



國立臺灣大學理學院數學系

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Master Thesis

Coleman 積分及 p -adic L -函數

On Coleman Integration and p -adic L -functions

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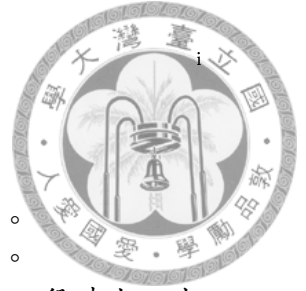
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誌謝

畢業在即，我想以一闕充分表述我目前心境的詞〈西江月〉起頭：

問訊湖邊春色，重來又是三年。東風吹我過湖船，楊柳絲絲拂面。
世路如今已慣，此心到處悠然。寒光亭下水如天，飛起沙鷗一片。

從研究所入學到完成這篇論文，歷時三年，是漫長而交織著酸甜苦辣的一段時光。本論文可以完稿，必須優先歸功於我的家人，尤其是雙親。您們在意志、情感、及物質上於我不曾間斷的支持與栽培，鼓勵我發掘並朝自己天賦和興趣邁進，是促成這篇論文的主要因素。

感謝指導教授 謝銘倫教授的提攜，提供我合適的論文題目。在過程中給予我對論文方向的指導同時，也讓我保有自由學習、摸索的空間，使我對數學和人生的體悟益發深刻，從您身上，我受益匪淺。

我自認現在的自己並不是善於面對挫折的人，所以難以忘懷在我求學過程遭受挫折時，熱心幫助過我的 于靖教授、朱樺教授、以及 江謝宏任教授，因為有您們在我腳步不穩時的攙扶，今日的我才有可能安穩地走在學術道路上。我也會時時勉勵自己，期盼不負您們的期待，成為足以獨當一面的數學家。

漫長的旅途中，相濡以沫的同伴們—無論是教學相長、共同勉勵、或分憂解勞，你們都是這篇論文之所以可以實現的遠因。也希望你們在人生或學術上的規劃，都能如願以償，「魚相與處於陸，不如相忘於江湖」，願共勉之。特別感謝伊婷，在這幾年的時間裡，經過了各種起伏，仍時刻陪伴在我身旁，在疲憊時支撐著我。

作為我學術生涯的一部分，這篇論文對我而言既不是開端、也不是終結，而是起著承先啓後的意義。我對於過去學思歷程中的種種滿懷感激，也同樣殷切期盼著在未來旅程上的青山綠水、即將滴下的汗水、和足以使人忘卻辛勞的豐碩果實。

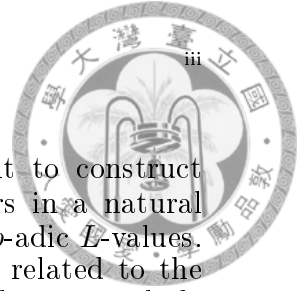
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中文摘要

本文探討在一維 p -adic 射影空間上的積分理論，並將其使用於建構 \mathbb{C}_p 上的對數 F -crystal，主要關注在過程中自然構造出的對數多項式函數 $l_k(z)$ 。對數多項式函數可實際應用在計算 p -adic 上的 L -函數特殊值；準確來說，對數多項式函數在分圓點上的取值和久保田-Leopold L -函數在正整數上的特殊值有連繫。文章以推導 Coleman (從 Koblitz 證明 $k = 1$ 的情形為推廣對象) 描述當 k 為正整數時， $L_p(k, \chi_{k-1})$ 以及 k 次對數多項式 $l_k(z)$ 關係的公式總結。

關鍵詞：Coleman 積分、 p -adic、對數 F -crystal、對數多項式、久保田-Leopold L -函數、 L -函數特殊值。



ABSTRACT

In this article we discuss the integration theory on $\mathbb{P}^1(\mathbb{C}_p)$, and apply it to construct the logarithmic F -crystal on \mathbb{C}_p , where polylogarithm functions $l_k(z)$ occurs in a natural development. The usage of polylogarithms realize in the computation for the p -adic L -values. To be precise, valuation of the polylogarithms at primitive roots of unity is related to the special values of the Kubota-Leopold L -function at positive integers. Eventually, we conclude by deriving a formula relating $L_p(k, \chi_{k-1})$ to the k^{th} -polylogarithm $l_k(z)$, which extends the formula by Koblitz, who proved the case $k = 1$.

Keywords : Coleman integral, p -adic, logarithmic F -crystal, polylogarithm, Kubota-Leopold L -function, special value of L -functions at positive integers.

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口試委員會審定書

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本論文係李宗儒君 (R02221030) 在國立臺灣大學數學所完成之
碩士學位論文，於民國 105 年 6 月 6 日承下列考試委員審查通過及口
試及格，特此證明

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ON COLEMAN INTEGRATION AND p -ADIC L -FUNCTIONS

CHUNG-RU LEE

ABSTRACT. In this article we discuss the integration theory on $\mathbb{P}^1(\mathbb{C}_p)$, and apply it to construct the logarithmic F -crystal on \mathbb{C}_p , where polylogarithm functions $l_k(z)$ occurs in a natural development. The usage of polylogarithms realize in the computation for the p -adic L -values. To be precise, valuation of the polylogarithms at primitive roots of unity is related to the special values of the Kubota-Leopold L -function at positive integers. Eventually, we conclude by deriving a formula relating $L_p(k, \chi_{k-1})$ to the k^{th} -polylogarithm $l_k(z)$, which extends the formula by Koblitz, who proved the case $k = 1$.

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1. INTRODUCTION

The main goal of the article is to introduce the Coleman integral and apply it to establish a formula for computing p -adic L -values. The Coleman integration is, vaguely speaking, to associate a closed 1-form $\theta \in \Omega(X)$ on X —a properly chosen space over non-archimedean fields—with a locally analytic primitive (or anti-derivative) $f_\theta \in \mathcal{L}(X)$, so that $df_\theta = \theta$. Integrating rigid analytic functions may be regarded as an analogy to the integration of complex or real analytic functions, except the abundance of locally constant functions over non-archimedean algebras made it difficult to determine the relation between local expressions.

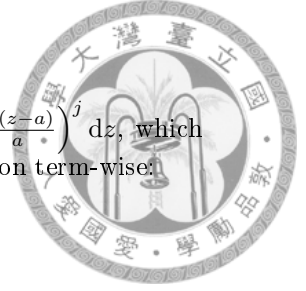
1.1. Coleman integral in brief.

The p -adic space is endowed with the ultrametric triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$, which made its topology rather simple. The extraordinariness of ultrametric triangle inequality is often characterized through two descriptions:

- (1) *Every point in a disc is the center of the disc,* and
- (2) *Any triangle is equilateral.*

To appreciate the difficulty to integration that hides behind the topology of p -adic spaces, observe an example:

$$X = \{z \in \mathbb{C}_p \mid \|z\| = 1\}, \text{ and } \theta = \frac{dz}{z} \in \Omega(X).$$



In order to seek for f_θ , it seems natural to write $\theta = \frac{d((z-a)+a)}{(z-a)+a} = \frac{dz}{(z-a)+a} = \frac{1}{a} \sum_{j=0}^{\infty} \left(\frac{-(z-a)}{a}\right)^j dz$, which applies on an open neighborhood about some $a \in X$ and simply integrate the local expression term-wise:

$$f_\theta = b - \frac{1}{a} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-(z-a)}{a}\right)^j.$$

The expression converges on $B(a, 1)$, and so the equality holds on $B(a, r)$ for $r \leq 1$ up to a constant $b = b(a)$. The choice of $a \in X$ is arbitrary, and one may duplicate the procedure for all points on X and obtain an abundant of local expressions about f_θ , up to a constant.

Until now the experience on \mathbb{R} or \mathbb{C} applies. In \mathbb{C} , in order to relate the local expressions and determine the constant b , the next step is to cover X with an open covering of connected nerve consisting of discs of the form $B(a, 1)$, and demand the valuation of the intersection coincide. In this way, we can adjust the local expression and find a constant $b = b(a)$ in a determined sense for each $a \in X$, actually $b(a)$ is constant module $2\pi i$, while winding around 0 creates branches for $\log(z)$ as we have known. On p -adic spaces, however, due to the fact (1) above, two discs of the same radius either coincide, or will be disjoint. Therefore the strategy above would not succeed.

A major property p -adic owns, while real or complex spaces do not, is the existence of the Frobenius morphisms, namely the automorphism $\phi : z \mapsto z^{p^n}$ on \mathbb{F}_{p^m} over \mathbb{F}_{p^n} , $m \geq n$. Coleman [Col82] developed a method referred by himself *analytic continuation along Frobenius morphism*. Consider the case above still, and $\phi : z \mapsto z^p$, which is Frobenius. Then

$$\phi^* \theta = p\theta.$$

Coleman imposed the condition that the integration f_θ should satisfies

$$\phi^* f_\theta - p f_\theta = b$$

for some global constant b . Vary f_θ by some non-zero constant we may set $b = 0$. Thus $\phi^* f_\theta - p f_\theta = f_\theta(z^p) - p f_\theta(z) = 0$. For any $a \in X$, there exists an $m \in \mathbb{N}$ so that the *Teichmüller point* in $B(a, 1) - \omega(a) = \lim_{n \rightarrow \infty} a^{p^{mn}}$ exists (and $(\omega(a))^{p^m} = \omega(a)$ follows), so $(1 - p^m) f_\theta(\omega(a)) = 0$. Thus the local expressions are determined uniquely near $\omega(a)$ by $b(\omega(a)) = 0$. Since for any $a \in X$, $a \in B(\omega(a), 1)$, the imposed condition indeed determined f_θ uniquely, up to a global constant.

To elaborate the method in a more thorough detail, suppose f_θ is constructed in the sense that it is unique module \mathbb{C}_p . Then $\int_x^y \theta = f_\theta(y) - f_\theta(x) \in \mathbb{C}_p$ is well-defined. Conversely, since $\theta \in \Omega = \text{Ad}z$, and $X = B[0, 1] \setminus B(0, 1)$ is covered by residue classes $X = \bigcup_{\|a\|=1} B(a, 1)$. Suppose we choose $f_\theta(a) = b$ and assume that there exists \mathbb{C}_p -linear

$$\int : \Omega(X) \rightarrow \text{A}(X) \quad \text{and} \quad d : \mathcal{L}(X) \rightarrow \Omega_{\mathcal{L}}(X)$$

that satisfy the following characterizations:

- (1) $d \circ \int : \Omega(X) \rightarrow \Omega_{\mathcal{L}}(X)$ is the canonical inclusion,
- (2) $\int \circ d : \text{A}(X) \rightarrow \mathcal{L}(X)/\mathbb{C}_p$ is a canonical inclusion (after omitted a global constant), and
- (3) $\phi^* \circ \int = \int \circ \phi^* \in \mathcal{L}(X)/\mathbb{C}_p$.

We may define $f_\theta(x)$ via

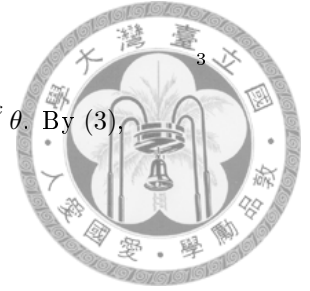
$$f_\theta(x) = f_\theta(a) + \int_a^x \theta.$$

If $x \in B(a, 1)$, by the local expression of $\int(\theta|_{B(a,1)}) \in \text{A}(B(a,1))/\mathbb{C}_p$ on $B(a, 1)$, $f_\theta(x)$ is uniquely determined. Now, for arbitrary $x \in X$, find $\omega(x)$, $\omega(a)$, and $m \in \mathbb{N}$ so that

$$\lim_{n \rightarrow \infty} (\phi^m)^n(a) = a \quad \text{and} \quad \lim_{n \rightarrow \infty} (\phi^m)^n(x) = x.$$

Then by definition,

$$\int_a^x \theta = \int_a^{\omega a} \theta + \int_{\omega a}^{\omega x} \theta + \int_{\omega x}^x \theta,$$



where the first and last term is known as discussed above. Thus it remains to compute $\int_{\omega a}^{\omega x} \theta$. By (3),

$$\int_{\omega a}^{\omega x} \theta = \int_{\phi^m(\omega a)}^{\phi^m(\omega x)} \theta = \int_{\omega a}^{\omega x} \phi^{*m} \theta = \phi^{*m} \int_{\omega a}^{\omega x} \theta = p^m \int_{\omega a}^{\omega x} \theta.$$

Thus $f_\theta(x)$ is uniquely determined by the choice of $f_\theta(a) \in \mathbb{C}_p$.

Remark. In the example we have worked with $\theta \in \Omega$ with $\phi^* \theta - p\theta = 0$. Actually replacement of p by any b which is not a primitive root of unity, and the right hand side by any exact form $dg \in dA$ can be dealt with using the same method.

In short, a \mathbb{C}_p -linear integration map satisfying the three conditions above may provide us with a unique solution to the differential equation

$$\begin{cases} \frac{d}{dz} f = g, \\ f(a) = b. \end{cases}$$

Coleman integration is mainly defined for this purpose, and hence is to satisfy the three conditions above, as we will see later in the passage.

1.2. The logarithmic F -crystal and p -adic L -functions.

The logarithmic F -crystal on a specific open set U is an $A(U)$ -module that lies between $A(U)$ and $\mathcal{L}(U)$. A logarithmic F -crystal consists of, roughly speaking, functions $f \in \mathcal{L}(U)$ so that $f dz$ may be integrated in the sense of the previous passage.

Coleman established a criterion of six examination to determine whether a $A(U)$ -submodule of $\mathcal{L}(U)$ is a logarithmic F -crystal. The *a posteriori* but astonishing result is that once we joint those integrals $\int f dz$ into the original logarithmic F -crystal, the sum (as an $A(U)$ -module) would still form a logarithmic F -crystal.

The logarithmic F -crystal, as we might observe in the very first example, is a natural module over $A(U)$ for the logarithmic function $\log(z)$ to occur (globally), and the fact that integration over a logarithmic F -crystal may be executed iteratively allows us to consider higher order differential equations, which arises the polylogarithmic functions $(\log z)^n$.

In practice, Coleman [Col82] defined $l_k(z)$ to be the solution to

$$\begin{cases} (z \frac{d}{dz})^k l_k = \frac{z}{z-1}, \\ \lim_{z \rightarrow 0} l_k(z) = 0. \end{cases}$$

Whose solution may be represented by the formal series

$$l_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

when close to 0. The significance of $l_1(z) = \log(1 - z)$ is well-established. Over the complex field \mathbb{C} , $l_k(z)$ has been related to the L -values at k . Let $\chi : \mathbb{C} \rightarrow \mathbb{C}^\times$ be a primitive Dirichlet character of conductor $d > 1$. If $\mathfrak{g}(\chi, \zeta)$ denote the *Gauss sum*: $\mathfrak{g}(\chi, \zeta) = \sum_{a=1}^d \chi(a) \zeta^{-a}$ for ζ being a primitive d^{th} -root of unity, and $l_k(0) = 0$ is the principle branch, then it is a well-known formula [Kob79] that

$$(1.1) \quad L(k, \chi) = \frac{\mathfrak{g}(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) l_k(\zeta^{-a}).$$

Also, when $\chi = 1$ is trivial, let $X = B(1, 1) \setminus \{1\}$, and $\mathcal{R} \xrightarrow{\pi} X$ denote the Riemann surface of $l_k|_X$. Consider a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{R}$ lying in finitely many sheets of \mathcal{R} , with $\lim_{n \rightarrow \infty} \pi(x_n) = 1$, then

$$(1.2) \quad L(k, 1) = \lim_{n \rightarrow \infty} l_k(x_n).$$

A purpose of this article is to apply the p -adic l_k constructed to formulate a p -adic analogue of the formulas above. Let ω denote the *Teichmüller character* on \mathbb{Z}_p^\times and χ a primitive Dirichlet character of conductor d ,



we will demonstrate

$$(1.3) \quad L_p(k, \chi \otimes \omega^{1-k}) = \left(1 - \frac{\chi(p)}{p^k}\right) \frac{\mathfrak{g}(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) l_{k,p}(\zeta^{-a})$$

$$(1.4) \quad L_p(k, \omega^{1-k}) = \left(1 - \frac{1}{p^k}\right) \lim_{x \rightarrow 1} l_k(x)$$

from [Col82]. Which are direct analogue for (1.1) and (1.2). For some of the notations not defined here, see section 6 and 7.

2. RIGID ANALYSIS ON PUNCTURED $\mathbb{P}^1(\mathbb{C}_p)$

2.1. Affinoid subspaces of \mathbb{C}_p .

Let \mathbb{C}_p be the complete non-archimedean algebraic closure of \mathbb{Q} above some finite place p , \mathcal{O} its ring of integer, \mathfrak{m} the maximal ideal of \mathcal{O} , and $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ the residue field.

Definition 2.1. (1) A *Tate algebra* over a non-archimedean field k is defined as

$$T_n(k) = k \langle z_1, \dots, z_n \rangle = \left\{ \sum_{|J| \geq 0} a_J z^J \mid a_J \in k, \|a_J\| \rightarrow 0 \text{ as } |J| \rightarrow \infty \right\},$$

with the *Gauss norm* $\| \sum_{|J| \geq 0} a_J z^J \| \triangleq \max_J \|a_J\|$ on it.

(2) A *k-affinoid algebra* is a k -algebra A which admits an isomorphism $A \simeq T_n/I$ for some ideal $I \subset T_n$. The maximal spectrum of a k -affinoid algebra A , along with the ring A , denoted $M(A) = (\text{Max} A, A)$, is said to be an *affinoid variety*.

Notice that the Tate algebra T_n is Noetherian, regular, factorial, and any ideal $I \subset T_n$ is closed. Therefore the ideal $I \subset T_n$ is always finitely generated, and $A = T_n/I$ possesses the residue norm $\|f\| \triangleq \inf_{g \in \tilde{f}} \|g\|$. An affinoid variety $M(A)$ is said to be connected if A is not a direct sum of two rings. Likewise, $M(A)$ is irreducible and reduced if A is, respectively.

For an affinoid algebra A , let $A_0 = \{f \in A \mid \|f\| \leq 1\}$, $A_1 = \{f \in A \mid \|f\| < 1\}$, and $\tilde{A} = A_0/A_1$. Note that \tilde{A} is a finitely generated polynomial ring over \mathbb{F} . The association $A \rightsquigarrow \tilde{A}$ is a covariant functor between the category of affinoid algebras and the category of rings. For an affinoid variety $X = M(A)$, we relate an $\tilde{X} = \text{Spec } \tilde{A}$. There is a natural reduction map

$$\begin{aligned} \text{red} : X &\rightarrow \tilde{X} \\ \mathfrak{m}_a &\mapsto [\mathfrak{m}_a \cap A_0], \end{aligned}$$

the last term indicates the ideal of residue classes in $A_0/A_1 = \tilde{A}$. X is said to have *good reduction* if \tilde{X} is smooth as an affine variety. The pre-image of a point $b \in \tilde{X}$ under the reduction map, namely $\text{red}^{-1}(b)$, is called a residue class in X . When k is algebraically closed, one can regard A as a k -valued function on $M(A)$ naturally. In this case it is known that $\|f\| = \sup_{a \in M(A)} \|f(a)\| = \max_{a \in M(A)} \|f(a)\|$.

In our notation, $k = \mathbb{C}_p$ unless specified. $B[a, r] = \{z \in \mathbb{C}_p \mid \|z - a\| \leq r\}$ and $B(a, r) = \{z \in \mathbb{C}_p \mid \|z - a\| < r\}$ are called *discs* in \mathbb{A}^1 , particularly we define $B[\infty, r] = \{z \in \mathbb{P}^1 \mid \|z\| \geq 1/r\} = \mathbb{P}^1 \setminus B(0, r^{-1})$ and $B(\infty, r) = \{z \in \mathbb{P}^1 \mid \|z\| > 1/r\}$. The *annuli* refer to sets of the form $A(a; r_1, r_2) = \{z \in \mathbb{A}^1 \mid r_1 < \|z - a\| < r_2, r_1, r_2 \in \|\mathbb{C}_p^\times\|\}$, in which case it is said to be an annulus about a . Similar to the definition of discs, we allow the center of an annulus to be ∞ . A *circle* is a set as $C(a, r) = A(a; r, r) = \{z \in \mathbb{C}_p \mid \|z - a\| = r\}$.

For this article, we consider the subspaces of $\mathbb{P}^1(\mathbb{C}_p)$ in the form $X = \{z \in \mathbb{P}^1(\mathbb{C}_p) \mid \|f(z)\| \leq 1, f \in F\}$, where $F \subset \mathbb{C}_p(z)$ is a finite subset. In this case if we consider $A(X)$ to be the completion of rational functions over \mathbb{C}_p that are regular on X , with respect to the supremum norm. $A(X)$ would be an affinoid algebra, thus X is an affinoid variety. Unless otherwise mentioned, we always assume that $X \subset \mathbb{P}^1(\mathbb{C})$, so $\dim_{\mathbb{C}_p} X = 1$. Higher dimensional affinoid varieties are not concerned in this article.

It is known that if X is connected, it is conformal to a set of form $B[0, 1] \setminus \bigcup_j B(a_j, r_j)$, that is $A(X) \simeq \mathbb{C}_p \langle z, x_1, \dots, x_s \rangle / (zx_j - t_j a_j \mid j = 1, \dots, s)$, where $0 < \|t_j\| = r_j \leq 1$. The form above is said to be a *standard*



subset of $B[0, 1]$. If further assume that X has good reduction, by a linear fractional transformation one may let $r_j = 1$ for $j = 1, \dots, s$. In this case the standard subset is called a *full subspace* of $B[0, 1]$, for further details, see Conrad [Con08] or Tate [Tat71].

2.2. The logarithm.

Definition 2.2. (1) For an arbitrary subset $V \subset \mathbb{P}^1$, let $\mathcal{L}(V)$ denote the set of *locally analytic functions* on V , and $A(V) \subset \mathcal{L}(V)$ be the set of functions f satisfying $f|_X \in A(X)$ for any affinoid subsets $X \subset V$.

(2) Let $V \subset \mathbb{P}^1$ be an open subset, we set $\Omega(V) = A(V)dz$, $\Omega_{\mathcal{L}}(V) = \mathcal{L}(V)dz$, and define the derivation $d : \mathcal{L} \rightarrow \Omega_{\mathcal{L}}$ in the canonical sense. $H^1(V)$ is defined as $\Omega(V)/dA(V)$.

We define a branch of the logarithm (usually referred to as the *p -adic Iwasawa logarithm*) by any locally analytic function l being a homomorphism $l : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p$ which satisfies $\frac{d}{dz}l(1) = 1$.

Lemma 2.3. *Let $l(z)$ denote a branch of the logarithm. Then $l(z)$ is analytic on $B(z, \|z\|)$ for any $z \in \mathbb{C}_p^\times$.*

PROOF. By definition, $l(xy) = l(x) + l(y)$, differentiation with respect to y and evaluate at $y = 1$ yields $l(1) = 0$ and $\frac{d}{dx}l(x) = \frac{1}{x}$, while $l(x)$ is analytic on an open neighborhood V around 1, its local expression can be written as

$$(2.1) \quad l(z) = - \sum_{n=1}^{\infty} \frac{(1-z)^n}{n},$$

which converges for $z \in B(1, 1)$. For any $z \in B(1, 1)$, there exists an $n \in \mathbb{N}$ so that $z^{p^n} \in V$. We thus have $p^n l(z) = l(z^{p^n})$, while the right hand side is analytic, leading to the analyticity of $l(z)$ on $B(1, 1)$. Finally, for any $x \in \mathbb{C}_p^\times$, consider $l(xz) = l(x) + l(z)$, which is analytic for $z \in B(1, 1)$. As a result $l(z)$ is analytic on $xB(1, 1) = B(x, \|x\|)$. \square

From the expression above, we have $d \log z = \frac{dz}{z}$ immediately. For any open subset $V \subset \mathbb{P}^1$ and $\log(z)$ a chosen branch of the logarithm, define $A_{\log}(V) = \tilde{A}(V) [\log f \mid f \in A(V)^\times]$.

A *wide open set* refers to a set $U = \{z \in \mathbb{P}^1 \mid \|f\| < r_f, f \in \mathcal{F}\}$, where $r_f = 1$ or ∞ , and $\mathcal{F} \subset \mathbb{C}_p(z)$ is a finite subset. If V is an annulus about $a \in \mathbb{A}^1$, a 1-form $\theta \in \Omega(V)$ can be written as $\sum_{j \in \mathbb{Z}} c_j(z-a)^j dz$, and we define the *residue* at a , denoted $\text{res}_a \theta$, to be c_{-1} . Observe that on any wide open annulus V around a , a 1-form $\theta \in dA(V)$ if and only if $\text{res}_a \theta = 0$.

Lemma 2.4. *If V is an annulus about a , and $f \in A(V)^\times$. If $\text{res}_a \frac{df}{f} = k$, then we have*

- (1) $k \in \mathbb{Z}$
- (2) f may be written as $f = c(z-a)^k(1+h)$, where $c \in \mathbb{C}_p$, $h \in A(V)$ and $\|h(z)\| < 1$ for any $z \in V$.

PROOF. Without loss of generality, let $a = 0$. It suffices to prove the lemma for each affinoid annulus about 0 contained in V , thus one may assume V is itself affinoid. That is, $V = A[0, r_1, r_2]$, and $r_1, r_2 \in \|\mathbb{C}_p\|$. Further exploit the advantage of a conformal map, we can assume that V is actually standard, which means $r_1 \leq 1$ and $r_2 = 1$.

- (1) Consider $V_1 = C(0, r_1) = \{z \in \mathbb{C}_p \mid \|z\| = r_1\}$. Choose $a_1 \in \mathbb{C}_p^\times$ so that $f_1 \triangleq a_1 f$ has $\|f_1\| = 1$. $\tilde{V}_1 \simeq \text{Spec} \frac{\mathbb{F}[z, y]}{(zy-1)}$, thus $\tilde{f}_1 \in A(\tilde{V}_1)^\times$ implies $\tilde{f}_1 = \tilde{b}_1 z^{k_1}$ for some $\tilde{b}_1 \in \mathcal{O}$, $\|\tilde{b}_1\| = 1$, and $k_1 \in \mathbb{Z}$. That is, $(\tilde{b}_1^{-1} z^{-k_1} f_1)^\sim = 1$. It follows that $\|\tilde{b}_1^{-1} z^{-k_1} f_1 - 1\| < 1$. From the previous lemma, this would mean that $\log \tilde{b}_1^{-1} z^{-k_1} f_1 \in A(V_1)$. Therefore $0 = \text{res}_0(d \log(\tilde{b}_1^{-1} z^{-k_1} f_1)) = \text{res}_0 \frac{d\tilde{f}_1}{\tilde{f}_1} - k_1$, and $\text{res}_0 \frac{df}{f} = \text{res}_0 \frac{d\tilde{f}_1}{\tilde{f}_1} = k_1 \in \mathbb{Z}$ follows.
- (2) Imitate the process in (1) for $V_2 = C(0, r_2)$. Let $b \in \mathbb{C}_p$ satisfies $\|bfz^{-k}\| = 1$. Observe that $(bz^{-k}f|_{V_j})^\sim = (ba_j^{-1}z^{-k}f_j)^\sim = (ba_j^{-1}b_j)^\sim$ are constants on V_j for $j = 1, 2$, respectively. Since the closure of $\tilde{V}_1 \cup \tilde{V}_2$ fills \tilde{V} , $(bz^{-k}f)^\sim = (ba_1b_1)^\sim = (ba_2b_2)^\sim = \tilde{b}\tilde{c}$ is a non-zero constant function on \tilde{V} for some $c \in \mathcal{O}$. Thus $f = cz^k(1+h)$ for some $h \in A(V)$ and $\|h\| < 1$. \square



Corollary 2.5. *By the results of Lemmas 2.3 and 2.4, for an annulus V about $a \in \mathbb{A}^1$,*

$$A_{\log}(V) = A(V) [\log(z - a)].$$

Let V be an open set and \mathcal{F} be a collection of \mathbb{C}_p -valued functions on V . \mathcal{F} is said to satisfy the *identity principle* on V if for any $f, g \in \mathcal{F}$, $f = g$ on V whenever $f = g$ on any open subset $U \subset V$. It is known that for a connected open subset $V \subset \mathbb{P}^1$, $A(V)$ satisfies the identity principle on V .

For an affinoid V , a point $a \in V$ and a function $f \in A(V)$, we define the *order* of f at a , denoted $\text{ord}_a f$, as when defined on an affine variety. For any rational function $h = f/g$, we let $\text{ord}_a h = \text{ord}_a f - \text{ord}_a g$. Notice that for a locally analytic function f , $\text{ord}_a f' \leq \text{ord}_a f$.

Proposition 2.6. *Let V be a wide open annulus about a . Then $A_{\log}(V) = A(V) [\log f \mid f \in A(V)^\times]$ satisfies the identity principle on V .*

PROOF. [Col82] Without loss of generality, let $a = 0$. Suppose the statement is false, let $n \in \mathbb{N}$ be the least possible integer so that there exists an $f \in A_{\log}(V)$, $f = \sum_{m=0}^n g_m(\log z)^m$ is non-zero, while $f = 0$ on some open $U \subset V$. $n > 0$ since $A(V)$ satisfies the identity principle. Consider

$$\begin{aligned} f' &= \sum_{m=0}^n g'_m(\log z)^m + \sum_{m=0}^{n-1} \frac{m+1}{z} g_{m+1}(\log z)^m \\ &= g'_n(\log z)^n + \sum_{m=0}^{n-1} (g'_m + \frac{m+1}{z} g_{m+1})(\log z)^m. \end{aligned}$$

$f' = 0$ on U , as well as $g'_n f - g_n f'$. While $\deg_{\log z} f > \deg_{\log z} (g'_n f - g_n f')$, $g'_n f - g_n f' = 0$ on V by assumption.

In particular, $g_n(g_{n-1} + \frac{n}{z} g_n) - g'_n g_{n-1} = 0$. Which indicates that $(\frac{g_{n-1}}{g_n})' = \frac{n}{z}$, and so $\frac{g_{n-1}}{g_n}$ is analytic on V since the only possible pole, 0, lies outside. $\frac{g_{n-1}}{g_n} \in A(V)$ contradicts with the fact that $\text{res}_0 d(\frac{g_{n-1}}{g_n}) = n$ is non-zero. \square

As a result, for a wide open annulus V , we have $\dim_{\mathbb{C}_p} H^0(A_{\log}(V)) = 1$. Moreover,

Lemma 2.7. *Let V be an wide open annulus, $H^1(A_{\log}(V)) = 0$.*

PROOF. Let $\theta = \sum_{j=0}^n h_j \log z^j \in \Omega_{\log}(V)$, where $h_j = \sum_{k \in \mathbb{Z}} a_{j,k} z^k \in A(V)$. We demonstrate by induction on n .

For $n = 0$, $\theta = h_0 = \sum_{k \in \mathbb{Z}} a_{0,k} z^k$. $(h_0 - a_{0,-1} z^{-1}) dz \in dA(V)$, while $a_{0,-1} \frac{dz}{z} = a_{0,-1} d \log z$.

As if $n > 0$, $h_n dz = a_{n,-1} \frac{dz}{z} + dg$ for some $g \in A(V)$ as in the previous case. So

$$\begin{aligned} h_n(\log z)^n dz &= a_{n,-1}(\log z)^n d \log z + (\log z)^n dg \\ &= \frac{a_{n,-1}}{n+1} d(\log z)^{n+1} + n(\log z)^{n-1} \frac{g}{z} dz - d(\log z^n g). \end{aligned}$$

Where the second last term lies in $dA_{\log}(V)$ by hypothesis. \square

3. THE DWORK PRINCIPLE

3.1. \mathbb{F}_q -Frobenius morphisms.

Definition 3.1. Let X be an affinoid over \mathbb{C}_p , \mathbb{F}_q denote the Galois field of order $q = p^n$, and $F : \tilde{A}(X) \rightarrow \tilde{A}(X)$ is a lifting of the absolute Frobenius automorphism of \mathbb{F}/\mathbb{F}_q to $\tilde{A}(X)$.

- (1) A *Frobenius morphism* refers to a morphism $\phi : X \rightarrow X$ so that the associated $\tilde{\phi}^* : \tilde{A}(X) \rightarrow \tilde{A}(X)$ is in the form of $\tilde{\phi}^*(f) = F^{-1}(f^q)$.
- (2) For $Y \subset \mathbb{P}^1$ a rigid analytic space and $X \subset Y$ be a subaffinoid. A pair (U, ϕ) of wide open neighborhood $X \subset U \subset Y$, along with a morphism $\phi : U \rightarrow Y$ is said to be a *Frobenius neighborhood* of X in Y if $\phi|_X$ is a Frobenius automorphism.
- (3) For any affinoid variety $X \subset \mathbb{P}^1$, and (U, ϕ) be an \mathbb{F}_q -Frobenius neighborhood of X in \mathbb{P}^1 . We define iteratively for $n \in \mathbb{N}$: $F_1(U) = U$, and $F_n(U) = \{z \in F_{n-1}(U) \mid \phi(z) \in F_{n-1}(U)\}$.



In the non-archimedean norm, we have the ultrametric triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. In particular, if $\|x\| > \|y\|$, then $\|x + y\| = \|x\|$.

Lemma 3.2. *Let X be a full subspace of $B[0, 1]$, then there exists some $q = p^n \in \mathbb{N}$ so that the map $\phi : z \mapsto z^q$ is a Frobenius automorphism of X . Thus, for any wide open neighborhood $\mathbb{P}^1 \supseteq U \supset X$, the pair (U, ϕ) is a Frobenius neighborhood of X .*

PROOF. Suppose X is of the form $B[0, 1] \setminus \cup_{j=1}^d B(a_j, 1)$. Since d is finite, a_j lies in finitely many residue classes. There exists some $q = p^m \in \mathbb{N}$ so that $\tilde{a}_j^q = \tilde{a}_j$ for $j = 1, \dots, d$. In other words, q may be chosen with $\|a_j^q - a_j\| < 1$.

For $z \in X$, we have $\|z - a_j\| = 1$, so $\tilde{z}^q - \tilde{a}_j^q = (\tilde{z} - \tilde{a}_j)^q \neq 0 \in \mathbb{F}$. Therefore $\|z^q - a_j^q\| = 1$, and $\|z^q - a_j\| = \max(\|z^q - a_j^q\|, \|a_j^q - a_j\|) = 1$. Since the argument applies for any j , $\phi : X \rightarrow X$ is an endomorphism, while it is Frobenius by direct computation. \square

The lemma above leads to the implication that any connected affinoid with good reduction has a Frobenius neighborhood.

Lemma 3.3. *Let $Y = \mathbb{A}^1 \setminus \{a_1, \dots, a_d\}$, $X \subset Y$ be a subaffinoid with good reduction, and (U, ϕ) an \mathbb{F}_q -Frobenius neighborhood of X in Y . Then there exists an $m \in \mathbb{N}$ and an \mathbb{F}_{q^m} -Frobenius neighborhood (V, φ) with $V \subset F_m(U)$ of X so that*

- (1) $\varphi(A \cap V) \subseteq A$ for each connected component A of $\mathbb{A}^1 \setminus X$.
- (2) $\varphi^* \log(z - a) - q^n \log(z - a) \in A(V)$.

PROOF. [Col82] By assumption, we may assume that X is a full subspace of Y , say, $X = B[0, 1] \setminus \cup_{j=1}^s B(a_j, 1)$, with $\|a_j\| \leq 1$ only when $j = 1, \dots, s$. In this way, the connected components of $\mathbb{A}^1 \setminus X$ are the residue classes $B(a_j, 1)$ and $\mathbb{A}^1 \setminus B[0, 1]$.

Consider the identity function $z \in B[0, 1]^\sim = T_1(k)^\sim = \mathbb{F}[z]$, which is simply $z : a \mapsto a$. Then

$$\phi(a)^\sim = \tilde{\phi}^*(z)|_{z=a} = z^q|_{z=a} = \tilde{a}^q.$$

Thus, $\|\phi(z) - z^q\| < 1$ for all $\|z\| \leq 1$. Define $W = \{z \in U \mid \|\phi(z) - z^q\| < 1\}$

$B[0, 1] \setminus X$ consists of only finitely many residue classes, one may choose $m \in \mathbb{N}$ so that $\|z^{q^m} - z\| < 1$ for any $z \in B[0, 1] \setminus X$. Let $\varphi = \phi^m$, then $\|\varphi(z) - z\| \leq \max(\|\varphi(z) - z^{q^m}\|, \|z^{q^m} - z\|) < 1$. Notice that for any $\|a\| > 1$, $\phi(a)^\sim = \tilde{a}^q \neq 0$, so $\|\phi(a)\| > 1$, resulting in $\|\varphi(a)\| = \|\phi^m(a)\| > 1$. Thus $\varphi|_A$ is an endomorphism for each connected component A of $\mathbb{A}^1 \setminus X$. In particular, the statement for (1) holds φ on any wide open neighborhood of X .

For (2), notice that for all $z \in X$,

$$(3.1) \quad \left\| 1 - \frac{\varphi(z) - a_j}{z^{q^m} - a_j} \right\| < 1 \quad \text{for } j \leq s,$$

$$(3.2) \quad \|z\| < \min(\|a_j\|, \|\varphi(a_j)\|) \quad \text{for } j > s.$$

The first inequality is true because for $z \in X$, $\varphi(z)^\sim = \tilde{z}^{q^n}$ and $\tilde{a}_j^{q^n} = \tilde{a}_j$. So $(\varphi(z) - a_j)^\sim = \tilde{z}^{q^n} - \tilde{a}_j^{q^n} = (\tilde{z} - \tilde{a}_j)^{q^n}$. The second holds since $\varphi : X \rightarrow X \subseteq \mathcal{O}$ is an endomorphism, while $\|a_j\| > 1$ for $j > s$.

By Lemma 2.3, the first inequality implies $\varphi^* \log(z - a_j) - q^n \log(z - a_j) = \log(\frac{\varphi(z) - a_j}{(z - a_j)^q})$ is analytic. Moreover, the second line says that $\log(z - a_j)$ and $\log(\varphi(z) - a_j)$ are analytic, therefore so is $\varphi^* \log(z - a_j) - q^n \log(z - a_j)$. Define V by

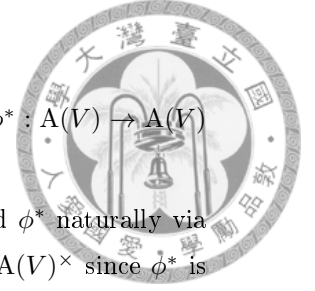
$$V = \{z \in F_m(U) \mid \left\| 1 - \frac{\varphi(z) - a_j}{z^{q^m} - a_j} \right\| < 1 \text{ for } j \leq s, \text{ and } \|z\| < \min(\|a_j\|, \|\varphi(a_j)\|) \text{ for } j > s\}$$

Then (V, φ) is a Frobenius neighborhood of $X \subset Y$ with (2) satisfied. \square

3.2. The Dwork Principle.

Let X be an affinoid with good reduction, ϕ a Frobenius morphism on X . It is known that for each residue class $R \subset X$ there exists an $n \in \mathbb{N}$, $z_R \in R$ so that $\lim_{m \rightarrow \infty} \phi^{nm}(z) = z_R$, in which case z_R is said to be the *Teichmüller point* of R in X with respect to ϕ^n .

Observe that z_R is a fixed point for ϕ^n , that is, $\phi^n(z_R) = z_R$.



Lemma 3.4. *Let V be a wide open annulus, and $\phi : V \rightarrow V$ be a rigid morphism. Then $\phi^* : \mathbb{A}(V) \rightarrow \mathbb{A}(V)$ may be extended to a map $\mathbb{A}_{\log}(V) \rightarrow \mathbb{A}_{\log}(V)$.*

PROOF. Note that by definition $\mathbb{A}_{\log}(V) = \mathbb{A}(V)[\log(f) \mid f \in \mathbb{A}(V)^\times]$, so one can extend ϕ^* naturally via $\phi^*(\sum_{j=1}^s g_j(z) \log f_j(z)) = \sum_{j=1}^s \phi^* g_j(z) \log \phi^* f_j(z)$, since $\phi^* g_j$ still lies in $\mathbb{A}(V)$, while $\phi^* f_j \in \mathbb{A}(V)^\times$ since ϕ^* is a homomorphism, thus $\phi^*(\sum_{j=1}^s g_j(z) \log f_j(z)) \in \mathbb{A}_{\log}(V)$ as desired. \square

Lemma 3.5. *Let X be an affinoid, $\phi : X \rightarrow X$ a Frobenius morphism, and f being locally constant. Suppose there exist some $a \in \mathbb{C}_p$ with $1 \notin \{a^n\}_{n \in \mathbb{N}}$ so that $\phi^* f - af = 0$. Then $f = 0$ on X .*

PROOF. $\phi^{n*} f = a^n f$ by direct computation. Let $R \subset X$ be a residue class, with z_R its Teichmüller point for ϕ^n . $f(z_R) = 0$ because $a^n f(z_R) = \phi^{n*} f(z_R) = f(z_R)$. f is locally constant, so that $f = 0$ for some open neighborhood $z_R \in V \subset R$. For any $z \in R$, chose m great enough so that $\phi^{nm}(z) \in V$. Then $a^{nm} f(z) = \phi^{nm*} f(z) = f(\phi^{nm}(z)) = 0$. Since the argument holds for any residue class R in X , $f = 0$ on X . \square

A few notations to be mentioned:

- (1) Let $\mathcal{L}_R(X) = \{f \in \mathcal{L}(X) \mid f|_R \in \mathbb{A}(R) \text{ for each residue class in } X\}$, and
- (2) $\Omega_R(X)$ is defined by $\Omega_R(X) \triangleq \mathcal{L}_R(X) \otimes_{\mathbb{A}(X)} \Omega(X)$.

Observe that if \tilde{X} is smooth, any residue class $R \subset X$ would be conformal to $B(0, 1)$, and $\phi(z_R) = z_{\phi(R)}$. Let R be a residue class, and $\phi : X \rightarrow X$ be a Frobenius morphism. Then $\phi(R)$ also lies in a residue class since $\|x - y\| < 1$ implies $(\phi(x) - \phi(y))^\sim = \tilde{x}^q - \tilde{y}^q = 0$. Suppose U lies in a residue class R , we set $n = n_{\phi, U} = \min_{m \in \mathbb{N}} \{m \mid \phi^m(R) \subseteq U\}$, the inequality $n_{\phi, R} \geq n_{\phi, \phi(R)}$ leads to

$$n_{\phi, R} \geq n_{\phi, \phi(R)} \geq \cdots \geq n_{\phi, \phi^n(R)} = n_{\phi, R}.$$

Which says $n_{\phi, R} = n_{\phi, \phi(R)}$ for any residue classes $R \subset X$.

Lemma 3.6. *Let X be an affinoid with good reduction, $\theta \in \Omega_R(X)$, and $\phi : X \rightarrow X$ be Frobenius satisfying $\phi^*(\theta) - a\theta \in d\mathcal{L}_R(X)$ with $1 \notin \{a^n\}_{n \in \mathbb{N}}$. Suppose $\phi^*(\theta) - a\theta = dg$ for some $g \in \mathcal{L}_R(X)$. Then there exists a unique $f \in \mathcal{L}_R(X)$ so that*

- (1) $df = \theta$, and
- (2) $\phi^*(f) - af = g$.

PROOF. Note first that if (2) hold for some $f \in \mathcal{L}_R(X)$, then f must also satisfies $\phi^{n*} f = a^n f_R + \sum_{j=0}^{n-1} a^{n-1-j} g \circ \phi^j$ for any $n \in \mathbb{N}$.

Any residue class $R \simeq B(0, 1)$ since \tilde{X} is smooth. In other words, we may let $\mathbb{A}(R) \simeq \mathbb{C}_p \langle z \rangle$. Choose $n = n_{\phi, R} \in \mathbb{N}$ be the minimal natural number so that $\phi^{nR}(R) \subseteq R$, and find the corresponding Teichmüller point z_R . Since $\theta \in \Omega_R(X)$, $\theta|_R \in \mathbb{A}(R)dz$. Hence there exist $f_R \in \mathbb{A}(R)$ so that $df_R = \theta|_R$. The f_R above is unique up to its constant term, and will be so after we demand that $f(z_R) = (1 - a^n)^{-1} \sum_{j=0}^{n-1} a^{n-1-j} g \circ \phi^j(z_R)$.

Define $f \in \mathcal{L}_R(X)$ to be the unique function satisfying $f|_R = f_R$, and f satisfies (1) by construction.

If we let $F = \phi^*(f) - af - g \in \mathcal{L}(X)$, $dF = 0$ implies F being locally constant. Notice that $F \in \mathcal{L}_R(X)$, so F is actually constant on each residue class. Since $n_{\phi, R} = n_{\phi, \phi(R)}$ as discussed above, we have $\phi^* f(z_R) =$



$(1 - a^n)^{-1} \sum_{j=0}^{n-1} a^{n-1-j} g \circ \phi^{j+1}(z_R) = (1 - a^n)^{-1} \sum_{j=1}^n a^{n-j} g \circ \phi^j(z_R)$. Thus

$$\begin{aligned} F(z_R) &= \phi^* f(z_R) - a f(z_R) - g(z_R) \\ &= \frac{1}{1 - a^n} \sum_{j=1}^n a^{n-j} g \circ \phi^j(z_R) - \frac{a}{1 - a^n} \sum_{j=0}^{n-1} a^{n-1-j} g \circ \phi^j(z_R) - g(z_R) \\ &= \frac{1}{1 - a^n} \left(\sum_{j=1}^n a^{n-j} g \circ \phi^j(z_R) - \sum_{j=0}^{n-1} a^{n-j} g \circ \phi^j(z_R) \right) - g(z_R) \\ &= \frac{1}{1 - a^n} (g(\phi^n(z_R)) - a^n g(z_R)) - g(z_R) \\ &= 0 \end{aligned}$$

Therefore $F = 0$ on X , which indicates that f satisfies (2). \square

Corollary 3.7. *The function f constructed above is manifestly unique.*

PROOF. Suppose there exists f_1 and f_2 in $\mathcal{L}_R(X)$ satisfying (1) and (2) simultaneously. Consider $F = f_1 - f_2$, which satisfies

- (1) $dF = 0$
- (2) $\phi^* F - aF = 0$

(1) implies that F is locally constant, and so (2) would lead to $F = 0$ on X by lemma 3.5. \square

4. THE LOGARITHMIC F -CRYSTALS

4.1. Definition of a logarithmic F -crystal on \mathbb{A}^1 .

Recall that a full subspace X in $B[0, 1]$ is an affinoid subset of the form $B[0, 1] \setminus \cup_{j=1}^d B(a_j, 1)$. A basic wide open set U about X is defined to be $U = \mathbb{A}^1 \setminus \cup_{j=1}^d B[a_j, r_{a_j}]$, where $r_{a_j} < 1$. For $j = 1, \dots, d$, let V_j denote the wide open annuli $V_j = A(a_j; r_{a_j}, 1)$. Notice that $X \cup \{V_j\}_{j=1}^d \cup A(\infty; 0, 1) = U$ is a disjoint covering of U . For $0 < r < 1$, let $U_r(X) \triangleq B(0, r^{-1}) \setminus \cup_{j=1}^d B[a_j, r]$. $U_r(X) \supset X$ is a wide open neighborhood, and $U_r(X) \subset U$ whenever r is sufficiently close to 1.

Now, U is a basic wide open set. For $z \in U$, set $r(z) \triangleq \min_{j=1}^d \{\|z - a_j\|\}$, $D = D(U) \triangleq \{(x, y) \in U \times U \mid \|x - y\| < r(x)\}$, and $p_j : D \rightarrow U$ to be the projection maps, $j = 1, 2$.

Suppose $U \subset \mathbb{A}^1$ is an open subset, and $M \subset \mathcal{L}(U)$ is an $A(U)$ -submodule. If $\iota : V \hookrightarrow U$, we let $M(V) = \iota^* M$, $\Omega_M(V) = \iota^* \Omega_M$ be defined by restriction. It follows that $M(V) = M \otimes_{A(U)} A(V) \subseteq \mathcal{L}(V)$ regarded as an $A(V)$ -module, and $\Omega_M(V) = M(V) \otimes_{A(V)} \Omega(V) \subseteq \Omega_{\mathcal{L}}(V)$. Thus M and Ω_M can be regarded as sheaves on U (though if not specified, M and Ω_M will denote $M(U)$ and $\Omega_M(U)$ respectively from now on). For any morphism between rigid spaces $f : U \rightarrow V$, $f^* M$ and $f^* \Omega_M$ will denote the inverse image sheaves (of A -modules), respectively. If further M satisfies $dM \subseteq \Omega_M(U)$, define $H^1(M(V)) \triangleq \Omega_M(V)/dM(V)$.

Definition 4.1 (Logarithmic F -crystals). Let $U = \mathbb{A}^1 \setminus \cup_{j=1}^d B[a_j, r_{a_j}]$ be a basic wide open set about a full subspace of $B[0, 1]$, say $X = B[0, 1] \setminus \cup_{j=1}^d B(a_j, 1)$. A *logarithmic F -crystal* on $U \subset \mathbb{A}^1$ is defined to be an $A(U)$ -module $M = M(U) \subset \mathcal{L}(U)$ with $A(U) \subset M$, satisfying the six conditions below:

- (A) For $j = 1, \dots, d$,
 - (A₁) $M(X) \subseteq \mathcal{L}_R(X)$, and
 - (A₂) $M(V_j) \subseteq A_{\log}(V_j)$.
- (B) $dM \subseteq \Omega_M(U)$.
- (C) $p_1^* M = p_2^* M$.
- (D) For any $0 < r < 1$ so that $U_r \subset U$, $M(U_r(X))$ satisfies the identity principle.
- (E) For any $0 < r < 1$ so that $U_r \subset U$, the natural map $\iota^* : H^1(M) \rightarrow H^1(M(U_r(X)))$ induced by restriction is an isomorphism.
- (F) There is an Frobenius neighborhood (V, ϕ) of X in U , and $\iota : V \rightarrow U$ the inclusion map, so that
 - (F₁) $\phi^*(M) \subseteq M(V) = \iota^*(M)$, and



(F₂) There exists a $b \in \mathbb{C}_p$ with $1 \notin \{b^n\}_{n=1}^\infty$ so that $\phi^*(\theta) - b\theta \in dM(V)$ for any $\theta \in \Omega_M(U)$.

Lemma 4.2. *For any Frobenius neighborhood (V, ϕ) of X in U satisfying **(F)**, $(F_n(V), \phi^n)$ also satisfies **(F)**, with ϕ^n and b^n in place of ϕ and b , respectively.*

PROOF. [Col82] Note that for any $n \in \mathbb{N}$, $\phi^{n-1}F_n(V) \subseteq V$. By (F₁),

$$\phi^{n*}M(F_n V) = \phi^*M(\phi^{n-1}F_n V) \subseteq \iota^*M(\phi^{n-1}F_n V) = \phi^{n-1*}M(F_n V),$$

so by induction, $\phi^{n*}M(F_n V) \subseteq M(F_n(V))$. Observe that $F_n(V)$ here may be replaced by any subset of itself, so (F₁) holds for $(F_n(V), \phi^n)$.

Suppose now $\phi^*\theta - b\theta = dg$ for some $g \in M(V)$. Since $\phi^*M(V) \subseteq \iota^*M(V) = M(V)$, by induction we have $\phi^{m*}g \in M(V)$. Thus $\phi^{m*}dg \in dM(V)$ for $m = 0, \dots, n-1$. Writing

$$\phi^{n*}\theta - b\theta = \sum_{j=1}^n (\phi^*)^{n-j} b^{j-1} (\phi^*\theta - b\theta) = \sum_{j=1}^n (\phi^*)^{n-j} b^{j-1} dg,$$

we see that (F₂) holds for $(F_n(V), \phi^n)$ □

From now on, we will always assume that the Frobenius neighborhood (V, ϕ) we choose for the logarithmic F -crystal M over U satisfies the statement of Lemma 4.2 and is of the form $V = U_r(X)$ for some $0 < r < 1$. In this manner, we have $V \cap V_j$ as an annulus for $j = 1, \dots, d$.

4.2. Integration on a logarithmic F -crystal.

The main theorem for this section is as follows:

Theorem 4.3. *There exists a unique minimal logarithmic F -crystal $M^1 \supset M$ over U so that*

$$dM^1 \supset \Omega_M(U).$$

That is, for any $\theta \in \Omega_M(U) = M(U)dz$, there exists an $f_\theta \in M^1$ so that $df_\theta = \theta$.

The proof for this theorem will be postponed until the end of this section.

Lemma 4.4. *Let M be a logarithmic F -crystal on $U \supset X$. There exists a \mathbb{F}_q -Frobenius neighborhood (W, φ) of X in U , $\varphi = \phi^n$ for some $n \in \mathbb{N}$ so that*

- (1) (W, ϕ) satisfies **(F)** with b replaced by $a = b^n$ in (F₂).
- (2) $\phi(W \cap V_j) \subset V_j$ for $j = 1, \dots, d$.
- (3) $\log\left(\frac{\phi(z) - a_j}{(z - a_j)^q}\right) \in A(W)$ for $j = 1, \dots, d$.

PROOF. [Col82] Let (V, ϕ) be the Frobenius neighborhood for M in **(F)**. By Lemma 3.3, there is an \mathbb{F}_q Frobenius neighborhood (V^1, ϕ^n) of X with $V^1 \in F_n(V)$ so that (3) is satisfied. Also, for any connected component $A \subset \mathbb{A}^1 \setminus X$, we have $\phi^n(A \cap V^1) \subset A$.

To see that (W, φ) satisfies (2), by shrinking V^1 , one may assume that each $A \cap V^1 \subset V_j$ for some $j = 1, \dots, d$. Set $W = \bigcap_{j=0}^n \{z \in V^1 \mid \phi^j(z) \in V^1\} = \bigcap_{j=1}^{n-1} F_j(V^1) = F_{n-1}(V^1)$, then for each connected component $A \subset \mathbb{A}^1 \setminus X$, since $W \subset V^1$, we have

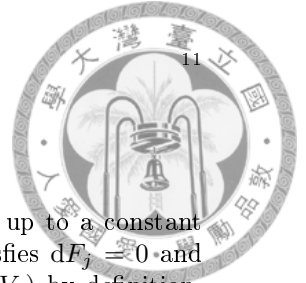
$$\phi^n(V_j \cap W) \subseteq \phi^n(A \cap W) \subseteq A \cap V_1 \subseteq V_j.$$

Since $W \subseteq F_n$, by Lemma 4.2, (W, φ) satisfies (1). □

Lemma 4.5. *Let $\theta \in \Omega_M(U)$ be a 1-form, (V, ϕ) be the Frobenius neighborhood in **(F)**. There exists an $f_\theta \in \mathcal{L}(U)$ and $W \subseteq V$ satisfying*

- (1) $df_\theta = \theta$,
- (2) $\phi^*f_\theta - bf_\theta \in M(V)$ after restriction onto V , and
- (3) $f_\theta|_X \in \mathcal{L}_R(X)$, $f_\theta|_{V_j} \in A_{\log}(V_j)$, where $j = 1, \dots, d$.

PROOF. [Col82] Let $\theta = h(z)dz$, we have $h|_X \in \mathcal{L}_R(X)$ by **(A)**, so $\theta|_X \in \Omega_R(X)$. Moreover, by (F₂), there exists some $g \in M(V)$ so that $\phi^*(\theta) - b\theta = dg$. Thus by Lemma 3.5, there exists a unique $f \in \mathcal{L}_R(X)$ with



- (a) $df|_X = \theta|_X$, and
- (b) $\phi^*(f) - bf = g$

on X .

Meanwhile for $j = 1, \dots, d$, on V_j we have $f_j \in A_{\log}$ with $df_j = \theta|_{V_j}$, which is unique up to a constant for each j . Again by (F_2) , we have $F_j = \phi^*f_j - bf_j - g \in M(V_j \cap V) \subseteq \mathcal{L}(V_j \cap V)$ satisfies $dF_j = 0$ and is locally constant. However, $\phi^*f_j \in \phi^*M(V \cap V_j) \subseteq M(V \cap V_j)$ by (F_1) , $bf_j \in M(V \cap V_j)$ by definition, $g|_{V \cap V_j} \in M(V \cap V_j)$, while $M(V \cap V_j) \subseteq A_{\log}(V_j \cap V)$ by **(A)**, we have $F_j \in A_{\log}(V_j \cap V)$. Since $V \cap V_j$ are annuli, by the identity principle of A_{\log} over annuli (Proposition 2.6), F_j is constant on $V \cap V_j$.

Notice that by Proposition 2.6, f_j may be differed only by a constant. Suppose $f_\theta|_{V_j} = f_j + b_j$, then $\phi^*(f_j + b_j) - b(f_j + b_j) - g = F_j + (1 - b)b_j$. Thus if we set $b_j = -(1 - b)^{-1}F_j$, that is, $f_\theta|_{V_j} = f_j - (1 - b)^{-1}F_j$, it would be $\phi^*f_\theta - bf_\theta = g$ on V_j .

Thus, we define f_θ by

- (a) $f_\theta|_{V_j} = f_j - (1 - b)^{-1}F_j$, $j = 1, \dots, d$, and
- (b) $f_\theta|_X = f$,

which satisfies the three conditions given in the statement. □

Respecting its own nature, we will denote the function f_θ constructed above as $\int \theta$ for the rest of this article. Justification of this notation will be done in the following passage.

Remark. Apparently the function satisfying the conditions in Lemma 4.5 above is not unique; however, when (V, ϕ) is replaced by any other Frobenius neighborhood, say $(F_n(V), \phi^n)$, the constant chosen would be different but will vary as a constant function on U by property **(D)**. We construct $f_\theta = \int \theta$ in the sense of proving Lemma 4.5, but any function satisfying the same conditions will not affect the validity of our results. One can simply regard the liberty by a constant as a generalization of the case in the indefinite integration over \mathbb{R} or finding primitive on \mathbb{C} . For convenience, We will denote $f \sim g$ if f and g differ by a constant.

We will now state some properties for primitive computation, some of which may be regarded as an analogy to the case on \mathbb{R} or \mathbb{C}

Corollary 4.6 (Linearity). *The association $\theta \mapsto \int \theta \pmod{\mathbb{C}_p}$ is a well-defined homomorphism. That is, let σ and ω be two 1-forms in $\Omega_M(U)$, and $a, b \in \mathbb{C}_p$, then we have*

$$\int (a\sigma + b\omega) \sim a \int \sigma + b \int \omega.$$

PROOF. Since $F = \int (a\sigma + b\omega) - a \int \sigma - b \int \omega \in M(U)$ satisfies $dF = 0$, it is a locally constant function. $\phi^*F - bF \in M(V)$, applying property **(D)** for r sufficiently close to 0 so that $U \supseteq U_r(X)$, we know that $\phi^*F - bF = \alpha \in \mathbb{C}_p$. Since $F|_X \in \mathcal{L}_R(X)$, consider $F^1 = F - \frac{\alpha}{1-b}$ on X , which satisfies $\phi^*F^1 - bF^1 = 0$, by Lemma 3.5, $F^1 = 0$, so $F = \frac{\alpha}{1-b}$ on X .

Since $F|_{V_j} \in A_{\log}(V_j)$ while it is locally constant, by Proposition 2.6, $F \in \mathbb{C}_p$ on any annulus containing V_j , with $(1 - b)F = \alpha$, thus $F = \frac{\alpha}{1-b}$ is constant on U . □

Corollary 4.7 (Fundamental Theorem of Calculus). *Let $\theta \in \Omega_M(U)$ be a closed 1-form, say, $\theta = dg$ for some $g \in \mathcal{L}(U_r)$, $0 < r < 1$. Then $\int \theta \sim g$ on U_r .*

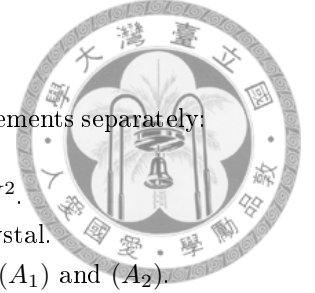
PROOF. Let $F = \int \theta - g$, so $dF = 0$. Choose $0 < r < 1$ sufficiently close to 1 so that on U_r , $\phi^*F - bF \in M(U_r)$. Therefore $\phi^*F - bF = \alpha \in \mathbb{C}_p$ as above. Similar to the prove above, $F|_X = \frac{\alpha}{1-b}$ is constant by Lemma 3.6 and its corollary. On the other hand, $(1 - b)F|_{V_j} = \alpha$, and the result can be extended to any annulus containing V_j by Proposition 2.6. □

Remark. By Corollary 4.7, for any $\theta \in \Omega_M$, one may define $\int_x^y \theta \triangleq f_\theta(y) - f_\theta(x) \in \mathbb{C}_p$, which is well-defined. The elimination of ambiguity may be regarded as an analogy to the definite integral over \mathbb{R} or contour integration for analytic function (which satisfies the *Cauchy theorem*) over \mathbb{C} .

One now define

$$\hat{M}^1 = M(U) + \sum_{\theta} A(C) \int \theta,$$

where the sum is taken over any $\theta \in \Omega_M(U)$ and $\int \theta = f_\theta$ satisfying the hypotheses in Lemma 4.5. $d\hat{M}^1 \supset \Omega_M$ is true from this definition.



PROOF OF THEOREM 4.3. [Col82] We will complete the proof for this theorem via two statements separately:

- (1) \hat{M}^1 is a logarithmic F -crystal over U with $\Omega_M(U) = Mdz \subseteq d\hat{M}^1$, and
- (2) For any logarithmic F -crystal M^2 over U so that $\Omega_M(U) \subseteq dM^2$, one has $\hat{M}^1 \subseteq M^2$.

For (1), we examine each of the properties provided in the definition of a logarithmic F -crystal.

- (A) By construction, any $f = \int \theta$ satisfies property (3) in Lemma 4.5, which is exactly (A₁) and (A₂).
- (B) $d\hat{M}^1 = d(M + \sum_{\theta} A(U) \int \theta) = dM + \sum_{\theta} (\theta A(U) + \int \theta dA(U))$. Observe that $dM \subseteq \Omega_M(U)$, $\theta \in \Omega_M(U)$ means $\theta A(U) \subseteq \Omega_M(U)$, and $\int \theta dA(U) \subseteq \hat{M}^1 dz = \Omega_{\hat{M}^1}$. Since $\Omega_M(U) \subseteq \Omega_{\hat{M}^1}(U)$, (B) holds for \hat{M}^1 .
- (C) Before proving (C), we need the following

Lemma 4.8. *If $f \in \hat{M}^1$, then f is analytic on $B(x, r(x))$ for $x \in U$. (For the definition of $r(x)$, review the second paragraph of this section.)*

PROOF. (a) For $x \in X$, $r(x) = 1$, while $B(x, 1) = R$ is a residue class in X . By construction $f|_X \in \mathcal{L}_R(X)$, thus $f|_R \in A(R)$.

- (b) If for some $j = 1, \dots, d$, $x \in V_j$, so $r(x) = \|x - a_j\|$. Since $f|_{V_j} \in A_{\log}(V_j) = A_{\log}[\log(z - a_j)]$ by Lemma 2.4 and its corollary, while $\log(z)$ is analytic on $B(x, \|x\|)$ by Lemma 2.3, we know that f is analytic for $z - a \in B(x - a, \|x - a\|)$, that is, $z \in B(x, r(x))$. □

Lemma 4.9. *Let $0 < r < 1$ so that $U_r(x) \supseteq U$. For any $f(x, y) \in A(D(U_r(X)))$, there exists a unique $F(x, y) \in A(D(U_r(X)))$ so that*

- (a) $\frac{\partial}{\partial y} F(x, y) = f(x, y)$, and
- (b) $F(x, x) = 0$.

PROOF. Note that $0 < r(x) \leq 1$, so $B(x, r(x))$ is conformal to $B(x, 1) \simeq M(T_1(k))$, which leads to $H^1(B(x, r(x))) = 0$. For any $x \in U_r(X)$, $p_1^{-1}(x) = \{(x, y) \in U \times U \mid \|y - x\| < r(x)\} \simeq \{y \in U \mid \|y - x\| < r(x)\} = B(x, r(x))$ and for any $x \in U_r(X)$, $f(x, y) \in A(B(x, r(x)))$ when regarded as a single-variable function in y . Thus there exists a unique $F_x(y) \in A(B(x, r(x)))$ so that $\frac{\partial}{\partial y} F_x(y) = f(x, y)$ and $F_x(x) = 0$ on $B(x, r(x))$. Suppose there exists a function $g(x, y)$ satisfying the statement of the lemma, then $g(x, y) = F_x(y)$ on $B(x, r(x))$ since $H^0(B(x, r(x))) = \mathbb{C}_p$, proving the existence and uniqueness.

Define $F(x, y) = F_x(y)$. For any affinoid Y contained in $U_r(X)$, let $h(x, t) = f(x, y)|_Y \in A(Y)$ by the change of variable $t = y - x$. Since $(x, x) \in D(U_r(X))$ for any $x \in U$, $h = \sum_{j=0}^{\infty} a_j(x)(y - x)^j$, where $a_j(x)$ lies in the completion of $\mathbb{C}_p(X)$. While $\frac{\partial}{\partial y} F(x, y) = h(x, y - x)$ on $p_1^{-1}(x)$ and $F(x, x) = 0$, by the identity principle on $A(B(x, r(x)))$ and $A(Y)$, we have $F(x, y) = \sum_{j=0}^{\infty} \frac{a_j(x)}{j+1} (y - x)^{j+1} \in A(Y)$. □

Lemma 4.10. *Let $\theta \in \Omega_M(U)$ and $f_{\theta} = \int \theta$, then*

$$p_1^* f_{\theta} - p_2^* f_{\theta} \in p_1^* M.$$

PROOF. Set $F(x, y) = (p_1^* f_{\theta})(x, y) - (p_2^* f_{\theta})(x, y) = f_{\theta}(x) - f_{\theta}(y)$. By Lemma 4.8, for each $x \in U$, $F(x, y)$ is analytic on $p_1^{-1}(x)$. $F(x, y)$ satisfies

- (a) $\frac{\partial}{\partial y} F(x, y) dy = p_2^*(\theta)$, and
- (b) $F(x, x) = 0$ for any $x \in U$.

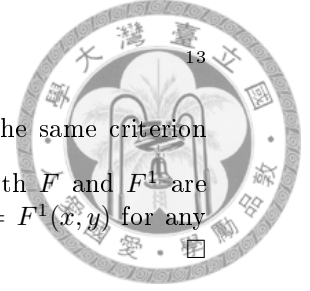
$\theta \in \Omega_M(U) = M \otimes_{A(U)} A(U) dz$ and so $p_2^* \theta \in p_2^* M \otimes_{A(D(U))} A(D(U)) dz$. Note that $p_2^* M = p_1^* M$ by

- (C) for M and $p_1^{-1}(U_r(X)) \subseteq p_2^{-1}U$ for some $0 < r < 1$. Thus, one may write $p_2^* \theta = \sum_{j=1}^n h_j(x, y) g_j(x)$

for some $h_j \in A(D)$ and $g_j \in M(U_r(X))$ for the $0 < r < 1$ chosen above.

By the previous lemma, for some restriction onto $U_r(X)$ for $0 < r < 1$, one can find $H_j(x, y) \in A(D)$ so that

- (a) $\frac{\partial}{\partial y} H_j(x, y) dy = p_2^*(\theta)$, and
- (b) $H_j(x, x) = 0$ for all $j = 1, \dots, n$.



If we let $F^1(x, y) = \sum_{j=1}^n H_j(x, y)g_j(x)$, then by direct computation F^1 satisfies the same criterion as F does and $F^1 \in p_1^*M$. For any $x \in U$, $p_1^{-1}(x) = \{x\} \times B(x, r(x))$, and both F and F^1 are analytic on $p_1^{-1}x$ as a function of y . By the criterion satisfies by both, $F(x, y) = F^1(x, y)$ for any $(x, y) \in \{x\} \times p_1^{-1}x$, and so on D . Thus $F \in p_1^*M$ as desired.

Since $p_1^*M = p_2^*M$, one immediately has $p_1^*f_\theta - p_2^*f_\theta \in p_2^*M$ as well. Therefore, for any $\theta \in \Omega_M(U)$, $p_2^* \int \theta \in p_1^*\hat{M}^1$ and $p_1^* \int \theta \in p_2^*\hat{M}^1$. This proves that \hat{M}^1 satisfies **(C)** since $\hat{M}^1 = M + \sum_{\theta} \int \theta \cdot A(U)$.

(D) First, note that for any affinoid X , it is known that $A(X)$ satisfies the *nullstellensatz*: for any ideal $I \subseteq A(X)$, $\{x \in X \mid f(x) = 0 \text{ for any } f \in I\} = \emptyset$ if and only if $I = A(X)$.

For any rigid space U so that $A(U)$ satisfies the identity principle, for any $x \in U$, $f \in A(U)$ non-zero, we have $(\frac{d}{dz})^m f(x) \neq 0$ for some $m \in \mathbb{N} \cup \{0\}$ (consider the local expression on some affinoid neighborhood $x \in X \subset U$ about x). Thus, the for any non-zero $f \in A(U)$, we know that

$$\sum_{m=0}^{\infty} ((\frac{d}{dz})^m f \cdot A(U)) = A(U).$$

For sake of a neat notation, we will use f_θ and $\int \theta$ interchangeably.

Now, **(D)** follows from

Proposition 4.11. *Let $0 < r < 1$ so that $U_r(X) \subset U$, and $\theta_1, \dots, \theta_n$ be in $\Omega_M(U)$ so that the congruent classes $[\theta_1], \dots, [\theta_n]$ defined in $H^1(M(U))$ are linearly independent. Suppose*

$$(4.1) \quad F(z) = f(z) + \sum_{j=1}^n h_j(z) \cdot \int \theta_j \in \hat{M}^1(U_r(X))$$

with $h_j \in A(U_r(X))$ and $f \in M(U_r(X))$. If F satisfies $F = 0$ on $V \subseteq U_r(X)$ for some non-empty open V . Then $h_j(z) = 0$ on $U_r(X)$.

PROOF. Suppose not; let $n \in \mathbb{N}$ be minimal so that for some $F \in \hat{M}^1$ as above, $F = 0$ while $h_j(z) \neq 0$. $dF = fdz + \sum_{j=1}^n (dh_j f_{\theta_j} + h_j \theta_j) = 0$. Define f^1 via $f^1 dz = fdz + \sum_{j=1}^n h_j \theta_j$, since

$\theta \in \Omega_M(U_r(X))$, $f^1 \in M(U_r(X))$. We have $F' = f^1 + \sum_{j=1}^n f_{\theta_j} h'_j$, so

$$h'_n F - h_n F' = (h'_n f - h_n f^1) + \sum_{j=1}^{n-1} f_{\theta_j} (h_n h'_j - h'_n h_j) = 0 \quad \text{on } U_r(X).$$

Notice that $h'_j \in M$ by **(B)**, and this function, lying in \hat{M}^1 , fit in the form in the statement of this proposition. By the minimality of n , we know that $h_n h'_j - h'_n h_j = 0$ for $j = 1, \dots, n-1$ (and, of course, for $j = n$). Thus, $d(\frac{h_j}{h_n}) = 0$, saying that $h_j/h_n \in A(U_r(X))$ (following an argument similar to that used in proving Lemma 2.6) and so $h_j = b_j h_n$ for some $b_j \in \mathbb{C}_p$. Thus, $h_j \int \theta_j = b_j h_n \int \theta_j = h_n \int (b_j \theta_j)$ by Corollary 4.6. Rewrite (4.1) as

$$F = f + h_n \int (\sum_{j=1}^n b_j \theta_j).$$

Let $\theta = \sum_{j=1}^n b_j \theta_j$, so $F = f + h_n \int \theta = 0$ on V . Differentiate to yield $F' dz = f' dz + h_n \theta + f_\theta \frac{d}{dz} h_n dz$. Define recursively $f^0 = f$, and $f^m dz = f' dz + (\frac{d}{dz})^{m-1} h_n \theta$. Then inductively we have

$$(\frac{d}{dz})^m F = f^m + (\frac{d}{dz})^m h_n f_\theta = 0 \quad \text{on } V.$$

By *nullstellensatz*, $g + f_\theta = 0$ on V for some $g \in M(U_r(X))$. By **(B)** and **(D)** we have $\theta = -dg$ on $U_r(X)$, so $[\theta] = 0$ in $H^1(MU_r(X))$. By **(E)** for M , the independence of $\theta_1, \dots, \theta_n \pmod{dM}$ is violated in $H^1(M)$, a contradiction. \square



Lemma 4.12. For any $1 > r > 0$, with $U_r(X) \subseteq U$, $F \in \hat{M}^1(U_r(X))$, one may always write F in form of

$$F = f(z) + \sum_{j=1}^n h_j(z) \int \theta_j,$$

with $h_j \in A(U_r(X))$, $f \in M(U_r(X))$, and $\{[\theta_j]\}_{j=1}^n$ being independent in $H^1(M(U_r(X)))$.

PROOF. It suffices to prove that for any finite sum $\sum_{j=1}^m h_j f_{\theta_j}$ as in the latter term, one has $\sum_{j=1}^m h_j f_{\theta_j} = f + \sum_{k=1}^n h_{j_k} f_{\theta_{j_k}}$, where $f \in M(U_r(X))$, $m \leq n$, and $\{[\theta_{j_k}]\}_{k=1}^n$ is independent in $H^1(M(U_r(X)))$. We proceed by induction:

For $m = 1$, if $d\theta_1 \neq 0$ we are finished. Otherwise, then $f_{\theta_1} \in \mathbb{C}_p$ and so $h_1 f_{\theta_1} \in A(U_r(X)) \subset M(U_r(X))$ may be joined into the first term, which leads to $n = 0$, a trivial case.

For $m > 1$, suppose $\sum_{k=1}^s b_k \theta_{j_k} \in dM$ for some $m \geq s$, $b_k \in \mathbb{C}_p^t$ times, say $\sum_{k=1}^s b_k \theta_{j_k} = df^1$. Since $\sum_{k=1}^s b_k \int \theta_{j_k} \sim f^1$ by Corollaries 4.6 and 4.7, and f^1 may be joined into the first term, we may simply assume that $\sum_{k=1}^s b_k \theta_{j_k} = 0$. However, then, $\sum_{k=1}^s b_k \cdot \int \theta_{j_k} = b \in \mathbb{C}_p$. Consider

$$\begin{aligned} \sum_{j=1}^m h_j f_{\theta_j} &= \frac{h_{j_1}}{b_1} \sum_{k=1}^s b_k \int \theta_{j_k} - \frac{h_{j_1}}{b_1} \sum_{k=1}^s b_k \int \theta_{j_k} + \sum_{j=1}^m h_j f_{\theta_j} \\ &= f + \sum_{\substack{j=1 \\ j \neq j_1}}^m g_j f_{\theta_j} \end{aligned}$$

with $f = \frac{h_{j_1}}{b_1} \sum_{k=1}^s b_k \int \theta_{j_k}$ and an proper choice of $g_j \in A(U_r(X))$. By induction hypothesis the proof is done. \square

Remark. Following the notation in the above lemma, note that when $\sum_{k=1}^s b_k \theta_{j_k} \in dM$, $b_k \in \mathbb{C}_p^\times$, one may rewrite the expression $\sum_{j=1}^m h_j f_{\theta_j}$ as

$$\sum_{j=1}^m h_j f_{\theta_j} = f_{(k)} + \sum_{\substack{j=1 \\ j \neq j_k}}^m g_j f_{\theta_j}$$

for any of $k = 1, \dots, s$, with $f_{(k)}$ depending on it. In short, one may choose to remove (in the above sense) any term from the expression that occurs in the dependence relation.

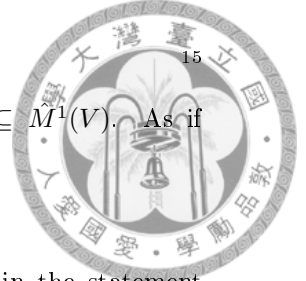
For any $F \in \hat{M}^1(U_r(X))$, $f = 0$ on V , write F in form of the lemma above. By the previous proposition, $m = 0$ and so $F = f \in M(U_r(X))$. By **(D)** for M , the same property is also satisfied by \hat{M}^1 .

(E) Starting with

Proposition 4.13. Let $V = U_r(X) \subset U$, the map

$$\begin{aligned} H^1(M(U)) \otimes H^1(A(V)) &\xrightarrow{\alpha} H^1(\hat{M}^1(V)) \\ \alpha([\theta], [\nu]) &= \left[\int \theta \cdot \nu \right] \end{aligned}$$

is a well-defined isomorphism.



PROOF. First, if $\theta \in dM$, $f_\theta \in M$ and so $[f_\theta \nu] = 0$ since $A(V) \subseteq M(V) \subseteq \hat{M}^1(V)$. As if $\nu = dg \in dA(V) \subseteq d\hat{M}^1(V)$, $\int \theta \in \hat{M}^1$ and

$$\int \theta \cdot \nu = \int \theta \cdot dg = d\left(\int \theta g\right) - (d \int \theta)g \in d\hat{M}^1 = d\left(\int \theta g\right) - \theta g \in d\hat{M}^1.$$

By Corollary 4.6, $\theta \mapsto \int \theta \pmod{\mathbb{C}_p}$ is a homomorphism. To prove the map in the statement unambiguous, it remains that for $b \in \mathbb{C}_p$, $b\nu \in \Omega_M(V) \subseteq d\hat{M}^1$.

α is an epimorphism since it is true on the form level. It suffices to prove that α is a monomorphism, that is, $\ker \alpha = 0$. Suppose now $\sum_{j=1}^n \theta_j \otimes \nu_j \in \ker \alpha$, that is, $\sum_{j=1}^n \int \theta_j \cdot \nu_j$. One may assume $[\theta_j] \in H^1(M)$ being independent: $\sum_{j=1}^n b_j \theta_j \in dM$ implies $\sum_{j=1}^n b_j f_{\theta_j} \in M$, so $\frac{1}{b_n} \sum_{j=1}^n b_j f_{\theta_j} \nu_n \in dM \subseteq d\hat{M}^1$, iterative subtraction with the original term guarantees independence. Let $\sum_{j=1}^n \int \theta_j \cdot \nu_j = dF$ for some $F \in \hat{M}^1$.

By Lemma 4.12, one may write $F = f + \sum_{j=1}^m h_j f_{\theta_j}$ with $f \in M$ and $h_j \in A(V)$. Notice that we did not require $\{[\theta_j]\}_{j=1}^m$ to be independent (see the remark following Lemma 4.12). So

$$\sum_{j=1}^n \int f_{\theta_j} \cdot \nu_j = dF = df + \sum_{j=1}^m (f_{\theta_j} h_j + h_j \theta_j),$$

and $\sum_{j=1}^m \int f_{\theta_j} (\nu_j - dh_j) = df + \sum_{j=1}^m \int h_j \theta_j \in \Omega_M = M \otimes_A \text{Adz}$ (for $j > n$, define $\nu_j = 0$). Choose the primitive in the way that, say, $\sum_{j=1}^m \int f_{\theta_j} (\nu_j - dh_j) = dg$ with $0 = g - \sum_{j=1}^m \int f_{\theta_j} (\nu_j - dh_j)$. By Proposition 4.11, we know that $\nu_j - dh_j = 0$, that is, $[\nu_j] = 0$ in $H^1(V)$, then $\sum_{j=1}^n [\theta_j] \otimes [\nu_j] = 0$. \square

Lemma 4.14. For $0 < r < 1$ so that $U_r(X) = V \subseteq U$, we have $H^1(A(U)) \xrightarrow{\sim} H^1(U_r(X))$ an isomorphism induced by ι^* .

PROOF. The map is well-defined homomorphism since restriction will not affect an 1-form being exact. Suppose $\theta|_V \in dA(V)$, then $(f\theta)|_V \in A(V)$, while by construction, $\int \theta \in \hat{M}^1$ satisfies $(f\theta)|_X \in \mathcal{L}_R(X)$ and $(f\theta)|_{V_j} \in A_{\log}(V_j)$. Since the $A_{\log}(V_j)$ possess the identity principle, and the local expressions on V_j would not change after passing to a annuli subset, one has $\int \theta \in A(U)$. Therefore $\theta = d \int \theta$, and $[\theta] = 0$ in $H^1(U)$. \square

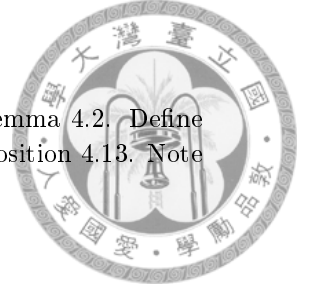
Thus, **(E)** follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(M) \times H^1(U) & \xrightarrow{\sim} & H^1(\hat{M}^1(U)) \\ \wr \downarrow & \circlearrowleft & \downarrow \wr \\ H^1(M) \times H^1(V) & \xrightarrow{\sim} & H^1(\hat{M}^1(V)) \end{array}$$

(F) (F_1) Let (V, ϕ) be a Frobenius neighborhood of X in U so that $M(U)$ satisfies (F_1) . By construction of \hat{M}^1 , for any $F \in \hat{M}^1(U)$, F may be written as $F = f + \sum_{j=1}^m h_j f_{\theta_j}$. Then

$$\phi^* F = \phi^* f + \sum_{j=1}^m (\phi^* h_j)(\phi^* f_{\theta_j}).$$

Note that $\phi^* f \in M(V)$ by (F_1) for M , $\phi^* h_j \in A(V)$ since a Frobenius morphism is rigid, and $\phi^* \int \theta_j - b_j \int \theta_j \in M(V) \subseteq \hat{M}^1$ from Lemma 4.5, so $\phi^* \int \theta_j \in \hat{M}^1$. Therefore \hat{M}^1 satisfies (F_1) .



(F₂) Let (V, ϕ) be a Frobenius neighborhood of X that satisfies the criterion in Lemma 4.2. Define $\alpha : H^1(M) \otimes H^1(A(V)) \rightarrow H^1(\hat{M}^1(V))$ to be the bilinear map defined in Proposition 4.13. Note that for $\theta \in \Omega_M$ and $\nu \in \Omega$,

$$\phi^* \alpha(\theta, \nu) = (\phi^* f_\theta) \phi^* \nu = f_{\phi^* \theta} \phi^* \nu$$

leads to

$$\begin{aligned} \phi^* \alpha(\theta, \nu) - qb \cdot \alpha(\theta, \nu) &= f_{\phi^* \theta} \phi^* \nu - qb \cdot f_\theta \nu \\ &= f_{\phi^* \theta} \phi^* \nu - b f_\theta \phi^* \nu + b f_\theta \phi^* \nu - qb \cdot f_\theta \nu \\ &= \alpha(\phi^* \theta - b\theta, \phi^* \nu) + b\alpha(\theta, \phi^* \nu - q\nu). \end{aligned}$$

As $\phi^* \theta - b\theta \in dM(V)$ by (F₂) for M , and $\phi^* \nu - q\nu \in dA(V)$ by Lemma 3.6 and identity principle for $A(V)$, $(\phi^* \theta - qb\theta) \otimes (\phi^* \nu - qb\nu) = \phi^* \theta \otimes \nu - qb \cdot \theta \otimes \nu \in \ker \alpha$. Since α is surjective and bilinear, \hat{M}^1 satisfies (F₂).

As for (2), for any M^2 so that $dM^2 \supseteq \Omega_M$, $f_\theta \in M^2$ and $dM \subseteq \Omega_M \subseteq dM^2$ by (B), so $\hat{M}^1 \subseteq M^2$ following 4.7. As a $A(U)$ -module, \hat{M}^1 is minimal, and we write $\hat{M}^1 = M^1$ afterwards. □

5. INTEGRATION THEORY FOR BASIC WIDE OPEN SETS

5.1. $A(U)$ as a logarithmic F -crystal.

In this section, we will prove that for a basic wide open set U , $A(U)$ is a logarithmic F -crystal and apply the conclusions from the last section to develop an integration theory on it. This process is essential because otherwise the discussion above would seem rather idiosyncratic.

we will let $U = \mathbb{P}^1 \setminus \{a_1, \dots, a_d\}$, $X = B[0, 1] \setminus \bigcup_{j=1}^d B(a_j, 1)$, and $U_r(X)$ defined as before.

Lemma 5.1. *Let $U \supset X$ be a basic wide open set. Then $A(U)$ satisfies criterion (F) above.*

PROOF. (F₁) holds for any rigid analytic morphism. Choose $q = p^n$ so that $z \mapsto z^q$ is a Frobenius morphism on X and satisfies the statements for Lemma 3.3. Then for each $\theta \in \Omega(U)$, since

$$\theta|_{V_j} - (\text{res}_{a_j} \theta) \frac{dz}{z} \in dA(V_j),$$

by condition (2) of Lemma 3.3, θ satisfies (F₂) on V_j for each $j = 1, \dots, d$. While $\theta|_X$ automatically satisfies so since Frobenius morphisms are rigid. □

Theorem 5.2. *Let $U \supset X$ be as above. Then $A(U)$ is a logarithmic F -crystal.*

PROOF. Note that X and V_j are affinoid subsets of U .

- (A) By definition of $A(U)$ for arbitrary open sets.
- (B) Consider the local expression on each subaffinoid, say, a residue class $R \subset X$. Since $H^1(R) = H^1(B(0, 1)) = 0$, (B) follows from (D).
- (C) It follows from the identity principle for $A(D)$: any $F(x, y) = p_1^* f(x, y) = f(x)$ is constant for $x \in U_r(X) \subset U$ on $p_1^{-1}(x) \simeq B(0, 1)$, which is a subaffinoid of D , thus it is a constant and therefore lies in $p_2^* A(U)$.
- (D) Uniqueness principle for analytic functions are satisfied on wide open sets.
- (E) See Lemma 4.14.
- (F) See Lemma 5.1.

□

We will denote $A^0(U) = A(U)$, define recursively $A^n(U) = (A(U))^1$, and let $\Omega^n = A^n \otimes \Omega$.

Lemma 5.3. *For any logarithmic F -crystal M ,*

$$\dim_{C_p} H^1(M^1) = d \cdot \dim_{C_p} H^1(M),$$

where d is the number of connected components in $A^1 \setminus X$.

PROOF. From Lemma 4.13, it suffices to prove that $\dim_{C_p} A(U) = d$. The latter follows from Lemma 2.7. □



Thus, inductively, $\dim_{\mathbb{C}_p} H^1(A^n(U)) = d^n$.

Corollary 5.4. *Let X be a standard subset of $B[0, 1]$ of the form $X = B[0, 1] \setminus \bigcup_{j=1}^d B(a_j, 1)$, $a_j \in \mathcal{O}$, each in distinct residue classes. $U \subset X$ a wide open neighborhood about as above. Then,*

$$A^1(U) = A(U) + \sum_{j=1}^d \log(z - a_j) \cdot A(U).$$

PROOF. It is not difficult to see that $\left\{ \left[\frac{dz}{z-a_j} \right] \right\}_{j=1}^d$ is a independent subset in $H^1(U)$. Compare the dimension of both sides to see that $H^1(U) = \sum_{j=1}^d \frac{dz}{z-a_j} \cdot dA(U)$. Thus

$$A^1(U) \subseteq A(U) + \sum_{j=1}^d \int \frac{dz}{z-a_j} \cdot A(U),$$

and equality holds since

$$\int \frac{dz}{z-a_j} = \log(z - a_j) \in A^1(U).$$

□

5.2. The structure of $A^n(U)$.

Lemma 5.5. *Let the notation be as above, then $A^n(V_j) = \sum_{k=0}^n (\log(z - a_j))^k \cdot A(V_j)$.*

PROOF. Without loss of generality, let $a_j = 0$ and write $V_j = V$. We proceed by induction, for $n = 1$, for any $\theta \in A^0(V)dz$, follow the prove of Lemma 5.1 and see that $\theta - (\text{res}_0 \theta) d \log(z) \in dA^0(V)$. By the uniqueness principle **(D)** for $A^1(V)$, $\int \theta \in (A(V) + \mathbb{C}_p \log(z))$. Since the conclusion holds for any 1-form θ ,

$$A^1(V) = A^0(V) + \sum_{\theta} \int \theta \cdot A(V) = A(V) + \log(z) \cdot A(V).$$

For $n > 1$, follow the computation of Lemma 2.7 to see that if $\theta \in A^{n-1}(V)$, written by induction hypothesis as $\theta = \sum_{j=0}^{n-1} h_j (\log z)^j$ with $h_j \in A(V)$, we have $\theta - \frac{1}{n} \text{res}_0(h_{n-1} dz) d(\log z)^n \in dA^{n-1}(V)$. Thus,

$$A^n(V) = A^{n-1}(V) + (\log z)^n \cdot A(V) = \sum_{k=0}^n (\log(z))^k \cdot A(V)$$

as desired.

□

Lemma 5.6. *Let $f \in \mathcal{L}(U)$, then $f \in A^n(U)$ if and only if*

- (1) $f|_X \in \mathcal{L}_R(X)$, and $f|_{V_j} = A_{\log}(V_j)$, for $j = 1, \dots, d$,
- (2) $df \in \Omega^n(U)$, and
- (3) there exists an \mathbb{F}_q -Frobenius neighborhood (V, ϕ) of X in U so that

$$\phi^*(f) - q^{n+1} f|_V \in A^n(V).$$

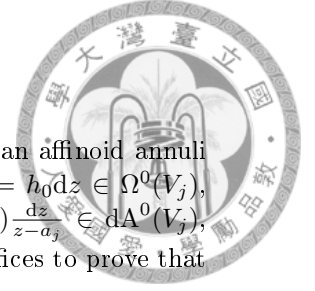
PROOF. (\Rightarrow) (1) $A^n(U)$ is a logarithmic F -crystal, thus satisfies **(A)**.

(2) It follows from condition **(B)**.

(3) Since by choosing any wide open neighborhood V with (V, ϕ) , $\phi : z \mapsto z^q$ satisfies (F_1) , both $\phi^*(f)$ and $q^{n+1} f|_V$ lies in $A^n(V)$.

(\Leftarrow) Note that (1) is necessary by definition. By (3), one may find g in $A^n(U)$ with $\phi^* f - q^{n+1} f = g$. Let b and a Frobenius neighborhood (V^1, ϕ^1) satisfies the conclusion of (F_2) for $A^n(U)$, then by (2) we can find $g^1 \in A^{n+1}(V)$ so that $df = dg^1$ and $\phi^* f - bf = g^1$. The problem here is whether we may choose the b in condition **(F)** for $A^n(U)$ as $b = q^{n+1}$, and $(V, \phi) = (V^1, \phi^1)$. If so, then $g^1 = g \in A^n(V)$ and therefore $df = dg^1$ implies $f \in A^n(V)$.

Let (V, ϕ) be an \mathbb{F}_q -Frobenius neighborhood that satisfies the conclusion of Lemma 3.3. We proceed



by induction on $n \in \mathbb{N}$, by **(D)** it suffices consider the expression on each of V_j :

For $n = 0$, $A(U)|_{V_j} = A(V_j) \simeq \mathbb{C}_p \langle z, y \rangle / (zy - 1)$ since the local expression on an affinoid annuli does not alter after restriction onto a subaffinoid annuli. Then for any $\theta = hdz = h_0dz \in \Omega^0(V_j)$, h may be written as a convergent Laurent series on V_j about a_j . $\theta - \text{res}_{a_j}(h_0dz) \frac{dz}{z-a_j} \in dA^0(V_j)$, while $\frac{dz}{z-a_j} = d \log(z - a_j)$. Hence, if we denote the operator $\phi^* - b = \mathcal{F}_b(\phi)$, it suffices to prove that $\mathcal{F}_q(\phi)(\frac{\log(z-a_j)}{z-a_j}dz) = \mathcal{F}_q(\phi)(d \log(z - a_j)) \in dA^0$, which holds by Lemma 3.3,

$$\mathcal{F}_q(\phi)d \log(z - a_j) = d(\mathcal{F}_q(\phi) \log(z - a_j)).$$

For heuristic purpose, we will proceed and discuss one more step in the induction. If $n = 1$, by Lemma 5.5, and $\theta \in \Omega^1(U)$ of the form $\theta = (h_0(z) + h_1(z) \log(z - a_j))dz$, from the discussion in Lemma 5.5, one know that $\theta - \text{res}_{a_j}(h_1(z)dz) \frac{1}{z-a_j} \log(z - a_j)dz \in dA(U)$. We need to prove that $\mathcal{F}_q(\phi)(\frac{\log(z-a_j)}{z-a_j}dz) = \mathcal{F}_q(\phi)(\frac{1}{2}d(\log(z - a_j))^2) \in dA^1$, which holds by Lemma 3.3,

$$\begin{aligned} \mathcal{F}_{q^2}(\phi)d(\log(z - a_j))^2 &= d(\mathcal{F}_{q^2}(\phi)(\log(z - a_j))^2), \text{ and} \\ \mathcal{F}_{q^2}(\phi) \log(z - a_j) &= (\phi^* \log(z - a_j) + q \log(z - a_j))\mathcal{F}_q(\phi \log(z - a_j)). \end{aligned}$$

where $\phi \log(z - a_j) = (\phi \log(z - a_j) - q \log(z - a_j)) + q \log(z - a_j) \in A^1(V \cap V_j)$, and the statement holds by induction hypothesis.

Suppose $n > 1$, again by Lemma 5.5, write $\theta = fdz \in \Omega^n(V)$ and $f(z) = f^1(z) + h_n(z)(\log z - a_j)^n$, with $f^1(z) = \sum_{j=0}^{n-1} h_j(z)(\log z - a_j)^j$ satisfies (3) for q^{n+1} . Since $\frac{(\log(z-a_j))^n}{z-a_j}dz \in dA^n(U)$, it is sufficient to prove that $\mathcal{F}_{q^{n+1}}(\phi)(\log(z - a_j))^n \in A^n$. Similar to the above case,

$$\mathcal{F}_{q^n}(\phi)(\log(z - a_j))^n = \sum_{j=0}^n (\phi^*(\log(z - a_j))^j q^{n-j} (\log(z - a_j))^{n-j}) \mathcal{F}_q(\phi \log(z - a_j)).$$

Since $\phi^*(\log(z-a_j))^{n-j} = (\phi^*(\log(z-a_j))^{n-j} - q^{n-j}(\log(z-a_j))^{n-j}) + q^{n-j}(\log(z-a_j))^{n-j} \in A^{n-j}(V \cap V_j)$ by induction hypothesis and Lemma 5.5, we have reached the conclusion that $\mathcal{F}_{q^{n+1}}(\phi)(\log(z - a_j))^n \in A^n$. □

Notice that $A^n(U)$ for $n > 0$ is in general not a ring; however, with the previous lemma,

Corollary 5.7. *The direct limit $\bar{A}(U) = \varinjlim_n A^n(U)$ of $A(U)$ -modules forms a ring, equipped with multiplication defined when regarded as a subset in $\mathcal{L}(U)$.*

PROOF. It follows from a simple observation: for $f \in A^n(U)$, $g \in A^m(U)$, $f \cdot g$ satisfies

$$\phi^*(f \cdot g) - q^{(n+1)(m+1)}(f \cdot g) = \phi^*f(\phi^*g - q^{m+1}g) + (\phi^*f - q^{n+1}f)q^{m+1}g.$$

So by the converse of the previous lemma, $f \cdot g \in \bar{A}(U)$. □

6. THE POLYLOGARITHMS

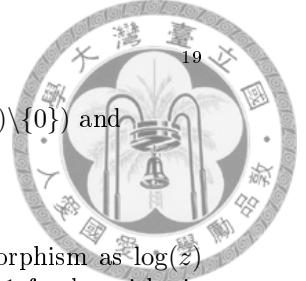
6.1. Definition and functional equations of the polylogarithms.

From now on, we will choose a specific affinoid subspace of \mathbb{P}^1 to work on. Let $U = \mathbb{P}^1 \setminus \{0, 1, \infty\} = A^1 \setminus \{0, 1\}$, and $X = B[0, 1] \setminus (B(0, 1) \cup B(1, 1)) = M(\frac{k(z, y_1, y_2)}{(zy_1 - 1, (z-1)y_2 - 1)})$.

Definition 6.1. Define the *polylogarithm functions* $l_k(z)$ recursively by:

- (1) $l_0(z) = \frac{z}{1-z} \in A(U)$,
- (2) $dl_k(z) = l_{k-1}(z) \frac{dz}{z}$, and
- (3) $\lim_{z \rightarrow 0} l_k(z) = 0$,

so that $l_k \in A^k(U)$ for $k \in \mathbb{N} \cup \{0\}$. l_k exists since U is a wide open set, and $A^{k-1}(U)$ are logarithmic F -crystals for $k \in \mathbb{N}$. Note that $l_0(z)$ is analytic for z near 0 and $l_0(0) = 0$. Precisely speaking, $l_0(z) = -\sum_{n=1}^{\infty} z^n$ on



$B(0, 1) \setminus \{0\}$, which is a wide open annulus. Thus by construction $l_k|_{B(0,1)\setminus\{0\}} \in A_{\log}(B(0, 1) \setminus \{0\})$ and

$$(6.1) \quad l_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

on $B(0, 1) \setminus \{0\}$ (compare (6.1) when $k = 1$ to (2.1), be aware that $l_1(z)$ is *not* a homomorphism as $\log(z)$ is), by Proposition 2.6, $l_k \in A(B(0, 1) \setminus \{0\})$. By the identity principal **(D)** for $0 < r < 1$ for logarithmic F -crystals, l_k extends to a locally analytic function on $\mathbb{P}^1 \setminus \{1, \infty\}$ (note that $l_k(0) = 0$ extends l_k as a locally analytic function to $z = 0$). Note that we also define l_k for $k < 0$; since $z \in A(U)^\times$, $l_k \in A(U) = A^0(U)$ for $k < 0$.

Proposition 6.2. *For $m \in \mathbb{N}$, let ζ be some m^{th} -root of unity. We have*

$$\frac{1}{m} \sum_{j=0}^{m-1} l_k(\zeta^j z) = \frac{l_k(z^m)}{m^k}.$$

PROOF. Note first that $\sum_{j=0}^{m-1} (\zeta^j)^n = \begin{cases} m & \text{if } n \in m\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$ since if otherwise, $\zeta^n \sum_{j=0}^{m-1} (\zeta^j)^n = \sum_{j=1}^m (\zeta^j)^n = \sum_{j=0}^{m-1} (\zeta^j)^n$. Due to the identity principle, one only has to consider the validity of this functional equation near 0, where $l_k = \sum_{n=1}^{\infty} z^n/n^k$. So

$$\frac{1}{m} \sum_{j=0}^{m-1} l_k(\zeta^j z) = \frac{1}{m} \sum_{n=1}^{\infty} \frac{z^n}{n^k} \sum_{j=0}^{m-1} (\zeta^j)^n = \sum_{d=1}^{\infty} \frac{z^{md}}{(md)^k} = \frac{l_k(z^m)}{m^k}.$$

□

For a simple generalization for (2), if $n \in \mathbb{Z}$, notice that $\frac{d}{dz} l_k(z^n) = \frac{l_{k-1}(z^n)}{z^n} \cdot n z^{n-1} = n \frac{l_{k-1}}{z}$, an identity which we will use repeatedly.

Observe that for X chosen in this section, since $\tilde{0} = 0$ and $\tilde{1} = 1$ both lies in $\mathbb{F}_p \subset \mathbb{F}$, for $s = 0, 1$, any point $a_s \in B(s, 1)$ satisfies $\|a_s^p - a_s\| < 1$. Hence $\phi : z \mapsto z^p$ is a Frobenius morphism of X , and any wide open neighborhood V about X is a Frobenius neighborhood of X in U ; furthermore, observe that any $a \in \{0, 1, \infty\}$, $\phi(a) = a$.

Define the *twisted polylogarithmic functions* by $l_{k,p} = l_k(z) - p^{-k} l_k(z^p)$,

Lemma 6.3. $l_{k,p} \in A^{k-1}(U_r(X))$ for $0 < r < 1$ sufficiently close to 1 so that $U_r(X) = V$ does not contain any primitive p^{th} -root of unity, say, $r = \|p\|^{\frac{1}{p-1}}$.

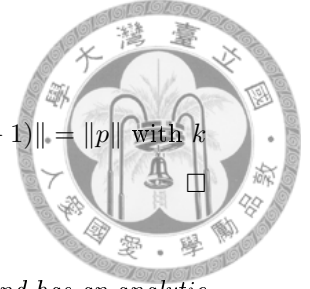
PROOF. Note that $z \in A(U)^\times$, and for $k > 1$, $\frac{d}{dz} l_k(z^p) = \frac{l_{k-1}(z^p)}{z^p} \cdot p z^{p-1} = p \frac{l_{k-1}(z^p)}{z}$. Thus,

$$z \frac{d}{dz} l_{k,p}(z) = l_{k-1}(z) - p^{-(k-1)} l_{k-1}(z^p).$$

So by construction, it suffices to prove that $l_1(z) - p^{-1} l_1(z^p) \in A(U_r(X))$. Since with $\phi : z \mapsto z^p$, (V, ϕ) is a Frobenius morphism of X , one may rewrite $l_1(z) - p^{-1} l_1(z^p) = -\frac{1}{p} ((\phi^* l_1)(z) - p \cdot l_1(z))$, and $d((\phi^* l_1)(z) - p \cdot l_1(z)) = (\frac{pz^{p-1}}{z^{p-1}} - \frac{p}{z-1}) dz$. On any residue class $R \simeq B(0, 1)$ of X , $(\frac{pz^{p-1}}{z^{p-1}} - \frac{p}{z-1}) \in A(R)$ and so $(\frac{pz^{p-1}}{z^{p-1}} - \frac{p}{z-1}) dz \in dA(R)$ since $H^1(B(0, 1)) = 0$. We then have $F \in \mathcal{L}_R(X)$ so that $F = \frac{1}{1-z^p} - \frac{p}{1-z}$ on each of the residue classes $R \subset X$. Since $\frac{1}{1-z^p} - \frac{p}{1-z} \in A(X)$, $F \in A(X)$. Thus $d((\phi^* l_1)(z) - p \cdot l_1(z)) \in dA(X)$ and $(l_1(z) - p^{-1} l_1(z^p))|_X \in A(X)$ follows.

For any $0 < r < 1$ so that $U_r(X) \cap \{z \in \mathbb{A}^1 \mid z^p = 1\} = \emptyset$, for either $s = 0$ or 1 , $(\frac{pz^{p-1}}{z^{p-1}} - \frac{p}{z-1}) dz \in \Omega(V_s)$ with $\text{res}_s(\frac{pz^{p-1}}{z^{p-1}} - \frac{p}{z-1}) = 0$, thus $(l_1(z) - p^{-1} l_1(z^p))|_{V_s} \in A(V_s)$. The facts above indicates that $(l_1(z) - p^{-1} l_1(z^p)) \in A(V)$ by **(D)** for $A^1(V)$.

At last, we prove that $r = \|p\|^{\frac{1}{p-1}}$ guarantees that $U_r(X) = V$ does not contain any primitive p^{th} -root of unity: $1 \notin V$ is direct, and if $z^p = 1$, $\tilde{z}^p - 1 = (\tilde{z} - 1)^p = 0$ implies $z \in B(1, 1)$. Now, $z^p - 1 = (z - 1) \sum_{j=0}^{p-1} z^j = 0$



and suppose $z \neq 1$. Write $z = 1 + \alpha$, then $\sum_{j=0}^{p-1} ((1 + \alpha)^j - 1) = p$ and $\|\alpha\| \leq \left\| \sum_{j=0}^{p-1} ((1 + \alpha)^j - 1) \right\| = \|p\|$ with k being non-archimedean. □

The lemma has a strengthened version:

Proposition 6.4. *Let $B = B \left[1, \|p\|^{\frac{1}{p-1}} \right]$. The twisted k -logarithm $l_{k,p}$ is analytic on $U \setminus B$ and has an analytic continuation onto $\mathbb{P}^1 \setminus B$ so that $l_{k,p}(0) = l_{k,p}(\infty) = 0$.*

PROOF. Note that $l_{k,p}(0) = 0$ by definition since $l_k(0) = 0$.

We will prove the proposition by induction on $k \in \mathbb{N} \cup \{0\}$ that $l_{k,p}$ can be extended with analyticity on $U \setminus B$.

For $k = 1$, $l_{0,p} = \frac{z}{1-z} - \frac{z^p}{1-z^p}$ is analytic on $\mathbb{P}^1 \setminus B$ since the poles of both terms are excluded, and $l_{0,p}(\infty) = 0$. By induction hypothesis, suppose $l_{k,p} \in A(U \setminus B)$. By Proposition 6.2,

$$\frac{1}{m} \sum_{j=0}^{m-1} l_{k,p}(\zeta^j z) = \frac{1}{m^k} l_k(z^m) - \frac{1}{m^k} p^{-k} l_k(z^{mp}) = \frac{1}{m^k} l_{k,p}(z^m).$$

Thus $l_{k,p}(\infty) = \frac{1}{m} \cdot m \cdot l_{k,p}(\infty) = m^{-k} l_{k,p}(\infty)$, hence $l_{k,p}(\infty) = 0$ when $k > 0$. $\frac{l_{k,p}}{z}$, then, is analytic on $\mathbb{P}^1 \setminus B$. In particular, $\theta_k \triangleq l_{k,p} \frac{dz}{z} \in \Omega(\mathbb{P}^1 \setminus B)$ and since $\mathbb{P}^1 \setminus B$ is a wide open disc $H^1(\mathbb{P}^1 \setminus B) = 0$. There exists an $F \in A(\mathbb{P}^1 \setminus B)$ so that $dF = \theta_k$. Then by the identity principle for $A(\mathbb{P}^1 \setminus B)$, $l_{k,p} \sim F$ is analytic on $\mathbb{P}^1 \setminus B$. □

6.2. The function $D(z)$ and its related identities.

Definition 6.5. (1) $\log(z)$ denotes a branch of the logarithm. Define $D(z) \in A^2(U)$ by

$$D(z) = l_2(z) + \frac{1}{2} \log(z) \log(1 - z).$$

(2) For any \mathbb{C}_p -values function f defined on $U \setminus \{a\}$, $a \in U$, define

$$\lim_{z \rightarrow a}^1 f(z) = \lim_{\substack{z \rightarrow a \\ z \in K}} f(z),$$

if any limits on the right exist and coincide, where K is an arbitrary finitely ramifies extension of \mathbb{Q}_p that contains a , or $a = \infty$.

Lemma 6.6. *Using the notation defined above, $\lim_{z \rightarrow 0}^1 \log(z) \log(1 - z) = 0$.*

PROOF. We may see that for any finite ramified extension $K \supseteq \mathbb{Q}_p$, $\log(z)$ is bounded near 0. Let $a \in K$, write $a = p^k u$ with $\|u\| = 1$. Since $\log : K^\times \rightarrow \mathbb{C}_p$ is a homomorphism, and $\{k \log p\}_{k \in \mathbb{Z}}$ is bounded, it suffices to prove that $\log|_{C(0,1) \cap K}$ is bounded. Since for any $u \in C(0,1) \cap K$ so that $u \neq 1$, $\|u - 1\| = 1$ and $\|u^q - 1\| < 1$ for some p -power q (we may choose q to be the order of K_0/K_1 , the residue class field). By the local expression of $\log(z)$ near 1, we know that $\{\log(u^q)\}_{u \in C(0,1) \cap K} = \{q \log(u)\}_{u \in C(0,1) \cap K}$ is then bounded. The existence of the restricted limit follows.

Since for z sufficiently close to 0, $\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ holds, we know that $\log(1 - z)/z$ is bounded on a punctured disc about 0. Thus, it suffices to prove that $\lim_{z \rightarrow 0}^1 z \log(z) = 0$. Suppose not, that is, $\lim_{z \rightarrow 0}^1 z \log(z) = a \in \mathbb{C}_p^\times$. $b \triangleq \log(a) \in \mathbb{C}_p$ is therefore defined. Since $\log(z) \in \mathcal{L}$, $b = \lim_{z \rightarrow 0}^1 \log(z \log(z)) = \lim_{z \rightarrow 0}^1 (\log z + \log(\log(z)))$. The last term does not exist in any finitely ramified extension $K \supset \mathbb{Q}_p$ by, say, $z_n = p^{p^n} \rightarrow 0$ because

$\log(z_n) + \log(\log(z_n)) = \log(p^{p^n}) + \log(\log(p^{p^n})) = (p^n \log p + \log p^n) + \log(\log p) = (p^n + n) \log p + \log(\log p)$ does not converge. Hence a contradiction is reached. □

Corollary 6.7. $\lim_{z \rightarrow 0}^1 D(z) = 0$

PROOF. This is a direct result from the lemma above and $l_2(0) = 0$ (the latter holds by construction). □

We adapt the convention that $\frac{1}{k!} = 0$ once $k < 0$ and $0! = 1$.



Proposition 6.8 (Coleman-Sinnott).

$$(6.2) \quad l_k(z) + (-1)^k l_k\left(\frac{1}{z}\right) = -\frac{1}{k!}(\log z)^k,$$

$$(6.3) \quad D(z) = -D\left(\frac{1}{z}\right), \text{ and}$$

$$(6.4) \quad D(z) = -D(1-z).$$

PROOF. [Col82]

(6.2): For $k = 0$,

$$l_0(z) + l_0(z^{-1}) = \frac{z}{1-z} + \frac{z^{-1}}{1-z^{-1}} = -1$$

holds; apply $z \frac{d}{dz}$ iteratively and observe that (6.2) holds for $k < 0$.

For $k \in \mathbb{N}$, we proceed by induction on k . Assume that (6.2) holds for some $k \geq 0$, differentiation yields

$$dz(l_{k+1}(z) + (-1)^{k+1}l_{k+1}\left(\frac{1}{z}\right)) = l_k(z) + (-1)^k l_k\left(\frac{1}{z}\right) \frac{dz}{z} = -\frac{1}{k!}(\log z)^k \frac{dz}{z}.$$

Note that the last term equals to $d\left(-\frac{1}{k!} \frac{(\log z)^{k+1}}{k+1}\right) = d\left(-\frac{1}{(k+1)!}(\log z)^{k+1}\right)$. Thus (6.2) holds for $k + 1$ module \mathbb{C}_p , namely

$$(6.5) \quad l_{k+1}(z) + (-1)^{k+1}l_{k+1}\left(\frac{1}{z}\right) = -\frac{1}{(k+1)!}(\log z)^{k+1} + b$$

for some $b \in \mathbb{C}_p$. Notice that

$$(\log z)^{k+1} = p^{-(k+1)}(\log z^p)^{k+1}.$$

Substituting z by z^p , we have

$$l_{k+1}(z^p) + (-1)^{k+1}l_{k+1}(z^{-p}) = -\frac{1}{(k+1)!}(\log z^p)^{k+1} + b = -\frac{p^{k+1}}{(k+1)!}(\log(z))^k + b.$$

The three equations above assemble together to

$$l_{k+1,p}(z) + (-1)^{k+1}l_{k+1,p}\left(\frac{1}{z}\right) = (1 - p^{k+1})b.$$

By Proposition 6.4, $l_{k+1,p}(0) = l_{k+1,p}(\infty) = 0$, giving $b = 0$ and therefore (6.2) holds for $k + 1$ by (6.5).

(6.3): By (6.2), and the definition of $D(z)$,

$$\begin{aligned} D(z^{-1}) &= l_2(z^{-1}) + \frac{1}{2} \log(z^{-1}) \log(1 - z^{-1}) \\ &= -l_2(z) - \frac{1}{2!}(\log z)^2 + \frac{1}{2}(\log(z^{-1}) \log(1 - z^{-1})) \\ &= -l_2(z) - \frac{\log z}{2}((\log z) + \log(1 - z^{-1})) \\ &= -l_2(z) - \frac{1}{2} \log z \log(z - 1) \\ &= -D(z). \end{aligned}$$

Note that $\log(-1) = \frac{1}{2} \log(1) = 0$.



(6.4): $l_1(z) = -\log(1-z)$, so $dl_2(z) = -\log(1-z)\frac{dz}{z} = -\log(1-z)d\log(z)$ and

$$\begin{aligned} dl_2\left(\frac{z}{z-1}\right) &= -\frac{z-1}{z}\log\left(\frac{1}{1-z}\right)d\left(\frac{z}{z-1}\right) \\ &= -\log\left(\frac{1}{z-1}\right)d(\log z - \log(z-1)) \\ &= \log(z-1)d(\log z - \log(z-1)) \\ &= -dl_2(z) - \log(z-1)d\log(z-1), \\ d\left(\log\left(\frac{z}{z-1}\right)\log\left(1-\frac{z}{z-1}\right)\right) &= -d((\log z - \log(z-1))\log(1-z)) \\ &= -d(\log(z)\log(1-z)) + 2\log(z-1)d\log(z-1). \end{aligned}$$

As a result,

$$\begin{aligned} dD(z) &= dl_z(z) + \frac{1}{2}d(\log(z)\log(1-z)) \\ &= (dl_z(z) + \log(z-1)d\log(z-1)) + \frac{1}{2}(d(\log(z)\log(1-z)) - 2\log(z-1)d\log(z-1)) \\ (6.6) \quad &= -dl_2\left(\frac{z}{z-1}\right) - d\left(\log\left(\frac{z}{z-1}\right)\log\left(1-\frac{z}{z-1}\right)\right) = dD\left(\frac{z}{z-1}\right). \end{aligned}$$

By (6.6), $D(z) + D\left(\frac{z}{z-1}\right)$ is constant. The constant is 0, considering $\lim_{z \rightarrow 0} (D(z) + D\left(\frac{z}{z-1}\right)) = 0$. \square

Corollary 6.9. $\lim_{z \rightarrow a}^1 D(z) = 0$ for $a = 0, 1$, or ∞ .

PROOF. $a = 0$ is proved before, by this result, $a = 1$ follows from (6.4) and $a = \infty$ from (6.3). \square

Proposition 6.10. For $k > 1$, the function:

$$l_k(z) - \frac{1}{k-1}\log(z)l_{k-1} \in A^k(U)$$

has an analytic continuation onto $B(0, 1)$.

PROOF. By induction on $k > 1$, for $k = 2$, by (6.8) and $\log(z)\log(1-z)$ being invariant under $z \mapsto 1-z$,

$$l_2(z) - \log(z)l_1(z) = l_2(z) + 2 \cdot \frac{1}{2}\log(z)\log(1-z) = l_2(1-z) \in A(B(0, 1)).$$

For $k > 2$, observe that

$$\begin{aligned} d\left(l_k - \frac{1}{k-1}\log(z)l_{k-1}(z)\right) &= l_{k-1}(z)\frac{dz}{z} - \frac{1}{k-1}l_{k-1}\frac{dz}{z} - \frac{1}{k-1}\log(z)l_{k-2}\frac{dz}{z} \\ &= \frac{k-2}{k-1}\left(l_{k-1} - \frac{1}{k-2}\log(z)l_{k-2}\right)\frac{dz}{z}, \end{aligned}$$

which lies in $\Omega(B(0, 1)) = A(B(0, 1))dz$ by induction hypothesis. Since $H^1(B(0, 1)) = 0$ and $A(B(0, 1))$ satisfies the uniqueness principle, $l_k - \frac{1}{k-1}\log(z)l_{k-1}(z) \in A(B(0, 1))$ to prove the statement. \square

7. RELATION BETWEEN THE POLYLOGARITHMS AND SPECIAL VALUES OF THE KUBOTA-LEOPOLD L-FUNCTION AT POSITIVE INTEGERS

7.1. An identity for the valuation of l_k .

For sake of a simplified notation, we will write $\sum_{(\zeta)} f(\zeta)$ to denote $\sum_{j=1}^{r-1} f(\zeta^j)$ when ζ is a non-trivial primitive r^{th} -root of unity.

Lemma 7.1. For $k > 1$, l_k extends to a \mathbb{C}_p -valued function that is

(1) continuous on any finitely ramified $K \supseteq \mathbb{Q}_p$ and,

(2) $\lim_{x \rightarrow 1}^1 l_k(x) = \frac{1}{r^{1-k}-1} \sum_{(\zeta)} l_k(\zeta)$

for any $r \in \mathbb{N}$ and $\zeta^r = 1$ non-trivial.



PROOF. (1) The only problematic points in this lemma are 0 and 1, continuity at $z = 0$ $\lim_{z \rightarrow 0} l_k(z) = \lim_{z \rightarrow 0} l_k(z) = 0$ exists by Definition 6.1. While the continuity at $z = 1$ for $k = 2$ follows from Lemma 6.6 and the functional equation

$$l_2(z) + \log(z) \log(1 - z) = l_2(1 - z).$$

For $k > 2$, the statement follows from Proposition 6.10 since $\log(1) = 0$ and the fact that rigid analytic functions are continuous.

(2) Proposition 6.2 may be revised as $r^k(\sum_{(\zeta)} l_k(\zeta z) + l_k(z)) = l_k(z^r)$. As

$$l_k(z^r) = \frac{r^{k-1}}{1 - r^{k-1}} \sum_{(\zeta)} l_k(\zeta z),$$

taking $z \rightarrow 1$ in $K \supseteq \mathbb{Q}_p$ proves (2). □

7.2. The p -adic L -function and its special values.

For $d \in \mathbb{N}$, let

$$X_d = \varprojlim_n \mathbb{Z}/dp^n\mathbb{Z}$$

as a topological space. For $a \in \mathbb{Z}$, $n \in \mathbb{N} \cup \{0\}$, there is a natural projection map $\pi_{d,n} : X_d \rightarrow \mathbb{Z}/dp^n\mathbb{Z}$. Construct the compact open sets $U_{d,n}(a) = \pi_{d,n}^{-1}(a) = \{x \in X_d \mid \pi_{d,j}(x) = a \pmod{dp^j\mathbb{Z}} \text{ for } j \leq n\} = a + dp^n\mathbb{Z}_p$. Notice that when we consider $U_{d,n}(a)$, the choice of a is actually independent module $dp^n\mathbb{Z}$. Let $X_{d,n}^* = \bigcup_{a=1}^{dp-1} U_{d,n}(a)$. If $z \in \mathbb{C}_p$ satisfying $z^{dp^n} \neq 1$ for all $n \in \mathbb{N}$, one may define a p -adic distribution $\mu_{d,z}$ (following Koblitz) on X_d :

$$\mu_{d,z}(U_{d,n}(a)) = \frac{z^a}{1 - z^{dp^n}}.$$

The measure $\mu_{d,z}$ defines a linear functional on locally constant functions $X_d \rightarrow \mathbb{C}_p$:

$$\int_{X_d} d\mu_{d,z} : f \mapsto \int_{X_d} f d\mu_{d,z} \triangleq \lim_{N \rightarrow \infty} \sum_{a=0}^{dp^N} f(a) \cdot \mu_{d,z}(U_{d,N}(a)).$$

If there is an ambiguity in the integration, we will denote $\int_{x \in X} d\mu$ to emphasize on the variable x being the integral variable. Note that $X_1 = \mathbb{Z}_p$ and $X_1^* = \mathbb{Z}_p \setminus \{0\}$. We now consider only $d = 1$ and henceforth drop out any subscript d by setting it 1.

- Definition 7.2.** (1) A *Teichmüller character* ω on $\mathbb{Z}_p \setminus \{0\}$ is a homomorphism characterized by $\omega(x) \in \mathbb{C}_p$ being the unique solution of $\omega(x) = \omega(x)^p$ in the residue class of x . It is also known that $\omega(x) = \lim_{n \rightarrow \infty} x^{p^n}$.
- (2) Let $\pi = \pi_1 : X \rightarrow \mathbb{Z}_p$, and for $x \in X$, let $\langle x \rangle = \pi(x)/\omega(\pi(x))$.
- (3) For any *Dirichlet character* $\chi : X \rightarrow \mathbb{C}_p$, $n \in \mathbb{N}$, let χ_n denote the *twisted Dirichlet character* $\chi \otimes \bar{\omega}^n$, where $\bar{\omega}(a) \triangleq \omega(a)^{-1}$.

For χ being a Dirichlet character of conductor d , it is known (by Koblitz) that for any $r \in \mathbb{N}$, $(r, pd) = 1$,

$$(7.1) \quad L_p(s, \chi) = \frac{1}{\langle r \rangle^{1-s} \chi(r) - 1} \int_{X^*} \langle x \rangle^{-s} \chi_1(x) d\mu,$$

where $\mu = \sum_{(\zeta)} \mu_\zeta$ with $\zeta^r = 1$ non-trivial.

Lemma 7.3. For $z \in \mathbb{C}_p \setminus B(1, 1)$, we have

$$l_{k,p}(z) = \int_{X^*} x^{-k} d\mu_z.$$



PROOF. (For the case when $k = 1$, see Koblitz [Kob79])

For $z \in B(0, 1)$,

$$\begin{aligned} \int_{X^*} x^{-k} d\mu_z &= \lim_{N \rightarrow \infty} \sum_{a=1}^{p^N-1} \frac{1}{a^k} \frac{z^a}{1-z^{p^N}} \\ &= \lim_{N \rightarrow \infty} \left(\sum_{a=1}^{p^N-1} \left(\frac{z^a}{a^k} \right) - \sum_{a=1}^{p^N-1-1} \left(\frac{z^{ap}}{(ap)^k} \right) \right) = l_k(z) - p^{-k} l_k(z^p) \end{aligned}$$

□

There is the p -adic analogy of the *Fourier inversion formula*: for χ being a primitive Dirichlet character of conductor d , $(r, dp) = 1$, and ξ, ζ primitive d^{th} -root and r^{th} -root of unity respectively, we have

$$\int_X \chi f d\mu_\zeta = \frac{\mathfrak{g}(\chi, \xi)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) \int_X f d\mu_{\xi^{-a}\zeta},$$

where $\mathfrak{g}(\chi, \xi) = \sum_{j=1}^{d-1} \chi(j)\xi^j$ is the *Gauss sum* and $\bar{\chi}(a) = \chi(a)^{-1}$.

Let $\pi : X \rightarrow \mathbb{Z}_p$ be the identification in the end of the first paragraph. Then

$$\int_X f \circ \pi d\mu_\zeta = \int_{\mathbb{Z}_p} f d\mu_{1, \zeta}.$$

In particular, choosing $f = x^{-k}$ and by the application of the Fourier inversion formula, for ξ, ζ being primitive d^{th} -root and r^{th} -root of unity respectively,

$$\begin{aligned} \int_{x \in X} \pi(x)^{-k} \chi d\mu_\zeta &= \frac{\mathfrak{g}(\chi, \xi)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) \int_{x \in \mathbb{Z}_p} x^{-k} d\mu_{\xi^{-a}\zeta} \\ (7.2) \quad &= \frac{\mathfrak{g}(\chi, \xi)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) \cdot l_{k,p}(\xi^{-a}\zeta) \\ &= \frac{\mathfrak{g}(\chi, \xi)}{d} \sum_{a=1}^{d-1} \bar{\chi}(a) \cdot (l_k(\xi^{-a}\zeta) - p^{-k} l_k((\xi^{-a}\zeta)^p)) \\ (7.3) \quad &= \frac{\mathfrak{g}(\chi, \xi)}{d} \left(\sum_{a=1}^{d-1} \bar{\chi}(a) \cdot l_k(\xi^{-a}\zeta) - \frac{\chi(p)}{p^k} \sum_{a=1}^{d-1} \bar{\chi}(a) l_k(\xi^{-a}\zeta) \right) \\ &= \frac{\mathfrak{g}(\chi, \xi)}{d} \left(1 - \frac{\chi(p)}{p^k} \right) \sum_{a=1}^{d-1} \bar{\chi}(a) \cdot l_k(\xi^{-a}\zeta). \end{aligned}$$

(7.2) follows from Lemma 7.3, and (7.3) holds by $(r, p) = 1$ thereby $\sum_{a=1}^{d-1} \bar{\chi}(ap) l_k(\xi^{-ap}\zeta^p) = \sum_{a=1}^{d-1} \bar{\chi}(a) l_k(\xi^{-a}\zeta)$.

Theorem 7.4. *Let the notation be as above, for χ a primitive, non-trivial Dirichlet character,*

$$(1.3) \quad L_p(k, \chi \otimes \omega^{1-k}) = \frac{\mathfrak{g}(\chi, \xi)}{d} \left(1 - \frac{\chi(p)}{p^k} \right) \sum_{(a)} \bar{\chi}(a) l_k(\xi^{-a}).$$



PROOF. [Col82] Notice that by definition, $\langle x \rangle^k \omega^k(x) = x^k$, by the computation above, for any integer $k > 1$,

$$\begin{aligned} L_p(k, \chi \otimes \omega^{1-k}) &= \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \int_{X^*} \langle x \rangle^{-k} (\omega^{(1-k)-1}(x) \otimes \chi(x)) d\mu \\ &= \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \int_{X^*} x^{-k} \chi(x) d\mu \\ &= \sum_{(\zeta)} \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \int_{X^*} x^{-k} \chi(x) d\mu_\zeta \\ &= \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \sum_{(\zeta)} \frac{\mathfrak{g}(\chi, \xi)}{d} \left(1 - \frac{\chi(p)}{p^k}\right) \sum_{a=1}^{d-1} \bar{\chi}(a) \cdot l_k(\xi^{-a} \zeta) \\ &= \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \frac{\mathfrak{g}(\chi, \xi)}{d} \left(1 - \frac{\chi(p)}{p^k}\right) \sum_{a=1}^{d-1} \bar{\chi}(a) \cdot \left(\frac{l_k(\xi^{-ak})}{r^{1-k}} - l_k(\xi^{-a})\right) \\ &= \frac{1}{\langle r \rangle^{1-k} \chi(r) - 1} \frac{\mathfrak{g}(\chi, \xi)}{d} \left(1 - \frac{\chi(p)}{p^k}\right) (\chi(r)r^{1-k} - 1) \sum_{a=1}^{d-1} \bar{\chi}(a) l_k(\xi^{-a}) \end{aligned}$$

The second last equation follows from Proposition 6.2, and the last follows from

$$\sum_{a=1}^{d-1} \bar{\chi}(a) (r^{1-k} l_k(\xi^{-ar})) = \chi(r)r^{1-k} \sum_{a=1}^{d-1} \bar{\chi}(a) l_k(\xi^{-a}).$$

Since the p -adic L -function represented in (7.1) is independent of r chosen, one may choose $r \in B(1, 1) \cap \mathbb{Z}$, so that $\omega(r) = 1$ and $\langle r \rangle = r$. We have then derived the desired formula. \square

Theorem 7.5. *When χ is chosen to be a trivial character, the above formula does not apply; however we have*

$$(1.4) \quad L_p(k, \omega^{1-k}) = \left(1 - \frac{1}{p^k}\right) \lim_{x \rightarrow 1} l_k(x).$$

PROOF. [Col82] For non-trivial Dirichlet character χ . Since $\chi = 1$,

$$\begin{aligned} L_p(k, \chi_{1-k}) &= L_p(k, \omega^{k-1}) = \frac{1}{\langle r \rangle^{1-k} - 1} \int_{X^*} \langle x \rangle^{-k} \omega(x)^{-k} d\mu \\ &= \frac{1}{\langle r \rangle^{1-k} - 1} \int_{X^*} x^{-k} d\mu \\ &= \frac{1}{\langle r \rangle^{1-k} - 1} \sum_{(\xi)} l_{k,p}(\xi) \end{aligned}$$

Choose $r \in B(1, 1) \cap \mathbb{Z}$ as above. By continuity of $l_k(x)$ at 1, $\lim_{x \rightarrow 1} l_k(x) = \lim_{x \rightarrow 1} l_k(x^p)$ and $\sum_{(\zeta)} l_{k,p}(\zeta) = \sum_{(\zeta)} (l_k(\zeta) - p^{-k} l_k(\zeta^p)) = (1 - p^{-k}) \sum_{(\zeta)} l_k(\zeta)$ if we choose ζ to be a r^{th} -root of unity with $(r, p) = 1$. Thus,

$$\lim_{x \rightarrow 1} l_{k,p}(x) = \left(1 - \frac{1}{p^k}\right) \lim_{x \rightarrow 1} l_k(x) = \left(1 - \frac{1}{p^k}\right) \frac{1}{r^{k-1} - 1} \sum_{(\zeta)} l_k(\zeta) = \frac{1}{r^{k-1} - 1} \sum_{(\zeta)} l_{k,p}(\zeta).$$

by Lemma 7.1, arises (1.4). \square

8. CONCLUSION AND SUGGESTIONS

We have now discussed the integration theory over connected affinoid subspaces of \mathbb{C}_p with good reduction. In this article a logarithmic F -crystal M is defined to be a A -submodule of \mathcal{L} with functions f so that $f dz$ can be integrated. Moreover, if we consider the smallest A -submodule of \mathcal{L} that contains M and $\int f dz$, which by definition would be $M^1 = M(U) + \sum_{\theta} A(C) \int \theta$ as discussed in section 4, is still a logarithmic F -crystal. We therefore applied it to define the solution to the recursive differential equations

$$(1) \quad l_0(z) = \frac{z}{1-z} \in A(U),$$



- (2) $dl_k(z) = l_{k-1}(z)\frac{dz}{z}$, and
 (3) $\lim_{z \rightarrow 0} l_k(z) = 0$.

The constructed solutions, namely the polylogarithmic functions $l_k(z)$, is locally of the form

$$l_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

As in the complex case, the polylogarithm are related to the special values of L -function at positive integers, namely through (1.3) and (1.4), partially contribute to the validity of the Coleman integral in moral.

Various differential equations whose solution exists in the complex field \mathbb{C} to possess arithmetic meaning, just as the one we discussed in this article. The author would suggests that thorough survey might lead to other analogy to the p -adic field C_p which involves deeper contents.

Another possible generalization is to discuss the integration theory on affinoid variety of dimensionality greater than 1, which Coleman partially discussed the case in his later-published articles.

REFERENCES

- [BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984, A systematic approach to rigid analytic geometry. MR 746961
- [CdS88] Robert Coleman and Ehud de Shalit, *p -adic regulators on curves and special values of p -adic L -functions*, Invent. Math. **93** (1988), no. 2, 239–266. MR 948100
- [Col82] Robert F. Coleman, *Dilogarithms, regulators and p -adic L -functions*, Invent. Math. **69** (1982), no. 2, 171–208. MR 674400
- [Col85] ———, *Torsion points on curves and p -adic abelian integrals*, Ann. of Math. (2) **121** (1985), no. 1, 111–168. MR 782557
- [Con08] Brian Conrad, *Several approaches to non-Archimedean geometry, p -adic geometry*, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 9–63. MR 2482345
- [Kob79] Neal Koblitz, *A new proof of certain formulas for p -adic L -functions*, Duke Math. J. **46** (1979), no. 2, 455–468. MR 534062
- [Tat71] John Tate, *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289. MR 0306196

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