R}^N} \Big(1-|1+v_\varepsilon|\Big)^2,
$$

thus, we have,

$$
\int_{\mathbb{R}^N} (1-|1+v_\varepsilon + f_\varepsilon(\cdot - a_\varepsilon)|^2)^2 = \int_{\mathbb{R}^N} \Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)^2 + \int_{\mathbb{R}^N} \Big(1-|1+v_\varepsilon|\Big)^2,
$$

combine all calculations :

$$
\begin{aligned}
E(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + |\nabla f_\varepsilon(\cdot - a_\varepsilon)|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \Big(1-|1+f_\varepsilon(\cdot - a_\varepsilon)|^2\Big)^2 + \frac{1}{4}\int_{\mathbb{R}^N} \Big(1-|1+v_\varepsilon|\Big)^2 \\
&= E(w_\varepsilon) + E(1+f_\varepsilon).
\end{aligned}
$$

Similar for p, we have,

$$
E(v) = E(w_\varepsilon) + E(1+f_\varepsilon), \text{ and } p(v) = p(w_\varepsilon) + p(1+f_\varepsilon) = \mathfrak{p} + \mathfrak{s} = \mathfrak{q}.
$$

It follows that,

$$
E_{\min}(\mathfrak{q}) \leq E(v) \leq E(w_\varepsilon) + \sqrt{2}\mathfrak{s} + \frac{\varepsilon}{2} \leq E_{\min}(\mathfrak{p}) + \sqrt{2}(\mathfrak{q} - \mathfrak{p}) + \varepsilon,
$$

taking $\varepsilon \longrightarrow 0$, we get the inequality.

Next, we claim the converse inequality,

$$E_{\min}(\mathfrak{p}) \leq E_{\min}(\mathfrak{q}) + \sqrt{2}(\mathfrak{q} - \mathfrak{p}).$$

Similarly, by Lemma 3.2 :

given $\delta > 0$, there exists $\tilde{w}_\delta = 1 + \tilde{v}_\delta$, where $\tilde{v}_\delta \in C_c^\infty(\mathbb{R}^N)$, such that

$$p(\tilde{w}_\delta) = \mathfrak{q}, \text{ and } E(\tilde{w}_\delta) \leq E_{\min}(\mathfrak{q}) + \frac{\delta}{2}.$$

Let $\mathfrak{s} = \mathfrak{q} - \mathfrak{p} > 0$ and by Lemma 3.4, we have the following,

there exists f_δ in $C_c^\infty(\mathbb{R}^N)$, such that $p(1 + f_\delta) = \mathfrak{s}$ and $E(1 + f_\delta) \leq \sqrt{2}\mathfrak{s} + \frac{\delta}{2}$.

Define $\check{f}_\delta(x_1, x_\perp) = f_\delta(-x_1, x_\perp) \in C_c^\infty(\mathbb{R}^N)$, where $x_\perp = (x_2, x_3, ..., x_N)$.

Set,

$$\tilde{v} = 1 + \tilde{v}_\delta + \check{f}_\delta(\cdot - b_\delta),$$

where b_δ is chosen, so that the support of $\tilde{v}_\delta$ and $\check{f}_\delta(\cdot - b_\delta)$, do not intersect.

Notice that we have,

$$E\big(1 + \check{f}_\delta(\cdot - b_\delta)\big) = E(1 + f_\delta) \leq \sqrt{2}\mathfrak{s} + \frac{\delta}{2},$$

and

$$p\big(1 + \check{f}_\delta(\cdot - b_\delta)\big) = -p(1 + f_\delta) = -\mathfrak{s},$$

so that, $p(\tilde{v}) = p(\tilde{w}_\delta) - \mathfrak{s} = \mathfrak{p}$. Hence, we have,

$$E_{\min}(\mathfrak{p}) \leq E(\tilde{v}) = E(\tilde{w}_\delta) + E\big(1 + \check{f}_\delta(\cdot - b_\delta)\big) \leq E_{\min}(\mathfrak{q}) + \sqrt{2}\mathfrak{s} + \delta,$$

taking $\delta \longrightarrow 0$. This completes the proof of Lemma 3.5. □

Lemma 3.6. *Let* $\mathfrak{p}, \mathfrak{q} \geq 0$. *Then,*

$$E_{\min}\Big(\frac{\mathfrak{p} + \mathfrak{q}}{2}\Big) \geq \frac{E_{\min}(\mathfrak{p}) + E_{\min}(\mathfrak{q})}{2}.$$

Proof. We use the reflexion argument.

For any $f \in W(\mathbb{R}^N)$, and $a \in \mathbb{R}$, we consider the map $T_a^\pm f$, defined by $T_a^\pm f = f \circ P_a^\pm$, where P_a^+ (resp. P_a^-) restricted to the set $\Gamma_a^+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N, x_N \geq a\}$

(resp. the set $\Gamma_a^- = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N, x_N \leq a\}$) is the identity, however its restriction to the set Γ_a^- (resp. Γ_a^+) is the symmetry with respect to the hyperplane of equation $x_n = a$. That is,

$$\begin{aligned} T_a^+ f(x_1, \ldots, x_N) =& f(x_1, \ldots, x_N) \text{ if } x_N \geq a, \\ T_a^+ f(x_1, \ldots, x_N) =& f(x_1, \ldots, 2a - x_N) \text{ if } x_N \leq a. \end{aligned}$$

Similarly, we can define $T_a^- f$,

$$\begin{aligned} T_a^- f(x_1, \ldots, x_N) =& f(x_1, \ldots, x_N) \text{ if } x_N \leq a, \\ T_a^- f(x_1, \ldots, x_N) =& f(x_1, \ldots, 2a - x_N) \text{ if } x_N \geq a. \end{aligned}$$

Notice that $T_a^{\pm} f$ belongs to $W(\mathbb{R}^N)$, and

$$E(T_a^{\pm} f) = 2E(f, \Gamma_a^{\pm}), \text{ and } p(T_a^{\pm} f) = 2\left(\frac{1}{2}\int_{\Gamma_a^{\pm}} \langle i\partial_1 f, f - 1\rangle\right).$$

Moreover,

$$p(T_a^+ f) + p(T_a^- f) = 2p(f),$$

also observe, that the function $a \mapsto p(T_a^+ f)$ is continuous, that is,

$$\text{given any } a_n \longrightarrow a, \text{ we have } p(T_{a_n}^+ f) \longrightarrow p(T_a^+ f).$$

This observation can be show by Lebesgue's theorem,

$$|p(T_{a_n}^+ f) - p(T_a^+ f)| = |\int_{x_N \geq a_n} \langle i\partial_1 f, f - 1\rangle - \int_{x_N \geq a} \langle i\partial_1 f, f - 1\rangle| \longrightarrow 0.$$

Moreover, by Lebesgue's theorem, we have the following,

$$\begin{cases} p(T_a^+ f) = \int_{x_N \geq a} \langle i\partial_1 f, f - 1\rangle = \int_{\mathbb{R}^N} \langle i\partial_1 f, f - 1\rangle (1 - \chi_{\{x_N < a\}}) \longrightarrow 0 \text{ as } a \longrightarrow +\infty, \\ p(T_a^+ f) = \int_{x_N \geq a} \langle i\partial_1 f, f - 1\rangle \longrightarrow 2p(f) \text{ as } a \longrightarrow -\infty. \end{cases}$$

Therefore, by continuity, for every $\alpha \in (0, p(f))$, there exists a number $a \in \mathbb{R}$, such that $p(T_a^+ f) = 2\alpha$ and by $p(T_a^+ f) + p(T_a^- f) = 2p(f)$, we have,

$$p(T_a^+ f) = 2\alpha, \text{ and } p(T_a^- f) = 2p(f) - 2\alpha.$$

Next, by Lemma 3.2, for any $\mathfrak{p}, \mathfrak{q} \geq 0$ and any $\delta > 0$, there exists a map $v \in W(\mathbb{R}^N)$, such that

$$p(v) = \frac{\mathfrak{p}+\mathfrak{q}}{2}, \text{ and } E(v) \leq E_{\min}\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) + \frac{\delta}{2},$$

taking $f = v$ and $\alpha = \frac{\mathfrak{p}}{2}$, there exists $a \in \mathbb{R}$, such that

$$p(T_a^+ v) = \mathfrak{p}, \text{ and } p(T_a^- v) = \mathfrak{q},$$

then, we get,

$$p(T_a^+ v) = \mathfrak{p} \implies E_{\min}(\mathfrak{p}) \leq E(T_a^+ v) \leq 2E(v, \Gamma_a^+),$$

and

$$p(T_a^- v) = \mathfrak{q} \implies E_{\min}(\mathfrak{q}) \leq E(T_a^- v) \leq 2E(v, \Gamma_a^-).$$

Adding these relations, we obtain,

$$E_{\min}(\mathfrak{p}) + E_{\min}(\mathfrak{q}) \leq 2E(v, \Gamma_a^-) + 2E(v, \Gamma_a^+) = 2E(v) \leq 2E_{\min}\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) + \delta,$$

taking $\delta \longrightarrow 0$, we obtain the results. □

Corollary 3.7. *The function $\mathfrak{p} \mapsto E_{\min}(\mathfrak{p})$ is concave and non-decreasing on $\mathbb{R}_+$.*

Proof. A continuous function f satisfying the inequality,

$$f\Big(\frac{\mathfrak{p}+\mathfrak{q}}{2}\Big) \geq \frac{f(\mathfrak{p}) + f(\mathfrak{q})}{2},$$

must be concave.

Also, a concave non-negative function on $\mathbb{R}_+$, is non-decreasing, thus, by Lemma 3.5 and 3.6, $E_{\min}$ is concave and non-decreasing on $\mathbb{R}_+$. □

Proof of Theorem 3 completed. Combine Lemma 3.5 and Corollary 3.7, finally we need to show $\Xi(\mathfrak{p}) \longrightarrow +\infty$, as $\mathfrak{p} \longrightarrow +\infty$ (the existence of $\mathfrak{p}_0$ being a consequence of the properties of $\Xi(\mathfrak{p}) \equiv \sqrt{2}\mathfrak{p} - E_{min}(\mathfrak{p})$).

$$E_{\min}(\mathfrak{p}) \leq 2\pi \ln(\mathfrak{p}) + K, \text{ as } \mathfrak{p} \longrightarrow +\infty. \tag{3.1}$$

For case $N = 3$,

$$E_{\min}(\mathfrak{p}) \sim \pi\sqrt{\mathfrak{p}} \ln(\mathfrak{p}), \text{ as } \mathfrak{p} \longrightarrow +\infty, \tag{3.2}$$

so that, $\Xi(\mathfrak{p}) \sim \sqrt{2}\mathfrak{p}$, as $\mathfrak{p} \longrightarrow +\infty$. □

3.2 Proof of Lemma 1 : Control the speed $c(u_{\mathfrak{p}})$

Given $\mathfrak{p} > 0$, and assume that $E_{\min}$ is achieved by a solution $u = u_{\mathfrak{p}}$. Then follows by Lagrange multiplier theorem, there exists speed $c = c(u_{\mathfrak{p}})$ s.t.

$$cDp(u_{\mathfrak{p}}) = DE(u_{\mathfrak{p}}),$$

where Dp and DE denote the Fréchêt differentials of p and E.
Notice that the equation (TWc) is the Euler-Lagrange equation for the constrained minimization problem $E_{\min}$.
For any $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$Dp(u_{\mathfrak{p}})(\psi) = \int_{\mathbb{R}^N} \langle i\partial_1 u_{\mathfrak{p}}, \psi\rangle \text{ and } DE(u_{\mathfrak{p}})(\psi) = -\int_{\mathbb{R}^N} \langle \Delta u_{\mathfrak{p}} + u_{\mathfrak{p}}(1 - |u_{\mathfrak{p}}|^2), \psi\rangle.$$

In order to use the Lagrange Multiplier theorem we need to claim $Dp(u_{\mathfrak{p}}) \neq 0$.
If we formally take $\psi_0 = u_{\mathfrak{p}} - 1$, then we can see $Dp(u_{\mathfrak{p}})(\psi_0) = 2\mathfrak{p} \neq 0$. But, ψ_0 does not belong to space $C_c^\infty(\mathbb{R}^N)$, so by density of smooth functions with compact support in $V(\mathbb{R}^N)$, the claim holds.
Thus we can let ψ_1 be a function in $C_c^\infty(\mathbb{R}^N)$, such that

$$Dp(u_{\mathfrak{p}})(\psi_1) = 1 \quad (\text{i.e. } DE(u_{\mathfrak{p}})(\psi_1) = c).$$

Consider the curve $\gamma : \mathbb{R} \mapsto W(\mathbb{R}^N)$, defined by $\gamma(t) = u_{\mathfrak{p}} + t\psi_1$. Since the functions E and p are smooth on $W(\mathbb{R}^N)$, using Taylor formula, we have,

$$p(\gamma(t)) = \mathfrak{p} + \mathfrak{s}, \text{ where } \mathfrak{s} = t + p(\psi_1)t^2,$$

$$E(\gamma(t)) = E(u_{\mathfrak{p}}) + DE(u_{\mathfrak{p}})(t\psi_1) + \underset{t\longrightarrow 0}{\mathcal{O}}(t^2) = E_{\min}(\mathfrak{p}) + ct + \underset{t\longrightarrow 0}{\mathcal{O}}(t^2),$$

since $p(\gamma(t)) = \mathfrak{p} + \mathfrak{s} \implies E_{\min}(\mathfrak{p} + \mathfrak{s}) \leq E(\gamma(t))$, so we have,

$$E_{\min}(\mathfrak{p} + \mathfrak{s}) - E_{\min}(\mathfrak{p}) \leq E(\gamma(t)) - E_{\min}(\mathfrak{p}) \leq c(u_{\mathfrak{p}})\mathfrak{s} + \underset{\mathfrak{s}\longrightarrow 0}{\mathcal{O}}(\mathfrak{s}^2).$$

For $\mathfrak{s} > 0$,

$$\frac{E_{\min}(\mathfrak{p} + \mathfrak{s}) - E_{\min}(\mathfrak{p})}{\mathfrak{s}} \leq c(u_{\mathfrak{p}}) + \underset{\mathfrak{s}\longrightarrow 0}{\mathcal{O}}(\mathfrak{s}).$$

Taking $\mathfrak{s} \longrightarrow 0^+$, since E_{min} is concave, thus, the left right derivatives both exist,

$$\frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq c(u_{\mathfrak{p}}).$$

Similarly, for $\mathfrak{s} < 0$ and take $\mathfrak{s} \longrightarrow 0^-$,

$$\frac{E_{\min}(\mathfrak{p}+\mathfrak{s}) - E_{\min}(\mathfrak{p})}{\mathfrak{s}} \geq c(u_{\mathfrak{p}}) + \underset{\mathfrak{s}\longrightarrow 0}{\mathcal{O}}(\mathfrak{s}),$$

$$c(u_{\mathfrak{p}}) \leq \frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big).$$

3.3 Proof of Lemma 2 : The property of affine energy $E_{min}(\mathfrak{p})$

We consider two numbers $0 \leq \mathfrak{p}_1 < \mathfrak{p}_2$ and assume throughout this section that $E_{\min}$ is "affine" on the interval $(\mathfrak{p}_1, \mathfrak{p}_2)$, that is,

$$E_{\min}\big(\theta\mathfrak{p}_1 + (1-\theta)\mathfrak{p}_2\big) = \theta E_{\min}(\mathfrak{p}_1) + (1-\theta)E_{\min}(\mathfrak{p}_2), \forall\theta \in [0,1].$$

Lemma 3.8. *Assume E_{min} is affine on $(\mathfrak{p}_1, \mathfrak{p}_2)$ and for some $0 \leq \mathfrak{p}_1 < \mathfrak{p} < \mathfrak{p}_2$, the infimum $E_{\min}(\mathfrak{p})$ is achieved by some function $u_{\mathfrak{p}}$. Then, we have,*

$$c(u_{\mathfrak{p}}) = \frac{E_{\min}(\mathfrak{p}_2) - E_{\min}(\mathfrak{p}_1)}{\mathfrak{p}_2 - \mathfrak{p}_1}.$$

Moreover, $0 < c(u_p) < \sqrt{2}$.

Proof. The identity in the equation is a direct consequence of Lemma 1. Also notice that, by Lemma 3.5 and the monotonicity of $E_{\min}$, we have the inequality $0 \leq c(u_p) \leq \sqrt{2}$.

Assume $c(u_{\mathfrak{p}}) = 0$, the concavity of $E_{\min}$ implies that, for all $\mathfrak{q} \geq \mathfrak{p}_2$,

$$\frac{E_{min}(\mathfrak{q}) - E_{min}(\mathfrak{p}_1)}{\mathfrak{q} - \mathfrak{p}_1} \leq \frac{E_{min}(\mathfrak{p}_2) - E_{min}(\mathfrak{p}_1)}{\mathfrak{p}_2 - \mathfrak{p}_1} = 0 \implies 0 \leq E_{\min}(\mathfrak{q}) \leq E_{\min}(\mathfrak{p}_1)$$

since E_{min} is Lipschitz on $\mathbb{R}_+$ implies continuous on $[0, \mathfrak{p}_2]$, so $E_{\min}$ is bounded on $\mathbb{R}_+$. But, this contradicts to (3.1) or (3.2), for $N = 2$ or 3.

Assume $c(u_{\mathfrak{p}}) = \sqrt{2}$, then by E_{min} is affine on $(\mathfrak{p}_1, \mathfrak{p}_2)$,

for any $\tilde{\mathfrak{p}} \in (\mathfrak{p}_1, \mathfrak{p}_2)$,

$$\frac{d}{d\mathfrak{p}}E_{min}(\tilde{\mathfrak{p}}) = \frac{E_{\min}(\mathfrak{p}_2) - E_{\min}(\mathfrak{p}_1)}{\mathfrak{p}_2 - \mathfrak{p}_1} = c(u_{\mathfrak{p}}) = \sqrt{2},$$

again, by the concavity of $E_{\min}$, the left and right derivatives of $E_{\min}$ are both decreasing as $\mathfrak{p}$ increases, so that, for any $\mathfrak{p} \in (0, \mathfrak{p}_1]$,

$$\frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big), \text{ and } \frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \geq \frac{d}{d\mathfrak{p}} E_{min}(\tilde{\mathfrak{p}}) = \sqrt{2},$$

also, by Theorem 3, $E_{min}(\mathfrak{p})$ has Lipschitz constant $\sqrt{2}$,

$$0 \leq \frac{d^+}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq \frac{d^-}{d\mathfrak{p}}\big(E_{\min}(\mathfrak{p})\big) \leq \sqrt{2},$$

that is, for $\mathfrak{p} \in (0, \mathfrak{p}_2)$,

$$\frac{dE_{min}(\mathfrak{p})}{d\mathfrak{p}} = \sqrt{2}.$$

By integration, we obtain that $E_{\min}(\mathfrak{p}) = \sqrt{2}\mathfrak{p}$ on $(0, \mathfrak{p}_2)$, but by Corollary 2.11, $\Sigma(u_{\mathfrak{p}}) = \sqrt{2}\mathfrak{p} - E(u_{\mathfrak{p}}) = 0 \implies u_{\mathfrak{p}}$ is constant, that is, $\mathfrak{p} = p(u_{\mathfrak{p}}) = 0$, which leads to a contradiction. □

Lemma 3.9. *Assume E_{min} is affine on $(\mathfrak{p}_1, \mathfrak{p}_2)$ and for some $0 \leq \mathfrak{p}_1 < \mathfrak{p} < \mathfrak{p}_2$, the infimum $E_{\min}(\mathfrak{p})$ is achieved by $u_{\mathfrak{p}}$. Let $\mathfrak{s}$ be such that $(\mathfrak{p} - \mathfrak{s}, \mathfrak{p} + \mathfrak{s}) \subset (\mathfrak{p}_1, \mathfrak{p}_2)$. Then, there exists some number $a(\mathfrak{s}) \in \mathbb{R}$, such that*

$$E\big(T^{\pm}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = E_{\min}(\mathfrak{p} \pm \mathfrak{s}), \text{ and } p\big(T^{\pm}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = \mathfrak{p} \pm \mathfrak{s},$$

that is, $E_{\min}(\mathfrak{p} \pm \mathfrak{s})$ is achieved by $T^{\pm}_{a(\mathfrak{s})} u_{\mathfrak{p}}$. Moreover, $\mathfrak{s} \mapsto a(\mathfrak{s})$ is decreasing.

Proof. By Lemma 3.6 and continuity of p, we can choose the value $a(\mathfrak{s})$, so that $p\big(T^{+}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = \mathfrak{p} + \mathfrak{s}$, also by $p\big(T^{+}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) + p\big(T^{-}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = 2p(u_{\mathfrak{p}}) = 2\mathfrak{p}$, we have,

$$p\big(T^{+}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = \mathfrak{p} + \mathfrak{s}, \;\; p\big(T^{-}_{a(\mathfrak{s})} u_{\mathfrak{p}}\big) = \mathfrak{p} - \mathfrak{s},$$

which yields a decreasing function $\mathfrak{s} \mapsto a(\mathfrak{s})$. It follows that,

$$E_{\min}(\mathfrak{p} \pm \mathfrak{s}) \leq E(T^{\pm}_{a(\mathfrak{s})} u_{\mathfrak{p}}) = 2E(u_{\mathfrak{p}}, \Gamma^{\pm}_{a(\mathfrak{s})}).$$

Adding two relations above, we obtain,

$$E_{\min}(\mathfrak{p} - \mathfrak{s}) + E_{\min}(\mathfrak{p} + \mathfrak{s}) \leq 2E(u_{\mathfrak{p}}, \Gamma^{-}_{a(\mathfrak{s})}) + 2E(u_{\mathfrak{p}}, \Gamma^{+}_{a(\mathfrak{s})}) = 2E(u_{\mathfrak{p}}) = 2E_{\min}(\mathfrak{p}).$$

On the other hand, by assumption E_{min} is affine, $E_{\min}(\mathfrak{p}-\mathfrak{s})+E_{\min}(\mathfrak{p}+\mathfrak{s})=2E(u_{\mathfrak{p}})$, which is only possible if we have "=" holds. That is,

$$E\big(T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}\big)=E_{\min}(\mathfrak{p}\pm\mathfrak{s}) \text{ and } p\big(T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}\big)=\mathfrak{p}\pm\mathfrak{s},$$

so that, $E_{\min}(\mathfrak{p}\pm\mathfrak{s})$ is achieved by $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$. □

Corollary 3.10. *Assume E_{min} is affine on $(\mathfrak{p}_1,\mathfrak{p}_2)$ and for some $0\le\mathfrak{p}_1<\mathfrak{p}<\mathfrak{p}_2$, the infimum $E_{\min}(\mathfrak{p})$ is achieved by some map $u_{\mathfrak{p}}$. Then,*
(i) *There exist real numbers $a_1\neq a_2$, such that*

$$\partial_N u_{\mathfrak{p}}=0, \ \ on\ \mathbb{R}^{N-1}\times(a_1,a_2).$$

(ii) *The infimum $E_{\min}(\mathfrak{p})$ is achieved for any $0\le\mathfrak{p}_1<\mathfrak{p}<\mathfrak{p}_2$.*

Proof. Since for every $\mathfrak{s}$ such that $(\mathfrak{p}-\mathfrak{s},\mathfrak{p}+\mathfrak{s})\subset(\mathfrak{p}_1,\mathfrak{p}_2)$, the infimum $E_{\min}(\mathfrak{p}\pm\mathfrak{s})$ is achieved by $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$ by Lemma 3.9, so $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$ solves (TWc) for some $c\ge 0$, and hence is smooth by Lemma 2.1.
Since $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$ is obtained through a reflexion of $u_{\mathfrak{p}}$ along the hyperplane of equation $x_n=a(\mathfrak{s})$, we have the following,

$$T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}} \text{ is in class } C^1 \text{ if and only if } \partial_N u_{\mathfrak{p}}=0 \text{ on } \mathbb{R}^{N-1}\times\{a(\mathfrak{s})\}.$$

For statement (i) :
Since $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$ is smooth, $T^{\pm}_{a(\mathfrak{s})}u_{\mathfrak{p}}$ is in class $C^1 \implies \partial_N u_{\mathfrak{p}}=0$ on $\mathbb{R}^{N-1}\times\{a(\mathfrak{s})\}$ for $\mathfrak{s}$ satisfies $(\mathfrak{p}-\mathfrak{s},\mathfrak{p}+\mathfrak{s})\subset(\mathfrak{p}_1,\mathfrak{p}_2)$. Let $\mathfrak{s}$ vary, we obtain the result.
For statement (ii) :
We notice that the infimum $E_{\min}(\mathfrak{q})$ is achieved for every $\mathfrak{q}\in(\mathfrak{p}-\mathfrak{s},\mathfrak{p}+\mathfrak{s})$, where $\mathfrak{s}$ satisfies $(\mathfrak{p}-\mathfrak{s},\mathfrak{p}+\mathfrak{s})\subset(\mathfrak{p}_1,\mathfrak{p}_2)$. The conclusion then follows from a continuity argument. □

Proof of Lemma 2 completed. Assume E_{min} is affine on $(\mathfrak{p}_1,\mathfrak{p}_2)$, and $E_{min}(\mathfrak{p})$ is achieved in $W(\mathbb{R}^N)$, for some $0\le\mathfrak{p}_1<\mathfrak{p}<\mathfrak{p}_2$, say $u_{\mathfrak{p}}\in W(\mathbb{R}^N)$. Also, by Lemma 3.1, we have $W(\mathbb{R}^N)\subset\mathcal{E}(\mathbb{R}^N)$, so $u_{\mathfrak{p}}$ has finite energy. By Corollary 3.10, there exist real numbers $a_1\neq a_2$, such that

$$\partial_N u_{\mathfrak{p}}=0 \text{ on } \mathbb{R}^{N-1}\times(a_1,a_2).$$

By Lemma 3.8 : $c < \sqrt{2}$, so we can apply Theorem 2.17 of analyticity of $u_{\mathfrak{p}}$, thus, $\partial_N u_{\mathfrak{p}}$ also analytic and bounded on $\mathbb{R}^N$ by Lemma 2.1, so by Liouville's theorem : $\partial_N u$ is constant on R^N, therefore, combine with previous, we have,

$$\partial_N u_{\mathfrak{p}} = 0 \text{ on } \mathbb{R}^N,$$

that is, $u_{\mathfrak{p}}$ does not depend on x_N, say $u_{\mathfrak{p}} = f(x_1, x_2, ..., x_{N-1})$.
Since the energy is finite,

$$\infty > \int_{\mathbb{R}^N} |\nabla u_{\mathfrak{p}}|^2 dx = \Big(\int_{\mathbb{R}^{N-1}} |\nabla f|^2 dx_1 d_2 ... dx_{N-1} \Big) \Big(\int_{\mathbb{R}} 1 dx_N \Big)$$

that means $\nabla f = 0$, so $u_{\mathfrak{p}}$ is constant, i.e. $\mathfrak{p} = 0$, which gives a contradiction. □

3.4 An upper bound for $E_{\min}(\mathfrak{p})$ and speed $c(u_{\mathfrak{p}}^n)$

In this subsection, we state the following Lemma which gives an upper bound for $E_{min}(\mathfrak{p})$ in dimension 2, and the Theorem which gives an upper bound for speed $c(v)$ on torus T_n^N. For more detail, see Béthuel [4, Lemma 3.9 and Theorem 4.2].

Lemma 3.11. *Assume $N = 2$. There exists some universal constant K_0, such that we have the upper bound,*

$$E_{\min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p} - \frac{48\sqrt{2}}{\mathcal{S}_{KP}^2}\mathfrak{p}^3 + K_0 \mathfrak{p}^4$$

for any $\mathfrak{p}$ sufficiently small. Here, $\mathcal{S}_{KP}$ denotes the action $S(w)$ of the ground-state solutions w to equation $\partial_1 w - w\partial_1 w - \partial_1^3 w + \partial_1^{-1}(\partial_2^2 w) = 0$.

Theorem 3.12. *Assume $N = 2$ or $N = 3$, and let $E_0 > 0$ and $\Sigma_0 > 0$ be given. Let v be a non-trivial finite energy solution to (TWc) in $X_n^2 \cap \mathcal{S}_n^0$, resp. X_n^3, with $c = c(v) \in \mathbb{R}$, $E_n(v) \leq E_0$ and*

$$0 < \Sigma_0 \leq \Sigma_n(v).$$

Then, there is some constant $n_0 \in \mathbb{N}$ depending only on E_0 and Σ_0, such that, if $n \geq n_0$, then

$$|c(v)| \leq K \frac{E_n(v)}{|\Sigma_n(v)|},$$

where $K > 0$ is some universal constant.

We will use Theorem 3.12 to control $c(u_{\mathfrak{p}}^n)$ in Proposition 1.

4 Proofs for the main results

4.1 Proof of Proposition 1 : The existence of a minimizer on $\mathbb{T}_n^N$

Step 1. we show that there exists a minimizer for $E_{min}^n(\mathfrak{p})$.

Since $E_{min}^n(\mathfrak{p}) = \inf\{E_n(u) | u \in \Gamma_n^N(\mathfrak{p})\}$, by definition of inf, there exists $w_k \in \Gamma_n^N(\mathfrak{p})$ s.t. $E_n(w_k) \longrightarrow E_{min}^n(\mathfrak{p})$ decreasingly as $k \to \infty$, that is, $0 \leq E_n(w_k) \leq E_n(w_1)$, which means $E_n(w_k)$ is bounded with respect to k, thus,

$$\{w_k\} \text{ is bounded in } H^1(\mathbb{T}_n^N).$$

Now, since the bounded sequence has a weak convergent subsequence, by passing possibly to a subsequence, we may say,

$$\exists u_{\mathfrak{p}}^n \in H^1(\mathbb{T}_n^N) \text{ s.t. } w_k \rightharpoonup u_{\mathfrak{p}}^n \text{ in } H^1(\mathbb{T}_n^N), \text{ as } k \to \infty,$$

by Rellich's compactness theroem : $H^1(\mathbb{T}_n^N) \hookrightarrow_{cp} L^2(\mathbb{T}_n^N)$, $H^1(\mathbb{T}_n^N) \hookrightarrow_{cp} L^4(\mathbb{T}_n^N)$, since $\{w_k\}$ is bounded in $H^1(\mathbb{T}_n^N)$, implies there exists a subsequence converge in L^2 and L^4, again up to subsequence, we may write w_k, then by direct calculation we have,

$$p_n(u_{\mathfrak{p}}^n) = \lim_{k \to \infty} p_n(w_k) = \mathfrak{p},$$

by weak lower semi-continuity, we have,

$$E_n(u_{\mathfrak{p}}^n) \leq \liminf_{k \longrightarrow \infty} E_n(w_k) = E_{min}^n(\mathfrak{p}).$$

The proof of N=2 and N=3 are very different.

Case 1. For dimension $N = 3$:

since $p_n(u_{\mathfrak{p}}^n) = \mathfrak{p}$, so $u_{\mathfrak{p}}^n \in \Gamma_n^N(\mathfrak{p})$ implies $E_n(u_{\mathfrak{p}}^n) \geq E_{min}^n(\mathfrak{p})$ (i.e.$E_n(u_{\mathfrak{p}}^n) = E_{min}^n(\mathfrak{p})$), that is, $u_{\mathfrak{p}}^n$ is the minimizer for $\mathcal{P}_n^N(\mathfrak{p})$, by Lagrange multiplier method (since $Dp_n(u_{\mathfrak{p}}^n) \neq 0$ on $\Gamma_n^N(\mathfrak{p})$), we have,

$$\exists c_{\mathfrak{p}}^n \in \mathbb{R} \text{ s.t. } DE_n(u_{\mathfrak{p}}^n) = c_{\mathfrak{p}}^n Dp_n(u_{\mathfrak{p}}^n),$$

also, the previous equation is exactly the weak formulation of the equation :

$$ic_{\mathfrak{p}}^n\partial_1 u_{\mathfrak{p}}^n + \Delta u_{\mathfrak{p}}^n + u_{\mathfrak{p}}^n(1-|u_{\mathfrak{p}}^n|^2) = 0 \text{ on } \mathbb{T}_n^N,$$

whose finite energy solutions are smooth by standard elliptic theory.

Case 2. For dimension $N=2$:

Our goal is to prove that $u_{\mathfrak{p}}^n$ is a minimizer for $(\mathcal{P}_n^2(\mathfrak{p}))$, it remains to show that,

$$u_{\mathfrak{p}}^n \in \mathcal{S}_n^0.$$

In order to prove this, we are going to show that a suitable choice of the minimizing sequence yields strong converge to $u_{\mathfrak{p}}^n$. This will yield the conclusion, in view of the closeness of $\mathcal{S}_n^0 \cap E_{n,\Lambda}$ for any fixed Λ, where $E_{n,\Lambda} = \{u \in H^1(\mathbb{T}_n^2) | E_n(u) \leq \Lambda\}$ is a sublevel set of $H^1(\mathbb{T}_n^2)$. The main tool is Ekeland's variational principle (see [12]). Consider some number Λ, such that $\Lambda > E_{\min}(\mathfrak{p})$. By Corollary 3.3, there exists some integer $n(\Lambda)$, such that $E_{\min}^n(\mathfrak{p}) < \Lambda$ for any $n \geq n(\Lambda)$.

Using Ekeland's variational principle, we can construct a minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $(\mathcal{P}_n^2(\mathfrak{p}))$, such that

$$E_{\min}^n(\mathfrak{p}) \leq E_n(w_k) < \Lambda, \forall k \in \mathbb{N}, \tag{4.1}$$

and

$$E_n(w_k) - E_n(w) \leq \frac{1}{k}\|w_k - w\|_{H^1(\mathbb{T}_n^2)}, \forall w \in \Gamma_n^2(\mathfrak{p}), \forall k \in \mathbb{N}^*. \tag{4.2}$$

Now, given $\delta > 0$, and $\psi \in H^1(\mathbb{T}_n^2)$, by Almeida [1, Theorem 6] and (4.1), the function $w_k - \delta\psi$ belongs to $E_{n,\Lambda} \cap \mathcal{S}_n^0$ for any δ sufficiently small, and any n sufficiently large. Moreover,

$$p_n(w_k - \delta\psi) = p_n(w_k) - \delta\int_{\mathbb{T}_n^2}\langle i\partial_1 w_k, \psi\rangle + \delta^2 p_n(\psi) \longrightarrow \mathfrak{p}, \text{ as } \delta \longrightarrow 0,$$

so that, the function $z_{k,\delta} = \sqrt{\frac{\mathfrak{p}}{p_n(w_k-\delta\psi)}}(w_k - \delta\psi)$ belongs to $\Gamma_n^2(\mathfrak{p})$ for δ sufficiently small. Setting $w = z_{k,\delta}$ in inequality (4.2), and taking the limit $\delta \longrightarrow 0$ after dividing by δ, we get,

$$\lambda_k Dp_n(w_k)(\psi) - DE_n(w_k)(\psi) \leq \frac{1}{k}\Big\|\frac{Dp_n(w_k)(\psi)}{2}w_k - \psi\Big\|_{H^1(\mathbb{T}_n^2)},$$

where $\lambda_k = \frac{1}{2\mathfrak{p}} DE_n(w_k)(w_k)$.

By (4.1) and $H^1(T_n^2)$–norm of w_k can be bounded by $E(w_k)$ up to some constant, this gives,

$$\Big|\lambda_k Dp_n(w_k)(\psi) - DE_n(w_k)(\psi)\Big| \leq \frac{K(\Lambda)}{k} \|\psi\|_{H^1(\mathbb{T}_n^2)},$$

where $K(\Lambda)$ is some constant, only depending on Λ. In particular, choosing $\psi = u_{\mathfrak{p}}^n$, we have,

$$\lambda_k Dp_n(w_k)(u_{\mathfrak{p}}^n) - DE_n(w_k)(u_{\mathfrak{p}}^n) \longrightarrow 0, \text{ as } k \longrightarrow +\infty.$$

Moreover, we have,

$$Dp_n(w_k)(u_{\mathfrak{p}}^n) \longrightarrow 2p_n(u_{\mathfrak{p}}^n) = 2\mathfrak{p}, \text{ as } k \longrightarrow +\infty,$$

and

$$DE_n(w_k)(u_{\mathfrak{p}}^n) \longrightarrow \int_{\mathbb{T}_n^2} \Big(|\nabla u_{\mathfrak{p}}^n|^2 - |u_{\mathfrak{p}}^n|^2(1 - |u_{\mathfrak{p}}^n|^2)\Big), \text{ as } k \longrightarrow +\infty,$$

so that,

$$\lambda_k \longrightarrow \frac{1}{2\mathfrak{p}} \int_{\mathbb{T}_n^2} \Big(|\nabla u_{\mathfrak{p}}^n|^2 - |u_{\mathfrak{p}}^n|^2(1 - |u_{\mathfrak{p}}^n|^2)\Big), \text{ as } k \longrightarrow +\infty,$$

also,

$$2\mathfrak{p}\lambda_k = DE_n(w_k)(w_k) = \int_{\mathbb{T}_n^2} |\nabla w_k|^2 - \int_{\mathbb{T}_n^2} |w_k|^2(1 - |w_k|^2).$$

Hence, by Rellich's compactness theorem, we have,

$$\int_{\mathbb{T}_n^2} |\nabla w_k|^2 \longrightarrow 2\mathfrak{p} \lim_{k \longrightarrow +\infty} (\lambda_k) + \int_{\mathbb{T}_n^2} |u_{\mathfrak{p}}^n|^2(1 - |u_{\mathfrak{p}}^n|^2) = \int_{\mathbb{T}_n^2} |\nabla u_{\mathfrak{p}}^n|^2, \text{ as } k \longrightarrow +\infty,$$

thus,

$$\int_{\mathbb{T}_n^2} |\nabla w_k - \nabla u_{\mathfrak{p}}^n|^2 \longrightarrow 0, \text{ as } k \longrightarrow +\infty,$$

this proves the strong H^1-convergence of the sequence $(w_k)_{k\in\mathbb{N}}$ to $u_{\mathfrak{p}}^n$. Moreover, $E_{n,\Lambda} \cap \mathcal{S}_n^0$ is closed by Almeida [1, Theorem 6] and $w_k \in E_{n,\Lambda} \cap \mathcal{S}_n^0$, so we may get $u_{\mathfrak{p}}^n \in E_{n,\Lambda} \cap \mathcal{S}_n^0 \subset \mathcal{S}_n^0$, so that $u_{\mathfrak{p}}^n$ is a minimizer for $(\mathcal{P}_n^2(\mathfrak{p}))$. Moreover, the set $\{u \in H^1(\mathbb{T}_n^2), \text{ s.t. } E_n(u) < \Lambda\} \cap \mathcal{S}_n^0$ is open by Almeida [1, Theorem 6], so that the Lagrange multiplier rule implies,

$$ic_{\mathfrak{p}}^n \partial_1 u_{\mathfrak{p}}^n + \Delta u_{\mathfrak{p}}^n + u_{\mathfrak{p}}^n(1 - |u_{\mathfrak{p}}^n|^2) = 0 \text{ on } \mathbb{T}_n^2,$$

for some $c_{\mathfrak{p}}^n \in \mathbb{R}$. Hence, $u_{\mathfrak{p}}^n$ is also smooth in the two-dimensional case.

Step 2. Show that the speed $c^n_{\mathfrak{p}}$ has uniform bound.

(Claim: $\exists K(\mathfrak{p})$ and $n(\mathfrak{p})$ s.t. $|c^n_{\mathfrak{p}}| \leq K(\mathfrak{p})$ for any $n \geq n(\mathfrak{p})$)

For any $\mathfrak{p} > 0$, by Corollary 3.3,

$$\limsup_{n \longrightarrow \infty} E^n_{min}(\mathfrak{p}) \leq E_{min}(\mathfrak{p}),$$

thus, we have the following,

$$\liminf_{n \longrightarrow \infty} \Sigma(u^n_{\mathfrak{p}}) \geq \sqrt{2} \liminf_{n \longrightarrow \infty} p(u^n_{\mathfrak{p}}) - \limsup_{n \longrightarrow \infty} E(u^n_{\mathfrak{p}}) \geq \sqrt{2}\mathfrak{p} - E_{min}(\mathfrak{p}) = \Xi(\mathfrak{p}) > 0,$$

by definition of lim inf and lim sup,

$$\exists n(\mathfrak{p}) \in \mathbb{N} \text{ s.t. } \Sigma(u^n_{\mathfrak{p}}) \geq \Sigma_0 \text{ for some } \Sigma_0 > 0, \forall n \geq n(\mathfrak{p}),$$

$$E_n(u^n_{\mathfrak{p}}) \leq E_{min}(\mathfrak{p}) + 1, \forall n \geq n(\mathfrak{p}),$$

by Theorem 3.12,

$$|c^n_{\mathfrak{p}}| = |c(u^n_{\mathfrak{p}})| \leq K \frac{E_n(u^n_{\mathfrak{p}})}{|\Sigma(u^n_{\mathfrak{p}})|} \leq K \frac{(E_{min}(\mathfrak{p}) + 1)}{\Sigma_0} \equiv K(\mathfrak{p}), \text{ for all } n \geq n(\mathfrak{p}).$$

Step 3. Show that the C^k-norm for solution $u^n_{\mathfrak{p}}$ is uniformly bounded.

Similar to the proof in Lemma 2.1, we have the following :

Let $n \in \mathbb{N}$ and let v be a finite energy solution to (TWc) on $\mathbb{T}^N_n$. There exist some constants $K(N)$ and $K(c, k, N)$ s.t.

$$\|1 - |v|\|_{L^\infty(\mathbb{T}^N_n)} \leq \max\{1, \frac{c}{2}\},$$

$$\|\nabla v\|_{L^\infty(\mathbb{T}^N_n)} \leq K(N)(1 + \frac{c^2}{4})^{\frac{3}{2}},$$

and more generally,

$$\|v\|_{C^k(\mathbb{T}^N_n)} \leq K(c, k, N), \forall k \in \mathbb{N}.$$

Now, since $u^n_{\mathfrak{p}}$ is a finite energy solution to (TWc) on $\mathbb{T}^N_n$, and $\{c^n_{\mathfrak{p}}\}$ is bounded, depending only on $\mathfrak{p}$ by Step 2, so, we finally have,

$$\exists K(k, c^n_{\mathfrak{p}}, \mathfrak{p}) \text{ s.t. } \|u^n_{\mathfrak{p}}\|_{C^k(\mathbb{T}^N_n)} \leq K(c^n_{\mathfrak{p}}, k, \mathfrak{p}) \leq K_k(\mathfrak{p}).$$

4.2 Proof of Proposition 2 : The existence of a finite energy solution

Step 1. Using diagonal argument to find a (TWc) solution on $\mathbb{R}^N$.

Since by Proposition 1, there exists sequence $\{u_{\mathfrak{p}}^n\}$ of finite energy solutions of $(TWc_{\mathfrak{p}}^n)$, with uniformly bounded energy, and such that $\{c_{\mathfrak{p}}^n\}$ is bounded, also,

$$\|u_{\mathfrak{p}}^n\|_{C^k(\mathbb{T}_n^N)} \leq K_k(\mathfrak{p}) \text{ for any } n \in \mathbb{N}.$$

Observe that, $|u_{\mathfrak{p}}^n(x) - u_{\mathfrak{p}}^n(y)| = |(u_{\mathfrak{p}}^n)'(\zeta)||x-y| \leq K_k(\mathfrak{p})|x-y|$ implies $\{u_{\mathfrak{p}}^n\}$ is equi-continuous.

By Ascoli's theorem, consider any compact ball $\overline{B(0,j)}$, where $j \in \mathbb{N}$, there exists a subsequence $\{u_{\mathfrak{p}}^n(j)\}_n = \{u_{\mathfrak{p}}^1(j), u_{\mathfrak{p}}^2(j), u_{\mathfrak{p}}^3(j), ...\} \subset \{u_{\mathfrak{p}}^n\}$ on $B(0,j)$ and a smooth map σ_j on $B(0,j)$, such that

$$u_{\mathfrak{p}}^n(j) \longrightarrow \sigma_j \text{ in } C^k(B(0,j)) \text{ as } n \to +\infty.$$

Now we let $j \to +\infty$ and use the diagonal argument,

$$\begin{array}{cccccccccc}
u_{\mathfrak{p}}^1(1) & u_{\mathfrak{p}}^2(1) & u_{\mathfrak{p}}^3(1) & u_{\mathfrak{p}}^4(1) & . & . & . & . & \sigma_1 & in\ B(0,1) \\
\cup & & & & & & & & & \\
u_{\mathfrak{p}}^1(2) & u_{\mathfrak{p}}^2(2) & u_{\mathfrak{p}}^3(2) & u_{\mathfrak{p}}^4(2) & . & . & . & . & \sigma_2 & in\ B(0,2) \\
\cup & & & & & & & & & \\
u_{\mathfrak{p}}^1(3) & u_{\mathfrak{p}}^2(3) & u_{\mathfrak{p}}^3(3) & u_{\mathfrak{p}}^4(3) & . & . & . & . & \sigma_3 & in\ B(0,3) \\
. & . & . & . & & & & & . & \\
. & . & . & . & & & & & . &
\end{array}$$

where $\{u_{\mathfrak{p}}^n(1)\} \supset \{u_{\mathfrak{p}}^n(2)\} \supset \{u_{\mathfrak{p}}^n(3)\} \ . \ . \ .$

(since $\{u_{\mathfrak{p}}^n(1)\}_{n=1}^{\infty}$ also uniform bounded and equi-continuous, so $\exists \{u_{\mathfrak{p}}^n(2)\}_{n=1}^{\infty} \subset \{u_{\mathfrak{p}}^n(1)\}_{n=1}^{\infty}$ s.t. $u_{\mathfrak{p}}^n(2) \longrightarrow \sigma_2$ in $C^k(B(0,2))$, and proceed in this way.)

First, let j fixed $\{u_{\mathfrak{p}}^n(j)\}_{n=1}^{\infty}$,

$$\text{for } \tfrac{1}{j} > 0\ \exists N_j \text{ s.t. } \forall i \geq N_j,\ \|u_{\mathfrak{p}}^i(j) - u_{\mathfrak{p}}^{N_j}(j)\|_{C^k(B(0,j))} < \tfrac{1}{j}, \text{ (Cauchy)}$$

let K be a compact set in $\mathbb{R}^N$, consider diagonal sequence $\{u_{\mathfrak{p}}^{N_l}(l)\}_{l=1}^{\infty}$.
(Claim: $\{u_{\mathfrak{p}}^{N_l}(l)\}_{l=1}^{\infty}$ converges in $C^k(K)$)
It is enough to show that $\{u_{\mathfrak{p}}^{N_l}(l)\}_{l=1}^{\infty}$ is Cauchy in $C^k(K)$,
given $\epsilon > 0$ $\exists N_0 \in \mathbb{N}$ s.t. $\forall l, m \geq N_0$, $\frac{1}{l} + \frac{1}{m} < \frac{\epsilon}{2}$ and $K \subset B(0, N_0)$,

$\|u_{\mathfrak{p}}^{N_l}(l) - u_{\mathfrak{p}}^{N_m}(m)\|_{C^k(K)}$

$\leq \|u_{\mathfrak{p}}^{N_l}(l) - u_{\mathfrak{p}}^{i}(l)\|_{C^k(K)} + \|u_{\mathfrak{p}}^{i}(l) - u_{\mathfrak{p}}^{j}(m)\|_{C^k(K)} + \|u_{\mathfrak{p}}^{j}(m) - u_{\mathfrak{p}}^{N_m}(m)\|_{C^k(K)}$

$\leq \|u_{\mathfrak{p}}^{N_l}(l) - u_{\mathfrak{p}}^{i}(l)\|_{C^k(B(0,l))} + \|u_{\mathfrak{p}}^{i}(l) - u_{\mathfrak{p}}^{j}(m)\|_{C^k(K)} + \|u_{\mathfrak{p}}^{j}(m) - u_{\mathfrak{p}}^{N_m}(m)\|_{C^k(0,m)}$

$\leq \frac{1}{l} + \frac{1}{m} + \|u_{\mathfrak{p}}^{i}(l) - u_{\mathfrak{p}}^{j}(m)\|_{C^k(K)}$ for any $i, j \geq \max\{N_l, N_m\}$

$\leq \frac{\epsilon}{2} + \|u_{\mathfrak{p}}^{i}(l) - u_{\mathfrak{p}}^{j}(m)\|_{C^k(K)}$ for any $i, j \geq \max\{N_l, N_m\}$

taking $i, j \longrightarrow \infty$

$$\|u_{\mathfrak{p}}^{N_l}(l) - u_{\mathfrak{p}}^{N_m}(m)\|_{C^k(K)} \leq \frac{\epsilon}{2} + \|\sigma_l - \sigma_m\|_{C^k(K)}.$$

Observe that,

$$\sigma_1 = \sigma_2 \ \ in \ \ B(0,1) \quad \sigma_2 = \sigma_3 \ \ in \ \ B(0,2) \quad ... \Longrightarrow \sigma_l = \sigma_m = \sigma_{N_0} \ \ in \ \ B(0, N_0),$$

that is, $\sigma_l = \sigma_m$ on K

$$\|u_{\mathfrak{p}}^{N_l}(l) - u_{\mathfrak{p}}^{N_m}(m)\|_{C^k(K)} \leq \frac{\epsilon}{2} < \epsilon,$$

thus, we have $u_{\mathfrak{p}}^{N_l}(l) \longrightarrow u_{\mathfrak{p}}$ in $C^k(K)$ and $u_{\mathfrak{p}} = \sigma_j$ in $B(0, j)$, for any $j \in \mathbb{N}$.
(using diagonal argument to define a solution $u_{\mathfrak{p}}$ on whole space $\mathbb{R}^N$, by σ_j.)

Step 2. Show that $u_{\mathfrak{p}}$ is a finite energy solution of (TWc) on $\mathbb{R}^N$.

Since $u_{\mathfrak{p}}^n$ satisfies the equation $(TWc_{\mathfrak{p}}^n)$ and the speed $c_{\mathfrak{p}}^n \equiv c(u_{\mathfrak{p}}^n)$ is bounded on $\mathbb{R}$, thus, it has a convergent subsequence, up to subsequence, we may assume from the beginning that $c_{\mathfrak{p}}^n \equiv c(u_{\mathfrak{p}}^n) \longrightarrow c$ for some c, so by taking the limit on equation, we can get σ_j, satisfies (TWc) with speed c. So $u_{\mathfrak{p}}$ is also a solution to (TWc) with speed c. Now, we show it has finite energy.

For any $\mathfrak{p} > 0$, by Corollary 3.3 and Theorem 3,

$$\limsup_{n \longrightarrow \infty} E_n(u_{\mathfrak{p}}^n) = \limsup_{n \longrightarrow \infty} E_{min}^n(\mathfrak{p}) \leq E_{min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p} < +\infty.$$

For any $j \in \mathbb{N}$, since $u_{\mathfrak{p}}^n(j) \longrightarrow \sigma_j$ in $C^k(B(0,j))$ as $n \to +\infty$, so we have

$$u_{\mathfrak{p}}^n(j) \longrightarrow \sigma_j \text{ and } \nabla u_{\mathfrak{p}}^n(j) \longrightarrow \nabla\sigma_j \text{ pointwisely on } B(0,j),$$

by Fatou's Lemma,

$$\int_{B(0,j)} |\nabla\sigma_j|^2 \leq \liminf_{n\longrightarrow\infty} \int_{B(0,j)} |\nabla u_{\mathfrak{p}}^n(j)|^2 \text{ and } \int_{B(0,j)} (1-|\sigma_j|^2)^2 \leq \liminf_{n\longrightarrow\infty} \int_{B(0,j)} (1-|u_{\mathfrak{p}}^n(j)|^2)^2,$$

also

$$\int_{B(0,j)} |\nabla u_{\mathfrak{p}}|^2 = \int_{B(0,j)} |\nabla\sigma_j|^2 \text{ and } \int_{B(0,j)} (1-|u_{\mathfrak{p}}|^2)^2 = \int_{B(0,j)} (1-|\sigma_j|^2)^2,$$

thus, we have,

$$E(u_{\mathfrak{p}}, B(0,j)) \leq \liminf_{n\longrightarrow\infty} E(u_{\mathfrak{p}}^n(j)) \leq \limsup_{n\longrightarrow\infty} E(u_{\mathfrak{p}}^n(j)) \leq \limsup_{n\longrightarrow\infty} E(u_{\mathfrak{p}}^n) \leq E_{min}(\mathfrak{p}),$$

by Monotone Convergence Theorem,

$$E(u_{\mathfrak{p}}) = \lim_{j\longrightarrow\infty} E(u_{\mathfrak{p}}, B(0,j)) \leq E_{min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p} < +\infty.$$

Thus, $u_{\mathfrak{p}}$ is also a finite energy solution to (TWc) with speed c.

Step 3. Now we show such solution $u_{\mathfrak{p}}$ is a non-trivial solution.

By the invariance of the problem on the torus $\mathbb{T}_n^N$. Without loss of generality, we may assume that the infimum of $|u_{\mathfrak{p}}^n|$ is achieved at the point 0, that is,

$$|u_{\mathfrak{p}}^n(0)| = \min_{x\in\mathbb{T}_n^N} |u_{\mathfrak{p}}^n(x)|.$$

For continuous map $v \in \Gamma_n^N(\mathfrak{p})$, $N = 2, 3$, we have $v \in X_n^2 \cap \mathcal{S}_n^0$, resp. $v \in X^3$ and $p_n(v) = \mathfrak{p} > 0$, also for n sufficiently large, by Béthuel [4, Lemma 4.2, 4.4], we have the lifting property for v on $\mathbb{T}_n^N$, i.e. $|v| \geq \frac{1}{2}$ on $\mathbb{T}_n^N$, and follow the argument in Lemma 3, we have,

$$\min_{x\in\mathbb{T}_n^N} |v(x)| \leq \sup\{\frac{1}{2}, 1 - \frac{\Sigma_n(v)}{\sqrt{2}p_n(v)}\},$$

let $v = u_{\mathfrak{p}}^n \in \Gamma_n^N(\mathfrak{p})$, so $\Sigma_n(v) = \sqrt{2}p_n(v) - E(v) = \sqrt{2}\mathfrak{p} - E(u_{\mathfrak{p}}^n) = \Xi(\mathfrak{p})$, and $p_n(v) = \mathfrak{p} > 0$,

$$|u_{\mathfrak{p}}^n(0)| = \min_{x\in\mathbb{T}_n^N} |u_{\mathfrak{p}}^n(x)| \leq \sup\{\frac{1}{2}, 1 - \frac{\Xi(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\}.$$

Taking lim sup on both sides, and by assumption $\Xi(\mathfrak{p}) > 0$:

$$|u_{\mathfrak{p}}(0)| \leq \limsup_{n \longrightarrow \infty} |u_{\mathfrak{p}}^n(0)| \leq \sup\{\frac{1}{2}, 1 - \frac{\Xi(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\} < 1.$$

Since $|u_{\mathfrak{p}}(0)| < 1$, so $|u_{\mathfrak{p}}|$ won't be 1, also $u_{\mathfrak{p}}$ is a finite energy solution, so $u_{\mathfrak{p}}$ won't be 0. That is, $u_{\mathfrak{p}}$ is a non-trivial solution to (TWc).

4.3 Proof of Proposition 3 : The concentration-compactness principle

We need two concentration-compactness results. Our purpose is to carry out the asymptotic analysis of the sequence. For more detail, see Béthuel [4, Theorem 5.1 and Theorem 5.2].

Theorem 4.1. *Assume $N = 2$ or $N = 3$, and let $(v^n)_{n\in\mathbb{N}^*}$ be a sequence of solutions of (TWc) in $X_n^2 \cap \mathcal{S}_n^0$, resp. X_n^3, satisfying*

$$E_n(v^n) \longrightarrow E,\ p_n(v^n) \longrightarrow \mathfrak{p},\ \text{and}\ c(v^n) \longrightarrow c,\ \text{as}\ n \longrightarrow +\infty.$$

Assume moreover, that $E > 0$ and $0 < c < \sqrt{2}$.
Then there exist an integer ℓ_0 depending only on c and E, ℓ non-trivial finite energy solutions $v_1, \ldots, v_\ell$ to (TWc) on $\mathbb{R}^N$ of speed c with $1 \leq \ell \leq \ell_0$, ℓ points $x_1^n, \ldots, x_\ell^n$, and a subsequence of $(v^n)_{n\in\mathbb{N}^*}$ still denoted $(v^n)_{n\in\mathbb{N}^*}$, such that

$$|x_i^n - x_j^n| \longrightarrow +\infty,\ \text{as}\ n \longrightarrow +\infty, \tag{4.3}$$

and

$$v^n(\cdot + x_i^n) \longrightarrow v_i(\cdot)\ \text{in}\ C^k(K),\ \text{as}\ n \longrightarrow +\infty, \tag{4.4}$$

for any $1 \leq i \neq j \leq \ell$, any $k \in \mathbb{N}$, and any compact set $K \subset \mathbb{R}^N$.
Moreover, we have the identities,

$$E = \lim_{n\longrightarrow+\infty} \big(E_n(v^n)\big) = \sum_{i=1}^{\ell} E(v_i),\ \text{and}\ \mathfrak{p} = \lim_{n\longrightarrow+\infty} \big(p_n(v^n)\big) = \sum_{i=1}^{\ell} p(v_i).$$

In Theorem 4.1, the tori $\mathbb{T}_n^N$ are identified with the subdomains Ω_n^N of $\mathbb{R}^N$, using possibly a suitable unfolding, so that convergence of (4.3) makes sense. We will also need a variant of Theorem 4.1 in the sonic case.

Theorem 4.2. *Assume $N = 3$, and let $(v^n)_{n\in\mathbb{N}^*}$ be as in Theorem 4.1 with assumption $c = \sqrt{2}$. Let $0 < \delta < 1$ be given.*
Then there exist an integer ℓ_0 depending only on E, ℓ non-trivial finite energy solutions $v_1, \ldots, v_\ell$ to (TWc) on $\mathbb{R}^N$ of speed $\sqrt{2}$ with $0 \le \ell \le \ell_0$, ℓ points $x_1^n, \ldots, x_\ell^n$, and a subsequence of $(v^n)_{n\in\mathbb{N}^*}$ still denoted $(v^n)_{n\in\mathbb{N}^*}$ such that (4.3) and (4.4) hold. Moreover, there exist real numbers $\mu \ge 0$ and ν such that we have the identities,

$$E = \lim_{n\longrightarrow+\infty} \big(E_n(v^n)\big) = \sum_{i=1}^{\ell} E(v_i) + \mu, \text{ and } \mathfrak{p} = \lim_{n\longrightarrow+\infty} \big(p_n(v^n)\big) = \sum_{i=1}^{\ell} p(v_i) + \nu,$$

and the inequality

$$|\mu - \sqrt{2}\nu| \le K\delta\mu,$$

where K is some universal constant.

Proof of Proposition 3 completed. **Claim : there exists ℓ finite energy solutions $u_1 = u_{\mathfrak{p}}, u_2, u_3, ..., u_\ell$ to (TWc) such that**

$$E_{min}(\mathfrak{p}) = \sum_{i=1}^{\ell} E(u_i) \qquad \mathfrak{p} = \sum_{i=1}^{\ell} p(u_i)$$

and u_i are minimizers of $E_{min}(\mathfrak{p}_i)$, where $\mathfrak{p}_i = p(u_i)$ and $0 < c(u_{\mathfrak{p}}) < \sqrt{2}$.
First, by Proposition 2, the sequence converge to a non-trivial finite energy solution $u_{\mathfrak{p}}$ of (TWc) with speed c, which satisfies in particular $\partial_1 u_{\mathfrak{p}} \neq 0$ (if not, similar to the proof in Lemma 2, $\partial_1 u_{\mathfrak{p}} = 0$ implies $u_{\mathfrak{p}}$ does not depend on x_1, but, since the energy is finite, $u_{\mathfrak{p}}$ must be constant, a contradiction).
Since,

$$ic_{\mathfrak{p}}^n \partial_1 u_{\mathfrak{p}}^n + \Delta u_{\mathfrak{p}}^n + u_{\mathfrak{p}}^n(1 - |u_{\mathfrak{p}}^n|^2) = 0 \text{ on } \mathbb{T}_n^N,$$

and

$$ic\partial_1 u_{\mathfrak{p}} + \Delta u_{\mathfrak{p}} + u_{\mathfrak{p}}(1 - |u_{\mathfrak{p}}|^2) = 0 \text{ on } \mathbb{T}_n^N,$$

also,

$$u_{\mathfrak{p}}^n \longrightarrow u_{\mathfrak{p}} \text{ in } C^k(K), \qquad \partial_1 u_{\mathfrak{p}} \neq 0, \qquad \{c(u_{\mathfrak{p}}^n)\} \text{ is bounded on } \mathbb{R},$$

thus,

$$c_{\mathfrak{p}}^n \equiv c(u_{\mathfrak{p}}^n) \longrightarrow c, \text{ as } n \to +\infty,$$

and from the result in Gravejat [19, Theorem 1], we know that, if $c > \sqrt{2}$ and $u_{\mathfrak{p}}$ is a finite energy solution of (TWc), then $u_{\mathfrak{p}}$ must be a constant solution, a contradiction to Proposition 2. Recall the Pohozaev's type inequality in the proof of Lemma 2.10.

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u_{\mathfrak{p}}|^2 + \frac{N}{4}\int_{\mathbb{R}^N} (1-|u_{\mathfrak{p}}|^2)^2 - c(u_{\mathfrak{p}})(N-1)p(u_{\mathfrak{p}}) = 0,$$

if $c = 0$, then $|u_{\mathfrak{p}}| = 1$ for $N = 2, 3$, with the same argument in Lemma 2.10, $u_{\mathfrak{p}}$ is a constant, contradict to Proposition 2, thus we have,

$$0 < c \leq \sqrt{2},$$

we may assume up to subsequence : $E(u_{\mathfrak{p}}^n) = E_{min}^n(\mathfrak{p}) \longrightarrow \limsup_{n\longrightarrow\infty} E_{min}^n(\mathfrak{p}) \equiv E$.
Case 1. $0 < c < \sqrt{2}$.
Since the sequence $(u_{\mathfrak{p}}^n)_{n\in\mathbb{N}^*}$ satisfies,

$$E_n(u_{\mathfrak{p}}^n) \longrightarrow E,\ p_n(u_{\mathfrak{p}}^n) = \mathfrak{p} \longrightarrow \mathfrak{p}, \text{ and } c(u_{\mathfrak{p}}^n) \longrightarrow c, \text{ as } n \longrightarrow +\infty,$$

if $E = 0$, in the proof of Proposition 1, and Fatou's Lemma,

$$E(u_{\mathfrak{p}}^n) \leq E_{min}^n(\mathfrak{p}) \implies E(u_{\mathfrak{p}}) \leq \liminf_{n\longrightarrow\infty} E(u_{\mathfrak{p}}^n) \leq \liminf_{n\longrightarrow\infty} E_{min}^n(\mathfrak{p}) \leq E = 0,$$

that is, $u_{\mathfrak{p}}$ is a constant, contradiction to Proposition 2.
We may assume $E > 0$ and by Theorem 4.1, we have,

$$\mathfrak{p} = \sum_{i=1}^{\ell} \mathfrak{p}_i \qquad \textit{and} \qquad E = \limsup_{n\longrightarrow\infty} E_{min}^n(\mathfrak{p}) = \sum_{i=1}^{\ell} E(u_i),$$

where $\mathfrak{p}_i = p(u_i)$ and u_i are non-trivial finite energy solutions to (TWc) on $\mathbb{R}^N$.
Also, since $c > 0$ by Lemma 2.10 that $\mathfrak{p}_i = p(u_i) > 0$, so $\mathfrak{p} = \sum_{i=1}^{\ell} \mathfrak{p}_i > 0$,

we may apply Corollary 3.3 : $\limsup_{n\longrightarrow\infty} E^n_{min}(\mathfrak{p}) \leq E_{min}(\mathfrak{p})$ and this yields,

$$\sum_{i=1}^{\ell} E_{min}(\mathfrak{p}_i) \leq \sum_{i=1}^{\ell} E(u_i) = \limsup_{n\longrightarrow\infty} E^n_{min}(\mathfrak{p}) \leq E_{min}(\mathfrak{p}).$$

Combine with Corollary 1, the inequalities become identities, that is

$$E(u_i) = E_{min}(\mathfrak{p}_i) \qquad \textit{and} \qquad \limsup_{n\longrightarrow\infty} E^n_{min}(\mathfrak{p}) = E_{min}(\mathfrak{p}).$$

Thus, u_i are minimizers of $E_{min}(\mathfrak{p}_i)$, where $\mathfrak{p}_i = p(u_i)$ and

$$E_{min}(\mathfrak{p}) = \sum_{i=1}^{\ell} E(u_i), \qquad \mathfrak{p} = \sum_{i=1}^{\ell} p(u_i).$$

Case 2. $c = \sqrt{2}$.

If $N = 2$, then by Gravejat [21, Theorem 1], there is no travelling wave exists.

So, we consider $N = 3$ and apply Theorem 4.2 to the sequence $(u^n_{\mathfrak{p}})_{n\in\mathbb{N}^*}$, with a parameter $\delta > 0$ to be determined later. There exists $\ell \geq 1$ of finite energy solutions u_i of speed $\sqrt{2}$ on $\mathbb{R}^N$, and $\mu \geq 0$, $\nu \geq 0$, such that

$$|\mu - \sqrt{2}\nu| \leq K\delta\mu, \tag{4.5}$$

where K is some universal constant, also,

$$E = \limsup_{n\longrightarrow+\infty} \Big(E^n_{\min}(\mathfrak{p})\Big) = \sum_{i=1}^{\ell} E(u_i) + \mu \quad \text{and} \quad \mathfrak{p} = \sum_{i=1}^{\ell} p(u_i) + \nu. \tag{4.6}$$

Again, since $c > 0$ by Lemma 2.10 that $\mathfrak{p}_i = p(u_i) > 0$, so $\mathfrak{p} > 0$, we then apply Corollary 3.3 to obtain,

$$\sum_{i=1}^{\ell} E_{\min}(\mathfrak{p}_i) + \mu \leq \limsup_{n\longrightarrow+\infty} \Big(E^n_{\min}(\mathfrak{p})\Big) \leq E_{\min}(\mathfrak{p}),$$

by Corollary 2.11, since u_i is non-constant so we have $\Sigma(u_i) < 0$, that is $E(u_i) > \sqrt{2}\mathfrak{p}_i$ implies ,

$$\sqrt{2}\Big(\sum_{i=1}^{\ell} \mathfrak{p}_i\Big) < \sum_{i=1}^{\ell} E(u_i).$$

Combining with (4.5) and (4.6), and noticing that $\mu \leq E_{min}(\mathfrak{p})$, we obtain,

$$\sqrt{2}\mathfrak{p} = \sqrt{2}\Bigg(\sum_{i=1}^{\ell} \mathfrak{p}_i + \nu\Bigg) \leq \sum_{i=1}^{\ell} E(u_i) + \sqrt{2}\nu \leq (E_{min}(\mathfrak{p}) - \mu) + \sqrt{2}\nu \leq E_{\min}(\mathfrak{p})(1 + K\delta),$$

that is,

$$\Xi(\mathfrak{p}) \leq K\delta E_{\min}(\mathfrak{p}).$$

Since δ was arbitrary, we may take $\delta \longrightarrow 0^+$, so that $\Xi(\mathfrak{p}) \leq 0$, which is a contradiction with assumption $\Xi(\mathfrak{p}) > 0$.

Hence, this case does not hold, so we deduce that $0 < c < \sqrt{2}$. □

Remark. *In the proof in Proposition 3, we have proved the identity,*

$$\limsup_{n \longrightarrow +\infty} \left(E_n(u_{\mathfrak{p}}^n)\right) = \limsup_{n \longrightarrow +\infty} \left(E_{\min}^n(\mathfrak{p})\right) = E_{\min}(\mathfrak{p}).$$

4.4 Proof of Theorem 4 : The existence of $E_{min}(\mathfrak{p})$ in $W(\mathbb{R}^N)$

Claim : if $\mathfrak{p} > \mathfrak{p}_0$ where $\mathfrak{p}_0 \geq 0$ is defined in Theorem 3, then $E_{min}(\mathfrak{p})$ is achieved by the map $u_{\mathfrak{p}} \in W(\mathbb{R}^N)$ constructed in Proposition 2.

If $\mathfrak{p} > \mathfrak{p}_0$, by Theorem 3 : $\Xi(\mathfrak{p}) > 0$, so apply Proposition 2,3.

In particular from Proposition 3, there exists integer $\ell \geq 1$ and $\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_\ell > 0$, such that $E_{min}(\mathfrak{p}_i)$ is achieved by some map u_i with,

$$\mathfrak{p} = \sum_{i=1}^{\ell} \mathfrak{p}_i, \qquad \textit{and} \qquad E_{min}(\mathfrak{p}) = \sum_{i=1}^{\ell} E_{min}(\mathfrak{p}_i).$$

(Claim : $\ell = 1$)

Assume $\ell \geq 2$, then by Corollary 1 : E_{min} is linear on $(0, \mathfrak{p})$, by Lemma 2 : $E_{min}(\mathfrak{q})$ is not achieved for any $\mathfrak{q} \in (0, \mathfrak{p})$, in particular, take $\mathfrak{q} = \mathfrak{p}_i$, so $E_{min}(\mathfrak{p}_\mathfrak{i})$ is not achieved, but this contradicts the fact that $E_{min}(\mathfrak{p}_i)$ is achieved by u_i, by Proposition 3.

Thus $\ell = 1$, and we have,

$$\mathfrak{p} = \mathfrak{p}_1 = p(u_1), \qquad \textit{and} \qquad E_{min}(\mathfrak{p}) = E_{min}(\mathfrak{p}_1) = E(u_1),$$

that is, $E_{min}(\mathfrak{p})$ is achieved by the map $u_1 = u_{\mathfrak{p}}$ which belongs to $W(\mathbb{R}^N)$, up to a multiplication by a constant of modulus one, by Corollary 2.7.

4.5 Proof of Main Theorem 1 : The existence results of (TWc) for N = 2

Proof of Theorem 1 completed. **(The existence results of (TWc) for $N = 2$)**

Claim : For $\mathfrak{p} > 0$ there exists a non-constant finite energy solution $u_{\mathfrak{p}} \in W(\mathbb{R}^N)$ to equation (TWc) , with $c = c(u_{\mathfrak{p}})$ s.t.

$$p(u_{\mathfrak{p}}) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_1 u_{\mathfrak{p}}, u_{\mathfrak{p}} - 1\rangle dx = \mathfrak{p}$$

and such $u_{\mathfrak{p}}$ is the solution of the minimization problem

$$E(u_{\mathfrak{p}}) = E_{min}(\mathfrak{p}) = \inf\{E(u)|u \in W(\mathbb{R}^N), p(u) = \mathfrak{p}\}.$$

By Theorem 4, we just need to show that (for $\mathfrak{p} > \mathfrak{p}_0 = 0$),

$$\mathfrak{p}_0 = 0 \text{ if } N = 2.$$

By Theorem 3, this is equivalent to show that,

$$\Xi(\mathfrak{p}) > 0, \ \forall \mathfrak{p} > 0.$$

Since the function Ξ is non-decreasing, it is sufficient to check such property for $\mathfrak{p}$ is sufficiently small.

By Lemma 3.11, for any $\mathfrak{p}$ sufficiently small,

$$\Xi(\mathfrak{p}) \geq \frac{48\sqrt{2}}{\mathcal{S}_{KP}^2}\mathfrak{p}^3 - K_0\mathfrak{p}^4 > 0,$$

thus, we get the result. □

4.6 Proof of Main Theorem 2 : The existence results of (TWc) for N = 3

Lemma 4.3. *Let $N = 3$. We have,*

$$\mathfrak{p}_0 \geq \frac{\mathcal{E}_0}{\sqrt{2}},$$

where $\mathcal{E}_0 > 0$ is the constant in Lemma 2.21 and $\mathfrak{p}_0$ is defined in Theorem 3.

Proof. If not, $\mathfrak{p}_0 < \frac{\mathcal{E}_0}{\sqrt{2}}$. For any $\mathfrak{p}_0 < \mathfrak{p} < \frac{\mathcal{E}_0}{\sqrt{2}}$, $E_{\min}(\mathfrak{p})$ is achieved for some map $u_{\mathfrak{p}}$ by Theorem 4, i.e. $E(u_{\mathfrak{p}}) = E_{min}(\mathfrak{p})$.

Also, by Theorem 3, $E_{\min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p}$, that is, for $\mathfrak{p}_0 < \mathfrak{p} < \frac{\mathcal{E}_0}{\sqrt{2}}$,

$$E(u_{\mathfrak{p}}) = E_{\min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p} < \mathcal{E}_0.$$

But, by Lemma 2,21, we have $E(u_{\mathfrak{p}}) \geq \mathcal{E}_0$, a contradiction. □

Lemma 4.4. *Given any* $\mathfrak{p} > \mathfrak{p}_0$*, let* $u_{\mathfrak{p}}$ *be a minimizer of* $E_{min}(\mathfrak{p})$ *given by Theorem 4. Then, there exists a function* $u_{\mathfrak{p}_0} \in W(\mathbb{R}^3)$*, such that* $u_{\mathfrak{p}} \longrightarrow u_{\mathfrak{p}_0}$ *in* $C^\infty_{\mathrm{loc}}(\mathbb{R}^3)$*, as* $\mathfrak{p} \longrightarrow \mathfrak{p}_0$*, with* $p(u_{\mathfrak{p}_0}) = \mathfrak{p}_0$*, and* $E(u_{\mathfrak{p}_0}) = \sqrt{2}\mathfrak{p}_0$*.*
In particular, $E_{\min}(\mathfrak{p}_0)$ *is achieved.*

Proof. First by Theorem 3, E_{min} is non-decreasing, by Theorem 4, $u_{\mathfrak{p}}$ is a non-trivial finite energy solution to (TWc) on $\mathbb{R}^3$, and by Corollary 2.23, there exists a universal constant $\tilde{K} > 0$ and $\alpha > 0$, such that for any $\mathfrak{p}$, $\mathfrak{p}_0 < \mathfrak{p} < \mathfrak{p}_0 + 1$, we have,

$$\|1 - |u_{\mathfrak{p}}|\|_{L^\infty(\mathbb{R}^3)} \geq \frac{\tilde{K}}{E_{\min}(\mathfrak{p})^\alpha} \geq \frac{\tilde{K}}{E_{\min}(\mathfrak{p}_0 + 1)^\alpha} \equiv K.$$

Without loss of generality, by the invariance of translation, we may assume that,

$$|u_{\mathfrak{p}}(0)| = \inf_{x \in \mathbb{R}^3} |u_{\mathfrak{p}}(x)|,$$

thus, we have,

$$|u_{\mathfrak{p}}(0)| \leq 1 - K < 1.$$

Similar to the proof of Proposition 2, there exists a non-trivial finite energy solution u_1 to (TWc) with $c = \limsup_{\mathfrak{p} \longrightarrow \mathfrak{p}_0} \big(c(\mathfrak{p})\big)$, such that up to a subsequence, we have,

$$u_{\mathfrak{p}} \longrightarrow u_1 \text{ in } C^k(K), \text{ as } \mathfrak{p} \longrightarrow \mathfrak{p}_0,$$

for any compact set K in $\mathbb{R}^3$ and any $k \in \mathbb{N}$.
Moreover, we have,

$$E(u_1) = \liminf_{\mathfrak{p} \longrightarrow \mathfrak{p}_0} (E(u_{\mathfrak{p}})) = E_{\min}(\mathfrak{p}_0) = \sqrt{2}\mathfrak{p}_0, \text{ and } |u_1(0)| \leq 1 - K < 1,$$

so that u_1 is non-trivial. (i.e. take $u_{\mathfrak{p}_0} = u_1$)
Assume $c = \sqrt{2}$, then we apply Theorem 4.2 to the sequence $(u_{\mathfrak{p}})_{\mathfrak{p} > \mathfrak{p}_0}$, with a parameter $\delta > 0$ to be determined later. So, there exists a number $\ell \geq 1$ of finite energy solutions u_i of (TWc) on $\mathbb{R}^3$, and numbers $\mu \geq 0$ and $\nu \geq 0$, such that

$$|\mu - \sqrt{2}\nu| \leq K\delta\mu,$$

$$\mathfrak{p}_0 = \sum_{i=1}^{\ell} p(u_i) + \nu = \sum_{i=1}^{\ell} \mathfrak{p}_i + \nu \quad \text{and} \quad \sqrt{2}\mathfrak{p}_0 = E_{\min}(\mathfrak{p}_0) = \sum_{i=1}^{\ell} E(u_i) + \mu,$$

by Theorem 3, we have,

$$\Xi(\mathfrak{p}_0 - \nu) = 0 \text{ and } \sqrt{2}\nu - (E_{\min}(\mathfrak{p}_0) - \sum_{i=1}^{\ell} E(u_i)) = \sqrt{2}\nu - \mu \leq K\delta\mu,$$

combine all the above,

$$E_{\min}(\mathfrak{p}_0 - \nu) = \sqrt{2}(\mathfrak{p}_0 - \nu) = E_{\min}(\mathfrak{p}_0) - \sqrt{2}\nu \geq \sum_{i=1}^{\ell} E(u_i) - K\delta\mu.$$

that is,

$$E_{min}(\sum_{i=1}^{\ell}\mathfrak{p}_i) \geq \sum_{i=1}^{\ell} E(u_i) - K\delta\mu$$

by Corollary 1, we have,

$$\sum_{i=1}^{\ell} E_{min}(\mathfrak{p}_i) \geq E_{min}(\sum_{i=1}^{\ell}\mathfrak{p}_i),$$

thus, we obtain,

$$E(u_i) \leq E_{\min}(\mathfrak{p}_i) + K\delta\mu \leq \sqrt{2}\mathfrak{p}_i + K\delta E_{\min}(\mathfrak{p}_0).$$

Taking $i = 1$ and letting $\delta \longrightarrow 0^+$, notice that when $i = 1$, $u_i = u_1$ and $\mathfrak{p}_1 = p(u_1)$ do not depend on δ, so we can take it tends to zero.

Hence, we obtain,

$$\Sigma(u_1) \geq 0,$$

by Corollary 2.11, we know that u_1 must be a constant trivial solution, a contradiction.

Hence, we may assume that $c < \sqrt{2}$, and apply Theorem 4.1 to the sequence $(u_{\mathfrak{p}})_{\mathfrak{p} > \mathfrak{p}_0}$. Similar to the proof of Theorem 4, we may obtain $\ell = 1$, $p(u_1) = \mathfrak{p}_0$, and $E_{\min}(\mathfrak{p}_0) = E(u_1)$. □

Remark. *Why do we need to estimate $|u_{\mathfrak{p}}(0)|$ again in Lemma 4.4?*

since in the previous we already have,

$$|u_{\mathfrak{p}}(0)| \leq \limsup_{n \longrightarrow \infty} |u_{\mathfrak{p}}^n(0)| \leq \sup\{\frac{1}{2}, 1 - \frac{\Xi(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\} < 1$$

but,

$$|u_1(0)| = \lim_{\mathfrak{p} \longrightarrow \mathfrak{p}_0} |u_{\mathfrak{p}}(0)| \leq \lim_{\mathfrak{p} \longrightarrow \mathfrak{p}_0} \sup\{\frac{1}{2}, 1 - \frac{\Xi(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\} = 1.$$

This cannot bring us the strictly inequality, $|u_1(0)| < 1$.

Proof of Theorem 2 completed. **(The existence results of (TWc) for $N = 3$)**
Claim : there exists some constant $\mathfrak{p}_0 > 0$ such that
For $\mathfrak{p} \geq \mathfrak{p}_0$, there exists a non-constant finite energy solution $u_\mathfrak{p} \in W(\mathbb{R}^N)$ to equation (TWc) , with $c = c(u_\mathfrak{p})$ s.t.

$$p(u_\mathfrak{p}) = \mathfrak{p} \qquad E(u_{\mathfrak{p}_o}) = E_{min}(\mathfrak{p}_o) = \sqrt{2}\mathfrak{p}_o$$

and, for any $\mathfrak{p}$ satisfies $\mathfrak{p} > \mathfrak{p}_0$,

$$E(u_\mathfrak{p}) = E_{min}(\mathfrak{p}) < \sqrt{2}\mathfrak{p}.$$

Moreover, we have,

$$E_{min}(\mathfrak{p}) = \sqrt{2}\mathfrak{p},$$

for any $\mathfrak{p}$ satisfies $0 < \mathfrak{p} < \mathfrak{p}_0$, and the infimum is not achieved in $W(\mathbb{R}^3)$.
According to Lemma 4.3, we have $\mathfrak{p}_0 > 0$, and by Theorem 4 we also have $E_{\min}(\mathfrak{p})$ is achieved for $\mathfrak{p} > \mathfrak{p}_0$, then from Lemma 4.4, $E_{\min}(\mathfrak{p}_0)$ is achieved. Moreover, $E_{\min}(\mathfrak{p}_0) = \sqrt{2}\mathfrak{p}_0$.
For any $\mathfrak{p}$ satisfies $\mathfrak{p} > \mathfrak{p}_0$, assume $E(u_\mathfrak{p}) = E_{min}(\mathfrak{p}) \geq \sqrt{2}\mathfrak{p}$, but we already have $E_{min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p} \implies E_{min}(\mathfrak{p}) = \sqrt{2}\mathfrak{p}$. Also by Theorem 3, E_{min} is concave and increasing, so we have $E_{\min}$ is affine on $(\mathfrak{p}_0, \mathfrak{p})$, then by Lemma 2, it is not achieved on $(\mathfrak{p}_0, \mathfrak{p})$, this contradicts to Theorem 4.
Now, we proof the last equality.
Similarly, by Theorem 3, $E_{\min}$ is affine on $(0, \mathfrak{p}_0)$, so that it is not achieved on $(0, \mathfrak{p}_0)$ by Lemma 2, and $E_{min}(\mathfrak{p}) = \sqrt{2}\mathfrak{p}$, for any $\mathfrak{p}$, $0 < \mathfrak{p} < \mathfrak{p}_0$. □

5 Future Study on Gross-Pitaevskii equation

Since for the Gross-Pitaevskii equation, there is an imaginary term in it, so a natural thought is to cancel such imaginary term by using change of variable.

$$i\partial_t \Psi = \Delta\Psi + \Psi(1 - |\Psi|^2)$$

(1) Consider $\Psi(x,t) = v((x_1 - ct), x_2, x_3)\exp(-i\frac{c}{2}x_1)$,
then, the profile v becomes,

$$\partial_{11}v - \frac{c^2}{4}v + \partial_{22}v + \partial_{33}v + v(1 - |v|^2) = 0.$$

We can find its Lagrangian,

$$\mathcal{L}[v] \;=\; \frac{1}{2}\int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{c^2}{8}\int_{\mathbb{R}^N} |v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\partial_2 v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\partial_3 v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2.$$

Let,

$$\mathcal{L}[v,\Omega] = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{c^2}{8}\int_\Omega |v|^2 + \frac{1}{4}\int_\Omega (1 - |v|^2)^2, \quad \text{and} \quad \Phi_j[v] = \mathcal{L}[v, B(0,j)].$$

For speed c given,
consider the minimizing problem : $E_{min}(j) = \inf\{\Phi_j[v] | v \in H^1(B_j)\}$, we can show that the inf is attachable.
By the definition of inf, there exists $w_k \in H^1(B_j)$ s.t. $\Phi_j[w_k] \longrightarrow E_{min}(j)$ decreasingly as $k \to \infty$, that is, $0 \le \Phi_j[w_k] \le \Phi_j[w_1]$, meaning $\Phi_j[w_k]$ is uniformly bounded with respect to k, so $\{w_k\}$ is bounded in $H^1(B_j)$.
Now, since a bounded sequence has a weak convergent subsequence,
by passing possibly to a subsequence, we may say,

$$\exists u_j \in H^1(B_j) \text{ s.t. } w_k \rightharpoonup u_j \text{ in } H^1(B_j), \text{ as } k \to \infty,$$

by Rellich's compactness theorem : $H^1(B_j) \hookrightarrow_{cp} L^2(B_j)$.
Since $\{w_k\}$ is bounded in $H^1(B_j)$, implies there exists a subsequence converge in L^2, (i.e. $\exists w_{k_j} \longrightarrow u_j$ in $L^2(B_j)$) again for convenience, we write w_{k_j} as w_k,
by weak lower semi-continuity,

$$\Phi_j[u_j] \le \liminf_{k \longrightarrow \infty} \Phi_j[w_k] = E_{min}(j),$$

that is,

$$\Phi_j[u_j] = E_{min}(j),$$

also, the previous equation is exactly the weak formulation of the equation,

$$\partial_{11}u_j - \frac{c^2}{4}u_j + \partial_{22}u_j + \partial_{33}u_j + u_j(1-|u_j|^2) = 0,$$

whose finite energy solutions are smooth by standard elliptic theory.

Now, using the diagonal argument, we can similarly find a solution u on the whole space $\mathbb{R}^N$, but, we don't know whether it is a non-trivial solution.

The advantage of this Lagrange is nonnegative, but notice that, this is not a travelling wave solution.

(2) Consider $\Psi(x,t) = u(c(x_1-ct), x_2, x_3)\exp(\frac{-ic(x_1-ct)}{2})$,

then the profile u becomes,

$$c^2\partial_{11}u + \frac{1}{4}c^2u + \partial_{22}u + \partial_{33}u + u(1-|u|^2) = 0.$$

We first find the original Lagrangian,

$$\mathcal{L}[u] \;=\; \frac{c^2}{2}\int_{\mathbb{R}^N}|\partial_1 u|^2 - \frac{c^2}{8}\int_{\mathbb{R}^N}|u|^2 + \frac{1}{2}\int_{\mathbb{R}^N}|\partial_2 u|^2 + \frac{1}{2}\int_{\mathbb{R}^N}|\partial_3 u|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(1-|u|^2)^2.$$

Notice that we can rewrite the equation,

$$c^2\partial_{11}u + \partial_{22}u + \partial_{33}u + u((1+\frac{c^2}{4}) - |u|^2) = 0,$$

so we can also rewrite the Lagrangian,

$$\mathcal{L}[u] \;=\; \frac{c^2}{2}\int_{\mathbb{R}^N}|\partial_1 u|^2 + \frac{1}{2}\int_{\mathbb{R}^N}|\partial_2 u|^2 + \frac{1}{2}\int_{\mathbb{R}^N}|\partial_3 u|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(|u|^2 - (1+\frac{c^2}{4}))^2.$$

Although, this Lagrange is nonnegative and is also a travelling wave solution, but, we cannot find a suitable admissible set for the minimizing problem.

(3) Consider $\Psi(x,t) = u(x_1-ct, x_2, x_3)\exp(\frac{-ic(x_1-ct)}{2})$,

then the profile u becomes,

$$\Delta u + \frac{c^2}{4}u + u(1-|u|^2) = 0,$$

or

$$\Delta u + u(1 + \frac{c^2}{4} - |u|^2) = 0.$$

Similarly, we can find its Lagrangian,

$$\mathcal{L}[u] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 + \frac{c^2}{4} - |u|^2)^2,$$

by observing $u(x_1, x_2, x_3)\exp(\frac{-ic(x_1)}{2}) \approx 1 \implies u \approx \exp(\frac{icx_1}{2})$, we decide to rewrite the Lagrangian again,

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{2}\int_{\mathbb{R}^N} |\partial_1(u - \exp\frac{icx_1}{2})|^2 - \frac{c^2}{2}\int_{\mathbb{R}^N} \frac{1}{4}|u - \exp\frac{icx_1}{2}|^2 \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} |\partial_2 u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\partial_3 u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(u - \exp\frac{icx_1}{2})|^2 - \frac{c^2}{8}\int_{\mathbb{R}^N} |u - \exp\frac{icx_1}{2}|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2,\end{aligned}$$

thus, consider the space $u \in W^{1,2}(\mathbb{R}^N) + e^{\frac{icx_1}{2}}$ may be a good choice for the admissible set of minimizing problem, but, the Lagrange could be negative.
For the equation,

$$\Delta u + \frac{c^2}{4}u + u(1 - |u|^2) = 0,$$

we may also consider the following Lagrangian,

$$\mathcal{L}[u] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{c^2}{8}\int_{\mathbb{R}^N} |u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2.$$

We set,

$$\begin{aligned}E(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2, \\ p(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (1 - |u|^2).\end{aligned}$$

Since $|u| \approx 1$, as $|x|$ is large, so we may not consider the momentum $p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2$, in this situation, such $p(u)$ won't be well-defined.
Moreover, our goal is to view c as a Lagrange multiplier, but in this situation, we need to control the Lagrange multiplier and make it positive, this is a problem we still need to conquer.

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