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王-丘擬局部質量之探討

A Note on Wang-Yau Quasi-Local Mass

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碩班兩年說長不長、說短不短，一路走來受到許多人幫忙及指引，讓我得以完成學業。首先要感謝指導教授崔茂培老師的提點，讓我從幾何的門外漢，一步一步爬上階梯到達幾何物理的登山口。崔老師不僅點出我學不深的毛病，也時時提醒我例子的重要性，想問題時應先看例子，才不會迷失方向；記得林琦焜老師說過:「定理來匆匆、去匆匆，但例子永流傳。」我想是很好的註記吧。再來是許多教過我的教授，尤其是王夏聲老師、陳子軒老師，王老師分享了許多故事給我們，也送了許多書給我；陳老師則不厭其煩地示範證明，讓我體會了自然的證明，這也是大學後逐漸丟失的。同時也謝謝王業凱以及王慕道老師的討論，讓我對擬局部質量有更多認識；蔡忠潤老師帶我進入微分幾何之門和莊武諺老師與我分享他在物理領域的所見所聞。此外謝謝吳宗堂學長分享對數學的看法，讓我跳脫自身貧乏的知識，而從宏觀的角度來看數學。當然還有天數五四四的各位以及吳彥慧，讓碩士生活不會如此枯燥。最後還有父母以及臺灣提供的環境，讓我順利走到今天。

最後很慶幸自己還記得來念數學的初衷—了解物理世界。我想數學與物理是相輔相成的，不管發展數學抑或物理，都需要向自然學習。往後依然會以此當目標，繼續往山的彼端邁進。

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摘要

本文回顧王-丘擬局部質量之物理背景與數學建構。最後在第三章討論王-丘擬局部質量之非負特性。

關鍵字: 王-丘擬局部質量

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Abstract

In this master thesis, we review the notion about the Wang-Yau quasi-local mass from the physical background to the mathematical construction. Finally, we discuss the positivity argument of Wang-Yau quasi-local mass in Wang-Yau's original paper.

Keyword: Wang-Yau quasi-local mass

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Chapter 1

Introduction

Mass is matter. It is an important quantity in its own right. It is also a fundamental ingredient in general relativity. Unfortunately, it is not possible to define a mass within an arbitrary region in general spacetimes [8]. However, in asymptotically flat spacetimes, one can do it. This leads to the Bondi mass [1, 13] at null infinity and the ADM mass at spacial infinity. Although the ADM mass is a physical quantity, its positivity is not straightforward as one expects. Here is the original conjecture.

Conjecture (Positive Mass Conjecture). *For any asymptotically flat initial data set satisfying the dominant energy condition, its ADM energy is positive except for the initial data set in the Minkowski spacetime.*

Finally, in around 1980, Schoen and Yau [14, 15] proved the above conjecture. Later, Witten [22] used the spinor method to give another approach.

Theorem 1.1 (Positive Mass Theorem). *Let* (Ω, g_{ij}, p_{ij}) *be a complete, asymptotically flat* 3*-manifold satisfying the dominant energy condition. Then the ADM mass of each end of* Ω *is non-negative.*

However there is no local notion of mass due to the equivalence principle. In any point, we can let Christoffel symbols to be 0. Hence the local energy density is meaningless [8, Section 20.4]. In 1982, Penrose [11] proposed a problem.

Problem. *Find a suitable quasi-local definition of energy-momentum in general relativity.*

After that, many suggestions were proposed. Some approach the problem through the Hamilton-Jacobi method, for instance, the Brown-York quasi-local mass [2] [3] and the Liu-Yau quasi-local mass [6]. For a detailed survey on the quasi-local mass, see [17].

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Definition 1.1 (Brown-York Quasi-Local Mass)**.** *Let Σ be a 2-surface which bounds a spacelike region Ω in a spacetime M. Suppose Σ has positive Gauss curvature. Then the Brown-York quasi-local mass is defined to be*

$$m_{BY} = \frac{1}{8\pi}\left(\int_{X(\Sigma)} k_0 d_{X(\Sigma)} - \int_{\Sigma} k dv_{\Sigma}\right)$$

where k is the mean curvature of Σ with respect to the outward normal of Ω and k_0 is the mean curvature of the isometric embedding $X : \Sigma \hookrightarrow \mathbb{R}^3$.

Definition 1.2 (Liu-Yau Quasi-Local Mass)**.** *Let Σ be an embedded 2-surface in a spacetime M. Suppose Σ has positive Gauss curvature. Then the Liu-Yau quasi-local mass is defined to be*

$$m_{LY} = \frac{1}{8\pi}\left(\int_{X(\Sigma)} k_0 dv_{X(\Sigma)} - \int_{\Sigma} |H| dv_{\Sigma}\right)$$

where H is the mean curvature vector in M and k_0 is the mean curvature of the isometric embedding $X : \Sigma \hookrightarrow \mathbb{R}^3$.

Remark 1.1. Brown and York used $\mathbb{R}^3$ as the reference while Liu and Yau took the norm of mean curvature vector in M.

For positivity of the Brown-York and Liu-Yau quasi-local mass, we have the following theorems.

Theorem 1.2. *[16] Suppose Ω has non-negative scalar curvature and $k > 0$. Then $m_{BY} \geq 0$. Moreover, $m_{BY} = 0$ if and only if Ω is flat.*

Theorem 1.3. *[6, 7] Suppose H is spacelike. Then $m_{LY} \geq 0$. Moreover, $m_{LY} = 0$ only if M is isometric to $\mathbb{R}^{3,1}$ along Σ.*

However there exists some cases in the Minkowski spacetime with strictly positive Brown-York quais-local mass as well as the Liu-Yau quasi-local mass [9]. In 2008, Mu-Tao Wang and Shing-Tung Yau used the momentum term to fix the problem [20].

Definition 1.3 (Wang-Yau Quasi-Local Mass)**.** *Let $X : \Sigma \hookrightarrow M$ be a spacelike embedding with spacelike mean curvature vector and assume the set of admissible functions is nonempty. Then the Wang-Yau quasi-local mass is defined to be*

$$m_{WY} = \inf_{\tau \in A} \{\mathfrak{H}(\Sigma, X_0, \tau) - \mathfrak{H}(\Sigma, X, \tau)\}$$

where X_0 is an isometric embedding into $\mathbb{R}^{3,1}$ and A is the set of all admissible functions. The definition of $\mathfrak{H}$ and the admissible function see Section 3.

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Remark 1.2. The Brown-York and Liu-Yau quasi-local mass involve an isometric embedding into $\mathbb{R}^3$, yet the Wang-Yau quasi-local mass involves an isometric embedding into the Minkowski space.

The rigidity of Wang-Yau quasi-local mass is assured by the following theorem (see Theorem 3.7).

Theorem A. *Let $X : \Sigma \hookrightarrow M$ be an embedding into a spacetime M. Suppose M satisfies the dominant energy condition and the mean curvature vector of $X(\Sigma)$ is spacelike. Then the Wang-Yau quasi-local mass is non-negative and the equality holds if X is isometric to $\mathbb{R}^{3,1}$ along $X(\Sigma)$.*

Remark 1.3. The Wang-Yau quasi-local mass is defined only when the mean curvature vector of Σ is spacelike. There are some important surfaces in general relativity that have not been well-studied. Also there are conjectures involving the timelike mean curvature vector, so it is important to know the timelike case.

Remark 1.4. Wang and Yau used the spacelike condition to find the suitable gauge.

The rest of this thesis is organized as follows. In section 2, we fix notations and review the action principle and the Hamilton formulation to motivate the quasi-local mass. In section 3, we define the Wang-Yau quasi-local mass and sketch the proof of its positivity. You can assume section 3.2 and 3.3 for the proof toward positivity.

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Chapter 2

Preliminary

Quasi-local mass is a physical notion, so it is helpful to know the physics background before jumping into the mathematical formulation. In this chapter, we first recall some notations and review the action principle and the Hamiltonian formulation of the Einstein equations.

2.1 Notations

Throughout this thesis, we adopt the Einstein convention and the units so that the speed of light and the gravitational constant are dimensionless and normalized to 1. The manifold Σ is preserved to denote a Riemannian 2-surface. Let M be a smooth n-manifold with metric g. In local coordinate $\{x^i\}$, the Riemann curvature tensor and the Ricci curvature tensor are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$Ric(X,Y) = g^{ij} \left\langle R\left(\partial_i, X\right) Y, \partial_j \right\rangle.$$

We denote

$$R_{ijkl} = \left\langle R\left(\partial_k, \partial_l\right) \partial_j, \partial_i \right\rangle$$

and

$$R_{ij} = Ric\left(\partial_i, \partial_j\right).$$

In the case of hypersurface, the mean curvature is associated with the outward normal vector unless otherwise specified. Suppose M is a 4-manifold with metric g. Then M is said to be a spacetime if g is of signature $(+++-)$. For any frame on M, index 4 is always assumed to be the timelike direction.

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2.2 Constraint equations and dominant energy condition

The important equation connecting the ambient and submanifold geometry is the Gauss-Codazzi equation. Let M be a spacetime with metric g and $\Omega \subset M$ be a spacelike hypersurface with timelike unit normal vector ν. Let $\{e_i\}_{i=1}^3$ be an orthonormal frame on Ω. Then the Gauss equation reads

$$R_{ijkl} = \widetilde{R}_{ijkl} + K_{lj}K_{ki} - K_{kj}K_{il}$$

where $\widetilde{R}$ is the Riemann curvature tensor on Ω and

$$K_{ij} = \langle \nabla_{e_i}\nu, e_j \rangle$$

is the second fundamental form while the Codazzi equation is

$$R_{4jkl} = -\nabla_k K_{lj} + \nabla_l K_{kj}$$

where we let ν be indexed 4. Contract both sides of the Gauss equation by the induced metric h on Ω, then we have

$$h^{ik}R_{ijkl} = R_{jl} + R_{4j4l} = \widetilde{R}_{jl} + KK_{jl} - K^i_j K_{il}$$

where K is the trace of the second fundamental form. Contract again, then we get

$$R_{44} + \frac{1}{2}R = \frac{1}{2}\left(\widetilde{R} + K^2 - K_{ij}K^{ij}\right). \tag{2.1}$$

On the other hand, contract both sides of the Codazzi equation, then we see that

$$R_{4k} = g^{jl}R_{4jkl} = -\nabla_k K + \nabla^j K_{kj} = \nabla^j\left(-Kg_{jk} + K_{kj}\right). \tag{2.2}$$

Equations (2.1) and (2.2) are called the constraint equations.

Here we focus on spacetimes that satisfy the Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij}$$

where T is the energy-momentum tensor. Let

$$\mu = \frac{1}{2}\left(\widetilde{R} + K^2 - K_{ij}K^{ij}\right)$$

and

$$J_i = \nabla^k\left(-Kg_{ik} + K_{ik}\right).$$

The dominant energy condition says that for every timelike vector V,

$$T^{ij}V_iV_j \geq 0$$

and that $T^{ij}V_i$ is non-spacelike. This means that the energy density is non-negative to any observer and that any energy flow can never be faster than light. It can be shown that

$$\mu \geq |J|$$

using the submanifold geometry and the dominant energy condition. For more introductions on the energy condition, see [4].

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2.3 Action principle

Through out this thesis, we assume that the matter field is vacuum. Given a compact spacetime M, it is known that the vacuum Einstein equations can be obtained from the Hilbert-Einstein action

$$S_{EH} = \frac{1}{16\pi} \int_M R dv_M.$$

Here $R\sqrt{-\det g}/16\pi$ is the Lagrangian density $\mathcal{L}_{EH}$. Let $g(t)$ be a smooth family of metrics on M such that $g(0) = g$. We calculate the variation of the action with respect to the family of metrics. Denote $\frac{\partial}{\partial t}\big|_{t=0}$ by δ.

Proposition 2.1. *The variation of S_{EH} with respect to the metric g, namely δS_{EH}, is given by*

$$-\frac{1}{16\pi} \int_M \left(R^{ij} - \frac{1}{2} g^{ij} R \right) \delta g_{ij} dv_M + \frac{1}{16\pi} \int_M \nabla^l \left(g^{ij} \nabla_j \delta g_{il} - g^{ij} \nabla_l \delta g_{ij} \right) dv_M. \tag{2.3}$$

Proof. In local coordinate, we have

$$\begin{aligned} \frac{d}{dt} \int_M R dv_M =& \frac{d}{dt} \int_M g^{ij} R_{ij} \sqrt{-\det g}\ dx \\ =& \int_M \left(\frac{\partial}{\partial t} g^{ij} \right) R_{ij} \sqrt{-\det g} + g^{ij} \left(\frac{\partial}{\partial t} R_{ij} \right) \sqrt{-\det g} \\ &+ R \frac{\partial}{\partial t} \sqrt{-\det g}\ dx. \end{aligned} \tag{2.4}$$

We point out the key identities. It is straightforward to see that

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} \frac{\partial}{\partial t} g_{kl}. \tag{2.5}$$

For the second term, contract the Riemann curvature tensor to derive

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{kl} \Gamma^l_{ij} - \Gamma^l_{ik} \Gamma^k_{lj}.$$

So

$$\begin{aligned} \frac{\partial}{\partial t} R_{ij} =& \partial_k \left(\frac{\partial}{\partial t} \Gamma^k_{ij} \right) - \partial_j \left(\frac{\partial}{\partial t} \Gamma^k_{ik} \right) + \left(\frac{\partial}{\partial t} \Gamma^k_{kl} \right) \Gamma^l_{ij} + \Gamma^k_{kl} \frac{\partial}{\partial t} \Gamma^l_{ij} - \left(\frac{\partial}{\partial t} \Gamma^l_{ik} \right) \Gamma^k_{lj} - \Gamma^l_{ik} \frac{\partial}{\partial t} \Gamma^k_{lj} \\ =& \left[\partial_k \left(\frac{\partial}{\partial t} \Gamma^k_{ij} \right) + \Gamma^k_{kl} \frac{\partial}{\partial t} \Gamma^l_{ij} - \Gamma^l_{ik} \frac{\partial}{\partial t} \Gamma^k_{lj} \right] - \left[\partial_j \left(\frac{\partial}{\partial t} \Gamma^k_{ik} \right) + \Gamma^k_{lj} \frac{\partial}{\partial t} \Gamma^l_{ik} - \Gamma^l_{ij} \frac{\partial}{\partial t} \Gamma^k_{kl} \right] \\ =& \left[\partial_k \left(\frac{\partial}{\partial t} \Gamma^k_{ij} \right) + \Gamma^k_{kl} \frac{\partial}{\partial t} \Gamma^l_{ij} - \Gamma^l_{ik} \frac{\partial}{\partial t} \Gamma^k_{lj} - \Gamma^l_{kj} \frac{\partial}{\partial t} \Gamma^k_{il} \right] \\ &- \left[\partial_j \left(\frac{\partial}{\partial t} \Gamma^k_{ik} \right) + \Gamma^k_{lj} \frac{\partial}{\partial t} \Gamma^l_{ik} - \Gamma^l_{ij} \frac{\partial}{\partial t} \Gamma^k_{kl} - \Gamma^l_{kj} \frac{\partial}{\partial t} \Gamma^k_{il} \right] \\ =& \nabla_k \left(\frac{\partial}{\partial t} \Gamma^k_{ij} \right) - \nabla_j \left(\frac{\partial}{\partial t} \Gamma^k_{ik} \right). \end{aligned}$$

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Hence

$$g^{ij}\frac{\partial}{\partial t}R_{ij} = \nabla_k\left(g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij}\right) - \nabla_j\left(g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ik}\right) = \nabla_k\left(g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij} - g^{ik}\frac{\partial}{\partial t}\Gamma^l_{il}\right)$$
$$= \left(\nabla^i\nabla^j - g^{ij}\nabla^k\nabla_k\right)\frac{\partial}{\partial t}g_{ij} = \left(g^{ik}g^{jl}\nabla_k\nabla_l - g^{ij}g^{kl}\nabla_k\nabla_l\right)\frac{\partial}{\partial t}g_{ij}$$
$$= \nabla^l\left(g^{ij}\nabla_j\frac{\partial}{\partial t}g_{il} - g^{ij}\nabla_l\frac{\partial}{\partial t}g_{ij}\right). \tag{2.6}$$

For the last term, use identities

$$\frac{\partial}{\partial t}\det g = \operatorname{tr}\left(\operatorname{adj}(g)\frac{\partial}{\partial t}g\right)$$

and

$$\operatorname{adj}\, g = (\det g)\, g^{-1}$$

to derive

$$\frac{\partial}{\partial t}\det\, g = (\det g)\, g^{ij}\frac{\partial}{\partial t}g_{ij}.$$

So

$$\frac{\partial}{\partial t}\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}\; g^{ij}\frac{\partial}{\partial t}g_{ij}. \tag{2.7}$$

Plug (2.5), (2.6) and (2.7) into (2.4), then we obtain the variation formula. □

Remark 2.1. The first term in the variation is the Einstein tensor

$$G^{ij} = R^{ij} - \frac{1}{2}Rg^{ij}.$$

If M is closed, then the second term in (2.3) is zero. We can directly read the vacuum Einstein equations from

$$\delta S_{EH} = -\frac{1}{16\pi}\int_M \left(R^{ij} - \frac{1}{2}g^{ij}R\right)\delta g_{ij}dv_M.$$

However, there are boundaries in general spacetimes, thus δS_{EH} has a nonzero divergence term. As a consequence, we can't derive the vacuum Einstein equations. Therefore once the spacetime has a boundary, it is appropriate to consider an action with a boundary term. York [23] proposed such a term by considering the induced metric h on ∂M. For convenience, assume that ∂M is spacelike and N is a future-directed timelike normal vector of ∂M. For any vector X along ∂M, we can decompose it into vector on ∂M and the normal part, that is

$$X = X^T - \langle X, N\rangle N.$$

So the induced metric h on ∂M is given by

$$h_{ij} = g_{ij} + n_i n_j$$

where n_i denotes $\langle \partial_i, N\rangle$.

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Remark 2.2. The tensor h_{ij} has 16 components.

Lemma 2.1. *Consider the variation of S_{EH} with respect to the metric g such that the metric is fixed on the boundary. Then*

$$\int_M \nabla^l \left(g^{ij}\nabla_j \delta g_{il} - g^{ij}\nabla_l \delta g_{ij}\right) dv_M = -2\int_{\partial M} \delta K dv_{\partial M}$$

where K is the mean curvature of ∂M associated with N.

Proof. By Stokes' theorem, we have

$$\int_M \nabla^l \left(g^{ij}\nabla_j \delta g_{il} - g^{ij}\nabla_l \delta g_{ij}\right) dv_M = \int_{\partial M} g^{ij}\left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) n^l dv_{\partial M}.$$

With the induced metric h, we get

$$g^{ij}\left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) n^l = \left(h^{ij} - n^i n^j\right)\left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) n^l.$$

Note that

$$n^l n^i n^j \left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right)$$

is anti-symmetric in i and l. So

$$n^l g^{ij}\left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) = h^{ij} n^l \left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right).$$

Since $\delta g = 0$ on ∂M, its tangential derivative is 0. Hence, in local coordinate $\{y^a\}$ on ∂M,

$$h^{ij}\nabla_j \delta g_{il} = \left(h^{ab}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\right)\nabla_j \delta g_{il} = h^{ab}\frac{\partial x^i}{\partial y^a}\left(\frac{\partial x^j}{\partial y^b}\nabla_j \delta g_{il}\right) = 0.$$

Then we have

$$h^{ij} n^l \left(\nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) = -h^{ij} n^l \nabla_l \delta g_{ij}.$$

Now consider the variation of the mean curvature K of ∂M, since $\delta g = 0$ on ∂M, we have

$$\begin{aligned}\delta K = \delta\left(h^{ij}\nabla_i n_j\right) &= \delta\left[h_{ij}\left(\partial_i n_j - \Gamma^k_{ij} n_k\right)\right] = -h^{ij}\left(\delta\Gamma^k_{ij}\right) n_k \\ &= -\frac{1}{2} h^{ij} g^{kl}\left(\nabla_i \delta g_{jl} + \nabla_j \delta g_{il} - \nabla_l \delta g_{ij}\right) n_k = \frac{1}{2} h^{ij} n^l \nabla_l \delta g_{ij}\end{aligned}$$

where we use the observations

$$h^{il}\nabla_l \delta g_{ij} = 0$$

and

$$0 = \delta\left(g^{ij} n_i n_j\right) = 2g^{ij} n_i \delta n_j = 2n^j \delta n_j.$$

The proof is completed by comparing expressions. $\square$

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Therefore let's introduce the boundary term to get the action

$$S_1 = S_{EH} + \frac{1}{8\pi} \int_{\partial M} K dv_{\partial M}.$$

Remark 2.3. The above boundary term leads to the action S^1 used by Brown-York [3].

Corollary 2.1. *The variation of S_1 is given by*

$$\delta S_1 = -\frac{1}{16\pi} \int_M \left(R^{ij} - \frac{1}{2} g^{ij} R \right) \delta g_{ij} dv_M.$$

Proof. Use Proposition 2.1 and Lemma 2.1. □

In particular, the above corollary holds when ∂M is nonempty. Note that we can directly read the vacuum Einstein equations from δS_1.

2.4 Hamiltonian formulation

In this section, we review the Hamiltonian formulation. To start, we need to decompose the spacetime (M, g) into space part and time part. Assume from now on that the spacetime M is foliated by a family of spacelike hypersurfaces Ω_t where t lies in the interval $[t_1, t_2]$ and e_4 be a future-directed timelike unit normal vector of Ω_t and $\partial\Omega_t = \Sigma_t$. This is a reasonable assumption since we believe there is a way to define time in the physical spacetime. Note that

$$\partial M = \Omega_{t_1} \cup \Omega_{t_2} \cup B^3$$

where B^3 is the union of Σ_t. Let e_3 be a spacelike unit outward normal vector along Σ_t such that it is orthogonal to e_4. Such t acts as a time function so that each Ω_t is of the same time. Now consider a vector T satisfying

$$\langle T, \nabla t \rangle = 1.$$

This T can be decomposed into the lapse function L and shift vector S as

$$T = Le_4 + S.$$

Here S is orthogonal to e_4. Let h be the induced metric on Ω_t. Denote e_3 and e_4 by v and u respectively. The spacetime M then look like Figure 1.

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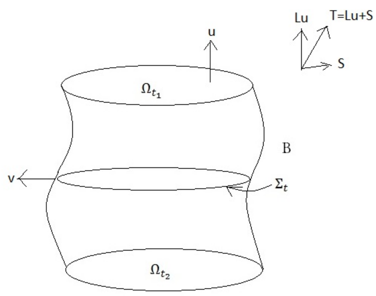

Figure 1. The spacetime M.

Before introducing the Hamiltonian density of the spacetime, we need to relate the geometry of the hypersurface to that of the spacetime.

Lemma 2.2.

$$R_{44} = K^2 - K_{ij}K^{ij} - \nabla_i \left(u^i \nabla_j u^j\right) + \nabla_i \left(u^j \nabla_j u^i\right).$$

Proof.

$$\begin{aligned} R_{44} &= R_{ij}u^i u^j \\ &= R^k_{jki}u^i u^j = u^j \left(\nabla_k \nabla_j - \nabla_j \nabla_k\right) u^k \\ &= \nabla_k \left(u^j \nabla_j u^k\right) - \left(\nabla_k u^j\right)\left(\nabla_j u^k\right) - \nabla_j \left(u^j \nabla_k u^k\right) + \left(\nabla_j u^j\right)\left(\nabla_k u^k\right) \\ &= K^2 - K_{jk}K^{jk} - \nabla_j \left(u^j \nabla_k u^k\right) + \nabla_k \left(u^j \nabla_j u^k\right). \end{aligned}$$

□

The Lagrangian density $\mathcal{L}_{EH}$ can be written as

$$\mathcal{L}_{EH} = \frac{1}{16\pi} R\sqrt{-\det g} = \frac{1}{8\pi}\left(R_{44} + \frac{1}{2}R - R_{44}\right)\sqrt{-\det g}.$$

Therefore by (2.1) and Lemma 2.2 we have

$$\mathcal{L}_{EH} = \frac{1}{16\pi}\left[\widetilde{R} + K_{ij}K^{ij} - K^2 + 2\nabla_i \left(u^i \nabla_j u^j\right) - 2\nabla_i \left(u^j \nabla_j u^i\right)\right] L\sqrt{\det h}.$$

So the action S_1 becomes

$$\frac{1}{16\pi}\int_M \widetilde{R} + K_{ij}K^{ij} - K^2 + 2\nabla_i \left(u^i \nabla_j u^j\right) - 2\nabla_i \left(u^j \nabla_j u^i\right) dv_M + \frac{1}{8\pi}\int_{\partial M} K dv_{\partial M}.$$

Recall that

$$\partial M = \Omega_{t_1} \cup \Omega_{t_2} \cup B$$

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where $B = \cup_{t\in[t_1,t_2]}\Sigma_t$. In particular,

$$\int_M \nabla_i \left(u^i \nabla_j u^j\right) dv_M = -\int_{\Omega_{t_1}} \nabla_j u^j dv_{\Omega_{t_1}} - \int_{\Omega_{t_2}} \nabla_j u^j dv_{\Omega_{t_2}} + \int_B K dv_B.$$

Similarly we can decompose $\int_{\partial M} K dv_{\partial M}$ into the corresponding parts, and one finds that the first two terms in the above are cancelled.

Corollary 2.2. *The action S_1 is expressed as*

$$\frac{1}{16\pi}\int_{t_1}^{t_2}\int_{\Omega_t} \widetilde{R} + K_{ij}K^{ij} - K^2 + 2\nabla_i\left(u^i\nabla_j u^j\right) - 2\nabla_i\left(u^j\nabla_j u^i\right) dv_{\Omega_t} L dt$$
$$+\frac{1}{8\pi}\int_{t_1}^{t_2}\int_{\Sigma_t} K L dv_{\Sigma_t} dt.$$

Note that

$$\int_{\Omega_t} \nabla_i\left(u^j\nabla_j u^i\right) dv_{\Omega_t} = -\int_{\Sigma_t} u^i u^j \nabla_j v_i dv_{\Sigma_t}$$

and

$$\int_{\Sigma_t} K dv_{\Sigma_t} = \int_{\Sigma_t} \left(g^{ij} - v^i v^j\right)\nabla_i v_i dv_{\Sigma_t}.$$

Hence

$$-\int_{\Omega_t} \nabla_i\left(u^j\nabla_j u^i\right) dv_{\Omega_t} + \int_{\Sigma_t} K dv_{\Sigma_t} = \int_{\Sigma_t} k dv_{\Sigma_t}$$

where k is the mean curvature of Σ_t associated with v. Therefore S_1 becomes

$$\frac{1}{16\pi}\int_{t_1}^{t_2}\left(\int_{\Omega_t} \widetilde{R} + K_{ij}K^{ij} - K^2 dv_{\Omega_t} + \int_{\Sigma_t} 2k dv_{\Sigma_t}\right) L dt.$$

Denote the Lagrangian of Ω_t and Σ_t by $\mathcal{L}$ and $\mathcal{L}_{\text{York}}$, that is

$$S_1 = \int_{t_1}^{t_2}\int_{\Omega_t} \mathcal{L} dx L dt + \int_{t_1}^{t_2}\int_{\Sigma_t} \mathcal{L}_{\text{York}} dy L dt.$$

So

$$\mathcal{L} = \frac{1}{16\pi}\left(\widetilde{R} + K_{ij}K^{ij} - K^2\right)\sqrt{\det h}. \tag{2.8}$$

Lemma 2.3.

$$K_{ij} = \frac{1}{2L}\left(\dot{h}_{ij} - \widetilde{\nabla}_i S_j - \widetilde{\nabla}_j S_i\right).$$

Here $\widetilde{\nabla}$ is the covariant derivative on Ω_t and $\dot{h}_{ij} = h_i^k h_j^l \mathfrak{L}_T h_{kl}$ where $\mathfrak{L}$ is the Lie derivative.

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Proof.

$$
\begin{aligned}
K_{ij} &= \frac{1}{2}\mathfrak{L}_u h_{ij} = \frac{1}{2}\left(u^k\nabla_k h_{ij} + h_{ik}\nabla_j u^k + h_{kj}\nabla_i u^k\right) \\
&= \frac{1}{2L}\left[Lu^k\nabla_k h_{ij} + h_{ik}\nabla_j\left(Lu^k\right) + h_{kj}\nabla_i\left(Lu^k\right)\right] \\
&= \frac{1}{2L}h_i^k h_j^l\left(\mathfrak{L}_T h_{kl} - \mathfrak{L}_S h_{kl}\right) = \frac{1}{2L}\left(\dot{h}_{ij} - \widetilde{\nabla}_i S_j - \widetilde{\nabla}_j S_i\right).
\end{aligned}
$$

Here we use the identity

$$
\begin{aligned}
h_i^k h_j^l \mathfrak{L}_S h_{kl} &= h_i^k h_j^l\left(S^p\nabla_p h_{kl} + h_{pl}\nabla_k S^p + h_{kp}\nabla_l S^p\right) \\
&= S^p\widetilde{\nabla}_p h_{ij} + \widetilde{\nabla}_i S_j + \widetilde{\nabla}_j S_i.
\end{aligned}
$$

□

The conjugate momentum to h is defined by

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}}.$$

Then by (2.8) and Lemma 2.3, we get

$$\pi^{ij} = \frac{1}{16\pi}\left(K^{ij} - h^{ij}K\right)\sqrt{\det h}.$$

Proposition 2.2. *The action S_1 can be written in the form*

$$\int_{t_1}^{t_2}\left[\int_{\Omega_t}\pi^{ij}\dot{h}_{ij} - LH - S_iH^i dx + \frac{1}{8\pi}\int_{\Sigma_t} Lk - S_i v_j\left(K^{ij} - Kh^{ij}\right)dv_{\Sigma_t}\right]dt$$

where

$$H = \frac{1}{16\pi}\left(K_{ij}K^{ij} - K^2 - \widetilde{R}\right)\sqrt{\det h}$$

and

$$H^i = -\frac{1}{8\pi}\widetilde{\nabla}_j\pi^{ij}.$$

Proof. Note that

$$\left(K^{ij} - h^{ij}K\right)\widetilde{\nabla}_j S_i + S_i\widetilde{\nabla}_j\left(K^{ij} - h^{ij}K\right) = \widetilde{\nabla}_i\left[S_j\left(K^{ij} - h^{ij}K\right)\right].$$

Then by direct calculation, we get the expression. □

The Hamiltonian density is obtained from the Legendre transformation [18, p460]

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L}.$$

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Proposition 2.3. *The Hamiltonian is given by*

$$\int_{\Omega_t} LH + S_i H^i dx - \frac{1}{8\pi}\int_{\Sigma_t} Lk - S_i v_j \left(K^{ij} - Kh^{ij}\right) dv_{\Sigma_t}$$

where

$$H = \frac{1}{16\pi}\left(K_{ij}K^{ij} - K^2 - \widetilde{R}\right)\sqrt{\det h}$$

and

$$H^i = -\frac{1}{8\pi}\widetilde{\nabla}_j \pi^{ij}.$$

Proof. It follows from the Legendre transformation and Proposition 2.2. □

The case $H = H_i = 0$, which is satisfied by the vacuum Einstein equations, gives us the notion of energy, that is, the surface Hamiltonian

$$\mathfrak{H}(T, e_4) = -\frac{1}{8\pi}\int_{\Sigma} Lk - S_i v_j \left(K^{ij} - Kh^{ij}\right) dv_{\Sigma}.$$

Note that we start from T and e_4, so the surface Hamiltonian depends on these two vectors. For convenience, we write it into the following form

$$\mathfrak{H}(T, e_4) = -\frac{1}{8\pi}\int_{\Sigma} Lk - S^i v^j \left(K_{ij} - Kg_{ij}\right) dv_{\Sigma}.$$

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Chapter 3

Wang-Yau Quasi-Local Mass

3.1 The definition

We know from the previous section, the surface Hamiltonian is given by

$$\mathfrak{H}(T, e_4) = -\frac{1}{8\pi}\int_{\Sigma} Lk - S^i v^j (K_{ij} - K g_{ij})\, dv_{\Sigma}. \tag{3.1}$$

The quasi-local energy of Σ is then defined by

$$\mathfrak{H}(T, e_4) - \mathfrak{H}(T_0, \breve{e}_4) \tag{3.2}$$

where $\mathfrak{H}(T, e_4)$ is the physical one and $\mathfrak{H}(T_0, \breve{e}_4)$ is the reference one. The issue here is the choice of the background reference and the vectors T, T_0, e_4 and $\breve{e}_4$. We recall an embedding theorem by Nirenberg [10] and Pogorelov [12].

Theorem 3.1. *Let σ be a metric on $\mathbb{S}^2$ with positive Gauss curvature. Then there exists a unique isometric embedding of σ into $\mathbb{R}^3$ up to Euclidean rigid motions.*

One natural reference is $\mathbb{R}^3$, we can use the above theorem to embed a 2-surface with positive Gauss curvature. However there is an example [9] such that the quasi-local mass is positive while the 2-surface lies in $\mathbb{R}^{3,1}$. Wang and Yau instead took $\mathbb{R}^{3,1}$ as the reference. But if we write down the components of such embedding as $\left(X^1, X^2, X^3, X^4\right)$, we only have three constrains, that is

$$\eta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b} = \sigma_{ab}$$

where σ is the induced metric of the 2-surface and η is the standard metric on $\mathbb{R}^{3,1}$. Thus we have one degree of freedom. Wang and Yau then fix T_0 to be any constant timelike Killing unit vector and proved the following embedding theorem associated with T_0 and the embedding satisfies some equation.

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Theorem 3.2. *Let (Σ, σ) be a Riemannian 2-manifold diffeomorphic to $\mathbb{S}^2$. Let λ be a function on Σ such that $\int_\Sigma \lambda dv_\Sigma = 0$ and τ be the solution of $\Delta\tau = \lambda$. Suppose*

$$\kappa + \left(1 + |\nabla\tau|^2\right)^{-1} \det\left(\nabla^2\tau\right) > 0 \tag{3.3}$$

where κ is the Gauss curvature of σ. Then there exists a unique spacelike isometric embedding $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ satisfying

$$\langle H_0, T_0 \rangle = -\Delta\tau \tag{3.4}$$

where H_0 is the mean curvature vector of $X(\Sigma)$. Here all differential operators are with respect to σ.

The above τ is the time function. Before we prove Theorem 3.2, we need some lemmas.

Lemma 3.1. *Let (Σ, σ) be a Riemannian 2-manifold and $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be an isometric embedding. Then the mean curvature vector H of $X(\Sigma)$ is ΔX.*

Proof. Given a local coordinate $\{x^i\}_{i=1}^2$ on Σ, we compute the second fundamental form

$$\mathrm{II}\left(\partial_i, \partial_j\right) = \left(\nabla_{\partial_i X}\partial_j X\right)^\perp = \nabla_{\partial_i X}\partial_j X - \widetilde{\nabla}_{\partial_i}\partial_j = \partial_i\partial_j X - \Gamma_{ij}^k \partial_k X.$$

Therefore

$$H = \sigma^{ij}\mathrm{II}\left(\partial_i, \partial_j\right) = \Delta X.$$

□

Lemma 3.2. *Let $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be an isometric embedding and $\widehat{X} = X - \tau T_0$ where $\tau = -\langle X, T_0 \rangle$. Then the Gauss curvature of $\widehat{X}$ is given by*

$$\left(1 + |\nabla\tau|^2\right)^{-1}\left[\kappa + \left(1 + |\nabla\tau|^2\right)^{-1} \det\left(\nabla^2\tau\right)\right].$$

Proof. Let $\breve{e}_3$ be the outward normal of $\widehat{X}(\Sigma)$ in $\mathbb{R}^3$. We can extend $\breve{e}_3$ parallelly along T_0 in $\mathbb{R}^{3,1}$. Since $\breve{e}_3$ is orthogonal to T_0, we have

$$\langle \breve{e}_3, \partial_i X \rangle = \left\langle \breve{e}_3, \partial_i \widehat{X} + \partial_i\tau T_0 \right\rangle = \left\langle \breve{e}_3, \partial_i \widehat{X} \right\rangle = 0.$$

Thus $\{\breve{e}_3, \breve{e}_4\}$ is an orthonormal basis for the normal bundle of $X(\Sigma)$. In fact, we can check directly that

$$\breve{e}_4 = \frac{1}{\sqrt{1 + |\nabla\tau|^2}}\left(T_0 + \nabla\tau\right).$$

Decompose the second fundamental form as

$$\mathrm{II}\left(\partial_i X, \partial_j X\right) = -h_{ij}^3 \breve{e}_3 + h_{ij}^4 \breve{e}_4$$

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where h_{ij}^3 and h_{ij}^4 are the second fundamental forms associated with $\breve{e}_3$ and $\breve{e}_4$ respectively. Since $\mathbb{R}^{3,1}$ is flat, by the Gauss equation, we have

$$\begin{aligned} 0 &= \widehat{R}_{1212} - h_{22}^3 h_{11}^3 + h_{22}^4 h_{11}^4 + h_{12}^3 h_{21}^3 - h_{12}^4 h_{21}^4 \\ &= \widehat{R}_{1212} - \det h^3 + \det h^4. \end{aligned} \tag{3.5}$$

By the definition of τ, we derive

$$\begin{aligned} h_{ij}^4 &= -\langle \partial_i \partial_j X, \breve{e}_4 \rangle = -\frac{1}{\sqrt{1+|\nabla\tau|^2}} \langle \partial_i \partial_j X, T_0 + \nabla \tau \rangle \\ &= \frac{1}{\sqrt{1+|\nabla\tau|^2}} \left(\partial_i \partial_j \tau - \Gamma_{ijk} \partial_k \tau \right) = \frac{\nabla_i \nabla_j \tau}{\sqrt{1+|\nabla\tau|^2}}. \end{aligned}$$

Note that the determinant of the induced metric of $\widehat{X}(\Sigma)$ is given by

$$\det \sigma \left(1 + |\nabla\tau|^2\right).$$

Divide both sides of (3.5) by $\det \sigma \left(1 + |\nabla\tau|^2\right)$, we get the desired expression. $\square$

Proof of Theorem 3.2. We first prove the uniqueness problem. Assume X_1 and X_2 be two embeddings given by Theorem 3.2. Let

$$\tau_i = -\langle X_i, T_0 \rangle.$$

Consider the projection $\widehat{X}_i = X_i - \tau_i T_0$ of Σ onto $\mathbb{R}^3$. By Lemma 3.2, the Gauss curvature of $\widehat{X}_i$ is

$$\left(1 + |\nabla\tau_i|^2\right)^{-1} \left[\kappa + \left(1 + |\nabla\tau_i|^2\right)^{-1} \det\left(\nabla^2 \tau_i\right)\right] > 0.$$

The metric on $\widehat{X}_i(\Sigma)$ can be computed as

$$\left\langle d\widehat{X}_i, d\widehat{X}_i \right\rangle = \langle dX, dX \rangle + d\tau_i^2.$$

On the other hand, by Lemma 3.1, we have

$$\langle \Delta (X_1 - X_2), T_0 \rangle = 0.$$

Hence $\langle X_1 - X_2, T_0 \rangle$ is a constant. So $d\tau_1 = d\tau_2$. As a consequence, the induced metrics of $\widehat{X}_1(\Sigma)$ and $\widehat{X}_2(\Sigma)$ are the same. Therefore by Theorem 3.1, $\widehat{X}_1(\Sigma)$ is the same as $\widehat{X}_2(\Sigma)$ up to Euclidean rigid motion. Now since $d\tau_1 = d\tau_2$, $X_1(\Sigma)$ and $X_2(\Sigma)$ are the same up to rigid motion.

For the existence, we solve $\lambda = \nabla\tau$. Consider the new metric $\sigma + d\tau^2$. The Gauss curvature is the same as before and positive by assumption. Hence Theorem 3.1 gives an embedding $\widehat{X}$. The isometric embedding into $\mathbb{R}^{3,1}$ is then given by $\widehat{X} + \tau T_0$.

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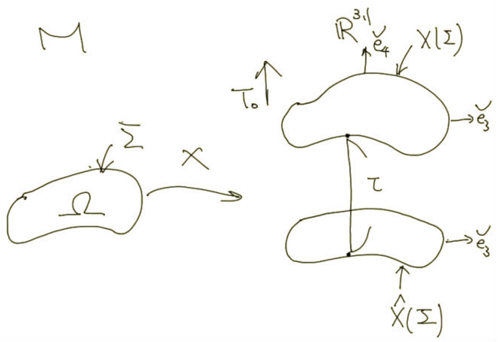

Figure 2. The picture of Proof of Theorem 3.2.

□

Remark 3.1. We use Theorem 3.1 to deal with the uniqueness in Theorem 3.2.

Corollary 3.1. *Let $\left(\mathbb{S}^2, \sigma\right)$ be a Riemannian manifold. Let τ be a function on $\mathbb{S}^2$ such that (3.3) holds. Then there exists a unique spacelike isometric embedding $X_0 : \mathbb{S}^2 \hookrightarrow \mathbb{R}^{3,1}$ and τ as the time function.*

Let $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be the isometric embedding in Theorem 3.2 and τ be the corresponding time function so that

$$\tau = -\langle X, T_0 \rangle . \tag{3.6}$$

We choose $\check{e}_4$ to be the unit normal vector in the normal direction of T_0. Use (3.6), one see that

$$\check{e}_4 = \frac{1}{\sqrt{1+|\nabla\tau|^2}} \left(T_0 + \nabla\tau\right).$$

So

$$\langle H_0, \check{e}_4 \rangle = \frac{-\Delta\tau}{\sqrt{1+|\nabla\tau|^2}}$$

where we use (3.4). At this moment, we can write down the reference Hamiltonian as

$$\mathfrak{H}\left(T_0, \check{e}_4\right) = -\frac{1}{8\pi}\int_{X(\Sigma)} L_0 k_0 - S_0^i v_0^j \left[\left(K_0\right)_{ij} - K_0 \left(g_0\right)_{ij}\right] dv_{X(\Sigma)}$$

where

$$L_0 = \sqrt{1+|\nabla\tau|^2}$$

and

$$S_0 = -\nabla\tau.$$

Here all the terms with 0 are in $\mathbb{R}^{3,1}$ and $T_0 = \sqrt{1+|\nabla\tau|^2}u_0 - \nabla\tau$. There are still two vectors unspecified, namely T and e_4.

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Now let's move on to the general spacetime. Given any basis $\{e_3, e_4\}$ for the normal bundle NΣ, Wang and Yau consider the vector

$$ke_4 - pe_3 - V$$

where V is the tangent part of W which satisfies

$$\langle W, X\rangle = \langle \nabla_X e_3, e_4\rangle$$

for any vector X and

$$p = K - K_{ij}v^i v^j.$$

This vector is closely related to the surface Hamiltonian. In fact, the expression (3.1) can be written as

$$\frac{1}{8\pi}\int_\Sigma \langle ke_4 - pe_3 - V, Le_4 + S\rangle \, dv_\Sigma.$$

To maintain consistency with Wang-Yau's notation, we denote $\langle \nabla_X e_3, e_4\rangle$ by $\alpha_{e_3}(X)$.

Definition 3.1. *Let $X : \Sigma \hookrightarrow M$ be a spacelike embedding into spacetime M. Let τ be a smooth function on Σ and e_3 be a spacelike normal vector. We denote*

$$\mathfrak{h}(\Sigma, X, \tau, e_3) = \sqrt{1 + |\nabla\tau|^2}\,\langle H, e_3\rangle + \alpha_{e_3}(\nabla\tau) \tag{3.7}$$

where H is the mean curvature vector of Σ in M.

Notice that $\mathfrak{h}$ defined above is different from the original paper [20] by a minus sign.

Remark 3.2. In term of the mean curvature vector H, we have

$$k = -\langle H, e_3\rangle, \; p = -\langle H, e_4\rangle.$$

Thus

$$\mathfrak{h}(\Sigma, X, \tau, e_3) = \left\langle ke_4 - pe_3 - V, \sqrt{1 + |\nabla\tau|^2} e_4 - \nabla\tau \right\rangle.$$

The reference Hamiltonian is then equal to

$$\frac{1}{8\pi}\int_\Sigma h(\Sigma, X, \tau, \breve{e}_3)\, dv_\Sigma = \frac{1}{8\pi}\int_\Sigma \sqrt{1 + |\nabla\tau|^2}\,\langle H, \breve{e}_3\rangle + \alpha_{\breve{e}_3}(\nabla\tau)\, dv_\Sigma.$$

Consider a coordinate transformation between bases $\{e_3, e_4\}$ and $\{\hat{e}_3, \hat{e}_4\}$, that is,

$$e_3 = \cosh\phi\ \hat{e}_3 - \sinh\phi\ \hat{e}_4, \; e_4 = -\sinh\phi\ \hat{e}_3 + \cosh\phi\ \hat{e}_4 \tag{3.8}$$

for some function ϕ.

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Proposition 3.1. *If the mean curvature vector of the embedding $X : \Sigma \hookrightarrow M$ is spacelike, then the integral*

$$\int_\Sigma \mathfrak{h}(\Sigma, X, \tau, e_3) dv_\Sigma$$

has a global maximum and such maximum is achieved by some vector $\bar{e}_4$ such that

$$\langle H, \bar{e}_4 \rangle = \frac{-\Delta\tau}{\sqrt{1 + |\nabla\tau|^2}}.$$

Proof. Let $\{e_1, e_2\}$ be a frame of $T\Sigma$. In terms of the transformation (3.8), we have

$$\begin{aligned}
\int_\Sigma \mathfrak{h}(\Sigma, X, \tau, e_3) &= \int_\Sigma \sqrt{1 + |\nabla\tau|^2} \langle H, e_3 \rangle + \alpha_{e_3}(\nabla\tau)\, dv_\Sigma \\
&= \int_\Sigma -\sqrt{1 + |\nabla\tau|^2} \sum_{a=1}^{2} (\cosh\phi \langle \nabla_{e_a} \hat{e}_3, e_a \rangle - \sinh\phi \langle \nabla_{e_a} \hat{e}_4, e_a \rangle) \\
&\quad + [\alpha_{\hat{e}_3}(\nabla\tau) + \nabla\tau \cdot \nabla\phi]\, dv_\Sigma.
\end{aligned}$$

Since H is spacelike, take

$$\hat{e}_3 = -\frac{H}{|H|}$$

and $\hat{e}_4$ to be the future-directed timelike normal vector such that $\{\hat{e}_3, \hat{e}_4\}$ is an orthonormal frame. Using the integration by parts, the integral becomes

$$F(\phi) = \int_\Sigma -\sqrt{1 + |\nabla\tau|^2} \cosh\phi |H| + \alpha_{\hat{e}_3}(\nabla\tau) - \phi\Delta\tau dv_\Sigma.$$

Note that the integral is a functional of ϕ. The first variation can be computed as

$$\left.\frac{d}{dt}\right|_{t=0} F(\phi + t\alpha) = \int_\Sigma -\left[\sqrt{1 + |\nabla\tau|^2} \sinh\phi |H| + \Delta\tau\right] \alpha dv_\Sigma.$$

So the critical point happens when

$$\sqrt{1 + |\nabla\tau|^2} \sinh\phi |H| + \Delta\tau = 0.$$

Also since the functional is concave, this point is a maximum. Hence there is a unique timelike unit normal vector $\bar{e}_4$ such that

$$\langle H, \bar{e}_4 \rangle = \frac{-\Delta\tau}{\sqrt{1 + |\nabla\tau|^2}}.$$

□

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Therefore we have a unique choice of $\bar{e}_4$ in the spacetime such that

$$\langle H, \bar{e}_4 \rangle = \langle H_0, \breve{e}_4 \rangle . \tag{3.9}$$

Finally, we assign T the vector $\sqrt{1 + |\nabla\tau|^2}\bar{e}_4 - \nabla\tau$.

Remark 3.3. The equality (3.9) is the equality (3) in [21]. This equation is to choose the unique gauge of the spacetime.

Definition 3.2. *Let $X : \Sigma \hookrightarrow M$ be a spacelike embedding. A smooth function τ on Σ is said to be admissible if the following conditions hold.*

1. *Σ bounds an spacelike hypersurface $\Omega \subset M$ such that the Jang-Schoen-Yau equation with the boundary data τ is solvable on Ω.*
2. $\kappa + \left(1 + |\nabla\tau|^2\right)^{-1} \det\left(\nabla^2\tau\right) > 0$.
3. $\mathfrak{h}\left(\Sigma, X, \tau, e_3'\right) < 0$.

Here κ is the Gauss curvature of Σ and e_3' is given by Theorem 3.4.

Definition 3.3. *Let $X : \Sigma \hookrightarrow M$ be a spacelike embedding into a spacetime M. The Wang-Yau quasi-local mass is defined to be the infimum among*

$$\mathfrak{H}(T, e_4) - \mathfrak{H}\left(T_0, \breve{e}_4\right)$$

where $\tau = -\langle X, T_0 \rangle$ is admissible and X_0 is the unique embedding into $\mathbb{R}^{3,1}$ associated with τ.

Remark 3.4. The physical and reference Hamiltonian are equal to

$$\mathfrak{H}\left(T, \bar{e}_4\right) = \int_\Sigma \sqrt{1 + |\nabla\tau|^2} \langle H, \bar{e}_3 \rangle + \alpha_{\bar{e}_3}\left(\nabla\tau\right) dv_\Sigma$$

and

$$\mathfrak{H}\left(T_0, \breve{e}_4\right) = \int_{X(\Sigma)} \sqrt{1 + |\nabla\tau|^2} \langle H_0, \breve{e}_3 \rangle + \alpha_{\breve{e}_3}\left(\nabla\tau\right) dv_{X(\Sigma)}$$

respectively.

3.2 Jang-Schoen-Yau equation

The Jang-Schoen-Yau equation was first proposed by Jang [5] to solve the positive mass conjecture. Schoen and Yau used the Jang-Schoen-Yau equation to prove the positive mass conjecture. In fact, their proof also reveals a deep relation between the solvability of the Jang-Schoen-Yau equation and the existence of black hole. In the case of the Wang-Yau's work, the Jang-Schoen-Yau equation is used to derive an equality between (3.11) and (3.12) in order to prove the positivity.

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Theorem 3.3. *Let (Ω, g, p) be an initial data set. Then (Ω, g, p) is Minkowski space if and only if there exists a function f and a flat metric k such that*

$$p_{ij} = \frac{D_i D_j f}{\sqrt{1+|Df|^2}}$$

and

$$g^{ij} = k^{ij} - D^i f D^j f.$$

Using the above theorem, we derive the Jang-Scheon-Yau equation

$$\left(g^{ij} - \frac{D^i f D^j f}{1+|Df|^2}\right)\left(\frac{D_i D_j f}{\sqrt{1+|Df|^2}} - p_{ij}\right) = 0. \tag{3.10}$$

Let (Ω, g, p) be an initial data set. We consider the Riemannian product manifold $(\Omega \times \mathbb{R}, \bar{g}, \bar{p})$ where $\bar{g}$ is the product metric and $\bar{p}$ is a symmetric tensor extended from p parallelly along the $\mathbb{R}$-direction, i.e. $\bar{p}(\cdot, v) = 0$ for the downward unit vector v in the $\mathbb{R}$-direction. The Jang-Schoen-Yau equation aims to find a hypersurface $\widetilde{\Omega} \subset \Omega \times \mathbb{R}$ defined by the graph of a function f so as the mean curvature of $\widetilde{\Omega}$ is the same as the trace of the restriction of $\bar{p}$ on $\widetilde{\Omega}$. It can be shown that the condition is the same in (3.10). We denote the Levi-Civita connection on $\Omega \times \mathbb{R}$ by $\widetilde{\nabla}$ where it reduces to the usual Levi-Civita connection on Ω and $\mathbb{R}$ by the virtue of the product metric.

Let τ be a smooth function on Σ. Denote the graph of τ on Σ by $\widetilde{\Sigma}$ and that of f on Ω by $\widetilde{\Omega}$. So it becomes a Dirichlet problem so that $f = \tau$ on the boundary. We consider the expression on $\widetilde{\Sigma}$

$$\tilde{k} - \left\langle \widetilde{\nabla}_{\tilde{e}_4} \tilde{e}_4, \tilde{e}_3 \right\rangle + \bar{p}\left(\tilde{e}_4, \tilde{e}_3\right\rangle \tag{3.11}$$

where $\tilde{k}$ is the mean curvature of $\widetilde{\Sigma}$.

Theorem 3.4. *Let $X : \Sigma \hookrightarrow M$ be a spacelike embedding. Let τ be a function on Σ and Ω be a spacelike hypersurface such that $\partial\Omega = \Sigma$. Suppose the Dirichlet problem of Jang-Schoen-Yau equation with boundary condition that $f = \tau$ on Σ is solvable. Then there exists a spacelike unit normal vector e_3' along Σ such that (3.11) at $\tilde{p} \in \widetilde{\Sigma}$ is equal to*

$$-\left\langle H, e_3' \right\rangle - \frac{1}{\sqrt{1+|\nabla\tau|^2}} \alpha_{e_3'}\left(\nabla\tau\right) \tag{3.12}$$

evaluated at $p \in \Sigma$ where $\tilde{p} = (p, \tau(p)) \in \widetilde{\Sigma}$.

Theorem 3.4 links the geometry of $\widetilde{\Sigma}$ with that of Σ.

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3.3 Shi-Tam inequality

The boundary term is crucial in the discussion of action principle and the notion of the quasi-local mass. Shi and Tam [16] proved an inequality relating the integral of the mean curvature of boundary to that of isometric embedding. To be precise, they showed

Theorem 3.5. *Let Ω be a compact Riemannian 3-manifold with nonnegative scalar curvature. Suppose $\Sigma = \partial\Omega$ has positive Gauss curvature and positive mean curvature k. Let $X : \Sigma \hookrightarrow \mathbb{R}^3$ be an isometric embedding and k_0 be the mean curvature of $X(\Sigma)$. Then*

$$\int_\Sigma k dv_\Sigma \leq \int_{X(\Sigma)} k_0 d_{X(\Sigma)}.$$

The equality holds if and only if Ω is in $\mathbb{R}^3$.

Wang and Yau generalized the Shi-Tam's work into the following theorem.

Theorem 3.6. *Let Ω be a compact Riemannian 3-manifold with boundary Σ. Suppose there exists a vector V on Ω such that*

$$R \geq 2|V|^2 - 2divV$$

on Ω where R is the scalar curvature of Ω and

$$k > \langle V, \nu \rangle$$

on Σ where ν is the outward normal vector of Σ and k is the mean curvature of Σ associated with ν. Suppose also the Gauss curvature of Σ is positive. Let $X : \Sigma \hookrightarrow \mathbb{R}^3$ be an isometric embedding. Then

$$\int_\Sigma k - \langle V, \nu \rangle \, dv_\Sigma \leq \int_{X(\Sigma)} k_0 dv_{X(\Sigma)}$$

where k_0 is the mean curvature of $X(\Sigma)$.

Remark 3.5. If $X = 0$, then Theorem 3.6 reduces to the result of Shi-Tam.

3.4 The positivity

This section shows the positivity of the Wang-Yau quasi-local mass. The moral is to compare the different forms of the integral (3.7) in various settings. We first link the geometry of the projection $\widehat{X}$ with that of the isometric embedding $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$.

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Proposition 3.2.

$$\int_{\widehat{X}(\Sigma)} \hat{k} dv_{\widehat{X}(\Sigma)} = \int_{X(\Sigma)} -\sqrt{1+|\nabla\tau|^2}\langle H_0, \breve{e}_3\rangle - \alpha_{\breve{e}_3}(\nabla\tau)\, dv_{X(\Sigma)}$$

where $\hat{k}$ is the mean curvature of $\widehat{X}(\Sigma)$ in $\mathbb{R}^3$.

Proof. Let $\{\hat{e}_1, \hat{e}_2\}$ be an orthonormal basis for $\widehat{X}(\Sigma)$. Note that

$$\hat{k} = \sum_{a=1}^{2} \langle \nabla_{\hat{e}_a}\breve{e}_3, \hat{e}_a\rangle = \sum_{a=1}^{2} \langle \nabla_{\hat{e}_a}\breve{e}_3, \hat{e}_a\rangle + \langle \nabla_{\breve{e}_3}\breve{e}_3, \breve{e}_3\rangle - \langle \nabla_{T_0}\breve{e}_3, T_0\rangle .$$

Therefore $\hat{k} = g^{ij}\langle \nabla_{e_i}\breve{e}_3, e_j\rangle$ for any frame $\{e_i\}$ of $\mathbb{R}^{3,1}$ where $g_{ij} = \langle e_i, e_j\rangle$. Choose $\{e_1, e_2\}$ to be the orthonormal basis for $X(\Sigma)$. Notice that $\{\breve{e}_3, \breve{e}_4\}$ forms an orthonormal basis for the normal bundle of $X(\Sigma)$. Hence we have

$$\hat{k} = \sum_{a=1}^{2} \langle \nabla_{e_a}\breve{e}_3, e_a\rangle - \langle \nabla_{\breve{e}_4}\breve{e}_3, \breve{e}_4\rangle = -\langle H_0, \breve{e}_3\rangle - \frac{1}{\sqrt{1+|\nabla\tau|^2}}\langle \nabla_{\nabla\tau}\breve{e}_3, \breve{e}_4\rangle .$$

The area forms of $X(\Sigma)$ and $\widehat{X}(\Sigma)$ have the following relation

$$dv_{X(\Sigma)} = \frac{1}{\sqrt{1+|\nabla\tau|^2}} dv_{\widehat{X}(\Sigma)}.$$

Combine all the terms, then we obtain the Proposition 3.2. □

Proposition 3.2 is related to the gravitational conservation law in the following sense. If we assume u and t are tangent to B, then the surface Hamiltonian can be simplified as [19]

$$\mathfrak{H}(T_0, \breve{e}_4) = \int_{\Sigma} (\pi_0)_{ij}\, u_0^i t_0^j dx$$

where $(\pi_0)_{ij}$ is the conjugate momentum to the induced metric on B. Let D be the region of the timelike hypersurface between $\widehat{X}(\Sigma)$ and $X(\Sigma)$. Since M is vacuum and t_0 is Killing, we have

$$\int_{\partial D} (\pi_0)_{ij}\, u_0^i t_0^j dv_{\partial D} = \int_D \nabla^i \left[(\pi_0)_{ij}\, t_0^j\right] dv_D = 0.$$

One have

$$\mathfrak{H}(T_0, \breve{e}_4) = -\int_{\widehat{X}(\Sigma)} \hat{k} dv_{\widehat{X}(\Sigma)}$$

by the conservation law and Proposition 3.2.

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Proposition 3.3. *Let $X : \Sigma \hookrightarrow M$ be an embedding into a spacetime M with the dominant energy condition and $X_0 : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be the isometric embedding given by Theorem 3.2. Suppose τ is admissible. Then*

$$\int_{X_0(\Sigma)} \mathfrak{h}\left(\Sigma, X_0, \tau, \breve{e}_3\right) dv_{X_0(\Sigma)} \leq \int_{X(\Sigma)} \mathfrak{h}\left(\Sigma, X, \tau, e_3'\right) dv_{X(\Sigma)}.$$

Proof. Since τ is admissible, $\mathfrak{h}\left(\Sigma, X, \tau, e_3'\right) < 0$. By Theorem 3.4,

$$\tilde{k} - \left\langle \widetilde{\nabla}_{\tilde{e}_4} \tilde{e}_4, \tilde{e}_3 \right\rangle + \bar{p}\left(\tilde{e}_4, \tilde{e}_3\right) > 0.$$

Take $\widetilde{X}$ to be the dual vector of $\left\langle \widetilde{\nabla}_{\tilde{e}_4} \tilde{e}_4, \cdot \right\rangle - \bar{p}\left(\tilde{e}_4, \cdot\right)$ on the graph of τ over Σ, namely $\widetilde{\Sigma}$. One see that $\widetilde{\Omega}$ satisfies the condition in Theorem 3.6 by [14].Then by Theorem 3.6,

$$\int_{\widehat{X}(\Sigma)} \hat{k} dv_{\widehat{X}(\Sigma)} \geq \int_{\widetilde{\Sigma}} \tilde{k} - \left\langle \widetilde{X}, \tilde{e}_3 \right\rangle dv_{\widetilde{\Sigma}}.$$

Finally, from Theorem 3.4 and Proposition 3.2, we have

$$\int_{X_0(\Sigma)} \mathfrak{h}\left(\Sigma, X_0, \tau, \breve{e}_3\right) dv_{X_0(\Sigma)} \leq \int_{X(\Sigma)} \mathfrak{h}\left(\Sigma, X, \tau, e_3'\right) dv_{X(\Sigma)}.$$

□

Theorem 3.7. *Let $X : \Sigma \hookrightarrow M$ be an embedding into a spacetime M. Suppose M satisfies the dominant energy condition and the mean curvature vector of $X(\Sigma)$ is spacelike. Then the Wang-Yau quasi-local mass is non-negative and the equality holds if X is isometric to $\mathbb{R}^{3,1}$ along $X(\Sigma)$.*

Proof. The following inequalities summarize the proof for the positivity.

$$\begin{aligned} \mathfrak{H}(T, \bar{e}_4) &= \int_\Sigma \mathfrak{h}\left(\Sigma, X, \tau, \bar{e}_3\right) dv_\Sigma \\ &\geq \int_\Sigma \mathfrak{h}\left(\Sigma, X, \tau, e_3'\right) dv_\Sigma \qquad (3.13) \\ &\geq \int_\Sigma \mathfrak{h}\left(\Sigma, X_0, \tau, \breve{e}_3\right) dv_\Sigma = \mathfrak{H}\left(T_0, \breve{e}_4\right). \end{aligned}$$

The first inequality in (3.13) follows from the definition of $\mathfrak{H}(T, \bar{e}_4)$ as the local maximum. The second inequality is from Proposition 3.3. The last one is from the definition. As a consequence of the above inequalities, the Wang-Yau quasi-local mass is non-negative. Now if X is isometric to $\mathbb{R}^{3,1}$, then we can take $X_0 : \Sigma \hookrightarrow \mathbb{R}^{3,1}$. □

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