國立臺灣大學理學院物理學系
碩士論文
Department of Physics
College of Science
National Taiwan University
Master Thesis

利用格林函數分析三維水波的繞射與散射
Three-dimensional Surface Water Wave Diffraction and Scattering Analysis Using Green's Function

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中華民國 105 年 6 月
June, 2016

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

國立臺灣大學碩士學位論文
口試委員會審定書

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Three-dimensional Surface Water Wave Diffraction and Scattering Analysis Using Green's Function

本論文係黃瀚賢君 R03222004 在國立臺灣大學物理學系、所完成之碩士學位論文，於民國 105 年 6 月 13 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

陳義裕

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曾文哲

doi:10.6342/NTU201600792

誌謝

感謝指導教授陳義裕教授這三年來的指導，老師給了我們很大的自由，讓我們的想法有很大的發揮空間。感謝所有學長和學弟，每個禮拜的 group meeting 報告都讓我學到很多東西。感謝研究室的朋友們，常常一起討論、聊天，一起增進知識、一起進步。最後一定要感謝家人，讓我無後顧之憂，完成碩士的學位。

doi:10.6342/NTU201600792

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摘要

在物理和工程學裡，水波的繞射與散射是常見且重要的問題，這篇論文利用了格林函數和其相關方法，對這兩種問題做了分析和研究。在水波繞射問題上，海更斯原理常常被當作一個類比，卻沒有數學基礎，我們利用格林函數的方法推導出水波的狹縫繞射公式，解釋了海更斯原理，我們的方法和光學裡的克希荷夫繞射公式的推導是類似的。而對於散射問題，我們假設水底幾乎是平坦的，但具有很小的起伏，我們同樣利用格林函數的方法，並結合微擾理論，推得出水波被地面散射的公式，為了驗證這個公式，我們利用其他方法來計算一個特別的例子，而其計算結果與我們的公式是一致的。

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

Abstract

Diffraction and scattering of water wave are common and important problems in physics and engineering. In this thesis, we use Green's function method to analysis the two problems. For diffraction, we apply Green's function method to derive the formula of slit diffraction to explain Huygens principle, which is usually just an analogy in water wave diffraction. We use the similar way that Kirchhoff derive his diffraction formula in optics. For scattering, we assume that the bottom of water is roughly flat with small variations. We use Green's function combined with perturbation method and obtain the formula of the scattering problem. We also use matching method to calculate a special case and then we find that the result is consistent with our formula.

doi:10.6342/NTU201600792

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List of Symbols

γ	Surface tension constant of water
p	Fluctuated pressure of water
ρ	The density of water
$\boldsymbol{u}$	Velocity field of water
g	Gravitational acceleration constant
t	The time variable
ω	Angular frequency
$\boldsymbol{k}$	Wave vector
$\boldsymbol{r}$	3-dim vector (x, y, z)
$\boldsymbol{x}$	2-dim vector (x, y)
$\zeta(\boldsymbol{x})$	The height of water surface
$h(\boldsymbol{x})$	The depth of water at postion $\boldsymbol{x}$
$\nabla_{\boldsymbol{r}}$	3-dim del operator
$\nabla_{\boldsymbol{x}}$	2-dim del operator
$J_n(r)$	Bessel function of first kind
$H_n^{(1)}(r)$	Hankel function of first kind
$\boldsymbol{r}_{PQ}$	Vector pointing from point P to Q
r_{PQ}	Distance between points P to Q
k_i	Every wave number that satisfies the dispersion relation (3.6)
M_{ij}	Matrix defined in (B.11)

doi:10.6342/NTU201600792

$\langle f, g \rangle_h$	Inner product of functions f and g, defined in (B.3)
$\phi_i(z)$	The orthonormal basis with inner product (B.3)

doi:10.6342/NTU201600792

Chapter 1

Introduction

Diffraction and scattering of water wave are important problems in engineering and oceanography. For instance, studying diffraction is crucial to harbor engineering, because it is necessary to understand the water wave patterns diffracted by breakwaters. Scattering by bottom topography is widely studied in offshore engineering and oceanography in that it is needed to know the effects of bottom or artifical stuctures to surface waves. Water surface wave problems are important but complicated to solve because the mechanism of water wave is nonlinear. Linear wave theory is a theory for water wave that linearize all the nonlinear terms and thus make the problems easier to solve, although sometimes it is still complicated and requires computer simulation.

A simple and intuitive explanation of diffraction waves is Huygens principle. It says that every point at wave front can be deemed as a source of wave. By Huygens principle, any diffraction of wave through an aperature can be calculated. In optics, Kirchhoff mathematically derived his diffraction formula and showed that Huygens principle is a direct consequence of his formula [4]. However, in fluid dynamics, Huygens principle of water wave diffraction is just an analogy. The mechanisms of water wave and optics are totally different. Therefore, in this thesis, we are going to deal with this problem.

Water surface wave scattering of bottom is extremely difficult to calculate. Most of the time, either some approximations or numerical computations are needed. In 1972, Berkhoff proposed mild-slope equation [2], which is well known and widely used today. Literally, this equation can be used in the case that the slope of bottom is small. Mild-slope equation can be derived from variational method of water wave. One degree of freedom is reduced by integrating the Lagrangian of water with respect to the vertical coordinate. The advantage of this method is that the equation depents only on the horizontal coordinates so it is simpler to solve and compute.

Another method to solve water wave scattering problem is Green's function method. Green's function method is commonly seen in physics. It can be used if the physical system has a linear differential equation. For water wave system, John, 1950, derived the Green's function and showed that it can be expanded into infinite series [6]. By applying

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Green's identity, the solution of water field is written as an integral of boundary conditions over the boundaries. Without any further approximations, it turns into an integral equation problem [9]. To solve the integral equation, numerical computation is necessary.

In this thesis, we discuss diffraction and scattering of three-dimensional water wave. First, like Kirchhoff's derivation of diffraction formula, we try to formulate the formula of flat-bottom water diffraction using Green's function. And then the phase shift of diffraction is discussed. Next, we use the already derived Green's function to further consider the case of uneven bottom. We apply perturbation method and obtain the formula for bottom scattering. Thoughout this thesis, we only consider linear wave system, and thus nonlinear terms are omitted. Besides, the effects of both gravity and surface tension on the free surface are maintained to keep generality.

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Chapter 2

Governing Eqautions

In this chapter, we formulate the governing equations of water wave. The derivation of governing equations can be found in textbooks of fluid dynamics like [1]. At the end of this chapter, the differential equation and the boundary conditions of water is provided for subsequent chapters to deal with.

Unlike other derivations using velocity potential Φ, we are going to use pressure as the main variable. Because we are using linear wave theory, we will see that velocity potential and pressure have the same differential equation and similar boundary conditions. However, pressure is a real physical quantity, so it can give us more insights.

Consider a body of water on a solid ground with depth $h(\boldsymbol{x})$. Assume that the water is static with a little fluctuation. Without loss of generality, the surface of water is defined to be at $z = 0$. So the pressure can defined as

$$-\rho g z + p(\boldsymbol{r}, t),$$

where $-\rho g z$ is the static pressure and $p(\boldsymbol{r}, t)$ is the fluctuation. Here, we use $\boldsymbol{x}$ to denote two-dimensional vector (x, y) and $\boldsymbol{r}$ to represent three-dimensional vector (x, y, z).

As usual, water is assumed to be an incompressible fluid, that is,

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{u} = 0, \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity field. The dynamics of fluid follows the Euler equation

$$\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\rho} \nabla_{\boldsymbol{r}} p.$$

By chain rule, $\frac{d\boldsymbol{u}}{dt}$ can be rewritten as

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla_{\boldsymbol{r}})\boldsymbol{u},$$

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which is nonlinear. Because we have assumed that the fluctuations are small, any higher order terms are negligible. In this case, the latter term can be discarded. After the elimination of nonlinear terms, we thus have the linearized Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla_{\boldsymbol{r}} p. \tag{2.2}$$

Combining (2.1) and (2.2), we can find that p satisfies Lapace equation, i.e.

$$\nabla_{\boldsymbol{r}}^2 p = 0. \tag{2.3}$$

The boundary of water has two parts. One is the free surface and the other is the boundary that water contacts the solids like bottom and walls. The boundary conditions should be considered separately.

For the boundary between water and solids, the velocity $\boldsymbol{u}$ must be parallel with the boundary because of the incapability for water particles to penetrate the solids. From the relationship between $\boldsymbol{u}$ and p in (2.2), it can be inferred that $\nabla_{\boldsymbol{r}} p$ must also parallel the boundary. Therefore, we have the boundary condition

$$\frac{\partial p}{\partial \boldsymbol{n}} = 0, \tag{2.4}$$

with $\boldsymbol{n}$ the normal vector of the boundary. More explicitly, on $z = -h(\boldsymbol{x})$, this equation can be written as

$$\nabla_{\boldsymbol{x}} h \cdot \nabla_{\boldsymbol{x}} p + \frac{\partial p}{\partial z} = 0. \tag{2.5}$$

For the free-surface boundary, it's more complicated because the position of the surface varies with time. Instead of finding the boundary condition on the real surface, we can just consider the plane $z = 0$, which is a fair approximation since the amplitude of water wave is assumed to be small. On $z = 0$ plane, by considering the effects of both gravity and sufrace tension, the pressure satisfies

$$p = -\gamma \nabla_{\boldsymbol{x}}^2 \zeta + \rho g \zeta, \tag{2.6}$$

where $\zeta(\boldsymbol{x})$ is the height of surface. For the places where $\zeta > 0$, p is the physical pressure. But for $\zeta < 0$, there is no water particle at $z = 0$, so the p is an imaginary pressure and is derived by extrapolation.

In addition, assume that the particles on the surface always stay on the surface, that is,

$$u_z = \frac{d\zeta}{dt},$$

with u_z the z-component of $\boldsymbol{u}$. Similar to the way we linearize (2.2), the nonlinear terms can be eliminated by replacing d with ∂. And then, by substituting the u_z into (2.2), we

doi:10.6342/NTU201600792

can obtain another boundary condition

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial^2 \zeta}{\partial t^2}. \tag{2.7}$$

In this thesis, we assume that the system is oscillating with angular frequency ω. Therefore, $p(\boldsymbol{r}, t)$ can be written as $p(\boldsymbol{r}, t) = \mathrm{Re}\left[p(\boldsymbol{r})\mathrm{e}^{-i\omega t}\right]$ and so are $\zeta(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{r}, t)$. For convenience, we solve the complex functions $p(\boldsymbol{r})$, $\zeta(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{r})$ rather that the real ones. Moreover, with this assumption, the second derivatives of any variables become $-\omega^2$, i.e. $\frac{\partial^2}{\partial t^2} = -\omega^2$. So the boundary condition (2.7) can be rewritten as

$$\frac{\partial p}{\partial z} = \rho\omega^2\zeta. \tag{2.8}$$

This assumption is made without any loss of generality because our system is totally linearized. We can get a more general solution with respect to time by making the superposition of all different frequencies ω.

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

Chapter 3

Flat Bottom Diffraction

In this chapter, we deal with the case that the bottom is flat. We let $h(\boldsymbol{x}) = h_0$ to be a constant. Therefore, the boundary condiditon (2.5) reduces to a simple form

$$\left.\frac{\partial p}{\partial z}\right|_{z=-h_0} = 0. \tag{3.1}$$

We will first introduction Green's method and then apply it to diffraction problems.

3.1 Green's Function Method

A Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ is a function that obeys the equation

$$\nabla^2_{\boldsymbol{r}'} G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}').$$

If $\boldsymbol{r}$ is in volume V, then, from Green's identity, we have

$$\begin{aligned} p(\boldsymbol{r}) &= \int_V p(\boldsymbol{r}')\delta(\boldsymbol{r} - \boldsymbol{r}')\, d\boldsymbol{r}' \\ &= \oint_{\partial V} \left(p\left(\boldsymbol{r}'\right) \frac{\partial G}{\partial \boldsymbol{n}'} - \frac{\partial p}{\partial \boldsymbol{n}'}(\boldsymbol{r}')G \right) da', \end{aligned} \tag{3.2}$$

where $\boldsymbol{n}'$ is the normal vector with respect to the boundary ∂V. With this formula, all we need to solve p is the value and derivative of p on the boundary.

In this section, we first apply this formula to a simple case, a plane wave. And later on, the method will be used on slit diffraction and uneven bottom problems. Now, assume that a plane water wave has the form

$$\zeta(\boldsymbol{x}) = A\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\boldsymbol{k}$ is an unknown constant we need to find.

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First we need to derive the Green's function for this problem. Considering the fact that $\nabla_{\boldsymbol{r}}^2 \left(\frac{-1}{4\pi r}\right) = \delta(\boldsymbol{r})$ and the boundary condition (3.1), we use the methed of image [5] by defining the Green's function as

$$G_1(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{4\pi}\left(\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} + \frac{1}{|\boldsymbol{r}' - \bar{\boldsymbol{r}}|}\right), \tag{3.3}$$

where $\bar{\boldsymbol{r}}$ is the mirror image of point $\boldsymbol{r}$ with respect to the bottom $z = -h_0$, i.e. $\bar{\boldsymbol{r}} = (x, y, -2h_0 - z)$. The integral domain V shoud be the whole space of water, that is, the region between $z = -h_0$ and $z = 0$. However, the boundary of infinity is not defined. Therefore, we let the integral domain V to be a vertical cylinder with radius R and bases

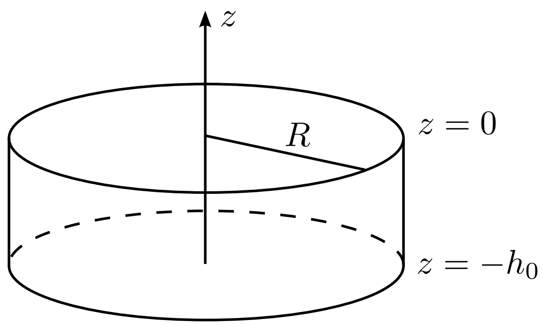

Figure 3.1: The domain of integration V

on $z = 0$ and $z = -h_0$ as in figure 3.1. Let R tend to infinity so that V appoaches the whole space of water. Applying the Green's identity (3.2), we have

$$p(\boldsymbol{r}) = \int_{z'=-h_0} + \int_{R\to\infty} + \int_{z'=0} \left(p\left(\boldsymbol{r}'\right)\frac{\partial G_1}{\partial \boldsymbol{n}'} - \frac{\partial p}{\partial \boldsymbol{n}'}(\boldsymbol{r}')G_1\right) da'. \tag{3.4}$$

Obviously, the integral over $z' = -h_0$ vanishes due to the symmetry of G_1 and the boundary condition (3.1) of p. And as for the integral of $R \to \infty$, although G_1 is of order $O(R^{-1})$ and the area of integration is $O(R)$, p is an oscillating function, so we assume that the integral also tends to zero when R approach infinity. And thus only the integral over $z' = 0$ remains. Substituting the boundary conditions (2.6) and (2.8) into (3.4), we get

$$\begin{aligned} p(\boldsymbol{r}) = -\frac{A}{4\pi}\int_{z'=0} \Bigg[\left(\gamma k^2 + \rho g\right)\frac{\partial}{\partial z'}\left(\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} + \frac{1}{|\boldsymbol{r}' - \bar{\boldsymbol{r}}|}\right) \\ - \rho\omega^2\left(\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} + \frac{1}{|\boldsymbol{r}' - \bar{\boldsymbol{r}}|}\right)\Bigg] \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}'} d^2\boldsymbol{x}'. \end{aligned} \tag{3.5}$$

By transformation of coordinate to polar coordinate with $\mu = |\boldsymbol{x} - \boldsymbol{x}'|$ and θ the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{x} - \boldsymbol{x}'$, we have

$$\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}'}}{|\boldsymbol{r}' - \boldsymbol{r}|} d^2\boldsymbol{x}' = \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{\mathrm{e}^{ik\mu\cos\theta}}{\sqrt{\mu^2 + (z - z')^2}}\, \mu d\mu\, d\theta$$

doi:10.6342/NTU201600792

Adopting the equations from integral tables,

$$\int_{-\pi}^{\pi} \mathrm{e}^{ix\cos(\tau)} d\tau = 2\pi J_0(x),$$
$$\int_0^{\infty} \frac{xJ_0(xy)}{\sqrt{a^2+x^2}} dx = \frac{\mathrm{e}^{-ay}}{y}, \quad \mathrm{Re}[a] > 0\,,$$

where $J_n(x)$ is the Bessel function of first kind, we can make the calculation

$$\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}'}}{|\boldsymbol{r}'-\boldsymbol{r}|} d^2\boldsymbol{x}' = 2\pi \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int_0^{\infty} \frac{\mu J_0(k\mu)}{\sqrt{\mu^2+(z-z')^2}} d\mu$$
$$= 2\pi \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\mathrm{e}^{-k|z-z'|}}{k}.$$

Substituting the result into (3.5), we can obtain

$$p(\boldsymbol{r}) = \frac{A}{2k} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[(\gamma k^2+\rho g)k + \rho\omega^2\right] \left(\mathrm{e}^{kz} + \mathrm{e}^{-k(2h_0+z)}\right).$$

We have to compare the equation to the orginal assumption $\zeta = A\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. By the boundary condition of water surface (2.6), the equation

$$\frac{A}{2k} \left[(\gamma k^2+\rho g)k + \rho\omega^2\right] \left(1 + \mathrm{e}^{-2kh_0}\right) = A(\gamma k^2+\rho g)$$

is shown. And then we derive the equation for k

$$\rho\omega^2 = (\gamma k^2+\rho g)k \tanh(kh_0). \tag{3.6}$$

This is called the dispersion relation of water wave. As a result, p has the form

$$p(\boldsymbol{r}) = A(\gamma k^2+\rho g)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\cosh(k\,(z+h_0))}{\cosh(kh_0)}. \tag{3.7}$$

3.2 Slit Diffraction

In this section, we are going apply Green's function method to solve slit diffraction problem. We will follow almost the same procedure in last section, but, because we don't know the form of the diffracted wave, unlike the calculations in last section, we will encounter an integral integration if we use the Green's function G_1 as in (3.3). Therefore, another proper Green's function is needed.

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3.2.1 The Green's Function

We define Green's function $G_2(\boldsymbol{r}, \boldsymbol{r}')$ that satisfies the boundary conditions

$$\begin{cases} \left(-\gamma\nabla^2_{\boldsymbol{x}'} + \rho g\right)\dfrac{\partial G_2}{\partial z'} - \rho\omega^2 G_2 = 0, & \text{for } z' = 0, \\ \dfrac{\partial G_2}{\partial z'} = 0, & \text{for } z' = -h_0. \end{cases} \tag{3.8}$$

So the integral of $p\frac{\partial G_2}{\partial z'} - \frac{\partial p}{\partial z'}G_2$ over an arbitrary area D on $z' = 0$ plane becomes

$$\begin{aligned} \int_D \left(p\frac{\partial G_2}{\partial z'} - \frac{\partial p}{\partial z'}G_2\right) da' &= \int_D \left(\left(-\gamma\nabla^2_{\boldsymbol{x}'} + \rho g\right)\zeta\frac{\partial G_2}{\partial z'} - \rho\omega^2\zeta G_2\right) da' \\ &= \int_D \left[-\gamma\nabla_{\boldsymbol{x}'}\cdot\left(\nabla_{\boldsymbol{x}'}\zeta\frac{\partial G_2}{\partial z'} - \zeta\nabla_{\boldsymbol{x}'}\frac{\partial G_2}{\partial z'}\right)\right] da' \\ &= -\gamma\int_{\partial D}\left(\frac{\partial \zeta}{\partial \boldsymbol{n}'}\frac{\partial G_2}{\partial z'} - \zeta\frac{\partial}{\partial \boldsymbol{n}'}\frac{\partial G_2}{\partial z'}\right) dl'. \end{aligned} \tag{3.9}$$

We can see that, with this Green's function, we don't need to guess the unknown wave form of ζ.

Now let's find the Green's function G_2 with boundary conditions (3.8). Because the delta function can written as the Fourier transform

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{(2\pi)^2}\int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} d^2\boldsymbol{s},$$

we let G_2 to be

$$G_2 = \frac{1}{(2\pi)^2}\int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\widetilde{G}_2\left(\boldsymbol{s}, z, z'\right) d^2\boldsymbol{s}, \tag{3.10}$$

which is just the Fourier transform of G_2. So the equation $\nabla^2_{\boldsymbol{r}'}G_2 = \delta(\boldsymbol{r}, \boldsymbol{r}')$ becomes

$$\frac{1}{(2\pi)^2}\int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\left(-s^2 + \frac{\partial^2}{\partial z'^2}\right)\widetilde{G}_2 d^2\boldsymbol{s}. = \delta(z - z')\frac{1}{(2\pi)^2}\int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} d^2\boldsymbol{s},$$

or

$$\left(-s^2 + \frac{\partial^2}{\partial z'^2}\right)\widetilde{G}_2 = \delta(z - z'), \tag{3.11}$$

with the B.C.

$$\begin{cases} \left(\gamma s^2 + \rho g\right)\dfrac{\partial \widetilde{G}_2}{\partial z'} - \rho\omega^2\widetilde{G}_2 = 0, & z' = 0, \\ \dfrac{\partial \widetilde{G}_2}{\partial z'} = 0, & z' = -h_0. \end{cases} \tag{3.12}$$

The equation (3.11) has a particular solution $\frac{1}{s}\sinh(s(z' - z))H(z' - z)$, where $H(z)$ is an unit step function, so we assume $\widetilde{G}_2$ to be the addition of the particular solution and

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homogeneous solutions as follows

$$\widetilde{G}_2 = \frac{1}{s}\sinh(s(z'-z))H(z'-z) + c_1\cosh(s(z'+h_0)) + c_2\sinh(s(z'+h_0)).$$

Substituting the B.C (3.12), we can obtain $c_2 = 0$ and

$$c_1 = -\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}.$$

So we have

$$\widetilde{G}_2 = \frac{1}{s}\Bigg[\sinh(s(z'-z))H(z'-z) - \frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\cosh(s(z'+h_0))\Bigg]. \quad (3.13)$$

G_2 is now an integral of a complicated fuction $\widetilde{G}_2$. It can be further expanded to infinite series, which is shown in appendix A.

3.2.2 Exapmle of a Simple Diffraction Problem

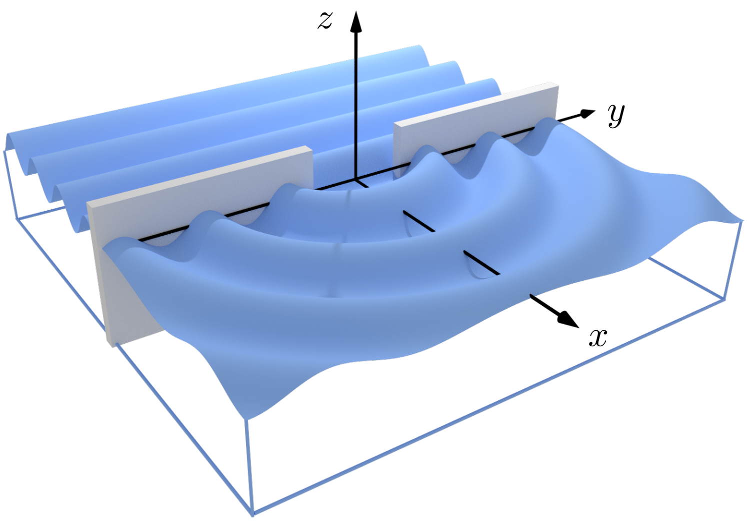

Figure 3.2: The setup for the slit diffraction

The setup of the diffraction problem is shown in figure 3.2. Although the Green's function method can be applied to any shape and geometry of walls, we consider only the simplest case here, i.e. vertical straight walls standing along $x = 0$ with some apertures open. The edges of apertures are straight . Assume that in $x < 0$ side, there is an incident

doi:10.6342/NTU201600792

wave $\zeta(\boldsymbol{x}) = \zeta_i(\boldsymbol{x})$ and, from (3.7), the incident pressure

$$p_i(\boldsymbol{r}) = \zeta_i(\boldsymbol{x})(\gamma k^2 + \rho g)\frac{\cosh(k(z+h_0))}{\cosh(kh_0)}. \tag{3.14}$$

In $x > 0$ space, the diffracted wave is the desired function that we want to solve.

The boundary conditions on the wall is the same as that on the bottom, $\frac{\partial p}{\partial \boldsymbol{n}} = 0$. For this reason, we define a new Green's function

$$G_3(\boldsymbol{r}, \boldsymbol{r}') = G_2(\boldsymbol{r}, \boldsymbol{r}') + G_2(\boldsymbol{r}, \bar{\boldsymbol{r}}'), \tag{3.15}$$

where $\bar{\boldsymbol{r}}'$ is the mirror image point of $\boldsymbol{r}'$ with respect to the $x' = 0$ plane, so that it satisfies

$$\left.\frac{\partial G_3}{\partial x'}\right|_{x'=0} = 0. \tag{3.16}$$

Now that we have the Green's function, it's time to apply (3.2) to calculate the diffracted wave. The integral domain V in (3.2) is selected as a half vertical cylinder with its center at the origin and radius R. As R tends to infinity, V will approach the whole space of water in $x > 0$. Therefore, (3.2) can be written as

$$p(\boldsymbol{r}) = \int_{z'=-h_0} + \int_{x'=0} + \int_{z'=0} + \int_{R\to\infty} \left(p\frac{\partial G_3}{\partial \boldsymbol{n}'} - \frac{\partial p}{\partial \boldsymbol{n}'} G_3 \right) da'. \tag{3.17}$$

The integral over $z' = -h_0$ vanishes because $\frac{\partial p}{\partial z'}$ and $\frac{\partial G_3}{\partial z'}$ are 0. The integral over $z' = 0$ can be replaced by equation (3.9), and then we get

$$p(\boldsymbol{r}) = \int_{x'=0} + \int_{R\to\infty} \left[\int_{-h_0}^{0} \left(p\frac{\partial G_3}{\partial \boldsymbol{n}'} - \frac{\partial p}{\partial \boldsymbol{n}'} G_3 \right) dz' \right.$$
$$\left. - \gamma \left(\frac{\partial \zeta}{\partial \boldsymbol{n}'}\frac{\partial G_3}{\partial z'} - \zeta \frac{\partial}{\partial \boldsymbol{n}'}\frac{\partial G_3}{\partial z'} \right)_{z'=0} \right] dl'. \tag{3.18}$$

We can argue that the integral of $R \to \infty$ will tend to 0 as R goes to ∞. Because in the diffraction region, $x > 0$, there is only outgoing wave, we must have the relation

$$\frac{\partial p}{\partial R} - ikp = O\left(R^{-\frac{3}{2}}\right), \tag{3.19}$$

for R very far away from the apertures. From the expanded Green's function (A.6) in appendix A, G_3 also has the following relation

$$\frac{\partial G_3}{\partial R} - ikG_3 = O\left(R^{-\frac{3}{2}}\right), \tag{3.20}$$

when $R \to \infty$. By combining (3.19) and (3.20), we can see that $p\frac{\partial G_3}{\partial R} - \frac{\partial p}{\partial R}G_3 = O(R^{-2})$.

doi:10.6342/NTU201600792

So the whole integral over $R \to \infty$ vanishes, and (3.18) becomes

$$p(\boldsymbol{r}) = \int_{\text{aperature}} \frac{\partial \zeta_i}{\partial x'} \left(\frac{\gamma k^2 + \rho g}{\cosh(kh_0)} \int_{-h}^{0} \cosh(k(z'+h_0)) G_3 \, dz' + \gamma \left. \frac{\partial G_3}{\partial z'} \right|_{z'=0} \right) dy'. \tag{3.21}$$

Expand G_3 with $\widetilde{G}_2$. From integral tables,

$$\int \cosh(\alpha)\cosh(\beta) dz = \frac{1}{a^2-b^2}\big(a\sinh(\alpha)\cosh(\beta) - b\cosh(\alpha)\sinh(\beta)\big), \text{ and}$$
$$\int \sinh(\alpha)\cosh(\beta) dz = \frac{1}{a^2-b^2}\big(a\cosh(\alpha)\cosh(\beta) - b\sinh(\alpha)\sinh(\beta)\big),$$

where $\alpha = az + c$ and $\beta = bz + d$, we can find out

$$\begin{aligned}
&\int_{-h}^{0} \cosh(k(z'+h_0)) \left. G_3 \right|_{x'=0} dz' \\
&= \frac{2}{(2\pi)^2} \int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \left[\int_{-h}^{0} \cosh(k(z'+h_0)) \widetilde{G}_2 \, dz' \right] d^2\boldsymbol{s} \\
&= \frac{2}{(2\pi)^2} \int \frac{\mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{s^2-k^2} \Bigg[-\cosh(k(z+h_0)) \\
&\qquad + \cosh(s(z+h_0)) \frac{(\gamma s^2 + \rho g)k\sinh(kh_0) - \rho\omega^2\cosh(kh_0)}{(\gamma s^2+\rho g)s\sinh(sh_0) - \rho\omega^2\cosh(sh_0)} \Bigg] d^2\boldsymbol{s}
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\left. \frac{\partial G_3}{\partial z'} \right|_{x',z'=0} &= \frac{2}{(2\pi)^2} \int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{\partial \widetilde{G}_2}{\partial z'} \, d^2\boldsymbol{s} \\
&= \frac{2}{(2\pi)^2} \int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{-\rho\omega^2\cosh(s(z+h_0))}{(\gamma s^2+\rho g)s\sinh(sh_0) - \rho\omega^2\cosh(sh_0)} \, d^2\boldsymbol{s}.
\end{aligned} \tag{3.23}$$

Finally, we can calculate the result by joining (3.22) and (3.23) with (3.21). Actually, most terms will be eliminated after combination, and only a simple form ramains:

$$p(\boldsymbol{r}) = \frac{-2}{(2\pi)^2}(\gamma k^2 + \rho g) \frac{\cosh(k(z+h_0))}{\cosh(kh_0)} \int_{\text{aperture}} \frac{\partial \zeta_i}{\partial x'} \int \frac{\mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{s^2-k^2} d^2\boldsymbol{s} \, dy'.$$

From integral table, the integral of $d^2\boldsymbol{s}$ becomes

$$\int \frac{\mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{s^2-k^2} d^2\boldsymbol{s} = 2\pi \int \frac{sJ_0(s|\boldsymbol{x}-\boldsymbol{x}'|)}{s^2-k^2} ds = i\pi^2 H_0^{(1)}(s|\boldsymbol{x}-\boldsymbol{x}'|), \tag{3.24}$$

where $H_n^{(1)}(z)$ is the Hankel function of first kind. As a result, we arrive at the diffraction formula of pressure,

$$p(\boldsymbol{r}) = -\frac{i}{2}(\gamma k^2 + \rho g) \frac{\cosh(k(z+h_0))}{\cosh(kh_0)} \int_{\text{aperture}} \frac{\partial \zeta_i}{\partial x'} H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|) dy', \tag{3.25}$$

doi:10.6342/NTU201600792

and surface height

$$\zeta(\boldsymbol{x}) = -\frac{i}{2}\int_{\text{aperture}} \frac{\partial \zeta_i}{\partial x'} H_0^{(1)}(k\,|\boldsymbol{x}-\boldsymbol{x}'|)dy'. \tag{3.26}$$

3.3 Phase Shift

In optics, Rubinowicz' diffraction theory [10] infers that there is a phase shift near the border of iluminated region. In this section, we are going to repeat the process in optics and find that there is also a phase shift in diffraction of water wave.

3.3.1 Rewrite the Diffraction Formula

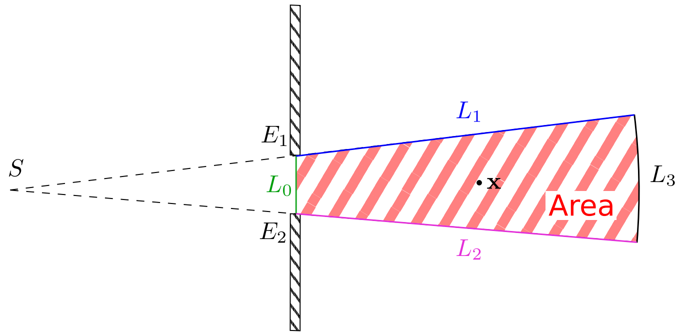

Figure 3.3: Diffraction of cylindrical wave

We consider the diffraction of a point source by a single slit. The diffraction setup is defined as in figure 3.3. Let S be the source of a cylindrical wave, generating the wave function

$$\zeta_i(\boldsymbol{x}) = AH_0^{(1)}(kr_{S\boldsymbol{x}}),$$

where the notation $\boldsymbol{r}_{PQ}$ represent the vector pointing form point P to Q and r_{PQ} is the length of the vector. E_1 and E_2 are the edge points of the slit. L_0 denotes the line segment of aperture. L_3 is an arc of a circle with its center at S. L_1 and L_2 are staight lines extented from the line segment of the source and the edges of aperture, so the Area bounded by curves L_is is the "illuminated" region of water wave.

We follow the method of Rubinowicz. Define a two dimentional Green's function to be as

$$G_4(\boldsymbol{x},\boldsymbol{x}') = -\frac{i}{4}\left(H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|) + H_0^{(1)}(k|\bar{\boldsymbol{x}}-\boldsymbol{x}'|)\right), \tag{3.27}$$

where $\bar{\boldsymbol{x}}$ is a symmetric point of $\boldsymbol{x}$ with respect to $x = 0$. This is a Green's function because $(\nabla_{\boldsymbol{x}}^2 + k^2)\left(-\frac{i}{4}H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|)\right) = \delta(\boldsymbol{x}-\boldsymbol{x}')$. Therefore, from Green's identity,

doi:10.6342/NTU201600792

we have

$$\int_{\text{Area}} \zeta_i(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')d^2\boldsymbol{x}' = \int_{\text{Area}} \zeta_i\left(\nabla^2_{\boldsymbol{x}'}+k^2\right)G_4\,d^2\boldsymbol{x}' \\ = \int_{L_0+L_1+L_2+L_3}\left(\zeta_i\frac{\partial G_4}{\partial \boldsymbol{n}'}-\frac{\partial \zeta_i}{\partial \boldsymbol{n}'}G_4\right)dl', \tag{3.28}$$

where we have used the equation $(\nabla^2_{\boldsymbol{x}}+k^2)\zeta_i(\boldsymbol{x})=0$.

The left-hand side of (3.28) is $\zeta_i(\boldsymbol{x})$ if $\boldsymbol{x}\in$ Area and 0 if $\boldsymbol{x}\notin$ Area. From the diffraction formula (3.26), the integral over L_0 is exactly the diffracted ζ:

$$\int_{L_0}\left(\zeta_i\frac{\partial G_4}{\partial \boldsymbol{n}'}-\frac{\partial \zeta_i}{\partial \boldsymbol{n}'}G_4\right)dl' = -\frac{i}{2}\int_{\text{aperture}}\frac{\partial \zeta_i}{\partial x'}H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|)dy' \\ = \zeta(\boldsymbol{x}).$$

And by applying the same argument in previous section, the integral over L_3 will vanish when the curve L_3 tends to be infinitely far from the aperture. In integrals of L_1 and L_2, $\frac{\partial\zeta_i}{\partial \boldsymbol{n}'}=0$ because the incident wave is a circular wave. As a result, (3.28) can be rearranged to be

$$\zeta(\boldsymbol{x}) = \begin{cases}\zeta_i(\boldsymbol{x}), & \boldsymbol{x}\in \text{Area},\\ 0, & \boldsymbol{x}\notin\text{Area},\end{cases} - \int_{L_1+L_2}\zeta_i\frac{\partial G_4}{\partial \boldsymbol{n}'}dl'. \tag{3.29}$$

Substituting (3.27) into the derivative of G_4, we can get

$$\frac{\partial G_4}{\partial \boldsymbol{n}'} = \nabla_{\boldsymbol{x}'}G_4\cdot\hat{\boldsymbol{n}}' \\ = -\frac{i}{4}\left(\hat{\boldsymbol{r}}_{\boldsymbol{x}\boldsymbol{x}'}\frac{dH_0^{(1)}(kr_{\boldsymbol{x}\boldsymbol{x}'})}{dr_{\boldsymbol{x}\boldsymbol{x}'}}+\hat{\boldsymbol{r}}_{\bar{\boldsymbol{x}}\boldsymbol{x}'}\frac{dH_0^{(1)}(kr_{\bar{\boldsymbol{x}}\boldsymbol{x}'})}{dr_{\bar{\boldsymbol{x}}\boldsymbol{x}'}}\right)\cdot\hat{\boldsymbol{n}}'.$$

Replace $\hat{\boldsymbol{r}}_{\boldsymbol{x}\boldsymbol{x}'}$ with $\frac{\boldsymbol{r}_{\boldsymbol{x}\boldsymbol{x}'}}{r_{\boldsymbol{x}\boldsymbol{x}'}}$, and then

$$\hat{\boldsymbol{n}}'\cdot\boldsymbol{r}_{\boldsymbol{x}\boldsymbol{x}'} = \hat{\boldsymbol{n}}'\cdot(\boldsymbol{r}_{\boldsymbol{x}E}+\boldsymbol{r}_{E\boldsymbol{x}'}) = \hat{\boldsymbol{n}}'\cdot\boldsymbol{r}_{\boldsymbol{x}E}.$$

Because $\hat{\boldsymbol{n}}_i$ is a normal vector perpendicular to line L_i, $\hat{\boldsymbol{n}}'\cdot\boldsymbol{r}_{E\boldsymbol{x}'}=0$. From figure 3.3, it can be easily seen that

$$\hat{\boldsymbol{n}}_1 = \hat{z}\times\hat{\boldsymbol{r}}_{SE_1}, \text{ and} \\ \hat{\boldsymbol{n}}_2 = -\hat{z}\times\hat{\boldsymbol{r}}_{SE_2}.$$

doi:10.6342/NTU201600792

Therefore, (3.29) turns to be

$$\zeta(\boldsymbol{x}) = \begin{cases} \zeta_i(\boldsymbol{x}), & \boldsymbol{x} \in \text{Area}, \\ 0, & \boldsymbol{x} \notin \text{Area}, \end{cases}$$
$$+\frac{i}{4}A\hat{z}\cdot\left((\hat{\boldsymbol{r}}_{SE_1} \times \boldsymbol{r}_{\boldsymbol{x}E_1})\int_{L_1}\frac{1}{r_{\boldsymbol{x}\boldsymbol{x}'}}H_0^{(1)}(kr_{S\boldsymbol{x}'})\frac{\partial H_0^{(1)}(kr_{\boldsymbol{x}\boldsymbol{x}'})}{\partial r_{\boldsymbol{x}\boldsymbol{x}'}}\,dl' + [\boldsymbol{x} \to \bar{\boldsymbol{x}}]\right) - [1 \to 2], \tag{3.30}$$

where $[a \to b]$ denotes the same term as the previous one but with a replaced by b.

3.3.2 Approximation of The Integral

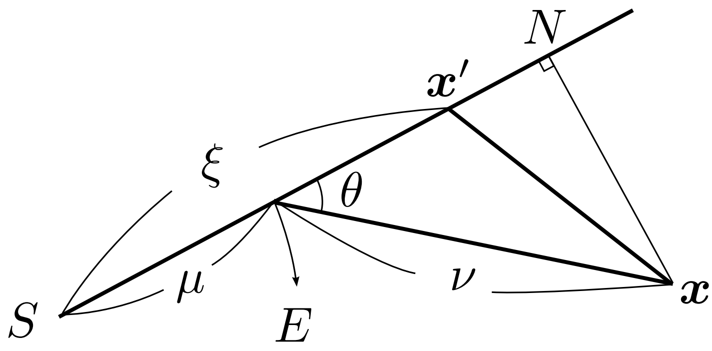

Figure 3.4: Symbols and parameters for solving integral (3.31)

The integral in (3.30) cannot be exactly solved, so we need some approximations. Let I be the integral we are focusing on:

$$I = \int_{\mu}^{\infty} H_0^{(1)}(k\xi)\frac{1}{r_{\boldsymbol{x}\boldsymbol{x}'}}\frac{dH_0^{(1)}(kr_{\boldsymbol{x}\boldsymbol{x}'})}{dr_{\boldsymbol{x}\boldsymbol{x}'}}d\xi, \tag{3.31}$$

where $\xi = r_{S\boldsymbol{x}'}$ and $\mu = r_{SE}$ as shown in figure 3.4. To make an approximation, observe that Hankel functions have the asympotic form

$$H_n^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}}\mathrm{e}^{i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)},$$

for $z \gg 1$. We assume that $k\mu \gg 1$ and $k\nu|\sin\theta| \gg 1$, so either k is very large or the measured point is far from the boundary, and we have the approximation of I:

$$I \approx \frac{2}{\pi}\int_{\mu}^{\infty}\frac{1}{\sqrt{\xi r_{\boldsymbol{x}\boldsymbol{x}'}^3}}\mathrm{e}^{ik(\xi + r_{\boldsymbol{x}\boldsymbol{x}'})}d\xi. \tag{3.32}$$

doi:10.6342/NTU201600792

We are still unable to exactly solve the integral, so we will further do some approximations. First, make the change of variable $\eta = \xi + r_{\boldsymbol{x}\boldsymbol{x}'}$ so that the exponent is in a simple form. By substituting these equations $\xi = \frac{1}{2}\frac{\eta^2 - r_{S\boldsymbol{x}}^2}{\eta - r_{SN}}$, $r_{\boldsymbol{x}\boldsymbol{x}'} = \frac{1}{2}\frac{(\eta - r_{SN})^2 + r_{N\boldsymbol{x}}}{\eta - r_{SN}}$ and $\frac{d\eta}{d\xi} = \frac{\eta - r_{SN}}{r_{\boldsymbol{x}\boldsymbol{x}'}}$ back to (3.32), the result of change of variable is

$$I \approx \frac{4}{\pi}\int_{\mu+\nu}^{\infty} \frac{\mathrm{e}^{ik\eta}}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)}} d\eta.$$

Next, let $g(\eta) = \frac{4}{\pi}\frac{1}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)}}$ be the integrand and let $\eta_0 = \mu + \nu$. By integration by parts,

$$\int_{\eta_0}^{\infty} g(\eta)\mathrm{e}^{ik\eta} d\eta = -g(\eta_0)\frac{\mathrm{e}^{ik\eta_0}}{ik} - \int_{\eta_0}^{\infty} g'(\eta)\frac{\mathrm{e}^{ik\eta}}{ik} d\eta.$$

Let the former term in the right hand side be denoted by I_0 and the latter term by ϵ. We will show that, in some circumstances, $I \approx I_0$ is a fair approximation. In other words, the error ϵ is small enough compared to I_0. To demonstrate the estimation of ϵ, integrate it by parts:

$$\epsilon = g'(\eta_0)\frac{\mathrm{e}^{ik\eta_0}}{(ik)^2} + \int_{\eta_0}^{\infty} g''(\eta)\frac{\mathrm{e}^{ik\eta}}{(ik)^2} d\eta.$$

Therefore,

$$\begin{aligned} |\epsilon| &\leq \left|\frac{g'(\eta_0)}{k^2}\right| + \left|\int_{\eta_0}^{\infty} g''(\eta)\frac{\mathrm{e}^{ik\eta}}{(ik)^2} d\eta.\right| \\ &\leq \frac{1}{k^2}\left|g'(\eta_0)\right| + \frac{1}{k^2}\int_{\eta_0}^{\infty} \left|g''(\eta)\right| d\eta. \end{aligned} \tag{3.33}$$

After direct calculation, we can find that

$$\begin{aligned} g'(\eta) &= \frac{-4}{\pi}\left(\frac{\eta}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)^3\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)}} + \frac{\eta - r_{SN}}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)^3}}\right) \\ &< 0, \\ g''(\eta) &= \frac{12}{\pi}\left(\frac{\eta^2}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)^5\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)}} + \frac{(\eta - r_{SN})^2}{\sqrt{(\eta^2 - r_{S\boldsymbol{x}}^2)\left((\eta - r_{SN})^2 + r_{N\boldsymbol{x}}^2\right)^5}}\right) \\ &> 0, \end{aligned}$$

for any $\eta \in [\eta_0, \infty)$. Now the integral in (3.33) can be calculated and the result is

$$|\epsilon| \leq -\frac{2}{k^2}g'(\eta_0).$$

doi:10.6342/NTU201600792

If we assume that $2|g'(\eta_0)| \ll k|g(\eta_0)|$, and then we must have $|\epsilon| \ll |I_0|$. To see in what condition it is the case, expand $\frac{2}{k}\left|\frac{g'(\eta_0)}{g(\eta_0)}\right| \ll 1$ in terms of μ, ν and θ:

$$\frac{\mu+\nu}{k\mu\nu(1-\cos\theta)} + \frac{1}{k\nu} \ll 1.$$

Since the assumption $k\nu|\sin\theta|$ has been made for the approximation (3.32), the inequality above becomes

$$1-\cos\theta \gg \frac{1}{k\mu} + \frac{1}{k\nu}.$$

In sum, the approximation

$$I \approx I_0 = \frac{2i}{\pi k}\frac{\mathrm{e}^{ik(\mu+\nu)}}{\sqrt{\mu\nu^3}(1-\cos\theta)}$$

is valid under the assumptions: $k\nu|\sin\theta| \gg 1$, $k\mu \gg 1$ and $1-\cos\theta \gg \frac{1}{k\mu}+\frac{1}{k\nu}$.

Finally, the diffraction formula (3.30) becomes

$$\begin{aligned}\zeta(\boldsymbol{x}) \approx &\begin{cases} \zeta_i(\boldsymbol{x}), & \boldsymbol{x} \in \text{Area}, \\ 0, & \boldsymbol{x} \notin \text{Area}, \end{cases} \\ &+ \frac{A}{2\pi}\hat{z}\cdot\left(\frac{\hat{\boldsymbol{r}}_{SE_1}\times\hat{\boldsymbol{r}}_{E_1\boldsymbol{x}}}{1+\hat{\boldsymbol{r}}_{SE_1}\cdot\hat{\boldsymbol{r}}_{E_1\boldsymbol{x}}}\frac{\mathrm{e}^{ik(r_{SE_1}+r_{E_1\boldsymbol{x}})}}{\sqrt{kr_{SE_1}}\sqrt{kr_{E_1\boldsymbol{x}}}} + [\boldsymbol{x}\to\bar{\boldsymbol{x}}]\right) - [1\to 2]\,.\end{aligned} \tag{3.34}$$

A phase shift near the boundary can be deduced from the result. For $\boldsymbol{x} \in$ Area, the phase of the diffracted wave is dominated by the phase of the incident wave ζ_i, i.e. $\mathrm{e}^{i(kr_{S\boldsymbol{x}}-\frac{\pi}{4})}$. However the phase in $\boldsymbol{x} \notin$ Area is approximately $\mathrm{e}^{ik(r_{SE}+r_{E\boldsymbol{x}})}$. Therefore, there is a $\frac{\pi}{4}$ phase shift near the boundary.

3.3.3 Simulations

To visualize the phase shift and to test our approximation, we are going to numerically compute the diffraction formula (3.26). But first, we apply the scale transformation $\zeta(\boldsymbol{x}) \to \zeta(k\boldsymbol{x})$ and $\boldsymbol{x} \to \frac{\boldsymbol{x}}{k}$ so we have

$$\zeta(\boldsymbol{x}) = -\frac{i}{2}\int_{\text{aperture}} \frac{\partial\zeta_i}{\partial x'} H_0^{(1)}(|\boldsymbol{x}-\boldsymbol{x}'|)dy'. \tag{3.35}$$

The mehtod of numerical integration we implement is trapezoidal rule. We divide the intevals by half until the error is less than 0.001. We set the upper edge of aperature E_1 to be at $(0,0)$. and let lower edge E_2 to be far away from E_1 in order that we can compare only the effect of E_1 instead of E_2. So in this case, we set it to be at $(0,-2000)$.

We consider three cases of source points: $(-100,0)$, $(-100,-50)$ and $(-100,50)$. The phase of diffracted wave with source point $(-100,0)$ is shown in figure 3.5. The line

doi:10.6342/NTU201600792

in the middle of the image is the boundary L_1.

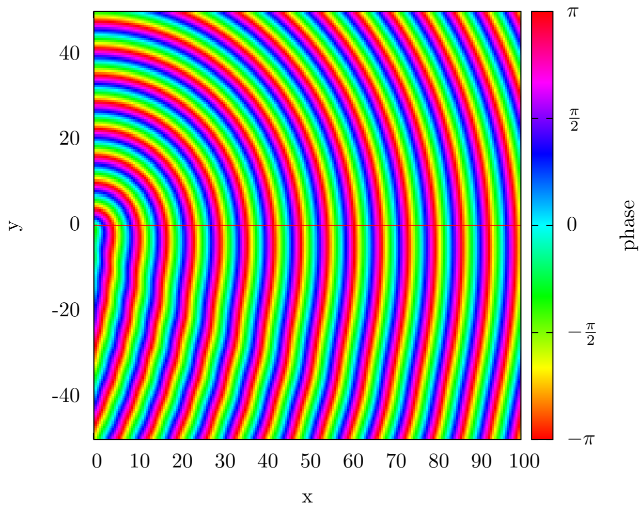

Figure 3.5: The phase of diffraction wave with source at $(-100, 0)$

To compare to (3.34), we subtract the computed phase by the length of path from the source to distination. So in the illuminated region, subtract it by $r_{S\boldsymbol{x}}$ and, in the outter region, by $r_{SE_1} + r_{E_1\boldsymbol{x}}$. We have the results shown in figure 3.6 to figure 3.8 with three different source points.

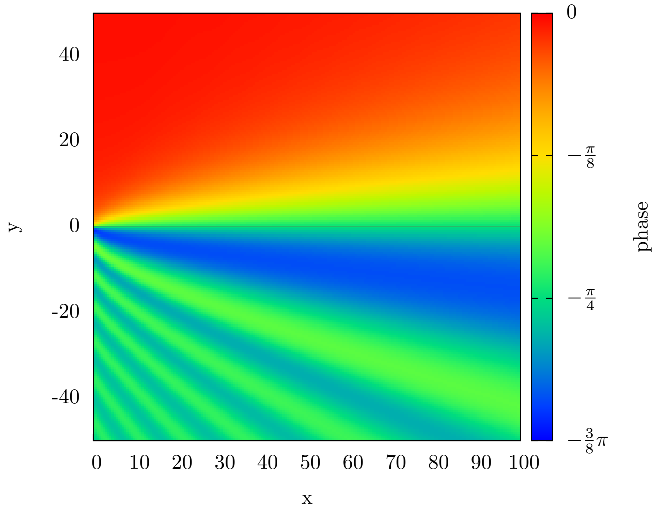

Figure 3.6: Phase shift with source at $(-100, 0)$

We can see that, for the outter region, the difference is approximately 0, and for the inner region, the difference is approximately $-\frac{\pi}{4}$. The difference near the boundary is quite large, but our approximation (3.34) is only valid at the points far from the boundary. So, indeed, there is an approximately $\frac{\pi}{4}$ phase shift.

doi:10.6342/NTU201600792

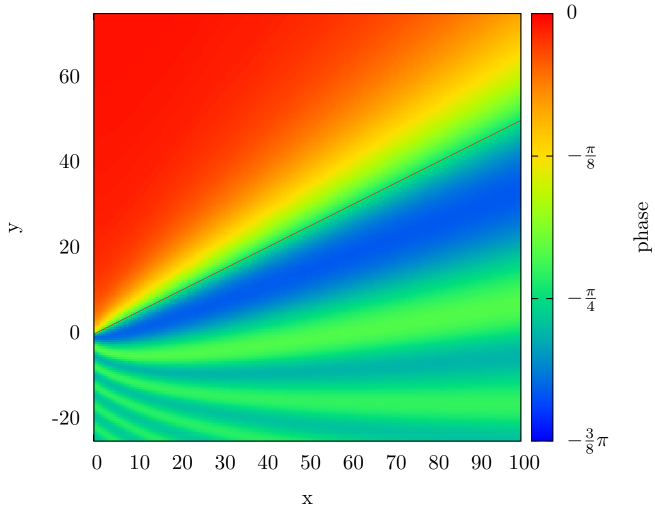

Figure 3.7: Phase shift with source at $(-100, -50)$

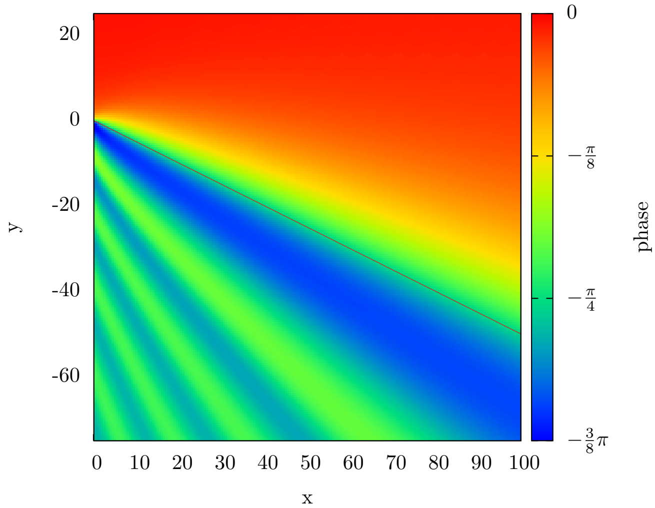

Figure 3.8: Phase shift with source at $(-100, 50)$

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Chapter 4

Uneven Bottom Scattering

In this chapter, we let the depth h of water be a function of $\boldsymbol{x}$ instead of a constant. However, we only deal with those problems that the variation of h is small, so we can use perturbation method to approximate. We use perturbation theory together with Green's function to find the formula of water wave scattering by the bottom.

4.1 Perturbation Method

We assume that the fucntion of the bottom is $z = -h(\boldsymbol{x}) = -h_0 + h_1(\boldsymbol{x})$, where h_0 is a constant. Let $p(\boldsymbol{r}) = p_0(\boldsymbol{r}) + p_1(\boldsymbol{r})$ and $\zeta(\boldsymbol{x}) = \zeta_0(\boldsymbol{x}) + \zeta_1(\boldsymbol{x})$, where p_0 and ζ_0 are the unperturbed terms, i.e. p_0 and ζ_0 are the solutions with $h(\boldsymbol{x}) = -h_0$. p_1 and ζ_1 are the perturbed terms. We must assume that the magnitude of p_1 and ζ_1 are smaller than that of p_0 and ζ_0.

From chapter 2, the differential equation and boundary conditons of p and ζ are

$$\begin{cases} \nabla_{\boldsymbol{r}}^2 p = 0, & \\ p = \left(-\gamma \nabla_{\boldsymbol{x}}^2 + \rho g\right) \zeta, & z = 0, \\ \dfrac{\partial p}{\partial z} = \rho \omega^2 \zeta, & z = 0, \\ \nabla_{\boldsymbol{x}} h \cdot \nabla_{\boldsymbol{x}} p + \dfrac{\partial p}{\partial z} = 0, & z = -h(\boldsymbol{x}). \end{cases} \tag{4.1}$$

But for p_0 , the only difference is the last boundary condtion, i.e.

$$\begin{cases} \nabla_{\boldsymbol{r}}^2 p_0 = 0, & \\ p_0 = \left(-\gamma \nabla_{\boldsymbol{x}}^2 + \rho g\right) \zeta_0, & z = 0, \\ \dfrac{\partial p_0}{\partial z} = \rho \omega^2 \zeta_0, & z = 0, \\ \dfrac{\partial p_0}{\partial z} = 0, & z = -h_0. \end{cases} \tag{4.2}$$

doi:10.6342/NTU201600792

By comparing the two collections of equations above, we can have, for p_1 and ζ_1,

$$\begin{cases} \nabla_{\boldsymbol{r}}^2 p_1 = 0, \\ p_1 = \left(-\gamma \nabla_{\boldsymbol{x}}^2 + \rho g\right) \zeta_1, & z = 0, \\ \dfrac{\partial p_1}{\partial z} = \rho \omega^2 \zeta_1, & z = 0. \\ \dfrac{\partial p_1}{\partial z} \approx \nabla_{\boldsymbol{x}} \cdot \left(h_1 \nabla_{\boldsymbol{x}} p_0\right), & z = -h_0. \end{cases} \tag{4.3}$$

The derivations of the first three equations are trivial but that of the last equation requires some additional assumptions and approximations.

From (4.1), the boundary condition on the bottom is

$$-\nabla_{\boldsymbol{x}} h_1 \cdot \nabla_{\boldsymbol{x}} p_1 + \frac{\partial p_1}{\partial z} = \nabla_{\boldsymbol{x}} h_1 \cdot \nabla_{\boldsymbol{x}} p_0 - \frac{\partial p_0}{\partial z}, \quad \text{at } z = -h_0 + h_1 \tag{4.4}$$

We assume that $|\nabla_{\boldsymbol{x}} h_1| \ll 1$ so, to the lowest order, the left-hand side of (4.4) is $\frac{\partial p_1}{\partial z}\big|_{z=-h_0}$.

For the right-hand side, from the mean value theorem,

$$\nabla_{\boldsymbol{x}} p_0|_{z=-h_0+h_1} = \nabla_{\boldsymbol{x}} p_0|_{z=-h_0} + h_1 \frac{\partial}{\partial z} \nabla_{\boldsymbol{x}} p_0 \bigg|_{z=-h_0+\chi}, \tag{4.5}$$

$$\frac{\partial p_0}{\partial z}\bigg|_{z=-h_0+h_1} = h_1 \frac{\partial^2 p_0}{\partial z^2}\bigg|_{z=-h_0} + \frac{h_1^2}{2} \frac{\partial^3 p_0}{\partial z^3}\bigg|_{z=-h_0+\chi'}, \tag{4.6}$$

where χ and χ' are some values between 0 and h_1. If p_0 is a plane wave or a cylindrical wave, we can write down

$$p_0 = \zeta_0 (\gamma k^2 + \rho g) \frac{\cosh(k(z+h_0))}{\cosh(kh_0)} = p_0|_{z=-h_0} \cosh(k(z+h_0)).$$

We add an assumption $|kh_1| \ll 1$, so the latter terms of the right-hand sides of (4.5) and (4.6) become

$$\begin{aligned} h_1 \frac{\partial}{\partial z} \nabla_{\boldsymbol{x}} p_0 \bigg|_{z=-h_0+\chi} &= h_1 \nabla_{\boldsymbol{x}} p_0|_{z=-h_0} \times k \sinh(k\chi) \\ &\approx (kh_1)^2 \left. \nabla_{\boldsymbol{x}} p_0 \right|_{z=-h_0}, \end{aligned}$$

and

$$\frac{h_1^2}{2} \frac{\partial^3 p_0}{\partial z^3}\bigg|_{z=-h_0+\chi'} \approx \frac{(kh_1)^2}{2} h_1 \frac{\partial^2 p_0}{\partial z^2}\bigg|_{z=-h_0}.$$

They are both of order $O((kh_1)^2)$. Therefore, for both (4.5) and (4.6), the latter terms are much smaller than the former terms in the right-hand sides.

doi:10.6342/NTU201600792

As a result, (4.4) becomes, at $z = -h_0$,

$$\begin{aligned} \frac{\partial p_1}{\partial z} &\approx \nabla_{\boldsymbol{x}} h_1 \cdot \nabla_{\boldsymbol{x}} p_0 - h_1 \frac{\partial^2 p_0}{\partial z^2} \\ &= \nabla_{\boldsymbol{x}} h_1 \cdot \nabla_{\boldsymbol{x}} p_0 + h_1 \nabla_{\boldsymbol{x}}^2 p_0 \\ &= \nabla_{\boldsymbol{x}} \cdot (h_1 \nabla_{\boldsymbol{x}} p) \,, \end{aligned} \tag{4.7}$$

which is the boundary condition (4.3).

With the unperturbed pressure p_0 given, (4.3) lets us find the first order correction. We can further apply Green's function method to the boundary conditions. We use the Green's function $G_2(\boldsymbol{r}, \boldsymbol{r'})$ defined in (3.8), Integrate the Green's idendity (3.2) over the whole space between $z = 0$ and $z = -h_0$ and assume the integral at infinity vanishes, and then we have

$$\begin{aligned} p_1 &= \int_{z'=-h_0} \frac{\partial p_1}{\partial z'} G_2(\boldsymbol{r}, \boldsymbol{r'}) \, d^2\boldsymbol{x'} \\ &= \int_{z'=-h_0} \nabla_{\boldsymbol{x'}} \cdot \big(h_1(\boldsymbol{x'}) \nabla_{\boldsymbol{x'}} p_0(\boldsymbol{r'}) \big) \, G_2(\boldsymbol{r}, \boldsymbol{r'}) \, d^2\boldsymbol{x'}. \end{aligned} \tag{4.8}$$

4.2 Scattering on Periodic Bottom

Now that we have the scattering formula, let's consider a simple problem. Let ζ_0 be an incident plane wave $A\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and thus the incident pressure $p_0 = A(\gamma k^2 + \rho g)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\cosh(k(z+h_0))}{\cosh(kh_0)}$. Let the bottom be $h_1 = B\cos(\boldsymbol{\kappa}\cdot\boldsymbol{x}) = \frac{B}{2}(\mathrm{e}^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}} + \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\boldsymbol{x}})$. Note that, according to the previous section, to make the approximation valid, $Bk \ll 1$ and $B\kappa \ll 1$ should be fulfilled.

From (3.10) and (3.13), we have the Green's function

$$G_2|_{z'=-h_0} = -\frac{1}{(2\pi)^2} \int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x'})} \frac{1}{s} \frac{(\gamma s^2 + \rho g) s \cosh(sz) + \rho\omega^2 \sinh(sz)}{(\gamma s^2 + \rho g) s \sinh(sh_0) - \rho\omega^2 \cosh(sh_0)} \, d^2\boldsymbol{s}.$$

Calculate $\mathrm{e}^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}}$ and $\mathrm{e}^{-i\boldsymbol{\kappa}\cdot\boldsymbol{x}}$ separetely. Substitute the Green's function above into the integral

$$\int_{z'=-h_0} \nabla_{\boldsymbol{x'}} \cdot \big(\mathrm{e}^{i\boldsymbol{\kappa}\cdot\boldsymbol{x'}} \nabla_{\boldsymbol{x'}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x'}} \big) \, G_2(\boldsymbol{r}, \boldsymbol{r'}) \, d^2\boldsymbol{x'}, \tag{4.9}$$

and integrate $\boldsymbol{x'}$ prior to integrating $\boldsymbol{s}$. There will be an integral that contains $\int \mathrm{e}^{i(\boldsymbol{\kappa}+\boldsymbol{k}-\boldsymbol{s})\cdot\boldsymbol{x'}} d^2\boldsymbol{x'}$, which has the result $(2\pi)^2\,\delta^2(\boldsymbol{\kappa} + \boldsymbol{k} - \boldsymbol{s})$. We can thus obtain that (4.9) is

$$\begin{aligned} &(\boldsymbol{k}\cdot(\boldsymbol{k}+\boldsymbol{\kappa})) \int \delta(\boldsymbol{\kappa}+\boldsymbol{k}-\boldsymbol{s}) \mathrm{e}^{i\boldsymbol{s}\cdot\boldsymbol{x}} \frac{1}{s} \frac{(\gamma s^2 + \rho g) s \cosh(sz) + \rho\omega^2 \sinh(sz)}{(\gamma s^2 + \rho g) s \sinh(sh_0) - \rho\omega^2 \cosh(sh_0)} \, d^2\boldsymbol{s} \\ &= (\boldsymbol{k}\cdot\boldsymbol{s}) \frac{1}{s} \frac{(\gamma s^2 + \rho g) s \cosh(sz) + \rho\omega^2 \sinh(sz)}{(\gamma s^2 + \rho g) s \sinh(sh_0) - \rho\omega^2 \cosh(sh_0)} \mathrm{e}^{i\boldsymbol{s}\cdot\boldsymbol{x}} \bigg|_{\boldsymbol{s}=\boldsymbol{k}+\boldsymbol{\kappa}} . \end{aligned}$$

doi:10.6342/NTU201600792

Finally, the perturbed pressure is

$$p_1 = \frac{AB}{2}\frac{\gamma k^2+\rho g}{\cosh(kh_0)} \times \left((\boldsymbol{k}\cdot\boldsymbol{s})\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\mathrm{e}^{i\boldsymbol{s}\cdot\boldsymbol{x}} \right)\Bigg|_{\boldsymbol{s}=\boldsymbol{k}+\boldsymbol{\kappa}} + \Bigg|_{\boldsymbol{s}=\boldsymbol{k}-\boldsymbol{\kappa}}, \tag{4.10}$$

and the surface is

$$\begin{aligned}\zeta_1 =& \frac{1}{\rho\omega^2}\frac{\partial p_1}{\partial z}\bigg|_{z=0} \\ =& \frac{AB}{2k\sinh(kh_0)} \\ &\times \left((\boldsymbol{k}\cdot\boldsymbol{s})\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\mathrm{e}^{i\boldsymbol{s}\cdot\boldsymbol{x}} \right)\Bigg|_{\boldsymbol{s}=\boldsymbol{k}+\boldsymbol{\kappa}} + \Bigg|_{\boldsymbol{s}=\boldsymbol{k}-\boldsymbol{\kappa}}.\end{aligned} \tag{4.11}$$

We can see that the first order correction of scattering wave has wave vector $\boldsymbol{k}\pm\boldsymbol{\kappa}$. This is called the class I Bragg condition [7]. In cases that $|\boldsymbol{k}+\boldsymbol{\kappa}|=k$ or $|\boldsymbol{k}-\boldsymbol{\kappa}|=k$, the denominator becomes 0 and so the amplitude tends to infinite large. This is called the class I Bragg resonance [7]. The value of the amplitude becomes unrealistic infinity because, in this case, the area of rippled bottom is infinite. If the rippled shape only occupies finite area of bottom, it can be calculated using the method introduced in next section and should have finite amplitude.

4.3 Scattering on Any Topography of Bottom

In this section, we are going to find the general formula of scattering whatever the form of h_1 is. The formula will be compared to the other method discussed in appendix B and we will find that they are consistent.

To make a formula that works for any h_1, substitute the Green's function (A.5) in appendix A into (4.8), and we can get

$$p_1 = \frac{i}{2}\int \nabla_{\boldsymbol{x}'}\cdot\left(h_1\nabla_{\boldsymbol{x}'}p_0|_{z'=-h_0}\right)\sum_{j=0}^{\infty}\frac{1}{k_j}\frac{dk_j}{dh_0}H_0^{(1)}(k_j|\boldsymbol{x}-\boldsymbol{x}'|)\cosh\left(k_j(z+h_0)\right)d^2\boldsymbol{x}',$$

where k_js are wave numbers that satisfy the dispersion relation (3.6). There are infinite k_js because most of them are imaginary numbers, except for two values: $\pm k$. Let k_0 be the only real positive value, k. If $h_1(\boldsymbol{x})$ decays faster than $O(r^{-\frac{1}{2}})$, by integration by parts,

doi:10.6342/NTU201600792

we have

$$p_1 = -\frac{i}{2}\int h_1 \nabla_{\boldsymbol{x}'} p_0|_{z'=-h_0} \cdot \sum_{j=0}^{\infty} \frac{1}{k_j}\frac{dk_j}{dh_0} \nabla_{\boldsymbol{x}'} H_0^{(1)}(k_j|\boldsymbol{x}-\boldsymbol{x}'|)\cosh\big(k_j(z+h_0)\big)\, d^2\boldsymbol{x}'. \tag{4.12}$$

By replacing p_0 with $\zeta_0(\gamma k^2+\rho g)\frac{\cosh(k(z+h_0))}{\cosh(kh_0)}$ and applying the relation $\zeta_1 = \frac{1}{\rho\omega^2}\frac{\partial p_1}{\partial z}\big|_{z=-h_0}$, the surface scattering wave appears to be

$$\zeta_1 = \frac{-i}{2k\sinh(kh_0)}\int h_1 \nabla_{\boldsymbol{x}'}\zeta_0 \cdot \sum_{j=0}^{\infty}\frac{dk_j}{dh_0}\nabla_{\boldsymbol{x}'} H_0^{(1)}(k_j|\boldsymbol{x}-\boldsymbol{x}'|)\sinh(k_j h_0)\, d^2\boldsymbol{x}'.$$

If the distance of $\boldsymbol{x}$ and $\boldsymbol{x}'$ are very far such that $|k_j(\boldsymbol{x}-\boldsymbol{x}')| \gg 1$ for any j, all terms but $j=0$ decay faster than exponential decay. Thus, only the term of k_0 remains:

$$p_1 = -\frac{i}{2k}\frac{dk}{dh_0}\cosh\big(k(z+h_0)\big)\int h_1\nabla_{\boldsymbol{x}'}p_0|_{z'=-h_0}\cdot\nabla_{\boldsymbol{x}'}H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|)\, d^2\boldsymbol{x}'.$$

And the suface is

$$\zeta_1 = -\frac{i}{2k}\frac{dk}{dh_0}\int h_1\nabla_{\boldsymbol{x}'}\zeta_0\cdot\nabla_{\boldsymbol{x}'}H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|)\, d^2\boldsymbol{x}'. \tag{4.13}$$

So far, we have converted a three-dimensional water wave scattering to a two-dimensional integral, which is much easier to calculate and numerically compute. And more importantly, its form is neat and simple.

We can compare our result (4.12) with the appendix B. In appendix B, we put a lot of effort using matching method to find scattering of water wave over a pillbox-shaped obstacle on uniform bottom. Under the assumptions that the incident wave is a plane wave $p_0 = A\mathrm{e}^{ikx}\cosh(k(z+h_0))$ and that the radius δR and height δh of the pillbox are small, the result is shown in (B.22). Here, we can use the formula developed in this section to this problem. From (4.12), by setting $h_1(\boldsymbol{x}) = \delta h$ in the region $|\boldsymbol{x}| \le \delta R$, we can get

$$\begin{aligned} p_1 &\approx -\frac{i}{2}A(\pi\delta R^2)\delta h\left(\nabla_{\boldsymbol{x}'}\mathrm{e}^{ikx'}\cdot\sum_{j=0}^{\infty}\frac{1}{k_j}\frac{dk_j}{dh_0}\nabla_{\boldsymbol{x}'}H_0^{(1)}(k_j|\boldsymbol{x}-\boldsymbol{x}'|)\cosh\big(k_j(z+h_0)\big)\right)\Bigg|_{\boldsymbol{x}'=0} \\ &\approx -\frac{i}{2}A(\pi\delta R^2)\delta h \\ &\quad\times\sum_{j=0}^{\infty}\frac{1}{k_j}\frac{dk_j}{dh_0}\left(\mathrm{e}^{ikx'}ik\hat{x}\cdot\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}k_jH_1^{(1)}(k_j|\boldsymbol{x}-\boldsymbol{x}'|)\right)\Bigg|_{\boldsymbol{x}'=0}\cosh(k_j(z+h_0)). \end{aligned}$$

Let the symbol φ denote the angle between vectors $\boldsymbol{x}-\boldsymbol{s}'$ and $\hat{x}$. And then we have

$$p_1 \approx \frac{A\pi k}{2}(\delta R)^2\delta h\sum_{j=0}^{\infty}\frac{dk_j}{dh_0}H_1^{(1)}(k_j|\boldsymbol{x}|)\cosh(k_j(z+h_0))\cos\varphi,$$

doi:10.6342/NTU201600792

which is exactly same as the scattered wave in (B.22). Therefore, the two methods are consistent.

Another thing worth mentioning is that we can find the term $\frac{dk_j}{dh_0}$ in a simple physical system, which is why we didn't expand this term explicitly. Assume that we now deal with the problem of a vibrating membrane. The displacement of the membrane follows Helmholtz equation $(\nabla^2 + k^2)\Psi = 0$. However, assume that on some parts of the membrane, k is not a constant but variates with position. The variation is little, so we let $k = k_0 + \delta k(\boldsymbol{x})$ in those regions with k_0 a constant, as shown in figure 4.1.

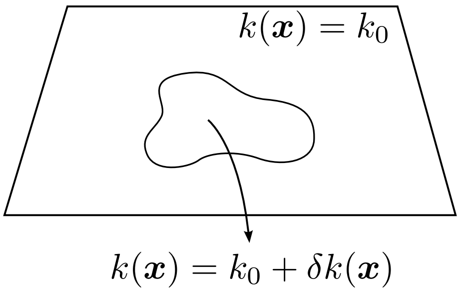

Figure 4.1: The two-dimensional membrane

We apply the perturbation method by setting $\Psi = \Psi_0 + \delta\Psi$, and thus the equations $(\nabla^2 + k_0^2)\Psi_0 = 0$ and $(\nabla^2 + (k_0 + \delta k)^2)\Psi$ should be satisfied. To lowest order, $\delta\Psi$ follows the equation

$$\left(\nabla^2 + k_0^2\right)\delta\Psi = -2k_0\,\delta k\Psi_0.$$

If Ψ_0 is given, $\delta\Psi$ can be solved by Green's function method. The Green's function of Helmholtz equation is $-\frac{i}{4}H_0^{(1)}(k_0|\boldsymbol{x} - \boldsymbol{x}'|)$, so

$$\delta\Psi = \frac{i}{2}\int k_0\,\delta k\Psi_0 H_0^{(1)}(k_0|\boldsymbol{x} - \boldsymbol{x}'|)d^2\boldsymbol{x}'.$$

After substituting $k_0^2\Psi_0 = -\nabla^2\Psi$ and integration by parts, it turns into

$$\delta\Psi = \frac{i}{2}\int \nabla'\Psi_0 \cdot \nabla'\left(\frac{\delta k}{k_0}H_0^{(1)}(k_0|\boldsymbol{x} - \boldsymbol{x}'|)\right)d^2\boldsymbol{x}'.$$

If the slope of δk is small so that $|\nabla\delta k| \ll k_0^2$, then we have

$$\delta\Psi = \frac{i}{2}\int \frac{\delta k}{k_0}\nabla'\Psi_0 \cdot \nabla' H_0^{(1)}(k_0|\boldsymbol{x} - \boldsymbol{x}'|)d^2\boldsymbol{x}'. \tag{4.14}$$

This result looks very similar to the equation (4.13). In fact, we can rewrite δk and show that

$$\delta k = \frac{dk}{dh}\Delta h = -\frac{dk}{dh_0}h_1.$$

And thus we can see that (4.13) is equivalent to (4.14).

doi:10.6342/NTU201600792

Chapter 5

Conclusion

In this thesis, we discussed diffraction and scattering of water surface wave. For diffraction, we assumed that the bottom of water is flat and we used Green's function method to find the formula of diffraction, which is similar to the derivation of Kirchhoff's diffraction formula in optics. We considered the problem of slit diffraction with the walls straight. By applying the tricks of Rubinowicz, we showed that there is a phase shift $\frac{\pi}{4}$ in diffraction region. And we numerically computed the diffraction wave and visually demonstrated the phase shift.

For scattering, we adopted perturbation method combined with Green's function method to obtain the formula of scattering. We calculated rippled bottom scattering and got the so called class I Bragg condition. We also derive the scattering formula for any topology of bottom. In the end, we used other method like matching method to compute a pillbox shaped bottom and showed that the two methods are equivalent.

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

Appendix A

Expansion of Green's function

In this appendix, we are going to expand the Green's function G_2, which is (3.10) together with (3.13), to infinite series. The first expansion of Green's function of water wave is derived by John, 1950 [6]. Here, our calculation includes surface tension.

First, we can rewrite the $\widetilde{G}_2$. For $z' < z$, the step function $H(z'-z)$ is 0, so

$$\widetilde{G}_2 = -\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\cosh(s(z'+h_0)).$$

And for $z' > z$, after a little calculations, we can find that

$$\begin{aligned}\widetilde{G}_2 &= \frac{1}{s}\left[\sinh(s(z'-z)) - \frac{(\gamma s^2+\rho g)s\cosh(sz)+\rho\omega^2\sinh(sz)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\cosh(s(z'+h_0))\right] \\ &= -\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz')+\rho\omega^2\sinh(sz')}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\cosh(s(z+h_0)).\end{aligned}$$

The above two expressions can be combined and become the expression below

$$\widetilde{G}_2 = -\frac{1}{s}\frac{(\gamma s^2+\rho g)s\cosh(sz_>)+\rho\omega^2\sinh(sz_>)}{(\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0)}\cosh(s(z_<+h_0)), \tag{A.1}$$

where $z_<$ denotes the smaller of z and z' while $z_>$ the larger one.

Next, we can expand the Green's function by its partial fraction expansion. Let

$$f(s) = (\gamma s^2+\rho g)s\sinh(sh_0)-\rho\omega^2\cosh(sh_0), \text{ and} \tag{A.2}$$

$$g(s) = \big((\gamma s^2+\rho g)s\cosh(sz_>)+\rho\omega^2\sinh(sz_>)\big)\cosh(s(z_<+h_0)). \tag{A.3}$$

Because $|z_>|+|z_<+h_0| < |h_0|$, $f(s)$ grows faster than $g(s)$ when s goes to infinity. And it can be proven that $\oint_C \frac{g(s)}{f(s)}\,ds < \infty$, for some closed loops C in complex plane that tends

doi:10.6342/NTU201600792

to the infinity. Therefore, $\frac{g(s)}{f(s)}$ can be expanded as

$$\frac{g(s)}{f(s)} = \sum_j \mathrm{PP}\left(\frac{g(s)}{f(s)}; s = k_j\right), \tag{A.4}$$

where PP $(F(z); z = z_0)$ denotes the principal part of $F(z)$ at $z = z_0$ and k_js are the poles of $\frac{g(s)}{f(s)}$ [8].

Because $g(s)$ has no sigularity, k_js are the roots of $f(s)$. $f(s)$ has one positive real root k, which statifies the dispersion relation (3.6). In addition, $f(s)$ also has infinite imaginary roots, which can be easily seen by setting $s = i\sigma$ and thus

$$f(i\sigma) = (\gamma\sigma^2 - \rho g)\sigma\sin(\sigma h_0) - \rho\omega^2\cos(\sigma h_0).$$

$\frac{g(s)}{f(s)}$ only has simple poles because the roots of $f(s)$ are all simple. Therefore, (A.4) turns into

$$\frac{g(s)}{f(s)} = \sum_j \frac{c_j}{s - k_j}, \quad \text{with } c_j = \frac{1}{2\pi i}\oint \frac{g(s)}{f(s)}\, ds.$$

As $f(s)$ is an even function, $-k_j$ is also a root of it. But $g(s)$ is an odd function, so $\frac{g(s)}{f(s)}$ is also an odd function and it can be rewritten as

$$\frac{g(s)}{f(s)} = \sum_{j=0}^{\infty} \frac{c_j}{s - k_j} + \frac{c_j}{s + k_j} = \sum_{j=0}^{\infty} \frac{2sc_j}{s^2 - k_j^2},$$

where the summation sums over all root pairs rather than every single root. Here, we define k_0 as the only positive real root, k, and the other k_js as the imaginary roots.

Using Cauchy's integral formula, we can find that

$$c_j = \frac{1}{2\pi i}\oint \frac{\frac{g(s)}{f(s)}(s - k_j)}{s - k_j} ds = \lim_{s\to k_j} \frac{g(s)}{f(s)}(s - k_j) = \frac{g(k_j)}{f'(k_j)}.$$

And thus we have

$$\widetilde{G}_2 = -\frac{1}{s}\sum_j \frac{g(k_j)}{f'(k_j)}\frac{2s}{s^2 - k_j^2}.$$

If we regard h_0 as a variable and define a function $f(s, h_0)$ that has the same expression as $f(s)$, then, by direct computation, we can see that

$$\begin{aligned} g(k_j) &= \left[(\gamma k_j^2 + \rho g)k_j\cosh(k_j h_0) - \rho\omega^2\sinh(k_j h_0)\right]\cosh(k_j(z + h_0))\cosh(k_j(z' + h_0)) \\ &= \frac{1}{k_j}\frac{\partial f(k_j, h_0)}{\partial h_0}\cosh(k_j(z + h_0))\cosh(k_j(z' + h_0)). \end{aligned}$$

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As a result, it leads to

$$\widetilde{G}_2 = -\sum_{j=0}^{\infty} \frac{1}{k_j} \frac{\frac{\partial f(k_j,h_0)}{\partial h_0}}{\frac{\partial f(k_j,h_0)}{\partial k_j}} \frac{2}{s^2 - k_j^2} \cosh(k_j(z' + h_0)) \cosh(k_j(z + h_0))$$
$$= \sum_{j=0}^{\infty} \frac{1}{k_j} \frac{dk_j}{dh_0} \frac{2}{s^2 - k_j^2} \cosh(k_j(z' + h_0)) \cosh(k_j(z + h_0)).$$

Finally, by the integral equation (3.24), we can obtain

$$\begin{aligned} G_2 &= \frac{1}{(2\pi)^2} \int \mathrm{e}^{i\boldsymbol{s}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \widetilde{G}_2 \, d^2\boldsymbol{s} \\ &= \frac{i}{2} \sum_{j=0}^{\infty} \frac{1}{k_j} \frac{dk_j}{dh_0} H_0^{(1)}(k_j|\boldsymbol{x} - \boldsymbol{x}'|) \cosh(k_j(z + h_0)) \cosh(k_j(z' + h_0)). \end{aligned} \tag{A.5}$$

For the case that $|\boldsymbol{x} - \boldsymbol{x}'|$ is very large, the term of k_0 becomes the dominant term becasue $H_0^{(1)}(i\sigma) \propto K_0(\sigma)$. $K_0(z)$ is the modified Bessel function of first kind, which decays faster than exponential function on real axis. Consequently, it can be approximated that

$$G_2 \approx \frac{i}{2k} \frac{dk}{dh_0} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{x}'|) \cosh(k(z + h_0)) \cosh(k(z' + h_0)), \tag{A.6}$$

for $k|\boldsymbol{x} - \boldsymbol{x}'| \gg 1$.

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

Appendix B

Pillbox Scattering

In this appendix, we consider the scattering problem of water wave over a pillbox on a flat bottom. We will use matching method to derive the scattered wave. However, it will be extremely difficult to exactly find the solution, so we will make some approximations.

The problem is defined as shown in figure B.1. A pillbox-shaped obstacle is lying in a water with constant depth h. The pillbox has radius R and the height of water above it is $\widehat{h}$. The space of water body can be split into two partitions: one is a cylinder right above

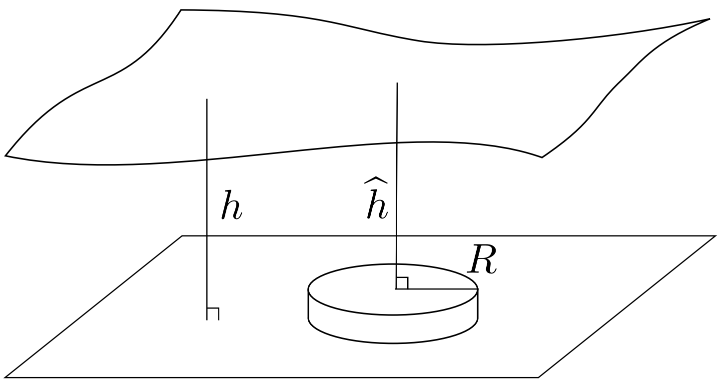

Figure B.1: Pillbox scattering

the pillbox and the other is the space outside the cylinder. Each partition of the water have the governing equation and boundary conditions described in (4.1), with $h(\boldsymbol{x}) = -h$ for outside and $h(\boldsymbol{x}) = -\widehat{h}$ for inside. If we adopt the cylindrical coordinate (ϱ, φ, z), then, on $\varrho = R$, where the inside and outside connect, we have these additional boundary conditions

$$\begin{cases} p\big|_{\varrho=R^+} = p\big|_{\varrho=R^-}, \quad \left.\dfrac{\partial p}{\partial \varrho}\right|_{\varrho=R^+} = \left.\dfrac{\partial p}{\partial \varrho}\right|_{\varrho=R^-}, & -\widehat{h} < z < 0, \\ \left.\dfrac{\partial p}{\partial \varrho}\right|_{\varrho=R^+} = 0, & -h < z < -\widehat{h}. \end{cases} \tag{B.1}$$

doi:10.6342/NTU201600792

We use separation of variable to solve the expressions of pressure of inside and outside cylinder respective and adopt matching method to connect them. Let $X(\boldsymbol{x})Z(z)$ be a solution of (4.1) with $h(\boldsymbol{x}) = h = \text{const}$. Substituting it into (4.1), we can get the eigenvalue problems $\nabla_{\boldsymbol{x}}^2 X_i = -k_i^2 X_i$ and $Z_i'' = k_i^2 Z_i$ and the boundary condtiions for Z_i

$$\begin{cases} \left(\gamma k_i^2 + \rho g\right) Z_i' = \rho\omega^2 Z_i, & z = 0, \\ Z_i' = 0, & z = -h. \end{cases} \tag{B.2}$$

Therefore, $Z_i \propto \cosh(k_i(z+h))$ with k_i satisfying its dispersion relation.

The boundary conditions contain eigenvalues so the Sterm-Liouville theorem cannot be applied to these eigenfunctions. Fortunely, this paper [11] finds that these kinds of eigenfunctions still have orthogonality by defining a new inner product. In this case, the inner product is difined as

$$\left\langle f(z), g(z) \right\rangle_h \equiv \int_{-h}^{0} f(z)g(z)dz + \frac{\gamma}{\rho\omega^2} f'(0)g'(0). \tag{B.3}$$

We can see the orthogonality by doing the calculations

$$\begin{aligned} (k_i^2 - k_j^2) \int_{-h}^{0} Z_i Z_j dz &= \int_{-h}^{0} \left(Z_i'' Z_j - Z_i Z_j''\right) dz \\ &= Z_i'(0)Z_j(0) - Z_i(0)Z_j'(0) \\ &= -\frac{\gamma}{\rho\omega^2}(k_i^2 - k_j^2) Z_i'(0) Z_j'(0). \end{aligned}$$

Adding the right-hand side back to the left-hand side, we can see that $\left\langle Z_i, Z_j \right\rangle = 0$ for $k_i \neq k_j$. But for $i = j$, $\left\langle Z_i, Z_i \right\rangle > 0$ is always positive. Therefore, we can define the orthonormal bases $\phi_i(z)$ and $\widehat{\phi}_i(z)$ and their corresponding inner products $\left\langle \cdot, \cdot \right\rangle_h$ and $\left\langle \cdot, \cdot \right\rangle_{\widehat{h}}$ respectively.

By further separation of variable, $X(\boldsymbol{x})$ can be expanded by products of Bessel functions and trigonometric functions Because p is the linear combination of the all possible solutions $X(\boldsymbol{x})Z(z)$, we can set p as

$$p(\varrho, \varphi, z) = \sum_{n=-\infty}^{\infty} P_n(\varrho, z) \mathrm{e}^{in\varphi},$$

while

$$P_n(\varrho, z) = A_n J_n(k\varrho)\phi_0(z) + \sum_{i=0}^{\infty} a_{ni} H_n^{(1)}(k_i \varrho)\phi_i(z), \quad \varrho > R, \tag{B.4}$$

$$P_n(\varrho, z) = \sum_{i=0}^{\infty} b_{ni} J_n(\widehat{k}_i \varrho)\widehat{\phi}_i(z), \qquad \varrho < R. \tag{B.5}$$

doi:10.6342/NTU201600792

Here, we assume that A_n is the coefficient of incident wave and $k_0 = k$ is the positive real eigenvalue.

Next, we need to find the coefficients a_{ni} and b_{ni}. The procedure below is inspired by [3]. Let $U_n(z) \equiv \frac{\partial P_n}{\partial \varrho}\Big|_{\varrho=R}$. Form the boundary condition (B.1), we have

$$U_n(z) = A_n k J_n'(kR)\phi_0(z) + \sum_{i=0}^{\infty} a_{ni} k_i H_n^{(1)\prime}(k_i R)\phi_i(z),$$
$$U_n(z) = \sum_{i=0}^{\infty} b_{ni}\widehat{k}_i J_n'(\widehat{k}_i R)\widehat{\phi}_i(z).$$

Still form (B.1), $U_n(z) = 0$ when $-h < z < -\widehat{h}$ and therefore $\langle \cdot, U_n\rangle_h = \langle \cdot, U_n\rangle_{\widehat{h}}$. After applying inner product on both sides,

$$\langle \phi_j, U_n\rangle_{\widehat{h}} = \langle \phi_j, U_n\rangle_h = A_n k J_n'(kR)\delta_{j0} + a_{nj}k_j H_n^{(1)\prime}(k_j R), \tag{B.6}$$
$$\langle \widehat{\phi}_j, U_n\rangle_{\widehat{h}} = b_{nj}\widehat{k}_j J_n'(\widehat{k}_j R). \tag{B.7}$$

If we expand U_n in basis $\widehat{\phi}_i$, that is $U_n = \sum_{i=0}^{\infty} c_{ni}\widehat{\phi}_i$, it can be seen that $\langle \phi_j, U_n\rangle_{\widehat{h}} = \sum_{i=0}^{\infty} c_{ni}\langle \phi_j, \widehat{\phi}_i\rangle_{\widehat{h}}$ and $\langle \widehat{\phi}_j, U_n\rangle_{\widehat{h}} = c_{nj}$, so $\langle \phi_j, U_n\rangle_{\widehat{h}} = \sum_{i=0}^{\infty}\langle \widehat{\phi}_i, U_n\rangle_{\widehat{h}}\langle \phi_j, \widehat{\phi}_i\rangle_{\widehat{h}}$. By substituting (B.6) and (B.7) into this equation, the relation of a_{nj} and b_{nj} comes out:

$$A_n k J_n'(kR)\delta_{j0} + a_{nj}k_j H_n^{(1)\prime}(k_j R) = \sum_{i=0}^{\infty} b_{ni}\widehat{k}_i J_n'(\widehat{k}_i R)\langle \phi_j, \widehat{\phi}_i\rangle_{\widehat{h}}. \tag{B.8}$$

Apply $\langle \widehat{\phi}_j, \cdot\rangle_{\widehat{h}}$ on both sides of $p|_{\varrho=R^+} = p|_{\varrho=R^-}$ from (B.1). And another relation of a_{ni} and b_{ni} appears:

$$A_n J_n(kR)\langle \widehat{\phi}_j, \phi_0\rangle_{\widehat{h}} + \sum_{i=0}^{\infty} a_{ni}H_n^{(1)}(k_i R)\langle \widehat{\phi}_j, \phi_i\rangle_{\widehat{h}} = b_{nj}J_n(\widehat{k}_j R). \tag{B.9}$$

Eliminate b_{ni} by combining (B.8) and (B.9). We finally obtain the equation of coffecient a_{ni}:

$$\begin{aligned} a_{ni}k_i H_n^{(1)\prime}(k_i R) - \sum_{j=0}^{\infty} a_{nj}H_n^{(1)}(k_j R)\left(\sum_{l=0}^{\infty}\frac{\widehat{k}_l J_n'(\widehat{k}_l R)}{J_n(\widehat{k}_l R)}\langle \phi_i, \widehat{\phi}_l\rangle_{\widehat{h}}\cdot\langle \widehat{\phi}_l, \phi_j\rangle_{\widehat{h}}\right) \\ = A_n J_n(kR)\sum_{l=0}^{\infty}\left(\frac{\widehat{k}_l J_n'(\widehat{k}_l R)}{J_n(\widehat{k}_l R)}\langle \phi_i, \widehat{\phi}_l\rangle_{\widehat{h}}\cdot\langle \widehat{\phi}_l, \phi_0\rangle_{\widehat{h}}\right) - A_n k J_n'(kR)\delta_{i0} \end{aligned} \tag{B.10}$$

The values of the coefficients can only be computed numerically. We are not going to do this so we make two approximations below. First, we let $h - \widehat{h} \equiv \delta h$ to be a small length. We can see that, if $\delta h = 0$, then $\langle \phi_i, \widehat{\phi}_j\rangle_{\widehat{h}} = \delta_{ij}$ and $a_{ni} = 0$, so a_{ni} is of order

doi:10.6342/NTU201600792

$O(\delta h)$. In addition, define M_{ij} such that

$$\langle \phi_i, \widehat{\phi}_j \rangle_{\widehat{h}} = \delta_{ij} + M_{ij}\delta h + O(\delta h^2). \tag{B.11}$$

Thus, to the lowest order of δh, (B.10) turns into

$$a_{ni} = \frac{A_n J_n(kR)}{H_n^{(1)}(k_i R)} \frac{\left(\frac{\widehat{k}J_n'(\widehat{h}R)}{J_n(\widehat{k}R)} - \frac{kJ_n'(kR)}{J_n(kR)}\right)\delta_{i0} + \left(\frac{\widehat{h}_i J_n'(\widehat{k}_i R)}{J_n(\widehat{k}_i R)} M_{0i} + \frac{\widehat{k}J_n'(\widehat{k}R)}{J_n(\widehat{k}R)} M_{i0}\right)\delta h}{\frac{k_i H_n^{(1)\prime}(k_i R)}{H_n^{(1)}(k_i R)} - \frac{\widehat{k}_i J_n'(\widehat{k}_i R)}{J_n(\widehat{k}_i R)}}. \tag{B.12}$$

The next assumption we would like to make is that R is small such that $|k_i R| \ll 1$. We set $R = \delta R$ in order to reminder ourself that it is a small value. Consider the fact that, for $n \neq 0$, $J_n(z) \propto z^n$ and $H_n^{(1)}(z) \propto z^{-n}$ when $z \ll 1$, so $\frac{J_n'(z)}{J_n(z)} \approx \frac{n}{z}$ and $\frac{H_n^{(1)\prime}(z)}{H_n^{(1)}(z)} \approx -\frac{n}{z}$. Therefore, (B.12) can be approximated as

$$a_{ni} \approx -\frac{1}{2} \frac{A_n J_n(k\delta R)}{H_n^{(1)}(k_i \delta R)} \left(M_{i0} + M_{0i}\right)\delta h, \quad n \neq 0. \tag{B.13}$$

However, for $n = 0$, $J_0(z) \propto 1 - \left(\frac{z}{2}\right)^2$ and $H_0^{(1)}(z) \propto \ln(z)$ when $z \ll 1$, and thus $\frac{J_0'(z)}{J_0(z)} \approx -\frac{z}{2}$ and $\frac{H_0^{(1)\prime}(z)}{H_0^{(1)}(z)} \approx \frac{1}{z\ln(z)}$. In this case,

$$a_{00} \approx \frac{A_0 J_0(k\delta R)}{H_0^{(1)}(k\delta R)} \delta R^2 \ln(k_0 \delta R) \left(\frac{1}{2}\frac{d(k^2)}{dh} - k^2 M_{00}\right)\delta h, \tag{B.14}$$

$$a_{0i} \approx -\frac{A_0 J_0(k\delta R)}{H_0^{(1)}(k_i \delta R)} \delta R^2 \ln(k_i \delta R) \frac{1}{2}\left(k_i^2 M_{0i} + k^2 M_{i0}\right)\delta h, \quad i \neq 0. \tag{B.15}$$

Now, we have the approximations of a_{ij}. The only thing that is left to be found is M_{ij}. From (B.11), we can calculate M_{ij} by

$$M_{ij} = -\left.\frac{\partial \langle \phi_i, \widehat{\phi}_j \rangle_{\widehat{h}}}{\partial \widehat{h}}\right|_{\widehat{h}=h}. \tag{B.16}$$

Let $\psi_i(z)$ be $c\phi_i(z)$, where c is an arbitrary constant. Likewise, $\widehat{\psi}_i$ a function proportional

doi:10.6342/NTU201600792

to $\widehat{\phi}_i$. The inner product of the two functions is calculated as follows:

$$\begin{aligned}
\langle \psi_i, \widehat{\psi}_j \rangle_{\widehat{h}} &= \int_{-\widehat{h}}^{0} \psi_i \widehat{\psi}_j \, dz + \frac{\gamma}{\rho\omega^2} \psi_i'(0) \widehat{\psi}_j'(0) \\
&= \frac{1}{k_i^2 - \widehat{k}_j^2} \int_{-\widehat{h}}^{0} \left(\psi_i' \widehat{\psi}_j - \psi_i \widehat{\psi}_j' \right)' dz + \frac{\gamma}{\rho\omega^2} \psi_i'(0) \widehat{\psi}_j'(0) \\
&= \frac{1}{k_i^2 - \widehat{k}_j^2} \left(\frac{\gamma}{\rho\omega^2} (\widehat{k}_j^2 - k_i^2) \psi_i'(0) \widehat{\psi}_j'(0) - \psi_i'(-\widehat{h}) \widehat{\psi}_j(-\widehat{h}) \right) + \frac{\gamma}{\rho\omega^2} \psi_i'(0) \widehat{\psi}_j'(0) \\
&= -\frac{1}{k_i^2 - \widehat{k}_j^2} \psi_i'(-\widehat{h}) \widehat{\psi}_j(-\widehat{h}).
\end{aligned} \tag{B.17}$$

For $i \neq j$, form (B.16) and (B.17),

$$\begin{aligned}
M_{ij} &= -\left. \frac{\partial \langle \phi_i, \widehat{\phi}_j \rangle_{\widehat{h}}}{\partial \widehat{h}} \right|_{\widehat{h}=h} \\
&= \frac{\partial}{\partial \widehat{h}} \left. \left(\frac{1}{k_i^2 - \widehat{k}_j^2} \phi_i'(-\widehat{h}) \widehat{\phi}_j(-\widehat{h}) \right) \right|_{\widehat{h}=h} \\
&= -\frac{1}{k_i^2 - k_j^2} \left(\phi_i''(-h) \widehat{\phi}_j(-h) \right) \\
&= -\frac{k_i^2}{k_i^2 - k_j^2} \phi_i(-h) \phi_j(-h).
\end{aligned}$$

Therefore, $M_{i0} + M_{0i} = -\phi_0(-h)\phi_i(-h)$ and $k_i^2 M_{ji} + k_j^2 M_{ij} = 0$. From (B.13), we have

$$a_{ni} \approx \frac{1}{2} \frac{A_n J_n(k\delta R)}{H_n^{(1)}(k_i \delta R)} \phi_0(-h) \phi_i(-h) \, \delta h, \quad n \neq 0, \tag{B.18}$$

and from (B.14),

$$a_{0i} = 0, \quad \text{for } i \neq 0.$$

For M_{ii}, because $\phi_i(z) = \frac{\psi_i(z)}{\sqrt{\langle \psi_i, \psi_i \rangle}}$, (B.16) can be rewritten as

$$\begin{aligned}
M_{ii} &= -\frac{\partial}{\partial \widehat{h}} \left. \left(\frac{\langle \psi_i, \widehat{\psi}_i \rangle_{\widehat{h}}}{\sqrt{\langle \psi_i, \psi_i \rangle_h} \sqrt{\langle \widehat{\psi}_i, \widehat{\psi}_i \rangle_{\widehat{h}}}} \right) \right|_{\widehat{h}=h} \\
&= -\frac{1}{\langle \psi_i, \psi_i \rangle_h} \left. \left(\frac{\partial}{\partial \widehat{h}} \langle \psi_i, \widehat{\psi}_i \rangle_{\widehat{h}} - \frac{1}{2} \frac{\partial}{\partial \widehat{h}} \langle \widehat{\psi}_i, \widehat{\psi}_i \rangle_{\widehat{h}} \right) \right|_{\widehat{h}=h}
\end{aligned}$$

doi:10.6342/NTU201600792

Because of the relation $\frac{d}{d\tau}\int_{g(\tau)}^{0} f(x,\tau)dx = -g'(\tau)f(g(\tau),\tau) + \int_{g(\tau)}^{0}\frac{\partial f}{\partial \tau}\,dx$, we have

$$\begin{aligned}\frac{\partial}{\partial \widehat{h}}\langle \psi_i, \widehat{\psi}_i\rangle_{\widehat{h}}\bigg|_{\widehat{h}=h} &= \frac{\partial}{\partial \widehat{h}}\left(\int_{-\widehat{h}}^{0}\psi_i\widehat{\psi}_i\,dz + \frac{\gamma}{\rho\omega^2}\psi_i'(0)\widehat{\psi}_i'(0)\right)\bigg|_{\widehat{h}=h} \\ &= \psi_i(-h)\psi_i(-h) + \left\langle \psi_i,\, \frac{\partial \widehat{\psi}_i}{\partial \widehat{h}}\bigg|_{\widehat{h}=h}\right\rangle.\end{aligned}$$

Similarily,

$$\frac{\partial}{\partial \widehat{h}}\langle \widehat{\psi}_i, \widehat{\psi}_i\rangle_{\widehat{h}}\bigg|_{\widehat{h}=h} = \psi_i(-h)\psi_i(-h) + 2\left\langle \psi_i,\, \frac{\partial \widehat{\psi}_i}{\partial \widehat{h}}\bigg|_{\widehat{h}=h}\right\rangle.$$

Therefore, substituting these results back into M_{ij}, we can get

$$M_{ii} = -\frac{1}{2}\frac{\psi_i(-h)^2}{\langle \psi_i, \psi_i\rangle_h} = -\frac{1}{2}\phi_i(-h)^2. \tag{B.19}$$

To find out what $\phi_i(-h)^2$ is, calculate $\langle \psi_i, \psi_i\rangle_h$,

$$\begin{aligned}\langle \psi_i, \psi_i\rangle_h &= \lim_{\widehat{h}\to h}\langle \psi_i, \widehat{\psi}_i\rangle_{\widehat{h}} \\ &= \lim_{\widehat{h}\to h}\frac{-\psi_i'(-\widehat{h})\widehat{\psi}_i(-\widehat{h})}{k_i^2 - \widehat{k}_i^2} \\ &= -\frac{k_i}{2}\frac{dh}{dk_i}\psi_i(-h)^2,\end{aligned}$$

where we have used (B.17) and L'Hopital's rule. so

$$\phi_i(-h)^2 = -\frac{2}{k_i}\frac{dk_i}{dh}. \tag{B.20}$$

We can easily calculate and derive, from (B.14) and (B.19),

$$a_{00} = 0$$

So far, we have found out $a_{0i} = 0$ for any i and a_{ni} as in (B.18). For all nonzero n, a_{ni} has order, in terms of δR, $O(\delta R^{2n})$. And thus, to lowest order of δR,

$$p \approx \sum_{n=-\infty}^{\infty} A_n J_n(k\varrho)\phi_0(z)\mathrm{e}^{in\varphi} + \sum_{n=\pm 1}\sum_{j=0}^{\infty} a_{nj}H_n^{(1)}(k_j\varrho)\phi_j(z)\mathrm{e}^{in\varphi},$$

where

$$a_{\pm 1j} = \frac{i\pi}{8}A_{\pm 1}kk_j(\delta R)^2\delta h\,\phi_0(-h)\phi_j(-h), \quad n \neq 0,$$

because $J_1(z) \approx \frac{z}{2}$ and $H_1^{(1)}(z) \approx -\frac{2i}{\pi z}$ when $z \ll 1$.

doi:10.6342/NTU201600792

We consider the case that the incident wave is a plane wave

$$p_0 = A\mathrm{e}^{ikx}\cosh(k(z+h)) = A\mathrm{e}^{ikx}\frac{\phi_0(z)}{\phi_0(-h)}. \tag{B.21}$$

Due to the expansion $\mathrm{e}^{ikx} = \sum_{n=-\infty}^{\infty} i^n J_n(k\varrho)\mathrm{e}^{in\varphi}$, we can see that $A_n = i^n \frac{A}{\phi_0(-h)}$, and then

$$p \approx A\mathrm{e}^{ikx}\frac{\phi_0(z)}{\phi_0(-h)} - \sum_{j=0}^{\infty}\frac{A\pi}{4}kk_j(\delta R)^2\delta h H_1^{(1)}(k_j\varrho)\phi_j(-h)\phi_j(z)\cos\varphi.$$

By using the equation of $\phi_j(-h)^2$ in (B.20), we finally have the approximation

$$\begin{aligned} p &\approx A\mathrm{e}^{ikx}\frac{\phi_0(z)}{\phi_0(-h)} + \sum_{j=0}^{\infty}\frac{A\pi}{2}k(\delta R)^2\delta h\frac{dk_i}{dh}\,H_1^{(1)}(k_j\varrho)\frac{\phi_j(z)}{\phi_j(-h)}\cos\varphi \\ &\approx A\mathrm{e}^{ikx}\cosh(k(z+h)) + \frac{A\pi k}{2}(\delta R)^2\delta h\sum_{j=0}^{\infty}\frac{dk_i}{dh}\,H_1^{(1)}(k_j\varrho)\cosh(k_j(z+h))\cos\varphi. \end{aligned} \tag{B.22}$$

doi:10.6342/NTU201600792

doi:10.6342/NTU201600792

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