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隨機多項式的一個普遍性

A Universality of Polynomials with Complex Random Roots

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國立臺灣大學碩士學位論文
口試委員會審定書

本論文係胡亦行君（R02221015）在國立臺灣大學數學系、所完成之碩士學位論文，於民國 106 年 01 月 12 日承下列考試委員審查通過及口試及格，特此證明

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希望我有資格引用這句話作結：

「所有的科學著作都應該是某種推理小說—這是一份追尋聖杯過程的報告書」

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ABSTRACT. Let $p_n(x)$ be a random polynomial of degree n and $\{Z_j^{(n)}\}_{j=1}^n$ and $\{X_j^{n,k}\}_{j=1}^{n-k}$, $k<n$, be the zeros of p_n and $p_n^{(k)}$, the kth derivative of p_n, respectively. We show that if the linear statistics $\frac{1}{a_n}\left[f\left(\frac{Z_1^{(n)}}{b_n}\right)+\cdots+f\left(\frac{Z_n^{(n)}}{b_n}\right)\right]$ associated with $\{Z_j^{(n)}\}$ has a limit as $n\to\infty$ at some mode of convergence, the linear statistics associated with $\{X_j^{n,k}\}$ converges to the same limit at the same mode. Similar statement also holds for the centered linear statistics associated with the zeros of p_n and $p_n^{(k)}$, provided the zeros $\{Z_j^{(n)}\}$ and the sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers satisfy some mild conditions.

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A UNIVERSALITY OF CRITICAL POINTS OF POLYNOMIALS WITH COMPLEX RANDOM ROOTS

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Abstract. Let $p_n(x)$ be a random polynomial of degree n and $\{Z_j^{(n)}\}_{j=1}^n$ and $\{X_j^{n,k}\}_{j=1}^{n-k}, k < n$, be the zeros of p_n and $p_n^{(k)}$, the kth derivative of p_n, respectively. We show that if the linear statistics $\frac{1}{a_n}\left[f\left(\frac{Z_1^{(n)}}{b_n}\right) + \cdots + f\left(\frac{Z_n^{(n)}}{b_n}\right)\right]$ associated with $\{Z_j^{(n)}\}$ has a limit as $n \to \infty$ at some mode of convergence, the linear statistics associated with $\{X_j^{n,k}\}$ converges to the same limit at the same mode. Similar statement also holds for the centered linear statistics associated with the zeros of p_n and $p_n^{(k)}$, provided the zeros $\{Z_j^{(n)}\}$ and the sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers satisfy some mild conditions.

1. Introduction

Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\{p_n(z)\}_{n=1}^\infty$ be a sequence of random polynomials such that deg $p_n = n$. We observe that the randomness of polynomials can be introduced in several different ways.

Type 1: Given a triangular array of random zeros $\{Z_j^{(n)}\}_{j=1}^n, n = 1, 2, \ldots$, let

$$p_n(z) = (z - Z_1^{(n)}) \cdots (z - Z_n^{(n)}). \tag{1.1}$$

Here the coefficient of z^n is set to be 1 for simplicity.

Type 2: Given a triangular array of random coefficients $\{a_j^{(n)}\}_{j=0}^n, n = 1, 2, \ldots$, let

$$p_n(z) = a_n^{(n)} z^n + \cdots + a_1^{(n)} z + a_0^{(n)}. \tag{1.2}$$

Type 3: Given a sequence of random matrices $\{A^{(n)}\}_{n=1}^\infty$, let

$$p_n(z) = \det(zI - A^{(n)}), \tag{1.3}$$

the characteristic polynomial of $A^{(n)}$. Here I, the identity matrix, and $A^{(n)}$ are square matrices of size n.

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No matter how p_n is constructed, we always denote the zeros of p_n by $Z_1^{(n)}, \dots, Z_n^{(n)}$. Next, for each positive integer $k < n$, let $p_n^{(k)}$ be the kth derivative of p_n, and $X_j^{n,k}$, $j = 1, 2, \dots, n-k$, be the zeros of $p_n^{(k)}$. In particular, when $k = 1$, $X_1^{n,1}, \cdots, X_{n-1}^{n,1}$ are called the critical points of p_n. The relation between the zeros and the critical points of polynomials has been much studied. For example, the Gauss-Lucas theorem asserts that all critical points of a non-constant polynomial f lie inside the closed convex hull formed by the zeros of f. It follows by induction that the zeros of $f^{(k)}, k < \deg f$, also lie inside the same closed convex hull. More refinements of Gauss-Lucas theorem can be found in [1] and the references therein. A recent paper [2] also discussed some related results and examples.

On the other hand, R. Pemantle and I. Rivin initiated a probabilistic study on the limit of the critical points of random polynomials of Type 1. Consider the following probability measures:

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{Z_j^{(n)}}, \qquad \text{and} \qquad \mu_n^{(k)} = \frac{1}{n-k}\sum_{j=1}^{n-k} \delta_{X_j^{n,k}}, 1 \le k < n, \tag{1.4}$$

where δ_z is the Dirac measure concentrated on z. μ_n and $\mu_n^{(k)}$ are the empirical measures associated with the zeros $\{Z_j^{(n)}\}_{j=1}^n$ and $\{X_j^{n,k}\}_{j=1}^{n-k}$, respectively. R. Pemantle and I. Rivin ([3]) showed that, if $Z_j^{(n)} = Z_j$ and $\{Z_j\}_{j=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables governed by a common law ν, then $\mu_n^{(1)} \xrightarrow{w} \nu$ almost surely (a.s.) as $n \to \infty$ provided ν satisfies certain energy condition. In this paper $\xrightarrow{w}$ means "*converges weakly*" or "*converges in distribution*". In the same i.i.d. setting without any further assumption on the probability law ν, Z. Kabluchko ([4]) proved in great generality that $\mu_n^{(1)} \xrightarrow{w} \nu$ in probability as $n \to \infty$. For the case of higher order derivatives, in the i.i.d. setting, if the probability measure ν is supported on the unit circle in $\mathbb{C}$, P. L. Cheung et. al. ([5]) showed that $\mu_n^{(k)} \xrightarrow{w} \nu$ a.s. as $n \to \infty$. Similar results for the zeros of the generalized derivatives of polynomials are also obtained in [5].

To state a result of Type 2 polynomials, recall that a polynomial is called a *Kac polynomial* ([6]) if it has the form $\sum_{j=0}^{n} \xi_j z^j$, where $\{\xi_j\}_{j=0}^{\infty}$ is a sequence of non-degenerate i.i.d. random variables. Furthermore, given a sequence of deterministic complex numbers $\{w_j\}_{j=0}^{\infty}$, a polynomial of the form $\sum_{j=0}^{n} \xi_j w_j z^j$ is called a *Littlewood-Offord random polynomial*. Clearly, any kth derivative of a Kac polynomial is a Littlewood-Offord random polynomial. Z. Kabluchko and D. Zaporozhets ([6] and Theorem 14 of [1]) proved that both sequences of the empirical measures $\{\mu_n\}$ and $\{\mu_n^{(k)}\}$ converge weakly to the uniform distribution on the unit circle of $\mathbb{C}$ centered at the origin in

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probability as $n \to \infty$, provided that $\mathbf{E}[\log(1 + |\xi_0|)] < \infty$. See [6] for the explicit statements of the theorems and examples.

Under various settings and assumptions, the eigenvalue statistics of random matrices/sample covariance matrices exhibits various interesting and important limit behavious, for example the circle law, semicircle law, Marčenko-Pastur law, central limit theorem, large deviations, and so on. See [7] and [19] for a systematic introduction. To name a result related to our work, consider a sequence of random Hermitian matrices $\{A^{(n)}\}_{n=1}^{\infty}$ with p_n its characteristic polynomial. S. O'Rourke ([1]) showed that the Lévy distance between μ_n and $\mu_n^{(1)}$ tends to zero almost surely as $n \to \infty$. This observation implies that a.s. $\{\mu_n^{(1)}\}$ converges weakly to the same semicircle law as $\{\mu_n\}$ does. Such phenomenon that $\{\mu_n^{(1)}\}$ converges weakly to the same law as that of $\{\mu_n\}$ (at some mode of convergence) was further demonstrated for several compact classical matrix groups by S. O'Rourke in the same paper. Check [1] (theorem 6, corollary 7, theorem 9, and remark 10) for explicit statements and references therein. Here we merely point out that all the limit laws of the eigenvalue statistics of the matrix models considered are compactly supported in $\mathbb{C}$.

In view of the fact that all the results concerning the relation between $\{\mu_n\}$ and $\{\mu_n^{(k)}\}$ reviewed above are of the type of law of large numbers, it is natural to ask how about other types of limit theorem? The goal of this paper is to show that if certain limit property, for example law of large numbers, central limit theorem, law of iterated logarithm, and so on, holds for the linear statistics of $\{Z_j^{(n)}\}_{j=1}^{n}$ (see below for the precise statement), then the same limit property passes to that of $\{X_j^{n,k}\}_{j=1}^{n-k}$ for any k, provided the zeros $\{Z_j^{(n)}\}_{j=1}^{n}$ satisfy some mild conditions which we now state.

Denote by $\Im z$ the imaginary part of a complex number z.

A1. There exists a non-negative constant $C_0 \geq 0$ independent of n such that

$$\sup_{n\in\mathbb{N}} \max_{1\leq j\leq n} |\Im Z_j^{(n)}| \leq C_0 \quad \text{a.s.} \tag{1.5}$$

That is, the imaginary parts of $\{Z_j^{(n)}\}$ are uniformly bounded with probability one. Since every zero of $p_n^{(k)}$ lies inside the closed convex hull of the zeros of $p_n^{(k-1)}$ by Gauss-Lucas theorem, we know by induction that

$$\sup_{n\in\mathbb{N}} \max_{1\leq k<n} \max_{1\leq j\leq n-k} |\Im X_j^{n,k}| \leq C_0 \quad \text{a.s.} \tag{1.6}$$

When the zeros are real numbers, we put $C_0 = 0$.

Recall that $\mu_n \xrightarrow{\text{w}} \nu$ is equivalent to $\lim_{n\to\infty} \int f\, d\mu_n = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(Z_j^{(n)}) = \int f\, d\nu$ for each bounded continuous function f. It is therefore quite common to study such sums (namely, linear statistics) for various categories of test functions. [9], [10], [11], [12], and [13] are a few examples. In particular, the issue of regularity conditions for

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the test functions is discussed in [14] and [15]. In this paper we restrict ourselves to regular test functions f such that $f, f' \in L^\infty$, $\hat{f}$ exists and $f(x) = \int \hat{f}(t)e^{itx}\,dt$, and

$$\int_{\mathbb{R}} |\hat{f}(t)|e^{3C_0|t|}\,dt < \infty, \tag{1.7}$$

where $i = \sqrt{-1}$, $\hat{f}(t) = \frac{1}{2\pi}\int f(u)e^{-itu}\,du$ is the Fourier transform of f, and C_0 is the same absolute constant appeared in (1.5).

To adapt to the different scalings in various limit theorems, we consider different sequences of positive numbers for different linear statistics to be defined later. Below we list three groups of assumptions to be used in the three main theorems of this paper, respectively.

A2. There exists a sequence of positive numbers $\{a_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} a_n = \infty, \tag{1.8}$$

$$\lim_{n\to\infty} \frac{n\,|a_{n-k} - a_{n-k-1}|}{a_{n-k}a_{n-k-1}} = 0, \text{ for each fixed } k < n-1, \tag{1.9}$$

$$\lim_{n\to\infty} \sup_{1\le j\le n} \left\{ \frac{|Z_j^{(n)}|}{a_{n-k}} \right\} = 0 \quad \text{a.s. for each fixed } k < n. \tag{1.10}$$

A3. There exists a sequence of positive numbers $\{b_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} b_n = \infty, \tag{1.11}$$

$$\lim_{n\to\infty} \frac{|b_{n-k} - b_{n-k-1}|}{b_{n-k}b_{n-k-1}} \left[\sum_{j=1}^n |Z_j^{(n)}| \right] = 0 \quad \text{a.s. for each fixed } k < n-1, \tag{1.12}$$

$$\lim_{n\to\infty} \sup_{1\le j\le n} \left\{ \frac{|Z_j^{(n)}|}{b_{n-k}} \right\} = 0 \quad \text{a.s. for each fixed } k < n. \tag{1.13}$$

A4. There exist two sequences of positive numbers $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty, \tag{1.14}$$

$$\lim_{n\to\infty} \frac{n\,|a_{n-k} - a_{n-k-1}|}{a_{n-k}a_{n-k-1}} = 0, \text{ for each fixed } k < n-1, \tag{1.15}$$

$$\lim_{n\to\infty} \frac{|b_{n-k} - b_{n-k-1}|}{a_{n-k-1}b_{n-k}b_{n-k-1}} \left[\sum_{j=1}^n |Z_j^{(n)}| \right] = 0 \quad \text{a.s. for each fixed } k < n-1, \tag{1.16}$$

$$\lim_{n\to\infty} \sup_{1\le j\le n} \left\{ \frac{|Z_j^{(n)}|}{a_{n-k}b_{n-k}} \right\} = 0 \quad \text{a.s. for each fixed } k < n. \tag{1.17}$$

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Now we are ready to introduce the key objects that we want to study. Consider the following three linear statistics associated with $\{Z_j^{(n)}\}$

$$\begin{aligned} L_{n,1}(f) &= \frac{1}{a_n}\left[f\left(Z_1^{(n)}\right)+\cdots+f\left(Z_n^{(n)}\right)\right], \\ L_{n,2}(f) &= f\left(\frac{Z_1^{(n)}}{b_n}\right)+\cdots+f\left(\frac{Z_n^{(n)}}{b_n}\right), \\ L_{n,3}(f) &= \frac{1}{a_n}\left[f\left(\frac{Z_1^{(n)}}{b_n}\right)+\cdots+f\left(\frac{Z_n^{(n)}}{b_n}\right)\right], \end{aligned}$$

and the three linear statistics associated with $\{X_j^{n,k}\}$

$$\begin{aligned} L_{n,1}^{(k)}(f) &= \frac{1}{a_{n-k}}\left[f\left(X_1^{n,k}\right)+\cdots+f\left(X_{n-k}^{n,k}\right)\right], \\ L_{n,2}^{(k)}(f) &= f\left(\frac{X_1^{n,k}}{b_{n-k}}\right)+\cdots+f\left(\frac{X_{n-k}^{n,k}}{b_{n-k}}\right), \\ L_{n,3}^{(k)}(f) &= \frac{1}{a_{n-k}}\left[f\left(\frac{X_1^{n,k}}{b_{n-k}}\right)+\cdots+f\left(\frac{X_{n-k}^{n,k}}{b_{n-k}}\right)\right]. \end{aligned}$$

It is also necessary to consider the centered (mean zero) linear statistics

$$\bar{L}_{n,\ell}(f) = L_{n,\ell}(f) - \mathbf{E}[L_{n,\ell}(f)], \quad \bar{L}_{n,\ell}^{(k)}(f) = L_{n,\ell}^{(k)}(f) - \mathbf{E}[L_{n,\ell}^{(k)}(f)],\ \ell = 1,2,3.$$

For example, $L_{n,1}(f)$ with $a_n = n$ and $\bar{L}_{n,1}(f)$ with $a_n = \sqrt{n}$ play the typical roles in law of large numbers and central limit theorem, respectively. In the random matrix models, one studies $L_{n,3}$ with $a_n = n, b_n = \sqrt{n}$ for law of large numbers results. In these cases **A2** and **A4** are valid obviously.

Theorem 1. *Let the random zeros $\{Z_j^{(n)}\}_{j=1}^n$ and the sequence $\{a_n\}$ of positive numbers be given as above such that they satisfy the assumptions* **A1** *and* **A2**. *If the linear statistics $L_{n,1}(f)$ has a limit as $n \to \infty$ at some mode of convergence, then, for each $k < n$, the linear statistics $L_{n,1}^{(k)}(f)$ converges to the same limit at the same mode of convergence. Similar statement holds for $\bar{L}_{n,1}(f)$ and $\bar{L}_{n,1}^{(k)}(f)$.*

Theorem 2. *Let the random zeros $\{Z_j^{(n)}\}_{j=1}^n$ and the sequence $\{b_n\}$ of positive numbers be given as above such that they satisfy the assumptions* **A1** *and* **A3**. *If the linear statistics $\bar{L}_{n,2}(f)$ converges weakly to some probability law $\nu = \nu_f$ as $n \to \infty$, then $\bar{L}_{n,2}^{(k)}(f) \xrightarrow{w} \nu_f$ for each fixed $k < n$.*

Theorem 3. *Let the random zeros $\{Z_j^{(n)}\}_{j=1}^n$ and the sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers be given as above such that they satisfy the assumptions* **A1** *and* **A4**. *If the linear statistics $L_{n,3}(f)$ has a limit as $n \to \infty$ at some mode of convergence,*

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then, for each $k < n$, the linear statistics $L_{n,3}^{(k)}(f)$ converges to the same limit at the same mode of convergence. Similar statement holds for $\bar{L}_{n,3}(f)$ and $\bar{L}_{n,3}^{(k)}(f)$.

Remark 1. Consider the **Type 1** random polynomials with $Z_j^{(n)} = Z_j$ and $\{Z_j\}$ being an i.i.d. sequence. In this case the uniform condition (1.10) in **A2** can be weakened to

$$\lim_{n\to\infty} \frac{|Z_1| + \cdots + |Z_n|}{a_{n-k}n} = 0 \quad \text{a.s. for each fixed } k < n.$$

Remark 2. Again consider the **Type 1** random polynomials with $Z_j^{(n)} = Z_j$ and $\{Z_j\}$ being an i.i.d. sequence. This is the setting studied in [3], [4], and [5]. Note that our Theorem 1 establishes, in addition to law of large numbers result, also central limit theorem, law of iterated logarithm, and so on, for the linear statistics of $\{X_j^{n,k}\}$. However, Theorem 2 is not as interesting since it does not include the central limit theorem of random matrix models. This is because the size of the zeros $Z_j^{(n)}$ is of order $\sqrt{n} = b_n$, and therefore the conditions (1.12) and (1.13) would not hold. Reasonable condition(s) should involve the centered quantities.

In Section 2 we establish a comparison identity. It is elementary, and yet crucial to our results. In Section 3 we prove three theorems and make some final remarks.

2. A comparison identity

First we state (with some modifications) a theorem of Cheung and Ng ([16]) which is the starting point of our argument.

Proposition 4 (Theorem 1.1 of [16]). *The set of all critical points of $p_n(z) = \prod_{k=1}^n (z - z_k), n \geq 2$, is the same as the set of all eigenvalues of the $(n-1) \times (n-1)$ matrix M_{n-1}:*

$$M_{n-1} = D_{n-1} + \frac{1}{n}\left(z_1 I_{n-1} - D_{n-1}\right) J_{n-1}, \tag{2.1}$$

where

$$D_{n-1} = \begin{pmatrix} z_2 & 0 & \cdots & 0 \\ 0 & z_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}, \qquad J_{n-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \tag{2.2}$$

and I_{n-1} is the identity matrix of size $n-1$.

Denote by $\operatorname{Tr} M$ the trace of a square matrix M. Our results rely on the following observation.

Lemma 5. *Let M_{n-1} and D_{n-1} be defined as in Proposition 2 and $i = \sqrt{-1}$. Then*

$$\operatorname{Tr}\left(e^{itM_{n-1}}\right) - \operatorname{Tr}\left(e^{itD_{n-1}}\right) = \frac{ic_{n-1}}{n} \operatorname{Tr}\left(\tilde{J}_{n-1}\, e^{itD_{n-1}}\right), \tag{2.3}$$

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where

$$\begin{aligned} c_{n-1} &= c_{n-1}(t) = \int_0^t \exp\left(iu\frac{\tilde{S}_{n-1}}{n} \right) du, \\ \tilde{S}_{n-1} &= (z_1 - z_2) + \cdots + (z_1 - z_n), \text{ and} \\ \tilde{J}_{n-1} &= \begin{pmatrix} z_1 - z_2 & 0 & \cdots & 0 \\ 0 & z_1 - z_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_1 - z_n \end{pmatrix} J_{n-1}\,. \end{aligned}$$

Proof. It can be proved straightforwardly by Taylor expansions. However, to derive the constant c_{n-1} in a more natural way, we first use the Duhamel formula

$$e^{(L_1+L_2)t} - e^{L_1 t} = \int_0^t e^{L_1(t-\tau)} L_2 e^{(L_1+L_2)\tau}\, d\tau$$

with $L_1 = iD_{n-1}$ and $L_1 + L_2 = iM_{n-1}$. In fact, $L_2 = \frac{i}{n}\tilde{J}_{n-1}$. Since $\mathrm{Tr}(e^{A+B}) = \mathrm{Tr}(e^A\, e^B)$ by $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, the left hand side of (2.3) equals

$$\frac{i}{n}\int_0^t \mathrm{Tr}\left(\tilde{J}_{n-1} e^{itD_{n-1}}\, e^{(iu\tilde{J}_{n-1})/n} \right) du. \tag{2.4}$$

One can show by induction that $\tilde{J}^k_{n-1} = \tilde{S}^{k-1}_{n-1}\tilde{J}_{n-1}, k \in \mathbb{N}$. After using this fact in the Taylor expansion of $e^{(iu\tilde{J}_{n-1})/n}$, the integrand within the integral of (2.4) can be simplified to

$$\mathrm{Tr}\left(\tilde{J}_{n-1} e^{itD_{n-1}} \right) e^{(iu\tilde{S}_{n-1})/n}. \tag{2.5}$$

This completes the proof. □

Remark 3. The D_{n-1} appeared in [16] is a diagonal matrix with diagonal entries $\{z_1, \ldots, z_{n-1}\}$, while we choose a different one given in (2.2) by the symmetry among the zeros. The M_{n-1} in (2.1) is modified accordingly.

Remark 4. Observe that the terms in (2.5) can be expressed in a more symmetric way:

$$\frac{\tilde{S}_{n-1}}{n} = z_1 - \frac{\sum_{j=1}^n z_j}{n}, \tag{2.6}$$

$$\frac{1}{n}\mathrm{Tr}\left(\tilde{J}_{n-1} e^{itD_{n-1}} \right) = z_1 \left(\frac{\sum_{j=1}^n e^{itz_j}}{n} \right) - \frac{\sum_{j=1}^n z_j e^{itz_j}}{n}. \tag{2.7}$$

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Suppose that $\{z_j\}_{j=1}^n$ satisfies the bound $\max_{1\le j\le n} |\Im z_j| \le C_0$, then it is easy get that

$$|c_{n-1}(t)| \le \frac{e^{2C_0|t|}-1}{2C_0}, \tag{2.8}$$

$$\left|\frac{1}{n}\mathrm{Tr}\left(\tilde{J}_{n-1}e^{itD_{n-1}}\right)\right| \le e^{C_0|t|}\left(|z_1| + \frac{\sum_{j=1}^n |z_j|}{n}\right). \tag{2.9}$$

When all zeros are real, $C_0 = 0$ and one simply gets $|c_{n-1}(t)| \le |t|$. These are used in the proofs of three theorems.

3. Proofs and concluding remarks

We now prove Theorem 1.

Proof. Consider the case of $L_{n,1}(f)$ and $L_{n,1}^{(1)}(f)$ ($k = 1$). To prove the theorem in this case, it suffices to show that $L_{n,1}(f) - L_{n,1}^{(1)}(f) \to 0$ a.s. as $n \to \infty$. Write $L_{n,1}(f) - L_{n,1}^{(1)}(f) = W_{n,1} + W_{n,2}$, where

$$\begin{aligned} W_{n,1} &= \frac{1}{a_n}\left[f\left(Z_1^{(n)}\right) + \cdots + f\left(Z_n^{(n)}\right)\right] - \frac{1}{a_{n-1}}\left[f\left(Z_2^{(n)}\right) + \cdots + f\left(Z_n^{(n)}\right)\right] \\ &= \frac{f\left(Z_1^{(n)}\right)}{a_n} + \frac{a_{n-1}-a_n}{a_{n-1}a_n}\sum_{j=2}^n f\left(Z_j^{(n)}\right), \end{aligned}$$

and

$$W_{n,2} = \frac{1}{a_{n-1}}\left[f\left(Z_2^{(n)}\right) + \cdots + f\left(Z_n^{(n)}\right)\right] - \frac{1}{a_{n-1}}\left[f\left(X_1^{n,1}\right) + \cdots + f\left(X_{n-1}^{n,1}\right)\right].$$

Clearly

$$|W_{n,1}| \le \frac{\|f\|_\infty}{a_n} + \frac{n\,|a_{n-1}-a_n|\|f\|_\infty}{a_{n-1}a_n} \to 0 \quad \text{a.s.}$$

by (1.8) and (1.9) of **A2**. Next we apply Lemma 3 with $Z_j^{(n)}$ in place of $z_j, j = 1, \ldots, n$, to obtain that

$$\begin{aligned} W_{n,2} &= \frac{1}{a_{n-1}}\int \hat{f}(t)\left[\mathrm{Tr}\left(e^{itD_{n-1}}\right) - \mathrm{Tr}\left(e^{itM_{n-1}}\right)\right] dt \\ &= \frac{1}{a_{n-1}}\int \hat{f}(t)\frac{c_{n-1}(t)}{n}\mathrm{Tr}\left(\tilde{J}_{n-1}\,e^{itD_{n-1}}\right) dt. \end{aligned}$$

Since we assume **A1**, the estimates (2.8) and (2.9) in Remark 4 can be used to yield that

$$|W_{n,2}| \le \frac{1}{2C_0}\left(\int |\hat{f}(t)|e^{3C_0|t|}\,dt\right)\left(\frac{|Z_1^{(n)}|}{a_{n-1}} + \frac{\sum_{j=1}^n |Z_j^{(n)}|}{na_{n-1}}\right). \tag{3.1}$$

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By the regularity condition (1.7) together with (1.8) and (1.10) of **A2**, we conclude that $|W_{n,2}| \to 0$ a.s. when $n \to \infty$.

For $k > 1$ we decompose $L_{n,1}(f) - L_{n,1}^{(k)}(f) = L_{n,1}(f) - L_{n,1}^{(1)}(f) + \sum_{j=1}^{k-1} L_{n,1}^{(j+1)}(f) - L_{n,1}^{(j)}(f)$ and need to show that each $|L_{n,1}^{(j)}(f) - L_{n,1}^{(j+1)}(f)| \to 0$ a.s. We simply follow the same strategy as above. Two facts can be useful when estimating the difference between the traces of two matrices. First one is the basic relation between roots and coefficients:

$$\frac{Z_1^{(n)} + \cdots + Z_n^{(n)}}{n} = \frac{X_1^{n,k} + \cdots + X_{n-k}^{n,k}}{n-k}, k < n.$$

The second relation can be found in [17] and [18] :

$$\frac{|X_1^{n,k}| + \cdots + |X_{n-k}^{n,k}|}{n-k} \leq \cdots \leq \frac{|X_1^{n,1}| + \cdots + |X_{n-1}^{n,1}|}{n-1} \leq \frac{|Z_1^{(n)}| + \cdots + |Z_n^{(n)}|}{n}.$$

The rest is easy and is omitted. When demonstrating the weak convergence of $\bar{L}_{n,1}^{(k)}$ from that of $\bar{L}_{n,1}$, the converging together lemma should be applied to complete the proof. □

The proofs of Theorem 2 and Theorem 3 are similar to that of Theorem 1. A mean value inequality and the boundedness of $|f'|$ can justify the transference of the scales from b_{n-k} to b_{n-k-1}. When estimating the terms like (3.1), the uniform condition (1.13) in **A3** and/or (1.16) in **A4** would be helpful.

Now prove Theorem 2.

Proof. Again, since

$$\begin{aligned}\tilde{L}_{n,2} - \tilde{L}_{n,2}^{(1)} &= (\sum_{k=1}^{n} f(\frac{Z_k^{(n)}}{b_n}) - \mathbf{E}f(\frac{Z_k^{(n)}}{b_n})) - (\sum_{k=2}^{n} f(\frac{Z_k^{(n)}}{b_{n-1}}) - \mathbf{E}f(\frac{Z_k^{(n)}}{b_{n-1}})) \\ &+ (\sum_{k=2}^{n} f(\frac{Z_k^{(n)}}{b_{n-1}}) - \mathbf{E}f(\frac{Z_k^{(n)}}{b_{n-1}})) - (\sum_{k=1}^{n-1} f(\frac{X_k^{n,1}}{b_{n-1}}) - \mathbf{E}f(\frac{X_k^{n,1}}{b_{n-1}})),\end{aligned}$$

where (again) first two summation terms vanishes by **A3** and $f, f' \in L^\infty$. The last two summation terms (again) are equal to

$$\int \hat{f}(t) \left[\mathrm{Tr}\left(e^{itD_{n-1}/b_{n-1}}\right) - \mathrm{Tr}\left(e^{itM_{n-1}/b_{n-1}}\right)\right] - \mathbf{E}\left[\mathrm{Tr}\left(e^{itD_{n-1}/b_{n-1}}\right) - \mathrm{Tr}\left(e^{itM_{n-1}/b_{n-1}}\right)\right] dt,$$

hence it is bounded by

$$\frac{1}{2C_0}\left(\int |\hat{f}(t)| e^{3C_0|t|}\, dt\right)\left[\left(\frac{|Z_1^{(n)}|}{b_{n-1}} + \frac{\sum_{j=1}^n |Z_j^{(n)}|}{nb_{n-1}}\right) + \mathbf{E}\left(\frac{|Z_1^{(n)}|}{b_{n-1}} + \frac{\sum_{j=1}^n |Z_j^{(n)}|}{nb_{n-1}}\right)\right]$$

which also vanishes by **A1** and **A3**. □

Finally prove Theorem 3.

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Proof. Similarly

$$L_{n,3} = \frac{1}{a_n}\sum_{k=1}^{n} f(\frac{Z_k^{(n)}}{b_n}) - \frac{1}{a_{n-1}}\sum_{k=2}^{n} f(\frac{Z_k^{(n)}}{b_{n-1}}) + \frac{1}{a_{n-1}}\sum_{k=2}^{n} f(\frac{Z_k^{(n)}}{b_{n-1}}) - \frac{1}{a_{n-1}}\sum_{k=1}^{n-1} f(\frac{X_k^{n,1}}{b_{n-1}}),$$

where (again) first two summation terms vanishes by **A4** and $f, f' \in L^\infty$. The last two summation terms (again) are equal to

$$\frac{1}{a_{n-1}}\int \hat{f}(t)\left[\mathrm{Tr}\left(e^{itD_{n-1}/b_{n-1}}\right) - \mathrm{Tr}\left(e^{itM_{n-1}/b_{n-1}}\right)\right] dt$$

hence it is bounded by

$$\frac{1}{2a_{n-1}C_0}\left(\int |\hat{f}(t)|e^{3C_0|t|}\, dt\right)\left(\frac{|Z_1^{(n)}|}{b_{n-1}} + \frac{\sum_{j=1}^{n}|Z_j^{(n)}|}{nb_{n-1}}\right)$$

which also vanishes by **A1** and **A4**. □

Remark 5. In the proofs of the theorems we deal with the strongest mode of convergence, namely the almost sure convergence, of $L_{n,\ell}(f) - L_{n,\ell}^{(k)}(f) \to 0, \ell = 1,2,3$. If the original $L_{n,\ell}(f), \ell = 1,3$, converges at a weaker mode, an alternative argument using weaker estimates should be considered. Consequently, it is conceivable that the assumptions weaker than **A1** and **A2** (or **A4**) might be sufficient for the theorems to hold, and the regularity conditions on the test functions used in the linear statistics might also be relaxed.

Remark 6. One can show that if the empirical measure associated with the zeros of p_n, under appropriate scaling, obeys a large deviations principle, so does the empirical measure associated with the zeros of $p_n^{(k)}$ for each k. This is treated elsewhere ([21]).

4. Large Deviations

But what can we say if we only have large deviation principle?

Observe that: if $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\lim_{n\to\infty} x_n - y_n = 0$, then

$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n$$

and

$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n.$$

So if

$$\lim_{n\to\infty} \Pr\{X_n \in A\} - \Pr\{Y_n \in A\} = 0$$

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for any A is Borel measurable and $\{X_n\}_{n\in\mathbb{N}}$ satisfy large deviation principle, then $\{Y_n\}_{n\in\mathbb{N}}$ also satisfy large deivation principle with the same rate function.

Obviously, the total variation

$$\sup_A |\Pr\{X_n \in A\} - \Pr\{Y_n \in A\}|$$

vanishes as $n \to \infty$ implies

$$\lim_{n\to\infty} \Pr\{X_n \in A\} - \Pr\{Y_n \in A\} = 0.$$

Nevertheless, even

$$\lim_{n\to\infty} X_n - Y_n = 0$$

almost surely does not imply the total variation vanishes as $n \to \infty$. One way to guarantee is that

Lemma 6. *If there is another measure m such that $\Pr\{X_n \in A\} = \int_A f_n dm$ and $\Pr\{Y_n \in A\} = \int_A g_n dm$ for all A is measurable, where f_n, g_n are integrable w.r.t. measure m, then*

$$\lim_{n\to\infty} \sup_A |\Pr\{X_n \in A\} - \Pr\{Y_n \in A\}| = 0 \Leftrightarrow \lim_{n\to\infty} \int |f_n - g_n| dm = 0.$$

Proof: One side is trivial since

$$\int |f_n - g_n| dm \geq \int_A |f_n - g_n| dm \geq |\int_A f_n - g_n dm|,$$

for any A is measurable. Conversely, because

$$\sup_A |\Pr\{X_n \in A\} - \Pr\{Y_n \in A\}| = max \left\{ \int_{\{f_n - g_n \geq 0\}} f_n - g_n dm, \quad -\int_{\{f_n - g_n < 0\}} f_n - g_n dm \right\} \to 0$$

as $n \to \infty$, we have

$$\int |f_n - g_n| dm = \int_{\{f_n - g_n \geq 0\}} f_n - g_n dm \quad - \int_{\{f_n - g_n < 0\}} f_n - g_n dm \to 0$$

as $n \to \infty$. Done.

This is the end of our main results.

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5. Appendix

Proof of Proposition 4: Since (here $p_n(z) = (z - Z_1)\cdots(z - Z_n)$)

$$0 = \frac{p_n'(w)}{p_n(w)} = \sum_{j=1}^{n} \frac{1}{w - Z_j}$$

if w is a critical point of p_n and w is not equal to any of roots Z_j. We have

$$\frac{1}{w - Z_n} = \sum_{j=1}^{n-1} \frac{1}{Z_j - w},$$

so

$$\sum_{j=1}^{n} \frac{Z_j - Z_n}{Z_j - w} = \sum_{j=1}^{n} \frac{Z_j - w + w - Z_n}{Z_j - w} = n.$$

Conversely this observation reveals that if a number $\lambda \notin \{Z_j\}_{j=1}^{n-1}$ and $\sum_{j=1}^{n} \frac{Z_j - Z_n}{Z_j - \lambda} = n$, then $\lambda \neq Z_n$ and $p_n'(\lambda) = 0$.

First we would show any critical point w of p_n, $w \notin \{Z_j\}_{j=1}^{n}$, is an eigenvalue of M with eigenvector

$$\left(\tfrac{Z_1 - Z_n}{n(Z_1 - w)}, \quad \cdots \quad , \tfrac{Z_{n-1} - Z_n}{n(Z_{n-1} - w)} \right)^T$$

via

$$M \begin{pmatrix} \frac{Z_1 - Z_n}{n(Z_1 - w)}, \\ \vdots \\ \frac{Z_{n-1} - Z_n}{n(Z_{n-1} - w)} \end{pmatrix} = \begin{pmatrix} \frac{Z_1(Z_1 - Z_n)}{n(Z_1 - w)} \\ \vdots \\ \frac{Z_{n-1}(Z_{n-1} - Z_n)}{n(Z_{n-1} - w)} \end{pmatrix} - \frac{1}{n^2} \sum_{j=1}^{n-1} \frac{Z_j - Z_n}{Z_j - w} \overline{D} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{Z_1(Z_1 - Z_n)}{n(Z_1 - w)} \\ \vdots \\ \frac{Z_{n-1}(Z_{n-1} - Z_n)}{n(Z_{n-1} - w)} \end{pmatrix} - \begin{pmatrix} \frac{Z_1(Z_1 - Z_n)}{n} \\ \vdots \\ \frac{Z_{n-1}(Z_{n-1} - Z_n)}{n} \end{pmatrix} = w \begin{pmatrix} \frac{Z_1(Z_1 - Z_n)}{n(Z_1 - w)} \\ \vdots \\ \frac{Z_{n-1}(Z_{n-1} - Z_n)}{n(Z_{n-1} - w)} \end{pmatrix}.$$

If w is equal to one of roots saying Z_i, then $w = Z_i = Z_j$, $i \neq j$. In particular if $i = n$, then $w = Z_n$ is an eigenvalue of M with eigenvector

$$e_j + \left(1, \quad \cdots \quad , 1\right)^T$$

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by showing

$$M\begin{pmatrix}1\\ \vdots\\ 2\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}Z_1\\ \vdots\\ 2Z_j\\ \vdots\\ Z_{n-1}\end{pmatrix} - \begin{pmatrix}Z_1 - Z_n\\ \vdots\\ 0\\ \vdots\\ Z_{n-1} - Z_n\end{pmatrix} = Z_n \begin{pmatrix}1\\ \vdots\\ 2\\ \vdots\\ 1\end{pmatrix},$$

where e_j is the standard basis, i.e. , 2 only occurs at j^{th} component.

For $w = Z_i = Z_j$, $i, j \leq n-1$, we have

$$M(e_i - e_j) = D(e_i - e_j) = w(e_i - e_j).$$

Therefore

$$\{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \subseteq \sigma(M).$$

Conversely if $\lambda \in \sigma(M)$ with eigenvector $\begin{pmatrix}v_1, & \cdots & , v_{n-1}\end{pmatrix}^T$, i.e.

$$\begin{pmatrix}(Z_1 - \lambda)v_1\\ \vdots\\ (Z_{n-1} - \lambda)v_{n-1}\end{pmatrix} = \frac{1}{n}(\sum_{j=1}^{n-1} v_j)\begin{pmatrix}Z_1 - Z_n\\ \vdots\\ Z_{n-1} - Z_n\end{pmatrix}.$$

If $\sum_{j=1}^{n-1} v_j = 0$, then at least two of $\{v_j\}_{j=1}^{n-1}$, saying v_1 and v_2, are non-zero. Hence $\lambda = Z_1 = Z_2$. Now assume $\sum_{j=1}^{n-1} v_j \neq 0$. If $\lambda = Z_j$, $j \leq n-1$, then $Z_n = Z_j = \lambda$. If $\lambda \notin \{Z_j\}_{j=1}^{n-1}$, then

$$\frac{v_j}{\sum_{i=1}^{n-1} v_i} = \frac{Z_j - Z_n}{n(Z_j - \lambda)}.$$

Summing up all j from 1 to $n-1$ we have

$$\sum_{j=1}^{n} \frac{Z_j - Z_n}{Z_j - \lambda} = n,$$

as a result,

$$\{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \supseteq \sigma(M).$$

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Adopt the proof above we have

Lemma 7 (Gauss-Lucas Theorem)**.** *Again* $p_n(x) := \prod_{k=1}^{n} (x - Z_k)$*, we have*

$$\{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \subseteq Conv\, \{Z_k\}_{k=1}^{n}.$$

Proof: Let $w \in \{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \setminus \{Z_k\}_{k=1}^{n}$ (otherwise it is trivial), then

$$0 = \frac{p_n'(w)}{p_n(w)} = \sum_{j=1}^{n} \frac{1}{w - Z_j} = \sum_{j=1}^{n} \frac{\overline{w} - \overline{Z_j}}{|w - Z_j|^2},$$

so we conclude that

$$w = \frac{\sum_{j=1}^{n} \frac{Z_j}{|w-Z_j|^2}}{\sum_{j=1}^{n} \frac{1}{|w-Z_j|^2}} \in Conv\, \{Z_k\}_{k=1}^{n}.$$

From above we can have

Proposition 8. *If* $\{Z_k\}_{k=1}^{n} \subseteq L^p$*,* $p \geq 1$*, then*

$$\{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \subseteq L^p.$$

Proof: Let $w \in \{z \,\epsilon\, \mathbb{C} \mid p_n'(z) = 0\} \setminus \{Z_k\}_{k=1}^{n}$ (otherwise it is trivial), then (through Jensen's inequality)

$$\mathbf{E}|w|^p = \mathbf{E}| \sum_{j=1}^{n} \frac{\frac{Z_j}{|w-Z_j|^2}}{\sum_{k=1}^{n} \frac{1}{|w-Z_k|^2}} |^p \leq \sup_{1 \leq k \leq n} \{\mathbf{E}|Z_k|^p\} < \infty.$$

The ideas below originally follow from [19]: First it is easy to see that

Proposition 9 ([19], Lemma 2.2 and Lemma 2.3)**.** *If every coefficients* a_k *of a random polynomial* $\sum_{k=0}^{n} a_k x^k$ *are in* L^p*,* $p \geq 1$*, then the polynomial is continuous in* L^p*, i.e.*

$$\lim_{x \to x_0} \mathbf{E}| \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} a_k x_0^k |^p = 0$$

and hence continuous in probability.

We also observe that

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Proposition 10. *Let $p_n(x) := \sum_{k=0}^{n} a_k x^k$ be a random polynomial, then*

$$\mathbf{E}\left[p_n(\beta) - p_n(\alpha) \mid \sigma\left\{p_n(x) \colon x \leq \alpha\right\}\right] = \mathbf{E}\left[\int_{\alpha}^{\beta} p_n'(x)dx \mid \sigma\left\{a_k\right\}_{k=1}^{n}\right],$$

hence $p_n(x)$ is a sub- or super-martingale on $[a,b] \subseteq \mathbb{R}$ if $p_n'(x) \geq$ or ≤ 0 on $[a,b]$.

Nevertheless we still need to answer a fundamental question: are roots of

$$p_n(x) := \sum_{k=0}^{n} a_k x^k$$

measurable if complex coefficients $\{a_k\}_{k=0}^{n}$ are measurable?

To answer the question, we need measurable selection Theorem due to Kuratowski and Ryll-Nardzewski, and the proof presented here follows from [20].

Theorem 11 ([20], Theorem 6.9.3)**.** *Let (Ω, Σ)be a measurable space, X be a Polish space and $F : \Omega \to X$ be a mapping with values in the family of non-empty closed subsets of X. If for every open subset $U \subset X$, we have*

$$\tilde{F}(U) := \{\omega \in \Omega : F(\omega) \cap G \neq \phi\} \in \Sigma.$$

Then F has a $\Sigma-$measurable selection $f : \Omega \to X$ such that

$$f(\omega) \in F(\omega), \forall \omega \in \Omega.$$

Proof: Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X. Define $f_0 : \Omega \to \{x_n\}_{n\in\mathbb{N}}$ as

$$f_0(\omega) := x_j,$$

where

$$j := \inf\{n \in \mathbb{N} : F(\omega) \cap B(x_n, 1) \neq \phi\}.$$

Hence f_0 is $\Sigma-$measurable since

$$f_0^{-1}\{x_j\} = \tilde{F}(B(x_j, 1)) \setminus \bigcup_{m=1}^{j-1} \tilde{F}(B(x_m, 1)) \in \Sigma.$$

Now we construct inductively measurable mappings $f_k : \Omega \to \{x_n\}_{n\in\mathbb{N}}$ such that

$$d(f_k(\omega), f_{k+1}(\omega)) < 2^{-k+1}$$

and

$$d(f_k(\omega), F(\omega)) < 2^{-k}.$$

Suppose f_k is already constructed. Fix k, then for all disjoint $f_k^{-1}\{x_i\}, i \in \mathbb{N}$, define

$$f_{k+1} : f_k^{-1}\{x_i\} \to \{x_n\}_{n\in\mathbb{N}}$$

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as

$$f_{k+1}(\omega) := x_j,$$

where

$$j := \inf\left\{n \in \mathbb{N} : \left(F(\omega) \cap B\left(x_i, 2^{-k}\right)\right) \cap B\left(x_n, 2^{-k-1}\right) \neq \phi\right\}$$

since $F(\omega) \cap B\left(x_i, 2^{-k}\right) \neq \phi$, $\forall\, \omega \in f_k^{-1}\{x_i\}$.

So

$$d\left(f_{k+1}(\omega), F(\omega)\right) < 2^{-k-1}$$

and

$$d\left(f_k(\omega), f_{k+1}(\omega)\right) < 2^{-k} + 2^{-k-1} < 2^{-k+1}.$$

As a result, we have a $\Sigma-$measurable mapping

$$\lim_{k\to\infty} f_k(\omega) =: f(\omega) \in F(\omega),$$

since $\{f_k(\omega)\}_{k\geq 0}$ is Cauchy and $\phi \neq F(\omega)$ is closed.

Now we are able to answer the question.

Theorem 12 ([19], Theorem 2.2)**.** *All roots of*

$$p_n(x) := \sum_{k=0}^{n} a_k x^k$$

are $\Sigma-$measurable if complex coefficients $\{a_k\}_{k=0}^{n}$ are $\Sigma-$measurable.

Proof: Let $F(\omega) := p_n^{-1}\{0\} \subset \mathbb{C}$, which is closed and non-empty in $\mathbb{C}$ by Fundamental Theorem of Algebra, and let D be countable dense subset of $\mathbb{C}$.

Observe that for every open subset $G \subset X$,

$$\tilde{F}(G)^c = \underset{x\in G}{\cup}\{\omega \in \Omega : p_n(x) \in \{0\}^c\} := \underset{x\in G}{\cup} A_x,$$

i.e.,

$$\omega \in A_x,\ x \in G \iff \sum_{k=0}^{n} a_k(\omega)\, x^k \neq 0.$$

Fix such $\omega \in A_x$ and $x \in G$, then

$$\exists\, \tilde{x} \in G \cap D \ni \sum_{k=0}^{n} a_k(\omega)\, \tilde{x}^k \in \{0\}^c$$

since $p_n(\omega)^{-1}\left(\{0\}^c\right)$ is open (then $\tilde{x}$ is in a neighbourhood of x contained in $p_n(\omega)^{-1}\left(\{0\}^c\right)$ such that $\sum_{k=0}^{n} a_k(\omega)\, \tilde{x}^k \neq 0$) and $\tilde{x} \in D \subset \mathbb{C}$ is dense.

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Hence

$$\tilde{F}(G)^c = \bigcup_{x \in G} A_x = \bigcup_{\tilde{x} \in G \cap D} A_{\tilde{x}} \in \Sigma.$$

Applying above measurable selection theorem, we have

$$Z_1(\omega) \in p_n^{-1}\{0\}$$

is $\Sigma-$measurable.

Take

$$p_{n-1}(x) := \frac{p_n(x)}{x - Z_1}$$

(we may assume $a_n \neq 0$ a.s.) and then the result follows by backward induction.

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