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論擴散-競爭型洛特卡-佛爾特拉方程組的行進波解之 N 型屏障最大值原理

On the N-barrier maximum principle for traveling wave solutions of diffusive competitive Lotka-Volterra systems

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中文摘要

N 型屏障最大值原理為一種估計一維擴散-競爭洛特卡-佛爾特拉方程組的行進波解之技術。這篇文章中，我們將 [4] 中考慮的雙物種之情況推廣到任意多物種。此外，我們將不再需要，為了得到更精細的估計，而在 [4] 中所考慮的切線法之限制條件。

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Abstract

The N-barrier maximum principle (NBMP) is a technique to estimate the total density of traveling wave solutions to one-dimensional diffusive competitive Lotka-Volterra systems. In this study, two-species cases, which are considered in [4], are generalized to multi-species cases. In addition, the constraints of the tangent line method proposed in [4] to obtain a refined estimate is released.

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1 Introduction

In mathematical biology, the diffusive competitive Lotka-Volterra models are second order partial differential systems often used to describe the dynamics of ecological systems such as the diffusion and the growth of each species, and the competitions between species. In population dynamics, how to estimate the total density is an important issue for understanding the ecological capacity of inhabitants. For one-dimensional cases, an upper bound and a lower bound of the total density of traveling waves can be obtained by employing the method of N-barrier.

This article studies the one-dimensional diffusive competitive Lotka-Volterra system:

$$\begin{cases} \tilde{u}_t = d_1\tilde{u}_{yy} + \tilde{u}(\sigma_1 - c_{11}\tilde{u} - c_{12}\tilde{v}), \ y \in \mathbb{R}, \ t > 0, \\ \tilde{v}_t = d_2\tilde{v}_{yy} + \tilde{v}(\sigma_2 - c_{21}\tilde{u} - c_{22}\tilde{v}), \ y \in \mathbb{R}, \ t > 0, \end{cases} \tag{1.0.1}$$

where $\tilde{u}(y,t)$ and $\tilde{v}(y,t)$ stand for the population density of two species $\tilde{u}$ and $\tilde{v}$, respectively; d_1, d_2 are diffusion rates, σ_1, σ_2 are intrinsic growth rates, c_{11}, c_{22} are intra-species competition rates, and c_{12}, c_{21} are inter-species competition rates. All of the coefficients are assumed to be positive.

When one species, say $\tilde{v}$, is absent in (1.0.1), the system is reduced to the Fisher-Kolmogorov equation

$$\tilde{u}_t = d_1\tilde{u}_{yy} + \tilde{u}(\sigma_1 - c_{11}\tilde{u}), \ y \in \mathbb{R}, \ t > 0. \tag{1.0.2}$$

For the case $d_1 = \sigma = c_{11} = 1$, Komolgorov, Petrovsky and Piskunov [5] proved that under the initial condition

$$\tilde{u}(y,0) = \begin{cases} 1 \ , \text{for } y < 0, \\ 0 \ , \text{for } y > 0, \end{cases}$$

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the solution $\tilde{u}(y,t)$ of (1.0.1) evolves to a traveling wavefront solution $u(x)$ with $x = y - \theta_{\min}t$, where $\theta_{\min} = 2$ is the minimum evolving speed which will be discussed more in the Appendix §7. In fact, there exists a function ψ such that

$$|\tilde{u}(y,t) - u(y - 2t - \psi(t))| \to 0 \text{ as } t \to 0$$

uniformly in x, and $\lim_{t\to\infty} \psi'(t) = 0$. This motivates us to study traveling wave solutions of (1.0.1):

$$(u(y,t), v(y,t)) = (u(x), v(x)),\ x = y - \theta t, \tag{1.0.3}$$

where θ stands for the wave velocity of the traveling wave. Substituting (1.0.3) into (1.0.1), the system becomes a nonlinear second-order ordinary differential system:

$$\begin{cases} d_1 u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v) = 0,\ x \in \mathbb{R}, \\ d_2 v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v) = 0,\ x \in \mathbb{R}. \end{cases} \tag{1.0.4}$$

There are four choices of the artificial boundary conditions $(u,v)(-\infty) = \mathbf{e}_-$ and $(u,v)(+\infty) = \mathbf{e}_+$ for (1.0.4):

$$\mathbf{e}_1 = (0,0),\ \mathbf{e}_2 = \left(\frac{\sigma_1}{c_{11}}, 0\right),\ \mathbf{e}_3 = \left(0, \frac{\sigma_2}{c_{22}}\right) \text{ and } \mathbf{e}_4 = \left(\frac{\sigma_1 c_{22} - \sigma_2 c_{12}}{c_{11}c_{22} - c_{12}c_{21}}, \frac{\sigma_2 c_{11} - \sigma_1 c_{21}}{c_{11}c_{22} - c_{12}c_{21}}\right),$$

which are the solutions to the algebraic equations:

$$\begin{cases} u(\sigma_1 - c_{11}u - c_{12}v) = 0,\ x \in \mathbb{R}, \\ v(\sigma_2 - c_{21}u - c_{22}v) = 0,\ x \in \mathbb{R}. \end{cases}$$

A typical boundary condition discussed in [4] is the $(\mathbf{e}_2, \mathbf{e}_3)$-boundary condition. That is,

$$(u,v)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0\right),\ (u,v)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}\right). \tag{1.0.5}$$

The boundary condition represents that u is dominant on the left region and v is dominant on the right region in $x \in \mathbb{R}$. In this situation, if we track back to the primitive equation (1.0.1), u will occupy the whole domain in $y \in \mathbb{R}$ eventually if

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$\theta > 0$ while v will occupy the whole domain eventually if $\theta < 0$.

In order to obtain a priori estimates for the total density, [1] considered the three-species case and used the classical elliptic maximum principle with the method of completing the square to obtain a priori estimates of $u + v + w$ under particular boundary conditions, the hypothesis that all the diffusion rates equal to 1 and other parametric assumptions. In [4], upper and lower bounds of $\alpha u + \beta v$, for arbitrary positive α and β, were obtained without any constraint on diffusion rates d_1, d_2 in our two-species case (1.0.4).

Sometimes (1.0.1) is rescaled for convenience as

$$\begin{cases} \tilde{u}_t = \tilde{u}_{yy} + \tilde{u}(1 - \tilde{u} - a_1\tilde{v}),\ y \in \mathbb{R},\ t > 0, \\ \tilde{v}_t = d\tilde{v}_{yy} + \sigma\tilde{v}(1 - a_2\tilde{u} - \tilde{v}),\ y \in \mathbb{R},\ t > 0. \end{cases} \tag{1.0.6}$$

where $\tilde{u}(y,t) = \frac{c_{11}}{\sigma_1}u((\frac{d_1}{\sigma_1})^{\frac{1}{2}}y, \frac{1}{\sigma_1}t)$, $\tilde{v}(y,t) = \frac{c_{22}}{\sigma_2}v((\frac{d_1}{\sigma_1})^{\frac{1}{2}}y, \frac{1}{\sigma_1}t)$, $a_1 = \frac{\sigma_2}{\sigma_1}\frac{c_{12}}{c_{22}}$, $a_2 = \frac{\sigma_1}{\sigma_2}\frac{c_{21}}{c_{11}}$, $d = \frac{d_2}{d_1}$ and $\sigma = \frac{\sigma_2}{\sigma_1}$. Then the corresponding traveling wave solution satisfies

$$\begin{cases} u'' + \theta u' + u(1 - u - a_1 v) = 0,\ x \in \mathbb{R}, \\ dv'' + \theta v' + \sigma v(1 - a_2 u - v) = 0,\ x \in \mathbb{R}, \\ (u,v)(-\infty) = (1,0),\ (u,v)(+\infty) = (0,1). \end{cases} \tag{1.0.7}$$

This thesis is organized in the following way. In §2, we review the N-barrier maximum principle for 2-species case and generalize the results to multi-species cases. As a corollary, the nonexistence theory is proposed in §3. A refined estimate is obtained explicitly in §4 by the improved tangent line method. Examples are shown in §5, summary and future researches are in §6, and the minimum wave speed is discussed in the Appendix §7.

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2 N-barrier Maximum Principle (NBMP)

In this section, the method of N-barrier is performed to obtain a lower bound in Theorem 2.1 with a complete proof proposed in [4]. Other results such as upper bound, generalized and multi-species cases are raised thereafter.

2.1 NBMP for 2-species

Theorem 2.1 (Lower bound). *Let $(u(x),v(x))$ be a nonnegative solution to*

$$\begin{cases} d_1u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v) \le 0, \ x \in \mathbb{R}, \\ d_2v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v) \le 0, \ x \in \mathbb{R}, \\ (u,v)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0\right), \ (u,v)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}\right). \end{cases} \tag{2.1.1}$$

Suppose that $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$, then for any $\alpha, \beta > 0$ we have the following lower bound:

$$\alpha u(x) + \beta v(x) \ge \min\left\{\alpha\frac{\sigma_2}{c_{21}}, \beta\frac{\sigma_1}{c_{12}}\right\}\frac{\min\{d_1,d_2\}}{\max\{d_1,d_2\}}.$$

Proof. For any given $a, b > 0$, we take the linear combination of the first two equations in (2.1.1) to get

$$\begin{aligned} 0 &\ge a[d_1u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v)] + b[d_2v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v)] \\ &= q''(x) + \theta p'(x) + F(u(x),v(x)), \end{aligned} \tag{2.1.2}$$

where

$$\begin{cases} q(x) = ad_1u(x) + bd_2v(x), \\ p(x) = au(x) + bv(x), \\ F(u,v) = au(\sigma_1 - c_{11}u - c_{12}v) + bv(\sigma_2 - c_{21}u - c_{22}v). \end{cases}$$

First of all, we leave the first two linear terms $q(x)$ and $p(x)$, and focus on the

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nonlinear term $F(u(x), v(x))$. Since the determinant of the quadratic curve $F(u, v) = 0$ is

$$
\begin{aligned}
(ac_{12} + bc_{21})^2 - 4ac_{11}bc_{22} = &\ a^2c_{12}^2 + 2abc_{12}c_{21} + b^2c_{21}^2 - 4abc_{11}c_{22} \\
> &\ a^2c_{12}^2 + 2abc_{12}c_{21} + b^2c_{21}^2 - 4ab\frac{\cancel{\sigma_1}c_{21}}{\cancel{\sigma_2}}\frac{\cancel{\sigma_2}c_{12}}{\cancel{\sigma_1}} \\
= &\ (ac_{12} - bc_{21})^2 \geq 0,
\end{aligned}
\tag{2.1.3}
$$

$F(u, v) = 0$ is a hyperbola. Here, we have used the parameter assumptions $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$.

Furthermore, by observing the signs of $F(u, v)$ on the u- and v-axes and the fact that $F(u, v) = 0$ passes through the three points $(0, 0)$, $(0, \frac{\sigma_2}{c_{22}})$ and $(\frac{\sigma_1}{c_{11}}, 0)$, we conclude that $(0, \frac{\sigma_2}{c_{22}})$ and $(\frac{\sigma_1}{c_{11}}, 0)$ lie on the same branch while $(0, 0)$ lies on the other branch (see Figure 1). Actually, $F(u, 0) = au(\sigma_1 - c_{11}u)$ and $F(0, v) = bv(\sigma_2 - c_{22}v)$ indicate that $F(u, 0) > 0$ for $0 < u < \frac{\sigma_1}{c_{11}}$, $F(u, 0) < 0$ for $u < 0$ or $u > \frac{\sigma_1}{c_{11}}$, $F(0, v) > 0$ for $0 < v < \frac{\sigma_2}{c_2}$, and $F(v, 0) < 0$ for $v < 0$ or $v > \frac{\sigma_2}{c_{22}}$.

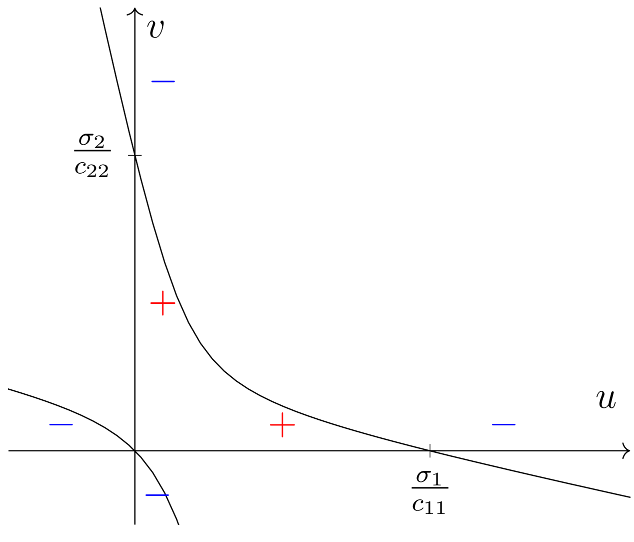

Figure 1: $F(u, v) = 0$ and the sign of F

We are ready to construct our N-barrier. Firstly, by the parameter assumptions $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$, the intersection of the two lines $\sigma_1 - c_{11}u - c_{12}v = 0$ and $\sigma_2 - c_{21}u - c_{22}v = 0$ is in the first quadrant, and it also lies on the quadratic curve $F(u, v) = au(\sigma_1 - c_{11}u - c_{12}v) + bv(\sigma_2 - c_{21}u - c_{22}v) = 0$. Therefore, the line segment

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between $(0, \frac{\sigma_1}{c_{12}})$ and $(\frac{\sigma_2}{c_{21}}, 0)$ lies underneath the quadratic curve $F(u,v) = 0$. This line segment together with the u-axis and v-axis form a right triangle $\mathcal{T}$ lies entirely below the curve $F(u,v) = 0$ in the first quadrant (see Figure 2). Thus $F(u,v) > 0$ for all $(u,v) \in \mathcal{T}$.

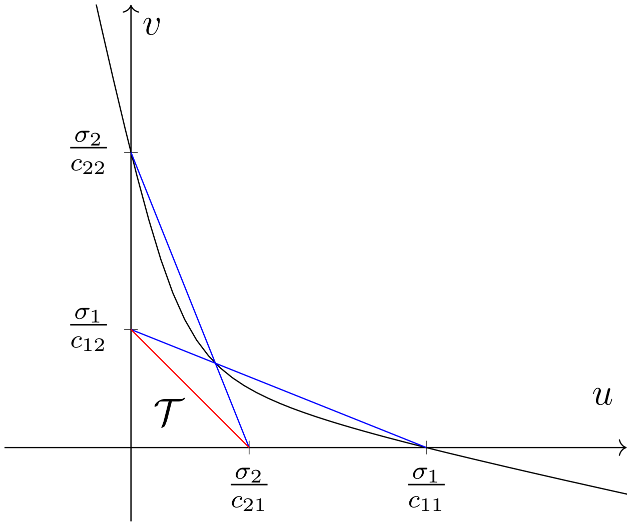

Figure 2: blue lines: $\sigma_1 - c_{11}u - c_{12}v = 0$ and $\sigma_2 - c_{21}u - c_{22}v = 0$; red line: the line segment between $(0, \frac{\sigma_1}{c_{12}})$ and $(\frac{\sigma_2}{c_{21}}, 0)$; and the right triangle $\mathcal{T}$

.

Let us denote

$$\mathcal{Q}_\lambda = \{(u,v) \mid ad_1u + bd_2v \le \lambda,\ u,v \ge 0\}$$

and

$$\mathcal{P}_\eta = \{(u,v) \mid au + bv \le \eta,\ u,v \ge 0\},$$

corresponding to the linear terms obtained in (2.1.2).

(i) The first line of the N-barrier is $q = ad_1u + bd_2v = \lambda_2$, where $\lambda_2 = \sup\{\lambda \mid \mathcal{Q}_\lambda \subset \mathcal{T}\}$. To be more specific, since the intercepts of $ad_1u + bd_2v = \lambda_2$ are $\left(\frac{\lambda_2}{ad_1}, 0\right)$ and $\left(0, \frac{\lambda_2}{bd_2}\right)$, and the condition $\mathcal{Q}_{\lambda_2} \subset \mathcal{T}$ requires that $\frac{\lambda_2}{ad_1} \le \frac{\sigma_2}{c_{21}}$ and $\frac{\lambda_2}{bd_2} \le \frac{\sigma_1}{c_{12}}$, hence $\lambda_2 = \min\left\{ad_1\frac{\sigma_2}{c_{21}}, bd_2\frac{\sigma_1}{c_{12}}\right\}$.

(ii) The second line of the N-barrier is $p = au + bv = \eta$, where $\eta = \sup\{\eta \mid \mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2}\}$. To be more specific, since the intercepts of $au + bv = \eta$ are $\left(\frac{\eta}{a}, 0\right)$ and

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$\left(0, \frac{\eta}{b}\right)$, and the condition $\mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2}$ requires that $\frac{\eta}{a} \leq \frac{\lambda_2}{ad_1}$ and $\frac{\eta}{b} \leq \frac{\lambda_2}{bd_2}$, so $\eta = \min\{\frac{\lambda_2}{d_1}, \frac{\lambda_2}{d_2}\} = \frac{\lambda_2}{\max\{d_1,d_2\}}$.

(iii) The third line of the N-barrier is $q = ad_1u + bd_2v = \lambda_1$, which is parallel to the first line, where $\lambda_1 = \sup\{\lambda \mid \mathcal{Q}_\lambda \subset \mathcal{P}_\eta\}$. To be more specific, since the intercepts of $ad_1u + bd_2v = \lambda_1$ are $\left(\frac{\lambda_1}{ad_1}, 0\right)$ and $\left(0, \frac{\lambda_1}{bd_2}\right)$, and the condition $\mathcal{Q}_{\lambda_1} \subset \mathcal{P}_\eta$ requires that $\frac{\lambda_1}{ad_1} \leq \frac{\eta}{a}$, $\frac{\lambda_1}{bd_2} \leq \frac{\eta}{b}$, therefore $\lambda_1 = \min\{d_1\eta, d_2\eta\} = \lambda_2 \frac{\min\{d_1,d_2\}}{\max\{d_1,d_2\}} = \min\left\{ad_1\frac{\sigma_2}{c_{21}}, bd_2\frac{\sigma_1}{c_{12}}\right\} \frac{\min\{d_1,d_2\}}{\max\{d_1,d_2\}}$.

The three lines established above form the N-barrier as Figure 3. It is easy to realize that the term 'N-barrier' comes from the resemblance to the English alphabet 'N', even the shape of the N-barrier may the reflection of the character 'N'.

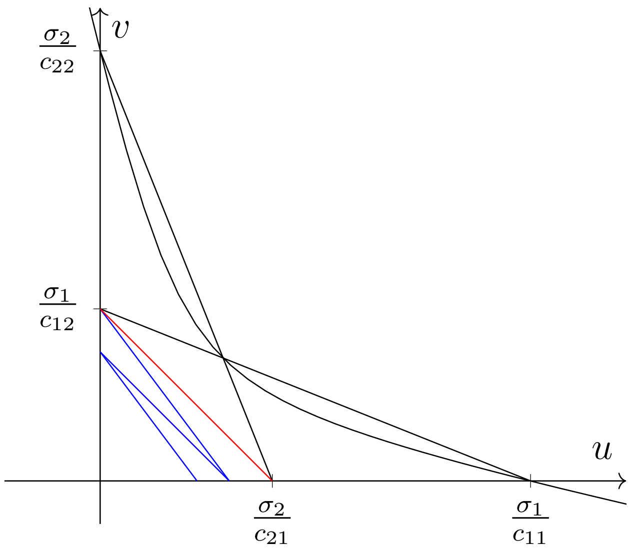

Figure 3: the N-barrier

Now, we show λ_1 is the lower bound as the following:

Claim. *$q(x) = ad_1u(x) + bd_2v(x) \geq \lambda_1$, for all $x \in \mathbb{R}$.*

Suppose contrary, then there exists $z_0 \in \mathbb{R}$ s.t. $q(z_0) = ad_1u(z_0) + bd_2v(z_0) < \lambda_1$. Since

$$\lambda_1 = \min\left\{ad_1\frac{\sigma_2}{c_{21}}, bd_2\frac{\sigma_1}{c_{12}}\right\} \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}} \leq \begin{cases} ad_1\frac{\sigma_2}{c_{21}} < ad_1\frac{\sigma_1}{c_{11}} \\ bd_2\frac{\sigma_1}{c_{12}} < bd_2\frac{\sigma_2}{c_{22}} \end{cases}$$

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and

$$q(-\infty) = ad_1\frac{\sigma_1}{c_{11}} > \lambda_1,\ q(+\infty) = bd_2\frac{\sigma_2}{c_{22}} > \lambda_1, \tag{2.1.4}$$

we may assume that $q(z_0) = \min_{x\in\mathbb{R}} q(x)$. Therefore, $q'(z_0) = 0$. Let z_1 and z_2 be the first point at which the solution $(u(x), v(x))$ intersects the first line $q = ad_1u + bd_2v = \lambda_2$ in the uv-plane when x moves from z_0 toward $-\infty$ and $+\infty$, respectively. That is,

$$z_1 = \inf\{z \in (-\infty, z_0) \mid q(x) = ad_1u(x) + bd_2v(x) < \lambda_2,\ \forall x \in (z, z_0)\}$$

and

$$z_2 = \sup\{z \in (z_0, +\infty) \mid q(x) = ad_1u(x) + bd_2v(x) < \lambda_2,\ \forall x \in (z_0, z)\}.$$

Hence, $q'(z_1) \le 0$ and $q'(z_2) \ge 0$. Furthermore, since $(u(z_0), v(z_0))$ lies underneath the second line $p = au + bv = \eta$ while $(u(z_1), v(z_1))$ and $(u(z_2), v(z_2))$ lie above which, $p(z_0) < \eta$ and $p(z_1), p(z_2) > \eta$ (see Figure 4).

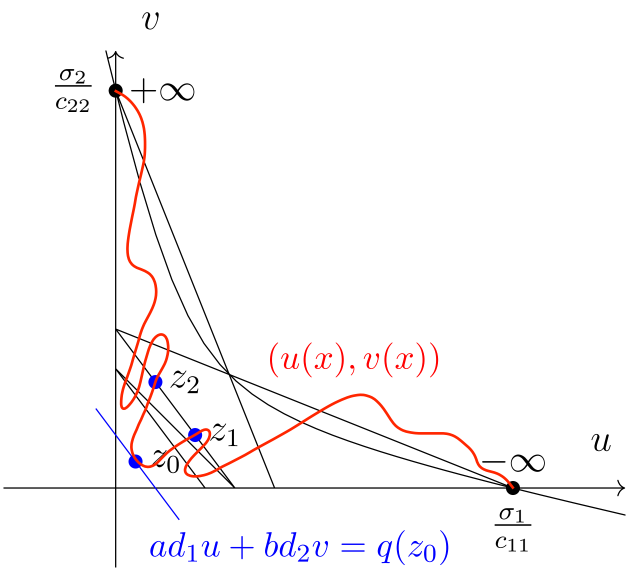

Figure 4: z_0, z_1, z_2 and solution curve $(u(x), v(x))$

In addition, since the arcs $\{(u(x), v(x)) \mid x \in (z_1, z_0)\}$ and $\{(u(x), v(v)) \mid x \in (z_0, z_2)\}$ lie in the right triangle $\mathcal{T}$ in Figure 2, $F(u(x), v(x)) > 0$ for all $x \in (z_1, z_0)$ or $x \in (z_0, z_2)$.

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For the case where $\theta \geq 0$, integrating (2.1.2) from z_0 to z_2 yields a contradiction:

$$\underbrace{q'(z_2)}_{\geq 0} - \cancelto{0}{q'(z_0)} + \theta(\underbrace{p(z_2)}_{>\eta} - \underbrace{p(z_0)}_{<\eta}) + \underbrace{\int_{z_0}^{z_2} F(u(x),v(x))dx}_{>0} \leq \underset{\rightarrow\leftarrow}{0} \tag{2.1.5}$$

For the other case where $\theta \leq 0$, we integrate (2.1.2) from z_1 to z_0 to obtain a contradiction:

$$\cancelto{0}{q'(z_0)} - \underbrace{q'(z_1)}_{\leq 0} + \theta(\underbrace{p(z_0)}_{<\eta} - \underbrace{p(z_1)}_{>\eta}) + \underbrace{\int_{z_1}^{z_0} F(u(x),v(x))dx}_{>0} \leq \underset{\rightarrow\leftarrow}{0} \tag{2.1.6}$$

Thus

$$d_1 a u(x) + d_2 b v(x) \geq \lambda_1 = \min\left\{ a d_1 \frac{\sigma_2}{c_{21}}, b d_2 \frac{\sigma_1}{c_{12}} \right\} \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}}.$$

By taking $a = \frac{\alpha}{d_1},\ b = \frac{\beta}{d_2}$, we obtain the desired result:

$$\alpha u(x) + \beta v(x) \geq \min\left\{ \alpha \frac{\sigma_2}{c_{21}}, \beta \frac{\sigma_1}{c_{12}} \right\} \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}}.$$

□

Note that if d_1 and d_2 are equal, the three lines of the N-barrier in the proof coincide. However, the proof above still works. In fact, the first line and second line of the N-barrier are used to deal with the two linear terms in (2.1.2), respectively. And the third line works in the proof of contradiction to show that $q(z_0) = \min_{x \in \mathbb{R}} q(x)$. Thus, if the three lines coincide, the two linear terms become the same, and we still have $q(z_0) = \min_{x \in \mathbb{R}} q(x)$.

Similarly, by constructing an N-barrier above the quadratic curve $F(u, v) = 0$ (see Figure 5), the corresponding upper bound can be obtained as follows.

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Theorem 2.2 (Upper bound). *Let $(u(x), v(x))$ be a nonnegative solution to*

$$\begin{cases} d_1 u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v) \geq 0, \ x \in \mathbb{R}, \\ d_2 v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v) \geq 0, \ x \in \mathbb{R}, \\ (u,v)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0\right), \ (u,v)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}\right). \end{cases}$$

Suppose that $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$, then for any $\alpha, \beta > 0$ we have the following upper bound:

$$\alpha u(x) + \beta v(x) \leq \max\left\{\alpha \frac{\sigma_1}{c_{11}}, \beta \frac{\sigma_2}{c_{22}}\right\} \frac{\max\{d_1, d_2\}}{\min\{d_1, d_2\}}.$$

Proof. We only show how to construct an N-barrier above $F(u, v) = 0$, and obtain λ_1. For checking λ_1 an upper bound for $\alpha u + \beta v$, the argument is same as which of Theorem 2.1 and is hence omitted.

(i) The first line of the N-barrier is $q = ad_1 u + bd_2 v = \lambda_2$. Since the intercepts of $ad_1 u + bd_2 v = \lambda_2$ are $\left(\frac{\lambda_2}{ad_1}, 0\right)$ and $\left(0, \frac{\lambda_2}{bd_2}\right)$, and we requires $\frac{\lambda_2}{ad_1} \geq \frac{\sigma_1}{c_{11}}$ and $\frac{\lambda_2}{bd_2} \geq \frac{\sigma_2}{c_{22}}$, hence $\lambda_2 = \max\left\{ad_1 \frac{\sigma_1}{c_{11}}, bd_2 \frac{\sigma_2}{c_{22}}\right\}$.

(ii) The second line of the N-barrier is $p = au + bv = \eta$. Since the intercepts of $au + bv = \eta$ are $\left(\frac{\eta}{a}, 0\right)$ and $\left(0, \frac{\eta}{b}\right)$, and we requires $\frac{\eta}{a} \geq \frac{\lambda_2}{ad_1}$ and $\frac{\eta}{b} \geq \frac{\lambda_2}{bd_2}$, so $\eta = \max\{\frac{\lambda_2}{d_1}, \frac{\lambda_2}{d_2}\} = \frac{\lambda_2}{\min\{d_1, d_2\}}$.

(iii) The third line of the N-barrier is $q = ad_1 u + bd_2 v = \lambda_1$, which is parallel to the first line. Since the intercepts of $ad_1 u + bd_2 v = \lambda_1$ are $\left(\frac{\lambda_1}{ad_1}, 0\right)$ and $\left(0, \frac{\lambda_1}{bd_2}\right)$, and we requires $\frac{\lambda_1}{ad_1} \geq \frac{\eta}{a}$ and $\frac{\lambda_1}{bd_2} \geq \frac{\eta}{b}$, therefore, $\lambda_1 = \max\{d_1 \eta, d_2 \eta\} = \lambda_2 \frac{\max\{d_1, d_2\}}{\min\{d_1, d_2\}} = \max\left\{ad_1 \frac{\sigma_1}{c_{11}}, bd_2 \frac{\sigma_2}{c_{22}}\right\} \frac{\max\{d_1, d_2\}}{\min\{d_1, d_2\}}$.

Similarly, taking $a = \frac{\alpha}{d_1}$ and $b = \frac{\beta}{d_2}$, we have the desired upper bound.

□

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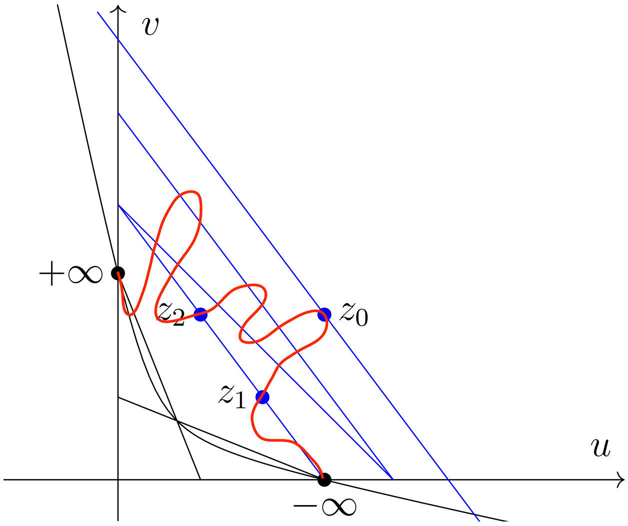

Figure 5: N-barrier for upper bound

2.2 Generalized NBMP

Recall that the parameter assumptions $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$ in Theorem 2.1 and Theorem 2.2 have been used to show that the quadratic curve $F(u,v) = 0$ is a hyperbola in (2.1.3), that the right triangle $\mathcal{T}$ in Figure 2 lies below the curve $F(u,v) = 0$ and that $q(-\infty) = ad_1\frac{\sigma_1}{c_{11}} > \lambda_1,\ q(+\infty) = bd_2\frac{\sigma_2}{c_{22}} > \lambda_1$ in (2.1.4).

First of all, without these parameter assumptions, the quadratic curve $F(u,v) = 0$ may be a parabola or a ellipse. Fortunately, using the positivity of the coefficients σ_i and c_{ij} (i,j=1,2), the signs of $F(u,v)$ can be shown as Figure 6.

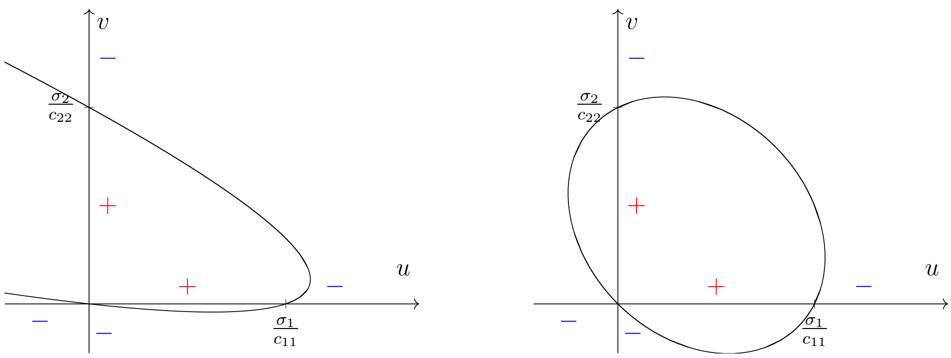

Figure 6: left: parabola; right: ellipse

Secondly, it is easy to see from Figure 6 that, for each case, there exists a right triangle $\mathcal{T}$ as in Figure 2 lies below the curve $F(u,v) = 0$ in the first quadrant so that

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$F(u,v) > 0$ for all $(u,v) \in \mathcal{T}$. Therefore, N-barriers can also be constructed in both cases.

Moreover, for (2.1.4), we only need to require the boundary conditions $\mathbf{e}_-$ and $\mathbf{e}_+$ lie above the first line $q = ad_1u + bd_2v = \lambda_1$.

Consequently, the assumptions $\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}$ and $\frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}}$ can be dropped. Actually, the proof also works for $F(u,v) = u^m f(u,v) + v^n g(u,v)$, for certain hypotheses on $f(u,v)$, $g(u,v)$ and the boundary conditions $\mathbf{e}_+$, $\mathbf{e}_-$, which will be explained in Theorem 2.3 and Theorem 2.4 (see Figure 7).

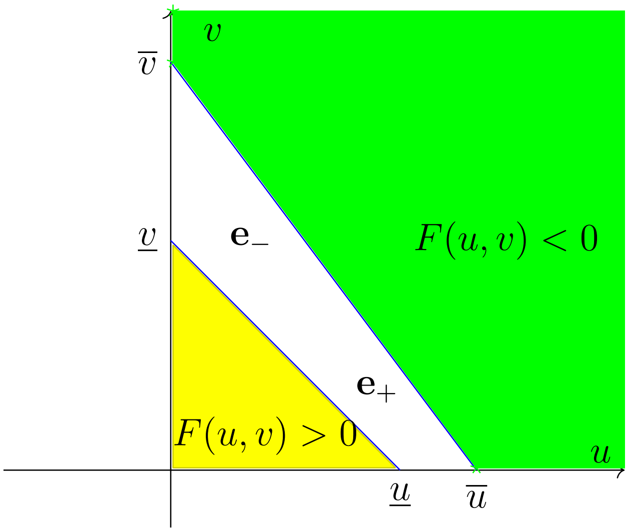

Figure 7: general conditions

Theorem 2.3 (Generalized lower bound)**.** *Let $(u(x),v(x))$ be a nonnegative solution to*

$$\begin{cases} d_1u'' + \theta u' + u^m f(u,v) \le 0, \ \ x \in \mathbb{R}, \\ d_2v'' + \theta v' + v^n g(u,v) \le 0, \ \ x \in \mathbb{R}, \\ (u,v)(-\infty) = \mathbf{e}_-, \ \ (u,v)(+\infty) = \mathbf{e}_+. \end{cases}$$

Suppose that there exists $\underline{u} > 0$ and $\underline{v} > 0$ s.t. $f(u,v) > 0$ and $g(u,v) > 0$, for all

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$(u,v) \in \underline{\mathcal{R}}$ and $\mathbf{e}_-, \mathbf{e}_+ \in [0,+\infty)^2 \setminus \underline{\mathcal{R}}$, where

$$\underline{\mathcal{R}} = \left\{ (u,v) \in [0,+\infty)^2 \,\middle|\, \frac{u}{\underline{u}} + \frac{v}{\underline{v}} < 1 \right\}.$$

Then for any $\alpha, \beta > 0$ we have the following lower bound:

$$\alpha u(x) + \beta v(x) \geq \min\{\alpha \underline{u}, \beta \underline{v}\} \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}}.$$

Theorem 2.4 (Generalized upper bound)**.** *Let $(u(x), v(x))$ be a nonnegative solution to*

$$\begin{cases} d_1 u'' + \theta u' + u^m f(u,v) \geq 0, \ x \in \mathbb{R}, \\ d_2 v'' + \theta v' + v^n g(u,v) \geq 0, \ x \in \mathbb{R}, \\ (u,v)(-\infty) = \mathbf{e}_-, \ (u,v)(+\infty) = \mathbf{e}_+. \end{cases}$$

Suppose that there exists $\overline{u} > 0$ and $\overline{v} > 0$ s.t. $f(u,v) < 0$ and $g(u,v) < 0$, for all $(u,v) \in \overline{\mathcal{R}}$ and $\mathbf{e}_-, \mathbf{e}_+ \in [0,+\infty)^2 \setminus \overline{\mathcal{R}}$, where

$$\overline{\mathcal{R}} = \left\{ (u,v) \in [0,+\infty)^2 \,\middle|\, \frac{u}{\overline{u}} + \frac{v}{\overline{v}} > 1 \right\}.$$

Then for any $\alpha, \beta > 0$ we have the following upper bound:

$$\alpha u(x) + \beta v(x) \leq \max\{\alpha \overline{u}, \beta \overline{v}\} \frac{\max\{d_1, d_2\}}{\min\{d_1, d_2\}}.$$

2.3 NBMP for Multi-species

By replacing all 'lines' in the argument of the two-species case discussed above by 'hyperplanes', the corresponding results for multi-species case rise.

Theorem 2.5 (Multi-species lower bound)**.** *Let $(u_1(x), \cdots, u_n(x))$ be a nonnegative solution to*

$$\begin{cases} d_i u_i'' + \theta u_i' + u_i^{m_i} f_i(u_1, \cdots, u_n) \leq 0, \ x \in \mathbb{R}, \ i = 1, \cdots, n, \\ (u_1, \cdots, u_n)(-\infty) = \mathbf{e}_-, \ (u_1, \cdots, u_n)(+\infty) = \mathbf{e}_+. \end{cases} \tag{2.3.1}$$

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Assume that for each $i = 1, \cdots, n$, *there exists* $\underline{u}_i > 0$ *s.t.* $f_i(u_1, \cdots, u_n) > 0$, *whenever* $(u_1, \cdots, u_n) \in \underline{\mathcal{R}}$, *and* $\mathbf{e}_-, \mathbf{e}_+ \in [0, +\infty)^n \setminus \underline{\mathcal{R}}$, *where*

$$\underline{\mathcal{R}} = \left\{ (u_1, \cdots, u_n) \in [0, +\infty)^n \,\middle|\, \sum_{i=1}^{n} \frac{u_i}{\underline{u}_i} < 1 \right\}.$$

Then for any $\alpha_i > 0$, *we have the following lower bound:*

$$\sum_{i=1}^{n} \alpha_i u_i(x) \geq \left(\min_{i=1,\cdots,n} \alpha_i \underline{u}_i \right) \frac{\min\limits_{i=1,\cdots,n} d_i}{\max\limits_{i=1,\cdots,n} d_i}, \;\; x \in \mathbb{R}.$$

Proof. For any given $a_1, \cdots, a_n > 0$, we take the linear combination of the n equations in (2.3.1), we obtain a single equation involving $p(x)$ and $q(x)$

$$q''(x) + p'(x) + F(u_1(x), u_2(x), ..., u_n(x)) \leq 0, \quad x \in \mathbb{R}, \tag{2.3.2}$$

where

$$\begin{cases} q(x) = \sum\limits_{i=1}^{n} a_i d_i u_i(x), \\ p(x) = \sum\limits_{i=1}^{n} a_i u_i(x), \\ F(u_1, u_2, ..., u_n) = \sum\limits_{i=1}^{n} \alpha_i \, u_i^{m_i} \, f_i(u_1, u_2, ..., u_n). \end{cases}$$

The construction of the N-barrier consists of determining λ_2, η, and λ_1 such that the three hyperplanes $\sum\limits_{i=1}^{n} a_i \, d_i \, u_i = \lambda_2$, $\sum\limits_{i=1}^{n} a_i \, u_i = \eta$ and $\sum\limits_{i=1}^{n} a_i \, d_i \, u_i = \lambda_1$ enjoy the property

$$\mathcal{Q}_{\lambda_1} \subset \mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2} \subset \underline{\mathcal{R}}, \tag{2.3.3}$$

where

$$\mathcal{Q}_\lambda = \left\{ (u_1, u_2, ..., u_n) \,\middle|\, \sum_{i=1}^{n} a_i \, d_i \, u_i \leq \lambda, \; u_1, u_2, ..., u_n \geq 0 \right\}; \tag{2.3.4}$$

$$\mathcal{P}_\eta = \left\{ (u_1, u_2, ..., u_n) \,\middle|\, \sum_{i=1}^{n} a_i \, u_i \leq \eta, \; u_1, u_2, ..., u_n \geq 0 \right\}. \tag{2.3.5}$$

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We follow the three steps below to construct the N-barrier:

(i) The first hyperplane of the N-barrier is $q = \sum\limits_{i=1}^{n} a_i d_i u_i = \lambda_2$, where $\lambda_2 = \sup\{\lambda \mid \mathcal{Q}_\lambda \subset \underline{\mathcal{R}}\}$. To be more specific, since the intercepts of $\sum\limits_{i=1}^{n} a_i d_i u_i = \lambda_2$ are $(\frac{\lambda_2}{a_1 d_1}, 0, ..., 0)$, $(0, \frac{\lambda_2}{a_2 d_2}, 0, ..., 0)$,..., and $(0, 0, ..., 0, \frac{\lambda_2}{a_n d_n})$, and the condition $\mathcal{Q}_{\lambda_2} \subset \underline{\mathcal{R}}$ requires that $\frac{\lambda_2}{a_i\, d_i} \leq \underline{u}_i$ for $i = 1, 2, ..., n$, hence $\lambda_2 = \min\limits_{i=1,\cdots,n} a_i d_i \underline{u}_i$.

(ii) The second hyperplane of the N-barrier is $p = \sum\limits_{i=1}^{n} a_i u_i = \eta$, where $\eta = \sup\{\eta \mid \mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2}\}$. To be more specific, since the intercepts of $\sum\limits_{i=1}^{n} a_i u_i = \eta$ are$(\frac{\eta}{a_1}, 0, ..., 0)$, $(0, \frac{\eta}{a_2}, 0, ..., 0)$,..., and $(0, 0, ..., 0, \frac{\eta}{a_n})$, and the condition $\mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2}$ requires that $\frac{\eta}{a_i} \leq \frac{\lambda_2}{a_i\, d_i}$ for $i = 1, 2, ..., n$, so $\eta = \min\limits_{i=1,\cdots,n} \frac{\lambda_2}{d_i} = \frac{\lambda_2}{\max\limits_{i=1,\cdots,n} d_i}$.

(iii) The third hyperplane of the N-barrier is $q = \sum\limits_{i=1}^{n} a_i d_i u_i = \lambda_1$, which is parallel to the first line, where $\lambda_1 = \sup\{\lambda \mid \mathcal{Q}_\lambda \subset \mathcal{P}_\eta\}$. To be more specific, since the intercepts of $\sum\limits_{i=1}^{n} a_i d_i u_i = \lambda_1$ are $(\frac{\lambda_1}{a_1\, d_1}, 0, ..., 0)$, $(0, \frac{\lambda_1}{a_2\, d_2}, 0, ..., 0)$,..., and $(0, 0, ..., 0, \frac{\lambda_1}{a_n\, d_n})$, and the condition $\mathcal{Q}_{\lambda_1} \subset \mathcal{P}_\eta$ requires that $\frac{\lambda_1}{a_i\, d_i} \leq \frac{\eta}{a_i}$ for $i = 1, 2, ..., n$, therefore $\lambda_1 = \min\limits_{i=1,\cdots,n} d_i \eta = \lambda_2 \frac{\min\limits_{i=1,\cdots,n} d_i}{\max\limits_{i=1,\cdots,n} d_i} = \min\limits_{i=1,\cdots,n} a_i d_i \underline{u}_i \frac{\min\limits_{i=1,\cdots,n} d_i}{\max\limits_{i=1,\cdots,n} d_i}$.

The three hyperplanes $\sum\limits_{i=1}^{n} \alpha_i\, d_i\, u_i = \lambda_2$, $\sum\limits_{i=1}^{n} \alpha_i\, u_i = \eta$ and $\sum\limits_{i=1}^{n} \alpha_i\, d_i\, u_i = \lambda_1$ constructed above form the N-barrier.

Now, we show λ_1 is the lower bound as the following:

Claim. $q(x) = \sum\limits_{i=1}^{n} a_i d_i u_i(x) \geq \lambda_1$*, for all* $x \in \mathbb{R}$.

Suppose that, contrary to our claim, there exists $z_0 \in \mathbb{R}$ such that $q(z_0) = \sum\limits_{i=1}^{n} a_i d_i u_i(z_0) < \lambda_1$. From $\mathbf{e}_-, \mathbf{e}_+ \in [0, +\infty)^n \setminus \underline{\mathcal{R}}$ and $\mathcal{Q}_{\lambda_1} \subset \underline{\mathcal{R}}$, we know that $q(\pm\infty) > \lambda_1$. So we may assume $\min\limits_{x\in\mathbb{R}} q(x) = q(z_0)$. Therefore, $q'(z_0) = 0$. Let z_1 and z_2 be the first point at which the solution $(u_1(x), \cdots, u_n(x))$ intersects first hyperplane $q = \sum\limits_{i=1}^{n} a_i d_i u_i = \lambda_2$ in the $u_1 \cdots u_n$-space when x moves from z_0 toward

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$-\infty$ and $+\infty$, respectively. That is,

$$z_1 = \inf\{z \in (-\infty, z_0) \mid q(x) = \sum_{i=1}^{n} a_i d_i u_i(x) < \lambda_2,\ \forall x \in (z, z_0)\}$$

and

$$z_2 = \sup\{z \in (z_0, +\infty) \mid q(x) = \sum_{i=1}^{n} a_i d_i u_i(x) < \lambda_2,\ \forall x \in (z_0, z)\}.$$

Hence, $q'(z_1) \le 0$ and $q'(z_2) \ge 0$. Furthermore, since $(u_1(z_0), \cdots, u_n(z_0))$ lies underneath the second hyperplane $p = \sum_{i=1}^{n} a_i\, u_i = \eta$ while $(u_1(z_1), \cdots, u_n(z_1))$ and $(u_1(z_2), \cdots, u_n(z_2))$ lie above which, $p(z_0) < \eta$ and $p(z_1)$, $p(z_2) > \eta$. In addition, since the arcs $\{(u_1(x), \cdots, u_n(x)) \mid x \in (z_1, z_0)\}$ and $\{(u_1(x), \cdots, u_n(x)) \mid x \in (z_0, z_2)\}$ lie in $\mathcal{Q}_{\lambda_2} \subset \underline{\mathcal{R}}$, $F(u_1(x), \cdots, u_n(x)) > 0$ for all $x \in (z_1, z_0)$ or $x \in (z_0, z_2)$.

For the case where $\theta \ge 0$, integrating (2.3.2) from z_0 to z_2 yields a contradiction:

$$\underbrace{q'(z_2)}_{\ge 0} - \cancelto{0}{q'(z_0)} + \theta(\underbrace{p(z_2)}_{>\eta} - \underbrace{p(z_0)}_{<\eta}) + \underbrace{\int_{z_0}^{z_2} F(u_1(x), \cdots, u_n(x))dx}_{>0} \le \underset{\rightarrow\leftarrow}{0}$$

For the other case where $\theta \le 0$, we integrate (2.3.2) from z_1 to z_0 to obtain a contradiction:

$$\cancelto{0}{q'(z_0)} - \underbrace{q'(z_1)}_{\le 0} + \theta(\underbrace{p(z_0)}_{<\eta} - \underbrace{p(z_1)}_{>\eta}) + \underbrace{\int_{z_1}^{z_0} F(u_1(x), \cdots, u_n(x))dx}_{>0} \le \underset{\rightarrow\leftarrow}{0}$$

Thus

$$\sum_{i=1}^{n} a_i d_i u_i(x) \ge \lambda_1 = \left(\min_{i=1,\cdots,n} a_i d_i \underline{u}_i\right) \frac{\displaystyle\min_{i=1,\cdots,n} d_i}{\displaystyle\max_{i=1,\cdots,n} d_i}.$$

By taking $a_i = \frac{\alpha_i}{d_i}$, we obtain the desired result:

$$\sum_{i=1}^{n} \alpha_i u_i(x) \ge \left(\min_{i=1,\cdots,n} \alpha_i \underline{u}_i\right) \frac{\displaystyle\min_{i=1,\cdots,n} d_i}{\displaystyle\max_{i=1,\cdots,n} d_i}.$$

□

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Note that if the diffusion rates d_i's are all equal, then the three hyperplanes of the N-barrier coincide. Nevertheless, the proof can still be accomplished as remarked after the proof of Theorem 2.1.

Theorem 2.6 (Multi-species upper bound). *Let $(u_1(x), \cdots, u_n(x))$ be a nonnegative solution to*

$$\begin{cases} d_i u_i'' + \theta u_i' + u_i^{m_i} f_i(u_1, \cdots, u_n) \geq 0, \ x \in \mathbb{R}, \ i = 1, \cdots, n, \\ (u_1, \cdots, u_n)(-\infty) = \mathbf{e}_-, \ (u_1, \cdots, u_n)(+\infty) = \mathbf{e}_+. \end{cases}$$

Assume that for $i = 1, \cdots, n$, there exists $\overline{u}_i > 0$ s.t. $f_i(u_1, \cdots, u_n) < 0$, whenever $(u_1, \cdots, u_n) \in \overline{\mathcal{R}}$, and $\mathbf{e}_-, \mathbf{e}_+ \in [0, +\infty)^n \setminus \overline{\mathcal{R}}$, where

$$\overline{\mathcal{R}} = \left\{ (u_1, \cdots, u_n) \in [0, +\infty)^n \,\middle|\, \sum_{i=1}^{n} \frac{u_i}{\overline{u}_i} > 1 \right\}.$$

Then for any $\alpha_i > 0$, we have the following upper bound:

$$\sum_{i=1}^{n} \alpha_i u_i(x) \leq \left(\max_{i=1,\cdots,n} \alpha_i \overline{u}_i \right) \frac{\displaystyle\max_{i=1,\cdots,n} d_i}{\displaystyle\min_{i=1,\cdots,n} d_i}.$$

Although the lower and upper bounds achieved above are used for the traveling wave solutions of the Lotka-Volterra systems, say (1.0.1), they actually can be applied to the steady state solutions for (1.0.1):

$$\begin{cases} d_1 u'' + u(\sigma_1 - c_{11} u - c_{12} v) = 0, \ x \in \mathbb{R}, \\ d_2 v'' + v(\sigma_2 - c_{21} u - c_{22} v) = 0, \ x \in \mathbb{R}. \end{cases} \tag{2.3.6}$$

Namely, $\theta = 0$ in the traveling wave version (1.0.4). In fact, $\theta = 0$ invalidates the affect of the linear term p in (2.1.5) and (2.1.6). Therefore, the first line of the N-barrier is the only line used in the *proof*, and the estimates of $\alpha u + \beta v$ in the steady states are better than which in the traveling waves. That is to say,

$$\min \left\{ \alpha \frac{\sigma_2}{c_{21}}, \beta \frac{\sigma_1}{c_{12}} \right\} \leq \alpha u + \beta v \leq \max \left\{ \alpha \frac{\sigma_1}{c_{11}}, \beta \frac{\sigma_2}{c_{22}} \right\},$$

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which is independent of d_1 and d_2, for the solutions to the steady state system (2.3.6).

3 Application: Nonexistence Results

The existence of traveling wave solutions of three-species diffusive competitive Lotka-Volterra systems is achieved in [2]. Chen, Hung, Mimura and Ueyama made an ansatz that

$$(u(x), v(x), w(x)) = (k_1(1 + \tanh x), k_2(1 + \tanh x)^2, k_3(1 + \tanh^2 x))$$

and verified it an exact solution under certain parameters and for suitable k_1, k_2 and k_3.

On the other hand, the nonexistence of traveling wave solutions of three-species diffusive competitive Lotka-Volterra systems can be achieved with the aid of Theorem 2.1.

Theorem 3.1 (Nonexistence of 3-species wave)**.** *Suppose that*

[H1] $\tilde{\sigma}_1 := \sigma_1 - c_{13}\frac{\sigma_3}{c_{33}} > 0,\ \tilde{\sigma}_2 := \sigma_2 - c_{23}\frac{\sigma_3}{c_{33}} > 0,$

[H2] $\min\{\frac{c_{31}\tilde{\sigma}_2}{c_{21}}, \frac{c_{32}\tilde{\sigma}_1}{c_{12}}\}\frac{\min\{d_1,d_2\}}{\max\{d_1,d_2\}} \geq \sigma_3.$

Then the three-species Lotka-Volterra system

$$\begin{cases} d_1u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v - c_{13}w) = 0,\ x \in \mathbb{R}, \\ d_2v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v - c_{23}w) = 0,\ x \in \mathbb{R}, \\ d_3w'' + \theta w' + w(\sigma_3 - c_{31}u - c_{32}v - c_{33}w) = 0,\ x \in \mathbb{R}, \\ (u,v,w)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, 0\right),\ (u,v,w)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}, 0\right) \end{cases} \tag{3.0.1}$$

has no positive solution $(u(x), v(x), w(x))$.

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Proof. Suppose contrary, then there exists a solution $(u(x), v(x), w(x))$, where $u(x)$, $v(x)$, and $w(x) > 0$, for all $x \in \mathbb{R}$. According to the boundary condition w(±∞)=0, there must be a $x_0 \in \mathbb{R}$ s.t. $w(x_0) = \max w$. At this point, the third equality of the system (3.0.1) becomes

$$\underbrace{d_3}_{>0} \underbrace{w''(x_0)}_{\leq 0} + \cancel{\theta w'(x_0)}^{0} + \underbrace{w(x_0)}_{>0}(\sigma_3 - c_{31}u(x_0) - c_{32}v(x_0) - c_{33}w(x_0)) = 0.$$

This shows that

$$\sigma_3 - c_{31}u(x_0) - c_{32}v(x_0) - c_{33}w(x_0) \geq 0. \tag{3.0.2}$$

Hence, $w(x) \leq w(x_0) \leq \frac{1}{c_{33}}(\sigma_3 - c_{31}u(x_0) - c_{32}v(x_0)) < \frac{\sigma_3}{c_{33}}$, $\forall x \in \mathbb{R}$. Substituting the upper bound for w into the first two equations in (3.0.1), the three-species case will be reduced into the two-species case:

$$\begin{cases} d_1u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v - c_{13}\frac{\sigma_3}{c_{33}}) < 0, \\ d_2v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v - c_{23}\frac{\sigma_3}{c_{33}}) < 0, \end{cases}$$

or

$$\begin{cases} d_1u'' + \theta u' + u(\underbrace{(\sigma_1 - c_{13}\frac{\sigma_3}{c_{33}})}_{\tilde{\sigma}_1} - c_{11}u - c_{12}v) < 0, \\ d_2v'' + \theta v' + v(\underbrace{(\sigma_2 - c_{23}\frac{\sigma_3}{c_{33}})}_{\tilde{\sigma}_2} - c_{21}u - c_{22}v) < 0. \end{cases}$$

Apply Theorem 2.1 with $\alpha = c_{31}, \beta = c_{32}$ and exploit the hypothesis [**H2**] to get the lower bound:

$$c_{31}u + c_{32}v \geq \min\left\{\frac{c_{31}\tilde{\sigma}_2}{c_{21}}, \frac{c_{32}\tilde{\sigma}_1}{c_{12}}\right\} \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}} \geq \sigma_3.$$

But from (3.0.2),

$$w(x_0) \leq \frac{1}{c_{33}}(\sigma_3 - c_{31}u(x_0) - c_{32}v(x_0)) \leq 0,$$

a contradiction. □

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Note that the steps of the *proof* for the nonexistence of three-species waves are that, first, reducing the three equations to two inequalities and, second, applying the lower bound obtained from the N-barrier maximum principle to reach a contradiction. Consequently, once the N-barrier maximum principle for multi-species Theorem 2.5 has been established, the nonexistence of multi-species waves follows:

Theorem 3.2 (Nonexistence result for multi-species)**.** *Suppose that*

$$\begin{cases} \tilde{\sigma}_i := \sigma_i - c_{in}\frac{\sigma_n}{c_{nn}} > 0, i = 1, \cdots, n-1, \\ \min\limits_{i=1,\cdots,n} \left\{ c_{ni} \min\limits_{j=1,\cdots,n} \frac{\tilde{\sigma}_i}{c_{ji}} \right\} \frac{\min\limits_{i=1,\cdots,n} d_i}{\max\limits_{i=1,\cdots,n} d_i} \geq \sigma_n. \end{cases}$$

Then the n-species Lotka-Volterra system

$$\begin{cases} d_i u_i'' + \theta u_i' + u_i \left(\sigma_i - \sum\limits_{j=1}^{n} c_{ij} u_j \right) = 0 \ , x \in \mathbb{R}, \ i = 1, \cdots, n, \\ (u_1, \cdots, u_n)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, \cdots, 0 \right), \\ (u_1, \cdots, u_n)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}, 0, \cdots, 0 \right) \end{cases}$$

has no positive solution $(u_1(x), \cdots, u_n(x))$.

For another application of the N-barrier maximum principle, the readers are referred to [3] for the existence of three-species waves under a different boundary condition. Hung first reduced the three equations to a single equation and employed the N-barrier maximum principle to construct a subsolution, then the method of supersolution-subsolution guarantee a solution.

4 Improved Tangent Line Method

In this section, a refined lower bound for $\alpha u + \beta v$ will be derived. For computational convenience, we consider the scaled system (1.0.7) with the bistable condition: a_1, $a_2 > 1$ and apply the improved tangent line method.

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In [4], it is shown that, under certain restrictions on the parameters, the lower bound in Theorem 2.1 can be improved by means of the tangent line method. However, the restrictions are redundant, since the idea is to enlarge λ_2 so that the N-barrier still lies underneath the quadratic curve:

$$F(u,v) = au(1-u-a_1v) + b\sigma v(1-a_2u-v) = 0.$$

To be more specific:

$$\mathcal{Q}_{\lambda_1} \subset \mathcal{P}_\eta \subset \mathcal{Q}_{\lambda_2} \subset \mathcal{R}, \tag{4.0.1}$$

where

$$\begin{aligned}
\mathcal{P}_\eta &= \Big\{(u,v) \;\Big|\; au+bv \le \eta,\ u,v \ge 0\Big\}; && (4.0.2)\\
\mathcal{Q}_\lambda &= \Big\{(u,v) \;\Big|\; au+dbv \le \lambda,\ u,v \ge 0\Big\}; && (4.0.3)\\
\mathcal{R} &= \Big\{(u,v) \;\Big|\; F(u,v) \ge 0,\ u,v \ge 0\Big\}. && (4.0.4)
\end{aligned}$$

In fact, λ_2 can be given by

$$\lambda_2 = \sup\{\lambda \mid \mathcal{Q}_\lambda \subset \mathcal{R}\}. \tag{4.0.5}$$

Replacing the first step for determining λ_2 in Theorem 2.1 by (4.0.5), a stronger lower bound than the one given by Theorem 2.1 can be found. In other words, the estimate can be refined without any further restriction on the parameters.

To calculate λ_2, we first solve v as a function of u in the hyperbola

$$F(u,v) = bkv^2 + (aa_1u + b\sigma(a_2u-1))v + au(u-1) = 0,$$

and choose the branch which does not pass through the origin. That is,

$$v(u) = \frac{-(aa_1u + b\sigma(a_2u-1)) + \sqrt{(aa_1u + b\sigma(a_2u-1))^2 - 4ab\sigma u(u-1)}}{2b\sigma}.$$

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So the tangent to the curve is

$$\frac{dv}{du}(u) = \frac{-(aa_1 + b\sigma a_2) + \frac{(aa_1u+b\sigma(a_2u-1))(aa_1+b\sigma a_2)-2ab\sigma(2u-1)}{\sqrt{(aa_1u+b\sigma(a_2u-1))^2-4ab\sigma u(u-1)}}}{2b\sigma}.$$

For given a, b and d, the slope of the line $au + dbv = \lambda_2$ is determined by $\frac{-a}{db}$. The supremum expression (4.0.5) shows that the line $au + dbv = \lambda_2$ should tangent to the hyperbola $F = 0$. However, since we are working in the first quadrant, there are two critical tangents: $\frac{dv}{du}(0) = \frac{-a(a_1-1)-b\sigma a_2}{b\sigma}$ and $\frac{dv}{du}(1) = \frac{-a}{aa_1+b\sigma(a_2-1)}$. Because the branch of the hyperbola we have chosen is convex (see Figure 1), the tangent to the curve $v(u)$ is increasing. Thus, there are three cases to be considered:

(i) $\frac{-a}{db} < \frac{-a(a_1-1)-b\sigma a_2}{b\sigma}$:

In this case, the first line of N-barrier $au+dbv = \lambda_2$ passes through the boundary $(0, 1)$, so λ_2 is determined as

$$\lambda_2 = a \cdot 0 + db \cdot 1 = db.$$

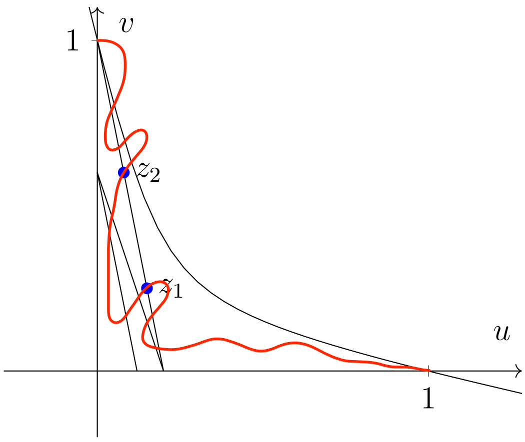

Figure 8: N-barrier for case (i)

Note that z_2 may be $+\infty$ in the proof of Theorem 2.1 in this case. In fact, $\lim\limits_{x\to+\infty} q'(x) \geq 0$, $\lim\limits_{x\to+\infty} p(x) > \eta$ and $\int_{z_0}^{+\infty} F(u(x), v(x))dx > 0$ still reach a contradiction as (2.1.5).

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(ii) $\frac{-a}{db} > \frac{-a}{aa_1+b\sigma(a_2-1)}$:

In this case, the first line of N-barrier $au+dbv = \lambda_2$ passes through the boundary $(1,0)$, so λ_2 is determined as

$$\lambda_2 = a \cdot 1 + db \cdot 0 = a.$$

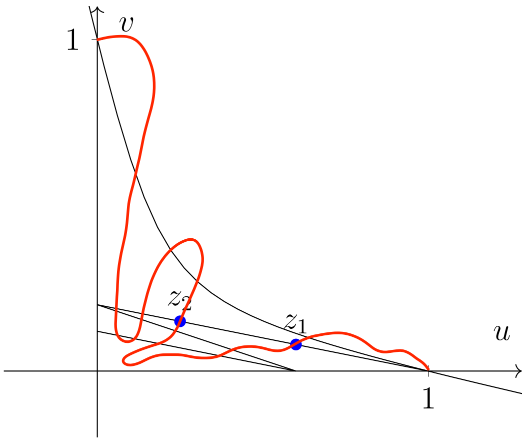

Figure 9: N-barrier for case (ii)

Note that z_1 may be $-\infty$ in the proof of Theorem 2.1 in this case. In fact, $\lim_{x\to-\infty} q'(x) \leq 0$, $\lim_{x\to-\infty} p(x) > \eta$ and $\int_{-\infty}^{z_0} F(u(x),v(x))dx > 0$ still reach a contradiction as (2.1.5).

(iii) $\frac{-a(a_1-1)-b\sigma a_2}{b\sigma} < \frac{-a}{db} < \frac{-a}{aa_1+b\sigma(a_2-1)}$:

In this case, the first line of N-barrier $au + dbv = \lambda_2$ is tangent to the curve $v(u)$.

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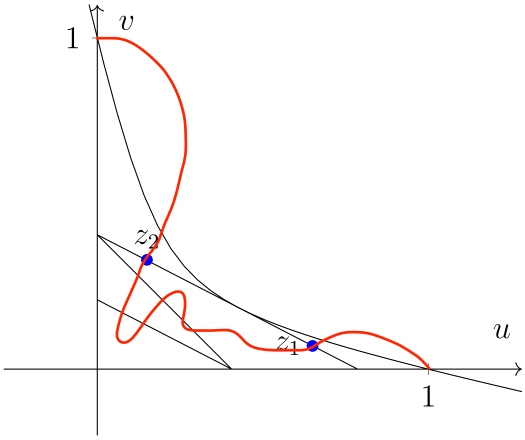

Figure 10: N-barrier for case (iii)

Therefore,

$$\frac{dv}{du}(u) = \frac{-(aa_1 + b\sigma a_2) + \frac{(aa_1 u + b\sigma(a_2 u - 1))(aa_1 + b\sigma a_2) - 2ab\sigma(2u-1)}{\sqrt{(aa_1 u + b\sigma(a_2 u - 1))^2 - 4ab\sigma u(u-1)}}}{2b\sigma} = \frac{-a}{db},$$

or

$$\frac{\left[(X^2 - 4ab\sigma)u + (-b\sigma X + 2ab\sigma)\right]^2}{(Xu - b\sigma)^2 - 4ab\sigma u(u-1)} = \left(X - \frac{2a\sigma}{d}\right)^2,$$

where $X = aa_1 + b\sigma a_2$. Then it becomes

$$\frac{Au^2 + Bu + C}{Du^2 + Eu + F} = G, \tag{4.0.6}$$

where

$$A = \left(X^2 - 4ab\sigma\right)^2,$$

$$B = 2(X^2 - 4ab\sigma)(-b\sigma X + 2ab\sigma),$$

$$C = (-b\sigma X + 2ab\sigma)^2,$$

$$D = X^2 - 4ab\sigma,$$

$$E = -2b\sigma X + 4ab\sigma,$$

$$F = b^2\sigma^2,$$

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$$G = \left(X - \frac{2a\sigma}{d}\right)^2.$$

Write (4.0.6) as

$$(A - DG)u^2 + (B - EG)u + (C - FG) = 0,$$

we have

$$u_0 = \frac{-(B - EG) \pm \sqrt{(B - EG)^2 - 4(A - DG)(C - FG)}}{2(A - DG)}.$$

Hence,

$$\begin{aligned}
\lambda_2 =& au_0 + dbv(u_0) \\
=& au_0 + db\frac{-(aa_1u_0 + b\sigma(a_2u_0 - 1)) + \sqrt{(aa_1u + b\sigma(a_2u_0 - 1))^2 - 4ab\sigma u_0(u_0 - 1)}}{2b\sigma},
\end{aligned}$$

where

$$u_0 = \frac{-(B - EG) \pm \sqrt{(B - EG)^2 - 4(A - DG)(C - FG)}}{2(A - DG)}$$

and the branch is chosen s.t.

$$0 < u_0 < 1$$

and

$$\frac{dv}{du}(u_0) = \frac{-X + \frac{(Xu_0 - b\sigma)X - 2abk(2u_0 - 1)}{\sqrt{(Xu_0 - b\sigma)^2 - 4ab\sigma u_0(u_0 - 1)}}}{2b\sigma} = \frac{-a}{db}.$$

Note that the branch of u_0 can not be determined unless the coefficients are given. Actually, by using the computer program MATLAB, we can show that for $a = 12$, $b = 2$, $a_1 = 5$, $a_2 = 3$, $d = 2$ and $\sigma = 6$, u_0 should be $\frac{-(B-EG)+\sqrt{(B-EG)^2-4(A-DG)(C-FG)}}{2(A-DG)}$; while for $a = 12$, $b = 1$, $a_1 = 12$, $a_2 = 2$, $d = 1$ and $\sigma = 12$, u_0 should be $\frac{-(B-EG)-\sqrt{(B-EG)^2-4(A-DG)(C-FG)}}{2(A-DG)}$.

Recall that in the *proof* of Theorem 2.1 we took $a = \frac{\alpha}{d_1}$ and $b = \frac{\beta}{d_2}$; while in this

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rescaled case, $a = \alpha$ and $b = \frac{\beta}{d}$. In conclusion, we have the following refined lower bound:

Theorem 4.1 (Refined estimate)**.** *Let $(u(x), v(x))$ be a nonnegative solution to (1.0.7), i.e.*

$$\begin{cases} u'' + \theta u' + u(1 - u - a_1 v) = 0, \ x \in \mathbb{R}, \\ dv'' + \theta v' + \sigma v(1 - a_2 u - v) = 0, \ x \in \mathbb{R}, \\ (u, v)(-\infty) = (1, 0), \ (u, v)(+\infty) = (0, 1), \end{cases}$$

where a_1, $a_2 > 1$. Then for any $\alpha, \beta > 0$ we have the following lower bound:

$$\alpha u(x) + \beta v(x) \geq \lambda_1 \frac{\min\{1, d\}}{\max\{1, d\}},$$

where

$$\lambda_1 = \begin{cases} \beta & , \text{ if } \frac{-\alpha}{\beta} < \frac{-\alpha(a_1-1) - \frac{\beta}{d}\sigma a_2}{\frac{\beta}{d}\sigma}, \\ \alpha & , \text{ if } \frac{-\alpha}{\beta} > \frac{-\alpha}{\alpha a_1 + \frac{\beta}{d}\sigma(a_2-1)}, \\ \alpha u_0 + \beta \frac{-(\alpha a_1 u_0 + \frac{\beta}{d}\sigma(a_2 u_0 - 1)) + \sqrt{(\alpha a_1 u_0 + \frac{\beta}{d}\sigma(a_2 u_0 - 1))^2 - 4\frac{\alpha\beta}{d}\sigma u_0(u_0-1)}}{2\frac{\beta}{d}\sigma} & , \text{ otherwise,} \end{cases}$$

in which $u_0 = \frac{-(B-EG) \pm \sqrt{(B-EG)^2 - 4(A-DG)(C-FG)}}{2(A-DG)}$, *where* $A = \left(X^2 - 4\frac{\alpha\beta}{d}\sigma\right)^2$, $B = 2(X^2 - 4\frac{\alpha\beta}{d}\sigma)(-\frac{\beta}{d}\sigma X + 2\frac{\alpha\beta}{d}\sigma)$, $C = \left(-\frac{\beta}{d}\sigma X + 2\frac{\alpha\beta}{d}\sigma\right)^2$, $D = X^2 - 4\frac{\alpha\beta}{d}\sigma$, $E = -2\frac{\beta}{d}\sigma X + 4\frac{\alpha\beta}{d}\sigma$, $F = \frac{\beta^2}{d^2}\sigma^2$ *and* $G = \left(X - \frac{2\alpha\sigma}{d}\right)^2$ *in which* $X = \alpha a_1 + \frac{\beta}{d}\sigma a_2$. *The branch is chosen s.t.* $0 < u_0 < 1$ *and* $\left(-X + \frac{(Xu_0 - \frac{\beta}{d}\sigma)X - 2\frac{\alpha\beta}{d}\sigma(2u_0-1)}{\sqrt{(Xu_0 - \frac{\beta}{d}\sigma)^2 - 4\frac{\alpha\beta}{d}\sigma u_0(u_0-1)}}\right) / (2\frac{\beta}{d}\sigma) = \frac{-\alpha}{\beta}$.

5 Examples

Exact solutions to two- and three-species Lotka-Volterra systems are proposed in [8] and [2], respectively. Both examples will be performed in this section. For the three-species exact solution, we will check that the upper and lower bounds in Theorem 2.5 and Theorem 2.6 are valid. For the two-species exact solution, we first compute the lower bound via the improved tangent line method Theorem 4.1 and

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check the lower bound valid. Then compare it with the lower bound obtained from the original N-barrier maximum principle Theorem 2.1.

5.1 An Example of NBMP for 3-species

By the ansatz that

$$\begin{cases} u(x) = k_1(1-\tanh x)^2, \\ v(x) = k_2(1+\tanh x), \\ w(x) = k_3(1-\tanh^2 x), \end{cases}$$

[2] provides exact solutions to the 3-species Lotka-Volterra system:

$$\begin{cases} d_1u'' + \theta u' + u(\sigma_1 - c_{11}u - c_{12}v - c_{13}w) = 0, \ x \in \mathbb{R}, \\ d_2v'' + \theta v' + v(\sigma_2 - c_{21}u - c_{22}v - c_{23}w) = 0, \ x \in \mathbb{R}, \\ d_3w'' + \theta w' + w(\sigma_3 - c_{31}u - c_{32}v - c_{33}w) = 0, \ x \in \mathbb{R}, \\ (u,v,w)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, 0\right), \ (u,v,w)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}}, 0\right) \end{cases} \tag{5.1.1}$$

for $k_1 = \frac{\sigma}{4}$, $k_2 = \frac{\sigma}{2}$, $d_1 = d_2 = d_3 = c_{11} = c_{22} = c_{33} = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$,

$$c_{21} = \frac{3c_{23}-1}{\sigma(-1+c_{23})}, \quad c_{12} = \frac{-8-3\sigma+c_{23}(3\sigma-24)}{\sigma(1-c_{23})}, \quad c_{13} = \frac{(\sigma-24)(c_{23}-1)}{16},$$

$$c_{32} = \frac{2(-\sigma-8c_{23}+\sigma c_{23})}{\sigma(-1+c_{23})}, \quad c_{31} = \frac{8(-1+3c_{23})}{\sigma(-1+c_{23})}, \quad k_3 = \frac{4}{-1+c_{23}},$$

$$\theta = \frac{-4+\sigma+20c_{23}-\sigma c_{23}}{2(-1+c_{23})}$$

and

$$\frac{16c_{13}}{-1+c_{13}} < \sigma < \frac{-8+24c_{13}}{-1+c_{13}} \text{ , if } 1 < c_{13} \leq 3,$$

$$24 < \sigma < \frac{-8+24c_{13}}{-1+c_{13}} \text{ , if } c_{13} > 3.$$

Take $\sigma = 28$ and $c_{23} = 4$ for example, then $c_{21} = \frac{22}{21}$, $c_{12} = \frac{37}{21}$, $c_{13} = \frac{3}{4}$, $c_{32} = \frac{26}{21}$,

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$c_{31} = \frac{22}{21}$, and $\theta = -\frac{4}{3}$. Thus, (5.1.1) becomes

$$\begin{cases} u'' + \theta u' + u(28 - u - \frac{37}{21}v - \frac{3}{4}w) = 0, \ x \in \mathbb{R}, \\ v'' + \theta v' + v(28 - \frac{22}{21}u - v - 4w) = 0, \ x \in \mathbb{R}, \\ w'' + \theta w' + w(28 - \frac{22}{21}u - \frac{26}{21}v - w) = 0, \ x \in \mathbb{R}, \\ (u, v, w)(-\infty) = (1, 0, 0)\,, \ (u, v, w)(+\infty) = (0, 1, 0)\,. \end{cases} \tag{5.1.2}$$

And the exact solution is

$$\begin{cases} u(x) = 7(1 - \tanh x)^2, \\ v(x) = 14(1 + \tanh x), \\ w(x) = \frac{4}{3}(1 - \tanh^2 x). \end{cases}$$

Also, we can choose $\overline{u}_1 = \overline{u} = 28$, $\underline{u}_1 = \underline{u} = \frac{28\cdot 21}{22}$, $\overline{u}_2 = \overline{v} = 28$, $\underline{u}_2 = \underline{v} = \frac{28\cdot 21}{37}$, $\overline{u}_3 = \overline{w} = \frac{28\cdot 4}{3}$, and $\underline{u}_3 = \underline{w} = 7$ in Theorem 2.5 and Theorem 2.6. Take $\alpha_1 = 1$, $\alpha_2 = \frac{1}{5}$, and $\alpha_3 = \frac{1}{2}$ in Theorem 2.5 and Theorem 2.6 for example, we have the following estimates:

$$\min\left\{\frac{28\cdot 21}{22}, \ \ \frac{1}{5}\cdot\frac{28\cdot 21}{37}, \ \ \frac{1}{2}\cdot 7\right\} \le u + \frac{1}{5}v + \frac{1}{2}w \le \max\left\{28, \ \ \frac{1}{5}\cdot 28, \ \ \frac{1}{2}\cdot\frac{28\cdot 4}{3}\right\},$$

or

$$\frac{1}{5}\cdot\frac{28\cdot 21}{37} \le u + \frac{1}{5}v + \frac{1}{2}w \le 28.$$

The upper bound and lower bound are shown in Figure 11.

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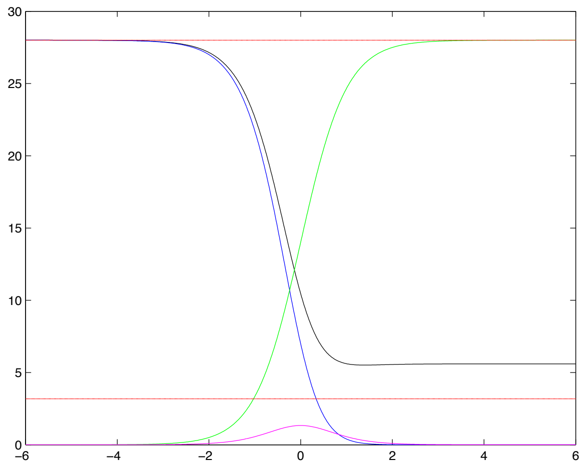

Figure 11: blue curve: u; green curve: v; pink curve: w; black curve: $u + \frac{1}{5}v + \frac{1}{2}w$; red line: upper and lower bounds

5.2 An Example of Improved Tangent Line Method

By the ansatz that $u' = \sum_{i=0}^{m} a_i u^i$ and $v' = \sum_{i=0}^{n} b_i u^i$ are both polynomials of u, [8] provides exact solutions of (1.0.7) under the parameter assumption:

$$d = 3\frac{a_2}{\sqrt{\sigma}}, \quad a_1\sqrt{\sigma} = 2 + \frac{5\sqrt{\sigma}}{3} - a_2, \quad \frac{\sqrt{\sigma}}{a_2} < \sqrt{\sigma} < a_1\sqrt{\sigma}, \theta \quad = \frac{-2 + a_2}{\sqrt{2a_2}}.$$

In particular, for $d = \frac{29}{5}$, $\sigma = 1$, $a_1 = \frac{26}{15}$ and $a_2 = \frac{29}{15}$,

$$\begin{aligned} u(x) &= \frac{1}{4}\left(1 - \tanh\left(\frac{x}{\sqrt{24}}\right)\right)^2 \\ v(x) &= \frac{1}{2}\left(1 + \tanh\left(\frac{x}{\sqrt{24}}\right)\right) \end{aligned}$$

is an exact solution.

Take $\alpha = 2$ and $\beta = \frac{1}{3}$ in Theorem 4.1 for example. Since

$$\frac{-\alpha}{\alpha a_1 + \frac{\beta}{d}\sigma(a_2 - 1)}(\approx -0.5) < \frac{-\alpha}{\beta}(= -6) < \frac{-\alpha(a_1 - 1) - \frac{\beta}{d}\sigma a_2}{\frac{\beta}{d}\sigma}(\approx -27.5),$$

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this is a tangent case, then λ_1 should be taken as

$$\lambda_1 = \alpha u_0 + \beta \frac{-(\alpha a_1 u_0 + \frac{\beta}{d}\sigma(a_2 u_0 - 1)) + \sqrt{(\alpha a_1 u_0 + \frac{\beta}{d}\sigma(a_2 u_0 - 1))^2 - 4\frac{\alpha\beta}{d}\sigma u_0(u_0 - 1)}}{2\frac{\beta}{d}\sigma}$$

in Theorem 4.1. If we choose $u_0 = \frac{-(B-EG)-\sqrt{(B-EG)^2-4(A-DG)(C-FG)}}{2(A-DG)}$, then $\frac{dv}{du}(u_0)(\approx -56.3) \neq \frac{-\alpha}{\beta}(= -6)$; while if the branch of u_0 is chosen as

$$u_0 = \frac{-(B-EG) + \sqrt{(B-EG)^2 - 4(A-DG)(C-FG)}}{2(A-DG)}, \tag{5.2.1}$$

then $\frac{dv}{du}(u_0) = -6 = \frac{-\alpha}{\beta}$. Thus, we choose u_0 as (5.2.1) in the expression of λ_1. Consequently, the lower bound in the improved tangent line method Theorem 4.1 becomes

$$2u(x) + \frac{1}{3}v(x) \geq \lambda_1 \frac{5}{29} \approx 0.05,$$

which is shown in Figure 12. However, from the original N-barrier maximum principle Theorem 2.1, the lower bound is

$$\min\left\{2 \cdot \frac{15}{29}, \frac{1}{3} \cdot \frac{15}{26}\right\} \frac{\min\left\{1, \frac{29}{5}\right\}}{\max\left\{1, \frac{29}{5}\right\}} = \frac{5 \cdot 5}{26 \cdot 29} \approx 0.03,$$

which is worse than the lower bound obtained via the improved tangent line method.

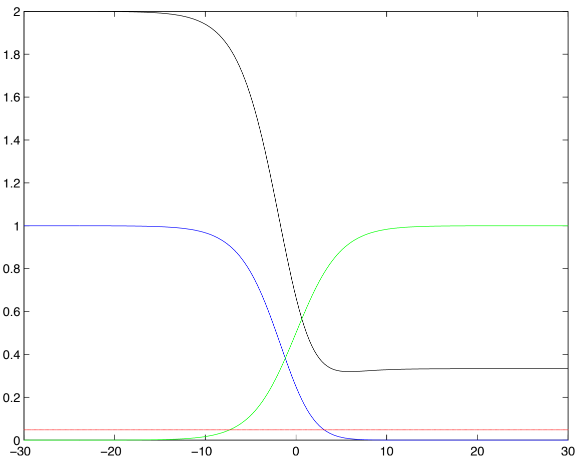

Figure 12: blue curve: u; green curve: v; black curve: $2u + \frac{1}{3}v$; red line: lower bound

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6 Conclusion and Future Studies

For one-dimensional multi-species diffusive competitive Lotka-Volterra systems, the N-barrier maximum principle still provides a priori estimates for the total density of traveling wave solutions. As a corollary, nonexistences of traveling wave solutions in one-dimensional diffusive Lotka-Volterra system of multiple competing species rise. Furthermore, the improved tangent line method ameliorates the lower bound explicitly.

One of the crucial assumptions is the uniformity of different species waves, in both velocity and direction. Based on this hypothesis, we can easily choose the reference coordinates as their wavefronts, while difficulties raise when the waves goes in various speed or direction. Another interesting and practical question is that whether there are some relevant results for the two- or higher-dimensional case.

7 Appendix: Minimal Wave Speed

In this section, we first investigate the minimal wave speed of the Fisher-Kolmogorov equation (1.0.2) by phase plane analysis as in Chapter 13 of [6], and then apply the same approach to the Lotka-Volterra systems.

Consider traveling wave solutions to the Fisher-Kolomogroff equation (1.0.2) with the boundary condition $u(-\infty) = \frac{\sigma_1}{c_{11}},\ u(+\infty) = 0$, we have

$$\begin{cases} d_1 u'' + \theta u' + u(\sigma_1 - c_{11} u) = 0,\ x \in \mathbb{R}, \\ u(-\infty) = \frac{\sigma_1}{c_{11}},\ u(+\infty) = 0. \end{cases} \tag{7.0.1}$$

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To be specific we assume $\theta > 0$. Denote the first derivative u' by U, then

$$\begin{pmatrix} u \\ U \end{pmatrix}' = \begin{pmatrix} U \\ \frac{1}{d_1}(-\theta U - u(\sigma_1 - c_{11}u)) \end{pmatrix}. \tag{7.0.2}$$

Around the singularity $(u, U) = (0, 0)$,

$$\begin{pmatrix} u \\ U \end{pmatrix}' \approx \begin{pmatrix} U \\ \frac{1}{d_1}(-\theta U - \sigma_1 u) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-\sigma_1}{d_1} & \frac{-\theta}{d_1} \end{pmatrix} \begin{pmatrix} u \\ U \end{pmatrix}. \tag{7.0.3}$$

The eigenvalues are

$$\lambda = \frac{-\frac{\theta}{d_1} \pm \sqrt{\left(\frac{\theta}{d_1}\right)^2 - 4\frac{\sigma_1}{d_1}}}{2}.$$

Then, if $\left(\frac{\theta}{d_1}\right)^2 < 4\frac{\sigma_1}{d_1}$, the eigenvalues would be complex numbers, and then (u, U) would be a stable spiral near $(0, 0)$. This violates the non-negativity of u. Therefore, $\left(\frac{\theta}{d_1}\right)^2 \geq 4\frac{\sigma_1}{d_1}$, or we have a lower bound for the wave velocity:

$$\theta \geq 2\sqrt{\sigma_1 d_1}. \tag{7.0.4}$$

Note that, in this case, the eigenvalues are real and non-positive, and (u, U) is a stable node near $(0, 0)$. On the other hand, at the other singularity $(u, U) = (\frac{\sigma_1}{c_{11}}, 0)$,

$$\begin{pmatrix} u \\ U \end{pmatrix}' \approx \begin{pmatrix} U \\ \frac{1}{d_1}(-\theta U + \sigma_1 u) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\sigma_1}{d_1} & \frac{-\theta}{d_1} \end{pmatrix} \begin{pmatrix} u \\ U \end{pmatrix}. \tag{7.0.5}$$

The eigenvalues are

$$\lambda = \frac{-\frac{\theta}{d_1} \pm \sqrt{\left(\frac{\theta}{d_1}\right)^2 + 4\frac{\sigma_1}{d_1}}}{2},$$

which are real and have different signs, hence it must be a saddle point. Furthermore, for the positive eigenvalue, u_+ and U_+ must have the same sign in the corresponding eigenvector $\begin{pmatrix} u_+ \\ U_+ \end{pmatrix}$, while for the negative eigenvalue, u_- and U_- must have different

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sign in the corresponding eigenvector $\begin{pmatrix} u_- \\ U_- \end{pmatrix}$.

Now, we show that there exists a solution for any $\theta \geq 2\sqrt{\sigma_1 d_1}$. Consider the triangle Γ in the u, U-plane formed by $u = \frac{\sigma_1}{c_{11}}$, $U + \frac{\theta}{2d_1}u = 0$ and the u-axis (c.f. Figure 13). We are now to show that Γ is a trapping zone. First of all, on the line segment $\overline{OA}$, $u' = U = 0$ and $U' = u'' = -\frac{1}{d_1}u(\sigma_1 - c_{11}u) < 0$. Furthermore, $|U'|$ decreases to 0 as u tends to 0 or tends to 1. Secondly, on $\overline{AB}$, $d_1 U' + \theta u' = 0$, hence $u' = U < 0$ and $U' = -\frac{\theta}{d_1}u' > 0$. In addition, $|u'|$ and $|U'|$ both increase as U moves from A to B.

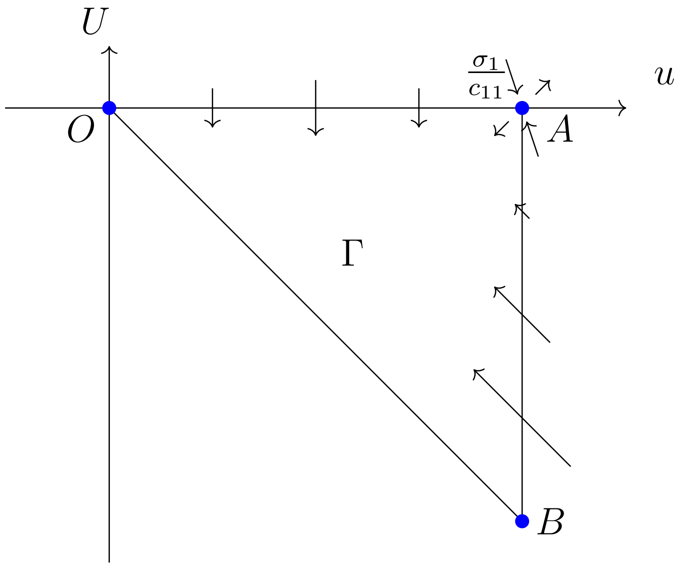

Figure 13: the trapping zone Γ

For $\overline{OB}$, if there is a trajectory cross $\overline{OB}$ from right to left at $s = s_0$, then

$$U'(s_0) + \frac{\theta}{2d_1}u'(s_0) < 0$$

and

$$U(s_0) + \frac{\theta}{2d_1}u(s_0) = 0.$$

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But, according to (7.0.1),

$$\begin{aligned}
U'(s_0)+\frac{\theta}{2d_1}u'(s_0) &= \frac{1}{d_1}[-\theta U(s_0)-u(s_0)(\sigma_1-c_{11}u(s_0))]+\frac{\theta}{2d_1}U(s_0)\\
&= -\frac{\theta}{2d_1}U(s_0)-\frac{1}{d_1}u(s_0)(\sigma_1-c_{11}u(s_0))\\
&= -\frac{\theta}{2d_1}(-\frac{\theta}{2d_1}u(s_0))-\frac{1}{d_1}u(s_0)(\sigma_1-c_{11}u(s_0))\\
&> u(s_0)\left[\left(\frac{\theta}{2d_1}\right)^2-\frac{\sigma_1}{d_1}\right]\\
&\geq 0 \qquad \text{(by the lower bound for the wave speed (7.0.4))},
\end{aligned}$$

which is a contradiction. This is to say, there is no trajectory cross $\overline{OB}$ from right to left. These facts certify that Γ is a trapping zone. Consequently, there must exists a trajectory from the saddle point $(u,U)=(\frac{\sigma_1}{c_{11}},0)$ to the stable node $(u,U)=(0,0)$ lying entirely in Γ. The existence of the trajectory is equivalent to the existence of solution to (7.0.1) for $\theta\geq 2\sqrt{\sigma_1 d_1}$, so these shows that $\theta_{\min}=2\sqrt{\sigma_1 d_1}$ is indeed the minimal wave speed.

For the Lotka-Volterra systems (1.0.7), [7] deal with the special case where $d=\sigma=1$, $a_1+a_2=2$, $a_1<1$, $a_2>1$ and $\theta>0$. They added the two equation in (1.0.7) together to obtain

$$q''+\theta q+q(1-q)=0,$$

where $q(\pm\infty)=1$. Then the classical maximum principle yields $q=1$ or $u+v=1$ for all $x\in\mathbb{R}$. In fact, suppose contrary there exists $x_0\in\mathbb{R}$ with $q(x_0)\neq 1$, say $q(x_0)>1$. Then for some $x_1\in\mathbb{R}$, we have $q(x_1)=\max\ q$. At this point, $q''(x_1)\leq 0$, $q'(x_1)=0$, and $q(x_1)>1$. Accordingly, $q''(x_1)+\theta q(x_1)+q(x_1)(1-q(x_1))<0$, which is a contradiction. The case $q(x_0)<1$ can be shown in a similar manner. Substituting $v=1-u$ into the first equation in (1.0.7) gives

$$u''+\theta u'+(1-a_1u(1-u))=0,$$

which is the Fisher-Kolomogroff equation. Employing the minimal wave speed for

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Fisher-Kolmogroff equation, $\theta^u_{\min} = 2\sqrt{1-a_1}$, i.e. $\theta \geq 2\sqrt{1-a_1}$. On the other hand, it is readily seen that v satisfies

$$v'' + \theta v + (a_2 - 1)v(1-v) = 0,$$

and the minimal wave speed is $\theta^v_{\min} = 2\sqrt{a_2 - 1} = 2\sqrt{1-a_1} = \theta^u_{\min}$.

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