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跡反常的古典有效理論和其在半古典量子重力的應用

Classical effective theory of trace anomaly and its application to semi-classical quantum gravity

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國立臺灣大學博士學位論文
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利用跡反常的古典有效理論和其在半古典量子重力的應用

The classical effective theory of trace anomaly and its application on semi-classical quantum gravity

本論文係沈哲民君（F97222024）在國立臺灣大學物理學系、所完成之博士學位論文，於民國104年12月19日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

我們在這篇論文中探討過去一直被忽略的二維與四維時空中跡反常作用量的邊界效應. 在引進了輔助純量場後化簡得到的局部跡反常作用量可以被利用來推導能量動量張量的量子期望值. 雖然輔助純量場的解中的自由度可以對應來描述物質場的不同量子態早已廣為人知, 但此兩者間的對應關係如何清楚的理解至今仍不甚清楚. 我們證明了在考慮了跡反常作用量的邊界效應後, 這個過去不清楚的對應關係被找到了. 從此，此考慮了邊界效應的跡反常作用量將可作為一成熟獨立的工具，用做為計算彎曲時空量子場論中重整化能量動量張量的一種選擇。同時, 我們也因此發現了跡反常作用量的額外使用限制條件, 那就是跡反常作用量只能被應用在歐拉特徵數為零的時空. 雖然在二維有邊界的時空歐拉特徵數總是為零, 但是在四維時空中卻不保證永遠成立. 最後, 我們把考慮了邊界效應的跡反常作用量應用在幾種常見的時空跟量子重力問題以供做應用的參考例子. 藉此展示把這套方法用來計算彎曲時空的量子效應時是如何的強效. 我們預期這套新方法可以成為一種相當有用的工具, 來研究某些有趣的量子重力問題.

這篇論文是建立 [1,2] 這兩篇工作上。其中 [1] 已經被發表，而 [2] 也將在近期投稿發表。

關鍵字：量子重力; 彎曲空間裡的量子場論; 黑洞的量子效應, 霍金輻射, 黑洞熱力學.

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Abstract

We discuss the boundary effect of anomaly-induced action in 2-dimensional and 4-dimensional spacetime, which is ignored in previous studies. Anomaly-induced action, which gives the stress tensor with the same trace as the trace anomaly, can be represented in terms of local operators by introducing an auxiliary scalar field. Although the degrees of freedom of the auxiliary field can in principle describe the quantum states of the original field, the correspondence between them was unclear. We show that, by considering the boundary effect, the missing correspondence will be restored. Therefore, from now on, this technique has become a mature and independent tool to calculate the renormalized stress tensor in curved spacetime. Also, we find that the anomaly-induced action can only be used for the spacetime with zero Euler characteristic which is in general not true in 4-dimension. As examples, we demonstrate our formalism via several different spacetime and famous quantum gravity issues to show how efficient and powerful this approach is. We expect that our new formalism can become an useful tool to study various interesting quantum gravity effects.

This thesis is based on the works [1, 2]. [1] is already published and [2] is about to be submitted for publication.

Key words: Quantum gravity; Quantum fields in curved spacetime; Quantum aspects of black holes, evaporation, thermodynamics.

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Chapter 1

Conventions and Abbreviations

In this thesis, the convention for the metric signature is $(-+++)$. Follow this convention, the Einstein equation have the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{1.1}$$

where the stress tensor (Energy-momentum tensor) is defined by

$$\frac{-2}{\sqrt{-g}}\frac{\delta S_M[g_{\mu\nu}]}{\delta g^{\mu\nu}} = T_{\mu\nu}, \tag{1.2}$$

where S_M is the action of matter fields.

Some abbreviations are used during the calculation of conformal transformation:

$$\sigma_\mu := \nabla_\mu \sigma = \partial_\mu \sigma, \tag{1.3}$$

$$\sigma^\mu := g^{\mu\nu}\nabla_\nu \sigma, \tag{1.4}$$

$$\Box\sigma := \nabla^2\sigma, \tag{1.5}$$

$$(\nabla\sigma)^2 := \sigma_\mu\sigma^\mu, \tag{1.6}$$

$$(\nabla\sigma)^{2N} := (\sigma_\mu\sigma^\mu)^N, \tag{1.7}$$

$$\sqrt{-g}T \pm \bar{...} := \sqrt{-g}T \pm \sqrt{-\bar{g}}\bar{T}, \tag{1.8}$$

where σ is the conformal transformation parameter to be introduced in Eq.(4.95), and T

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is any arbitrary tensor. Also the abbreviation below,

$$h_1(x)\mathcal{L}h_2(x) \pm h_2(x)\mathcal{L}h_1(x) := h_1(x)\mathcal{L}h_2(x) \pm (h_1(x) \leftrightarrow h_2(x)) \tag{1.9}$$

has been used in Ch.(C), where $\mathcal{L}$ is an arbitrary differential operator and h_1, h_2 are arbitrary scalar functions. There are some other abbreviations we used as follows:

$$A_{\underset{\sqcup}{ab}} := \frac{1}{2!}(A_{ab} - A_{ba}), \tag{1.10}$$

$$A_{\underset{\sqcup\sqcup}{abcd}} = \frac{1}{2!2!}(A_{abcd} - A_{bacd} - A_{abdc} + A_{badc}), \tag{1.11}$$

$$O(\varepsilon;\eta) := O(\varepsilon) + O(\eta), \tag{1.12}$$

$$n^\mu \nabla_\mu \phi := \nabla_n \phi, \tag{1.13}$$

$$\epsilon := n^\mu n_\mu = \pm 1. \tag{1.14}$$

where n_μ is the unit normal vector of the boundaries of the manifolds we considered whose norm, ϵ, is $+1$ and -1 for timelike and spacelike boundaries individually.

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Chapter 2

Introduction

In the absence of a complete theory of quantum gravity, quantum field theory in (classical) curved spacetimes, a well-developed semiclassical approach, has been applied widely to study quantum corrections to general relativity [3]. In this semiclassical approach, the quantum divergences of matter fields can be covariantly renormailzed and gives the (one-loop) effective action. The expectation value of the stress tensor of quantum matter fields can also be derived with this procedure. The result suggests that, even in conformal field theories, a nonzero trace of stress tensor arises due to the symmetry break from the renormalization. This nonzero trace of stress tensor is called the trace (or conformal) anomaly [3, 5, 6].

In principle, we can obtain the expectation value of the stress tensor of quantum matter fields in this semiclassical approach (i.e. the quantum field theory in curved spacetimes). However, we have a practical problem; the calculation is so complicated that there is no explicit expression of the effective stress tensor in general background spacetimes. We need to derive the effective stress tensor individually in each spacetime that we are interested in. Because of the complicated calculations, we usually rely on, for instance, numerical and/or approximation approaches to get the result, even in simple common spacetimes such as Schwarzschild spacetime [7]. One way to tackle with this problem is using the corresponding "anomaly-induced action" [8–10].

The anomaly-induced action which is used to describe the effective action is rebuilt from the divergent terms in the effective action which leads to trace (or conformal) anomaly.

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Although there is no rigorous proof which shows the anomaly-induced action and the effective action from the original semiclassical approach are exactly the same. It can be expected and has been checked in some specific cases that in two-dimensional spacetime the anomaly-induced action can exactly describe the stress tensor of quantum field in vacuum state [11]. This approach has been applied widely to study the quantum stress tensor in curved spacetimes [12, 13], black-hole physics [11–14] and cosmology [15, 16]. The anomaly-induced action is naturally built in non-local form, and can be further localized by introducing an additional auxiliary scalar field [17, 18].

Different solutions of the auxiliary scalar fields could describe the effects of different quantum states of the original conformal matter field. Although there are attempts to find the correspondence between the quantum states of the original field and the solutions of the auxiliary scalar field, so far people have not known the general principle behind it. Therefore, the anomaly action approach is hard to use for deriving the quantum effect from any specific vacuum state although it can be used as an alternative way to do double check.

In our recent work [1, 2], we take into consideration the boundary effect in the discussion of anomaly-induced action, which has been neglected in the previous works [8–12, 19]. Our main discovery from this work is that "boundary contribution dictates the vacuum"! After including the boundary effect, additional boundary constraints for the auxiliary scalar field appears and it indeed recovers the missing reference between the quantum states of the original field and the solutions of the auxiliary scalar field. In other words, by using our modified version of anomaly-induced action, we can derive the exact solutions for a given vacuum state by choosing the corresponding boundaries. Therefore, the anomaly-induced action with boundary effect now really become an independent formalism which can be used to calculate the stress tensor for any specific vacuum state. It thus becomes a powerful and efficient tool to calculate the stress tensor for various quantum gravity problems.

Moreover, after we get the 2-dim anomaly-induced action with boundary effect, recently we generalize this formalism to 4-dim. It turns out that we only need to introduce

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one auxiliary scalar field to get the 4-dim anomaly-induced action instead of two auxiliary scalar fields which is suggested by the previous researches. Also, we find that the anomaly-induced action can only used in the spacetime with zero Euler characteristics. Although this requirement satisfies automatically in bounded 2-dim spacetime, it is in general not true in 4-dim spacetime. Therefore, the usage of 4-dim anomaly-induced action is limited and we should be careful when using the 4-dim one.

The structure of this paper is organized as follows. In Ch.3, we give a short review of general relativity. In Ch.4, we briefly review the quantum field theory in curved spacetime and the knowledge related to trace anomaly. In Ch.5, at first, we introduce the original 2-dim anomaly-induced action. After that, based on our recent work, we propose the modified version of 2-dim anomaly-induced action which contains boundary effect. In Ch.6, we first derive the 4-dim anomaly-induced action without boundary contribution. Then similarly, we rederive 4-dim anomaly-induced action with boundary effect. Finally, we give a summary and discussion.

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Chapter 3

Basis of General Relativity

General Relativity (GR) is Einstein's theory of gravitation. It is a classical theory which does not involve with any quantum effect. The essential idea of it is straightforward: while most forces of nature are represented by a lot of kinds of matter fields defined on spacetime, e.g. electromegnatic field , gravity is inherent in spacetime itself. In the context of GR, rather than introducing some other additional field propagating through spacetime, the dynamical field used to describe gravitation is the metric tensor associated to the curvature of spacetime itself. In other word, the gravity we experience is just a manifestation of the curvature of spacetime. Therefore, "Gravity is Geometry." Follow this insight, Einstein then introduced the field equation of the metric, the so-called Einstein equation, to describe how other forces (matter fileds) interact with gravity. What John Wheeler said before presicely summarizes the key spirit of GR, "spacetime tells matter how to move; matter tells spacetime how to curve". Einstein's GR opens a new door to the study of gravitation, and thus stimulates the development of many direction of Physics research, such as black hole, cosmology and so on. In the following, we will briefly review the basic knowledge of GR.

3.1 The Metric

At first, assuming the spacetime used to describe our world is a 4-dimensional Riemannian differentiable manifold, each point of spacetime can be labelled by a coordinate

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x^k with $k = 0, 1, 2, 3$. Every Riemannian manifold is equipped with a metric tensor $g_{\mu\nu}$, which defines the length of line elements:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.1}$$

For example, in the 3-dimensional Euclidean space, the line element is $ds^2 = dx^2 + dy^2 + dz^2$, the metric tensor is thus $g_{ij} = \text{diag}(1, 1, 1)$; similarly, in the theory of special relativity, Minkowski spacetime is assumed whose the line element is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and the metric tensor is thus $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. In GR, arbitrary metric is allowed, only with a few requirements, e.g. the metric tensor should be symmetric and (usually, but not always) nondegenerate, i.e. its determinant should satisfies $g = \det(g_{\mu\nu}) \neq 0$. This allows us to define the inverse metric $g^{\mu\nu}$ via

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma, \tag{3.2}$$

where δ^μ_σ is the Kronecker delta which is defined $\delta^\mu_\sigma := \text{diag}(1, 1, 1, 1)$. The symmetry of $g_{\mu\nu}$ implies that $g^{\mu\nu}$ is also symmetric. The same as in special relativity, the metric and its inverse can be used to raise or lower indices on tensors. Given two vectors V^μ and W^ν, we can define the inner product of them by

$$g(V, W) = g_{\mu\nu} V^\mu W^\nu. \tag{3.3}$$

A simple common example of a nontrivial metric is as follows,

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{3.4}$$

This metric describes an isotropic and homogeneous expanding universe. This is a special case of a Robertson-Walker metric, which is conformally flat.

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3.2 Geodesics

Given a generic metric $g_{\mu\nu}$ for a manifold, one can define the proper time for a test particle whose trajectory (worldline) is parameterized by λ as $x^{\mu}(\lambda)$. The proper time (for a time-like path) is defined by the functional:

$$\tau = \int \left(-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{1/2} d\lambda, \tag{3.5}$$

where the integral is over the path. Take variation of the functional, one can obtain

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} = 0. \tag{3.6}$$

which is the geodesic equation. In other words, the geodesic equation is resulted from the extremum of the proper time. The quantity, $\Gamma^{\mu}_{\rho\sigma}$, is called the Christoffel symbols, which is important in defining the connection of a metric. It is straightforward to solve the Christoffel symbols in terms of the metric as:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha} (\partial_{\rho} g_{\sigma\alpha} + \partial_{\sigma} g_{\rho\alpha} - \partial_{\alpha} g_{\rho\sigma}). \tag{3.7}$$

Next, we will introduce the idea of covariant derivatives which is generalization of partial derivatives in the flat space. An covariant derivative is an operator that reduces to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitrary manifold. The reason for why we need covariant derivative is obvious; equations such as energy conservation law, $\partial_{\mu} T^{\mu\nu} = 0$, must be generalized to curved space in some way. To begin with, we require a covariant derivative ∇ to be a map from (k, l) tensor to $(k, l+1)$ tensor which has the following tow properties: (1) Linearity: $\nabla(T + S) = \nabla T + \nabla S$; (2) Leibnitz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$. If ∇ is going to obey the Leibnitz rule, it can always be written as the partial derivative plus some linear transformation. It means that, for each direction μ, the covariant derivative ∇_{μ} will be given by the partial derivative ∂_{μ} plus a correction specified by a set of $n \times n$

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matrices $(\Gamma_\mu)^\rho{}_\sigma$. For a vector V^ν, we therefore have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{3.8}$$

Notice that in the second term the index originally on V has moved to Γ, and a new index is summed over. If this is the expression for the covariant derivative of a vector, we should be able to determine the transformation property of $\Gamma^\nu_{\mu\rho}$ by demanding that the lhs of eq. (3.7) to be a (1,1) tensor. That is, by requiring the transformation law,

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu, \tag{3.9}$$

and combine with eq.(3.1) and eq.(3.2), we can obtain the transformation rule for the connection coefficients:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{3.10}$$

It should be careful that this is not a tensor transformation law because the second term on the rhs spoils it. The connection coefficients are not the components of a tensor. They are constructed in such a way that the combination of eq.(3.1) transforms like a tensor, therefore the extra terms in the transformation law of the partial derivative cancels exactly with the Γ's. If we further demand the covariant derivative to have additional two properties: (3) it commutes with contractions: $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$, and (4) it reduces to the partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$, one can then derive the covariant derivative of a one-form ω_ν by using the fact that $\omega_\lambda V^\lambda$ is a scalar and $\nabla_\mu(\omega_\lambda V^\lambda) = \partial_\mu(\omega_\lambda V^\lambda)$ as:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda. \tag{3.11}$$

Note that the covariant derivative is not unique in a manifold, i.e., given a Riemannian manifold with a metric $g_{\mu\nu}$, there are still many choices of connection coefficients which result in well-defined covariant differentiation. However, if we require that (5) the covariant derivative to be torsion-free: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, and (6) metric compatible: $\nabla_\rho g_{\mu\nu} = 0$, then

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there is an unique covariant derivative. The unique set of connection coefficients which satisfies conditions (1)-(6) is called the “Levi-Civita” connection. It is straightforward to solve the Levi-Civita connection coefficients in terms of the metric tensor components as:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{3.12}$$

Note that the Levi-Civita connection coefficients are exactly the same as the Christoffel symbol (Γ’s) in the geodesic equation, eq.(3.6), that is the reason we use the same symbol for these two coefficients.

Now we can define the directional covariant derivative of a given curve $x^\mu(\lambda)$ to be

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu. \tag{3.13}$$

This operator which is defined only along the path, maps a (k, l) tensor to a (k, l) tensor. One can then define parallel transport of the tensor T along the path $x^\mu(\lambda)$. That is, the covariant derivative of T along the path vanishes:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l} = 0. \tag{3.14}$$

This equation is well-defined and known as the equation of parallel transport. For a vector it takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0. \tag{3.15}$$

Consider the tangent vector of the path $x^\mu(\lambda)$, i.e. $V^\mu = dx^\mu/d\lambda$, then if the tangent vector is parallel transported along the curve, it should satisfy the following condition:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{3.16}$$

This is exactly the geodesic equation, eq.(3.6). Hence, a curve is geodesic if it parallel-transports its own tangent vector. This property is commonly taken as the alternative definition of a geodesics.

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3.3 Curvature

Roughly speaking, the concept of curvature is to measure how "non-flat" a manifold is. In fact, parallel transport around a closed loop leaves a vector unchanged in a "flat" manifold. However, parallel transport of a vector around a closed loop in a curved space will lead to a rotation of the vector in general, and this rotation depends on the total curvature enclosed by the loop. It would be more useful to have a local description of the curvature at each point, which is what the Riemann curvature tensor is supposed to provide. Given two vector fields A^μ and B^ν, imagine that we take parallel transport of a vector V^μ to move along the direction A^μ first, and then along B^ν. After that we move it backward along A^μ, and then B^ν, to return to the starting point. This action is indeed coordinate independent, so there must be a tensor which describe how the vector changes after it comes back to its starting point. It will be a linear transformation on a vector, and thus we expect that this linear map, the change of this vector, δV^ρ, will depend on A, B, and V which can be written as:

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu, \tag{3.17}$$

where $R^\rho{}_{\sigma\mu\nu}$ is a $(1, 3)$ tensor known as the Riemann tensor. Recall that the covariant derivative of a vector along a certain direction measures how much the vector changes after parallel transport. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the vector along first direction then the other, and the opposite ordering. Therefore, one obtains that

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_\mu \nu \nabla_\lambda V^\rho, \tag{3.18}$$

$$= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho, \tag{3.19}$$

where we identify the first term as the Riemann tensor,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{3.20}$$

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and the second term as the torsion tensor,

$$T^{\lambda}{}_{\mu\nu} = 2\Gamma^{\lambda}_{[\mu\nu]}. \tag{3.21}$$

We can see that the Riemann tensor measures the part of the commutator of covariant derivatives that is proportional to the vector field while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field and the second derivative doesn't involve at all. For the torsion-free Levi-Civita connection, the torsion tensor simply vanishes. Take the Riemann tensor as a map from three vector fields to a forth one, we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3.22}$$

where $\nabla_X = X^{\mu}\nabla_{\mu}$. Similarly, take the torsion tensor as a map from two vector fields to a third one, we have

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{3.23}$$

There are some properties of the Riemann tensor. Consider $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$, then the (lower-index) Riemann tensor is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \tag{3.24}$$

It is antisymmetric in its first and last two indices:

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} = -R_{\sigma\rho\mu\nu}. \tag{3.25}$$

The sum of cyclic permutations of the last three indices vanishes:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0, \tag{3.26}$$

which is equivalent to

$$R_{\rho[\sigma\mu\nu]} = 0. \tag{3.27}$$

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With some effort, we can prove further

$$R_{[\rho\sigma\mu\nu]} = 0. \tag{3.28}$$

With these useful properties, we can check that the number of independent components within Riemann tensor is $\frac{1}{12}n^2(n^2-1)$. Therefore, in four dimensions the Riemann tensor has 20 independent components.

In addition to the algebraic symmetries, the Riemann tensor also obeys a differential identity,

$$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0, \tag{3.29}$$

which is the so-called Bianchi identity. Take trace of the first and third indices of the Riemann tensor, we can define the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{3.30}$$

The Ricci tensor associated with the Levi-Civita connection is automatically symmetric: $R_{\mu\nu} = R_{\nu\mu}$. Finally, the trace of the Ricci tensor is callled the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}{}_{\mu}. \tag{3.31}$$

The Ricci tensor and Ricci scalar contain all information of the trace of the Riemann tensor, leaving us the trace-free parts. The trace free part of the Riemann tensor is called the Weyl tensor, which is defined by

$$C_{\rho\sigma\mu\nu} := R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R. \tag{3.32}$$

This complex definition is designed to make sure all possible contractions of $C_{\rho\sigma\mu\nu}$ vanish,

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while it retains the symmetry of the Riemann tensor:

$$C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}, \tag{3.33}$$

$$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]}, \tag{3.34}$$

$$C_{\rho[\sigma\mu\nu]} = 0. \tag{3.35}$$

The Weyl tensor is only defined in three or more dimensions, and in three dimension it vanishes identically. One of the most important property of the Weyl tensor is that it is invariant inder conformal transformations. For this reason, it is often known as the conformal tensor.

An especially important form of the Bianchi identity comes from contracting twice of the eq.(3.29):

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R. \tag{3.36}$$

By combining the Einstein tensor which is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{3.37}$$

with the Bianchi identity, we will get

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{3.38}$$

Therefore, one can expect that the Einstein tensor is related to the energy conservation law and it will play the key role in GR.

3.4 Einstein's Equation

Consider a (classical) field theory in which the dynamical variables are a set of fields ϕ_i, with the action S which is expressed as the integral of a lagrangian $\mathcal{L}$ as,

$$S = \int \mathcal{L}(\phi_i, \nabla_{\mu}\phi_i)\sqrt{-g} d^n x. \tag{3.39}$$

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For example, a scalar field theory ϕ in the curved spacetime can be written as

$$S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] \sqrt{-g} d^n x, \tag{3.40}$$

which would lead to an equation of motion

$$\Box \phi - \frac{dV}{d\phi} = 0, \tag{3.41}$$

where the covariant d'Alembertian is $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = \nabla^\mu \nabla_\mu$.

To construct the action for general relativity, note that the dynamical variable is now the metric $g_{\mu\nu}$. Since we know one can choose a coordinate such that the metric is in its canonical form and its first derivatives vanish at each point, the lagrangian scalar should contain at least second order derivatives of the metric for a non-trivial field theory. Because the Riemann tensor itself is second order derivative in the metric and the Ricci scalar is the only independent scalar which can be constructed from the it, the simplest independent scalar which is resulted from the metric and no higher than second in its derivatives, is the Ricci scalar. Hilbert proposed this simplest possible choice of the lagrangian for GR as:

$$S_H = \int \sqrt{-g} R \, d^n x, \tag{3.42}$$

which is known as the Hilbert action (or Einstein-Hilbert action). The equation of motion for the Hilbert action come from variation the action with the metric. By using the facts, $g^{\mu\nu} \delta g_{\nu\rho} = -g_{\mu\nu} \delta g^{\nu\rho}$, the trace formula,

$$\det(M) = \exp \mathrm{Tr}(\ln(M)), \tag{3.43}$$

where M is arbitrary matrix, and the variation of the Christoffel symbol:

$$\delta \Gamma^\sigma_{\mu\nu} = -\frac{1}{2} [2 g_{\lambda(\mu} \nabla_{\nu)} (\delta g^{\lambda\sigma}) - g_{\mu\alpha} g_{\nu\beta} \nabla^\sigma (\delta g^{\alpha\beta})], \tag{3.44}$$

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we can obtain the variation of the Hilbert action with respect to the metric:

$$\delta S_H = \int d^n x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta g^{\mu\nu}. \tag{3.45}$$

Therefore, the equation of motion of the Hilbert action, i.e. the Einstein equation in vacuum, is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{3.46}$$

We derived the Einstein equation in ”vacuum” because we only considered the gravitational part inside the action without any matter contribution. To get full Einstein equation, let’s consider

$$S = \frac{1}{16\pi G} S_H + S_M, \tag{3.47}$$

where S_M is the action for matter (fields). By varying the action with respect to the metric,

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}},$$

one can get the full Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = G_{\mu\nu} = 8\pi G \, T_{\mu\nu}, \tag{3.48}$$

where the energy-momentum tensor for matter is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.49}$$

Note that the conservation law $\nabla^\mu T_{\mu\nu} = 0$ now is consistent with the result come from the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$.

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Chapter 4

Quantum Field Theory in Curved Spacetime

4.1 Quantum Field Theory in Flat Space

In this section, we will review the quantum field theory in flat space briefly. At first, for simplicity we consider scalar field in the following. The Lagrangian density of a scalar field with mass m in D-dimensional Minkowski spacetime takes the form,

$$\mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{4.1}$$

which results in the well-known Klein-Gordon equation,

$$\Box\phi - m^2\phi = 0. \tag{4.2}$$

From the equation of motion above, we can solve the general solutions of ϕ. One of the complete set of solution modes is

$$\{u_{\vec{k}}(x)|u_{\vec{k}}(x) = u_{\vec{k}}(t,\vec{x}) \propto e^{i\vec{k}\vec{x}-i\omega t}, k \in \mathbb{R}\} \tag{4.3}$$

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where $w = \sqrt{k^2 + m^2}$, and $k := |\vec{k}|$. If the modes satisfies

$$\frac{\partial}{\partial_t} u_k(t, \vec{x}) = -i\omega u_k(t, \vec{x}), \ \ \omega > 0, \tag{4.4}$$

it is said to be positive frequency w.r.t. t. Next, given two arbitrary scalar fields, $\phi_1(x)$ and $\phi_1(x)$, we can define the inner product of them as:

$$(\phi_1(x), \phi_2(x)) = -i \int \{\phi_1(x)\partial_t\phi_2^*(x) - \partial_t\phi_1(x)\phi_2^*(x)\} d^{n-1}x, \tag{4.5}$$

$$:= -i \int_{\Sigma_t} \{\phi_1(x) \overleftrightarrow{\partial_t} \phi_2^*(x)\} d^{n-1}x, \tag{4.6}$$

where Σ_t is a spacelike hypersurface at instant t. By using the Klein-Gordon equation, it can be shown that the value of the inner produce is indepedent of t. [1]

Follow the definition of inner product, the solutions in (4.3) can be normalised as:

$$u_k(x) = \frac{1}{\sqrt{2\omega(2\pi)^{n-1}}} e^{i\vec{k}\vec{x} - i\omega t}. \tag{4.7}$$

This make sure solution modes are orthogonal, $(u_{\vec{k}}, u_{\vec{k}'}) = \delta^{(n-1)}(\vec{k} - \vec{k}')$, and thus the modes given in (4.7) now is a orthogonal complete solution set.

In the following, we will quickly go through the standard procedure of canonical quantization. At first, let's impose the equal time commutation relations:

$$[\phi_k(t, \vec{x}), \phi_k(t, \vec{x})] = 0,$$

$$[\pi_k(t, \vec{x}), \pi_k(t, \vec{x})] = 0,$$

$$[\phi_k(t, \vec{x}), \pi_k(t, \vec{x})] = i\delta^{n-1}(\vec{x} - \vec{x}') \tag{4.8}$$

where π is the canonical momentum of ϕ defined by

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} = \partial_t \phi. \tag{4.9}$$

[1] We assume here that the set $\{\Sigma_t | t \in \mathbb{R}\}$ is a foliation of the D-dim Minkowski space.

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Next, we can expand ϕ by the modes (4.7), and quantize this field (Second quantization) as:

$$\phi \to \hat{\phi} = \int d^{n-1}k \ \hat{a}_{\vec{k}} u_{\vec{k}}(x) + \hat{a}^{\dagger}_{\vec{k}} u^{*}_{\vec{k}}(x), \tag{4.10}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the so-called creation and annihilation oparators. Then the equal time commutation relations for ϕ and π are equivalent to

$$\begin{aligned} \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] &= 0, \\ \left[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}\right] &= 0, \\ \left[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}\right] &= \delta^{n-1}(\vec{k} - \vec{k}'). \end{aligned} \tag{4.11}$$

The vacuum is thus defined by

$$\hat{a}_{\vec{k}} \left|0\right\rangle = 0, \ \text{for all } \vec{k}, \tag{4.12}$$

and the number operator is

$$\hat{N}_{\vec{k}} := \hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}}. \tag{4.13}$$

From the Lagrangian (4.1), we can derive the (classical) energy-momentum tensor,

$$T_{\mu\nu}\left[\phi(x), \phi(x)\right] = \frac{-2}{\sqrt{-g}} \frac{\delta \mathscr{L}}{\partial g^{\mu\nu}} = \left(\partial_{\mu}\phi \partial_{\nu}\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\delta}\partial_{\lambda}\phi\partial_{\delta}\phi - \frac{1}{2}m^2\phi^2\eta_{\mu\nu}\right) \tag{4.14}$$

In order to consider the quantum version of $T_{\mu\nu}$, i.e. the expectation value of $\hat{T}_{\mu\nu}$, we should promote it to the operator form[2]

$$T_{\mu\nu}\left[\phi(x), \phi(x')\right] \to \hat{T}_{\mu\nu}\left[\hat{\phi}(x), \phi(\hat{x}')\right], \tag{4.15}$$

[2]It is fine to set $x' = x$ now. However, I prefer to write it in this way here because it is convenient when we consider the (point-splitting) renormalization in next section for curved spacetime.

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and we can then define the Hamitonian and Momentum operators as:

$$\hat{H} \equiv \int_{\Sigma_t} d^{n-1}x \, \hat{T}_{tt} \left[\hat{\phi}(x), \hat{\phi(x)} \right] = \frac{1}{2} \int_{\vec{k}} \Big(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} \Big) \omega, \tag{4.16}$$

$$\hat{P}_i \equiv \int_{\Sigma_t} d^{n-1}x \hat{T}_{ti} \left[\hat{\phi}(x), \hat{\phi(x)} \right] = \int_{\vec{k}} \Big(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} \Big) k_i. \tag{4.17}$$

When we try to compute the expetation value of $\hat{H}$, $\hat{P}_i$, or $\hat{N}_{\vec{k}}$ in a specific physical quantum state, e.g. the vacuum state we defined in Eq.(4.12), we will face the divergence problem. The most common way people developed to tackle with this problem is the normal ordering which is defined as

$$: \hat{a}_k \hat{a}^{\dagger}_k :\equiv \hat{a}^{\dagger}_k \hat{a}_k. \tag{4.18}$$

Take Hamitonian as an example, consider the original form of Hamitoinian operator as follows:

$$\hat{H} = \int_{\vec{k}} \Big(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} \Big) \omega \Rightarrow \langle 0 | \hat{H} | 0 \rangle = 0 + \underbrace{\infty}_{\propto \delta(0)}, \tag{4.19}$$

we find there is a divergent term. However, if we regularize it by normal ordering:

$$: \hat{H} := \int_{\vec{k}} \Big(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} \Big) \omega \Rightarrow, \langle 0 | : \hat{H} : | 0 \rangle = 0 \tag{4.20}$$

the divergent term vanishes and the expectation value of Hamiltonian now become zero which coincides with our expectation for the ”vacuum” state.

4.2 Quantum Field Theory in Curved Space

At first, consider the Lagrangian density of a scalar field in a general D-dimensional curved spacetime which in general takes the form:

$$\mathscr{L} = \sqrt{-g} \, \frac{1}{2} \{ -g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - (m^2 + \xi R) \phi^2 \} \tag{4.21}$$

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which leads to the equation of motion of ϕ,

$$\Box_x \phi - (m^2 + \xi R)\phi = 0 \tag{4.22}$$

where m is mass, ξ is the coupling constant and R is Ricci scalar. Similar to the flat case, we can solve the field equation above and get arbitrary complete sets of solution modes. However, the choice of solution modes in general is not unique and thus for example we can have a solution set as

$$\{u_{\vec{k}}(x) | \vec{k} \in \mathbb{R}^3\} \tag{4.23}$$

or as

$$\{v_{\vec{k}}(x) | \vec{k} \in \mathbb{R}^3\} \tag{4.24}$$

where $u_{\vec{k}}(x) \neq v_{\vec{k}}(x)$. This is because in curved spacetime, there is in general no unique way to define "time" (or time-slicing). [3] Similarly, we can define a modified version of inner product for curved spacetime as

$$(\phi_1, \phi_2) = -i \int \sqrt{\gamma} n^\mu \{\phi_1 \nabla_\mu \phi_2 - \phi_2^* \nabla_\mu \phi_1\} d^{n-1}x \tag{4.25}$$

$$:= -i \int_{\Sigma_t} \sqrt{\gamma} n^\mu \{\phi_1(x) \overleftrightarrow{\nabla}_\mu \phi_2^*(x)\} d^{n-1}x \tag{4.26}$$

where Σ_t is a spacelike hypersurface corresponding to an arbitrary time-slicing (foliation). By the definition of inner product, $\{u_{\vec{k}}(x)\}$, $\{v_{\vec{k}}(x)\}$ can adjusted to the orthogonal solution modes, i.e.

$$(u_{\vec{k}}, u_{\vec{k}'}) = \delta^{(n-1)}(\vec{k} - \vec{k}'), \tag{4.27}$$

$$(v_{\vec{k}}, v_{\vec{k}'}) = \delta^{(n-1)}(\vec{k} - \vec{k}'). \tag{4.28}$$

[3]Although sometimes people prefer to use timelike killing vector to define time, it is in general not necessary. Note that a killing vector η^μ satisfies $\mathcal{L}_\eta g_{\mu\nu} = 0$.

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Next, in order to perform canonical quantization, we impose the similar equal time commutation relations

$$
\begin{aligned}
&[\phi_k(x), \phi_k(x)] = 0, \\
&[\pi_k(x), \pi_k(x)] = 0, \\
&[\phi_k(x), \pi_k(x)] = \frac{i}{\sqrt{-g}}\delta^{(n-1)}(\vec{x} - \vec{x}'),
\end{aligned} \tag{4.29}
$$

where the canonical momentum $\pi = \frac{\partial \mathscr{L}}{\partial(\nabla_0\phi)} = \sqrt{-g}\nabla_0\phi$. We can choose different modes to expand the scalar field and perform the second quantization

$$\hat{\phi} = \int d^{n-1}k\ \hat{a}_{\vec{k}} u_{\vec{k}}(x) + \hat{a}^\dagger_{\vec{k}} u^*_{\vec{k}}(x), \tag{4.30}$$

$$\text{or } = \int d^{n-1}k\ \hat{b}_{\vec{k}} v_{\vec{k}}(x) + \hat{b}^\dagger_{\vec{k}} v^*_{\vec{k}}(x), \tag{4.31}$$

and get the commutation relations equivalent to (4.29) as

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] = 0, \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = 0, \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = \delta^{(n-1)}(\vec{k} - \vec{k}'), \tag{4.32}$$

$$\text{or } \left[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}\right] = 0, \left[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}\right] = 0, \left[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}\right] = \delta^{(n-1)}(\vec{k} - \vec{k}'). \tag{4.33}$$

Also, the different ”vacuum” states and ”number” operators will be defined as

$$\hat{a}_{\vec{k}} \left|0_A\right\rangle = 0, \text{ for all } \vec{k}, \tag{4.34}$$

$$\text{or } \hat{b}_{\vec{k}} \left|0_B\right\rangle = 0, \text{ for all } \vec{k}, \tag{4.35}$$

and

$$\hat{N}^{(A)}_{\vec{k}} = \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}, \tag{4.36}$$

$$\text{or } \hat{N}^{(B)}_{\vec{k}} = \hat{b}^\dagger_{\vec{k}} \hat{b}_{\vec{k}}. \tag{4.37}$$

Similar to flat space, in order to consider quantum effect, we should promote the clas-

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sical stress tensor obtained from Lagrangian (4.21),

$$
\begin{aligned}
T_{\mu\nu} = & (1-2\xi)\nabla_\mu\phi\nabla_\nu\phi + (2\xi - \frac{1}{2})g_{\mu\nu}g^{\lambda\delta}\nabla_\lambda\phi\nabla_\delta\phi - 2\xi\phi\nabla_\mu\nabla_\nu\phi \\
& - \frac{2}{n}\xi g_{\mu\nu}\phi\Box\phi + \xi\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{2(n-2)}{n}\xi R g_{\mu\nu}\right]\phi^2 - 2\left[\frac{1}{4} - (1-\frac{1}{n})\xi\right]m^2\phi^2 g_{\mu\nu},
\end{aligned}
\tag{4.38}
$$

to operator and then compute quantum expectation of it. In curved spacetime, the divergent problem still exists, i.e. $\langle 0|\hat{T}_{\mu\nu}\left[\phi(\hat{x}),\phi(\hat{x})\right]|0\rangle = \infty$, and unfortunately it turns out that the ”normal ordering” process is unable to handle it in curved spacetime. Therefore, during the past decades, many spent a lot effort to deal with it and developed many regularization techniques for it, such as point-splitting regularization, Zeta function regularization, or dimensional regularization and so on. Here, we will briefly introduce one of these methods, the point-splitting regularization:

Let’s consider the two point function depend on x and x' as follows, and take the limit $x' \to x$,

$$
\lim_{x'\to x}\langle 0|\hat{T}_{\mu\nu}\left[\phi(\hat{x}),\phi(\hat{x}')\right]|0\rangle = \lim_{x'\to x}\frac{1}{\epsilon}(\ldots) + \underbrace{(\ldots)}_{:=\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle_{ren}}, \tag{4.39}
$$

where $x' = (t(\epsilon),\vec{x}(\epsilon)), x = (t(0),\vec{x}(0))$ and x, x' should be connected by geodesic to make all formula covariant. We can thus analyse the divergence behavior by expanding (4.39) w.r.t. ε. By removing the divergent part and we will get the finite part which can be used to define $\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle_{ren}$ as shown in (4.39). [4]

4.3 Bogoliubov Transformation

As we mentioned in the last section, there in general exists no unique way to define solution modes and thus results in different choices of $\hat{a}^\dagger$, $\hat{a}$, and “vacuum” states in curved spacetime. Because each solution mode is complete set, we can thus expend one set in

[4] It can be checked that the renormalized stress tensor satisfies the energy conservation law, i.e. $\nabla^\mu\langle T_{\mu\nu}\rangle_{ren} = 0$

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terms of the other one as follows,

$$v_{\vec{k}}(x) = \int_{\vec{k}'} \alpha_{\vec{k}\vec{k}'} u_{\vec{k}'} + \beta_{\vec{k}\vec{k}'} u^*_{\vec{k}'} \tag{4.40}$$

$$u_{\vec{k}}(x) = \int_{\vec{k}'} \alpha^*_{\vec{k}'\vec{k}} v_{\vec{k}'} - \beta_{\vec{k}'\vec{k}} v^*_{\vec{k}'}. \tag{4.41}$$

The transformation from one basis to another is called Bogoliubov Transformation. By using the orthogonal property of the basis modes, the functions $\alpha_{\vec{k}\vec{k}'}$ and $\beta_{\vec{k}\vec{k}'}$, which are called Bogoliubov coefficients, can be expressed as

$$\alpha_{\vec{k}\vec{k}'} = (v_{\vec{k}}, u_{\vec{k}'}), \tag{4.42}$$

$$\beta_{\vec{k}\vec{k}'} = -(v_{\vec{k}}, u^*_{\vec{k}'}). \tag{4.43}$$

Then, we can calculate the expectation value of the ”B” number operator, $\hat{N}^{(B)}$, in the “A”-vacuum,

$$\langle 0_A | \hat{N}^{(B)} | 0_A \rangle_{ren} = \int_{\vec{k}'} |\beta_{\vec{k}\vec{k}'}|^2. \tag{4.44}$$

We can notice that the expectation considered above is in general nonzero, it means that an “empty” vacuum defined by “A”-frame is not a “empty” vacuum from B-frame’s perspective.

4.3.1 Unruh Effect

As an example of the application of Bogoliubov transformation, we will introduce the famous Unruh Effect in this section. At first, for simplicity lets consider a massless, minimal coupling scalar field in 2-dim Minkowski space. We list two different coordinates, ”inertial” frame (t, x), and Rindler frame (T_R, R_R) for 2-dim Minkowski space,

$$ds^2 = -dt^2 + dx^2 = -dU_f dV_f \tag{4.45}$$

$$= -\rho^2 du_f dv_f = -\rho^2(-dT_R{}^2 + dR_R{}^2) \left(= -\rho^2 dT_R{}^2 + d\rho^2\right) \tag{4.46}$$

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where

$$U_f := t - x, \qquad V_f := t + x, \tag{4.47}$$

$$\rho := (x^2 - t^2)^{1/2}, \quad u_f := -\log(-U_f), \quad v_f := \log V_f, \tag{4.48}$$

$$T_R := \frac{1}{2}(v_f + u_f), \quad R_R := \frac{1}{2}(v_f - u_f), \tag{4.49}$$

[5] The solution modes of the scalar field which satisfies the field equation,

$$\Box\phi = 0 \tag{4.50}$$

is

$$u_k(x) = \frac{1}{\sqrt{4\pi|\vec{k}|}} e^{-ik(t-x)}, \tag{4.51}$$

$$\text{or } v_k(x) = \frac{1}{\sqrt{4\pi|\vec{k}|}} e^{-ik(T_R - R_R)}. \tag{4.52}$$

$$\tag{4.53}$$

As mentioned in the previous section, we can quantize the scalar field by expanding it with different solution modes,

$$\hat{\phi} = \int dk \; \hat{a}_{\vec{k}} u_{\vec{k}} + \hat{a}^{\dagger}_{\vec{k}} u^{*}_{\vec{k}}, \tag{4.54}$$

$$\text{or } = \int dk \; \hat{b}_{\vec{k}} v_{\vec{k}} + \hat{b}^{\dagger}_{\vec{k}} v^{*}_{\vec{k}}, \tag{4.55}$$

[5]By considering the trajectory ($T_R, R_R =$ constant), $x^\mu(\tau)$, we can compute the acceleration of it, $a^\mu := \frac{D^2 x^\mu}{d\tau^2} = \frac{d^2 x^\mu}{d\tau^2}$. It turns out that the amplitude of acceleration along this trajectory, $a \equiv \sqrt{a^\mu a_\mu}$, is constant. Therefore, people commonly call the observer moving along this trajectory as ”constantly accelerating” observer and the Rindler frame as the ”accelerating” frame.

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and thus define the different vacua, the so-called ”Minkowski vacuum” and ”Rindler vacuum”, as

$$\hat{a}_k |0_M\rangle = 0, \tag{4.56}$$

$$\hat{b}_k |0_R\rangle = 0. \tag{4.57}$$

We can then calculate the expectation value of the ”Rindler” number operator, $\hat{N}_k^{(R)}$, in the ”Minkowski” vacuum, and it results in

$$\langle 0_M | N_k^R | 0_M \rangle \propto \frac{1}{e^{2\pi\frac{\omega}{a}} - 1}. \tag{4.58}$$

This result is the same as the blackbody spectrum with a temperature

$$T = \frac{a}{2\pi}. \tag{4.59}$$

It means that an accelerating observer feels itself in a thermal bath in Minkowski vacuum state, and this fact is the famous “Unruh effect”.

4.3.2 Hawking Radiation

Similarly, let’s consider the two-dimensional Schwarzschild spacetime in three different frames, (t, r^*), (T_H, R_H) and (T_U, R_U):

$$\begin{aligned} ds^2 &= -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 = (1-\frac{2M}{r})(-dt^2 + dr^{*2}) = -(1-\frac{2M}{r})dudv & (4.60) \\ &= -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV = -\frac{32M^3}{r}e^{-\frac{r}{2M}}(-dT_H{}^2 + dR_H{}^2) & (4.61) \\ &= -\frac{8M^2}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}dUdv = \frac{8M}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}(-dT_U{}^2 + dR_U{}^2), & (4.62) \end{aligned}$$

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where

$$r^* := r + 2M\ln(\frac{r}{2M} - 1), \qquad u := t - r^*, \qquad v := t + r^*, \tag{4.63}$$
$$U := -e^{\frac{-u}{4M}}, \qquad V := e^{\frac{v}{4M}}, \qquad T_H := \frac{1}{2}(V + U), \qquad R_H := \frac{1}{2}(V - U), \tag{4.64}$$
$$T_U := \frac{1}{2}(v + U), \qquad R_U := \frac{1}{2}(v - U). \tag{4.65}$$

We can then quantize the scalar field by using the solution modes corresponding to the three frames,

$$\hat{\phi} = \int_{-\infty}^{\infty} dk\ \hat{a}_{\vec{k}} u_{\vec{k}} + \hat{a}^\dagger_{\vec{k}} u^*_{\vec{k}} \equiv \int_0^{\infty} dk\ \left\{ \hat{A}_{\vec{k}} e^{-iku} + \hat{A}^\dagger_{\vec{k}} e^{-iku} + \hat{B}_{\vec{k}} e^{-ikv} + \hat{B}^\dagger_{\vec{k}} e^{-ikv} \right\}, \tag{4.66}$$

$$\text{or } = \int dk\ \hat{b}_{\vec{k}} v_{\vec{k}} + \hat{b}^\dagger_{\vec{k}} v^*_{\vec{k}}, \tag{4.67}$$
$$\text{or } = \int dk\ \hat{c}_{\vec{k}} s_{\vec{k}} + \hat{c}^\dagger_{\vec{k}} s^*_{\vec{k}}, \tag{4.68}$$

where

$$u_k(x) = \frac{1}{\sqrt{4\pi|\vec{k}|}} e^{-ik(t-r^*)}, \tag{4.69}$$
$$v_k(x) = \frac{1}{\sqrt{4\pi|\vec{k}|}} e^{-ik(T_H-R_H)}, \tag{4.70}$$
$$s_k(x) = \frac{1}{\sqrt{4\pi|\vec{k}|}} e^{-ik(T_U-R_U)}. \tag{4.71}$$

and thus define the three vacua, Boulware, Hartle-Hawing and Unruh vacua:

$$\hat{a}_k \left|0_B\right\rangle = 0, \tag{4.72}$$
$$\hat{b}_k \left|0_H\right\rangle = 0, \tag{4.73}$$
$$\hat{c}_k \left|0_U\right\rangle = 0. \tag{4.74}$$

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From the similar calculation,

$$\langle H|\hat{A}^\dagger \hat{A}|H\rangle = \langle U|\hat{A}^\dagger \hat{A}|U\rangle \propto \frac{1}{e^{8\pi M\omega}-1}, \tag{4.75}$$

we find that when Hartle-Hawing(or Unruh) vacuum is considered, for "static" observer, there is an thermal radiation emitting to outside from black with a temperature

$$T = \frac{1}{8\pi M} \tag{4.76}$$

which is the so-called Hawing temperature. One of the alternative way to realize Hawking radiation is by computing the expectation value for stress tensor in Unruh vacuum state, and the result is

$$\langle U|\hat{T}_{\mu\nu}|U\rangle_{ren} = \frac{1}{24\pi}\begin{pmatrix} \frac{1}{32M^2}+\frac{7M^2}{r^4}-\frac{4M}{r^3} & -\frac{1}{32M^2}\left(1-\frac{2M}{r}\right)^{-1} \\ -\frac{1}{32M^2}\left(1-\frac{2M}{r}\right)^{-1} & \left(1-\frac{2M}{r}\right)^{-2}\left(\frac{-M^2}{r^4}+\frac{1}{32M^2}\right)\end{pmatrix} \tag{4.77}$$

Note that the T_{tr} is nonzero and thus Hawking radiation indeed takes energy out from black hole continuously. [6]

4.4 The Analysis of the Quantum Divergent Behavior through Path-integral Quantization and One-loop Effective Action

Start with the lagrangian of the scalar field I introduced previously:

$$\mathcal{L}_{cl} = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2}(m^2+\xi R)\phi^2, \tag{4.79}$$

[6] For 4-dim case,although the calculation is much more complicated. However a similar result can also be obtained as [20]

$$\langle U|\hat{T}_{\mu\nu}|U\rangle \propto \frac{1}{4\pi r^2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{4.78}$$

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and we can then obtain the one-loop effective action from it by path-integral [7]

$$e^{-S_{eff}} := \int [D\phi] e^{-S_{cl}[\phi]} \tag{4.80}$$

which can lead to the expectation value of $\hat{T}_{\mu\nu}$

$$\frac{-2}{\sqrt{-g}} \frac{\delta S_{eff}[g_{\mu\nu}]}{\delta g^{\mu\nu}} = \frac{\left\langle out, 0 \middle| \hat{T}_{\mu\nu} \middle| 0, in \right\rangle}{\langle out, 0 \,|\, 0, in \rangle} \tag{4.81}$$

where

$$S_{eff} \equiv \int d^n x \sqrt{-g} L_{eff}(x). \tag{4.82}$$

The effective lagrangian including the one-loop contribution for this scalar field can be derived as [3]

$$L_{eff}(x) = \frac{i}{2} \lim_{x \to x'} \int_{m^2}^{\infty} dm'^2 G_F^{DS}(x, x'; m'^2), \tag{4.83}$$

where G_F^{DS} is the DeWitt-Schwinger representation of the Feynman propagator G_F which provides a way to expand $G_F(x, x')$. By the DeWitt-Schwinger expansion, this effective Lagrangian can be expressed as

$$L_{eff}(x) \overset{\text{eq.(6.41) in [3]}}{\approx} \frac{1}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) m^{n-2j} \Gamma(j - \frac{n}{2}), \tag{4.84}$$

[7] In [3], the author use W_{eff} instead of S_{eff} we used here.

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where n is the dimension, a_j's are geometric quantities which is written in : [8]

$$a_0(x) = 1, \tag{4.85}$$

$$a_1(x) = (\frac{1}{6} - \xi)R, \tag{4.86}$$

$$\begin{aligned} a_2(x) &= \frac{1}{2}(\frac{1}{6} - \xi)^2 R^2 + \frac{1}{180}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu}) + \frac{1}{6}(\frac{1}{5} - \xi)\Box R \\ &\overset{\xi=\frac{1}{6}}{=} \frac{1}{120}(\frac{-1}{3}E + F + \frac{2}{3}\Box R) \end{aligned} \tag{4.87}$$

$$\cdots,$$

where E and F are the the Gauss-Bonnet term and the square (contraction) of Weyl tensor which are defined as:

$$E : \; = \; R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{4.88}$$

$$F : \; = \; R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{4.89}$$

$$= \; C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}. \tag{4.90}$$

[9]

The gamma function $\Gamma(j - n/2)$ within the effective Lagrangian diverges when its argument is naught or a negative integer, and thus when considering it in even dimension, some counter terms should be introduced to renormalize these divergent parts which will result in trace anomaly. We will explain it in the following sections. From now on, let us focus on conformal scalar field theory and continue to analyse the divergence part of S_{eff}, i.e. S_{div} for later use. In 2-dim, the first two terms of S_{eff} are divergent which can

[8] Note that the coefficient of $\Box R$ in a_2 here differ from [3] by a minus sign.

[9] Only in exactly 4-dimensional spacetime, the definition of F, eq.(4.89) is equal to eq.(4.90). When we deal with S_{div} in $4+\epsilon$ dimensional spacetime by dimensional regularization, like in eq.(4.102) later, what we should use to describe F is eq.(4.89) instead of eq.(4.90). That is because $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ is only equal to $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ up to leading order in $(4+\epsilon)$-dimensional spacetime.

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be derived as:

$$
\begin{aligned}
L_{div}(x) &= \lim_{n\to 2}\frac{1}{2(4\pi)^{n/2}}m^{n-2}[m^2\Gamma(-\frac{n}{2})a_0(x)+m^0\Gamma(1-\frac{n}{2})a_1(x)],\\
&= \lim_{n\to 2}\frac{1}{2(4\pi)}m^{n-2}[m^2[\frac{2}{n-2}+(\gamma-1)+O(n-2)]a_0(x)+m^0[-\frac{2}{n-2}-\gamma+O(n-2)]a_1(x)],\\
&\xlongequal{m\to 0,\xi=0,(4.92)} \lim_{n\to 2}(-\frac{1}{24\pi}\frac{R}{n-2}-\frac{\gamma R}{48\pi})
\end{aligned}
\tag{4.91}
$$

where γ is Euler-Mascheroni constant which appears from the following expansion of Gamma function near pole,

$$
\begin{cases}
\Gamma(-\frac{n}{2}) = \frac{2}{n-2}+(\gamma-1)+O(n-2),\\
\Gamma(1-\frac{n}{2}) = -\frac{2}{n-2}-\gamma+O(n-2).
\end{cases}
\tag{4.92}
$$

Similarly, in 4-dim, the first three terms of S_{eff} are divergent and it can be derived as:

$$
\begin{aligned}
L_{div}(x) &= \lim_{n\to 4}\frac{1}{2(4\pi)^{n/2}}m^{n-4}[m^4\Gamma(-\frac{n}{2})a_0(x)+m^2\Gamma(1-\frac{n}{2})a_1(x)+m^0\Gamma(2-\frac{n}{2})a_2(x)],\\
&\xlongequal{m\to 0,\xi=\frac{1}{6},(4.94)} \lim_{n\to 4}m^{n-4}(-\frac{1}{16\pi^2}\frac{a_2(x)}{n-4}-\frac{\gamma a_2(x)}{32\pi^2}),
\end{aligned}
\tag{4.93}
$$

where we used the expansion near the pole $n=4$ as follows:

$$
\begin{cases}
\Gamma(-\frac{n}{2}) = -\frac{1}{n-4}+(\frac{3}{4}-\frac{\gamma}{2})+O(n-4),\\
\Gamma(1-\frac{n}{2}) = \frac{2}{n-4}+(\gamma-1)+O(n-4),\\
\Gamma(2-\frac{n}{2}) = -\frac{2}{n-4}-\gamma+O(n-4).
\end{cases}
\tag{4.94}
$$

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4.5 Traceless Stress Tensor in Conformal Invariant Theory

Let's introduce the definition of conformal transformation: [10]

$$\begin{cases} g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = e^{-2\sigma(x)} g_{\mu\nu}(x) = \Omega^{-2}(x) g_{\mu\nu}, \\ g^{\mu\nu}(x) \to \bar{g}^{\mu\nu}(x) = e^{2\sigma(x)} g^{\mu\nu}(x) = \Omega^{2}(x) g_{\mu\nu}, \end{cases} \tag{4.95}$$

which lead to the relations

$$\Rightarrow \delta\bar{g}^{\mu\nu} \equiv (\Omega + \delta\Omega)^2(x) g_{\mu\nu} - \Omega^2(x) g_{\mu\nu} = 2\bar{g}^{\mu\nu}\Omega^{-1}\delta\Omega \overset{\frac{\delta\Omega}{\Omega}=\delta\sigma}{=} 2\bar{g}^{\mu\nu}\delta\sigma. \tag{4.96}$$

By considering the infinitesimal conformal transformation of an action as follows:

$$\begin{aligned} S[\bar{g}_{\mu\nu}(x)] &= S[g_{\mu\nu}(x)] + \int d^n x \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta \bar{g}^{\alpha\beta}} \delta\bar{g}^{\alpha\beta}(x), \\ &= S[g_{\mu\nu}(x)] + \int d^n x \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta \bar{g}^{\alpha\beta}} 2\bar{g}^{\alpha\beta}\delta\sigma, \\ &= S[g_{\mu\nu}(x)] - \int d^n x \sqrt{-\bar{g}}\, T^{\lambda}_{\lambda}[\bar{g}_{\mu\nu}(x)]\delta\sigma, \end{aligned} \tag{4.97}$$

we can then get a relation between the variation of action and the trace of stress tensor:

$$\Rightarrow T^{\lambda}_{\lambda}[\bar{g}_{\mu\nu}] = \frac{-1}{\sqrt{-\bar{g}}}\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\sigma} = \frac{-\Omega}{\sqrt{-\bar{g}}}\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\Omega}, \tag{4.98}$$

$$\Rightarrow T^{\lambda}_{\lambda}[g_{\mu\nu}] = \frac{-1}{\sqrt{-g}}\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\sigma}|_{\sigma=0} = \frac{-\Omega}{\sqrt{-g}}\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\Omega}|_{\Omega=1}. \tag{4.99}$$

Therefore, when the theory we are interested in is conformal invariant, i.e. $\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\sigma}|_{\sigma=0} = 0$, the corresponding stress tensor would be traceless.

[10] The definition of σ here differs from the convention used in [3] by a minus sign.

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4.6 Trace (Conformal) Anomaly due to Renormalization

In order to renormalize the effective Lagrangian, we need to introduce the corresponding counter terms to cancel the divergent parts. The renormalized effective Lagrangian is defined as $L_{ren} := L_{eff} - L_{ct} = L_{eff} - L_{div}$, [11] where L_{ct} is the Lagrangian of the counter terms. After renormalization, the renormalized Lagrangian L_{ren} can be used to derived the renormalized stress tensor $\langle T_{\mu\nu} \rangle_{ren}$, and by the calculation below,

$$\left\langle T^{\lambda}_{\lambda} \right\rangle_{ren} = \frac{-2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta S_{ren}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta (S_{eff} - S_{div})}{\delta g^{\mu\nu}} \overset{\xi=\xi(n)}{=} \frac{+2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta S_{div}}{\delta g^{\mu\nu}} = - \left\langle T^{\lambda}_{\lambda} \right\rangle_{div}, \tag{4.100}$$

we find that the trace of $\left\langle T^{\lambda}_{\lambda} \right\rangle_{ren}$ is equal to $- \left\langle T^{\lambda}_{\lambda} \right\rangle_{div}$, because in a conformal theory, S_{eff} is conformal invariant. In the remaining part of this section, we will continue to show that in 2-dim and 4-dim, $\left\langle T^{\lambda}_{\lambda} \right\rangle_{div}$ is actually nonzero! Therefore, although start from a (classical) conformal theory, the quantum effect actually breaks conformal symmetry and results in a nonzero trace of renormalized stress tensor which is the so-called trace (conformal) anomaly.

In the previous section, (4.4), we already get the divergence part of effective action in 2-dim,

$$S_{div}[g] \overset{(4.91)}{=} -\frac{1}{4\pi} \lim_{n\to 2} \frac{\int d^2x \sqrt{-g} a_1(x)}{n-2} = -\frac{1}{24\pi} \lim_{n\to 2} \frac{\int d^2x \sqrt{-g} R}{n-2}, \tag{4.101}$$

and in 4-dim,

$$S_{div}[g] \overset{(4.93)}{=} -\frac{1}{(4\pi)^2} \lim_{n\to 4} \frac{\int d^4x \sqrt{-g} a_2(x)}{n-4} = -\frac{1}{16\pi^2} \lim_{n\to 4} \frac{\int d^4x \sqrt{-g} [\frac{1}{120}(\frac{-1}{3}E + F + \frac{2}{3}\Box R)]}{n-4}. \tag{4.102}$$

According to eq.(4.99), we can now derive the trace anomaly due to the divergent part of

[11] In fact, L_{ct} can differ from L_{div} by some finite terms, and it results in additional contribution in the (lhs of) semiclassical Einstein equation. However, we will ignore the possible differences in the discussion of this thesis.

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effective action [12]. By the result derived in Sec.(A.2), we have the relation

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^nx\sqrt{-g}R|_{\sigma=0} = -(n-2)R \tag{4.103}$$

and thus get the trace anomaly in 2-dim:

$$\begin{aligned}\left\langle T^{\lambda}_{\lambda}\right\rangle_{div} &= -\frac{R}{24\pi} = -\frac{a_1(x)}{4\pi},\\ \Rightarrow \left\langle T^{\lambda}_{\lambda}\right\rangle_{ren} &= +\frac{R}{24\pi} = +\frac{a_1(x)}{4\pi}.\end{aligned} \tag{4.104}$$

Similarly, in 4-dim, by using those relations

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^nx\sqrt{-g}F|_{\sigma=0} = \frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx\sqrt{-g}F|_{\sigma=0} = -(n-4)(F+\frac{2}{3}\Box R), \tag{4.105}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^nx\sqrt{-g}E|_{\sigma=0} = \frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx\sqrt{-g}E|_{\sigma=0} = -(n-4)E, \tag{4.106}$$

[13] we thus get the trace anomaly in 4-dim:

$$\begin{aligned}\left\langle T^{\lambda}_{\lambda}\right\rangle_{div} &= \frac{-1}{(4\pi)^2}\frac{1}{120}(\frac{-1}{3}E + F + \frac{2}{3}\Box R) = -\frac{a_2(x)}{16\pi^2},\\ \Rightarrow \left\langle T^{\lambda}_{\lambda}\right\rangle_{ren} &= +\frac{a_2(x)}{16\pi^2}.\end{aligned} \tag{4.107}$$

It can further be proved that trace anomaly in any even dimension ($n = 2k$, $k \in \mathbb{N}$) is:

$$\left\langle T^{\lambda}_{\lambda}\right\rangle_{ren} = \frac{-1}{\sqrt{-g}}\frac{\delta S_{ren}[g_{\mu\nu}]}{\delta\sigma}|_{\sigma=0} = -\frac{-1}{\sqrt{-g}}\frac{\delta S_{div}[g_{\mu\nu}]}{\delta\sigma}|_{\sigma=0} = -\left\langle T^{\lambda}_{\lambda}\right\rangle_{div} = +\frac{a^k(x)}{(4\pi)^k}. \tag{4.108}$$

[12] It mentions in eq.(6.107)(6.108) of [3] or eq.(3.36) of [4]

[13] We correct an error appear in [3]. The first result we get here is $-\varepsilon(F+\frac{2}{3}\Box R)$ instead of $-\varepsilon(F-\frac{2}{3}\Box R)$ in [3].

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4.7 The Effective Action for Conformal Fields with Boundary

Due to conformal symmetry, the action for conformal scalar field with boundary contribution (for Neumann boundary condition) should be written as

$$S_{cl}[g] = -\frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\left[\nabla^\mu\phi\nabla_\mu\phi + \xi(n)R\phi^2\right] - \int_\Sigma d^{n-1}x\sqrt{-\gamma}\xi(n)K\phi^2. \quad (4.109)$$

From the variation of this action, we can show the boundary condition needed for ϕ is

$$\left[n^\mu\nabla_\mu - 2\xi(n)K\right]\phi = 0. \quad (4.110)$$

The effective action of quantum fields with various boundary conditions have been studied a lot in the previous works [21–23]. From the results, we know that S_{div} for 2-dim conformal scalar field with Neumann boundary condition (same as Dirichlet boundary condition) is

$$S_{div}[g] = \frac{-1}{24\pi}\frac{\int_{\mathcal{M}} d^2x\sqrt{-g}R + 2\epsilon\int_\Sigma d^1x\sqrt{-\gamma}K}{n-2}, \quad (4.111)$$

which will be used to derive the 2-dim anomaly-induced action in Ch.5.

Also, S_{div} for 4-dim conformal fields with different spins and boundary conditions is

$$\begin{aligned} S_{div}[g] &= \frac{\int_{\mathcal{M}} d^4x\sqrt{-g}\left[b'E + b(F + \frac{2}{3}\Box R)\right] + \epsilon\int_\Sigma d^3x\sqrt{-\gamma}\left[b'E^B - \frac{2}{3}bn^\mu\nabla_\mu R + 8bj_1 + q_2 j_2\right]}{n-4}, \\ &= \frac{\int_{\mathcal{M}} d^4x\sqrt{-g}\left[b'E + bF\right] + \epsilon\int_\Sigma d^3x\sqrt{-\gamma}\left[b'E^B + 8bj_1 + q_2 j_2\right]}{n-4}, \end{aligned} \quad (4.112)$$

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where

$$E^B = -4(RK - 2R_{ab}\gamma^{ab}K + 2R_{acbd}n^a n^b K^{cd} - \frac{2}{3}K^3 + 2KK_2 - \frac{4}{3}K_3), \quad (4.113)$$

$$j_1 = C_{abcd}n^a n^c K^{bd}, \quad (4.114)$$

$$j_2 = K_3 - KK_2 + \frac{2}{9}K^3, \quad (4.115)$$

$$K_2 = K_{ab}K^{ab}, \quad (4.116)$$

$$K_3 = K_a{}^b K_b{}^c K_c{}^a. \quad (4.117)$$

j_1 and j_2 are conformal invariant scalar curvatures and the coefficients b, b', q_2 depend on the number of matter fields with different spin and boundary conditions as:

$$b = -\frac{1}{16\pi^2}\left[\frac{1}{120}(N_S^R + N_S^D) + \frac{1}{20}N_F^m + \frac{1}{10}(N_V^a + N_V^r)\right], \quad (4.118)$$

$$b' = \frac{1}{16\pi^2}\left[\frac{1}{360}(N_S^R + N_S^D) + \frac{11}{360}N_F^m + \frac{31}{180}(N_V^a + N_V^r)\right], \quad (4.119)$$

$$q_2 = \frac{1}{16\pi^2}\left[\frac{2}{35}N_S^R + \frac{2}{45}N_S^D + \frac{2}{7}N_F^m + \frac{16}{35}(N_V^a + N_V^r)\right], \quad (4.120)$$

where N_S^R, N_S^D, N_F^m, N_V^a, and N_V^r are the number of fields with different spin and boundary conditions (B.C.): S (spin-0), F (spin-$\frac{1}{2}$), V (spin-0), R (Robin B.C.), D (Dirichlet B.C.), m (mixed B.C.), a (absolute B.C.), r (relative B.C.).

Notice that the numerator of Eq.(4.111), and Eq.(4.112) are in fact conformal invariant. However, by multiplying it with the infinite term $\frac{1}{\varepsilon} := 1/(n-2)$ for 2-dim (or $1/(n-4)$ for 4-dim), the S_{div} is no long a conformal invariant term and thus remove it will break conformal symmetry. That is indeed the essence of conformal anomaly and the formalism we propose in this thesis is surrounded with it.

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Chapter 5

2-dimensional Anomaly-induced Action

On a curved spacetime, the conformal anomaly appears through the renormalization of the stress tensor. The expectation value of the stress tensor diverges even for the linear field theory[1] and the renormalization is required. The counter terms represented by geometric forms are introduced for the renormalization and, in even-dimansional spacetime, the anomalous contribution appears in the gravitational equation. This contribution violates the conformal symmetry even if the original action for the fields possesses the symmetry, and thus it is called the conformal (or trace) anomaly [3, 5, 6]. Anomaly-induced action, the action rebuilt from this anomalous contribution, is written by the nonlocal and geometric functions and can be further expressed in a local form by introducing scalar fields. The local form of anomaly-induced action is a useful formalism to calculate the stress tensor of various vacua and thermal states. Therefore it can be applicable to the fields such as cosmology, black hole physics and so on.

In this chapter, we will first briefly review the idea of the effective local action for the 2-dim trace anomaly [12, 17]. Next, in our work based on the published paper [1], we will further propose the new version of the anomaly-induced action which is corrected by the boundary effect. It turns out that the choice of vacuum state is naturally linked

[1] This divergence appears even in the flat spacetime, which is the vacuum energy. In a theory without gravity we can just ignore it, because it is coupled only with gravity. In a gravitational theory, however, since it can be the source of gravity, we need to renormalize it. Usually, we assume that the renormalized vacuum energy is tiny, which might explain the acceleration of the Universe. Nevertheless, there is no natural reason for its smallness, which is the well-known cosmological constant problem. This issue is beyond the scope of this paper and we will not dwell on it further.

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to the boundary effect (constraint). By taking the boundary effect into consideration, the modified anomaly-induced action really become an independent formalism to derive stress tensor for specific vacuum states whereas the original one cannot. Therefore, the modified anomaly-induced action become more very powerful and efficient formalism to derive stress tensor of vacuum and thermal states than the original anomaly action. Finally, we will apply our new formalism to rederive various well-known problems to demonstrate the powerful utility of this formalism.

5.1 2-dimensional Anomaly-induced Action without Boundary

In this section, we will review the derivation of the 2-dim anomaly-induced action [12, 17]. We start with the derivation of the non-local action for this anomalous contribution. The Wess-Zumino (WZ) action is useful for this derivation, which is defined as

$$\Gamma_{WZ}[\bar{g}, \sigma] := S[\bar{g}] - S[g], \tag{5.1}$$

with

$$\bar{g}_{\mu\nu} := \exp(-2\sigma) g_{\mu\nu}. \tag{5.2}$$

Due to the conformal symmetry, before introducing counter terms, the action is conformally invariant, i.e. we have $S_{eff}[\bar{g}] = S_{eff}[g]$. This makes the relation of the renormalized WZ action to that for the counter terms as

$$\begin{aligned} \Gamma_{WZ}[\bar{g}, \sigma] &= S_{ren}[\bar{g}] - S_{ren}[g] \\ &= \Big[S_{eff}[\bar{g}] - S_{eff}[g]\Big] - \Big[S_{ct}[\bar{g}] - S_{ct}[g]\Big] \\ &= 0 - \Big[S_{ct}[\bar{g}] - S_{ct}[g]\Big]. \end{aligned} \tag{5.3}$$

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From the WZ action we could read the form of the renormalized action S_{ren}. However, the renormalized action derived from the WZ action has ambiguity; adding conformally invariant terms S_{conf} to the obtained action S_{ren}, the new action $S_{ren}+S_{conf}$ still gives the same WZ action. Meanwhile, all information of the trace anomaly is definitely included in S_{ren}, and thus the renormalized action that we can read from the WZ action is called the anomaly-induced action S_{anom}, i.e.

$$\Gamma_{WZ}[\bar{g},\sigma] = S_{anom}[\bar{g}] - S_{anom}[g]. \tag{5.4}$$

As introduced in Sec.(4.4), the divergent parts of the effective Lagrangian for a conformally coupled scalar field in two-dimensional spacetime, is written as

$$S_{ct} = \frac{-1}{24\pi}\lim_{n\to 2}\int d^2x\sqrt{-g}\frac{R}{(n-2)}. \tag{5.5}$$

As mentioned above, the effective action is conformal invariant. However, this counter term designed in $2+\varepsilon$ dimension is not conformal invariant (even for $n\to 2$)! Therefore after renormalization, the renormalised action is no longer conformal invariant and results in a nonzero trace of energy-momentum tensor [3]. Substituting this counter term into eq. (5.3), we can derive the WZ action as

$$\begin{aligned}
\Gamma_{WZ}[\bar{g},\sigma] &= \frac{1}{24\pi}\lim_{n\to 2}\left[\frac{\int d^2x\sqrt{-\bar{g}}\bar{R} - \int d^2x\sqrt{-g}R}{n-2}\right] \\
&\overset{(A.19)}{=} -\frac{1}{24\pi}\int d^2x\sqrt{-\bar{g}}[\sigma\bar{R} - \sigma\bar{\Box}\sigma] \\
&= \frac{1}{96\pi}\int d^2x\int d^2x'\sqrt{-\bar{g}}\sqrt{-\bar{g}'}\bar{R}(x)\bar{D}_2(x,x')\bar{R}(x') \\
&\quad -\frac{1}{96\pi}\int d^2x\int d^2x'\sqrt{-g}\sqrt{-g'}R(x)D_2(x,x')R(x')
\end{aligned} \tag{5.6}$$

From eq.(5.4) and eq.(5.6), we can now write down

$$S_{anom}[g] = \frac{1}{96\pi}\int d^2x\int d^2x'\sqrt{-g}\sqrt{-g'}R(x)D_2(x,x')R(x'), \tag{5.7}$$

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where D_2 is the inverse operator of D'Alembert operator, i.e.

$$\Box D_2(x, x') = -\frac{\delta(x - x')}{\sqrt{-g}}. \tag{5.8}$$

In the last equality of eq.(5.6), we have imposed the symmetric condition of D_2, i.e. $D_2(x, x') = D_2(x', x)$, and used the relations:

$$2\sqrt{-g}\Box\sigma = 2\sqrt{-\bar{g}}\bar{\Box}\sigma = \sqrt{-\bar{g}}\bar{R} - \sqrt{-g}R, \tag{5.9}$$

$$\Rightarrow \sigma(x) \equiv \frac{1}{2}\int d^2x' D_2(x, x')(\sqrt{-g'}R' - \sqrt{-\bar{g}'}\bar{R}') \tag{5.10}$$

which is obtained in Sec.(A.1.2).

This non-local anomaly-induced action can be localized by introducing a real scalar field φ which is defined as

$$\varphi(x) := \int d^2x' D_2(x, x')R(x'). \tag{5.11}$$

Operating this auxiliary scalar field by the D'Alembert operator, we can obtain its field equation as follows:[2]

$$\Box\varphi = -R. \tag{5.12}$$

Now the localized version of the anomaly-induced action (5.7) can be expressed in terms of the auxiliary scalar field φ as

$$S_{anom}[g, \varphi] = \frac{-1}{96\pi}\int d^2x\sqrt{-g}[g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - 2\varphi R]. \tag{5.13}$$

[2] Although the definition (5.11) seems equivalent to (5.12), for the derivation of eq.(5.11) from eq.(5.12) we need the double integrations. Therefore eq.(5.11) has the information of eq.(5.12) and two integration constants, i.e. using a specific inverse function D_2 is indeed equivalent to choosing a specific particular solution for eq.(5.12) here. Meanwhile, within the previous works for four-dimensional case [17, 18], in order to derive the localized anomaly action, two auxiliary scalar fields which should share the same green function (inverse operator) analogous to D_2 should be introduced. However, according to our recent work [2] which will be introduced in Sec.6, we find that introducing only one auxiliary scalar field is enough to construct the localized S_{anom} in 4-dim.

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We can check that eq.(5.12) can also be obtained by varying this action w.r.t. φ, and thus it is consistent with the action above (5.7). Also this action is reduced to the anomaly action (5.7) after substituting eq.(5.11), and thus it gives the same dynamics as the non-local action (5.7). The corresponding stress tensor can be obtained by the variation w.r.t. the metric $g_{\mu\nu}$, and its explicit form is

$$\mathrm{T}^{anom}_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{anom}}{\delta g^{\mu\nu}} \tag{5.14}$$

$$= \frac{-1}{24\pi}[(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\varphi - \nabla_\mu\nabla_\nu\varphi + g_{\mu\nu}\Box\varphi - \frac{1}{2}(\nabla_\mu\varphi)(\nabla_\nu\varphi) + \frac{1}{4}g_{\mu\nu}(\nabla_\alpha\varphi)(\nabla^\alpha\varphi)]. \tag{5.15}$$

This trace obtained from the stress tensor above consists with the well-known trace anomaly,

$$g^{\mu\nu}\mathrm{T}^{anom}_{\mu\nu} = \frac{-1}{24\pi}\Box\varphi = \frac{1}{24\pi}R. \tag{5.16}$$

Therefore, it is concluded that scalar field action (5.13) describes the anomalous contribution.

5.2 2-dimensional Anomaly-induced action with Boundaries

In this section, we will propose the modified anomaly-induced action with boundary effect which is based on our published work [1]. We introduce the surface terms (i.e. the boundary effect) into anomaly action, which was ignored in the previous works [8–12,19], and find a natural relation between the quantum states of the original scalar field and boundary constraint of the auxiliary scalar field. Also, it turns out that the boundary effect is important not only for bounded spacetimes but also for unbounded ones. For a spacetime with a boundary, e.g. a horizon, the surface term fixes the boundary condition on the boundary. Meanwhile, for an unbounded spacetime, the surface term constrains on the asymptotic behavior at the (spatial) infinity.

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5.2.1 Operator-Modified Method

In this section, we will review the way we developed in the work [1] to include the boundary effect in anomaly-induced action, which we call operator-modified method in this thesis.

At first, let's consider the boundary part of action of the conformally coupled scalar filed. Due to conformal symmetry, the action with boundary contribution should be written as

$$S_{cl}[g] = -\frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\left[\nabla^\mu\phi\nabla_\mu\phi + (m^2+\xi R)\phi^2\right] - \epsilon\int_{\Sigma} d^{n-1}x\sqrt{-\epsilon\gamma}\xi K\phi^2, \quad (5.17)$$

From variation of this action, we can show the boundary condition needed for ϕ is Neumann boundary condition, i.e. $n^\mu\nabla_\mu\phi = 0$. The divergent part of effective action including the boundary contribution corresponding to Neumann boundary condition or Dirichlet boundary condition is derived in [21,22]. Therefore in order to renormalize the divergent effective action with boundary contribution via dimensional regularization, in additional to eq.(5.5) we should introduce the associated boundary term. The overall counter term in 2-dim thus now has the following form:

$$S_{ct}[g] := \frac{-1}{24\pi}\lim_{n\to 2}\frac{\int_{\mathcal{M}} d^2x\sqrt{-g}R + 2\epsilon\int_{\Sigma} d^1x\sqrt{-\epsilon\gamma}K}{n-2}, \quad (5.18)$$

where $\mathcal{M}$ is two-dimensional spacetime and Σ is the timelike boundary. We indeed need to take only timelike boundaries into consideration. That is because in the standard way to derive classical equation of motion, we take the variation of the action while fixing the initial and final states. Even if the surface terms on the spacelike boundaries (i.e. the initial and final hypersurfaces) are introduced, the final form of the stress tensor derived by the variation of the action would not be affected. We thus ignore the contribution of spacelike boundaries in the following. The boundary term in the numerator is the Gibbons-Hawking term [24, 25] and thus the numerator overall is a conformal invariant term in two-dimension. However, in a way similar to the previous case without boundary, the counter term introduced here is defined in $2+\varepsilon$ dimension and is not conformal invariant

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which is the origin of conformal anomaly.

For the convenience in the later discussion, we introduce an arbitrary scalar function $f(x)$ which is unity on the boundary and arbitrary elsewhere, i.e.

$$f(x) = 1, \qquad x \in \Sigma. \tag{5.19}$$

Using this scalar function, we rewrite the action of the counter terms in

$$S_{ct}[g] = \frac{-1}{24\pi} \lim_{n\to 2} \frac{\int d^2x\sqrt{-g}R + 2\epsilon \int d^1x\sqrt{-\epsilon\gamma}fK}{n-2}. \tag{5.20}$$

As in eq.(5.6), the corresponding WZ action can be shown as the form [26]:

$$\begin{aligned}
\Gamma_{WZ}[\bar{g},\sigma] &= \frac{1}{24\pi}\lim_{n\to 2}[\frac{(\int d^2x\sqrt{-\bar{g}}\bar{R} + 2\epsilon\int d^1x\sqrt{-\bar{\gamma}}\bar{K}) - (\int d^2x\sqrt{-g}R + 2\epsilon\int d^1x\sqrt{-\epsilon\gamma}K)}{n-2}] \\
&\overset{(A.19),(A.22)}{=} -\frac{1}{24\pi}\{\int d^2x\sqrt{-\bar{g}}[\sigma\bar{R} - \sigma\bar{\Box}\sigma] + \epsilon\int d^1x(\sqrt{-\epsilon\gamma}\sigma K + \sqrt{-\epsilon\bar{\gamma}}\sigma\bar{K})\} \\
&=: S_{anom}[\bar{g}] - S_{anom}[g].
\end{aligned} \tag{5.21}$$

From Sec.(A.1.2), by expanding the conformal transformation of R and K to the first order in ε $(:= n-2)$, we will have the following relations:

$$\begin{cases}
\left[\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R}\right] = -2\sqrt{-g}\Box\sigma + O(\varepsilon) = -\left[\sqrt{-g}\Box\sigma + \bar{...}\right] + O(\varepsilon) \\
\left[\sqrt{-\epsilon\gamma}K - \sqrt{-\epsilon\bar{\gamma}}\bar{K}\right] = \sqrt{-\epsilon\gamma}n^\mu\sigma_\mu + O(\varepsilon) = \frac{1}{2}\left[\sqrt{-\epsilon\gamma}n^\mu\sigma_\mu + \bar{...}\right] + O(\varepsilon)
\end{cases} \tag{5.22}$$

and it results in:

$$\begin{aligned}
\left\{\sqrt{-g}\left[R + 2\nabla_a(fn^aK)\right] - \bar{...}\right\} &= -2\sqrt{-g}(\Box\sigma - \nabla_a f n^a n^b \nabla_b\sigma) + O(\varepsilon) \\
&= -[\sqrt{-g}(\Box - \nabla_a f n^a n^b \nabla_b)\sigma + \bar{...}] + O(\varepsilon).
\end{aligned} \tag{5.23}$$

Here, n^μ is the unit normal vector on boundary and does not need to be fixed elsewhere.

From above relation, we realize that the D'Alembert operator, $\Box$, which is the unique 2-dim conformal invariant operator for the case without boundary should be modified to a new hermitian(self-adjoint) and conformal invariant operator for the case with boundary,

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L_2^f, which can be naturally constructed as: [3]

$$L_2^f := (-\Box + \nabla^\mu f n_\mu n_\nu \nabla^\nu). \tag{5.24}$$

We thus have a relation analogous to eq.(5.9) [4]

$$2\sqrt{-g}L_2^f\sigma = 2\sqrt{-\bar{g}}\bar{L}_2^f\sigma = \sqrt{-g}[R + 2\nabla^\mu(n_\mu f K)] - \sqrt{-\bar{g}}[\bar{R} + 2\bar{\nabla}_\mu(\bar{n}^\mu f\bar{K})]. \tag{5.25}$$

Using the relation (5.25), we can read the non-local anomaly-induced action from eq. (5.21)

$$S_{anom}[g] = \frac{1}{96\pi}\Bigg[\int d^2x \int d^2x'\sqrt{-g}\sqrt{-g'}(R(x) + 2\nabla^\mu(n_\mu f K))D_f(x,x') \cdot(R(x') + 2\nabla'^\mu(n'_\mu f' K')) - 4\int d^2x\sqrt{-g}fK^2\Bigg]. \tag{5.26}$$

Here, D_f is the symmetric inverse operator of L_2^f, which is defined by

$$L_2^f D_2^f(x,x') = -\frac{\delta(x-x')}{\sqrt{-g}}, \qquad D_2^f(x,x') = D_2^f(x',x). \tag{5.27}$$

As the derivation of the local anomaly-induced action in the previous section, we will introduce a real auxiliary scalar field φ which is defined by

$$\varphi := \int d^2x'\sqrt{-g}D_f(x,x')[R' + 2\nabla'_\mu(n'^\mu f' K')]. \tag{5.28}$$

Operating L_2^f to this equation, we have

$$L_2^f\varphi = R + 2\nabla^\mu(n_\mu f K). \tag{5.29}$$

[3] Proved in Sec.(A.1.2) and Sec.(C.1).

[4] $\Box\sigma \neq \nabla nn\nabla\sigma$, but $\int_M \Box\sigma = \epsilon\int_{\partial M} n^\mu\nabla_\mu\sigma = \int_{\partial M}\underbrace{(n_\mu n^\mu)}_{=\epsilon}(n^\nu\nabla_\nu\sigma) = \epsilon\int_M \nabla_\mu n^\mu n^\nu\nabla_\nu\sigma$.

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Similar to the action (5.13), again the localized version of the anomaly-induced action (5.26) can be expressed in terms of the auxiliary scalar field φ as

$$S_{anom}[g,\varphi] = \frac{-1}{96\pi}\left\{\int d^2x\sqrt{-g}[\varphi L_2^f\varphi - 2\varphi(R(x)+2\nabla^\mu(n_\mu fK))] - 4\int d^2x\sqrt{-g}fK^2\right\} \tag{5.30}$$

$$= \frac{-1}{96\pi}\left\{\int d^2x\sqrt{-g}(-\varphi\Box\varphi - 2\varphi R) + \int d^1x\sqrt{\gamma}(\varphi n_\mu\nabla^\mu\varphi - 4\varphi K) + \int d^2x\sqrt{-g}f[(n_\mu\nabla^\mu\varphi)(-n_\nu\nabla^\nu\varphi + 4K) - 4K^2]\right\}, \tag{5.31}$$

this action gives eq. (5.29) and, by substituting eq. (5.29) into it, this action indeed reduce to the non-local action (5.26). Therefore, this is the localized anomaly action we want.

Recall that f is an arbitrary function except that it should be unity on the boundary. In the following, we will choose f to approach step function to grasp the boundary effect. In order to tackle with this explicitly, let us condider the following f function:

$$f_\delta(\lambda) := \begin{cases} \frac{1}{2}[\cos(\frac{\lambda\pi}{\delta})+1], & (0<\lambda\leq\delta) \\ 0, & (\lambda\geq\delta) \end{cases} \tag{5.32}$$

where λ is the affine parameter[5] for the geodesic orthogonal to the boundary, and δ is a positive constant. By taking the limit $\delta\to 0$, the anomaly-induced action (5.31) becomes

$$S_{anom}[g,\varphi] \overset{\delta\to 0}{\to} \frac{-1}{96\pi}\{\int d^2x\sqrt{-g}(-\varphi\Box\varphi - 2\varphi R) + \int d^1x\sqrt{-\epsilon\gamma}(\varphi n_\mu\nabla^\mu\varphi - 4\varphi K)\}. \tag{5.33}$$

It turns out that we have exactly the same action as the previous one (5.13) expect for the additional boundary terms. The boundary terms have no additional contribution on the stress tensor except on the boundary, and thus the obtained stress tensor in the bulk, $\mathcal{M}$, is still the same as that one obtained from the case without the boundary term. Meanwhile, the boundary terms affect the boundary condition for the scalar field φ. Equation (5.29)

[5] We set the affine parameter λ to be zero on the boundary.

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can be rewritten in

$$-\Box\varphi + (n^\mu\nabla_\mu f_\delta)(n^\nu\nabla_\nu\varphi) + f_\delta\nabla_\mu n^\mu(n^\nu\nabla_\nu\varphi) = R + 2(n^\mu\nabla_\mu f_\delta)K + 2f_\delta\nabla_\mu n^\mu K. \tag{5.34}$$

Taking the limit $\delta \to 0$, we find the equations for φ

$$\Box\varphi = -R, \tag{5.35}$$

with the boundary conditions[6]

$$n^\nu\nabla_\nu\varphi = 2K, \qquad x \in \Sigma. \tag{5.36}$$

This means that there is the additional boundary constraint on φ which was not taken into consideration in the previous works.

5.2.2 Green's Function-Modified Method

Instead of the operator-modified method we just showed in the previous section, in this section, we will propose another equivalent method, which we call the "Green's Function-Modified Method", to derive anomaly-induced action with boundary effect. It turns out that the Green's function-modified method is easier to generalise to 4-dimensional case than operator-modified method.

At first, lets consider the identity

$$\int_{\mathcal{M}} d^2x\sqrt{-g}[u(x)\Box v(x) - v(x)\Box u(x)] = \epsilon\int_\Sigma d^1x\sqrt{-\epsilon\gamma}[u(x)\nabla_n v(x) - v(x)\nabla_n u(x)], \tag{5.37}$$

where $u(x)$ and $v(x)$ are arbitrary scalar functions, also $\Box$ and its boundary-associated

[6]These equations can be also obtained from the action (5.33) directly. Note that because $-n^\nu\nabla_\nu f_\delta$ becomes Dirac delta function in the limit $\delta \to 0$, the terms proportion to it in the lhs and rhs of eq.(5.34) should be balanced.

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operator ∇_n are conformally invariant operators which satisfies

$$\begin{cases} \Box u = e^{-2\sigma}\bar{\Box}u, \\ \nabla_n u = e^{-\sigma}\bar{\nabla}_n u. \end{cases} \tag{5.38}$$

The Green's function, i.e. the inverse operator, of D'Alembert operator is defined by

$$\begin{cases} \Box G(x,x') = -\dfrac{\delta^{(2)}(x,x')}{\sqrt{-g}}, \\ \nabla_n G(x,x') = w(x), \end{cases} \tag{5.39}$$

where $w(x)$ is an arbitrary function which should satisfy[7]

$$\epsilon \int_\Sigma d^1x\sqrt{-\epsilon\gamma}w(x) = -1. \tag{5.40}$$

From eq.(5.22), we know that in 2-dimensional spacetime the conformal transformation parameter σ must satisfy the following two differential equations:

$$\begin{cases} \Box\sigma = \dfrac{-1}{2\sqrt{-g}}\left[\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R}\right], \\ \nabla_n\sigma = \dfrac{1}{\sqrt{-\epsilon\gamma}}\left[\sqrt{-\epsilon\gamma}K - \sqrt{-\epsilon\bar{\gamma}}\bar{K}\right]. \end{cases} \tag{5.41}$$

By using eq.(5.37) and eq.(5.41), we can expresses σ in terms of $G(x,x')$ as:

$$\begin{aligned} \sigma(x) = &\int_{\mathcal{M}} d^2x' G(x',x)\left\{\frac{1}{2}[\sqrt{-g'}R' - \sqrt{-\bar{g}'}\bar{R}']\right\} + \epsilon\int_\Sigma d^1x' G(x',x)[\sqrt{-\epsilon\gamma'}K' - \sqrt{-\epsilon\bar{\gamma}'}\bar{K}'] \\ &+ \langle\sigma\rangle_{w(x)}, \end{aligned} \tag{5.42}$$

where $\langle u(x)\rangle_{w(x)} := -\epsilon\int_\Sigma d^1x'\sqrt{-\epsilon\gamma'}w(x')u(x')$ for any scalar function u(x).

Similar to the last section, we will introduce a real auxiliary scalar field φ which is defined by

$$\varphi(x) = \int_{\mathcal{M}} d^2x'\sqrt{-g'}G(x',x)R' + 2\epsilon\int_\Sigma d^1x'\sqrt{-\epsilon\gamma'}G(x',x)K' + \langle\varphi\rangle_{w(x)}. \tag{5.43}$$

[7] In order to guarantee $\int_{\mathcal{M}} d^2x\Box G(x,x') = \epsilon\int_\Sigma d^1x\sqrt{-\epsilon\gamma}\nabla_n G(x,x')$.

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Note that by this definition, the scalar field φ automatically satisfies the following equation of motion and boundary condition:

$$\begin{cases} \Box\varphi = -R, \\ n^{\nu}\nabla_{\nu}\varphi = 2K, \qquad x \in \Sigma, \end{cases} \tag{5.44}$$

which are exactly the same as eq.(5.35) and eq.(5.36). Together with the relation $G(x, x') = \bar{G}(x, x')$ which is resulted from the conformal symmetry of $\Box$ and ∇_n, i.e. eq.(5.38), σ can thus be expressed in terms of φ as

$$\sigma(x) = \frac{1}{2}[\varphi(x) - \bar{\varphi}(x)]. \tag{5.45}$$

In the setting of two-manifolds with boundary, the D,Alembert operator and its boundary associated operator, ∇_n are related to a "cocycle" functional $\mathcal{F}$ which is defined by

$$\mathcal{F}[u(x)] := \int_{\mathcal{M}} d^2x\sqrt{-g}[u\Box u + Ru] + \epsilon \int_{\Sigma} d^1x\sqrt{-\epsilon\gamma}[-u\nabla_n u + 2Ku], \tag{5.46}$$

where $u(x)$ is an arbitrary scalar function. It can be checked that this functional $\mathcal{F}$ satisfies the following "cocycle" condition

$$\mathcal{F}[\sigma + u] = \bar{\mathcal{F}}[u]. \tag{5.47}$$

Next, with the help of the cocycle functional $\mathcal{F}$, the WZ action can be derived as fol-

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lows:

$$
\begin{aligned}
\Gamma_{WZ}[\bar{g},\sigma] &= \frac{1}{24\pi}\lim_{n\to 2}[\frac{(\int d^2x\sqrt{-\bar{g}}\bar{R}+2\epsilon\int d^1x\sqrt{-\bar{\gamma}}\bar{K})-(\int d^2x\sqrt{-g}R+2\epsilon\int d^1x\sqrt{-\epsilon\gamma}K)}{n-2}] \\
&= -\frac{1}{24\pi}\left[\frac{1}{2}\int d^2x(\sqrt{-g}R+\sqrt{-\bar{g}}\bar{R})\sigma+\epsilon\int d^1x(\sqrt{-\epsilon\gamma}K+\sqrt{-\epsilon\bar{\gamma}}\bar{K})\sigma\right] \\
&= -\frac{1}{24\pi}\left[\int d^2x\sqrt{-g}(\sigma R+\sigma\Box\sigma)+\epsilon\int d^1x\sqrt{-\epsilon\gamma}(-\sigma\nabla_n\sigma+2\sigma K)\right] \\
&= -\frac{1}{24\pi}\mathcal{F}[\sigma] \\
&= -\frac{1}{24\pi}(\mathcal{F}[\frac{1}{2}\varphi]-\bar{\mathcal{F}}[\frac{1}{2}\bar{\varphi}]) \\
&=: S_{anom}[\bar{g}]-S_{anom}[g], \qquad (5.48)
\end{aligned}
$$

where we have used eq.(5.45) and eq.(5.47) to get the last second equality. Therefore, we can immediately read the anomaly-induced action from eq.(5.48) as

$$
\begin{aligned}
S_{anom}[g,\varphi] &= \frac{1}{24\pi}\mathcal{F}[\frac{1}{2}\varphi] \\
&= \frac{1}{96\pi}\{\int d^2x\sqrt{-g}(\varphi\Box\varphi+2\varphi R)+\int d^1x\sqrt{-\epsilon\gamma}(-\varphi n_\mu\nabla^\mu\varphi+4\varphi K)\}.
\end{aligned}
\qquad (5.49)
$$

Note that this result is exactly the same as eq.(5.33). Therefore, we proved that the Green-function modified method gives the same result as the operator-modified method. Also, by the Green-function modified method, we don't need to assume the symmetric property of Green's function in the derivation of the anomaly action. In the next chapter, we will see that this alternative approach is much easier to generalize to 4-dimensional case.

One more remark is that by adding up the lhs and rhs of (5.44) respectively, we will have

$$\int_{\mathcal{M}} d^2x\sqrt{-g}\Box\phi-\int_{\Sigma} d^1x\sqrt{-\gamma}\nabla_n\phi=0, \qquad (5.50)$$

$$\int_{\mathcal{M}} d^2x\sqrt{-g}R+\int_{\Sigma} d^1x\sqrt{-\gamma}2K=\chi_2[\mathcal{M}], \qquad (5.51)$$

where $\chi_2[\mathcal{M}]$ is the Euler-characteristic of the bounded 2-dim manifold $\mathcal{M}$ and we have used the 2-dim Gauss–Bonnet theorem to get the second equality. In order to make sure

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the two relations satisfy at the same time, we must have the following equality:

$$\int_{\mathcal{M}} d^2x\sqrt{-g}R + \int_{\Sigma} d^1x\sqrt{-\gamma}2K = \chi_2[\mathcal{M}] = 0. \tag{5.52}$$

It means that the anomaly-induced action can only be used in the spacetime with zero Euler-characteristics. Although this requirement satisfies automatically in bounded 2-dim spacetime, later we will find that it is important for 4-dim generalization.

5.3 Applications

Since we have construct the anomaly-induced action with boundary effect, in the following, we will apply this formalism to several examples. Then we can appreciate how powerful and efficient this formalism is when one need to solve the expectation of stress tensor of quantum vacuum and thermal states.

5.3.1 General Analysis for 2-dimensional Spacetime

Since any two-dimensional spacetime can be described by the conformally-flat metric, in this section, we first apply our result to conformal flat spacetime and then use the result to analyse several common 2-dimensional spacetimes, which are the flat, 2-dimensional Schwarzschild, and de Sitter spacetimes.

Any metric of 2-dimensional spacetime can be written in the conformal flat form:

$$ds^2 = F(t, r)(-dt^2 + dr^2). \tag{5.53}$$

We consider the case in which the boundaries exist on $r = r_1$ and $r = r_2 = r_1 + L(> r_1)$. The normal vector on the boundary is written in

$$n^\mu = \begin{pmatrix} 0, & F^{-\frac{1}{2}} \end{pmatrix}. \tag{5.54}$$

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The Ricci scalar and extrinsic curvature on the boundary are, respectively,

$$R = F^{-1}(-\partial_t^2 \ln F + \partial_r^2 \ln F), \tag{5.55}$$
$$K = \frac{1}{2} F^{-\frac{3}{2}} \partial_r F. \tag{5.56}$$

With the metric (5.53), eq. (5.35) can be rewritten as

$$F^{-1}(-\partial_t^2 \varphi + \partial_r^2 \varphi) = F^{-1}(-\partial_t^2 \ln F + \partial_r^2 \ln F). \tag{5.57}$$

A particular solution of this equation is $\ln F (=: \varphi_p)$, and thus the general solution for φ is derived as

$$\varphi = \varphi_p + \varphi_h, \tag{5.58}$$
$$\varphi_h := A_1 r + A_2 t + A_3 + A_0 r t + \int_{-\infty}^{\infty} d\omega [c_\pm(\omega) e^{i\omega t} e^{\pm i\omega r}] + \int_{-\infty}^{\infty} d\omega [d_\pm(\omega) e^{\omega t} e^{\pm \omega r}], \tag{5.59}$$

where φ_h is the homogeneous solution satisfying $\Box \varphi_h = 0$. A_0, A_1, A_2, A_3 are real constants, $c_\pm(\omega)$ are constant functions satisfy $c_\pm(\omega) = c_\pm^*(-\omega)$, and $d_\pm(\omega)$ are real functions.

The boundary equation (5.36) becomes

$$F^{-\frac{1}{2}} \partial_r \varphi = F^{-\frac{3}{2}} \partial_r F. \tag{5.60}$$

With this boundary condition, the solution (5.58) is constrained as

$$\varphi = \varphi_p + \varphi_0, \tag{5.61}$$
$$\varphi_0 := A_2 t + A_3 + \sum_{n=-\infty}^{\infty} c_n \cos(\omega_n r) e^{i\omega_n t}, \tag{5.62}$$

where $\omega_n = \frac{\pi n}{L}$, $n \in \mathbb{N}$, c_n are constants satisfy $c_n = c_{-n}^*$.

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The stress tensor of the trace anomaly (5.15) can be transformed as

$$\mathrm{T}^{anom}_{\mu\nu}[\varphi = \varphi_p + \varphi_0; g_{\mu\nu}] = T^{\varphi_p}_{\mu\nu} + T^{\varphi_0}_{\mu\nu}, \tag{5.63}$$

$$T^{\varphi_p}_{\mu\nu} := \frac{1}{24\pi}[g_{\mu\nu}\Box\varphi_p + \frac{1}{4}g_{\mu\nu}(\nabla_\alpha\varphi_p)(\nabla^\alpha\varphi_p) - \frac{1}{2}(\nabla_\mu\varphi_p)(\nabla_\nu\varphi_p) - \nabla_\mu\nabla_\nu\varphi_p], \tag{5.64}$$

$$T^{\varphi_0}_{\mu\nu} := \frac{1}{24\pi}[\frac{1}{4}g_{\mu\nu}(\nabla_\alpha\varphi_0)(\nabla^\alpha\varphi_0) - \frac{1}{2}(\nabla_\mu\varphi_0)(\nabla_\nu\varphi_0) - \partial_\mu\partial_\nu\varphi_0]. \tag{5.65}$$

Note that there is no coupling term between φ_p and φ_0, i.e. $T_{\mu\nu}$ can be separated into φ_p part and φ_0 part. As we will see later, φ_p part indeed describes the vacuum polarization, while φ_0 part seems related to the excitations. Since all φ's in the stress tensor have at least one derivative, A_3 does not affect the stress tensor. Therefore, without loss of generality, hearinafter we set A_3 to be naught. Furthermore, if we restrict φ_0 to be A_2t, the φ_0 part of stress tensor ($T^{\varphi_0}_{\mu\nu}$) would become stationary. [8] This contribution is expected to be that of the thermal state.

5.3.2 Conformal Vacuum Solutions

As mentioned in the previous sections, people expected that the different solutions of auxiliary scalar field corresponds to different choices of quantum states of the original conformal scalar field. Therefore, we should also expect that the general solutions of it contain the information of all choices of vacuum states. In the following, we will quickly show that the general solutions, Eq.(5.61) indeed contains all information of all (conformal) vacuum states.

At first, by comparing (t, x) with another general conformal transformed coordinate (w, s) as: [9]

$$ds^2 = F_1(t, x)(-dt^2 + dx^2) = F_2(w, s)(-dw^2 + ds^2), \tag{5.66}$$

[8] $T^{\varphi_p}_{\mu\nu}$ might not be stationary in general because of time dependence of $F(t, r)$.

[9] If another coordinate (w, s) introduced here exists or not is nontrivial. We prove the existence of it in Ch.B. According to this prove, it also means that there is no unique conformal vacuum for 2-dim spacetime.

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where $F_2 := F_1 \cdot c^2$, and the expression of c^2 is given in (B.19). According to the result from last section, by considering the (t, x) frame, we obtain the general particular and homogeneous solutions written as

$$\begin{cases} \varphi_p^1 = \ln F_1, \\ \varphi_0^1 = A_2 t + \sum_n c_n \cos(n\pi x) e^{in\pi t}, \end{cases} \tag{5.67}$$

and by considering the (w, s) frame, we get another solutions as

$$\begin{cases} \varphi_p^2 = \ln F_2 = \ln F_1 + \ln c^2, \\ \varphi_0^2 = B_2 w + \sum_n d_n \cos(n\pi s) e^{in\pi w}. \end{cases} \tag{5.68}$$

By using Taylor expansion of $\ln c^2$ and Eq.(B.18), it can be shown that $\varphi_0^1 + \varphi_p^1$ is indeed equal to $\varphi_0^2 + \varphi_p^2$. Therefore, the two expression of the solutions of φ are equivalent.

Also, because $\varphi_p^1 = \ln F_1$ corresponds to the conformal vacuum based on (t, x) frame and $\varphi_p^2 = \ln F_2$ corresponds to the conformal vacuum based on (w, s) frame, we now understand the general solution Eq.(5.61) indeed includes (and only includes) the information of all (conformal) vacuum states.

Minkowski (Flat) Spacetime

By using the result from the general analysis for 2-dim spacetime, we can immediately get the solution of stress tensor for any given metric. Let us first consider the Minkowski spacetime. There are two famous vacua of Minkowski spacetime; the Minkowski vacuum (which based on the Cartesian coordinate) and the Rindler vacuum. The vacuum based on the Cartesian coordinate is defined in the full region of Minkowski spacetime (see FIG. 5.1), and thus we expect that the boundaries exist at two spatial infinities. Meanwhile, the Rindler vacuum is defined in the Rindler wedge (see FIG. 5.2). One boundary exists on the Rindler horizon and the other is at the spatial infinity. Moreover, in order to compare with the Unruh vacuum of the two-dimensional Schwarzschild spacetime that we will discuss later, we consider another vacuum of Minkowski spacetime, which we call the

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Unruh-like vacuum. This is just the analog of the Unruh vacuum in the two-dimensional Schwarzschild spacetime; one of the boundaries is the white hole horizon, and the other is spatial infinity. The corresponding region is the sum set of the Rindler patch and the future Milne patch (see FIG. 5.3).

To describe each region, we write the Minkowski metric in various forms:

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 = -dU_f dV_f && (5.69)\\ &= -\rho^2 du_f dv_f = -\rho^2(-dT_R{}^2 + dR_R{}^2)\left(= -\rho^2 dT_R{}^2 + d\rho^2\right) && (5.70)\\ &= -V_f dU_f dv_f = V_f(-dT_U{}^2 + dR_U{}^2), && (5.71)\end{aligned}$$

where

$$U_f := t - x, \qquad V_f := t + x, \tag{5.72}$$

$$\rho := (x^2 - t^2)^{1/2}, \quad u_f := -\log(-U_f), \quad v_f := \log V_f, \tag{5.73}$$

$$T_R := \frac{1}{2}(v_f + u_f), \quad R_R := \frac{1}{2}(v_f - u_f), \tag{5.74}$$

$$T_U := \frac{1}{2}(v_f + U_f), \qquad R_U := \frac{1}{2}(v_f - U_f). \tag{5.75}$$

The metric forms (5.69), (5.70) and (5.71) describe the regions of whole, Rindler patch and the sum set of the Rindler patch and the future Milne patch of Minkowski spacetime. They are corresponding to the Minkowski vacuum, Rindler vacuum and Unruh-like vacuum respectively.

Minkowski Vacuum

The Minkowski vacuum is the lowest energy state defined in whole of Minkowski spacetime. Therefore, we consider coordinate (5.69) with boundaries at $x = x_\pm$ and take the limit $x_\pm \to \pm\infty$.

From eq.(5.61), the general solution can be written in

$$\varphi = A_2 t + \int d\omega\, c(\omega)\cos[\omega x] e^{i\omega t}. \tag{5.76}$$

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The stationary stress tensor can be obtained be setting $c(\omega) = 0$ as

$$T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} \frac{A_2^2}{4} & 0 \\ 0 & \frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (t, x) \text{ coordinate.} \tag{5.77}$$

As $A_2 = 0$, $T_{\mu\nu}$ becomes the same as that of the Minkowski vacuum, i.e. all components become zero. Meanwhile, A_2 characterizes the temperature of the thermal equilibrium state.

Rindler Vacuum

The Rindler patch is described by the metric (5.70). We consider the case where boundaries exist at $R_R = R_\pm$ and take the limit $R_\pm \to \pm\infty$. Then the solution for φ becomes

$$\varphi = 2R_R + A_2 T_R + \int d\omega\, c(\omega)\cos[\omega R_R] e^{i\omega T_R}. \tag{5.78}$$

The stationary stress tensor (with respect to the Rindler time) is realized if $c(\omega) = 0$, and the corresponding stress tensor is:

$$T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} -1 + \frac{A_2^2}{4} & 0 \\ 0 & -1 + \frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (T_R, R_R) \text{ coordinate,} \tag{5.79}$$

$$= \frac{1}{24\pi}\begin{pmatrix} \frac{\left(A_2^2-4\right)\left(x^2+t^2\right)}{4(x^2-t^2)^2} & -\frac{\left(A_2^2-4\right)xt}{2(x^2-t^2)^2} \\ -\frac{\left(A_2^2-4\right)xt}{2(x^2-t^2)^2} & \frac{\left(A_2^2-4\right)\left(x^2+t^2\right)}{4(x^2-t^2)^2} \end{pmatrix} \quad \text{in } (t, x) \text{ coordinate.} \tag{5.80}$$

For $A_2 = 0$, the result is the same as that corresponding to the Rindler vacuum state, and A_2 characterizes the temperature of the "thermal equilibrium state" based on the Rindler vacuum. The condition $A_2 = 2$ gives the same result as that corresponding to the Minkowski vacuum state, and thus the vacuum of the Cartesian coordinate is a thermal state based on the Rindler vacuum. This is consistent with the Unruh effect; the Rindler observer feels the thermal radiation in the Minkowski vacuum state.

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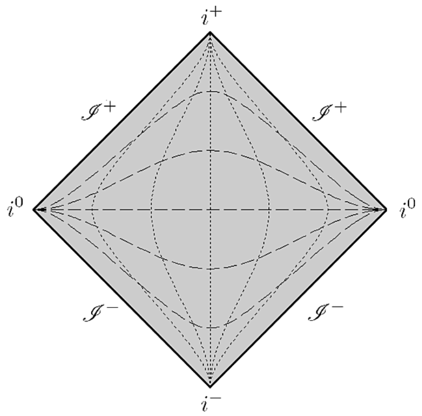

Figure 5.1: Region corresponding to the Minkowski vacuum: The (t, x) coordinate covers the whole Minkowski spacetime where $t =$ constant and $r =$ constant curves are drawn in dashed and dotted lines respectively.

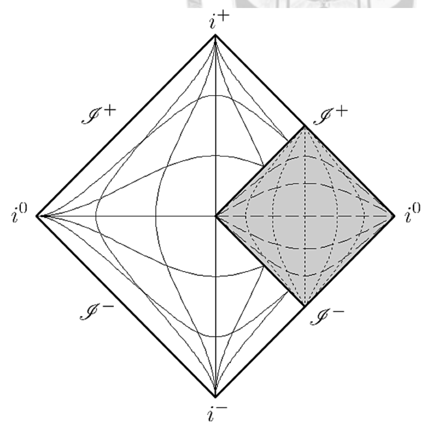

Figure 5.2: Region corresponding to the Rindler vacuum: The (T_R, R_R) coordinate covers only one quarter of Minkowski spacetime (Rindler wedge) where $T_R =$ constant and $R_R =$ constant curves are drawn in dashed and dotted lines respectively.

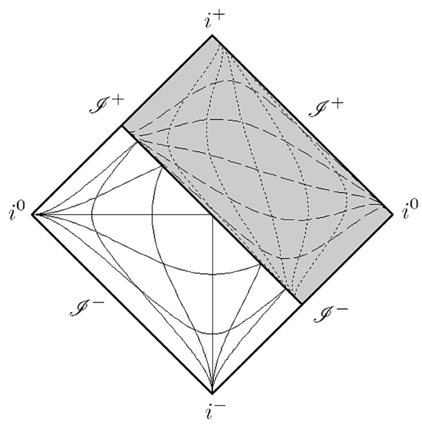

Figure 5.3: Region corresponding to the Unruh-like vacuum: The (T_U, R_U) coordinate covers a half of Minkowski spacetime where $T_U =$ constant and $R_U =$ constant curves are drawn in dashed and dotted lines respectively.

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Unruh-like Vacuum

In the Schwarzchild black hole spacetime, we are sometimes interested in the vacuum state defined in the sum set of the outer region and the future trapped region, which gives the Unruh state. To see the correspondence between the Minkowski spacetime and the two-dimensional Schwarzschild spacetime that we will discuss later, it is useful to consider the corresponding situation. That is, we consider the sum set of the Rindler patch and the future Milne patch, which is described by the metric (5.71). The boundaries are set at $R_U = R_\pm$ and we take the limit $R_\pm \to \pm\infty$. Then the solution for φ becomes

$$\varphi = \ln(t+x) + A_2 T_U + \int d\omega\, c(\omega) \cos[\omega R_U] e^{i\omega T_U}. \tag{5.81}$$

The stress tensor of the thermal state is expected to be obtained with the condition $c(\omega) = 0$:

$$T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} -\frac{1}{2}+\frac{A_2^2}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}+\frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (T_U, R_U) \text{ coordinate,} \tag{5.82}$$

$$= \frac{1}{24\pi}\begin{pmatrix} \frac{-4+A_2^2\left((x+t)^2+1\right)}{8(x+t)^2} & \frac{-4-A_2^2\left((x+t)^2-1\right)}{8(x+t)^2} \\ \frac{-4-A_2^2\left((x+t)^2-1\right)}{8(x+t)^2} & \frac{-4+A_2^2\left((x+t)^2+1\right)}{8(x+t)^2} \end{pmatrix} \quad \text{in } (t,x) \text{ coordinate.} \tag{5.83}$$

The terms depending on A_2 appear in the diagonal part in (T_U, R_U) coordinate, and it is traceless. This implies that its energy flows along ∂_{T_U}, and thus, a thermal gas comoves along ∂_{T_U}. The case with $A_2 = 0$ is expected to be the vacuum state of the region that we consider. The stress tensor has the off-diagonal term in (t, x) coordinate. This means that we have energy flow in the vacuum state, which is corresponding to the Hawking radiation in the Unruh state of the black hole spacetimes.

2-dimensional Schwarzschild Spacetime

The vacuum polarization in the black hole spacetime is one of the major interests in the quantum field theory on curved spacetimes. As a simplified toy model, the two-dimensional Schwarzschild spacetime is often invoked, where one considers the same

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metric as the time and radial components of the four-dimensional Schwarzschild spacetime. This geometry is not a solution of a gravity theory,[10] but it is fixed by hand. The artificial spacetime is enough for the discussion of the renormalized stress tensor. The causal structure in this two-dimensional Schwarzschild spacetime is the same as that in the four-dimensional Schwarzschild spacetime, and thus qualitatively we can expect that similar features of the vacuum polarization, such as the Hawking radiation, appear. Here, we study the vacuum polarization of the three familiar states; the Boulware, Hartle-Hawing and Unruh states.

In order to describe the corresponding regions to the three states, we write the two-dimensional Schwarzschild spacetime in various descriptions:

$$
\begin{aligned}
ds^2 &= -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 = (1-\frac{2M}{r})(-dt^2 + dr^{*2}) && (5.84)\\
&= -(1-\frac{2M}{r})dudv = -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV \\
&= -\frac{32M^3}{r}e^{-\frac{r}{2M}}(-dT_H{}^2 + dR_H{}^2) && (5.85)\\
&= -\frac{8M^2}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}dUdv = \frac{8M}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}(-dT_U{}^2 + dR_U{}^2), && (5.86)
\end{aligned}
$$

where

$$
\begin{aligned}
& r^* := r + 2M\ln(\frac{r}{2M}-1), \qquad u := t - r^*, \qquad v := t + r^*, && (5.87)\\
& U := -e^{\frac{-u}{4M}}, \qquad V := e^{\frac{v}{4M}}, \\
& T_H := \frac{1}{2}(V+U), \qquad R_H := \frac{1}{2}(V-U), && (5.88)\\
& T_U := \frac{1}{2}(v+U), \qquad R_U := \frac{1}{2}(v-U). && (5.89)
\end{aligned}
$$

The coordinates (5.84), (5.85) and (5.86) describe the outside of the black hole (see FIG. 5.5), whole spacetime (see FIG. 5.4) and the sum set of the outside and the future trapped region (see FIG. 5.6), and they are corresponding to the Boulware, Hartle-Hawking and Unruh states, respectively. Comparing these coordinates (5.84), (5.85), (5.86) and the transformations (5.87), (5.88), (5.89) with those of the Minkowski spacetime (5.69), (5.70), (5.71),

[10] In two dimensional spacetime, general relativity is not well-defined.

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(5.72), (5.74) and (5.75), we can read the analog of the Boulware, Hartle-Hawing and Unruh vacua to the Rindler, Minkowski and Unruh-like vacua in the Minkowski spacetime, respectively.

Hartle-Hawking Vacuum

The energy momentum tensor of the Hartle-Hawking state [27, 28] is defined in the whole spacetime, which is regular even at horizons and infinity, and thus state can be defined everywhere. Therefore, the metric (5.85) is the corresponding metric, which is regular everywhere. We set the boundaries at $R_H = R_\pm$ and take the limit $R_\pm \to \pm\infty$. Then the general solution (5.58) can be written in

$$\varphi = ln(1-\frac{2M}{r}) - \frac{1}{2M}r^* + A_2 T_H + \int d\omega\, c(\omega)\cos[\omega R_H]e^{i\omega T_H}. \tag{5.90}$$

Stationary stress tensor (in (T_H, R_H)-coordinate sense) is achieved if $c(\omega)$ vanishes, and it becomes

$$T_{T_H T_H} = \frac{1}{24\pi}[-\frac{64M^4}{r^4}\mathrm{e}^{-\frac{r}{2M}} + \left(\frac{48M^4}{r^4}+\frac{16M^3}{r^3}+\frac{4M^2}{r^2}\right)\mathrm{e}^{-\frac{r}{M}}\left({R_H}^2+{T_H}^2\right)+\frac{{A_2}^2}{4}], \tag{5.91}$$

$$T_{R_H R_H} = \frac{1}{24\pi}[\frac{64M^4}{r^4}\mathrm{e}^{-\frac{r}{2M}} + \left(\frac{48M^4}{r^4}+\frac{16M^3}{r^3}+\frac{4M^2}{r^2}\right)\mathrm{e}^{-\frac{r}{M}}\left({R_H}^2+{T_H}^2\right)+\frac{{A_2}^2}{4}], \tag{5.92}$$

$$T_{T_H R_H} = T_{R_H T_H} = \frac{-1}{24\pi}\left(\frac{96M^4}{r^4}+\frac{32M^3}{r^3}+\frac{8M^2}{r^2}\right)\mathrm{e}^{-\frac{r}{M}}\left(T_H R_H\right), \tag{5.93}$$

in (T_H, R_H) coordinate, and

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$$
\begin{aligned}
T_{tt} &= \frac{1}{24\pi}[\left(\frac{7M^2}{r^4}-\frac{4M}{r^3}+\frac{1}{16M^2}\right)+\frac{{A_2}^2}{64M^2}\left({R_H}^2+{T_H}^2\right)], && (5.94)\\
T_{rr} &= \frac{1}{24\pi}[\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{1}{16M^2}-\frac{M^2}{r^4}\right)+\frac{{A_2}^2}{64M^2}\left(1-\frac{2M}{r}\right)^{-2}\left({R_H}^2+{T_H}^2\right)], && (5.95)\\
T_{tr} &= T_{rt}=\frac{1}{24\pi}[\frac{{A_2}^2}{32M^2}\left(1-\frac{2M}{r}\right)^{-1}(T_HR_H)], && (5.96)
\end{aligned}
$$

in (t,r) coordinate.

For $A_2 = 0$, the energy density is constant for the Killing observer (whose trajectory is tangent to ∂_t) outside the black hole, and the stress tensor is the same as that of the Hartle-Hawking vacuum state. A_2 characterizes the thermal excitation based on the Hartle-Hawking vacuum.

Boulware Vacuum

The Boulware vacuum [29] has the same asymptotic behavior as the Minkowski vacuum, while the stress tensor diverges on the horizon. Thus, the state (and the quantum theory) is defined only outside the horizons. The metric (5.84) is the corresponding one. We set the boundaries at $r^* = r^*_\pm$ and take the limit $r^*_\pm \to \pm\infty$. The form of general solution (5.58) becomes

$$\varphi = \ln(1-\frac{2M}{r}) + A_2 t + \int d\omega\, c(\omega)\cos[\omega r^*]e^{i\omega t}. \tag{5.97}$$

Imposing the stationary condition of the stress tensor, $c(\omega)$ should vanishes and the stress

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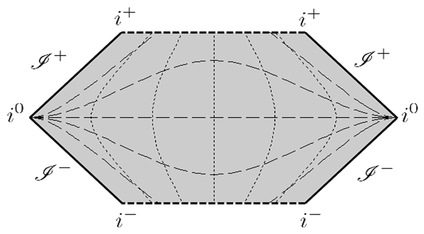

Figure 5.4: Region corresponding to the Hartle-Hawking vacuum: The (T_H, R_H) coordinate covers the whole two-dimentional Schwarzschild spacetime where T_H = constant and R_H = constant curves are drawn in dashed and dotted lines respectively.

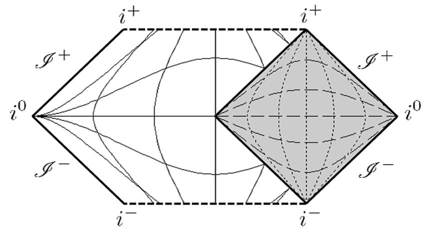

Figure 5.5: Region corresponding to the Boulware vacuum: The (t, r) coordinate covers one quarter of two-dimentional Schwarzschild spacetime where t = constant and r = constant curves are drawn in dashed and dotted lines respectively.

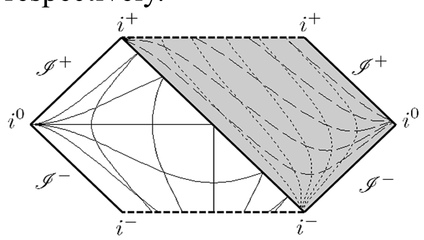

Figure 5.6: Region corresponding to the Unruh vacuum: The (T_U, R_U) coordinate covers a half of two-dimentional Schwarzschild spacetime where T_U = constant and R_U = constant curves are drawn in dashed and dotted lines respectively.

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tensor is derived as

$$
\begin{aligned}
T_{\mu\nu} &= \frac{1}{24\pi}\begin{pmatrix} \frac{-4Mr+7M^2}{r^4}+\frac{A_2^2}{4} & 0 \\ 0 & -\frac{M^2}{r^4}+\frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (t,r^*) \text{ coordinate,} && (5.98)\\
&= \frac{1}{24\pi}\begin{pmatrix} \frac{-4Mr+7M^2}{r^4}+\frac{A_2^2}{4} & 0 \\ 0 & -\frac{M^2}{r^2(r-2M)^2}+\frac{A_2^2r^2}{4(r-2M)^2} \end{pmatrix} \quad \text{in } (t,r) \text{ coordinate.} && (5.99)
\end{aligned}
$$

For $A_2 = 0$, the energy density has the minimum value, which corresponds to the Boulware vacuum state. A_2 characterizes the temperature of the thermal equilibrium state based on the Boulware vacuum. Similar to the relation between the Minkowski and Rindler vacua, for $A_2 = \pm 1/(2M)$, the resulting stress tensor is the same as that of the Hartle-Hawking vacuum state. That is, Hartle-Hawking vacuum state is a thermal equilibrium state on Boulware vacuum.

Unruh Vacuum

In the Unruh vacuum state [30], we take the Minkowski vacuum state at the past null infinity, while the stress tensor is regular on the black hole horizon but not on white hole horizon. We can extend the state to the inside of the black hole but not of the white hole. Therefore, the corresponding region is the sum of the outside of horizon and inside of black hole, which is described with the metric (5.86). We set the boundaries at $R_U = R_\pm$ and take the limit $R_\pm \to \pm\infty$. Then, the general solution is written in

$$\varphi = \ln(1-\frac{2M}{r}) + \frac{1}{4M}(t-r^*) + A_2 T_U + \int d\omega\, c(\omega)\cos[\omega R_U]e^{i\omega T_U}. \qquad (5.100)$$

The stationary stress tensor (in (T_U,R_U) sense) is obtained if $c(\omega)$ vanishes and it is written

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as

$$T_{T_U T_U} = \frac{1}{24\pi}\left[\frac{1}{r^4}\left(-Mr + \frac{3}{2}M^2 - 16M^3 \mathrm{e}^{\frac{t-r}{4M}}\sqrt{\frac{r}{2M}-1}\right.\right.$$
$$\left.\left. +2M^2\mathrm{e}^{\frac{t-r}{2M}}\left(\frac{r}{2M}-1\right)\left(r^2+4Mr+12M^2\right)\right) + \frac{{A_2}^2}{4}\right], \tag{5.101}$$

$$T_{R_U R_U} = \frac{1}{24\pi}\left[\frac{1}{r^4}\left(-Mr + \frac{3}{2}M^2 + 16M^3 \mathrm{e}^{\frac{t-r}{4M}}\sqrt{\frac{r}{2M}-1}\right.\right.$$
$$\left.\left. +2M^2\mathrm{e}^{\frac{t-r}{2M}}\left(\frac{r}{2M}-1\right)\left(r^2+4Mr+12M^2\right)\right) + \frac{{A_2}^2}{4}\right], \tag{5.102}$$

$$T_{T_U R_U} = T_{R_U T_U} = \frac{M}{48\pi r^4}\left(-2r + 3M - 2\mathrm{e}^{\frac{t-r}{2M}}\left(r-2M\right)\left(r^2+4Mr+12M^2\right)\right), \tag{5.103}$$

in (T_U, R_U) coordinate, and

$$T_{tt} = \frac{1}{24\pi}[\left(\frac{1}{32M^2} + \frac{7M^2}{r^4} - \frac{4M}{r^3}\right) + \frac{{A_2}^2}{8}\left(1 + \frac{r-2M}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)], \tag{5.104}$$

$$T_{rr} = \frac{1}{24\pi}[\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{-M^2}{r^4} + \frac{1}{32M^2}\right) + \frac{{A_2}^2}{8}\left(\left(1-\frac{2M}{r}\right)^{-2} + \left(1-\frac{2M}{r}\right)^{-1}\frac{r}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)], \tag{5.105}$$

$$T_{tr} = T_{rt} = \frac{1}{24\pi}[-\frac{1}{32M^2}\left(1-\frac{2M}{r}\right)^{-1} + \frac{{A_2}^2}{8}\left(\left(1-\frac{2M}{r}\right)^{-1} - \frac{r}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)], \tag{5.106}$$

in (t, r) coordinate.

The lowest energy state with respect to (T_U, R_U)-coordinate is realized for $A_2 = 0$, and then the stress tensor is the same as that of Unruh vacuum state. A_2 describes the thermal excitation for the Unruh observer (whose trajectory is tangent to ∂_{T_U}).

de Sitter Spacetime

Here, we consider the stress tensor in de Sitter spacetime. In cosmology, de Sitter spacetime approximately describes the beginning part of the Universe, i.e. inflation. Meanwhile, de Sitter spacetime has the maximal symmetry, and thus has intriguing features. Therefore, de Sitter spacetime is interesting in both phenomenological and theoret-

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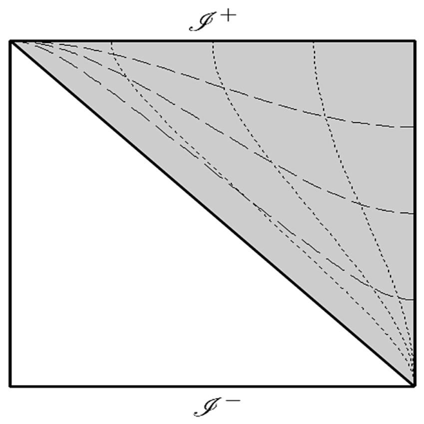

Figure 5.7: Region corresponding to Bunch-Davies vacuum: The (t_f, r_f) coordinate covers a half of de Sitter spacetime where $t_f =$ constant and $r_f =$ constant curves are drawn in dashed and dotted lines respectively.

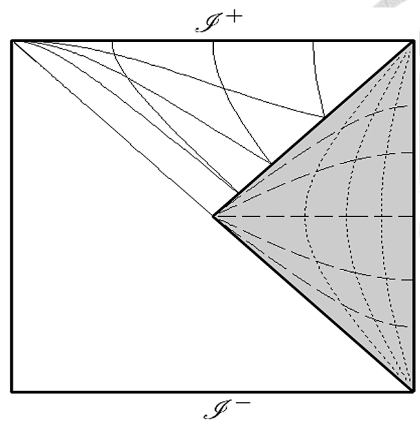

Figure 5.8: Region of the static chart: The (t_s, r_s) coordinate covers one quarter of de Sitter spacetime where $t_s =$ constant and $r_s =$ constant curves are drawn in dashed and dotted lines respectively.

ical viewpoints.

In de Sitter spacetime, two vacua, the vacuum of the static chart and the Bunch-Davies vacuum, are often discussed. We describe de Sitter spacetime with two different coordinates,

$$
\begin{aligned}
ds^2 &= -(1-H^2r_s^2)dt_s^2 + (1-H^2r_s^2)^{-1}dr_s^2 = (1-H^2r_s^2)(-dt_s^2 + dr_s^{*2}) \quad (5.107)\\
&= -dt_f^2 + e^{2Ht_f}dr_f^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dr_f^2), \quad (5.108)
\end{aligned}
$$

where

$$
\begin{aligned}
&r_s^* := \frac{\tanh^{-1}(Hr_s)}{H}, \quad (5.109)\\
&r_f := re^{-Ht_f}, \qquad \eta := -\frac{e^{-Ht_f}}{H}, \qquad t_f := t_s + \frac{1}{2H}\log\left[H^{-1}\left(1 - H^2r_s^2\right)\right], \quad (5.110)
\end{aligned}
$$

and "s" and "f" mean the static and flat slicing charts, respectively. The vacua with the coordinates (5.107) and (5.108) are corresponding to the vacuum of the static chart and the Bunch-Davies vacuum, respectively.

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Bunch-Davies Vacuum

The vacuum state of the flat chart (5.108) is the so-called Bunch-Davies state [31]. The flat chart (5.108) describes the region shown in FIG. 5.7. We set the boundaries at $r_f = r_\pm$ and take the limit $r_\pm \to \pm\infty$. Then, the general solution (5.58) becomes

$$\varphi = -2\ln(H\eta) + A_2\eta + \int d\omega\, c(\omega)\cos[\omega r_f]e^{i\omega\eta}. \tag{5.111}$$

The stationary stress tensor with respect to the conformal time η is obtained for $c(\omega) = 0$ as

$$T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} -H^2 + \frac{A_2^2}{4}e^{-2Ht_f} & 0 \\ 0 & e^{2Ht_f}H^2 + \frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (t_f, r_f) \text{ coordinate,} \tag{5.112}$$

$$= \frac{1}{24\pi}\begin{pmatrix} -e^{2Ht_f}H^2 + \frac{A_2^2}{4} & 0 \\ 0 & e^{2Ht_f}H^2 + \frac{A_2^2}{4} \end{pmatrix} = \frac{1}{24\pi}\begin{pmatrix} -\eta^{-2} + \frac{A_2^2}{4} & 0 \\ 0 & \eta^{-2} + \frac{A_2^2}{4} \end{pmatrix}$$
$$\text{in } (\eta, r_f) \text{ coordinate.} \tag{5.113}$$

The lowest energy state is realized for $A_2 = 0$, and then the stress tensor becomes the same as that of Bunch-Davies vacuum. A_2 describes the thermal state with respect to the conformal time ∂_η.

Static Vacuum

The static chart (5.107) describes the region shown in FIG. 5.8. We set boundary at $r_s = r_\pm$ and take the limit $r_\pm = \pm\infty$. Then the general solution (5.58) becomes

$$\varphi = \ln(1 - H^2 r_s^2) + A_2 t_s + \int d\omega\, c(\omega)\cos[\omega r_s^*]e^{i\omega t_s}. \tag{5.114}$$

The stationary stress tensor with respect to the Killing direction ∂_{t_s} is obtained for

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$c(\omega) = 0$, and it is derived as

$$T_{\mu\nu} = \frac{1}{24\pi} \begin{pmatrix} -2H^2 + H^4 r_s^2 + \frac{A_2^2}{4} & 0 \\ 0 & -H^4 r_s^2 + \frac{A_2^2}{4} \end{pmatrix} \quad \text{in } (t_s, r_s^*) \text{ coordinate,} \tag{5.115}$$

$$= \frac{1}{24\pi} \begin{pmatrix} -2H^2 + H^4 r_s^2 + \frac{A_2^2}{4} & 0 \\ 0 & \frac{-H^4 r_s^2 + A_2^2/4}{(1-H^2 r_s^2)^2} \end{pmatrix} \quad \text{in } (t_s, r_s) \text{ coordinate.} \tag{5.116}$$

Imposing $A_2 = 0$, the minimum energy state is realized and the stress tensor becomes the same as that of the vacuum state in static chart. A_2 describes the thermal excitation on the static chart. For $A_2 = \pm 2H$, the resulting stress tensor is the same as that of the Bunch-Davies vacuum state. That is, Bunch-Davies vacuum state is a thermal equilibrium state based on static vacuum.

5.3.3 Dynamical Casimir effect

In the following, let's consider the one moving mirror problem in flat spacetime. In this problem, there exists two boundaries $x = 0$ and $x = L(t)$ in flat space. Next let us introduce "conformal-flat" coordinate (w, s) where the corresponding boundaries are $s = 0$ and $s = 1$,

$$\begin{cases} x = 0 \Leftrightarrow s = 0, \\ x = 0 \Leftrightarrow s = 0. \end{cases} \tag{5.117}$$

Also the corresponding line element is

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 = -dudv = -f^{'}(\bar{u})g^{'}(\bar{v})d\bar{u}d\bar{v} \\ &= \underbrace{f^{'}(w-s)g^{'}(w+s)}_{\equiv F(w,s) \text{ or } F(\bar{u},\bar{v})}(-dw^2 + ds^2), \end{aligned} \tag{5.118}$$

where $\begin{cases} t - x = u = f(\bar{u}) = f(w-s) \\ t + x = v = g(\bar{v}) = g(w+s) \end{cases}$ and $\begin{cases} \bar{u} = w - s \\ \bar{v} = w + s \end{cases}$.

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Because there exists only one d.o.f., i.e. $L(t)$ in this problem, we can thus find a function R [12] satisfy $f = g = R^{-1}$. For later convenience, we also define a function Q,

$$Q(\bar{u}) \equiv \frac{d}{d\bar{u}} R^{-1}(\bar{u}) = \frac{1}{R'[R^{-1}(\bar{u})]}, \tag{5.119}$$

and then

$$\begin{aligned} Q' &= \frac{d}{d\bar{u}} Q = -\frac{R''(R^{-1}(\bar{u}))}{R'^3(R^{-1}(\bar{u}))}, \\ Q'' &= \frac{\mathrm{d}^2}{d\bar{u}^2} Q = \frac{3R''^2}{R'^3} - \frac{R'''}{R'^4}. \end{aligned} \tag{5.120}$$

After the set up for this problem, we can now immediately calculate the solution of φ for this spacetime (5.118):

$$\varphi = \varphi_p + \varphi_0, \tag{5.121}$$

where

$$\varphi_p = l_n F = l_n Q(\bar{u}) + l_n Q(\bar{v}). \tag{5.122}$$

φ_p is responsible for the vacuum solution and by substituting it into the stress tensor formula below

$$24\pi T^{\varphi_p}_{\mu\nu} = g_{\mu\nu}\Box\varphi_p + \frac{1}{4} g_{\mu\nu}\nabla_\alpha\varphi_p\nabla^\alpha\varphi_p - \frac{1}{2}\nabla_\mu\varphi_p\nabla_\nu\varphi_p - \nabla_\mu\nabla_\nu\varphi_p, \tag{5.123}$$

[11] The mapping of these functions: $f, g, R^{-1} : \begin{smallmatrix}\bar{u} \\ \bar{v}\end{smallmatrix} \to \begin{smallmatrix}u \\ v\end{smallmatrix}$; $R : \begin{smallmatrix}u \\ v\end{smallmatrix} \to \begin{smallmatrix}\bar{u} \\ \bar{v}\end{smallmatrix}$.

[12] Be careful that the R here is not Ricci scalar, we choose R to keep the convention the same as the previous works.

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we can get the renormalised stress tensor in $(\bar{u}, \bar{v})$ frame as:

$$\begin{cases} T^{\varphi_p}_{\bar{u}\bar{u}} = \bar{h}(\bar{u}), \\ T^{\varphi_p}_{\bar{v}\bar{v}} = \bar{h}(\bar{v}), \\ T^{\varphi_p}_{\bar{u}\bar{v}} = T^{\varphi_p}_{\bar{v}\bar{u}} = 0, \end{cases} \tag{5.124}$$

and in (u, v) frame as:

$$\Rightarrow \begin{cases} T^{\varphi_p}_{uu} = (R^{'}(u))^2 T_{\bar{u}\bar{u}} = h(u), \\ T^{\varphi_p}_{vv} = (R^{'}(v))^2 T_{\bar{v}\bar{v}} = h(v), \\ T^{\varphi_p}_{uv} = T^{\varphi_p}_{vu} = 0, \end{cases} . \tag{5.125}$$

where[13]

$$\bar{h}(\bar{u}) \equiv \frac{1}{2Q^2}[Q^{'2} - 2QQ^{''} \underbrace{\frac{F_u}{F}}_{=\frac{Q^{'}}{Q}}] = \frac{3[Q^{'}(\bar{u})]^2 - 2Q(\bar{u})Q^{''}(\bar{u})}{2[Q(\bar{u})]^2}$$
$$= \frac{3}{2}(\frac{R^{''}}{R^{'2}})^2 - (\frac{3R^{''2}}{R^{'4}} - \frac{R^{'''}}{R^{'3}}) = -\frac{3}{2}(\frac{R^{''}}{R^{'2}})^2 + \frac{R^{'''}}{R^{'3}} = \frac{1}{R^{'2}}[-\frac{3}{2}(\frac{R^{''}}{R^{'}})^2 + \frac{R^{'''}}{R^{'}}]. \tag{5.126}$$

Finally by using the formalism we develop, we get the same result from dynamical Casimir problem as the works before [32, 33].

[13] The $R^{'2}$ below means $[R^{'}(R^{-1}(\bar{u}))]^2$.

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Chapter 6

4-dimensional Anomaly-induced Action

Although we have proposed the 2-dim anomaly-induced action and applies it to investigate some physical issues, it can only be treated as a toy model used to mimic our real world which is (at least) 4-dimensional spacetime. Therefore, in order to investigate the real physics in our world, we need to generalize the anomaly-induced action method to 4-dim spacetime.

In this chapter, we will first derive the 4-dim anomaly-induced action without boundary effect. Although the 4-dim anomaly-induced action (without boundary effect) has been widely used before, the result we proposed here slightly differs from what in the previous works [9, 10, 12] in several aspects: First, it turns out that introducing one auxiliary scalar field is already sufficient to obtain the local anomaly-induced action instead of two. Second, we restore the missing d.o.f of the conformally invariant terms which has been ignored before back to the anomaly-induced action. Finally the stress tensor from the 4-dim anomaly-induced action is slightly revised and it thus results in the correct 4-dim trace anomaly whereas the result from the literature before seems to be problematic.

In the next section, we will derive the 4-dim anomaly-induced action with boundary effect based on the "Green's function-modified method". It turns out that by using this method, it is much easier to generalize the anomaly-induced action to 4-dim than "operator-modified method". Also, we learned from our final result that by this method, it is not necessary to require Green's function to be symmetric in order to obtain the 4-dim anomaly-induced action and the 4-dim anomaly-induced action must be limited to the

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spacetime with zero Euler characteristic.

6.1 4-dimensional Anomaly-induced Action Without Boundary Effect

Same as 2-dim case, in order to derive 4-dim S_{anom}, we start with the counter terms for 4-dim case which is introduced by Eq.(4.102),

$$S_{div}[g] = -\frac{1}{16\pi^2}\lim_{n\to 4}\frac{\int_{\mathcal{M}} d^4x\sqrt{-g}[\frac{1}{120}(\frac{-1}{3}E + F - \frac{2}{3}\Box R)]}{n-4}, \tag{6.1}$$

$$= \lim_{n\to 4}\frac{\int_{\mathcal{M}} d^4x\sqrt{-g}\left[b'E + b(F + \frac{2}{3}\Box R)\right]}{n-4}. \tag{6.2}$$

Similar to 2-dim case, the effective action $a_2(x)$ for conformal scalar field is conformal invariant [1]. However, this counter term designed in $4+\varepsilon$ dimension is not conformal invariant. Therefore after dimensional-regularization, the renormalised action is no longer conformal invariant and results in a nonzero trace of stress tensor, i.e. trace anomaly. Substituting this counter term into eq.(5.3), we can derive the WZ action as

$$\begin{aligned}\Gamma_{WZ}[\bar{g},\sigma] &= \int_M d^4x \frac{\left\{\sqrt{-g}\left[bF + b'\left(E + \frac{2}{3}\Box R\right)\right] - \overline{\dots}\right\}}{\varepsilon} \\ &= \int_M d^4x \Bigg\{ \frac{1}{2}\left\{\sqrt{-g}\left[bF + b'\left(E + \frac{2}{3}\Box R\right)\right] + \overline{\dots}\right\}\sigma \\ &\quad + \left(-\frac{b}{18} - \frac{b'}{18}\right)\left[\sqrt{-g}R^2 - \overline{\dots}\right] + c\left[\sqrt{-g}F - \overline{\dots}\right]\Bigg\}\end{aligned} \tag{6.3}$$

where c is an arbitrary constant. [2] From eq.(A.38), we can get the following relation:

$$\left\{\sqrt{-g}\left[bF + b'\left(E - \frac{2}{3}\Box R\right)\right] - \overline{\dots}\right\} = 2b'\left(\sqrt{-g}\Delta_4\sigma + \overline{\dots}\right) = 4b'\sqrt{-g}\Delta_4\sigma, \tag{6.4}$$

where $\Delta_4 := \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu$ is the unique 4th-order conformal

[1] Although the last term $-\frac{2}{3}\Box R$ in a_2 is not conformal invariant, we will ignore it because in this section we neglect all boundary contributions.

[2] Because $[\sqrt{-g}F - \overline{\dots}] = 0 + O(\varepsilon)$, it leads to an additional d.o.f to add the last term with an arbitrary coefficient.

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invariant differential operator in 4-dim spacetime. Then, we can define $D_4(x, x')$, the inverse operator of Δ_4, which satisfies

$$\Delta_4 D_4(x, x') = \frac{\delta^{(4)}(x - x')}{\sqrt{-g}}, \tag{6.5}$$

and thus express σ in terms of $D_4(x, x')$ as

$$\sigma = \frac{1}{4b'} \int d^4x' D_4(x, x') \left\{ \sqrt{-g} \left[bF + b' \left(E - \frac{2}{3} \Box R \right) \right] - \bar{...} \right\}. \tag{6.6}$$

By substituting eq.(6.6) into eq.(6.3) and imposing the symmetric condition of D_4, i.e. $D_4(x, x') = D_4(x', x)$, we will get

$$\Gamma_{\omega z} = \frac{1}{8b'} \int d^4x \int d^4x' \left\{ \sqrt{-g} \left[bF + b' \left(E - \frac{2}{3} \Box R \right) \right] + \bar{...} \right\} D_4$$
$$\left\{ \sqrt{-g'} \left[bF' + b' \left(E' - \frac{2}{3} \Box' R' \right) - \bar{...} \right] \right\} + \int d^4x \left[-\frac{b + b'}{18} \left(\sqrt{-g} R^2 - \bar{...} \right) + c \left(\sqrt{-g} F - \bar{...} \right) \right]. \tag{6.7}$$

From eq.(5.4) and eq.(6.7), we can now write down the nonlocal anomaly-induced action:

$$S_{anom} = -\frac{1}{8b'} \int d^4x \int d^4x' \sqrt{-g} \left[bF + b' \left(E - \frac{2}{3} \Box R \right) \right] D_4 \sqrt{-g'} \left[bF' + b' \left(E' - \frac{2}{3} \Box' R' \right) \right]$$
$$+ \int d^4x \sqrt{-g} \left[\frac{b + b'}{18} R^2 - cF \right]. \tag{6.8}$$

This non-local anomaly-induced action can be localized by introducing a real auxiliary scalar field φ which is defined as

$$\varphi(x) \equiv \int d^4x \sqrt{-g} \left\{ -\varphi \Delta_4 \varphi + 2 \left[bF + b' \left(E - \frac{2}{3} \Box R \right) \right] \varphi \right\}. \tag{6.9}$$

Operating this auxiliary scalar field by the Δ_4 operator, we can obtain its field equation as

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follows:

$$\Delta_4\varphi = bF + b'\left(E - \frac{2}{3}\Box R\right). \tag{6.10}$$

Now the localized version of the anomaly-induced action, eq.(6.8), can be expressed in terms of the auxiliary scalar field φ as

$$S_{anom} = -\frac{1}{8b'}\int d^4x\sqrt{-g}\{-\varphi\Delta_4\varphi + 2[bF + b'(E - \frac{2}{3}\Box R)]\varphi\} + \int d^4x\sqrt{-g}\left[\frac{b+b'}{18}R^2 - cF\right]. \tag{6.11}$$

We can check that eq.(6.10) can also be obtained by varying this local action w.r.t. φ, and thus it is consistent with the action, eq.(6.8). After substituting eq.(6.9) into it, this action is reduced to the anomaly action, eq.(6.8), and thus it gives the same dynamics as the non-local action.

The corresponding stress tensor can be obtained by varying the localized action w.r.t. the metric $g_{\mu\nu}$, and its explicit form is

$$T_{ab}^{anom}[g;\varphi] := \frac{-2}{\sqrt{-g}}\frac{\delta S_{anom}}{\delta g^{ab}} = \frac{1}{8b'}(2A_{ab} + 4b'B_{ab} + 4bC_{ab}) - \frac{b+b'}{9}D_{ab} + 2cE_{ab}, \tag{6.12}$$

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where A_{ab}, B_{ab}, C_{ab}, D_{ab} and E_{ab} are defined as

$$\begin{aligned}
A_{ab}[g;\varphi] \quad &:= \frac{1}{\sqrt{-g}}\frac{\delta[\int d^4x\sqrt{-g}(-\varphi\Delta_4\varphi)]}{\delta g^{ab}} \\
&= g_{ab}\{+\frac{1}{2}(\Box\varphi)^2 - R^{cd}(\nabla_c\varphi)(\nabla_d\varphi) + \frac{1}{3}Rg^{cd}(\nabla_c\varphi)(\nabla_d\varphi) - \frac{1}{6}\Box[(\nabla_c\varphi)(\nabla^c\varphi)]\} \\
&+ 2(\nabla_{(a}\Box\varphi)(\nabla_{b)}\varphi) + 4R^c{}_{(a}(\nabla_{b)}\varphi)(\nabla_c\varphi) - 2\nabla_c[(\nabla_a\nabla_b\varphi)(\nabla^c\varphi)] \\
&- \frac{2}{3}(\nabla_a\varphi)(\nabla_b\varphi)R - \frac{2}{3}(\nabla_c\varphi)(\nabla^c\varphi)R_{ab} + \frac{2}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\varphi)], \qquad (6.13)\\
B_{ab}[g;\varphi] \quad &:= \frac{1}{\sqrt{-g}}\frac{\delta\int d^4x(E-\frac{2}{3}\Box R)\varphi}{\delta g^{ab}} = +\frac{2}{3}\nabla_a\nabla_b\Box\varphi + 4C_{a\ b}^{\ c\ d}\nabla_c\nabla_d\varphi \\
&+ 4R^c_{(a}\nabla_{b)}\nabla_c\varphi - \frac{8}{3}R_{ab}\Box\varphi - \frac{4}{3}R\nabla_a\nabla_b\varphi + \frac{2}{3}\left(\nabla_{(a}R\right)\nabla_{b)}\varphi \\
&- \frac{1}{3}g_{ab}\left\{2\Box^2\varphi + 6R^{cd}\nabla_c\nabla_d\varphi - 4R\Box\varphi + (\nabla^cR)\nabla_c\varphi\right\}, \qquad (6.14)\\
C_{ab}[g;\varphi] \quad &:= \frac{1}{\sqrt{-g}}\frac{\delta\int d^4xF\varphi}{\delta g^{ab}} = 4\nabla^c\nabla^d\left(C_{cabd}\varphi\right) + 2C_{cabd}R^{cd}\varphi, \qquad (6.15)\\
D_{ab}[g] \quad &:= \frac{1}{\sqrt{-g}}\frac{\delta\int d^4xR^2}{\delta g^{ab}} = -\frac{1}{2}g_{ab}R^2 + (2R)R_{ab} - \nabla_b\nabla_a(2R) + g_{ab}\Box(2R), \\
&\qquad (6.16)\\
E_{ab}[g] \quad &:= \frac{1}{\sqrt{-g}}\frac{\delta\int d^4xF}{\delta g^{ab}} = 4\nabla^c\nabla^d\left(C_{cabd}\right) + 2C_{cabd}R^{cd}. \qquad (6.17)
\end{aligned}$$

By checking the trace of each component

$$\begin{aligned}
A^a_a &= 0, \qquad (6.18)\\
B^a_a &= -2\Delta_4\varphi \overset{Eq.(6.10)}{=} -2\left[bF + b'\left(E - \frac{2}{3}\Box R\right)\right], \qquad (6.19)\\
C^a_a &= 0 \qquad (6.20)\\
D^a_a &= 6\Box R, \qquad (6.21)\\
E^a_a &= 0, \qquad (6.22)
\end{aligned}$$

we will see that the trace of the stress tensor is exactly the 4-dim trace anomaly:

$$g^{\mu\nu}T^{anom}_{\mu\nu} = -\left[bF + b'\left(E - \frac{2}{3}\Box R\right)\right] - \frac{2}{3}(b+b')\Box R = -\left[b\left(F + \frac{2}{3}\Box R\right) + b'E\right]. \qquad (6.23)$$

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6.2 4-dimensional Anomaly-induced Action With Boundary Effect Based on Green's Function-Modified Method

Because it is much easier to generalize the Green's function-modified method to 4-dim case, in this section, we will derive the 4-dim anomaly-induced action with boundary effect by this way. At first, Similar to the 2-dim case, lets consider the identity [34]:

$$
\begin{aligned}
&\int_{\mathcal{M}} d^4x\sqrt{-g}(u\Delta_4 v - v\Delta_4 u)\\
&= \epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\left\{(u\Delta_3 v - v\Delta_3 u) - [(\nabla_n u)(\Box v) - (\nabla_n v)(\Box u)]\right\}\\
&= -\epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\left\{[(\mathbf{B}_0 u)(\mathbf{B}_3 v) + (\mathbf{B}_1 u)(\mathbf{B}_2 v)] - [u\leftrightarrow v]\right\}\\
&+ \epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\Bigg\{\bigg\{ - 2\mathcal{D}_\mu\left[u(K\mathcal{D}^\mu - K^{\mu\nu}\mathcal{D}_\nu)v\right] + \frac{2}{3}\mathcal{D}_\mu\left[uK\mathcal{D}^\mu v\right] - 2\mathcal{D}_\mu(u\mathcal{D}^\mu\nabla_n v)\\
&- 2\mathcal{D}_\mu\left[(\nabla_n u)\mathcal{D}^\mu v\right]\bigg\} - \{u\leftrightarrow v\}\Bigg\}\\
&\approx -\epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\left\{[(\mathbf{B}_0 u)(\mathbf{B}_3 v) + (\mathbf{B}_1 u)(\mathbf{B}_2 v)] - [u\leftrightarrow v]\right\},
\end{aligned}
\tag{6.24}
$$

where u and v are arbitrary scalar functions, also Δ_4 and its boundary-associated operators, $\mathbf{B}_i$, $i = 0 \sim 3$ [34] are conformal invariant operators which are defined by

$$\mathbf{B}_0 u := u, \tag{6.25}$$

$$\mathbf{B}_1 u := \nabla_n u, \tag{6.26}$$

$$\mathbf{B}_2 u := (\Box - 2\mathcal{D}^2 - \frac{2}{3}\epsilon K\nabla_n)u, \tag{6.27}$$

$$
\begin{aligned}
\mathbf{B}_3 u := \Big[&-(\nabla_n\Box + 2n_\mu G^{\mu\nu}\nabla_\nu + \frac{1}{3}R\nabla_n) - 2\mathcal{D}_\mu(K\mathcal{D}^\mu - K^{\mu\nu}\mathcal{D}_\nu)\\
&+\frac{2}{3}\mathcal{D}^\mu(K\mathcal{D}_\mu) - 2\mathcal{D}^2\nabla_n\Big]\, u.
\end{aligned}
\tag{6.28}
$$

There conformal invariant operators satisfy the following relations

$$\Delta_4\sigma = e^{-4\sigma}\bar{\Delta}_4\sigma, \tag{6.29}$$

$$\mathbf{B}_i\sigma = e^{-i\sigma}\bar{\mathbf{B}}_i. \tag{6.30}$$

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Note that we have ignored the 2-dimensional boundary terms to get the last line of eq. (6.24).

Similar to 2-dim case, the Green's function, i.e. the inverse operator, of Δ_4 is now defined by

$$\Delta_4 G(x,x') = \frac{\delta^{(4)}(x,x')}{\sqrt{-g}} \tag{6.31}$$

$$\mathbf{B_3} G(x,x') = w(x) \tag{6.32}$$

$$\mathbf{B_2} G(x,x') \tag{6.33}$$

$$\mathbf{B_1} G(x,x') \tag{6.34}$$

where $w(x)$ is an arbitrary function which should satisfies[3]

$$\epsilon \int_\Sigma d^3x \sqrt{-\epsilon\gamma}\, w(x) = -1. \tag{6.35}$$

It can be shown that in 4-dim spacetime the conformal transformation parameter σ and these conformal invariant operators is related to the following four equations:

$$\Delta_4 \sigma = E_Q - e^{-4\sigma}\bar{E}_Q, \tag{6.36}$$

$$\mathbf{B}_i \sigma = T_i - e^{-i\sigma}\bar{T}_i, \tag{6.37}$$

where E_Q [4] and T_i, $i = 1 \sim 3$ [34] are defined by

$$E_Q := \frac{1}{4}(E - \frac{2}{3}\Box R), \tag{6.38}$$

$$T_1 := \frac{K}{3}, \tag{6.39}$$

$$T_2 := \frac{1}{4}{}^\gamma R - \frac{1}{2}\epsilon R_{nn} + \frac{1}{12}R + \frac{1}{18}\epsilon K^2 = \frac{1}{3}R - \epsilon R_{nn} + \frac{11}{36}\epsilon K^2 - \frac{1}{4}\epsilon K_2, \tag{6.40}$$

$$T_3 := \frac{1}{4}(E^B + \frac{2}{3}\nabla_n R) - \frac{2}{3}\mathcal{D}^2 K. \tag{6.41}$$

[3] In order to guarantee $\int_\mathcal{M} d^4x \Delta_4 G(x,x') = \epsilon \int_\Sigma d^3x \sqrt{-\epsilon\gamma}\mathbf{B_3} G(x,x')$.

[4] The E_Q defined here is indeed the Q curvature [35–38] which is associated to the Δ_4 operator.

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By using eq.(6.24) and eq.(6.37), we can expresses σ in terms of $G(x,x')$ as:

$$\sigma(x) = \int_{\mathcal{M}} d^4x' \sqrt{-g'} G(x',x)\Delta_4'\sigma' + \epsilon \int_{\Sigma} d^3x' \sqrt{-\epsilon\gamma'}\, [-(\mathbf{B}_3' G(x',x))(\mathbf{B}_0'\sigma')$$
$$+G(x',x)(\mathbf{B}_3'\sigma') - (\mathbf{B}_2' G(x',x))(\mathbf{B}_1'\sigma') + (\mathbf{B}_1' G(x',x))(\mathbf{B}_2'\sigma')] \tag{6.42}$$
$$= \int_{\mathcal{M}} d^4x' G(x',x)[\sqrt{-g'}E_Q' - \sqrt{-\bar{g}'}\bar{E}_Q'] + \langle\sigma\rangle_w$$
$$+ \epsilon \int_{\Sigma} d^3x' \Big\{ G(x',x)[\sqrt{-\epsilon\gamma'}T_3' - \sqrt{\epsilon\bar{\gamma}'}\bar{T}_3'] - [\sqrt{-\epsilon\gamma'}(\mathbf{B}_2' G(x',x))T_1'$$
$$- \sqrt{\epsilon\bar{\gamma}'}(\bar{\mathbf{B}}_2' G(x',x))\bar{T}_1'] + [\sqrt{-\epsilon\gamma'}(\mathbf{B}_1' G(x',x))T_2' - \sqrt{\epsilon\bar{\gamma}'}(\bar{\mathbf{B}}_1' G(x',x))\bar{T}_2'] \Big\} \tag{6.43}$$

where $\langle u(x)\rangle_{w(x)} := -\epsilon \int_\Sigma d^3x' \sqrt{-\epsilon\gamma'} w(x')u(x')$ for any scalar function u(x).

Similar to the 2-dimensional case, we introduce a real auxiliary scalar field φ which is defined by

$$\phi(x) := \int_{\mathcal{M}} d^4x' \sqrt{-g'} G(x',x) E_Q' + \langle\phi\rangle_w \tag{6.44}$$
$$+ \epsilon \int_{\Sigma} d^3x' \sqrt{-\epsilon\gamma'}\, [G(x',x)T_3' - (\mathbf{B}_2' G(x',x))T_1' + (\mathbf{B}_1' G(x',x))T_2] \tag{6.45}$$

Note that by the definition above, the scalar field φ automatically satisfies the following equation of motion and boundary condition:

$$\begin{cases} \Delta_4\phi = E_Q, \\ \mathbf{B}_i\phi = T_i, \qquad x\in\Sigma, i = 1 \sim 3. \end{cases} \tag{6.46}$$

Together with the relation $G(x,x') = \bar{G}(x,x')$ which is resulted from the conformal symmetry of Δ_4 and $\mathbf{B}_i$, i.e. eq.(6.30), σ can thus be expressed in terms of φ as

$$\sigma(x) = \varphi(x) - \bar{\varphi}(x). \tag{6.47}$$

In the setting of 4-manifolds with boundary, Δ_4 and its boundary associated operators,

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$\mathbf{B}_i$, are related to two "cocycle" functionals, $\mathcal{F}$ and $\mathcal{G}$, which are defined by

$$\mathcal{F}[u(x)] := \int_{\mathcal{M}} d^4x\sqrt{-g}[u\Delta_4 u - 2E_Q u] + \epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}[u\mathbf{B_3}u - 2T_3 u$$
$$+ (\nabla_n u)\mathbf{B_2}u - 2T_2(\mathbf{B_1}u)], \tag{6.48}$$
$$\mathcal{G}[u(x)] := \epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}[(\mathbf{B_1}u)(\mathbf{B_2}u) - T_2(\mathbf{B_1}u) - T_1(\mathbf{B_2}u)], \tag{6.49}$$

where $u(x)$ is an arbitrary scalar function. It can be checked that these functionals, $\mathcal{F}$ and $\mathcal{G}$, satisfy the following "cocycle" condition:

$$\mathcal{F}[\sigma + u] = \bar{\mathcal{F}}[u], \tag{6.50}$$
$$\mathcal{G}[\sigma + u] = \bar{\mathcal{F}}[u]. \tag{6.51}$$

Next, with the help of these cocycle functionals, the first part of the WZ action can be derived as follows:

$$\frac{\int_{\mathcal{M}} d^4x[\sqrt{-g}E - \sqrt{-\bar{g}}\bar{E}] + \epsilon\int_{\Sigma} d^3x[\sqrt{-\epsilon\gamma}E^B - \sqrt{\epsilon\bar{\gamma}}\bar{E}^B]}{\varepsilon}$$
$$= \frac{1}{\varepsilon}\epsilon\int_{\Sigma} d^3x\left\{-4[\sqrt{-\epsilon\gamma}\mathcal{D}_\mu(K\mathcal{D}^\mu\sigma - K^{\mu\nu}\mathcal{D}_\nu\sigma) + \bar{...}]\right\}$$
$$+ \epsilon\int_{\Sigma} d^3x\Big\{-4[\sqrt{-\epsilon\gamma}\mathcal{D}_\mu(K\mathcal{D}^\mu\sigma - K^{\mu\nu}\mathcal{D}_\nu\sigma) + \bar{...}] + 2[\sqrt{-\epsilon\gamma}\mathcal{D}_\mu(\sigma K\mathcal{D}^\mu\sigma$$
$$- \sigma K^{\mu\nu}\mathcal{D}_\nu\sigma) - \bar{...}] + \frac{4}{3}[\sqrt{-\epsilon\gamma}\mathcal{D}_\mu(\sigma\mathcal{D}^\mu K - K\mathcal{D}^\mu\sigma) + \bar{...}]\Big\}$$
$$+ \int_{\mathcal{M}} d^4x\left\{-\frac{1}{18}[\sqrt{-g}R^2 - \bar{...}] + \frac{1}{2}[\sqrt{-g}(E - \frac{2}{3}\Box R) + \bar{...}]\sigma\right\}$$
$$+ \epsilon\int_{\Sigma} d^3x\Big\{\frac{1}{2}[\sqrt{-\epsilon\gamma}(E^B + \frac{2}{3}\nabla_n R - \frac{8}{3}\mathcal{D}^2 K) + \bar{...}]\sigma + \frac{4}{3}[\sqrt{-\epsilon\gamma}K\mathcal{D}^2\sigma + \bar{...}]$$
$$- \frac{1}{3}[\sqrt{-\epsilon\gamma}R\nabla_n\sigma + \bar{...}] - [\sqrt{-\epsilon\gamma}(\nabla_n\sigma)(\nabla\sigma)^2 + \bar{...}] + \frac{2}{3}\epsilon[\sqrt{-\epsilon\gamma}(\nabla\sigma)^3 + \bar{...}]\Big\} \tag{6.52}$$
$$\approx -\frac{1}{18}\int_{\mathcal{M}} d^4x[\sqrt{-g}R^2 - \bar{...}] - 2\mathcal{F}[\sigma] + 4\mathcal{G}[\sigma]$$
$$+ \epsilon\int_{\Sigma} d^3x\left\{-\frac{2}{9}[\sqrt{-\epsilon\gamma}KR - \bar{...}] - \frac{8}{81}\epsilon[\sqrt{-\epsilon\gamma}K^3 - \bar{...}]\right\} \tag{6.53}$$
$$= \left\{\left[-2\mathcal{F}[\phi] + 4\mathcal{G}[\phi] - \frac{1}{18}\int_{\mathcal{M}} d^4x[\sqrt{-g}R^2 - \bar{...}] - \epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}(\frac{2}{9}KR + \frac{8}{81}\epsilon K^3)\right] - [\bar{...}]\right\}$$
$$=: \bar{S}^A_{anom}[\bar{g};\bar{\varphi}] - S^A_{anom}[g;\varphi], \tag{6.54}$$

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where

$$S^{A}_{anom}[g,\varphi] = 2\mathcal{F}[\varphi] - 4\mathcal{G}[\varphi] + \frac{1}{18}\int_{\mathcal{M}} d^4x\sqrt{-g}R^2 + \int_{\Sigma} d^3x\sqrt{-\gamma}(\frac{2}{9}KR + \frac{8}{81}K^3), \tag{6.55}$$

Also, we have used eq.(6.47) ,eq.(6.51) and ignored 2-dimensional boundary terms to get the last second line. The remaining parts of the WZ action are easies to derive and they satisfy the following relations:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathcal{M}} d^4x[\sqrt{-g}F - \bar{\dots}] \\
&= \int_{\mathcal{M}} d^4x\left\{\frac{1}{2}(\sqrt{-g}F + \sqrt{-\bar{g}}\bar{F})\sigma + \frac{5}{18}\left[\sqrt{-g}R^2 - \bar{\dots}\right] - \left[\sqrt{-g}R^2_{ab} - \bar{\dots}\right]\right\} \\
&= \int_{\mathcal{M}} d^4x\left\{(\sqrt{-g}F\varphi - \bar{\dots}) + \frac{5}{18}\left[\sqrt{-g}R^2 - \bar{\dots}\right] - \left[\sqrt{-g}R^2_{ab} - \bar{\dots}\right]\right\} \\
&=: \bar{S}^{B}_{anom}[\bar{g};\bar{\varphi}] - S^{B}_{anom}[g;\varphi],
\end{aligned} \tag{6.56}$$

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Sigma} d^3x[\sqrt{-\epsilon\gamma}\epsilon j_1 - \bar{\dots}] \\
&= \int_{\Sigma} d^3x\left\{\frac{1}{2}(\sqrt{-\epsilon\gamma}\epsilon j_1 + \bar{\dots})\sigma - \left[\sqrt{-\epsilon\gamma}(\frac{1}{4}K^{ab}R_{ab} + \frac{1}{4}\epsilon n^a n^b R_{ab}K - \frac{5}{36}KR) - \bar{\dots}\right]\right\} \\
&= \int_{\Sigma} d^3x\left\{(\sqrt{-\epsilon\gamma}\epsilon j_1\varphi - \bar{\dots}) - \left[\sqrt{-\epsilon\gamma}(\frac{1}{4}K^{ab}R_{ab} + \frac{1}{4}\epsilon n^a n^b R_{ab}K - \frac{5}{36}KR) - \bar{\dots}\right]\right\} \\
&=: \bar{S}^{C}_{anom}[\bar{g};\bar{\varphi}] - S^{C}_{anom}[g;\varphi],
\end{aligned} \tag{6.57}$$

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Sigma} d^3x[\sqrt{-\epsilon\gamma}\epsilon j_2 - \bar{\dots}] \\
&= \int_{\Sigma} d^3x\left\{\frac{1}{2}(\sqrt{-\epsilon\gamma}\epsilon j_2 + \bar{\dots})\sigma + \frac{1}{3}\epsilon\left[\sqrt{-\epsilon\gamma}K^2\nabla_n\sigma + \bar{\dots}\right]\right. \\
&\qquad\left. - \frac{1}{2}\epsilon\left[\sqrt{-\epsilon\gamma}K_2\nabla_n\sigma + \bar{\dots}\right] - \frac{1}{4}\epsilon\left[\sqrt{-\epsilon\gamma}(\nabla_n\sigma)^3 + \bar{\dots}\right]\right\} \\
&= \int_{\Sigma} d^3x\left\{\frac{1}{2}(\sqrt{-\epsilon\gamma}\epsilon j_2 + \bar{\dots})\sigma + \epsilon\left[\sqrt{-\epsilon\gamma}(\frac{4}{27}K^3 - \frac{1}{3}KK_2) - \bar{\dots}\right]\right\} \\
&= \int_{\Sigma} d^3x\left\{(\sqrt{-\epsilon\gamma}\epsilon j_2\varphi - \bar{\dots}) + \epsilon\left[\sqrt{-\epsilon\gamma}(\frac{4}{27}K^3 - \frac{1}{3}KK_2) - \bar{\dots}\right]\right\} \\
&=: \bar{S}^{D}_{anom}[\bar{g};\bar{\varphi}] - S^{D}_{anom}[g;\varphi],
\end{aligned} \tag{6.58}$$

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where

$$S^B_{anom}[g;\varphi] := -\int_{\mathcal{M}} d^4x\sqrt{-g}(F\varphi + \frac{5}{18}R^2 - R^2_{ab}), \tag{6.59}$$

$$S^C_{anom}[g;\varphi] := -\epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\left[j_1\varphi - (\frac{1}{4}\epsilon K^{ab}R_{ab} + \frac{1}{4}n^a n^b R_{ab}K - \frac{5}{36}\epsilon KR)\right], \tag{6.60}$$

$$S^D_{anom}[g;\varphi] := -\epsilon\int_{\Sigma} d^3x\sqrt{-\epsilon\gamma}\left[j_2\varphi + (\frac{4}{27}K^3 - \frac{1}{3}KK_2)\right]. \tag{6.61}$$

Then, from eq.(4.112) and the result above, we can immediately read the anomaly-induced action as

$$\Gamma_{WZ}[\bar{g},\sigma] = S_{div}[g] - S_{div}[\bar{g}] = b' S^A_{anom}[g] + bS^B_{anom}[g] + 8bS^C_{anom}[g] + q_2 S^D_{anom}[g]. \tag{6.62}$$

Similar to the 2-dim case, we have one more remark here. By adding up the lhs and rhs of (6.46) respectively, we will have

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\Delta_4\phi + \int_{\Sigma} d^3x\sqrt{-\gamma}\mathbf{B_3}\phi = 0, \tag{6.63}$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}E_Q + \int_{\Sigma} d^3x\sqrt{-\gamma}T_3 = \chi_4[\mathcal{M}], \tag{6.64}$$

where $\chi_4[\mathcal{M}]$ is the Euler-characteristic of the bounded 4-dim manifold $\mathcal{M}$ and we have used the 4-dim Gauss–Bonnet theorem to get the second equality. In order to make sure the two relations satisfy at the same time, we must have the following equality:

$$\int_{\mathcal{M}} d^4x\sqrt{-g}E_Q + \int_{\Sigma} d^3x\sqrt{-\gamma}T_3 = \chi_4[\mathcal{M}] = 0. \tag{6.65}$$

It means that the anomaly-induced action can only be used in the spacetime with zero Euler-characteristics. Unlike to the 2-dim spacetime, this requirement is in general not true in 4-dim spacetime. Therefore, the usage of 4-dim anomaly-induced action is limited and we should be careful when using the 4-dim version of the anomaly-induced action.

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Chapter 7

Conclusion

In this thesis, we have derived the anomaly-induced action with the boundary effect for 2-dim and 4-dim cases by restoring the corresponding boundary terms to the Lagrangian for the counter terms. Although the boundary action seems not to alter the stress tensor in the region within boundary after including the boundary effect, there are indeed additional boundary constraints for the auxiliary field φ. Therefore, even though the functional form of the stress tensor is the same as that without the boundary effect, due to the additional constraints, the solution of stress-tensor is restricted. The most important discovery from it is that the correspondence between the quantum states of the original field and the solutions of the auxiliary scalar field is naturally restored due to this boundary constraint. Therefore, the anomaly-induced action with boundary effect can be used to derive the stress tensor for any specific vacuum state. This fact has not been noticed before and it would significantly increase the capability of this formalism.

Moreover, by analyzing the field equation and boundary constraint of φ, we find that the anomaly-induced action can only be used in the spacetime with zero Euler characteristics. Although this requirement is satisfied automatically in bounded 2-dim spacetime, it is in general not the case in 4-dim spacetime. Therefore, one must be careful about the topological structure of spacetime when using the anomaly-induced action in 4-dim spacetime.

There also exists some other new features in our result. By using the Green's function-modified method to derive the anomaly-induced action, we find that it is indeed not nec-

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essary to assume the symmetric Green's function. That means the formalism that has been widely used is in fact a limited version with a redundant requirement (symmetric Green's function). In addition, although two auxiliary scalar fields have been introduced to get the localised 4-dim anomaly-induced action in the previous works [9, 10, 12], it turns out that introducing only one auxiliary scalar field is already sufficient to obtain the 4-dim anomaly-induced action. We correct this mistake in our recent work [2] and thus the right formula which we obtained differs slightly from the form found in the previous literature (even when no boundary effect is involved).

As examples, we have applied our result to several familiar spacetimes (flat, two-dimensional Schwarzchild, and de Sitter spacetimes), and rederive various well-known quantum gravity phenomena (the dynamical Casimir effect, Unruh effect, and Hawking temperature). Although these are already well-known knowledge, by using our result to rederive these problems again, we can appreciate how efficient the formalism is.

Now since we know the correct relation between quantum states of the original field and the solution of the auxiliary field, we can deal with the quantum effects on curved spacetime as the classical dynamics of the auxiliary field φ. It can be expected that by using the classical anomaly-induced action, many important quantum gravity issues maybe investigated more easily. For instances, it will be interesting to apply our method to the extremal black hole which is indeed a manifold with zero Euler characteristic. Another interesting direction is to study the backreaction problems in semi-classical approaches without bothering with complicated calculations arisen from the renormalization scheme. In summary, we expect that the anomaly-induced action in 4-dim spacetime would be a powerful tool to investigate various physically interesting semi-classical problems in cosmology, semi-classical physics on black hole spacetime, and so on. We leave these interesting explorations as future works.

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Appendix A

Conformal Transformation (CT) and Some Useful Relations

The conformal transformation, which abbreviated as CT in the following, is defined by:

$$\begin{cases} \bar{g}_{\mu\nu}(x) = e^{-2\sigma(x)} g_{\mu\nu}(x) \\ \bar{g}^{\mu\nu}(x) = e^{2\sigma(x)} g^{\mu\nu}(x). \end{cases} \tag{A.1}$$

The relation for the normal vector of boundaries after conformal transformation is defined as follows:

$$\begin{cases} \bar{n}_a = n_a e^{-\sigma} \\ \bar{n}^a = n^a e^{\sigma} \end{cases} \tag{A.2}$$

and thus the norm of n and $\bar{n}$ are the same, i.e.

$$\bar{n}_a \bar{n}_b \bar{g}^{ab} = n_a n_b g^{ab} = \pm 1 =: \epsilon, \tag{A.3}$$

where the sign of the spacelike (timelike) normal vector n is $+(-)$ and thus corresponds to timelike(spacelike) boundary.

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A.1 Conformal Transformation of Geometrical Quantities

From the definition of CT (A.1), we can derive the following CT relations:

CT of determinant of metric:

$$\sqrt{-g} = e^{D\sigma}\sqrt{-\bar{g}}, \tag{A.4}$$

CT of Christoffel symbols:

$$\begin{aligned}\Gamma^c_{ab} &= \frac{1}{2}g^{c\lambda}(\partial_a g_{b\lambda} + \partial_b g_{\lambda a} - \partial_\lambda g_{ab}) = \bar{\Gamma}^c_{ab} + \bar{g}^{c\lambda}\bar{g}_{b\lambda}\sigma_a + \bar{g}^{c\lambda}\bar{g}_{\lambda a}\sigma_b - \bar{g}^{c\lambda}\bar{g}_{ab}\sigma_\lambda \\ &= \bar{\Gamma}^c_{ab} + \sigma_b\delta^c_a + \sigma_a\delta^c_b - \sigma_d\bar{g}^{cd}\bar{g}_{ab},\end{aligned} \tag{A.5}$$

CT of Riemann tensor:

$$\begin{aligned}R^a{}_{bcd} = \bar{R}^a_{bcd} &- \bar{g}_{db}\bar{\nabla}_c\bar{\nabla}^a\sigma + \bar{g}_{cb}\bar{\nabla}_d\bar{\nabla}^a\sigma + \delta^a_d\bar{\nabla}_c\bar{\nabla}_b\sigma - \delta^a_c\bar{\nabla}_b\bar{\nabla}_b\sigma \\ &+ \delta^a_c\sigma_b\sigma_d - \delta^a_c\bar{g}_{bd}(\bar{\nabla}\sigma)^2 - \bar{\sigma}^a\bar{g}_{cb}\bar{\sigma}_d \\ &- \delta^a_d\sigma_c\sigma_d + \delta^a_d\bar{g}_{cb}(\bar{\nabla}\sigma)^2 + \bar{\sigma}^a\bar{g}_{bd}\bar{\sigma}_c\end{aligned} \tag{A.6}$$

$$\begin{aligned}\Rightarrow e^{-2\sigma}R_{abcd} &= \bar{R}_{abcd} - \bar{g}_{db}\bar{\nabla}_c\bar{\nabla}_a\sigma + \bar{g}_{cb}\bar{\nabla}_d\bar{\nabla}_a\sigma + \bar{g}_{ad}\bar{\nabla}_c\bar{\nabla}_b\sigma - \bar{g}_{ac}\bar{\nabla}_b\bar{\nabla}_d\sigma \\ &\quad + \bar{g}_{ac}\bar{\sigma}_b\bar{\sigma}_d - \bar{g}_{cb}\bar{\sigma}_a\bar{\sigma}_d - \bar{g}_{ad}\bar{\sigma}_c\bar{\sigma}_b + \bar{g}_{bd}\bar{\sigma}_a\bar{\sigma}_c - (\bar{g}_{ac}\bar{g}_{bd} + \bar{g}_{ad}\bar{g}_{bc})(\bar{\nabla}\sigma)^2 \\ &= \bar{R}_{abcd} - 4\bar{g}_{ac}\bar{\nabla}_b\bar{\nabla}_d\sigma + \bar{g}_{ac}\bar{\sigma}_b\bar{\sigma}_d - 2\bar{g}_{ac}\bar{g}_{bd}(\bar{\nabla}\sigma)^2\end{aligned} \tag{A.7}$$

$$\Rightarrow e^{2\sigma}\bar{R}_{abcd} = R_{abcd} + 4g_{ac}\nabla_b\nabla_d\sigma + g_{ac}\sigma_b\sigma_d - 2g_{ac}g_{bd}(\nabla\sigma)^2 \tag{A.8}$$

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CT of Ricci tensor:

$$R_{ab} = \bar{R}_{ab} - \bar{g}_{ab}(\bar{\nabla}^2\sigma) + (D-2)[\sigma_a\sigma_b - \bar{\nabla}_a\bar{\nabla}_b\sigma - \bar{g}_{ab}(\bar{\nabla}\sigma)^2] \tag{A.9}$$

$$\Rightarrow \bar{R}_{ab} = R_{ab} + g_{ab}(\Box\sigma) + (D-2)\nabla_a\nabla_b\sigma + (D-2)\sigma_a\sigma_b - (D-2)g_{ab}(\nabla\sigma)^2 \tag{A.10}$$

$$\begin{aligned}\Rightarrow \bar{R}^{ab} &= \bar{g}^{aa'}\bar{g}^{bb'}\bar{R}_{a'b'} \\ &= e^{4\sigma}g^{aa'}g^{bb'}\left[R_{a'b'} + g_{a'b'}(\Box\sigma) + (D-2)\nabla_{a'}\nabla_{b'}\sigma + (D-2)\sigma_{a'}\sigma_{b'} - (D-2)g_{a'b'}(\nabla\sigma)^2\right] \\ &= e^{4\sigma}\left[R^{a'b'} + g^{a'b'}(\Box\sigma) + (D-2)\nabla^{a'}\nabla^{b'}\sigma + (D-2)\sigma^{a'}\sigma^{b'} - (D-2)g^{a'b'}(\nabla\sigma)^2\right]\end{aligned} \tag{A.11}$$

CT of Ricci scalar:

$$e^{2\sigma}R = \bar{R} - 2(D-1)(\bar{\Box}\sigma) - (D-1)(D-2)(\bar{\nabla}\sigma)^2 \tag{A.12}$$

$$\Rightarrow e^{-2\sigma}\bar{R} = R + 2(D-1)(\Box\sigma) - (D-1)(D-2)(\nabla\sigma)^2 \tag{A.13}$$

The CT of tensors associated with boundaries:

$$\sqrt{-\gamma} = e^{(D-1)\sigma}\sqrt{-\bar{\gamma}} \tag{A.14}$$

$$\begin{aligned}\nabla_a n_b &= \partial_a n_b - \Gamma^c_{ab}n_c = \partial_a(e^\sigma\bar{n}_b) - (\bar{\Gamma}^c_{ab}n_c + \sigma_b\delta^c_a + \sigma_a\delta^c_b - \sigma_d\bar{g}^{cd}\bar{g}_{ab})(e^\sigma\bar{n}_c) \\ &= e^\sigma(\bar{\nabla}_a\bar{n}_b - \sigma_b\bar{n}_a + \sigma_d\bar{g}^{cd}\bar{g}_{ab}\bar{n}_c)\end{aligned} \tag{A.15}$$

$$K = \gamma^{ab}K_{ab} = \gamma^{ab}\nabla_a n_b = (e^{-2\sigma}\bar{\gamma}^{ab})e^\sigma(\bar{\nabla}_a\bar{n}_b - \sigma_b\bar{n}_a + \sigma_d\bar{g}^{cd}\bar{g}_{ab}\bar{n}_c) = e^{-\sigma}[\bar{K} + (D-1)\sigma_a\bar{n}^a] \tag{A.16}$$

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A.1.1 The CT Relations Used for Solving 2-dimensional S_{anom}:

Derive $\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R}$:

$$\overset{(A.12)}{\Rightarrow} \sqrt{-g}R = e^{(D-2)\sigma}\sqrt{-\bar{g}}[\bar{R} - 2(D-1)\bar{\Box}\sigma - (D-1)(D-2)\sigma_\mu\sigma^\mu] \tag{A.17}$$

$$= \sqrt{-\bar{g}}[\bar{R} - 2(1+\varepsilon)\bar{\Box}\sigma - (1+\varepsilon)\varepsilon\bar{\sigma}_\mu\bar{\sigma}^\mu + \varepsilon\sigma\bar{R} - 2(1+\varepsilon)\varepsilon\sigma\bar{\Box}\sigma - (1+\varepsilon)\varepsilon^2\sigma\bar{\sigma}_\mu\bar{\sigma}^\mu] \tag{A.18}$$

$$\Rightarrow [\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R}] = -2\sqrt{-g}\Box\sigma + \varepsilon\sqrt{-g}[\sigma R + 2\sigma\Box\sigma - 2\Box\sigma + (\nabla\sigma)^2]$$

$$= -[\sqrt{-g}\Box\sigma + \bar{\ldots}] + \varepsilon\{\frac{1}{2}[\sqrt{-g}R + \bar{\ldots}]\sigma + \underbrace{[\sqrt{-g}\sigma\Box\sigma - \bar{\ldots}]}_{=O(\varepsilon)}$$

$$- [\sqrt{-g}\Box\sigma + \bar{\ldots}] + \frac{1}{2}\underbrace{[\sqrt{-g}(\nabla\sigma)^2 - \bar{\ldots}]}_{=O(\varepsilon)}\}$$

$$= -[\sqrt{-g}\Box\sigma + \bar{\ldots}] + \varepsilon\left\{\frac{1}{2}[\sqrt{-g}R + \bar{\ldots}]\sigma - [\sqrt{-g}\Box\sigma + \bar{\ldots}]\right\} \tag{A.19}$$

Derive $\sqrt{-\gamma}K - \sqrt{-\bar{\gamma}}\bar{K}$:

$$\overset{(A.16)}{\Rightarrow} \sqrt{-\gamma}K = e^{(D-2)\sigma}\sqrt{-\bar{\gamma}}[\bar{K} + (D-1)\sigma_a\bar{n}^a]$$

$$= \sqrt{-\bar{\gamma}}\{[\bar{K} + (1+\varepsilon)\sigma_a\bar{n}^a] + \varepsilon[\sigma\bar{K} + (1+\varepsilon)\sigma\sigma_a\bar{n}^a]\} \tag{A.20}$$

$$\Rightarrow [\sqrt{-\gamma}K - \sqrt{-\bar{\gamma}}\bar{K}] = \frac{1}{2}[\sqrt{-\gamma}\nabla_n\sigma + \bar{\ldots}] \tag{A.21}$$

$$+ \varepsilon\left\{\frac{1}{2}[\sqrt{-\gamma}K + \bar{\ldots}]\sigma + \frac{1}{2}[\sqrt{-\gamma}\nabla_n\sigma + \bar{\ldots}] - \frac{1}{2}\underbrace{[\sqrt{-\gamma}\sigma\nabla_n\sigma - \bar{\ldots}]}_{=O(\varepsilon)}\right\} \tag{A.22}$$

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A.1.2 Conformally Invariant Differential Operator Associated to the 2-dimensional S_{anom}

By considering the equalities due to the CT of Ricci scalar R bellow,

$$\begin{aligned}
&\sqrt{-\bar{g}}\bar{R} - \sqrt{-g}R \\
&\overset{(A.18)}{=} -\sqrt{-\bar{g}}\{-2(1+\varepsilon)\bar{\Box}\sigma + \varepsilon[-(1+\varepsilon)\bar{\sigma}_\mu\bar{\sigma}^\mu + \sigma\bar{R} - 2(1+\varepsilon)\sigma\bar{\Box}\sigma - (1+\varepsilon)\varepsilon\sigma\bar{\sigma}_\mu\bar{\sigma}^\mu]\} \\
&\overset{n=2}{=} \sqrt{-\bar{g}}(2\bar{\Box}\sigma) = \sqrt{-g}(2\Box\sigma) \qquad \text{(A.23)}
\end{aligned}$$

we can find the conformally invariant operator, $\Box$, which is naturally related to Ricci scalar in 2-dimentional manifold.

Similarly, by considering the CT of Ricci scalar R together with the related boundary term $2K$, the Gibsson-Hawking term as follows,

$$\begin{aligned}
\sqrt{-g}[R + 2\nabla^\mu(n_\mu K)] &= \sqrt{-g}R + 2\partial_\mu(\sqrt{-g}n^\mu K) = \sqrt{-g}R + 2\partial_\mu(N\sqrt{-\gamma}n^\mu K) \\
&= \sqrt{-\bar{g}}[\bar{R} - 2\bar{\Box}\sigma] + 2\partial_\mu[Nn^\mu\sqrt{-\bar{\gamma}}(\bar{K} + \sigma_a\bar{n}^a)] \\
&= \sqrt{-\bar{g}}[\bar{R} - 2\bar{\Box}\sigma] + 2\partial_\mu[\bar{N}\bar{n}^\mu\sqrt{-\bar{\gamma}}(\bar{K} + \sigma_a\bar{n}^a)] \\
&= \sqrt{-\bar{g}}[\bar{R} - 2\bar{\Box}\sigma] + 2\partial_\mu[\sqrt{-\bar{g}}\bar{n}^\mu(\bar{K} + \sigma_a\bar{n}^a)] \\
&= \sqrt{-\bar{g}}[\bar{R} - 2\bar{\Box}\sigma] + 2\sqrt{-\bar{g}}\bar{\nabla}_\mu[\bar{n}^\mu(\bar{K} + \sigma_a\bar{n}^a)] \\
&= \sqrt{-\bar{g}}[\bar{R} + 2\bar{\nabla}_\mu\bar{n}^\mu\bar{K}] - 2\sqrt{-\bar{g}}[\bar{\Box}\sigma - \bar{\nabla}_\mu\bar{n}^\mu\bar{n}^a\sigma_a], \qquad \text{(A.24)}
\end{aligned}$$

we can naturally find a conformally invariant boundary-associated operator L_2 which satisfies the following relation:

$$\Rightarrow 2\sqrt{-\bar{g}}\bar{L}_2\sigma := 2\sqrt{-\bar{g}}[-\bar{\Box}\sigma + \bar{\nabla}_\mu\bar{n}^\mu\bar{n}^a\bar{\nabla}_\alpha\sigma] = \sqrt{-g}[R + 2\nabla^\mu(n_\mu K)] - \sqrt{-\bar{g}}[\bar{R} + 2\bar{\nabla}_\mu(\bar{n}^\mu\bar{K})], \qquad \text{(A.25)}$$

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where we used $\sqrt{-g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{-g}V^\mu)$ and

$$\begin{cases} N = \sqrt{-g}/\sqrt{-\gamma} \\ \bar{N} = \sqrt{-\bar{g}}/\sqrt{-\bar{\gamma}} \end{cases} \Rightarrow N = \bar{N}e^{\sigma}. \tag{A.26}$$

during the derivation above.

A.1.3 The relations used for solving 4-dim S_{anom}:

The relations used in Sec.(A.1.4) for solving 4-dim S_{anom}:

(B1): Calculate $\sqrt{-g}R^{abcd}R_{abcd} - \sqrt{-\bar{g}}\bar{R}^{abcd}\bar{R}_{abcd}$:

$$\begin{aligned} &\overset{(A.8)}{\Rightarrow} e^{2\sigma}\bar{R}_{ab}^{cd} = R_{ab}{}^{cd} + 4\delta_a{}^c\nabla_b\nabla_d\sigma + \delta_a{}^c\sigma_b\sigma_d - 2\delta_a{}^c\delta_b{}^d(\nabla\sigma)^2 \\ &\Rightarrow e^{2\sigma}R^{ab}{}_{cd}\bar{R}_{ab}^{cd} = R^{ab}{}_{cd}R_{ab}{}^{cd} + 4R^{\mu\nu}\nabla_\mu\nabla_\nu\sigma + 4R^{\mu\nu}\sigma_\mu\sigma_\nu - 2R(\nabla\sigma)^2 \end{aligned} \tag{A.27}$$

where we used $R^{abcd}R_{abcd} = R^{abcd}R_{abcd}$ in the last equality.

$$\begin{aligned} \Rightarrow (\sqrt{-g}\sqrt{-\bar{g}})^{\frac{1}{2}}R_{cd}^{ab}\bar{R}_{ab}^{cd} &= \sqrt{-g}\,\underbrace{e^{(\frac{4-D}{2})\sigma}}_{=1-\frac{1}{2}\varepsilon\sigma}[R^{abcd}R_{abcd} + 4R_{ab}\nabla_a\nabla_b\sigma + 4R_{ab}\sigma_a\sigma_b - 2R(\nabla\sigma)^2] \\ &= \sqrt{-\bar{g}}\,\underbrace{e^{(\frac{D-4}{2})\sigma}}_{=1+\frac{1}{2}\varepsilon\sigma}[\bar{R}^{abcd}\bar{R}_{abcd} - 4\bar{R}_{ab}\bar{\nabla}_a\bar{\nabla}_b\bar{\sigma} + 4\bar{R}_{ab}\bar{\sigma}_a\bar{\sigma}_b - 2\bar{R}(\bar{\nabla}\bar{\sigma})^2] \end{aligned} \tag{A.28}$$

$$\begin{aligned} &\Rightarrow \sqrt{-g}R^{abcd}R_{abcd} - \sqrt{-\bar{g}}\bar{R}^{abcd}\bar{R}_{abcd} = -4(R^{ab}\nabla_a\nabla_b\sigma + \bar{\cdots}) - 4(R^{ab}\sigma_a\sigma_b - \bar{\cdots}) + 2[R(\nabla\sigma)^2 - \bar{\cdots}] \\ &+ \varepsilon\left\{\frac{1}{2}(R^{abcd}R_{abcd}\sigma + \bar{\cdots}) + 2[R^{ab}(\nabla_a\nabla_b\sigma)\sigma - \bar{\cdots}] + 2(R^{ab}\sigma_a\sigma_b\sigma + \bar{\cdots}) - [R(\nabla\sigma)^2\sigma - \bar{\cdots}]\right\} \end{aligned} \tag{A.29}$$

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(B2): Calculate $\sqrt{-g}R^{ab}R_{ab} - \sqrt{-\bar{g}}\bar{R}^{ab}\bar{R}_{ab}$:

$$
\begin{aligned}
&\overset{(A.9)}{\Rightarrow} (\sqrt{-g}\ \underbrace{\sqrt{-\bar{g}}}_{\sqrt{-g}e^{-D\sigma}}\ )^{\frac{1}{2}} R^a_b \quad \underbrace{\bar{R}^b_a}_{e^{2\sigma}(R^b_a+\ldots)} \\
&= \sqrt{-g}\,\underbrace{e^{(\frac{4-D}{2})\sigma}}_{=1-\frac{1}{2}\varepsilon\sigma}[R^{ab}R_{ab} + R\Box\sigma + (D-2)R^{ab}\nabla_a\nabla_b\sigma + (D-2)R^{ab}\sigma_a\sigma_b - (D-2)R(\nabla\sigma)^2] \\
&= \sqrt{-\bar{g}}\,\underbrace{e^{(\frac{D-4}{2})\sigma}}_{=1+\frac{1}{2}\varepsilon\sigma}[\bar{R}^{ab}\bar{R}_{ab} - \bar{R}\bar{\Box}\sigma - (D-2)\bar{R}^{ab}\nabla_a\nabla_b\sigma + (D-2)\bar{R}^{ab}\sigma_a\sigma_b - (D-2)\bar{R}(\bar{\nabla}\sigma)^2]
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
&\Rightarrow \sqrt{-g}R^{ab}R_{ab} - \sqrt{-\bar{g}}\bar{R}^{ab}\bar{R}_{ab} \\
&= -(R\Box\sigma + \bar{R}\bar{\Box}\sigma) - 2(R^{ab}\nabla_a\nabla_b\sigma + \bar{\cdots}) - 2(R^{ab}\sigma_a\sigma_b - \bar{\cdots}) - 2[R(\nabla\sigma)^2 - \bar{\cdots}] \\
&+ \varepsilon\Big\{\frac{1}{2}(R^{ab}R_{ab}\sigma + \bar{\cdots}) + \frac{1}{2}[R(\Box\sigma)\sigma - \bar{\cdots}] + [R^{ab}(\nabla_a\nabla_b\sigma)\sigma - \bar{\cdots}] + (R^{ab}\sigma_a\sigma_b\sigma + \bar{\cdots}) \\
&- [R(\nabla\sigma)^2\sigma + \bar{\cdots}] - [R^{ab}(\nabla_a\nabla_b\sigma) + \bar{\cdots}] - [R^{ab}\sigma_a\sigma_b - \bar{\cdots}] + [R(\nabla\sigma)^2 - \bar{\cdots}]\Big\}
\end{aligned}
\tag{A.31}
$$

(B3): Calculate $\sqrt{-g}R^2 - \sqrt{-\bar{g}}\bar{R}^2$:

$$
\begin{aligned}
\Rightarrow (\sqrt{-g}\sqrt{-\bar{g}})^{\frac{1}{2}} R\bar{R} &= \sqrt{-g}\,\underbrace{e^{(\frac{4-D}{2})\sigma}}_{=1-\frac{1}{2}\varepsilon\sigma}[R^2 + 2\underbrace{(D-1)}_{3+\varepsilon}R\Box\sigma - \underbrace{(D-1)}_{3+\varepsilon}\underbrace{(D-2)}_{2+\varepsilon}R(\nabla\sigma)^2] \\
&= \sqrt{-\bar{g}}\,\underbrace{e^{(\frac{D-4}{2})\sigma}}_{=1+\frac{1}{2}\varepsilon\sigma}[\bar{R}^2 - 2(D-1)\bar{R}\bar{\Box}\sigma - (D-1)(D-2)\bar{R}(\bar{\nabla}\sigma)^2]
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
&\Rightarrow \sqrt{-g}R^2 - \sqrt{-\bar{g}}\bar{R}^2 = -6(R\Box\sigma + \bar{R}\bar{\Box}\sigma) + 6[R(\nabla\sigma)^2 - \bar{R}(\bar{\nabla}\sigma)^2] \\
&+ \varepsilon\{\frac{1}{2}(R^2\sigma + \bar{R}^2\sigma) + 3[R(\Box\sigma)\sigma - \bar{R}(\bar{\Box}\sigma)\sigma] - 3[R(\nabla\sigma)^2\sigma + \bar{R}(\bar{\nabla}\sigma)^2\sigma] - 2[R\Box\sigma + \bar{R}\bar{\Box}\sigma] \\
&+ 5[R(\nabla\sigma)^2 - \bar{R}(\bar{\nabla}\sigma)^2]\}
\end{aligned}
\tag{A.33}
$$

(B4): Calculate $\sqrt{-g}\Box R - \sqrt{-\bar{g}}\bar{\Box}\bar{R}$:

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At first, by considering conformal transformation of $\Box R$,

$$\begin{aligned}
\sqrt{-\bar{g}}\bar{\Box}\bar{R} &= \partial_a(\sqrt{-\bar{g}}\bar{\nabla}^a\bar{R}) = \partial_a(e^{(2-D)\sigma}\sqrt{-g}\nabla^a\bar{R}) = \sqrt{-g}\nabla_a[e^{(2-D)\sigma}\nabla^a\bar{R}] \\
&= \sqrt{-g}\nabla_a\left[e^{(2-D)\sigma}\nabla^a[e^{2\sigma}(R-2(D-1)\Box\sigma-(D-1)(D-2)(\nabla\sigma)^2)]\right] \\
&\overset{D=4+\varepsilon}{=} \sqrt{-g}\nabla_a\left[e^{-2\sigma}e^{-\varepsilon\sigma}\nabla^a\left[[e^{2\sigma}\left(R-2(3+\varepsilon)\Box\sigma-(3+\varepsilon)(2+\varepsilon)(\nabla\sigma)^2\right)\right]\right] \\
&= \sqrt{-g}\nabla_a\{e^{-2\sigma}\{\nabla^a\left[e^{2\sigma}[R+6\Box\sigma-6(\nabla\sigma)^2]\right]-(\varepsilon\sigma)\nabla^a\left[e^{2\sigma}[R+6\Box\sigma-6(\nabla\sigma)^2]\right] \\
&+\varepsilon\nabla^a\left[e^{2\sigma}[2\Box\sigma-5(\nabla\sigma)^2]\right]\}\} \\
&= \sqrt{-g}\nabla_a\Big\{[\nabla^aR+6\nabla^a\Box\sigma\underbrace{-6\nabla^a(\nabla\sigma)^2}_{-12\sigma_b(\nabla^a\nabla^b\sigma)}]+(2\nabla^a\sigma)[R+6\Box\sigma-6(\nabla\sigma)^2] \\
&+\varepsilon\Big\{-\sigma[\nabla^aR+6\nabla^a\Box\sigma-6\nabla^a(\nabla\sigma)^2]-\sigma(2\nabla^a\sigma)[R+6\Box\sigma-6(\nabla\sigma)^2] \\
&+[2\nabla^a\Box\sigma-5\nabla^a(\nabla\sigma)^2]+(2\nabla^a\sigma)[2\Box\sigma-5(\nabla\sigma)^2]\Big\}\Big\}
\end{aligned} \tag{A.34}$$

we then can get

$$\begin{aligned}
&\sqrt{-g}\Box R-\sqrt{-\bar{g}}\bar{\Box}\bar{R} \\
&= \sqrt{-g}\nabla_a\{-6\nabla^a\Box\sigma+12\sigma_b\nabla^a\nabla^b\sigma-2\sigma^aR-12\sigma^a\Box\sigma+12\sigma^a(\nabla\sigma)^2\} \\
&+\varepsilon\sqrt{-g}\nabla_a\Big\{+\sigma\nabla^aR+6\sigma\nabla^a\Box\sigma-6\sigma\nabla^a(\nabla\sigma)^2+2\sigma\sigma^aR+12\sigma\sigma^a\Box\sigma-12\sigma\sigma^a(\nabla\sigma)^2 \\
&= \ldots \\
&= \sqrt{-g}\nabla_a\{-3[\nabla^a\Box\sigma+\bar{\ldots}]-[\sigma^aR+\bar{\ldots}]-3[\sigma^a(\nabla\sigma)^2+\bar{\ldots}]\} \\
&+\varepsilon\sqrt{-g}\nabla_a\Big\{3[\sigma\sigma_b\nabla^a\nabla^b\sigma+\bar{\ldots}]-3[\sigma\sigma^a\Box\sigma+\bar{\ldots}]+\frac{1}{2}[\sigma\nabla^aR+\bar{\ldots}]-[\nabla^a\Box\sigma+\bar{\ldots}]-\frac{5}{2}[\sigma^a(\nabla\sigma)^2+\bar{\ldots}]\Big\}.
\end{aligned} \tag{A.35}$$

A.1.4 Conformal transformation for the derivation of 4-dim S_{anom}

In this section, we will show the conformal transformation of the Gauss-Bonet term (E) and the Weyl square term (F) , and thus construct the unique conformal invariant operator, Δ_4 (without boundary) and $_4$ (with boundary)

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Comformal transformation of Gauss-Bonet term(E):

$$\sqrt{-g}E - \sqrt{-\bar{g}}\bar{E} = \sqrt{-g}(R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}) - \sqrt{-\bar{g}}(\bar{...}) \overset{(A.33),(A.31),(A.29)}{=} ...$$
$$= 4[\sqrt{-g}\nabla_a(G^{ab}\nabla_b\sigma) + \bar{...}] - 2[\sqrt{-g}\nabla^a(\sigma_a\sigma^b\sigma_b) + \bar{...}]$$
$$+ \varepsilon\{\frac{1}{2}[(E - \frac{2}{3}\Box R) + \bar{...}]\sigma - \frac{1}{18}[R^2 - \bar{...}] - 4[\nabla^a(\sigma_a\sigma_b\sigma^b) + \bar{...}] + 2[\sqrt{-g}\nabla^a(\sigma\sigma_a\sigma^b\sigma_b) - \bar{...}]$$
$$+ 4[\sqrt{-g}\nabla_a(G^{ab}\nabla_b\sigma) + \bar{...}] - 2\left[\nabla_a[G^{ab}(\nabla_b\sigma)\sigma] - \bar{...}\right] - \frac{1}{3}\left[\nabla^a(R\sigma_a) + \bar{...}\right] + \frac{1}{3}\left[\nabla^a[(\partial_a R)\sigma] + \bar{...}\right]\} \tag{A.36}$$

Comformal transformation of Weyl-Square term(F):

$$\sqrt{-g}F - \sqrt{-\bar{g}}\bar{F} = \sqrt{-g}(\frac{1}{3}R^2 - 2R^{ab}R_{ab} + R^{abcd}R_{abcd}) - \sqrt{-\bar{g}}(\bar{...}) \overset{(A.33),(A.31),(A.29)}{=} ...$$
$$= \varepsilon\left\{\frac{1}{2}(F + \bar{...})\sigma - \frac{1}{18}[R^2 - \bar{...}] - \frac{1}{2}[\nabla_a(\sigma^a\sigma_b\sigma^b) + \bar{...}]\right\} \tag{A.37}$$

Comformal transformation of $E - \frac{2}{3}\Box R$:

$$[\sqrt{-g}(E - \frac{2}{3}\Box R) - \bar{...}]$$
$$= 4[\sqrt{-g}\nabla_a(G^{ab}\nabla_b\sigma) + \bar{...}] + \frac{2}{3}[\sqrt{-g}\nabla_a(\sigma^a R) + \bar{...}] + 2[\sqrt{-g}\Box^2\sigma + \bar{...}] + O(\varepsilon)$$
$$= 2\{\sqrt{-g}[\Box^2\sigma + 2\nabla_a(G^{ab}\nabla_b\sigma) + \frac{1}{3}\nabla_a(\sigma^a R)] + [\bar{...}]\} + O(\varepsilon)$$
$$= 2[\sqrt{-g}\Delta_4\sigma + \bar{...}] + O(\varepsilon) \tag{A.38}$$

where $\Delta_4 := \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu)\nabla_\mu$. Therefore, a (exactly) 4-dim conformal invariant (withour boundary) operator Δ_4 can naturally be found.

Similar to 2-dim, by considering the conformal transformation of E together with its corresponding boundary term E^B, we can have:

$$\left\{\left[\int d^4x\sqrt{-g}(E - \frac{2}{3}\Box R) - \int d^3x\sqrt{-\gamma}\left(\nabla_a(n^a E^B) - \frac{2}{3}\nabla_a n^a n^b\nabla_b R\right)\right] - \bar{...}\right\}$$
$$= 2[\int d^4x\sqrt{-g}(\Delta_4\sigma - \Delta_4^B\sigma) + \bar{...}] + O(\varepsilon)$$
$$:= 2[\int d^4x\sqrt{-g}L_4\sigma + \bar{...}] + O(\varepsilon) \tag{A.39}$$

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where $L_4 := \Delta_4 - \Delta_4^B$ and

$$\Delta_4^B := \{ \left[\nabla_a n^a n^b \nabla_b \Box + \Box \nabla_a n^a n^b \nabla_b - \left(\nabla_a n^a n^b \nabla_b\right)^2\right] + 2\left[\nabla_a n^a n^b G_b^c \nabla_c + \nabla_a G_b^a n^b n^c \nabla_c - \nabla_a n^a n^b G_{bc} n^c n^d \nabla_d\right] + \frac{1}{3}\left(\nabla_a n^a R n^b \nabla_b\right)\} \tag{A.40}$$

A.2 Derivation of Trace anomaly from S_{div}

Derive the 2-dim trace anomaly:

$$\overset{(A.19)}{\Rightarrow} \frac{1}{\sqrt{-g}}[\sqrt{-\bar{g}}\bar{R} - \sqrt{-g}R] = -\sigma\varepsilon R + \nabla_a[...] + O(\sigma^2;\varepsilon)$$
$$\Rightarrow \frac{2}{\sqrt{-g}} g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} \int d^n x\sqrt{-g}R|_{\sigma=0} \overset{(4.99)}{=} \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^n x\sqrt{-g}R|_{\sigma=0} = -(D-2)R \tag{A.41}$$

where $\varepsilon = n-2$.

Derive the 4-dim trace anomaly:

$$\frac{1}{\sqrt{-g}}[\sqrt{-\bar{g}}\bar{R}^2 - \sqrt{-g}R^2]$$
$$\overset{(A.33)}{=} 12R\Box\sigma + O(\sigma^2;\varepsilon) = 12\sigma\Box R + \nabla_a[R\sigma^a - \sigma\nabla^a R] + O(\sigma^2;\varepsilon)$$
$$= 12\sigma\Box R + \nabla_a[...] + O(\sigma^2;\varepsilon)$$
$$\Rightarrow \frac{2}{\sqrt{-g}} g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} \int d^n x\sqrt{-g}R^2|_{\sigma=0} \overset{(4.99)}{=} \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^n x\sqrt{-g}R^2|_{\sigma=0} = 12\Box R \tag{A.42}$$

$$\frac{1}{\sqrt{-g}}[\sqrt{-\bar{g}}\bar{E}^2 - \sqrt{-g}E^2]$$
$$\overset{(A.36)}{=} \frac{-\varepsilon}{\sqrt{-g}}\{\frac{1}{2}[\sqrt{-g}(E - \frac{2}{3}\Box R) + \bar{...}]\sigma - \frac{1}{18}[\sqrt{-g}R^2 - \bar{...}]\} + \nabla_a[...] + O(\sigma^2;\varepsilon^2)$$
$$= -\varepsilon\{(E - \frac{2}{3}\Box R)\sigma + \frac{12}{18}\sigma\Box R\} + \nabla_a[...] + O(\sigma^2;\varepsilon^2) = -\varepsilon\sigma E + \nabla_a[...] + O(\sigma^2;\varepsilon^2)$$
$$\Rightarrow \frac{2}{\sqrt{-g}} g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} \int d^n x\sqrt{-g}E|_{\sigma=0} \overset{(4.99)}{=} \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^n x\sqrt{-g}E|_{\sigma=0} = -\varepsilon E \tag{A.43}$$

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$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}[\sqrt{-\bar{g}}\bar{F}-\sqrt{-g}F] \\
&\overset{(A.37)}{=}\frac{-\varepsilon}{\sqrt{-g}}\{\frac{1}{2}[\sqrt{-g}F+\bar{...}]\sigma-\frac{1}{18}[\sqrt{-g}R^2-\bar{...}]\}+\varepsilon\nabla_a[...]+O(\sigma^2;\varepsilon^2) \\
&=\frac{-\varepsilon}{\sqrt{-g}}\{\frac{1}{2}[\sqrt{-g}F+\bar{...}]\sigma+\frac{12}{18}\sigma\Box R\}\}+\varepsilon\nabla_a[...]+O(\sigma^2;\varepsilon^2) \\
&\Rightarrow\frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx\sqrt{-g}F|_{\sigma=0}\overset{(4.99)}{=}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma}\int d^nx\sqrt{-g}F|_{\sigma=0}=-\varepsilon(F+\frac{2}{3}\Box R)
\end{aligned}
\tag{A.44}
$$

where $\varepsilon=n-4$.

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Appendix B

There is no unique conformal vacuum for $1+1$ dim, bounded spacetime

WLOG we start with a flat spacetime covered by the ”Minkowski” coordinate (t, x) which is bounded by 2 timelike hyperspace $x = 0$ and $x = 1$. Next we first assume that there exists another conformally related, orthogonal coordinate (w, s) satisfies

$$-dt^2 + dx^2 = c^2(w, s)(-dw^2 + ds^2) \tag{B.1}$$

and we require the two boundaries in this frame now are corresponding to $s = 0$ and $s = 1$ individually, that means:

$$\begin{cases} x = 0 \Leftrightarrow s = 0 \\ x = 1 \Leftrightarrow s = 1 \end{cases} \tag{B.2}$$

Let us consider (t, x) in terms of (w, s) as:

$$\begin{cases} t = t(w, s) \\ x = x(w, s) \end{cases}, \tag{B.3}$$

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and thus the boundaries conditions becomes:

$$\overset{Eq.(B.2)}{\Rightarrow}\begin{cases} x(w,0)=0 \\ x(w,1)=1 \end{cases}. \tag{B.4}$$

By substituting the coordinate transformation relations

$$\begin{cases} dt=\partial_w t dw+\partial_s t ds \\ dx=\partial_x t dw+\partial_s x ds \end{cases} \tag{B.5}$$

into line element and compare the result with the assumption Eq.(B.1) in the beginning, we get

$$\begin{aligned} &-\mathrm{d}t^2+\mathrm{d}x^2 \\ &=[-(\partial_w t)^2+(\partial_w x)^2]dw^2+[-(\partial_s t)^2+(\partial_s x)^2]ds^2+2(-\partial_w t\partial_s t+\partial_w x\partial_s x)dwds \\ &=c^2(w,s)(-dw^2+ds^2) \end{aligned} \tag{B.6}$$

and thus the constraints:

$$-(\partial_w t)^2+(\partial_w x)^2=(\partial_s t)^2-(\partial_s x)^2 \tag{B.7}$$

$$\partial_w t\partial_s t=\partial_w x \tag{B.8}$$

From Eq.(B.7) and Eq.(B.8), we can derive the following relations:

$$\begin{aligned} Eq.(B.7) &\overset{Eq.(B.8)}{\Rightarrow} \underbrace{-(\partial_w t)^2}_{=(\frac{\partial_s x}{\partial_s t}\partial_w x)^2}+(\partial_w x)^2=(\partial_s t)^2-(\partial_s x)^2 \\ &\Rightarrow [-(\frac{\partial_s x}{\partial_s t})^2+1](\partial_w x)^2=(\partial_s t)^2-(\partial_s x)^2 \\ &\Rightarrow [-(\partial_s x)^2+(\partial_s t)^2](\partial_w x)^2=[(\partial_s t)^2-(\partial_s x)^2](\partial_s t)^2 \\ &\overset{\text{as}((\partial_s t)^2-(\partial_s x)^2)\neq 0}{\Longrightarrow} (\partial_s x)=\pm(\partial_s t) \end{aligned} \tag{B.9}$$

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$$\overset{Eq.(B.8)}{\Rightarrow} \frac{\partial_w t}{\partial_s t} = \frac{\partial_w x}{\partial_s t} \overset{Eq.(B.9)}{=} \pm 1 \tag{B.10}$$

$$\Rightarrow \begin{cases} \partial_w t = \pm \partial_s x \\ \partial_w x = \pm \partial_s t \end{cases} \tag{B.11}$$

$$\Rightarrow \partial_w^2 x = \pm \partial_w \partial_s t = \underbrace{(\pm)^2}_{=1} \partial_w^2 x \Rightarrow (\partial_w^2 - \partial_s^2) x = 0 \tag{B.12}$$

Finally, we can write down the general expression for x:

$$\Rightarrow x = \int dk B_\pm(k) e^{iks} e^{\pm iks} + \int dl D_\pm(l) e^{lw} e^{\pm ls} + A_0 ws + A_1 s + A_2 w + A_3 \tag{B.13}$$

and then by using boundary condition

$$\begin{cases} x(w,0) = 0 \Rightarrow A_2 = A_3 = 0 \\ x(w,1) = 1 \Rightarrow A_1 = 1, A_0 = 0 \end{cases} \tag{B.14}$$

we can get the solution of x in terms of (w, s)

$$\Rightarrow x = s + \sum_{n=-\infty}^{\infty} \alpha_n \sin(n\pi s) e^{in\pi w}, \tag{B.15}$$

where $\alpha_n = -\alpha_{-n}^*$.

By substitute the solution into Eq.(B.11), we can derive the relation:

$$\begin{cases} \partial_s x = 1 + \sum_n \alpha_\mathrm{n} \cdot n\pi \cdot \cos(n\pi s) e^{in\pi w} = \pm \partial_w t \\ \partial_w x = \sum_n \alpha_\mathrm{n} \cdot in\pi \cdot \sin(n\pi s) e^{in\pi w} = \pm \partial_s t \end{cases} \tag{B.16}$$

and thus get the solution of t as

$$t = t_0 + w + \sum_n = idn \cos(n\pi s) e^{in\pi w}, \tag{B.17}$$

where WLOG we consider $(+)$ case.

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Finally, we find the general solutions of (t, x) in terms of (w, s):

$$\begin{cases} x = s + \sum_{n=-\infty}^{\infty} \alpha_n \sin(n\pi s) e^{in\pi w} \\ t = s + \sum_{n=-\infty}^{\infty} -i\alpha_n \cos(n\pi s) e^{in\pi w} \end{cases} . \tag{B.18}$$

Therefore, we concluded that for the bounded flat spacetime, there exists infinite choices of orthogonal coordinate and thus infinite choices of conformal vacua! [1]

Substitute Eq.(B.18) in to the line element,

$$\begin{aligned} -dt^2 + dx^2 &= [(\partial_w t)^2 - (\partial_w x)^2](-dw^2 + ds^2) \\ &= \underbrace{[(1 + \sum_n \alpha_\mathrm{n} \cdot n\pi \cos(n\pi s) e^{in\pi w})^2 - (\sum_n \alpha_\mathrm{n} \cdot in\pi \cdot \sin(n\pi s) e^{in\pi w})^2]}_{\equiv c^2(w,s)}(-dw^2 + ds^2) \end{aligned} \tag{B.19}$$

we can solve c^2 in terms of (w, s).

[1] According to this, we can easily generalise our result here to any $1 + 1$ dim conformally flat spacetime.

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Appendix C

Hermitian Property of the Conformal Invariant Operator L_2^f and L_4^f

C.1 Proof of Hermitian Operator L_2^f

In this section, we will show that after taking boundary into consideration, L_2^f is a hermitian operator which is an important property during the derivation of S_{anom}.

Recall the definition: $L_2^f := \Box - \Box^{Bf}$, where $\Box^{Bf} := \nabla_a f n^a n^b \nabla_b$.

WLOG we will assume $n^a n_a = 1$ through the following derivation:

$$\begin{aligned}
&\int_M d^2x\sqrt{-g}\,(h\Box g - g\Box h) = \int_M d^2x\sqrt{-g}\nabla_a\left(h\nabla^a g - g\nabla^a h\right) = \int_{\partial M} d^1x\sqrt{\gamma}n^b\left(h\nabla_b g - g\nabla b h\right) \\
&= \int_{\partial M} d^1x\sqrt{\gamma}f\left(n_a n^b\right)n^b\left(h\nabla_b g - g\nabla_b h\right) = \int_M d^2x\sqrt{-g}\nabla_a\left(fn^a n^b h\nabla_b g\right) - \nabla_a\left(fn^a n^b g\nabla_b h\right) \\
&= \int_M d^2x\sqrt{-g}h\nabla_a fn^a n^b\nabla_b g + f\left(\nabla_a h\right)n^a n^b\left(\nabla_b g\right) - g\nabla_a fn^a n^b\nabla_b h - f\left(\nabla_a g\right)n^a n^b\left(\nabla_b h\right) \\
&= \int_M d^2x\sqrt{-g}\left(h\nabla_a fn^a n^b g - g\nabla_a fn^a n^b\nabla_b h\right)
\end{aligned} \tag{C.1}$$

Therefore, we know that

$$\begin{aligned}
&\int_M d^2x\sqrt{-g}\left(hL_2^f g - gL_2^f h\right) \\
&= \int_M d^2x\sqrt{-g}\,(h\Box g - g\Box h) - \int_M d^2x\sqrt{-g}\left(h\nabla_a fn^a n^b g - g\nabla_a fn^a n^b\nabla_b h\right) = 0
\end{aligned} \tag{C.2}$$

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and thus show that L_2^f is a Hermitian operator.

C.2 Proof of Hermitian operator L_4^f

In this section, we will show that with boundary, L_4^f is a hermitian operator. The operator L_4^f is defined as $L_4^f := \Delta_4 - \Delta_4^{Bf}$, where

$$\Delta_4 := \underbrace{\Box^2}_{...(I)} + \underbrace{2\nabla_a\left(G^{ab}\nabla_b\right)}_{...(II)} + \frac{1}{3}\underbrace{\nabla_a\left(R\nabla^a\right)}_{...(III)}, \tag{C.3}$$

and the operator Δ_4^{Bf} is defined as:

$$\Delta_4^{Bf} := \{ \left[\nabla_a f n^a n^b \nabla_b \Box + \Box \nabla_a f n^a n^b \nabla_b - \left(\nabla_a f n^a n^b \nabla_b\right)^2\right] \\ + 2\left[\nabla_a f n^a n^b G_b^c \nabla_c + \nabla_a f G_b^a n^b n^c \nabla_c - \nabla_a f n^a n^b G_{bc} n^c n^d \nabla_d\right] + \frac{1}{3}\left(\nabla_a f n^a R n^b \nabla_b\right)\} \tag{C.4}$$

Let us prove L_4^f is a hermitian operator step by step by considering following operators:

(I): Now let show $\Box^2 - [\nabla f n n \nabla \Box + \Box \nabla f n n \nabla - (\nabla f n n \nabla)^2]$ is a Hermitian operator:

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$$
\begin{aligned}
&\int_M d^4x\sqrt{-g}\left[h\Box^2 g - g\Box^2 h\right] \\
&\overset{(C.6)}{=} \int_M d^4x\sqrt{-g}\left[\nabla_a\left(h\nabla^a\Box g\right) - (h\leftrightarrow g)\right] - \left\{\nabla_a\left[\left(\nabla^a h\right)\Box g\right] - (h\leftrightarrow g)\right\} + \left[\left(\Box h\right)\left(\Box g\right) - (h\leftrightarrow g)\right] \\
&= \int_{\partial M} d^3x\sqrt{\gamma}\left\{\left[n_a h\nabla^a\Box g - (h\leftrightarrow g)\right] - \left[n_a\left(\nabla^a h\right)\left(\Box g\right) - (h\leftrightarrow g)\right]\right\} \\
&= \int_{\partial M} d^3x\sqrt{\gamma}\left\{f\left(n_a n^a\right)\left[n^b h\nabla_b\Box g - (h\leftrightarrow g)\right] - f\left(n_a n^a\right)\left[n^b\left(\nabla_b h\right)\Box g - (h\leftrightarrow g)\right]\right\} \\
&= \int_M d^4x\sqrt{-g}\left\{\left[\nabla_a\left(fn^a n^b h\nabla_b\Box g\right) - \nabla_a\left(fn^a nb\left(\nabla_b h\right)\left(\Box g\right)\right)\right] - (h\leftrightarrow g)\right\} \\
&= \int_M d^4x\sqrt{-g}\{[h\nabla_a fn^a n^b\nabla_b\Box g + \left(\nabla_a h\right)fn^a n^b\nabla_b\Box g - \left(\nabla_a h\right)fn^a n^b\nabla_b\Box g \\
&- \underbrace{\left(\nabla_a fn^a n^b\nabla_b h\right)\left(\Box g\right)]}_{=(C.11)} - (h\leftrightarrow g)\} \\
&= \int_M d^4x\sqrt{-g}\{\left[h\nabla_a fn^a n^b\nabla_b\Box g - (h\leftrightarrow g)\right] - \left[\left(\nabla_a fn^a n^b\nabla_b h\right)\left(\nabla_c fn^c n^d\nabla_d g\right) - (h\leftrightarrow g)\right] \\
&- \left[g\Box\left(\nabla_c fn^c n^d\nabla_d h\right) - (h\leftrightarrow g)\right] + \left[g\nabla_a fn^a n^b\nabla_d\left(\nabla_c fn^c n^d\nabla_d h\right) - (h\leftrightarrow g)\right]\} \\
&= \int_M d^4x\sqrt{-g}\{\left[h\nabla_a fn^a n^b\nabla_b\Box g - (h\leftrightarrow g)\right] + \left[h\Box\left(\nabla_c fn^c n^d\nabla_d g\right) - (h\leftrightarrow g)\right] \\
&- \left[h\nabla_a fn^a n^b\nabla_b\left(\nabla_c fn^c n^d\nabla_d g\right) - (h\leftrightarrow g)\right]\} \\
&= \int_M d^4x\sqrt{-g}\left\{h\left[\nabla_a fn^a n^b\nabla_b\Box + \Box\nabla_c fn^c n^d\nabla_d - \nabla_a fn^a n^b\nabla_b\left(\nabla_c fn^c n^d\nabla_d\right)\right]g - (h\leftrightarrow g)\right\}
\end{aligned}
\tag{C.5}
$$

where we used the following relations:

$$
h\Box^2 g = \nabla_a\left(h\nabla^a\Box g\right) - \left(\nabla_a h\right)\left(\nabla^a\Box g\right) = \nabla_a\left(h\nabla^a\Box g\right) - \nabla_a\left[\left(\nabla^a h\right)\Box g\right] + \left(\Box h\right)\left(\Box g\right)
\tag{C.6}
$$

$$
\begin{aligned}
&\int_M d^4x\sqrt{-g}\nabla_a\left[\left(n^a n^b\nabla_b h\right)\left(\nabla_c fn^c n^d\nabla_d g\right)\right] = \int_{\partial M} d^3x\sqrt{\gamma}\left[\underbrace{\left(n_a n^a\right)}_{=1} n^b\nabla_b h\right]\left(\nabla_c fn^c n^d\nabla_d g\right) \\
&= \int_M d^4x\nabla_b\left[\left(\nabla^b h\right)\left(\nabla_c fn^c n^d\nabla_d g\right)\right] = \int_M d^4x\sqrt{-g}\left(\Box h\right)\left(\nabla_c fn^c n^d\nabla_d g\right) + \left(\nabla^b h\right)\left(\nabla_c fn^c n^d\nabla_d g\right)
\end{aligned}
\tag{C.7}
$$

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$$
\begin{aligned}
&\int_M d^4\sqrt{-g}\nabla_a\left[n^a n^b h\nabla_b\left(\nabla_c f n^c n^d\nabla_d g\right)\right] = \int_{\partial M} d^3x\sqrt{\gamma}\left(n_a n^a\right)n^b h\nabla_b\left(\nabla_c f n^c n^d\nabla_d g\right)\\
&= \int_M d^4x\sqrt{-g}\nabla_b\left[h\nabla^b\left(\nabla_c f n^c n^d\nabla_d g\right)\right]\\
&= \int_M d^4x\sqrt{-g}\left[\left(\nabla_b h\right)\nabla^b\left(\nabla_c f n^c n^d\nabla_d g\right) + h\Box\left(\nabla_c f n^c n^d\nabla_d g\right)\right]
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
&\int_M d^4x\sqrt{-g}\left(\nabla_a f n^a n^b\nabla_b h\right)\left(\nabla_c f n^c n^d\nabla_d g\right)\\
&= \int_M d^4x\sqrt{-g}\left\{\nabla_a\left[f\left(n^a n^b\nabla_b h\right)\left(\nabla_c f n^c n^d\nabla_d g\right)\right] - f\left(n^a n^b\nabla_b h\right)\nabla_a\left(\nabla_c f n^c n^d\nabla_d g\right)\right\}\\
&= \underbrace{\int_M d^4x\sqrt{-g}\nabla_a\left[\left(n^a n^b\nabla_b h\right)\left(\nabla_c f n^c n^d\nabla_d g\right)\right]}_{\overset{(C.7)}{=}\int_M d^4x\sqrt{-g}\left[(\Box h)\left(\nabla_c f n^c n^d\nabla_d g\right)+(\nabla_b h)\nabla^b\left(\nabla_c f n^c n^d\nabla_d g\right)\right]} - \underbrace{\int_M d^4x\sqrt{-g}\nabla_a\left[f n^a n^b h\nabla_b\left(\nabla_c f n^c n^d\nabla_d g\right)\right]}_{\overset{(C.8)}{=}\int_M d^4x\sqrt{-g}\left[(\nabla_b h)\nabla^b\left(\nabla_c f n^c n^d\nabla_d g\right)+h\Box\left(\nabla_c f n^c n^d\nabla_d g\right)\right]}\\
&+ \int_M d^4x\sqrt{-g}h\nabla_a f n^a n^b\nabla_b\left(\nabla_c f n^c n^d\nabla_d g\right)
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
&\int_M d^4x\sqrt{-g}\left[(\Box h)\left(\nabla_c f n^c n^d\nabla_d g\right)\right]\\
&\overset{(C.7)}{=} \int_M d^4x\sqrt{-g}\left(\nabla_a f n^a n^b\nabla_b h\right)\left(\nabla_c f n^c n^d\nabla_d g\right) - h\nabla_a f n^a n^b\nabla_b\left(\nabla_c f n^c n^d\nabla_d g\right)
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
&\left(\nabla_a f n^a n^b\nabla_b h\right)(\Box g) \overset{(C.11)}{=}\\
&\int_M d^4x\sqrt{-g}\left\{\left(\nabla_a f n^a n^b\nabla_b h\right)\left(\nabla_c f n^c n^d\nabla_d g\right) + g\Box\left(\nabla_a f n^a n^b\nabla_b h\right) - g\nabla_a f n^a n^b\nabla_b\left(\nabla_c f n^c n^d\nabla_d h\right)\right\}.
\end{aligned}
\tag{C.11}
$$

Therefore, we have shown that

$$
\int_M d^2x\sqrt{-g}\left\{h\left[\Box^2 - \left[\nabla f nn\nabla\Box + \Box\nabla f nn\nabla - (\nabla f nn\nabla)^2\right]\right]g - (h\leftrightarrow g)\right\} = 0
\tag{C.12}
$$

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and thus $\Box^2 - \left[\nabla fnn\nabla\Box + \Box\nabla fnn\nabla - (\nabla fnn\nabla)^2\right]$ is a Hermitian operator.

(II): Next consider the operator $\nabla_a G^{ab}\nabla_b - (\nabla fnnG\nabla + \nabla fGnn\nabla - \nabla fnnGnn\nabla)$:

By calculating

$$\int_M d^4x\sqrt{-g}\left[\underbrace{h\nabla_a\left(G^{ab}\nabla_b g\right)}_{=\nabla_a\left(hG^{ab}\nabla_b g\right)-(\nabla_a h)G^{ab}(\nabla_b g)} - (h\leftrightarrow g)\right] = \int_M d^4x\sqrt{-g}\left[\nabla_a\left(hG^{ab}\nabla_b g\right) - (h\leftrightarrow g)\right]$$
$$= \int_{\partial M} d^3x\sqrt{\gamma}\left[\left(n_a hG^{ab}\nabla_b g\right) - (h\leftrightarrow g)\right] \overset{a\to b\to c}{=} \int_{\partial M} d^3x\sqrt{\gamma}\left[\left(n_a hG^{bc}\nabla_c g\right) - (h\leftrightarrow g)\right]$$
$$= \int_{\partial M} d^3x\sqrt{\gamma}\left[(n_a n^a)\, n^b hG^c_b\nabla_c g - (h\leftrightarrow g)\right] = \int_M d^4x\sqrt{-g}\left[\nabla_a\left(fn^a n^b hG^c_b\nabla_c g\right) - (h\leftrightarrow g)\right]$$
$$\overset{(C.15)}{=} \int_M d^4x\sqrt{-g}\{\left[h\nabla_a\left(fn^a n^b G^c_b\nabla_c g\right) - (h\leftrightarrow g)\right] - \left[g\nabla_c\left(fG^c_b n^b n^a\nabla_a h\right) - (h\leftrightarrow g)\right]$$
$$+\left[g\nabla_a fn^a n^b G_{bc}n^c n^d\nabla_d h - (h\leftrightarrow g)\right] + \left[f\,(\nabla_a g)\, n^a n^d\,(\nabla_d h)\left(n^b G_{bc}n^c\right) - (h\leftrightarrow g)\right]\}$$
$$= \int_M d^4x\sqrt{-g}\left[\left(h\nabla_a fn^a n^b G^c_b\nabla_c g + h\nabla_a fG^a_b n^b n^c\nabla_c g - h\nabla_a fn^a n^b G_{bc}n^c n^d\nabla_d g\right) - (h\leftrightarrow g)\right]$$
(C.13)

where we used the following relations:

$$\int_M d^4x\sqrt{-g}\nabla_c\left[(\nabla_a h)\, fn^a n^b G^c_b g\right]$$
$$= \int_{\partial M} d^3x\sqrt{\gamma}\left(n_d n^d\right) fn_c\,(\nabla_a h)\, n^a n^b G^c_b g = \int_M d^4x\sqrt{-g}\nabla_d\left[fn^d n^c\,(\nabla_a h)\, n^a n^b G_{bc} g\right]$$
$$= \int_M d^4x\sqrt{-g}\left[(\nabla_d g)\,(\nabla_a h)\, fn^a n^d n^b G_{bc}n^c + g\nabla_d fn^d n^c G_{bc}n^b n^a\nabla_a h\right] \qquad \text{(C.14)}$$

$$\int_M d^4x\sqrt{-g}\left(fn^a n^b hG^c_b\nabla_c g\right)$$
$$= \int_M d^4x\sqrt{-g}\left\{h\nabla_a\left(fn^a n^b G^c_b\nabla_c g\right) + \underbrace{(\nabla_a h)\left(fn^a n^b G^c_b\nabla_c g\right)}_{=\int_M d^4x\sqrt{-g}\left\{\nabla_c\left[(\nabla_a h)fn^a n^b G^c_b g\right]-g\nabla_c\left[fG^c_b n^b n^a(\nabla_a h)\right]\right\}}\right\}$$
$$\overset{(C.14)}{=} \int_M d^4x\sqrt{-g}\{h\nabla_a\left(fn^a n^b G^c_b\nabla_c g\right) + (\nabla_d g)\,(\nabla_a h)\, fn^a n^d n^b G_{bc}n^c + g\nabla_d fn^d n^c G_{bc}n^b n^a\nabla_a h$$
$$- g\nabla_c\left[fG^c_b n^b n^a\,(\nabla_a h)\right]\} \qquad \text{(C.15)}$$

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Therefore, we know that

$$\int_M d^2x\sqrt{-g}\left\{h\left[\nabla_a G^{ab}\nabla_b - (\nabla fnnG\nabla + \nabla fGnn\nabla - \nabla fnnGnn\nabla)\right]g - (h \leftrightarrow g)\right\} = 0 \tag{C.16}$$

and thus $\nabla_a G^{ab}\nabla_b - (\nabla fnnG\nabla + \nabla fGnn\nabla - \nabla fnnGnn\nabla)$ is a Hermitian operator.

(III): Similarly, by calculating

$$\begin{aligned}
&\int_M d^4x\sqrt{-g}\left[\underbrace{h\nabla_a\left(R\nabla^a g\right)}_{=\nabla_a(hR\nabla^a g)-R(\nabla_a h)(\nabla^a g)} - g\nabla_a\left(R\nabla^a h\right)\right] = \int_M d^4x\sqrt{-g}\left[\nabla_a\left(hR\nabla^a g\right) - (h \leftrightarrow g)\right] \\
&= \int_{\partial M} d^3x\sqrt{\gamma}\left[n_b hR\nabla^b g - (h \leftrightarrow g)\right] = \int_{\partial M} d^3x\sqrt{\gamma}\left[\left(n_a n^a\right)n^b hR\nabla_b g - (h \leftrightarrow g)\right] \\
&= \int_M d^4x\sqrt{-g}\left[\underbrace{\nabla_a\left(fn^a n^b Rh\nabla_b g\right)}_{=h\nabla_a fn^a Rn^b\nabla_b g + f(\nabla_a h)n^a Rn^b(\nabla_b g)} - (h \leftrightarrow g)\right] \\
&= \int_M d^4x\sqrt{-g}\left[h\nabla_a fn^a Rn^b\nabla_b g - (h \leftrightarrow g)\right],
\end{aligned} \tag{C.17}$$

we can find that $\nabla_a R\nabla^a - [\nabla_a fn^a Rn^b\nabla_b]$ is a hermitian operator

By combining the results from (I), (II), (III) above, we proved that L_4^f is a hermitian operator.

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