國立臺灣大學理學院物理學研究所

碩士論文

Department of Physics

College of Science

National Taiwan University

Master Thesis

在量子力學和量子場論中的 Resurgence

Resurgence in Quantum Mechanics and Quantum Field Theory

陳宗彥

Tsung-Yen Chen

指導教授:細道和夫 博士

Advisor: Kazuo Hosomichi, Ph.D.

中華民國 106 年 5 月

May 2017

doi:10.6342/NTU201700825

摘要

本論文總結了關於應用 resurgence trans-series 在量子力學和量子場論的知識。我們重現了在微擾級數中的虛數項和瞬子-反瞬子對中的虛數項的相消。結合所有來自非微擾半古典組態的貢獻之後,就可以構築出所謂的trans-series。因為紅外重整子的存在,在量子場論中構建trans-series被認為是無法實現的。近年來，因為在某些被緊湊化後的量子場論中發現了新的半古典組態，這些半古典組態被認為可能對應到紅外重整子。利用 trans-series 來給部分量子場論在弱藕荷常數極限下一個非微擾的定義可能是可以實現的。

關鍵詞: Resurgence,Borel resummation,瞬子, 重整子,微擾展開,trans-series

doi:10.6342/NTU201700825

Abstract

This thesis reviews the idea of resurgence trans-series and its application on Quantum mechanics and Quantum field theory. The cancellation between the imaginary part of the perturbative series and the imaginary part of the instanton-anti-instanton configuration in quantum mechanics is shown. Combining all the contribution from the non-perturbative semiclassical configuration, the so-called trans-series can be constructed. Because of the existence of IR renormalon, the construction of trans-series in QFT was not thought to be valid. In recent years, new semiclassical configurations have been found in some compactified theories. Those configurations are thought to be corresponded to the elusive IR renormalons. A non-perturbative definition of a class of field theories by the trans-series in the weak coupled limit may be possible.

Keywords: Resurgence, Borel resummation, instanton, renormalon, perturbative expansion, trans-series

doi:10.6342/NTU201700825

Contents

1 **Introduction** **1**

2 **Perturbation series** **3**

2.1 Borel resummation 5

2.2 Source of divergence 7

2.2.1 Renormalon divergence 9

2.3 Trans-series approach 13

3 **Resurgence in QM** **16**

3.1 Quantum mechanics in the anharmonic oscillator 17

3.2 Quantum mechanics in the double well potential 19

3.2.1 One instanton contribution 21

3.2.2 Multi instanton contribution 25

4 **Resurgence in QFT** **31**

4.1 QCD(adj) on $\mathbb{R}^3 \times S^1$ 32

4.1.1 4-d theory 33

4.1.2 3-d effective theory 34

4.1.3 Topological configurations on $\mathbb{R}^3 \times S^1$ 39

4.1.4 Topological molecules and IR renormalons in QCD(adj) on $\mathbb{R}^3 \times S^1$ 42

4.2 CP^{N-1} model on $\mathbb{R}^1 \times S^1$ 44

4.2.1 CP^{N-1} model 45

4.2.2 Gauge holonomy 46

4.2.3 Free energy 48

4.2.4 Topological configurations on $\mathbb{R}^1 \times S^1$ 48

4.2.5 IR renormalons and bions 56

5 **Conclusion and Future directions** **61**

Appendices **62**

A **Ambiguity of Borel resummation** **62**

doi:10.6342/NTU201700825

B Computation of the functional determinant in Quantum Mechanics 64
Appendix B.A Gelfand-Yaglom method 66

Bibliography 70

doi:10.6342/NTU201700825

List of Figures

1 The poles of the function $B_P(g)$ 6

2 Renormalons and the instanton poles. Renormalons are usually closer to the origin. 8

3 The fermion bubble diagrams we choose for the Adler function. The number of fermion loops inserted to the single gluon line can be infinity. 9

4 The shape of the anharmonic potential $V(q)$ 18

5 The shape of the potential $V(q)$ with the sign change $g \to -g$ 18

6 The shape of the double well potential $V(q)$ 20

7 An instanton configuration and an anti-instanton configuration for $g = 1$, both at $t_0 = 0$. 20

8 The poles on the Borel t plane for QCD(adj) on different manifold. . 43

9 Lateral Borel sums, the different choices of the path give us different results. 58

10 Upper figure: The conjectured pole structure of the Borel plane for CP^{N-1} on $\mathbb{R}^2$. Lower figure: The semiclassical singularities corresponding to the topological molecules in CP^{N-1} on small $\mathbb{R} \times S^1$. There is an extra singularity closer to the origin for $N_f = 0$. For $N_f \geq 1$, the pole structure of the topological molecules and the IR renormalons concide. Although the analysis is done in weakly coupled region and the IR renormalons come from strongly coupled region, we conjectured that the pole structure does not change significantly. This is an evidence to show that bion-anti-bion pairs on $\mathbb{R}^1 \times S^1$ may be the semiclassical realization of the IR renormalons on $\mathbb{R}^2$. 59

11 The pole on the Borel complex t plane 63

12 The different choice of paths 63

doi:10.6342/NTU201700825

1 Introduction

Searching a good way to define quantum field theory

Qunatum field theory (QFT) is a framework which combines quantum physics and special relativity. The Standard Model (SM) is the success of QFT which meets the experimental results very well. When we use QFT to reproduce experimental results, we often use perturbation theory. We choose our model to be simple so we can solve exactly, then we turn on the interaction and make the theory become what we want. After renormalization, we can do perturbative expansion with respect to the interaction parameter, say g. So we can compute the contribution of g order by order and we can write down our observable as a power series in g,

$$O(g) = a_0 + a_1 g + a_2 g^2 + \cdots = \sum_{n=0}^{\infty} a_n g^n. \tag{1.1}$$

Here a_0 is the value of the observable when $g = 0$ which we can solve exactly. All higher order terms a_n can be computed based on the original simple model. This perturbation expansion works very well when we compute observables in some QFT and QM, like the anomalous magnetic moment of electron in QED or the ground state energy for some quantum mechanics problem. But in all these cases, we only take the first few terms to do the prediction and we neglect the contribution from the higher order terms. We think the contribution from the higher order terms are so small that we can throw them away.

If all the higher order terms are smaller and smaller then this is harmless, but in fact this is not the case. Higher order terms in perturbation series can become very large thus the precondition of perturbation theroies breaks down and the series do not make any sense, see [1, 2, 3, 4, 5]. The perturbative series is generically factorially divergent and have zero radius of convergence. So what we are actually computing and why the first few terms make good prediction to the experiments become a serious question.

A useful mathmatical approach to this kind of divergent series problem is known as *Borel-Écalle resummation.* This method can redifine the series and extract the information from the divergent series. Borel resummation can reveal the underlying non-perturbative strructure in the perturbative series. This can tell us there are

doi:10.6342/NTU201700825

at least how many non-perturbative structures are hidden in our theory. Borel resummation can be applied to all the divergent series, but not all series are Borel summable. For a Borel summable series, Borel resummation gives us a well-defined answer, but for a non-Borel summable series, because there are poles lying on the integration contour, we need to do analytic continuation to change the contour. The value of the series depends on the integration contour we have chosen and the value is ambiguous.

When one do Borel resummation to the perturbation series in QFT or QM, one sometimes find it is non-Borel summable, thus the series is ambiguous. It was believed that the resolution to this phenomenon is the non-perturbative effects in QFT. It was found that the series is non-Borel summable is simply because we only consider the expansion around single saddle point. If we also consider the other saddle points and doing expansion around them, the combination of the expansions around different saddle points is Borel summable. The large order terms of the perturbation series become meaningless because of the non-perturbative effects. When we do perturbation expansions, we can not only consider the trivial vacuum background, but also need to consider the nontrivial saddle points background (instantons, bions, etc) because they are invisible in perturbation series ($O(e^{-1/g})$).

For example, considering all the saddle points, the semiclassical weak-coupling expansion of the partition function is given by[1],

$$Z(g^2) = \int D[\phi]\ e^{-\frac{1}{g^2}S[\phi]} \approx \sum_{saddles\ n} P_n(g^2)\ e^{-\frac{1}{g^2}S_n}, \tag{1.2}$$

$P_n(g^2)$ is the perturbative series (quantum fluctuations) around the saddle point n and S_n is the action of the confiugration of the saddle point n . n contains trivial vacuum ($n = 0$) background and non-perturbative sectors. The perturbative series $P_n(g^2)$ (for all saddles) are also in general divergent, and can be either Borel summable or non-summable, but it was believed if we can really find all the saddle points in our series and do this expansion correctly, this expansion is well defined (unique, ambiguity-free, finite) at weak coupling limit. Though the perturbative series for single saddle points may be ill-defined (non-Borel summable), but the combination of all the series is well-defined (Borel summable).

[1]We use the Wick rotated expression

doi:10.6342/NTU201700825

The idea of resurgence is that the perturbation series around different saddle points are related in a nontrivial way. This idea comes from quantum mechanics problem with degenerate vacua [6, 7, 8]. The ambigiuty of the perturbative series of ground state energy can be cured by the ambiguity of the instanton-anti-instanton configuration, which is known as the Bogomolny Zinn-Justim mechanism. This gives us a hint that there may be a general relation between different saddles. The expansion around trivial saddle points ecncodes the information around non-trivial saddle points. There are some evidence of this relation in quantum mechanics [9, 10, 11, 12] but it is still unclear whether the same relation holds in QFT [13, 14]. The final goal is that we can only consider the perturbative expansions around finite saddles then we can understand the perturbative expansions around all other saddles. We can capture all the information of the function by only considering finite saddle points. If we know this resurgence relation precisely, we can make a unique ambiguity-free definition for physical observables by only considering the perturbative expansion around only some saddle points. This may serve as a non-perturbative definition of QFT and also reveal the underlying structure of QFT.

This thesis reviews the idea of resurgence trans-series in quantum mechanics and shows that it may be applied to quantum field theory. Section 2 reviews some basic knowledge about divergent series, Borel resummation, renormalons and trans-series. Section 3 introduces the appplication of resurgence trans-series to quantum mechanics. Section 4 discusses new semiclassical configuration in compactified field theory and shows they may be related to the renormalon singularities. Section 5 is some comments about the construction of trans-series in quantum mechanics and the problems we would encounter in quantum field theory.

2 Perturbation series

When one wants to compute the expectation value of some operator A in quantum field theory. One writes down the path integral and do perturbative expansion around vacuum.

$$\langle A \rangle = \int D[\phi]\ A\ e^{\frac{i}{\hbar}\int d^4x L[\phi]} \cong \sum_{n=0}^{\infty} a_n g^n, \tag{2.1}$$

doi:10.6342/NTU201700825

where a_n is the coefficient of the perturbation series and g is the coupling constant. When we deal with this kind of expansions, we need to ask some questions.

1.Can the series be summed? Can we give this series a numerical value to define them?

2.What is the relation between the expansions and the function A?

3.Can we give the function A an exact definition by the series?

Since we are doing perturbative expansions, we know there must be non-perturbative parts ($O(e^{\frac{-1}{g}})$) in the function A that we can not see in this expansion. So the answer to the second question is the perturbative expansion is only part of A. With regard to the third question, we can say if we can find all the missing non-perturbative part of A, they we can combine all the parts to define A. For the first question, the answer is Yes. We will discuss this later.

This kind of series in QM and QFT are in general divergent. Actually, for $n \to \infty$, the perturbative series in QM takes the following asymptotic form [15].

$$a_n \sim C_n A^{-n} n^{b-1} n!, \tag{2.2}$$

n is the number of loops, the factor A depends on the action of the semiclassical solution. For different theories, the coefficients C_n, A, b have different value, but there is a $n!$ factor in almost all the cases.

Thus the series has zero radius of convergence. No matter how small g is, it diverges eventually. There is a strong physical reason for this phenomenon. Dyson argued that this kind of series can not converge. If this series is convergent for some small $|g|$ and gives us some value, then it can be analytically continued to negative g and describes the physics for small $g < 0$. But for $g < 0$, the situation is completely different. The theory become unstable and of course can not be described by the value we have obtained. This is known as the Dyson argument. The series diverge simply because it is not analytic at $g = 0$.

For an unstable potential, the vacuum is not stable and it will eventually decay by tunneling effect. So if the physcis of small positive g and small negative g are related by analytic continuation. There must be a deep connection between the tunneling effect and the zero radius of convergence.

doi:10.6342/NTU201700825

2.1 Borel resummation

There is one useful way to redefine the asymptotic divergent series and extract the information from the asymptotic divergent series, which is known as the *Borel-Écalle resummation.* First, we introduce the Borel transform. The original series is,

$$P(g) = \sum_{n=0}^{\infty} a_n g^n. \tag{2.3}$$

Assume $a_n \sim B^{-n} n!$ or other factorially diverge form. This series has zero radius of convergence. We define the Borel transform by,

$$B_P(g) \equiv \sum_{n=0}^{\infty} \frac{a_n}{n!} g^n. \tag{2.4}$$

The n-th term is divided by $n!$ and this series can be summed. Then this function now has a finite radius of convergence, it can be summed when $|g| < |B|$. So we can compute $B_P(g)$ at least for some value of g in principle. The function (2.4) is defined from (2.3) by the inverse Laplace transformation. So we can recover the original series by doing a Laplace transform,

$$P(g) = \int_0^\infty dt\ e^{-t}\ B_P(gt). \tag{2.5}$$

This procedure is called Borel resummation. You can also think we just write,

$$n! = \int_0^\infty dt\ e^{-t}\ t^n. \tag{2.6}$$

Then we still get the same result. The integral (2.5) is not always doable. It depends on the complex structure of the Borel transform $B_P(g)$. If $B_P(g)$ has no singularity on the real positive axis, then (2.5) is well-defined and we call $P(g)$ is Borel summable. This is the case when $P(g)$ is sign-alternating, $a_n \sim (-1)^n$.

When $P(g)$ is sign-nonalternating, there are singularities lying on the positive real axis of $B_P(g)$, see Fig. 1, the integral (2.5) is ambiguous. When there are singularities, we can slightly change the integration contour $C_\pm$ to just go above or below the singularities.

$$P_\pm(g) = \int_{C_\pm} dt\ e^{-t} B_P(gt). \tag{2.7}$$

This is known as the lateral Borel sum. Different choice of contour would give us different result, $P_\pm = \mathrm{Re}\, P(g) \pm i\, \mathrm{Im}\, P(g)$. We say this kind of series are non-Borel summable and ambiguous.

doi:10.6342/NTU201700825

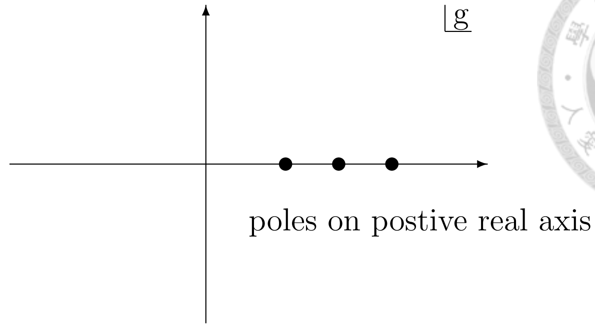


Figure 1: The poles of the function $B_P(g)$

Let's see some simple examples for Borel resummation. Consider two factorially divergent series,

$$\begin{aligned} P_1(g) &= \sum_{n=0}^{\infty}(-1)^n n! g^n, \\ P_2(g) &= \sum_{n=0}^{\infty} n! g^n. \end{aligned} \tag{2.8}$$

After Borel transformed, they become,

$$\begin{aligned} B_{P_1}(g) &= \sum_{n=0}^{\infty}(-1)^n g^n = \frac{1}{1+g}, \\ B_{P_2}(g) &= \sum_{n=0}^{\infty} g^n = \frac{1}{1-g}. \end{aligned} \tag{2.9}$$

Then,

$$\begin{aligned} P_1(g) &= \int_0^{\infty} dt\ e^{-t}\frac{1}{1+gt}, \\ P_2(g) &= \int_0^{\infty} dt\ e^{-t}\frac{1}{1-gt}. \end{aligned} \tag{2.10}$$

The integration of $P_1(g)$ can be done, but the integration of $P_2(g)$ is ambiguous. In order to avoid the singularities, we need to change the integration contour and the choice of contour leads to the ambiguity. We can choose to go above the pole or to go below the pole and different paths give us different imaginary part. The ambiguous imaginary part of $P_2(g)$ is $\pm\frac{i\pi}{g}e^{-1/g}$, see Appendix A.

The order of the ambiguity $\operatorname{Im} P(g)$ is $e^{-1/g}$, it cannot be resolved by the original perturbation series only. We say the perturbation series itself is ambiguous and ill-defined. We need more information which we have neglected when we did the perturbation theory.

doi:10.6342/NTU201700825

If one wants to compute the ground state energy in QM, one can do perturbative expansions around the vacuum. For unstable potential, the perturbation series around the vacuum is non-Borel summable, the imaginary part means the system is unstable and it can decay. For stable potential without degenerate vacua, the perturbation series is Borel summable and well-defined. But for stable potential with degenerate vacua, the perturbation series is still non-Borel summable. Why is there an ambiguity in stable potential? The resolution is non-perturbative effect (instanton effect). In potential with degenerate vacua, there are instanton effects which we did not consider.

In quantum mechanics, non-perturbative effect can cure the ambiguity. In double well potential quantum mechanics problem, the contribution from instanton-anti-instanton is also ambiguous, but if we combine this with the expansion around vanumm, we can get a ambiguity free result. The ambiguity of the instanton-anti-instanton would cancel the ambiguity of the perturbative series of vacumm[7, 16]. If we also consider the instanton effect, we can get a well-defined and ambiguous result. This is known as the BZJ (Bogomonly, Zinn Justin) prescription. We will see how does this work in section 3.

2.2 Source of divergence

The poles on the Borel plane and the divergence of the series are related. When we apply Borel resummation on QM and QFT, their are two kinds of poles lying on the Borel plane. The first kind of pole is the instanton pole, it appears in both QM and QFT. The second kind of pole is known as the "Renormalon" , it appears in QFT and is related to renormalization.

These two kinds of singularities have been thought to be very different. We have known a lot on the instanton pole, they appear because of the factorial growth of the number of Feynman diagrams. The number of n-loop Feynman diagrams grows as $n!$. This makes the coefficient of the series a_n also grows as $n!$. For n-th order contribution, even though each graph only gives us contribution of order 1, all the diagrams give us of order $n!g^n$. When $n \cong \frac{1}{g} \gg 1$ the multiloop graphs can be seen as soft fields which are semiclassical field configurations, in this case, instantons. Instansons are related to the factorial growth of the Feynman diagrams.

doi:10.6342/NTU201700825

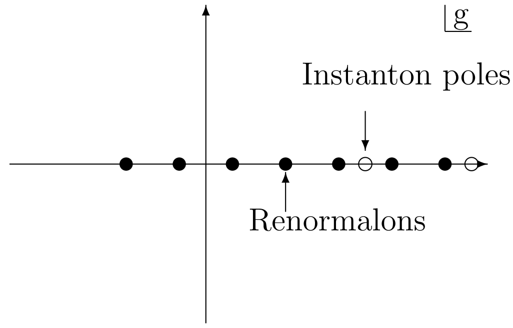


Figure 2: Renormalons and the instanton poles. Renormalons are usually closer to the origin.

This relation can also be seen by taking the ‘t Hooft’s limit,

$$Ng \quad \text{fixed}, \quad N \to \infty, \tag{2.11}$$

N is the number of colors and g is the coupling constant. At weak coupling limit, the instanton contribution is of order $e^{-\frac{8\pi}{g}}$. Under ‘t Hooft’s limit,

$$\frac{8\pi}{g} \sim N \to \infty, \tag{2.12}$$

the contribution of instanton is suppressed. On the other hand, it has been found under this limit, the number of n-loop Feynman diagrams do not grow as $n!$, they grow as C^n [17], where C is some numerical constant. It is a one-to-one correspondence between the instanton and the factorial growth of the diagrams. Because of the semiclassical origin of the instanton poles, we have already known how to deal with them. The instanton pole can be resolved by considering the expansion around the instanton background. We know the non-perturbative effect behind the instanton poles is just the instantons.

However, there is another kind of pole on the Borel plane, the “Renormalon”. This object appears in asymptotic free field theories, like QCD. Unlike instantons pole is related to the factorial growth of various graphs at higher loop, renormalon comes from a single graph with n loops which is factorially large, the so called ”bubble diagrams”. In field theory, there are two kinds of renormalons appear on the Borel plane, UV renormalons and IR renormalons. They are located on the different axis, if IR renormalons appear on the positive real axis, then UV renormalons appear on the negative real axis. For a good introduction for renormalons, see [15, 18]. There is a good example to see what is renormalon divergence.

doi:10.6342/NTU201700825

2.2.1 Renormalon divergence

Consider the correlation functions of two vector currents of massless quarks in QCD. The number of flavor is N_f and

$$\Pi_{\mu\nu}(q) = i\int d^4x e^{-iqx} \langle 0|\, T[j_\mu(x) j_\nu(0)]\, |0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2), \tag{2.13}$$

$$j_\mu = \bar{\psi}\gamma_\mu\psi, \tag{2.14}$$

where ψ is the quark field and $Q^2 = -q^2$. The Adler function is defined by,

$$D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}. \tag{2.15}$$

We are going to compute the contribution of the fermion bubble diagrams shown in Fig. 3 to the Adler function. The set of diagrams we will choose is gauge-invariant, we need to notice that it is not the only set of diagrams that resulting in renormalon. Of course at this order there are many other diagrams contribute to the Adler function, but we only focus on those diagrams which are enough to present the renormalon divergence.

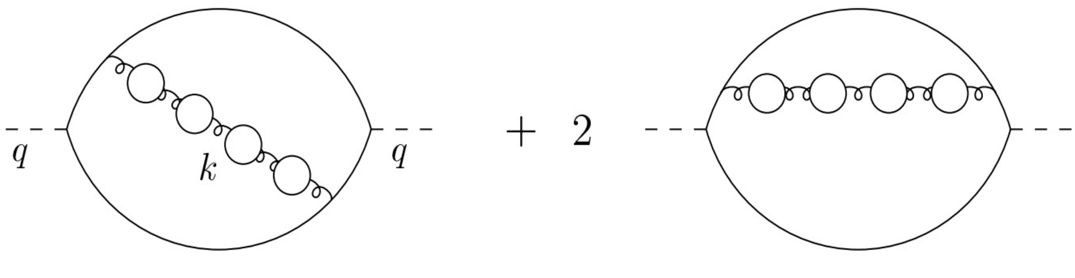


Figure 3: The fermion bubble diagrams we choose for the Adler function. The number of fermion loops inserted to the single gluon line can be infinity.

Since the coupling constant runs, we must choose a particular value of the coupling constant to do the expansion. The Adler function is labeled by the external momentum Q^2. It seems we should choose $\alpha_s(Q^2)$ as our parameter. However, this is a good choice only when the contribution of the loop diagrams comes mostly from $k \sim Q$. This is true as long as the order of the perturbation expansion n is small. However, when n become large, the naive estimate $k \sim Q$ is not valid.

doi:10.6342/NTU201700825

We can take a closer look at these graphs. Collecting all fermion bubble insertions in the gluon line [19], no bubble, 1 bubble, 2 bubbles and so on[2],

$$D = CQ^2 \int dk^2 \frac{k^2 \alpha_s(k^2)}{(k^2+Q^2)^3}, \tag{2.16}$$

where C is some overall constant and $\alpha_s(k^2)$ is the coupling constant. The coupling constant is defined by,

$$\alpha_s = \frac{g^2}{4\pi}. \tag{2.17}$$

The running coupling can be expressed by,

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 - \frac{\beta_0 \alpha_s(Q^2)}{4\pi} \ln(Q^2/k^2)}, \tag{2.18}$$

where $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$ is the first coefficient of the β-function. Focusing on the IR domain and omitting the constant, (2.16) becomes,

$$D(Q^2) \approx \frac{1}{Q^4}\alpha_s \sum_{n=0}^{\infty} (\frac{\beta_0 \alpha_s}{4\pi})^n \int dk^2\ k^2 (\ln \frac{Q^2}{k^2})^n, \quad \alpha_s = \alpha_s(Q^2), \tag{2.19}$$

which can be rewritten as,

$$D(Q^2) \approx \frac{\alpha_s}{2} \sum_{n=0}^{\infty} (\frac{\beta_0 \alpha_s}{8\pi})^n \int dy\ y^n e^{-y}, \quad y = 2\ln\frac{Q^2}{k^2}. \tag{2.20}$$

The y integration gives rise to $n!$. The characteristic value of y saturating this integral is,

$$y \sim n, \quad k^2 \sim Q^2 \exp\Big(-\frac{n}{2}\Big). \tag{2.21}$$

One can see that if Q^2 is fixed and n become large enough, a factorial divergence is created from the integral in the IR region. Naively we do not think $k^2 \ll Q^2$ would contribute to the Adler function, but this is not true. When we do the perturbative expansion to high orders, the momentum at IR contributes to the expansion and leads to the factorial divergence. This is the infrared renormalon.

Let's next focus on the UV region. When k^2 is large,

$$D(Q^2) \approx Q^2 \alpha_s \sum_{n=0}^{\infty} (\frac{\beta_0 \alpha_s}{4\pi})^n (-1)^n \int dk^2\ \frac{1}{k^4} (\ln \frac{k^2}{Q^2})^n. \tag{2.22}$$

[2] This expression is simplified, the exact result of fixed k^2 is in [20]

doi:10.6342/NTU201700825

Again, $\alpha_s = \alpha_s(Q)$ and β_0 is $\frac{11}{3}N_c - \frac{2}{3}N_f$. This can be rewritten as,

$$D(Q^2) \approx \alpha_s \sum_{n=0}^{\infty}(-1)^n(\frac{\beta_0\alpha_s}{4\pi})^n \int dy\ y^n e^{-y}, \quad y = \ln\frac{k^2}{Q^2}. \tag{2.23}$$

The characteristic value saturating the y integration is,

$$y \sim n, \quad k^2 \sim Q^2 \exp(n). \tag{2.24}$$

There is also a factorial divergence which comes from the integral of the UV region. This series is sign-alternating and it is Borel-summable. Unlike IR renormalons, the corresponding poles on the Borel plane is on the real negative axis. These are the UV renormalons. After Borel transform on the perturbative series,

$$\begin{aligned} B_D(\alpha_s) &\sim \frac{1}{1-\frac{\beta_0\alpha_s}{8\pi}},\ \text{IR},\\ B_D(\alpha_s) &\sim \frac{1}{1+\frac{\beta_0\alpha_s}{4\pi}},\ \text{UV}, \end{aligned} \tag{2.25}$$

and the location of the poles are,

$$\begin{aligned} \alpha_s &= \frac{8\pi}{\beta_0},\ \text{IR},\\ \alpha_s &= -\frac{4\pi}{\beta_0},\ \text{UV}. \end{aligned} \tag{2.26}$$

Note the location of the renormalon is closer than the instanton pole in general (The closest instanton-anti-instanton pole is at 4π in QCD). One can see the fermion bubble diagrams contain factorial divergence, this divergence comes from only one set of diagrams. Obviously, renormalon divergence are related to the IR and UV behavior of the theory. UV renormalons are located on the negative real axis thus it does not lead to ambiguity when we do Borel resummation. On the other hand, IR renormalons give us ambiguity. This is also reasonable since it is related to the momentum integration at IR region, at that scale, QCD is strong coupled.

There is another interesting phenomenon, the location of the UV renormalon on the Borel plane is closer to the origin than the IR renormalon. The leading large order behavior of the series is determined by the pole closest to the origin on the Borel plane. For example, consider a function which is a combination of two factorially divergent series,

$$R(g) = \sum_{n=0}^{\infty} g^n a^n n! + \sum_{m=0}^{\infty} g^m b^m m!, \tag{2.27}$$

doi:10.6342/NTU201700825

the Borel transform is,

$$B_R(g) = \frac{1}{1-ga} + \frac{1}{1-gb}, \tag{2.28}$$

if $a > b$, the series of a is the leading contribution to the large order behavior of $R(a,b)$. The position of the pole on the Borel plane is at $g = \frac{1}{a}$, which is closer to the origin than the other pole $g = \frac{1}{b}$. So the leading large order behavior in the previous case is determined by the UV renormalon.

Let's put our eye on (2.20) again. The integral of k^2 contains the momentum at IR region, we know this is meaningless when the theory is strong coupled. At small $k^2 \sim \Lambda^2$, we should cut off the integral. Λ is the value when,

$$\frac{\beta_0 \alpha_s(Q^2)}{4\pi} \ln \frac{Q^2}{\Lambda^2} = 1. \tag{2.29}$$

So the integral becomes,

$$D(Q^2) \approx \frac{1}{Q^4}\alpha_s \sum_{n=0}^{\infty} (\frac{\beta_0\alpha_s}{4\pi})^n \int_{\Lambda^2}^{\infty} dk^2\ k^2 (\ln\frac{Q^2}{k^2})^n. \tag{2.30}$$

Changing the integration variable to y,

$$D(Q^2) \approx \frac{\alpha_s}{2} \sum_{n=0}^{\infty} (\frac{\beta_0\alpha_s}{8\pi})^n \int^{n_*} dy\ y^n e^{-y}, \quad n_* = 2\ln\frac{Q^2}{\Lambda^2}. \tag{2.31}$$

So the integral over y is cut off, it do not give us factorial divergence anymore. The factorial growth is suppressed when $n > n_*$,

$$D(Q^2) \sim \frac{\alpha_s}{2} \sum_{n=0}^{n_*} (\frac{\beta_0\alpha_s}{8\pi})^n n!, \tag{2.32}$$

the series is truncated. Notice that at $n = n_*$, the asymptotic series reaches its highest accuracy. Truncating at $n = n_*$ create an error of order $e^{-\frac{8\pi}{\beta_0\alpha_s}} \sim \frac{\Lambda^4}{Q^4}$. The same order to the ambiguity of the Borel lateral sum and is also a non-perturbative effect.

Renormalon is obviously related to the running of the coupling constant. It is generated from single set of diagrams, and appear when the momentum is at the scale of UV or IR, unlike instanton poles come from the growth of the number of the various diagrams at high loop. There has been no known semiclassical configuration corresponding to renormalon (the location of the poles are so close to origin that we must find instanton-like configurations with less action than the instantons). But in

doi:10.6342/NTU201700825

recent years, it has been discovered that by compactifying the theory, there are new semiclassical configurations appearing in the theory [13, 14]. The ambiguity of the renormalon may be recovered by those new configurations, this may be a resolution to the renormalon puzzle.

2.3 Trans-series approach

In quantum mechanics, the cancellation between perturbative part and non-perturbative part gives us some hint about perturbation series. We need to consider all possible hidden non-perturbative effects when we do expansion. This is called trans-series expansion. The trans-series expansion for quantum mechanics is given by,

$$f(g) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} c_{n,k,l}\; g^n[\exp\left(-\frac{S}{g}\right)]^k[\ln\left(\pm\frac{1}{g}\right)]^l. \tag{2.33}$$

The term $\exp\left\{-\frac{S}{g}\right\}$ comes frrom the contribution of (anti)instantons and the term $\ln\left(\pm\frac{1}{g}\right)$ comes from the quasi zero modes (instanton-anti-instanton pairs) integration. $c_{n,k,l}$ is the coefficient, k is the number of instanons or anti-instantons, l is the number of quasi zero modes and $+/-$ states for the interaction of the quasi zero mode is repulsive/attractive.

There are some properties for the trans-series,

1.It encodes all the information for the function $f(g)$, perturbative or non-perturbative

2.It is well-defined under analytic continuation.

3.The coefficients $c_{n,k,l}$ are correlated in a non-trivial way.

The trans-series unifies the perturbative and non-perturbative sectors. It was believed that the trans-series contains all possible information of the function in an encoding form. If we want to express some unknown function by a series expansion, we should use trans-series expansion because it does not lose any information

There is a simple example to get the idea of trans-series expansion [13]. Consider a 0-dimensional partition function $Z_1 = \mathrm{tr}\, e^{-V_1}$ with potential $V_1(x) = \frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}x)$. This integral is related to the modified Bessel function of the second kind K_0,

$$Z_1(\lambda) = \int_{\infty}^{\infty} dx\; e^{-\frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}x)} = \frac{1}{\sqrt{\lambda}}e^{\frac{1}{4\lambda}}K_0(\frac{1}{4\lambda}). \tag{2.34}$$

doi:10.6342/NTU201700825

The asymptotic perturbative expansion for $K_0(\frac{1}{4\lambda})$ gives,

$$Z_1(\lambda) \sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^n (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\Gamma(\frac{1}{2})^2}, \quad \lambda \to 0^+. \tag{2.35}$$

This is a factorially divergent series, the coefficients are sign alternating and it is Borel summable. On the other hand, if we change $\lambda \to -\lambda$, the potential becomes to $V_2(x) = \frac{1}{2\lambda}\sin^2(\sqrt{\lambda}x)$. The partition function $Z_2 = \mathrm{tr}\, e^{-V_2}$ is related to the modified Bessel function of the first kind I_0.

$$Z_2(\lambda) = \int_0^{\pi/\sqrt{\lambda}} dx\ e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda}x)} = \frac{\pi}{\sqrt{\lambda}} e^{-\frac{1}{4\lambda}} I_0(\frac{1}{4\lambda}) \tag{2.36}$$

$$\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\Gamma(\frac{1}{2})^2}, \quad \lambda \to 0^+. \tag{2.37}$$

This is another factorially divergent series, the coefficients are not sign alternating and it is not Borel summable. This gives us a mystery, it seems we can conclude that $Z_1(-\lambda) = Z_2(\lambda)$ by the perturbative expansion, but the perturbative series for $Z_2(\lambda)$ is not Borel summable and this makes a contradiction. The periodic potential $V_2(x)$ is stable and the partition function should be real, but if we define $Z_2(\lambda)$ by its perturbative expansion, we have an ambiguous imaginary part in $Z_2(\lambda)$. We can not define $Z_2(\lambda)$ by its asymptotic perturbative expansion, it does not contain all the information of the function, there are missing non-perturbative parts inside. There are several ways to recover this non-perturbative part, one can find the instantons inside and compute their amplitude, or one can use analytic continuation to change $\lambda \to \lambda e^{\pm i\pi}$ and the choice of the direction gives us another ambiguity to cancel the one in the asymptotic expansion, or one can write down the integral representation of $I_0(z)$ and do the correct expansion for it. The main point is that the perturbative expansion does not define the function. The perturbative series of $Z_1(\lambda)$ can be summed, it can be expressed by the hypergeometric function,

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2\lambda} \int_0^{\infty} dt\ e^{-\frac{t}{2\lambda}}\ {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; -t). \tag{2.38}$$

The hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; -t)$ has a cut $(-\infty, -1)$ along the negative t axis. This integral is well defined for $\lambda > 0$, whereas for $\lambda < 0$, the integration contour meets the branch cut. We need to define the integral for $\lambda < 0$ by analytic continuation from $Z_1(\lambda)$. Rotating the phase of λ by an angle θ, $\lambda \to \lambda e^{i\theta}$, when

doi:10.6342/NTU201700825

θ approaches to $\pm\pi$, the sign alternation series for $Z_1(\lambda)$ become to the sign non-alternating series for $Z_2(\lambda)$. However, the choice of the direction of rotation is important, the different choice of the phase creates an ambiguous imaginary part. The difference can be obtained by,

$$Z_1(e^{i\pi}\lambda) - Z_1(e^{-i\pi}\lambda) = \sqrt{\frac{\pi}{2}}\frac{1}{2\lambda}\int_1^\infty dt\ e^{-\frac{t}{2\lambda}}[{}_2F_1(\frac{1}{2},\frac{1}{2},1,t-i\epsilon) - {}_2F_1(\frac{1}{2},\frac{1}{2},1,t+i\epsilon)] \tag{2.39}$$

$$= -2i\sqrt{\frac{\pi}{2}}\frac{1}{2\lambda}\int_0^\infty dt\ e^{-\frac{t}{2\lambda}}{}_2F_1(\frac{1}{2},\frac{1}{2},1,-t) \tag{2.40}$$

$$= -2ie^{-\frac{1}{2\lambda}}Z_1(\lambda). \tag{2.41}$$

Thus, combine these results, we can conclude that,

$$Z_1(e^{\pm i\pi}\lambda) = Z_2(\lambda) \mp ie^{-\frac{1}{2\lambda}}Z_1(\lambda). \tag{2.42}$$

The correct expression for $Z_2(\lambda)$ is,

$$Z_2(\lambda) = Z_1(e^{\pm i\pi}\lambda) \pm ie^{-\frac{1}{2\lambda}}Z_1(\lambda). \tag{2.43}$$

This $Z_2(\lambda)$ is a real function, and if we do the perturbative expansions for it, the expansion of $Z_1(e^{\pm i\pi}\lambda)$ parts gives an imaginary part which cancels another imaginary part.

$$Z_2(\lambda) = \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(2\lambda)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\Gamma(\frac{1}{2})^2} \pm ie^{-\frac{1}{2\lambda}}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(2\lambda)^n(-1)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\Gamma(\frac{1}{2})^2}. \tag{2.44}$$

This is a two-term trans-series expansion for the function $Z_2(\lambda)$. We can observe some features in this simple case. First, the asymptotic perturbative expansion for the function does not define the function, we need to do the trans-series expansion. Second, the coefficients in the expansions are related, in this case, they differ simply by a factor $(-1)^n$.

There is an important phenomenon which gives us a hint of the cancellation of the imaginary part from different saddle points. In QM, it has been observed that for a theory which has instanton solution with the action of order $e^{\frac{-S_I}{g}}$. The large order behavior of the coefficient of the observable we want to compute will be $c_n \sim \frac{n!}{(2S_I)^n}$, this asymptotic behavior would result in a pole located on the Borel plane at $g = 2S_I$. This creates an ambiguity of the Borel resummation $\pm i\frac{2S_I\pi}{g}\exp\left[-\frac{2S_I}{g}\right]$. This is of

doi:10.6342/NTU201700825

the same order as the 2 instanton contribution. The imaginary part in the two instanton contribution would cancel the imaginary part in the Borel resummation of the trival vacuum. While it was believed that the resurgence framework can be done in QM, there is no enough evidence to show that this also works in QFT because of the existence of the renormalons.

3 Resurgence in QM

In quantum mechanics, the ground state energy can be either computed by Schrödinger equation or the path integral approach. We can solve the Schrödinger equation and find the ground state energy exactly. We can also use path integral approach. In the path integral approach, we use the thermal partition funcion. The partition function is,

$$Z(\beta) = \operatorname{tr} e^{-\beta H}. \tag{3.1}$$

The ground state energy is,

$$E_0 = -\lim_{\beta \to \infty} \frac{1}{\beta} \log Z(\beta). \tag{3.2}$$

The partition can be expressed in the path integral formulation,

$$Z(\beta) = \int D[q(t)] e^{-S_E(q)}, \tag{3.3}$$

$S_E(q)$ is the Euclidean action for the theory, and the path integral is performed under the periodic boundary condition.

$$S_E(q) = \int_{\beta/2}^{\beta/2} dt\ L_E(t) = \int_{\beta/2}^{\beta/2} dt\ [\frac{1}{2}p^2 - V(q)], \tag{3.4}$$

$$q(\frac{-\beta}{2}) = q(\frac{\beta}{2}), \tag{3.5}$$

where $p = \frac{\partial}{\partial q}$. Note the Euclidean Lagrangian is just the Lagrangian with inverse potential.

The path integral itself is exact and well defined. So if we can compute the path integral exactly, we can find the ground state energy and compare the result to the result computed by the Schrödinger equation. They should be exactly the same.

doi:10.6342/NTU201700825

But in most cases, we can only use perturbation expansion to compute the path integral, non-perturbative information is lost. In order to construct the trans-series, we need to find the non-perturbative saddles in the action and doing path integral over them. We need to find the instanton contribution to the partition function. The ground state energy should be expressed in this form.

$$E_0(g) = \sum_k \sum_l E_0^{(k,l)}(g), \tag{3.6}$$

$E_0^{(k,l)}$ should be realized as a power series in g, which is the expansion around the saddle (k, l). k is the number of instanton and l is the number of quasi zero modes. This expression also depends on the specific potential. For example, if the potential is unstable, then we know there must be an imaginary part in the energy which represents the decay time. Thus the expression of the energy is not Borel summable. If the potential is stable, then the energy should be real and the expression must be Borel summable. For the stable potential which contains (anti)instanton, for single perturbation series $E_0^{(k,l)}(g)$ it is non-Borel summable, but the combination $E_0(g)$ of those series is Borel summable.

Potential	Energy of single saddle $E_0^{(k,l)}(g)$	Energy $E_0(g)$
Unstable	Non-Borel summable	Non-Borel summable
Stable	Borel summable	Borel summable
Stable with instanton	Non-Borel summable	Borel summable

3.1 Quantum mechanics in the anharmonic oscillator

We first consider the stable potential with no instanton exist. The Hamiltonian is,

$$H = \frac{1}{2}p^2 + V(q), \quad V(q) = \frac{1}{2}\omega^2 q^2 + gq^4. \tag{3.7}$$

We want to compute the ground state energy of this model. Using the standard perturbation theory,

$$E_0 = \frac{\omega}{2}(1 + c_1 g + c_2 g^2 + O(g^3)). \tag{3.8}$$

The asymptotic behavior of the coefficients is [5],

$$c_n \sim (-1)^n B^{-n} n!, \quad n \gg 1, \tag{3.9}$$

doi:10.6342/NTU201700825

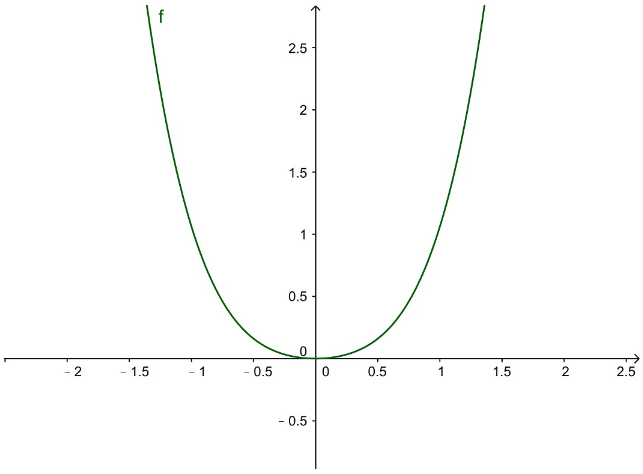


Figure 4: The shape of the anharmonic potential $V(q)$

where $B = \frac{1}{3}\omega^3$ in this potential, we also subtracted the constant C_n in (2.2). The Borel transform of the series is,

$$B_{E_0}(g) = \frac{\omega}{2}\sum_{n=0}^{\infty}\frac{c_n}{n!}g^n = \frac{\omega}{2}\frac{B}{B+g}. \tag{3.10}$$

So the Borel resummation of the ground state energy is,

$$E_0 = \frac{\omega}{2}\int_0^{\infty} dt\ e^{-t}\frac{B}{B+tg}, \tag{3.11}$$

this integral is well defined since B_{E_0} has no pole on the real positive axis. The ground state energy has no ambiguous imaginary part. This is what we expected, the potential is stable and the ground state energy is well defined.

Now let's change the sign of g in the potential, $g \to -g$. The perturbation series

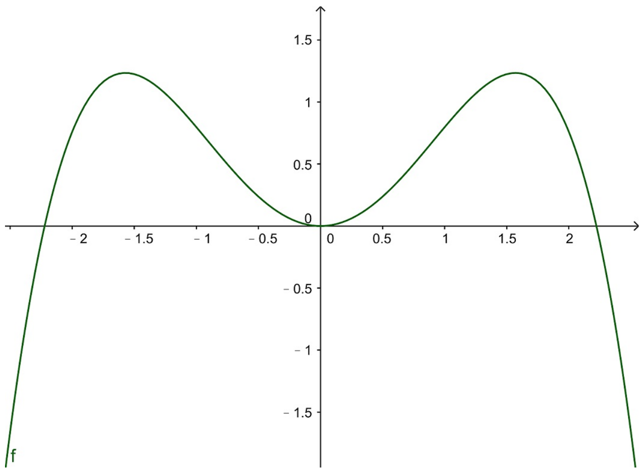


Figure 5: The shape of the potential $V(q)$ with the sign change $g \to -g$

doi:10.6342/NTU201700825

now is not sign-alternating.

$$c_n \sim B^{-n} n!, \quad n \gg 1. \tag{3.12}$$

It means the Borel transform of the ground state energy has a pole on the positive real axis.

$$B_{E_0}(g) = \frac{\omega}{2}\frac{B}{B-g}. \tag{3.13}$$

The Borel resummation is not well defined. In order to avoid the pole, we need to do the lateral Borel resummation. The integral has an ambiguity, so the ground energy has an imaginary part.

$$\operatorname{Im} E_0 = \pm\pi\frac{\omega}{2}\frac{B}{g}\exp\left\{-\frac{B}{g}\right\}. \tag{3.14}$$

This is also reasonable because the potential after we changed the sign of g is unstable. The imaginary part of the energy represents the decay rate.

We can see that whether the series is summable or non-summable depends on the sign of the coupling constant. We also know that the perturbation series for stable potential should be summable, for unstable potnetial should be non-summable. But for the potential for which instantons exist, though the potential is stable, the expansion around trival vacuum is still non-summable. We need to take instantons into account and it would cancel the imaginary part of the energy.

3.2 Quantum mechanics in the double well potential

The following computation of this secion is based on [3, 21]. Consider a quantum mechanics system with double well potential. The Hamiltonian is given by

$$H = \frac{1}{2}p^2 + V(q), \quad V(q) = \frac{q^2}{2}(1-q\sqrt{g})^2. \tag{3.15}$$

The Euclidean action is,

$$S_E(q) = \int_{-\beta/2}^{\beta/2} dt\ [\frac{1}{2}\dot{q}(t)^2 - V(q(t))]. \tag{3.16}$$

In perturbation theory there are two degenerate ground states sharing the same energy at,

$$q = 0, \quad q = \frac{1}{\sqrt{g}}. \tag{3.17}$$

doi:10.6342/NTU201700825

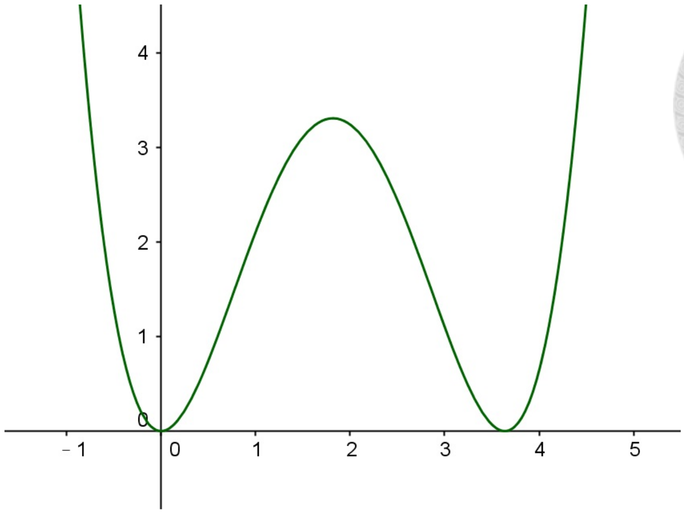
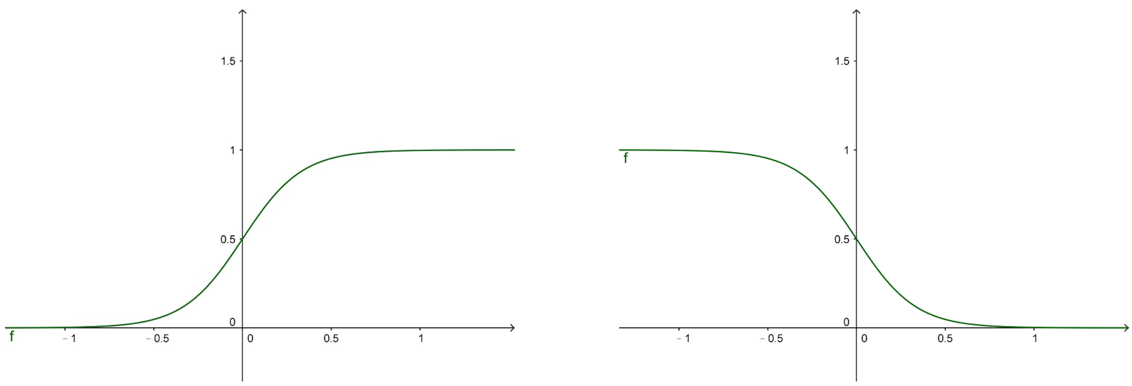


Figure 6: The shape of the double well potential $V(q)$

In the large β limit, there are instanton and anti-instanton solutions connecting these two vacua. The solutions are,

$$q_c(t) = \frac{1}{\sqrt{g}} \frac{1}{1 + e^{\pm(t-t_0)}}, \tag{3.18}$$



(a) instanton

(b) anti-instanton

Figure 7: An instanton configuration and an anti-instanton configuration for $g = 1$, both at $t_0 = 0$.

t_0 is the location of the (anti)instanton, sometimes it is called the collective coordinate, modulus or the zero modes. These trajectories are the classical solution of the equation of motion. They satisfy the energy conservation constraint.

$$\frac{1}{2}\dot{q}^2 + V(q) = E(\beta). \tag{3.19}$$

For $\beta \to \infty$, $E(\beta) \to 0$. For more details of instanton calculation in Quantum mechanics, it can be found in [3, 7]. What we want to compute is the multi-instanton

doi:10.6342/NTU201700825

contribution to the partition function. But it is better to consider the one instanton contribution in the double well first.

3.2.1 One instanton contribution

It seems the one (anti)instanton configuration has no contribution to the ground state energy at the first sight because the path integral of the ground state energy is over all the periodic trajectories and the (anti)instanton trajectory dose not satisfy. Actually this is not true. We will see how instanton effect split the two degenerate states in the double well potential.

In the double well potential with the Hamiltonian (3.15), the Hamiltonian is invariant under the exchange

$$q \leftrightarrow \frac{1}{\sqrt{g}} - q. \tag{3.20}$$

So the Hamiltonian commutes with the parity operator P which exchanges q and $\frac{1}{\sqrt{g}} - q$. The eigenfunctions of H are labeled by the qunatum numbers, ϵ and N, where ϵ stands for the parity and N is the N-th energy level.

$$H\psi_{\epsilon,N}(q) = E_{\epsilon,N}\psi_{\epsilon,N}(q), \quad P\psi_{\epsilon,N}(q) = \epsilon\psi_{\epsilon,N}(q). \tag{3.21}$$

Where $\epsilon = \pm 1$ and N goes from 0 to ∞. In perturbation theory, the energy for different parity is equal. $E_{1,0} = E_{-1,0}$.

$$E_{\epsilon,N} = N + \frac{1}{2} + O(g). \tag{3.22}$$

But the instanton effect breaks this degeneracy. In order to see this effect, we consider the "twisted" partition function which is defined by,

$$Z_t(\beta) = \mathrm{Tr}\big(Pe^{-\beta H}\big), \tag{3.23}$$

P is the parity operator. At the large β and small g limit, (3.23) become,

$$\begin{aligned} Z_t(\beta) &\approx e^{-\beta E_+} - e^{-\beta E_-} \approx -2\sinh[\frac{\beta(E_+ - E_-)}{2}]e^{-\beta(E_+ + E_-)/2} \\ &\beta \to \infty, \ g \to 0. \end{aligned} \tag{3.24}$$

Where E_+ and E_- are $E_{-1,0}$ and $E_{1,0}$. They are the ground state energy with different parity. Since $E_+ - E_-$ is 0 in perturbation theory, $E_+ - E_-$ is small, the r.h.s can be rewritten by,

$$Z_t(\beta) \approx -\beta(E_+ - E_-)e^{\frac{-\beta}{2}}[1 + O(g, e^{-\beta})]. \tag{3.25}$$

doi:10.6342/NTU201700825

$Z_t(\beta)$ can be written in path integral formulation with twisted boundary condition,

$$Z_t(\beta) = \int_{q(\beta/2)=P(q(-\beta/2))} D[q(t)] \exp[-S(q(t))]. \tag{3.26}$$

With,

$$S(q) = \int_{-\beta/2}^{\beta/2} [\frac{1}{2}\dot{q}^2(t) + \frac{1}{2}q^2(t)(1-\sqrt{g}q(t))^2]dt. \tag{3.27}$$

The boundary condition is,

$$q(\beta/2) + q(-\beta/2) = \frac{1}{\sqrt{g}}. \tag{3.28}$$

The path integral formulation of $\mathrm{Tr}\big(e^{-\beta H}\big)$ is dominated by the two trivial saddle points,

$$q(t) = 0, \quad q(t) = \frac{1}{\sqrt{g}}. \tag{3.29}$$

These are the location of the two degenerate ground states. This gives us the usual perturbative expansion. However, these do not contirbute to the twisted path integral (3.26) because they do not satisfy the boundary condition. We need to find the solutions of the equation of motion which satisfy the boundary condition (3.28). In the infinite β limit, the solutions are the (anti)instanton solutions (3.18). The path integral around these non-trivial saddle points leads to the split of the ground state energy.

Let's expand the action around $q_c(t)$,

$$q(t) = q_c(t) + q_f(t), \quad q_c(t) = \frac{1}{\sqrt{g}}\frac{1}{1+e^{\pm(t-t_0)}}, \tag{3.30}$$

$q_f(t)$ is the fluctuations around the classical path. In quadratic order of $q_f(t)$

$$S(q) \approx S(q_c) + \frac{1}{2}\int dt_1 dt_2\ q_f(t_1)M(t_1,t_2)q_f(t_2), \tag{3.31}$$

the action of instanton is,

$$S(q_c) = \frac{1}{6g}, \tag{3.32}$$

the operator M is given by,

$$M(t_1,t_2) = \frac{\delta^2 S}{\delta q_c(t_1)\delta q_c(t_2)} = [-(\frac{d}{dt_1})^2 + V''(q_c(t_1))]\delta(t_1-t_2). \tag{3.33}$$

doi:10.6342/NTU201700825

The path integral around this path become,

$$\int D[q(t)]e^{-S(q)} \approx e^{-S(q_c)} \int D[q_f(t)] \exp\left[-\frac{1}{2}\int dt_1 dt_2 q_f(t_1) M(t_1, t_2) q_f(t_2)\right]. \tag{3.34}$$

The boundary condition of $q_f(t)$ depends on the boundary condition of $q(t)$, since

$$q_c(\beta/2) + q_c(-\beta/2) + q_f(\beta/2) + q_f(-\beta/2) = \frac{1}{\sqrt{g}}, \tag{3.35}$$

and

$$q_c(\beta/2) + q_c(-\beta/2) = \frac{1}{\sqrt{g}}, \tag{3.36}$$

The boundary condition is,

$$q_f(\beta/2) = -q_f(-\beta/2). \tag{3.37}$$

So we need to integrate over all anti-periodic trajectories. The Gaussian integration for the bosonic modes gives,

$$\int D[q_f(t)] \exp\left[-\frac{1}{2}\int dt_1 dt_2 q_f(t_1) M(t_1, t_2) q_f(t_2)\right] = (\det M)^{-\frac{1}{2}}. \tag{3.38}$$

Thus, if we want to find the instanton contribution for the path integral at the leading order, what we need to do is to find the classical path and doing expansion around it, find the operator M and finally calculate the determinant of the operator M. The twisted partition function $Z_t(\beta)$ now can be expressed as,

$$Z_t(\beta) \approx \mathcal{N} e^{-S(q_c)} (\det M)^{-\frac{1}{2}}, \tag{3.39}$$

$\mathcal{N}$ is an overall normalization which is independent of the potential, it can be easily eliminated by divided to other reference partition function. However, the determinant of M may have zero modes and negative modes, we need to be careful when we deal with them. The detail computation of $\det M$ is explained in the appendix. $V''(q)$ is,

$$V''(q) = 6gq_c(t)^2 - 6\sqrt{g}q_c(t) + \frac{1}{2}. \tag{3.40}$$

Substituting the value of $q_c(t)$, the operator M is given by,

$$M(t_1, t_2) = [-(\frac{d}{dt_1})^2 + 1 - \frac{3}{2\cosh^2(t_1/2)}]\delta(t_1 - t_2). \tag{3.41}$$

doi:10.6342/NTU201700825

The operator M in this case has one zero mode and has no negative modes. The zero mode comes from the freedom of the position of instanton. Because of the zero modes, we need to remove it in the calculation of determinant. The integration over the zero mode gives us a factor $\frac{\beta S_c^{1/2}}{\sqrt{2\pi}}$. The partition function become,

$$Z_t(\beta) \approx \mathcal{N} e^{-S_c} \frac{\beta S_c^{1/2}}{\sqrt{2\pi}} (\det{}' M)^{-\frac{1}{2}}, \tag{3.42}$$

$\det' M$ is the determinant after removing the zero mode. S_c is $S(q_c)$ which is $\frac{1}{6g}$.

Let's choose the reference partition function Z_G to be the harmonic oscillator with $\omega = 1$, for large β limit, we have

$$Z_G(\beta) = \mathcal{N}(\det M_0)^{-\frac{1}{2}} \approx e^{-\beta/2}, \tag{3.43}$$

where

$$M_0(t_1, t_2) = [-(\frac{d}{dt_1})^2 + 1]\delta(t_1 - t_2). \tag{3.44}$$

After dividing by a good reference partition function,

$$Z_t(\beta) = 2Z_G(\beta) e^{-S_c} \frac{\beta S_c^{1/2}}{\sqrt{2\pi}} (\frac{\det' M}{\det M_0})^{-\frac{1}{2}}, \tag{3.45}$$

the additional 2 factor comes from instanton and anti-instanton, they should give us the same contribution. Now we only need to compute the fraction of the functional determinant. There are several methods to compute this, see section 2.5 and 2.6 of [21] or [22, 23]. The result is,

$$[\frac{\det' M}{\det M_0}] = \frac{1}{12}. \tag{3.46}$$

Finally, we find the leading order contribution of the instanton to the twisted partition function.

$$\mathrm{Tr}\big(Pe^{-\beta H}\big) = Z_t(\beta) \approx \frac{2}{\sqrt{\pi g}} \beta e^{-\beta/2} e^{-\frac{1}{6g}} (1 + O(g)). \tag{3.47}$$

The energy difference is,

$$E_+ - E_- = -\frac{2}{\sqrt{\pi g}} e^{-\frac{1}{6g}} (1 + O(g)). \tag{3.48}$$

The ground state energy degeneracy is splitted by instanton effect,

$$E_{\epsilon,0} = \frac{1}{2} + O(g) - \frac{\epsilon}{\sqrt{\pi g}} e^{-1/6g} (1 + O(g)), \tag{3.49}$$

ϵ is the eigenvalue of parity, 1 means parity even and -1 means parity odd.

doi:10.6342/NTU201700825

3.2.2 Multi instanton contribution

Let's come back to our main subject, the resurgent structure of the double well. We have learned that the ground state energy should be represented by the trans-series. Consider the ground state energy expansion around the trival vacuum.

$$E_0 = \sum_{n=0}^{\infty} c_n g^n, \tag{3.50}$$

where the asymptotic behavior is[3]

$$c_n \sim -3^n n!, \quad n \gg 1. \tag{3.51}$$

The Borel transform of this series is,

$$B_{E_0}(g) = \frac{-1}{1-3g}. \tag{3.52}$$

So the resulting singularity on the Borel plane is at $g = \frac{1}{3}$, the ambiguity of this pole is,

$$\operatorname{Im} E_0 = \mp\pi \frac{1}{3g} \exp\left\{-\frac{1}{3g}\right\}. \tag{3.53}$$

This ambiguity should be canceld by other saddles. Recall that the action of single instanton in the double well is $S_c = \frac{1}{6g}$. The order of the ambiguity gives us a hint that this is a two instanton configuration effect and actually, it is.

We first consider the instanton-anti-instanton pair configuration. Since we are considering the ground state energy, we need to integrate over all the periodic paths, so the two instanton or the two anti-instanton configuration do not contribute. In fact, an instanton-anti-instanton pair (IA) pair is not a classical path. It is just a approximate saddle point. Only when the distance between the instanton and anti-instanton is very large, this configuration can be seen as a saddle point (dilute instanton approximation). We are looking for a configuration which is a sum of the instanton and anti-instanton but separated by a distance θ. It is given by,

$$q_c^{\theta}(t) = \frac{1}{\sqrt{g}}\left(\frac{1}{1+e^{t-\theta/2}} + \frac{1}{1+e^{-t-\theta/2}} - 1\right), \tag{3.54}$$

[3]We have suppressed the constant C_n, which is $\frac{3}{\pi}$

doi:10.6342/NTU201700825

we want to compute the action of this path. It is convenient to introduce the following notation,

$$
\begin{aligned}
q_+^{\theta/2}(t) &= \frac{1}{\sqrt{g}}\frac{1}{1+e^{-t-\theta/2}},\\
q_-^{\theta/2}(t) &= \frac{1}{\sqrt{g}}\frac{1}{1+e^{t-\theta/2}},\\
u(t) &= q_-^{\theta/2}(t),\\
v(t) &= u(t+\theta) = q_-^{-\theta/2}(t),
\end{aligned} \tag{3.55}
$$

the path can be rewritten by,

$$
q_c^{\theta}(t) = q_-^{\theta/2}(t) + q_+^{\theta/2}(t) - 1 = q_-^{\theta/2}(t) - q_-^{-\theta/2} = u(t) - v(t). \tag{3.56}
$$

The action of this path is,

$$
\begin{aligned}
S(q_c^{\theta}) &= \int dt\ (\frac{1}{2}\dot{q_c}^2 + V(q_c))\\
&= \int dt\ [(\frac{\dot{u}}{2} + V(u)) + (\frac{\dot{v}}{2} + V(v)) - \dot{u}\dot{v} + V(u-v) - V(u) - V(v)]\\
&= \frac{1}{3g} + \int dt\ [-\dot{u}\dot{v} + V(u-v) - V(u) - V(v)],
\end{aligned} \tag{3.57}
$$

the first tem is just the leading contribution of two instantons. Since the path is an even function, we can change the integral to twice the integral from 0 to ∞. After integration by part,

$$
S(q_c^{\theta}) = \frac{1}{3g} + 2v(0)\dot{u}(0) + 2\int_0^{\infty} dt\ [v\ddot{u} + V(u-v) - V(u) - V(v)], \tag{3.58}
$$

then we can expand the integral in powers of v^2, we stop at v^2 because we only want to compute the leading term of θ. We also use the equation of motion of u. We find,

$$
S(q_c^{\theta}) = \frac{1}{3g} + 2v(0)\dot{u}(0) + 2\int dt\ [\frac{v^2V''(u)}{2} - \frac{v^2V''(0)}{2}], \tag{3.59}
$$

the function v decay from origin very fast, so the integral saturated when t is around zero, where $u = 1+O(e^{\theta/2})$ there. Because $V''(u) \sim V''(1) = V''(0)$, the terms inside the integral cancel. And,

$$
v(0)\dot{u}(0) \sim -\frac{e^{-\theta}}{g}. \tag{3.60}
$$

The action at leading θ contribution is,

$$
S(q_c^{\theta}) = \frac{1}{g}[\frac{1}{3} - 2e^{-\theta} + O(e^{-2\theta})]. \tag{3.61}
$$

doi:10.6342/NTU201700825

Now we can consider n instanton configuration separated by distance θ_i with the constraint,

$$\sum_{i=1}^{n} \theta_i = \beta. \tag{3.62}$$

We only need to keep the interaction between the nearest neighbour instantons at the leading order. The action of n-instanton configuration is just a sum of two instanton configuration.

$$S(\theta_i) = \frac{1}{g}[\frac{n}{6} - 2\sum_{i=1}^{n} e^{-\theta_i} + O(e^{-\theta_i - \theta_j})]. \tag{3.63}$$

We want to compute the n-instanton contribution to the partition, so we need to calculate the quantum fluctuation around this action. Although this is not a classical path of the equation of motion, at the large θ_i limit, it can be seen as a approximate classical path. Expand the fluctuation to second order, we need to compute the determinant of the operator M. At large θ_i limit, the spectrum of the operator M can be seen as the same spectrum of the operator at the single instanton problem but n times degenerate. The determinant of M is just the $n = 1$ case but with power n. Thus, the n instanton contribution to the partition function is,

$$Z_{\epsilon}^{(n)}(\beta) = e^{-\frac{\beta}{2}} \frac{\beta}{n} (\epsilon \frac{e^{-1/6g}}{\sqrt{\pi g}})^n \int_{\theta_i \geq 0} \delta(\sum_{i=1}^{n} \theta_i - \beta) \prod_i d\theta_i \exp\left[\frac{2}{g} \sum_{i=1}^{n} e^{-\theta_i}\right]. \tag{3.64}$$

The $e^{-\beta/2}$ is the ground state energy of harmonic oscillator, overall β comes from the global time translation, the factor $1/n$ is because the configuration is invariant under a cylic permutation. The integral over θ_i is because we need to include all possible value of θ_i. Note for n odd, the instanton contributes to odd parity $Z_{-1}^{(n)}$ and for n even , the instanton contributes to even parity $Z_1^{(n)}$.

Now we need to do the integral of θ_i. However, the interatcion between the instantons is attractive when g positive. For $g \to 0^+$, the integral is saturated when the distance θ_i between instantons are small. When instanton and anti-instanton are close to each other, we can not distinguish that it is an IA pair or just the fluctuations around vacuum. This breaks our assumption of dilute instanton configuration. We need to do regularization to this integral, we can first do the integral for $g < 0$, then we do analytic continuation to $g > 0$. For negative g, the interaction between instantons is repulsive and the dilute instanton approximation is preserved. The

doi:10.6342/NTU201700825

interesting thing is, the choice of the direction of analytic continuation leads to ambiguity. This is the same phenomenon when we compute the Borel resummation of the ground state energy expansion around vacuum. If we sum all instanton contribution to the ground state energy, it would finally become ambiguity free. This is the resurgence in quantum mechanics.

In order to do the integral, it is convenient to introuduce some notation.

$$\lambda(g) = \frac{\epsilon}{\sqrt{\pi g}} e^{-1/6g}, \tag{3.65}$$

$$\mu = -\frac{2}{g}, \tag{3.66}$$

we also use the integral representation of the delta function,

$$\delta(\sum_{i=1}^{n} \theta_i - \beta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \ \exp\left[-s(\beta - \sum_{i=1}^{n} \theta_i)\right]. \tag{3.67}$$

Define the function

$$I(s) = \int_0^\infty d\theta \ \exp\left[s\theta - \mu e^{-\theta}\right], \tag{3.68}$$

the partition can be rewritten as

$$Z_\epsilon^{(n)}(\beta) \sim \frac{\beta e^{-\beta/2}}{2\pi i} \frac{\lambda^n}{n} \int_{i\infty}^{i\infty} ds \ e^{-\beta s} I(s)^n. \tag{3.69}$$

To evaluate $I(s)$, we set

$$\mu e^{-\theta} = t, \tag{3.70}$$

the integral becomes,

$$I(s) = \int_0^\mu \frac{dt}{t} (\frac{\mu}{t})^s e^{-t} = \int_0^\infty \frac{dt}{t} (\frac{\mu}{t})^s e^{-t} + O(e^{-\mu}/\mu). \tag{3.71}$$

For μ positive and large, $g \to 0^-$, the corrections are small. We find,

$$I(s) \sim \mu^s \Gamma(-s). \tag{3.72}$$

We want to sum all instanton contribution,

$$Z(\beta) = e^{-\beta/2} + \sum_{n=1}^{\infty} Z_\epsilon^{(n)}(\beta), \tag{3.73}$$

doi:10.6342/NTU201700825

using the expression of $I(s)$,

$$\begin{aligned} Z(\beta) &= e^{-\beta/2}(1 + \frac{\beta}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{-\beta s}\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\mu^{ns}\Gamma(-s)^n) \\ &= e^{-\beta/2}(1 - \frac{\beta}{2\pi i}\int_{-i\infty}^{i\infty} ds\ e^{-\beta s}\ln[1-\lambda\mu^s\Gamma(-s)]), \end{aligned} \tag{3.74}$$

we set,

$$E = s + \frac{1}{2}, \quad \phi(E) = 1 - \lambda\mu^{E-1/2}\Gamma(1/2 - E). \tag{3.75}$$

After integrate $\beta e^{-\beta s}$ by parts, we find,

$$Z(\beta) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dE\ e^{-\beta E}\frac{\phi'(E)}{\phi(E)}. \tag{3.76}$$

The asymptotic behavior of Gamma function makes the integral converge. The contour can also be deformed to enclose the postive half plane $\mathrm{Re}(E) > 0$. So this integral is a sum of all residues,

$$Z(\beta) = \sum_{N\geq 0} e^{-\beta E_N}. \tag{3.77}$$

Where E_N is the N-th state energy, they are also the solutions of the equation,

$$\phi(E) = 1 - \lambda\mu^{E-1/2}\Gamma(1/2 - E) = 0. \tag{3.78}$$

At the weak coupling limit, λ is very small, so the zero of $\phi(E)$ is close to the pole of $\Gamma(1/2 - E)$.

$$E_N = N + \frac{1}{2} + O(\lambda). \tag{3.79}$$

We can expand the N-th state energy in power of λ,

$$E_N(g) = \sum_{n=0}^{\infty} E_N^{(n)}(g)\lambda^n. \tag{3.80}$$

This is the multi-instanton contribution to all the energy levels at leading order. The coefficient of order n can be found by solving (3.78). It can be rewritten as,

$$-i\epsilon = \frac{e^{-1/6g}}{\sqrt{2\pi}}(-\frac{2}{g})^E\Gamma(\frac{1}{2} - E). \tag{3.81}$$

The imaginary part comes from the square root of g. Using the Euler's reflection formula, we obtain,

$$\frac{\cos\pi E}{\pi} = i\epsilon\frac{e^{-1/6g}}{\sqrt{2\pi}}(-\frac{2}{g})^E\frac{1}{\Gamma(\frac{1}{2}+E)}. \tag{3.82}$$

doi:10.6342/NTU201700825

Compare LHS and RHS in order of λ, we can find $E_N^{(n)}(g)$ order by order. For example, $E_N^{(1)}(g)$ is,

$$E_N^{(1)}(g) = -\frac{\epsilon}{N!}(\frac{2}{g})^N(1+O(g)), \tag{3.83}$$

and $E_N^{(n)}(g)$ is,

$$E_N^{(2)}(g) = \frac{1}{(N!)^2}(\frac{2}{g})^{2N}[\ln\left(-\frac{2}{g}\right) - \psi(N+1) + O(g\ln g)], \tag{3.84}$$

where ψ is the logarithmic derivative of the gamma function. For n-th order contribution, it can in general be computed. It takes the form at leading order,

$$E_N^{(n)}(g) = -(\frac{2}{g})^{nN}\{P_n^N(\ln\left(-\frac{g}{2}\right)) + O(g(\ln g)^{n-1}\}, \tag{3.85}$$

where $P_n^N(a)$ is a polynomial of degree $n-1$. For example, for $N=0$, one can find,

$$P_2(a) = a+\gamma, \quad P_3(a) = \frac{3}{2}(a+\gamma)^2 + \frac{\pi^2}{12}, \tag{3.86}$$

here γ is the Euler's constant, $\gamma = -\psi(1) = 0.57721\ldots$. Remember we are computing the multi instanton contribution at leading order to the N-th energy level. We changed the coupling constnat g to negative thus we can define the multi instanton configuration. At the same time, the Borel sums of the energy without any instanton is summable since the coupling is negative. Now we want to change the coupling from negative to positive by analytic continuation, two things happen. The Borel sums become non-summable and get an ambiguous imaginary part of order two instanton. At the same time, the function $\ln\left(-\frac{2}{g}\right)$ also gets an ambiguous imaginary part $\pm i\pi$. These imaginary parts would cancel each other and the energy is still ambiguity free. The imaginary part of the ground state energy without instanton is (3.53). The imaginary part P_2 is,

$$\operatorname{Im} E_0^{(2)}(g) = \frac{1}{\pi g}e^{(-1/3g)}\operatorname{Im}[P_2(\ln(-g/2))] \sim \pm\frac{1}{g}e^{-1/3g}. \tag{3.87}$$

The same order as the imaginary part of the ground state energy (3.53) without instanton. In fact, they cancel each other. Similar cancellation appear at all order. The leading imaginary part of $E^{(1)}(g)$ is canceled by $E^{(3)}(g)$. For the sub imaginary part of $E^{(0)}(g)$, they are canceled by $E^{(4)}(g)$, $E^{(6)}(g)$ and to all order for n even. Thus we can construct a rather complicated expression to the energy of double well

doi:10.6342/NTU201700825

potential.

$$E_N(g) = \sum_n E_{N,n} g^n + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} (\frac{e^{-1/6g}}{\pi g})^k (\ln\left(-\frac{2}{g}\right))^l c_{N,n,k,l} g^n, \quad (3.88)$$

where n is the order of the perturbative expansion, k is the number of instanton and l is to sum over the polynomial P_n, it also represents the number of the quasi zero modes (IA pairs). This is the trans-series of the double well potential. The construction of the trans-series is verified in several different potential and can also be extended to supersymmetry case [7, 24, 25].

4 Resurgence in QFT

In Quantum field theory, whether it is possible to construct the trans-series or not is an interesting question. There are renormalons in asymptotically free theories[4]. The infrared renormalons appear on the positive real axis of Borel plane have less action than instanton, but we can not find such classical solutions coressponding to those singularities. We also noticed in previous section that renormalon is related to the running coupling and is a strong coupled effect, therefore it is a quantum effect and it seems very hard to realize it classically. There are some conjectures of the IR renormalons in asymptotic free theories, some claim they correspond to the OPE vev takes nonzero value, others think that one may need to do second renormalization of the theory. Although there are many different conjectures, no one really gives a concrete argument for them.

Recent years, it has been discovered by Argyres and Ünsal [14, 29] that in the compactified gauge theory with adjoint fermions, there are new semiclassical configurations, e.g., bion-anti-bion events, which may correspond to the infrared renormalons. They conjectured that these new saddle points are the leading singularities in the Borel plane and they are the incarnation of the renormalons in the weak coupling limit. If this is true, it may be possible to construct the trans-series of QFT and give it a non-perturbative definition at weak coupling limit. Since renormalon comes from the infrared or ultraviolet momentum contribution, these saddles should

[4] It has been proved that there are no renormalons in ϕ^4 theory and QED in 4d because of the asymptotic behaivor of the beta function. [26, 27, 28]

doi:10.6342/NTU201700825

also relate to the different scales of the theory. However, the relation between the bion-anti-bion configuration and the different scales is still unclear. Futhermore, the position of the bion-anti-bion on the Borel plane is at $\frac{8\pi}{N_c/4}$, where the position of the closest IR renormalon is $\frac{8\pi}{\beta_0}$[5], which do not coincide. There are still no enough evidence to conclude these new semiclassical configurations correspond to the IR renormalons.

After several months, Dunne and Ünsal published another paper [13] which discusses the resurgence relation may be carried out in the compactified CP^{N-1} model. By spatial compactification and periodic boundary condition on the fermions, there are new semiclassical configurations, the kink-instantons. Kink-instantons has action of order $e^{S_I/N}$, which is the instanton action divided by a factor N. The kink-anti-kink forms a bion and the bion-anti-bion configuration has ambiguous imaginary parts. The position of the bion-anti-bion and the position of the closest IR renormalon is the same order. This is a hint that IR renormalons may be realized semiclassically.

The aim of this section is to show that when some theories are compactified in a particular way, there are new semiclassical configurations and no significantly phase transition during the compactified radius change. We want to show that these new semiclassical configurations may correspond to the IR renormalons in the non-compactified theory.

4.1 QCD(adj) on $\mathbb{R}^3 \times S^1$

We are going to discuss the four dimensional gauge theory with $G = SU(N)$ gauge group and N_f adjoint fermions. The theory is compactified to $\mathbb{R}^3 \times S^1$ with periodic boundary condition for the fermions. This is the so called QCD(adj) with time direction compactified by a spatial circle with circumference L. With this compactification, the theory has no center symmetry changing phase transition when the radius is varied [30]. At small circle size, the theory is weakly coupled and the semiclassical analysis is valid. The gauge holonomy around S^1 takes nozero value and behaves as a Higgs field, the gauge group abelianizes at long distances, $G \to U(1)^r$ where r is the rank of the gauge group. There are new semiclassical configurations,

[5] β_0 is the first coefficient of the beta function, which is $\frac{11}{3}N_c$ in pure bosonic QCD

doi:10.6342/NTU201700825

monopole instantons, appearing in the theory. Instead of the 4-d BPST instantons, monopole instantons (or 3-d instantons, twisted instantons) $M_i, i = 1, \ldots, r+1$ make the leading contribution to the semiclassical expansion. A monopole instanton anti-monopole instanton pair is a bion $B = [M\bar{M}]$. There are two types of bion, magnetic bion $B_{ij} = [M_i\bar{M}_j]$ and neutral bion $B_{ii} = [M_i\bar{M}_i]$, where magnetic bion carries magnetic charge and neutral bion does not. What we are interested in is the neutral bions since they do not carry any topological charge (magnetic charge, instanton number) and may be related to the renormalon on the Borel plane. The bion-anti-bion $[B\bar{B}]$ pair has the same quantum number as the perturbative vacuum, just like the IA pair in quantum mechanics. Using a generalized version of the technique when calculating the IA pair in QM double well, we can compute the imaginary part of the $[B\bar{B}]$ pair. Argyres and Ünsal found the location of the $[B\bar{B}]$ pole is qualitatively of the same order as the IR renormalons and claim they correspond to the elusive IR renormalons on the Borel plane.

4.1.1 4-d theory

The Largrangian for the 4-d theories with general gauge group G with Lie algebra $\mathcal{G}$ and N_f massless adjoint fermions is,

$$L = \frac{1}{2g^2}(F_{\mu\nu}, F_{\mu\nu}) + \frac{2i}{g^2}(\bar{\psi}_f, \bar{\sigma}^\mu D_\mu \psi_f) + \frac{i\theta}{16\pi^2}(F_{\mu\nu}, \tilde{F}_{\mu\nu}), \tag{4.1}$$

where $(\cdot,\cdot)$ is the gauge invariant Killing form on $\mathcal{G}$, $f = 1, \ldots, N_f$ is the flavor index of the Weyl fermions and $\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. We will focus on the gauge group $G = SU(N_c)$ case later, but now we can do some general discussion. Without mass term insertion, we can use a chiral rotation to set the theta angle equal to zero, $\theta = 0$. The coupling constant is the following function of the energy scale μ at one loop in perturbation theory.

$$\exp\left\{-\frac{8\pi^2}{g^2(\mu)}\right\} = (\frac{\Lambda}{\mu})^{\beta_0}, \tag{4.2}$$

where Λ is the strong coupling scale and β_0 is the coefficient of one loop beta function, which is given by

$$\beta_0 = h^\vee \frac{11 - 2N_f}{3}, \tag{4.3}$$

doi:10.6342/NTU201700825

when the fermions are in the adjoint representation, $h^\vee$ is the dual Coxeter number of the Lie algebra. The condition for asymptotic freedom is $N_f < 5$, we can not add too many fermions. We want to put the theory on $\mathbb{R}^3 \times S^1$ with the S^1 of size L in the x^4 direction, and we impose periodic boundary condition on fermions. We also assume the inverse radius $L^{-1} \gg \Lambda$ so the theory is weakly coupled at the scale of the compactification, $g(L^{-1}) \ll 1$. We study the dynamics of the effective 3-d theory at a scale μ, $\Lambda \ll g/L \ll \mu \ll 1/L$.

4.1.2 3-d effective theory

Integrating the theory along the x^4 direction gives us the 3-d effective Lagrangian, which can also be seen as a dimensional reduction. We also assume the fields do not depend on the compactified direction x^4. We find,

$$L_{3d} = \frac{L}{g^2}[\frac{1}{2}F_{mn}F^{mn} + |D_m A_4|^2 + 2i\bar{\psi}_f \not{D} \psi_f - \bar{\psi}_f \bar{\sigma}^4 A_4 \psi_f]. \tag{4.4}$$

We can use gauge transformations to rotate A_4 to its Cartan subalgebra (CSA) with generators $H_i \subset \mathcal{G}$ (i.e, we can diagonalize A_4). We define the 3-d gauge fields by,

$$A_4(x) := \frac{2\pi}{L}\phi^i(x)H_i, \qquad H_i \in CSA,\ i = 1, \dots, r, \tag{4.5}$$

$$A_m(x) := a^i_m(x)H_i + W^\alpha_m(x)E_\alpha, \quad E_\alpha \in CSA^\perp, \alpha = 1, \dots, N_c^2 - 1 - r, \tag{4.6}$$

where H_i are the basis of the Cartan subalgebra and E_α are the roots, r is the rank of the gauge group G, ϕ^i are scalar fields which are related to the gauge holonomy, the a_m photons are massless bosons since they are in the CSA, the W^α fields are charged by their roots. We denote $\phi := \phi^i H_i$ later. With these definition, keeping only quadratic terms, the Lagrangian is,

$$L_{3d} = \frac{L}{2g^2}(f_{mn} + d_{[m}W_{n]})^2 + \frac{4\pi^2}{g^2 L}(\partial_m \phi + \alpha(\phi)W^\alpha_m E_\alpha)^2 + \frac{2L}{g^2}\psi_f[i\not{d} - \frac{2\pi}{L}\bar{\sigma}^4\lambda(\phi)]\psi^\lambda_f. \tag{4.7}$$

The field strength is defined by $f_{mn} := \partial_{[m}a_{n]}$ and the covariant derivative is $d_m := \partial_m + ia_m$. The charges of the W_m boson and ψ_f fermions are the roots α of G, and weights λ of fundamental representation of G.

The different vacua are parameterized by different choice of $\langle\phi\rangle$, the gauge inequivalent choices of ϕ corresponds to points on the affine Weyl chamber. The affine

doi:10.6342/NTU201700825

Weyl chamber, sometimes called the gauge cell, has a simple description. We denote the affine Weyl chamber by $\hat{T}$,

$$\hat{T} := [\phi|\alpha_i(\phi) \geq 0, i = 1, \ldots, r, \quad \text{and} \quad \alpha_0(\phi) \geq -1], \tag{4.8}$$

where α_i are a basis of simple roots, and α_0 is the lowest root of this basis. The points of $\hat{T}$ correspond to the gauge inequivalent choices of ϕ. At interior points of $\hat{T}$, the gauge group is Higgsed to abelian factors,

$$\phi : G \to U(1)^r \quad \text{for} \quad \phi \in \text{interior}(\hat{T}). \tag{4.9}$$

This holds for general gauge group G. Once one compactifies the theory along one direction, ϕ behaves like a Higgs field and Higgses the theory[6]. So now our 3-d effective theory become a theory with gauge group $U(1)^r$. We only want to focus on the massless content of our theory, so we integrate out all the charged fields like the W- bosons and those fermions not in CSA. The 3-d classical effective Lagrangian for the massless modes become,

$$L_{3d} = \frac{L}{2g^2}(f_{mn}, f_{mn}) + \frac{4\pi^2}{g^2 L}(\partial_m\phi, \partial_m\phi) + i\frac{2L}{g^2}(\bar{\psi}_f, \partial\!\!\!/ \psi_f). \tag{4.10}$$

Where $f_{mn} = \partial_m a_n - \partial_n a_m$ stands for the $U(1)^r$ field strength. This is a 3-d $U(1)^r$ gauge theory with r real, massless, neutral scalars and Weyl fermions. This is the case when we choose the interior points of $\hat{T}$ as our vacuum, while at the boundaries of $\hat{T}$, the gauge symmetry is not completely broken and leaving nonabelian factors, some of W^α-bosons and ψ^α fermions also become massless. But it is not what we are interested in now.

Electric and Magnetic charges

The electric λ and magnetic μ charges in the 4-d $U(1)^r$ theory are defined by,

$$\lambda := \int_{S^2_\infty} *f, \quad \mu := \frac{1}{2\pi}\int_{S^2_\infty} f, \tag{4.11}$$

[6]At generic points on the affine Weyl chamber, the gauge symmetry is broken down to $U(1)^r$ while r is the rank of the gauge group. However, the effective potential of ϕ may preserve the gauge symmetry. Only when the point on the affine Weyl chamber is at the minimum of the effective potential, it is quantum mechanically stable. Actually, only for SU(N) gauge group, the gauge symmetry is completely broken down to U(1), while for other gauge group, there are still non-abelian symmetry survive.

doi:10.6342/NTU201700825

where the 2-form $U(1)^r$ field strength is defined by $f := \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$, while the dual field strength is $*f := \frac{1}{2} \tilde{f}^*_{\mu\nu} dx^\mu \wedge dx^\nu$. Here

$$\tilde{f} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma} \tag{4.12}$$

Our original gauge group is G and all fields transfrom in some rep of G. So the electric charges, λ, of the fields $U(1)^r \subset G$ live on the gauge lattice Γ_G. But now all the dynamical fields in our theory are in the adjoint representation, thus the electric charges are in the root lattice $\Gamma_r = \Gamma_{adj}$ of G. By the Dirac quantization condition, the allowed magnetic charges μ are in the co-weight lattice $\Gamma_w^\vee$. But if we add new massive fields inside which are not in the adjoint representation, the massive sources can have charge in the larger weight lattice Γ_w, then by Dirac quantization, the allowed magnetic charges are in the co-root lattice $\Gamma_w^\vee$, which is smaller than the co-weight lattice.

We can define the corresponding electric and magnetic point operator. The Wilson loop operator wrapping the S^1 direction at a point $P \subset \mathbb{R}^3$ represents the 3-d effective electric point operator. The Wilson loop can be represented by a sum of a set of operators with different charges.

$$\mathrm{Tr}\, P \exp\left[i \int_{S^1} A_4 dx^4 \right] = \sum_\lambda \exp[2\pi i \lambda(\phi)]. \tag{4.13}$$

The eletirc operator at point P with charge λ is then,

$$E[\lambda, P] := \exp[2\pi i \lambda(\phi)](P). \tag{4.14}$$

For the magnetic operator, the t'Hooft line operator on $\mathbb{R}^3 \times S^1$ wrapping the S^1 direction represents a monopole operator at some point in$\mathbb{R}^3$. The monopole operator at point $P \subset \mathbb{R}^3$ with charge μ is given by,

$$M[\mu, P] \text{ creates a gauge field singularity at P,with } \int_S f = 2\pi\mu, \tag{4.15}$$

for any closed surface S, f is the $U(1)^r$ 2-form field strength.

The 3-d dual photon

Since the $U(1)^r$ field strengths (f_{mn}, f_{mn}) decouple from all the other fields content, we can replace them by the dual photon fields $\sigma(x)$ because they give us the same equation of motion [31, 32]. Consider a theory contanins the 3-d $U(1)^r$ gauge

doi:10.6342/NTU201700825

fields a_m, vector field b_m and a scalar field σ. The partition fucntion is given by,

$$Z = \int [da_m][db_m][d\sigma] \exp\Big[-\int d^3x \ L\Big], \tag{4.16}$$

where the Lagrangian L is,

$$L := \frac{g^2}{4L}(\partial_m \sigma + b_m)^2 + \frac{i}{2}\epsilon_{mnp} b_m (f_{np}). \tag{4.17}$$

We input $U(1)^r$ gauge invariance for a_m and additional symmetry to σ and b_m,

$$\sigma \to \sigma + \sigma', \quad b_m \to b_m - \partial_m \sigma'. \tag{4.18}$$

If we fix this symmetry by setting $\sigma = 0$ and integrating out b_m fields, we get

$$Z = \int [da_m] \exp\Big[-\frac{L}{2g^2}\int d^3x \ (f_{mn}, f_{mn})\Big]. \tag{4.19}$$

which is the $U(1)^r$ field strength part of (4.10). If we integrate out a_m and set $b_m = 0$ to fix the gauge symmetry, we find,

$$Z = \int [d\sigma] \exp\Big[-\frac{g^2}{4L}\int d^3x \ (\partial_m \sigma, \partial_m \sigma)\Big]. \tag{4.20}$$

This is the dual photon expression of the original $U(1)^r$ gauge field a_m. Finally, our 3-d effective Lagrangian becomes,

$$L_{3d} = \frac{g^2}{4L}(\partial_m \sigma, \partial_m \sigma) + \frac{4\pi^2}{g^2 L}(\partial_m \phi, \partial_m \phi) + i\frac{2L}{g^2}(\bar{\psi}_f, \partial\!\!\!/ \psi_f). \tag{4.21}$$

The σ and ϕ fields are dimensionless, where the ψ_f fermions have dimension $\frac{3}{2}$. There are r real scalar bosons σ, r real real scalar bosons ϕ and r Weyl fermions ψ_f, all the fields are massless. Under this duality, the magnetic point operator becomes a local operator,

$$M[\mu, P] := \exp[2\pi i \sigma(\mu)](P). \tag{4.22}$$

This is equivalent to inserting in an gauge invariant operator $e^{2\pi i \sigma(\mu)}(P) \cdot e^{2\pi i \int_C b(\mu)}$ with the Dirac string C ending at P into the path integral. Integrating out b_m fields and fixing the gauge by setting $\sigma = 0$ gives us the original field strength integration with the boundary condition (4.15). Integrating out a_m fields and setting $b_m = 0$ to fix the gauge gives us the monopole operator.

Effective potential for ϕ

doi:10.6342/NTU201700825

Up to now, it seems the choice of ϕ is arbitrary, but this is not true in quantum theory. Actually, when we do path integral and integrate out all other fields, it will generate an effective potential for ϕ. For $n_f = 1$, there is a supersymmetry in our theory and it prohibits the potential of ϕ be generated. For $n_f \neq 1$, when we integrated all the massive modes in loops, it would generate an effective potential for ϕ [33]. Note that we only need to integrate out those modes with masses greater than μ since we are considering the theory at scales $\mu \leq L^{-1}$. The effective potential for ϕ takes the form,

$$V_{pert}(\phi) = L^{-3}(v_0(\phi) + g^2 v_2(\phi) + g^3 v_3(\phi) + \dots). \tag{4.23}$$

where $v_n(\phi)$ are dimensionless functions of ϕ. For analytic 1-loop contribution to the effective potential, v_0, we can expand it around the minimum,

$$v_0 \sim (\phi - \phi_0)^\vee \cdot v_{0,2} \cdot (\phi - \phi_0) + O(\phi - \phi_n)^3, \tag{4.24}$$

where ϕ_0 is the position of the minimum of ϕ, $v_{0,2}$ is a positive-definite matrix. The higher order terms only give corrections of positive order of g and they would not change the minimum of ϕ significantly. So at 1-loop level, the ϕ fields take a unique vacuum value.

The effective potential of ϕ at one loop is obtained by intergrating out all other fields at one loop order,

$$V_{pert}(\phi) = -\frac{1}{\mathbb{V}} \ln \left(\frac{\prod_f \det\left(-D^2_{R_f}\right)}{\det\left(-D^2_{adj}\right) \prod_b \det\left(-D^2_{R_b}\right)} \right). \tag{4.25}$$

This is for the gauge theory with fermions in representation R_f and bosons in representation R_b, $\mathbb{V}$ is the volume of $\mathbb{R}^3$. For there are N_f adjoint massless fermions with $SU(N_c)$ gauge group, the effective potential is given by [14],

$$V_{pert}(\phi) = \frac{8(N_f - 1)}{\pi^2 L^3} \sum_{1 \leq i \leq j \leq N_c} g(\phi_i - \phi_j) \text{ with } \sum_{1 \leq i \leq N_c} \phi_i = 0, \tag{4.26}$$

the minimum of ϕ is,

$$\phi_j = \frac{N_c + 1 - 2j}{2N_c}, \quad j = 1, \dots, N_c, \tag{4.27}$$

where ϕ_j is the j-th element of the diagonal matrix ϕ. This minimum is at the interior point of $\hat{T}$ so the analysis we have done is reliable. We can see for $SU(N_c)$, the eigenvalues of the holonomy are equally separated, which preserve the Z_{N_c} center symmetry.

doi:10.6342/NTU201700825

4.1.3 Topological configurations on $\mathbb{R}^3 \times S^1$

Now we will focus on the case with gauge group $SU(N_c)$. We want to study the semiclassical configurations inside our 3-d effective theory and we will find out that there appear monopole-instantons in our theory. The monopole-instantons are classified by two quantum numbers, the magnetic charge μ and the topological charge ν.

$$\mu := \frac{1}{2\pi}\int_{S^2_\infty} f, \quad \nu := \frac{1}{16\pi^2}\int_{\mathbb{R}^3\times S^1}(F_{\mu\nu}, \tilde{F}_{\mu\nu}), \tag{4.28}$$

f is the 3-d effective $U(1)^{N_c-1}$ 2-form field strength, $(\ , \ )$ is the gauge invariant Killing form. The topological charge (instanton number) is in terms of the microscopic 4-d field strength. The Killing form is normalized to make sure the smallest instanton number on $\mathbb{R}^4$ is 1. The self-dual equation for the 4-d BPST instanton is,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{4.29}$$

After dimensional reduction to 3-d and we assume the gauge fields are independent of x^4, the equation becomes,

$$F_{mn} = \epsilon_{mnp}D_pA_4 = \frac{2\pi}{L}\epsilon_{mnp}D_p\phi. \tag{4.30}$$

This equation gives us the solutions of N_c types of elementary monopole-instantons M_j, $j = 0, \ldots, N_c,$. In general, we expect that there are N_c-1 elementary monopole-instantons related to the Higgsing to $U(1)^{N_c-1}$. We have one more monopole-instanton because the adjoint Higgs field is compact. The monopole-instantons are in ono-to-one correspondence to the simple roots α_j, the additional monopole-instanton corresponds to the lowest root α_0. Each monopole-instanton M_j carries magnetic charge $\mu^{(j)}$ and topological charge $\nu^{(j)}$. The magnetic charges are given by,

$$\mu^{(j)} = \alpha_j^\vee, \quad j = 0, \ldots, N_c - 1, \tag{4.31}$$

where $\alpha_j^\vee$ are the affine or simple co-roots. The topological charges are,

$$\nu^{(j)} = \frac{1}{N_c}, \tag{4.32}$$

doi:10.6342/NTU201700825

which is independent of the index j in $SU(N_c)$ case. The action for a single monopole-instanton is,

$$S_{(j)} = \frac{S_I}{N_c}, \tag{4.33}$$

where S_I is the action of the usual 4d BPST instanton, which is $8\pi/g^2$ for QCD. The anti monopole-instanton $\bar{M}_j$ carries opposite magnetic and topological charges. Under this dimensional reduction, the 4-d BPST instantons are just a combination of the monopole-instantons. They are not the leading contribution in the semiclassical analysis. The monopole operator carries fermion zero modes when $N_f \geq 1$, accordging to Nye-Singer index theorem [34, 35], the monopole has $2N_f$ fermionic zero modes. The monopole operator in QCD(adj) on $\mathbb{R}^3 \times S^1$ is given by,

$$M_j = C_j \exp\left[-S_j(\phi) + 2\pi i \sigma(\alpha_j^\vee)\right] \det_{f,f'}[\alpha_j(\psi_f) \cdot \alpha_j(\psi_{f'})] \tag{4.34}$$

this is an operator in the effective 3d theory, if we choose the value of gauge holonomy ϕ to be the minimum of the effective potential of ϕ, $S_j = \frac{S_I}{N_c}$. σ are the dual photon fields and ψ_f are the N_f fermions. The coefficient C_j should be determined by the zero-mode integral.

Amplitude of the topological molecules

We call a pair of monopole-instanton-anti-monopole-instanton $[M\bar{M}]$ a bion. There are two types of bion, one is called the magnetic bion, another one is called neutral bion. For a magnetic bion, we denote them by $[M_i\bar{M}_j]$. It carries magnetic charges and has no topological charge,

$$(\mu, \nu) = (\alpha_i^\vee - \alpha_j^\vee, 0). \tag{4.35}$$

For a neutral bion, we denote it by $[M_i\bar{M}_j]$. It carries no magnetic charge and topological charge,

$$(\mu, \nu) = (0, 0). \tag{4.36}$$

For a general bion amplitude, we can obtain it by inserting a monopole and an anti-monopole into the path integral with a distance r,

$$B_{ij} := \int d^3r \left\langle M_i(R + \frac{1}{2}r)\bar{M}_j(R - \frac{1}{2}r)\right\rangle. \tag{4.37}$$

doi:10.6342/NTU201700825

If the separation between the two monopole is large enough, i,e, the integral is dominated when $r \gg r_b$ where r_b is some length scale large enough for us to claim that there is a bion exist, then this bion is well defined in the theory. Using the explicit form of the monopole operators, we find the bion amplitude is in the following form,

$$B_{ij} = C_{ij} e^{-S_i - S_j} e^{2\pi i \sigma(\alpha_i^\vee - \alpha_j^\vee)}, \tag{4.38}$$

where

$$C_{ij} = C_i C_j (\frac{g^2}{2L})^{2N_f} \frac{(\alpha_i, \alpha_j)^{2N_f}}{(2\pi)^{2N_f}} \int d^3 r \; e^{-V_{\text{eff}}^{ij}(r)}, \tag{4.39}$$

and

$$V_{\text{eff}}^{ij}(r) = -(\alpha_i^\vee, \alpha_j^\vee) \frac{2\pi}{g^2} (1 + e^{-m_\phi r}) \frac{L}{r} + 4N_f \ln(r). \tag{4.40}$$

Here m_ϕ is the mass of the field ϕ in the perturbative vacuum, C_i are normalization factors. The mass term is $m_\phi \sim g/L$ for $N_f > 1$, which decouples at the small radius limit. So we can use 0 to replace $e^{-m_\phi r}$. For $N_f = 1$, since the theory is supersymmetric, there is no effective potential for ϕ so it is massless. We can use 1 to replace $e^{-m_\phi r}$.

The effective potential $V_{\text{eff}}^{ij}(r)$ describes the interaction between the monopole and the anti-monopole. The second term in the potential is attractive which is induced by fermion zero mode exchange, while the first term is the Coloumb interaction which is repulsive for $(\alpha_i, \alpha_j) < 0$ and attractive for $(\alpha_i, \alpha_j) > 0$. This interaction term is repulsive for magnetic bion $i \neq j$ buy attractive for neutral bion $i = j$. Therefore, the amplitude is well defined for magnetic bion but requires regularization for neutral bion. After rescaling the variables, the quasi zero mode integral $\int e^{-V_{\text{eff}}}$ for magnetic bion takes the following form,

$$L(g^2, N_f) = \int_0^\infty dz \;\; \exp\left[-\frac{1}{g^2 z} - (4N_f - 2) \ln z \right] = (\frac{1}{g^2})^{3-4N_f} \Gamma(4N_f - 3). \tag{4.41}$$

This is an ambiguity free result. On the other hand, the same integral for neutral bion is,

$$\tilde{L}(g^2, N_f) = \int_0^\infty dz \;\; \exp\left[\frac{1}{g^2 z} - (4N_f - 2) \ln z \right] \tag{4.42}$$

This integral is dominated by the small separation region and we need to do analytic continuation to regularize it. By rotating $g^2 \to -g^2$, we find $\tilde{L}(-g^2, N_f) = L(g^2, N_f)$

doi:10.6342/NTU201700825

is well defined. Next, we come back to positive g^2,

$$\tilde{L}(g^2, N_f) \to L(-g^2, N_f) = (-\frac{1}{g^2})^{3-4N_f}\Gamma(4N_f - 3) = -I(g^2, N_f). \quad (4.43)$$

While this equality is only valid for integer N_f, if N_f is fractional, the branch cut would make the integral ambiguous. This result is also ambiguity free because the monopole and anti-monopole both carry fermionic zero modes.

4.1.4 Topological molecules and IR renormalons in QCD(adj) on $\mathbb{R}^3 \times S^1$

We want to focus on those topological molecules which can produce ambiguous imaginary part, just like the quantum mechanical case. The interaction of the magnetic bion is repulsive when the separation between the monopole-anti-monopole is small and is attractive when the separation is large, thus this configuration has no imaginary part and is well defined. Also, since it carries magnetic charges, it cannot lead to the pole on the Borel plane. The neutral bions have the same quantum number as the perturbative vacuum, they may correspond to the singularities on the Borel plane.

The interaction for a neutral bion is attractive, the path integral needs to be regularized by changing the coupling to negatvie first, $g^2 \to -g^2$, then analytically continuing back to positive. However, the amplitude for a neutral bion is also ambiguity free, this is because the monopole-instanton has a fermionic zero mode, the neutral bion dose not have an imaginary part for N_f integer. So no imaginary part is generated by the magnetic bion and neutral bion, we need to see the next leading order, the two bion amplitude $[B_{ij}B_{kl}]$. We want to focus on those two bion configurations without magnetic charge and topological charge. They are the bion-anti-bion $[B_{ij}B_{ji}] = [B\bar{B}]$ configurations. There is no fermionic zero mode for a single bion, so the $[B\bar{B}]$ configuration has an ambiguity at the order $\exp\left[-4\frac{S_I}{N_c}\right]$. Sicne the bion-anti-bion is the leading semi-classical configuration which has ambiguity, it should correspond to the singularity which is closest to the orgin on the Borel plane and determine the leading large order behavior of the expansion around perturbative vacuum.

Let's consider QCD(adj) on $\mathbb{R}^4$. The 4-d BPST instanton-anti-instanton config-

doi:10.6342/NTU201700825

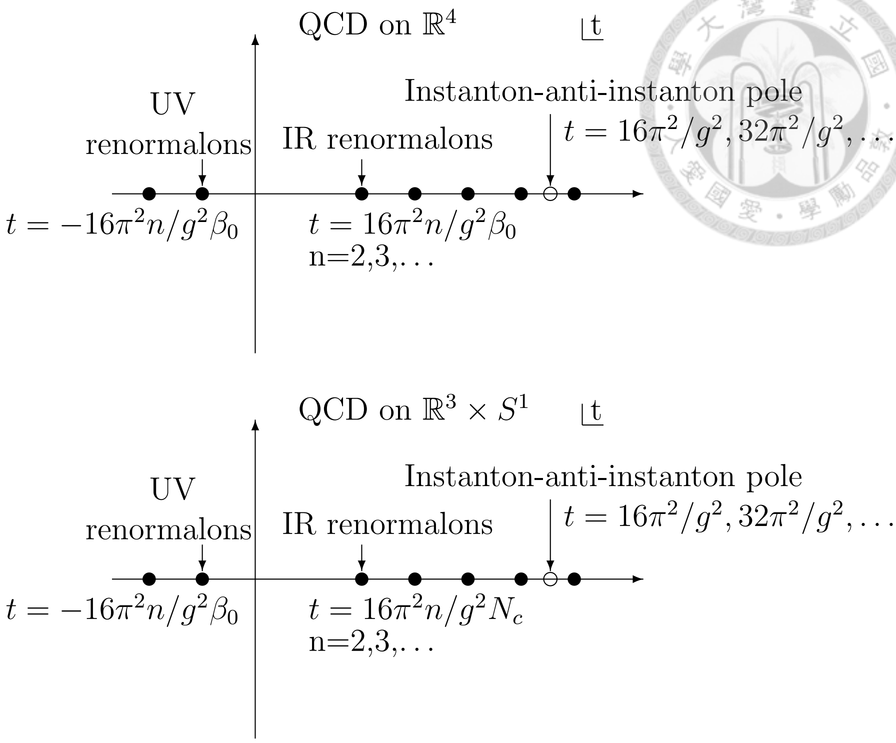


Figure 8: The poles on the Borel t plane for QCD(adj) on different manifold.

urations correspond to poles on the Borel t plane at,

$$t = 2nS_I = \frac{16n\pi^2}{g^2}, n = 1, 2, \ldots. \tag{4.44}$$

But there are IR renormalons in QCD-like asymptotic free theories, the IR renormalons for QCD(adj) on $\mathbb{R}^4$ are at,

$$t = \frac{16\pi^2}{g^2\beta_0}n = \frac{48\pi^2}{g^2 N_c(11 - 2N_f)}, n = 2, 3, \ldots, \tag{4.45}$$

The IR renormalons are closer to the origin by a factor β_0.

It was known that no semiclassical confiugration can cure the IR renormalons. However, we have found that when QCD(adj) on $\mathbb{R}^4$ is compactified to $\mathbb{R}^3 \times S^1$, new topological molecules which have no topological charge and can generate imaginary parts, like $[B_{ij}B_{ji}]$, $[B_{ij}B_{jk}B_{ki}]$ appear, corresponding to the poles on the Borel t plane at,

$$t = \frac{16\pi^2}{g^2 N_c}n, \quad n = 2, 3 \ldots, \tag{4.46}$$

The resulting singularities are shown in fig.8. The position of those singularities produced by two bion molecules or three bion molecules is the same order as the

doi:10.6342/NTU201700825

position of IR renormalons. It has been conjectured that when the 4-d theory is compactified to 3-d, the positions of the poles on the Borel t plane change continuously with the compactified radius. If this is true, the bion-anti-bion configurations may be the elusive IR renormalons in QCD and the resurgence trans-series can also be applied to QCD.

4.2 CP^{N-1} model on $\mathbb{R}^1 \times S^1$

There is another interesting model that resurgence relation may be carried out, the CP^{N-1} model. This 2-d theory is a complex version of $O(N)$ non-linear sigma model. It has some important properties which also appear in QCD, like asymptotic freedom, gauge symmetry, instantons, confinement and mass gap, which is a useful toy model to realize QFT. It has been found that there are IR renormalons in CP^{N-1} model on $\mathbb{R}^2$, but just like in most quantum field theories which have renormalons, we can not find the semiclassical configurations corresponding to the IR renormalons. Dunne and Ünsal found that if one compactifies the theory, something interesting happens.

Since we want to understand CP^{N-1} model on $\mathbb{R}^2$, this is equivalent to CP^{N-1} model on $\mathbb{R}^1 \times S^1$ with the the circumference of S^1, $L \to \infty$. If we want to do semiclassical analysis on the compactified theory, we need to take the length L to be small, thus the theory is weakly coupled and the analysis is reliable.

If we want to apply the result we have obtained at small L to big L, there must be no phase transition in our theory. If there is a phase transition, then what we obtained in small L region are meaningless at large L. However, when CP^{N-1} model is thermal compactified (anti-periodic boundary condition on fermions), there is a phase transition between the confined phase and deconfined phase. On the other hand, if we use spatial compactification (peridoic boundary condition on fermions) on CP^{N-1} model, there are not any phase transition at $N \to \infty$ or rapid-crossovers at N finite. Therefore, we can do semiclassical analysis on it by taking L small, then change $L \to \infty$ to go back to $\mathbb{R}^2$, since there is no significant phase transition, we deduce the results we obtain at small L is still reliable.

When the CP^{N-1} model is spatial compactified by a circumference $L = 2\pi R$ with periodic boundary condition on fermions, new semiclassical configurations ap-

doi:10.6342/NTU201700825

pear, just like in the QCD(adj) case! These new configurations are called kink-instantons. These kink-instantons can combine with anti-kink-instantons and form a bion. Again, the bion-anti-bion $[B\bar{B}]$ configurations may correspond to the closest IR renormalon in CP^{N-1} model.

4.2.1 CP^{N-1} model

Consider the CP^{N-1} model defined on 2-d Euclidean space-time with one dimension is spatial (not thermal) compactified, $\mathbb{R}^1 \times S^1$. We use corrdinates (x_1, x_2), where x_2 is the compactified dimension of length $L = 2\pi R$. The classical action is,

$$S = \frac{2}{g^2}\int d^2x\ (D_\mu n)^\dagger D_\mu n. \tag{4.47}$$

The field n is an N-component complex vector of norm 1,

$$n = (n_1, n_2, \ldots, n_N)^T, \quad \sum_{i=1}^{N} |n_i|^2 = 1. \tag{4.48}$$

The covariant derivative is $D_\mu = \partial_\mu + iA_\mu$, where A_μ is determined by its equation of motion,

$$A_\mu = \frac{i}{2}(n^\dagger \partial_\mu n - \partial_\mu n^\dagger n). \tag{4.49}$$

There is a U(1) gauge symmetry,

$$n(x) \to e^{i\alpha(x)} n(x), \quad A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x). \tag{4.50}$$

The degree of feedom of this theory is $2N-2$, there are $2N$ real fields and two constraints, the norm constraint and the gauge symmetry. The topological theta term is,

$$S_\theta = i\Theta Q, \tag{4.51}$$

where Q is the topological charge,

$$Q = -\frac{1}{2\pi}\int d^2x\ \epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{4.52}$$

CP^{N-1} theories are asymptotic free, the first coefficient of the beta function is independent of the fermion number N_f. The runnging coupling is,

$$\exp\left\{-\frac{4\pi}{g^2(\mu)}\right\} = (\frac{\Lambda}{\mu})^{\beta_0}, \quad \beta_0 = N, \tag{4.53}$$

doi:10.6342/NTU201700825

where Λ is the strong coupling scale, β_0 is the first coefficient of the 1-loop beta function and μ is the energy scale of the coupling $g^2(\mu)$. We need to consider theories with fermions, the fermions can induce the potential for the gauge holonomy. Consider the CP^{N-1} model with N_f Dirac fermions, the action of the fermion is,

$$S_f = \frac{2}{g^2}\int d^2x\ [-i\bar{\psi}_f\gamma_\mu D_\mu\psi_f + \frac{1}{4}((\bar{\psi}_f\psi_f)^2 + (\bar{\psi}_f\gamma_3\psi_f)^2 - (\bar{\psi}_f\gamma_\mu\psi_f)^2)]. \qquad (4.54)$$

For $N_f = 1$, the theory is $\mathcal{N} = (2, 2)$ supersymmetric.

We want to do semiclassical analysis to CP^{N-1} model, so we need to take the compactification length L small to make the theory weak coupling.

4.2.2 Gauge holonomy

The gauge fields can take nonzero values along the compactified dimension. It is convenient to consider the holonomy,

$$\Omega(x_1) = \exp\left[i\int_0^L A_2(x_1, x_2)dx_2\right], \qquad (4.55)$$

which is a $N \times N$ matrix. It acts on the bosonic fields and the fermionic fields as a global U(N) transformation.

$$n(x_1, x_2 + L) = \Omega(x_1)n(x_1, x_2), \quad \psi(x_1, x_2 + L) = \Omega(x_1)\psi(x_1, x_2), \qquad (4.56)$$

where $\Omega(x_1) \subset U(N)$. For a general choice of value of the background gauge fields, the global U(N) symmetry breaks down to $U(1)^N$, in addition to the relation $n^\dagger n = 1$, we have $\det\Omega(x_1) = 1$. The matrix can be diagonalized,

$$\Omega(x_1) = \begin{pmatrix} e^{2\pi\mu_1+\phi_{f_1}(x_1)} & 0 & \dots & 0 \\ 0 & e^{2\pi\mu_2+\phi_{f_2}(x_1)} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{2\pi\mu_N+\phi_{f_N}(x_1)} \end{pmatrix}, \quad 0 \le \mu_1 \le \mu_2 \le \dots \le \mu_N \le 1. \qquad (4.57)$$

Here μ_i is the classical background value of the gauge fields, ϕ_{f_i} is the quantum fluctuation around the classical background. This is the gauge inequivalent choice of the gauge holonomy. The choice of μ_i is arbitary classically but it receives quantum corrections. The effecitve potential for μ_i just like the effective potential for ϕ in the QCD(adj) case, and is obtained by integrating out all other fields (KK modes)

doi:10.6342/NTU201700825

with energy scale higher then μ [33]. (we can get the one-loop effective potential for the gauge holonomy). Whether the boundary condition for fermion is thermal or spatial changes the effective potential significantly. The one-loop effective potential for the gauge holonomy for different compactification is given by,

$$V_{-}[\Omega] = \frac{2}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1+(-1)^n N_f)(|\operatorname{tr}\Omega^n|-1), \quad \text{(themal)} \tag{4.58}$$

$$V_{+}[\Omega] = (N_f-1)\frac{2}{\pi L^2}\sum_{n^2}^{\infty}(|\operatorname{tr}\Omega^n|-1), \quad \text{(spatial)} \tag{4.59}$$

The minimum of the potential in the thermal case is,

$$\Omega_{\text{thermal}} = e^{i\frac{2\pi k}{N}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \tag{4.60}$$

where k labels the position of eigenvalues. In the thermal case, the eigenvalues of the holonomy attract each other. This breaks the Z_N center symmetry.

In the spatial case, the minimum of the potential is,

$$\Omega_{\text{spatial}} = \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix}, \tag{4.61}$$

all the eigenvalues of the holonomy repel each other, this result is just like the gauge holonomy in the QCD(adj) case. The Z_N center symmetry is preserved under this holonomy. We will focus on the spatial compactification case later. The one-loop potential has a dependence on the number of flavors. For $N_f > 1$, the minumum takes the form (4.61). For $N_f = 1$, because of the theory has a $\mathcal{N} = (2,2)$ supersymmetry, the effective potential vanishes. For $N_f = 0$, since there is no difference between the thermal and spatial compactification, the center symmetry is broken and the minimum takes the form (4.60). But we can still recover the center symmetry in $N_f = 0$ and $N_f = 1$ case, if we put heavy fermions inside and integrate out them, we can get the $N_f = 0$ and $N_f = 1$ theroy with center stability.

doi:10.6342/NTU201700825

4.2.3 Free energy

Since we want to understand the theory with large L by looking at the theory with L small. We do not want the theory has a rapid phase transition when L change. The leading order of the free energy density is given by the minimum of the effective potential of gauge holonomy. For thermal case, the minimum is at $\Omega_{\text{thermal}} = \mathbf{1}$, the free energy is

$$F_{\text{thermal}} = \frac{2}{\pi\beta^2}(N-1)\sum_{n=1}^{\infty}\frac{1}{n^2}(-1+(-1)^n N_f) = -(2N-2)\frac{\pi}{6\beta^2}(1+\frac{N_f}{2}), \quad (4.62)$$

so when β is large, the theory is expected be in the confined phase with $O(N^0)$ free energy, when β is small, the theory is at the deconfined phase with $O(N^1)$ free energy. The change of the free energy is finite at finite-N and becomes a sharp phase transition at $N \to \infty$. The analysis we have done at small β does not help us to understand large β.

One the other hand, the free energy for the spatial case is,

$$F_{\text{spatial}} = (N_f - 1)\frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(|\operatorname{tr}\Omega^n| - 1). \quad (4.63)$$

The minimum of Ω is obtained by minimizing the value of $|\operatorname{tr}\Omega|$, then $|\operatorname{tr}\Omega^2|$ and all the way to $|\operatorname{tr}\Omega^{\frac{N}{2}}|$. The highest order we need to consider is $\frac{N}{2}$, higher order will not change the value of Ω. Since we minimize all the value of $|\operatorname{tr}\Omega^n|$, the summation appearing in the free energy give us a quantity of $O(N^0)$, therefore, unlike in the thermal case, there is not a rapid change of free energy.

4.2.4 Topological configurations on $\mathbb{R}^1 \times S^1$

The leading semiclassical configuraions in CP^{N-1} on $\mathbb{R}^2$ are the 2d instantons. The equations of the 2d instantons can be obtained by completing the square in the Lagrangian,

$$L = (D_\mu n)^\dagger D_\mu n = |(D_\mu \pm i\epsilon_{\mu\nu}D_\nu)n|^2 \mp i\epsilon_{\mu\nu}\partial_\mu(n^\dagger\partial_\nu n). \quad (4.64)$$

The self-dual equations are,

$$D_\mu n = \mp i\epsilon_{\mu\nu}D_\nu n. \quad (4.65)$$

doi:10.6342/NTU201700825

The instanton solutions saturate the BPS bound,

$$S = \frac{2}{g^2} | \mp i\epsilon_{\mu\nu} \int (D_\mu n)^\dagger D_\nu n| \geq \frac{4\pi}{g^2}|Q|, \tag{4.66}$$

and satisfy the constraint,

$$D_\mu n = 0, \quad \text{at } |x| \to \infty, \tag{4.67}$$

where Q is the topological charge which we have defined in (4.52). For an instanton solution with topological charge 1, the actions is $S_I = \frac{4\pi}{g^2}$. It is convenient to use homogeneous coordinates for the fields and complex coordinates to describe the instantons.

$$n = \frac{v}{|v|}, \quad A_\mu = \frac{i}{2}(\frac{v^\dagger \partial_\mu - \partial_\mu v^\dagger v}{v^\dagger v}), \quad z = x_1 + ix_2. \tag{4.68}$$

Equation (4.65) becomes to

$$\partial_\mu v(x_1, x_2) = \mp i\epsilon_{\mu\nu}\partial_\nu v(x_1, x_2) \Longrightarrow \partial_z v(z, \bar{z}) = 0, \ \partial_{\bar{z}} v(z, \bar{z}) = 0, \tag{4.69}$$

which means the (anti)instanton solution is (anti)holomorphic. In general case, the instantons are polynomials in z, with maximal degrees m, where m is the topological charge and with no common roots (the solutions with common roots are gauge equivalent with no common roots). For CP^1 on $\mathbb{R}^2$, the single instanton solution is,

$$v = \begin{pmatrix} 1 \\ (z - z_0)/\rho \end{pmatrix}, \tag{4.70}$$

which has topological charge 1. There are two complex moduli parameteres, z_0 is the location of the instanton, $|\rho|$ is the size of the instanton and $\arg(\rho)$ is the phase of the instanton. In general CP^{N-1} case, there are $2N$ real moduli parameteres for 1 instanton which are correspond to $2N$ zero modes. Two parameters are the position, one is the size and $2N-3$ are the internal orientational phase of the instanton. They represent to translation, dilatation and orientation respectively.

Kink-instantons in CP^1

In the compactified theory, new semiclassical configurations appear, the kink-instantons. We first discuss the kink-instantons in CP^1.

We only have two complex fields in CP^1, n_1, n_2. Where n_1, n_2 acquire a phase by holonomy when $x_2 \to x_2 + L$.

$$n(x_1, x_2 + L) = \Omega(x_1) n(x_1, x_2), \tag{4.71}$$

doi:10.6342/NTU201700825

We can use θ and ϕ to parameterize the fields.

$$n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e^{i\frac{2\pi\mu_1 x_2}{L}} & \\ & e^{i\frac{2\pi\mu_2 x_2}{L}} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin\frac{\theta}{2} \end{pmatrix}, \tag{4.72}$$

the $\theta(x_1, x_2)$ field and the $\phi(x_1, x_2)$ field are periodic in x_2. So we can write down the action,

$$S = \frac{2}{g^2} \int_{\mathbb{R}^1 \times S^1} |D_\mu n_i|^2 = \frac{1}{2g^2} \int_{\mathbb{R}^1 \times S^1} (\partial_\mu \theta)^2 + \sin^2\theta (\partial_\mu \phi + \zeta \delta_{\mu 2})^2, \tag{4.73}$$

where we denote $\zeta = \frac{2\pi}{L}(\mu_2 - \mu_1)$. Because ϕ and θ are both periodic, we can fourier decompose them and reduce the theory to a 1d theory.

$$\theta(x_1, x_2) = \sum_n \theta(x_1) e^{i\frac{2\pi n x_2}{L}}, \quad \phi(x_1, x_2) = \sum_n \phi(x_1) e^{i\frac{2\pi n x_2}{L}}. \tag{4.74}$$

We only need to keep the zero Kluza-Klein modes since we can take L very small, higher modes decoupled. However, the first Kluza-Klein mode is important even when $L \to 0$, we would discuss this later. The action become,

$$S_{\text{zero}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_1 \theta)^2 + \sin^2\theta (\partial_1 \phi)^2 + \zeta^2 \sin^2\theta. \tag{4.75}$$

The equations of motions are,

$$\partial_1^2 \theta - \sin\theta \cos\theta ((\partial_1 \phi)^2 + \zeta^2) = 0, \tag{4.76}$$

$$\partial_1^2 \phi + 2\partial_1 \phi \cot\theta = 0. \tag{4.77}$$

We can set $\phi =$ constant to solve the second equation. The first equaiton reduces to a 1d non-linear differential equation. This is the kink solution.

$$\partial_1^2 \theta - \zeta^2 \sin\theta \cos\theta = 0. \tag{4.78}$$

The solution of this is the Jacobi amplitude function,

$$\theta(x_1) = -\text{am}(\sqrt{-(\zeta^2 - c_1)(t + c_2)^2} | \frac{a^2}{a^2 - c_1}). \tag{4.79}$$

With the gauge holonomy we have found in previous section, we can find the action of this configuration.

$$K_1 : \quad S_1 = \frac{4\pi}{g^2} \times (\mu_1 - \mu_2) = \frac{S_I}{2}. \tag{4.80}$$

doi:10.6342/NTU201700825

This is a kink-instanton connecting from $\theta = 0$ to $\theta = \pi$. It also carries topological charge $Q = \frac{1}{2}$. The action of the kink is half of the 2d instanton. In CP^{N-1} case, the action of kink is $1/N$ of the 2d instanton. The action of the kink is determined by the eigenvalues of the gauge holonomy Ω. So we can only find this kind of kink soutions under spatial compactification. The kink is an interpolation between $x_1 = -\infty$ to $x_1 = \infty$.

$$K_1: \quad \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}(\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.81}$$

Where $\tilde{n}$ is defined by the periodic part of n.

$$\tilde{n}_j = e^{-i\frac{2\pi\mu_j x_2}{L}} n_j, \quad \tilde{n}(x_1, x_2 + L) = \tilde{n}(x_1, x_2). \tag{4.82}$$

The kink-instanton has two real moduli parameters, one is the position and another one is the phase. We denote the anti-kink-instanton by $\bar{K_1}$, which has opposite topological charge.

Affine Kink-instantons in CP^1

In CP^1 case, besides the kink-instanton which arises from the zero KK-modes, there is another kink-instanton which arises from the first KK-modes, which is called the affine kink-instanton.

In order to see the affine kink-instanton, we keep the first KK-mode in the fields,

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} e^{i(-\frac{\phi}{2} + \frac{2\pi\mu_1 x_2}{L})} \cos\frac{\theta}{2} \\ e^{i(\frac{\phi}{2} + \frac{2\pi(-1+\mu_2)x_2}{L})} \sin\frac{\theta}{2} \end{pmatrix} \tag{4.83}$$

. It carries an extra unit of KK-momentum in x_2 direction. After substitute this into the CP^1 action, we do dimensional reduction to 1d theory. We get,

$$S_{\text{first}} = \frac{L}{2g^2} \int_{\mathbb{R}} (\partial_t \theta)^2 + \sin^2\theta (\partial_t \phi)^2 + \zeta'^2 \sin^2\theta, \quad \zeta' = \frac{2\pi(\mu_2 - \mu_1 - 1)}{L}. \tag{4.84}$$

We can also use the equation of motion to find the kink-instanton solution, the solution interpolates from $\theta = \pi$ to $\theta = 0$,

$$K_2: \quad \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix}(\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.85}$$

and has the same topological charge and action as K_1,

$$Q = 1 - \mu_2 + \mu_1 = \frac{1}{2}, \quad S_2 = \frac{4\pi}{g^2} \times (1 - \mu_2 + \mu_1) = \frac{S_I}{2}. \tag{4.86}$$

doi:10.6342/NTU201700825

K_1 and K_2 have the same topological charge and action, but they interpolate $-\infty$ and ∞ in an opposite way. It is important to note that K_2 is not $\bar{K}_1$, K_2 carries positive topological charge.

The differences between K_1 and K_2 can be seen easily by changing the value of $\mu_1 - \mu_2$. The actions of K_1 and $\bar{K}_1$ are the same, the actions of K_2 and $\bar{K}_2$ are also the same, but they are not equal to each other any more. We have 2 types of kinks in CP^1 model, in CP^{N-1} model, we have N types of kinks.

Kink-instantons in CP^{N-1}

The kink-instantons in CP^{N-1} model can be constructed by embedding the kink-instantons in CP^1 to CP^{N-1}. We use the complexified hyper-spherical coordinates to describe the fields,

$$\tilde{n} = \begin{pmatrix} e^{i\phi_1} & 0 & \dots & 0 \\ 0 & e^{i\phi_2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i\phi_N} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \vdots \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\dots\sin\frac{\theta_{N-1}}{2} \end{pmatrix}, \tag{4.87}$$

where $\phi_i \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$. In CP^{N-1} model, there are N complex fields, so there should be $2N$ real fields. Because we have the constraint $n^\dagger n = 1$ and the gauge symmetry, the degree of freedom of CP^{N-1} model is $2N - 2$. Here the fields $(\theta_1 \dots \theta_{N-1})$ are independent, and one of $(\phi_1 \dots \phi_N)$ can be gauged away. Including the effect of the gauge holonomy, the fields are,

$$n = \begin{pmatrix} e^{i\frac{2\pi\mu_1 x_2}{L}} & 0 & \dots & 0 \\ 0 & e^{i\frac{2\pi\mu_2 x_2}{L}} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i\frac{2\pi\mu_N x_2}{L}} \end{pmatrix} \tilde{n}. \tag{4.88}$$

The fundamental and affine kinks in CP^{N-1} are corresponding to the simple roots and affine roots of $SU(N)$ algebra. By choosing the fields to be the follwing values, we can see the existence of the kink-instantons in CP^1 in CP^{N-1} model.

$$\theta_1 = \dots = \theta_{k-1} = \pi,\ \theta_k = \theta_k(x_1),\ \theta_{k+1} = 0,\ \theta_{k+2} = \dots = \theta_{N-1} = 0. \tag{4.89}$$

Substituting this into the action (4.73), then the do dimensional reduction, we get,

$$S = \frac{L}{2g^2}\int_{\mathbb{R}} (\partial_1\theta_k)^2 + \sin^2\theta_k[\partial_1(\phi_{k+1} - \phi_k)]^2 + (\frac{2\pi(\mu_{k+1} - \mu_k)}{L})^2 \sin^2\theta_k, \tag{4.90}$$

doi:10.6342/NTU201700825

which is the same as (4.75). The field θ is replaced by θ_k, $(\phi_{k+1} - \phi_k) = \phi$ and $\frac{2\pi(\mu_{k+1}-\mu_k)}{K} = \zeta$. The K_k kink configuration is,

$$K_k: \quad \tilde{n}(-\infty) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \equiv e_k, \quad \tilde{n}(\infty) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \equiv e_{k+1}, \tag{4.91}$$

where $e_{N+1} \equiv e_1,\ k = 1, \ldots, N.$ $e_{k+1} - e_k$ are associated with the simple roots and the affine root of the $SU(N)$ algebra. The kink events can be characterized by them,

$$K_k: \quad \tilde{n}(\infty) - \tilde{n}(-\infty) = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_{k+1} - e_k \equiv \alpha_k. \tag{4.92}$$

Also, the action of the kink is given by,

$$K_k: \quad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N}, \tag{4.93}$$

where we used the spatial compactified center stable gauge holonomy (4.61). The action of the kinks is $\frac{1}{N}$ of the 2d instantons in CP^{N-1} model. This is an important point, since the renormalon singularities are closer to the origin then the instanon singularities by a factor $\frac{1}{\beta_0}$ on the Borel t plane. The first coefficient β_0 of the β function is N in CP^{N-1} model. The kink-instanton configurations can give the same order of singularities on the Borel plane as the renormalons. We deduce that the kinks(or the combination of the kinks) are the semiclassical configurations which may correspond to the renormalons in CP^{N-1} model.

Topological molecules' amplitude

We have seen that the kink events can be labeled by,

$$K_j: \tilde{n}(\infty) - \tilde{n}(-\infty) = \alpha_j, \quad \alpha_j \in \Gamma_r^\vee \tag{4.94}$$

doi:10.6342/NTU201700825

where $\Gamma_r^\vee$ is the co-root latice of $SU(N)$. The amplitudes of these kink events are given by,

$$K_j = \exp\left[-\frac{S_I}{N}\right], \quad S_I = \frac{4\pi}{g^2} - i\Theta, \quad j = 1, \ldots, N. \tag{4.95}$$

Since the theta angle Θ is of period 2π, $\frac{\Theta}{N}$ has N different values. For a single kink-instanton amplitude, it is a multi-branched quantity,

$$K_j = \exp\left[-\frac{4\pi}{g^2 N} + i\frac{\Theta + 2\pi k}{N}\right], \quad k = 1, \ldots, N. \tag{4.96}$$

We can include the interactions by writing the kink amplitudes as,

$$K_j = e^{-\alpha_j \cdot Y}, \quad j = 1, \ldots, N-1, \tag{4.97}$$

$$K_N = \eta e^{-\alpha_N \cdot Y}, \quad \eta = e^{-\frac{4\pi}{g^2} + i\Theta}, \tag{4.98}$$

where

$$\langle K_j \rangle = \left\langle e^{-\alpha_j \cdot Y} \right\rangle = e^{-\frac{4\pi}{g^2}(\mu_{j+1} - \mu_j)}. \tag{4.99}$$

Y is a N-component complex field, it can be expressed by its real part and imaginary part,

$$Y(x_1) = \operatorname{Re} Y(x_1) - i \operatorname{Im} Y(x_1), \tag{4.100}$$

the real part $\operatorname{Re} Y(x_1)$ is the N-component sigma model connection,

$$\operatorname{Re} Y(x_1) = \frac{4\pi}{g^2}\{\mu_1, \mu_2, \ldots, \mu_N\}. \tag{4.101}$$

The imaginary part $\operatorname{Im} Y(x_1)$ accounts for the induced interactions. When the theory has $N_f \geq 1$, the kink amplitudes will be modified, each elementary kink-instanton carries $2N_f$ fermionic zero modes [34, 35]. The amplitudes of the kink-instantons are given by,

$$K_j = e^{-\alpha_j \cdot Y} \det_{f,f'}[(\alpha_j \cdot \psi_-^f)(\alpha_j \cdot \psi_+^{f'})], \quad j = 1, \ldots, N-1, \tag{4.102}$$

$$K_N = \eta e^{-\alpha_N \cdot Y} \det_{f,f'}[(\alpha_N \cdot \psi_-^f)(\alpha_N \cdot \psi_+^{f'})], \quad \eta = e^{-\frac{4\pi}{g^2} + i\Theta}. \tag{4.103}$$

A kink-anti-kink event is a bion, which is labeled by $K_i \bar{K}_i = [B_{ii}]$, it has no topological charge and can not be distinguished from the vacuum. We should take them

doi:10.6342/NTU201700825

into account when we do path integral. For a kink K_i and an anti-kink $\bar{K}_i$ separated by a distance τ, the interatcion term induced by bosonic exchange is given by,

$$S_{\text{int}}(\tau) = -8\zeta \frac{\alpha_i \cdot \alpha_i}{g^2} e^{-\zeta\tau}, \quad \zeta \equiv \frac{2\pi(\mu_{i+1} - \mu_i)}{L} = \frac{2\pi}{LN}, \tag{4.104}$$

where for the fermion zero-mode exchange, the interaction is,

$$\left\langle \prod_{f=1}^{N_f} [\alpha_i(\psi_f)]^2 (t - \tau/2) \prod_{f=1}^{N_f} [\alpha_i(\bar{\psi}_f)]^2 (t + \tau/2) \right\rangle = (\frac{\alpha_i \cdot \alpha_i}{2})^{2N_f} (\frac{g^2}{2L})^{2N_f} e^{-2N_f \zeta\tau}. \tag{4.105}$$

Therefore, the bion amplitude is given by,

$$A_{ii} = A_i A_i (\frac{\alpha_i \cdot \alpha_i}{2})^{2N_f} (\frac{g^2}{2L})^{2N_f} 2 \int_0^\infty d\tau \; e^{-V_{\text{eff}}^{ii}(\tau)}, \tag{4.106}$$

which involves the integral over the quasi-zero mode τ. The effective potential between the two kinks is,

$$V_{\text{eff}}^{ii}(\tau) = -8\zeta \frac{\alpha_i \cdot \alpha_i}{g^2} e^{-\zeta\tau} + 2N_f \zeta\tau. \tag{4.107}$$

We use the Lie algebra convention $\alpha_i \cdot \alpha_i = 1$, the qusai zero mode integral is given by,

$$I(g^2) = \int_0^\infty d\tau \; e^{-V_{\text{eff}}^{ii}(\tau)} = \int_0^\infty d\tau \; \exp\left[\frac{8\zeta}{g^2} e^{-\zeta\tau} - 2N_f \zeta\tau\right]. \tag{4.108}$$

However, the interaction between the kinks is attractive, the integral is dominated at small τ and the kink-anti-kink configuration become meaningless. As we have encountered in QM case, we need to do analytic continuation to rotate $g^2 \to g^2 e^{\pm i\pi}$. Then the interaction has a repulsive component. We do the integration for $-g^2$ and rotate back to the original g^2. The result is,

$$I(g^2, N_f) \to \tilde{I}(-g^2, N_f) = (-\frac{g^2 N}{8\pi})^{2N_f} \Gamma(2N_f), \tag{4.109}$$

where $\tilde{I}(g^2, N_f)$ is,

$$\tilde{I}(g^2, N_f) = \int_0^\infty d\tau \; \exp\left[-\frac{8\zeta}{g^2} e^{-\zeta\tau} - 2N_f \zeta\tau\right] = (\frac{g^2 N}{8\pi})^{2N_f} \Gamma(2N_f). \tag{4.110}$$

For $N_f \geq 1$, the result is real and unambiguous, but for $N_f = 0$, taking $N_f = \epsilon \to 0$

doi:10.6342/NTU201700825

and subtracting the pole term[7], we find,

$$\tilde{I}(g^2, N_f = \epsilon) = (\frac{g^2 N}{8\pi})^{2\epsilon}\Gamma(2\epsilon) = \frac{1}{2\epsilon} + \log\left(\frac{g^2 N}{8\pi} - \gamma\right) + O(\epsilon) \to \log\left(\frac{g^2 N}{8\pi} - \gamma\right) \tag{4.111}$$

$$I(g^2, N_f = 0) = \log\left(-\frac{g^2 N}{8\pi} - \gamma\right) = \tilde{I}(g^2) \pm i\pi. \tag{4.112}$$

Therefore, for $N_f = 0$, the bion amplitude has an ambiguity,

$$[B_{ii}]_{\theta^\pm} = [K_i \bar{K}_i]_{\theta^\pm} = (\log\left(\frac{g^2 N}{8\pi}\right) - \gamma) 2A_i^2 e^{-2S_k} \pm i\pi 2A_i^2 e^{-2S_k}, \tag{4.113}$$

where $\theta^\pm$ denotes the direction we chose to do analytic continuation, $S_k = \frac{S_I}{N}$ is the action of the kink-instanton and γ is the Euler-Mascheroni constant. So there is a non-perturbative ambiguous imaginary part arising from the bion confiugration in the pure bosonic theory, whereas for $N_f \geq 1$, there is no ambiguity in the bion configuration.

Let's focus on the next leading topological molecules, the bion-anti-bion molecules $[B_{ij}B_{ji}]$. The quasi-zero mode integrals is given by,

$$I(g^2) = \int_0^\infty d\tau \ \exp(V(\tau)), \quad V(\tau) = (\mu_B, \mu_B)\frac{8\zeta}{g^2} e^{-\zeta\tau}, \tag{4.114}$$

where $\mu_B := \alpha_i - \alpha_j \in \Gamma_r^\vee$. Again, this integral is dominated by small τ and we need to do regularization. It gives us an amplitude of the form,

$$[B_{ij}\bar{B}_{ij}]_{\theta^\pm} = \mathrm{Re}[B_{ij}\bar{B}_{ij}] + i\,\mathrm{Im}[B_{ij}\bar{B}_{ij}]_{\theta^\pm} \sim e^{-4S_k} \pm i\pi e^{-4S_k}. \tag{4.115}$$

This is the leading ambiguity in the theory with $N_f \geq 1$ fermions.

4.2.5 IR renormalons and bions

We have seen that when QCD(adj) is compactified in a paticular way, there are new semiclassical configurations in it. Those new semiclassical configurations (monopole-instantons) form new topological molecules (bions, bion-anti-bion pairs). Some of the topological molecules ($[B_{ij}B_{ji}]$, $[B_{ij}B_{jk}B_{ki}]$) have ambiguous imaginary parts, just like the instanton-anti-instanton pair in QM. We have conjectured that these

[7] Actually, the quasi-zero mode integral is divergent. The divergent term is due to the large separation of the kink and anti-kink, the double-counting of the uncorrelated $[K] - [\bar{K}]$ events. It should be subtracted off.

doi:10.6342/NTU201700825

imaginary parts can cancel the imaginary parts of the IR renormalons in non-compactified QCD(adj). However, the order of the imaginary part of the IR renormalon is not the same as the topological molecules, so it is impossible for them to cancel each other.

In CP^{N-1} model, by similar procedure, we found new semiclassical configurations (kinks). These kinks can form bions ($[B_{ii}] = K_i\bar{K}_i$) and the combinations of the bions have ambiguous imaginary parts. They give us imaginary parts of order $e^{-\frac{S_I}{N}}$ and is of the same order as the imaginary parts of the IR renormalons. Therefore, the ambiguity we have found in compactified theory may cancel the ambiguity of the IR renormalons in non-compact theory.

Let's start from the perturbation series in CP^{N-1} model. We denote asymptotic perturbative series in CP^{N-1} model by $P(g^2)$,

$$P(g^2) = \sum_{n=0}^{\infty} a_n g^{2n}. \tag{4.116}$$

We define the Borel transform of $P(g^2)$ by $B_P(g^2)$,

$$B_P(g^2) := \sum_{n=0}^{\infty} \frac{a_n}{n!} g^{2n}. \tag{4.117}$$

And we define the Borel resummation of $P(g^2)$ by $\mathbb{B}(g^2)$[8],

$$\mathbb{B}(g^2) = \int_0^{\infty} e^{-t} B_P(g^2 t) dt \tag{4.118}$$

If the function $B_P(g^2 t)$ has no pole on the positive real axis, then this integration can be done and $\mathbb{B}(g^2)$ is real. If there are poles lying on the positive real axis, the series is non-Borel summable. If we still want to define the sum, we need to change the integration contour to avoid the poles. The integration has ambiguities, this is called the lateral Borel sum. The ambiguities depend on whether the choice of the integration path is above or below the poles,

$$\mathbb{B}(g^2)_{\theta\pm} = \operatorname{Re}\mathbb{B}(g^2) \pm i \operatorname{Im}\mathbb{B}(g^2). \tag{4.119}$$

[8]We often define the divergent series $P(g^2)$ as its Borel resummation $\mathbb{B}(g^2)$, so $P(g^2)$ and $\mathbb{B}(g^2)$ are the same thing

doi:10.6342/NTU201700825

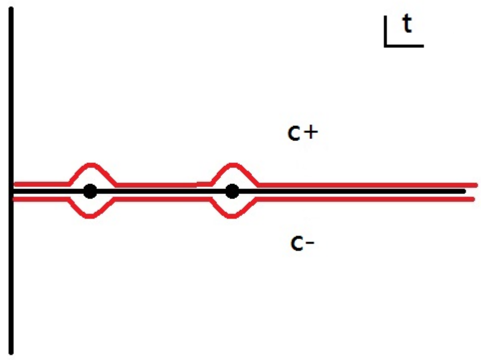


Figure 9: Lateral Borel sums, the different choices of the path give us different results.

In CP^{N-1} model on $\mathbb{R}^2$, the leading imaginary part of the Borel sum is of the form [36, 37],

$$\operatorname{Im}\mathbb{B}(g^2) \sim e^{-\frac{2nS_I}{\beta_0}} \sim \pi e^{-\frac{2nS_I}{N}}, \quad n = 2, 3, \ldots, \tag{4.120}$$

which comes from the IR renormalons. Let's recall the semiclasscial configurations we have found in CP^{N-1} on $\mathbb{R} \times S^1$; we have N types of kinks $K_i,\ i = 1, 2 \ldots N$. A kink-anti-kink pair forms a bion,

$$K_i \bar{K}_i = [B_{ii}], \tag{4.121}$$

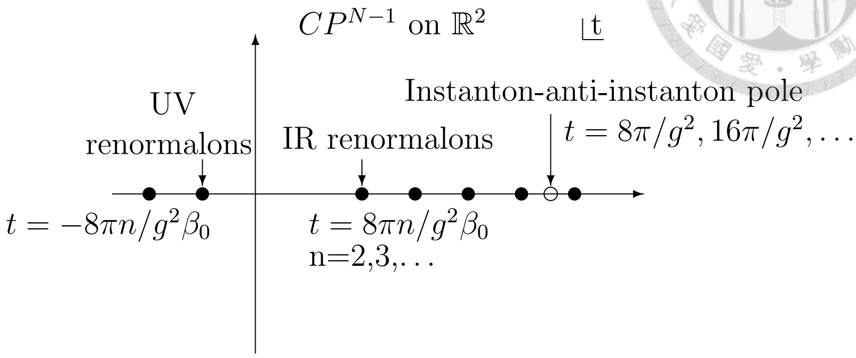
For $N_f \geq 1$, since kink configuration carries fermionic zero modes, the bion amplitude is real and it has no ambiguity. The leading ambiguity appears at the 4-th order, the bion-anti-bion amplitude. The ambiguity of the bion-anti-bion amplitude is of order $e^{-\frac{4S_I}{N}}$, and the leading IR renormalon is also of order $e^{-\frac{4S_I}{N}}$. So if the imaginary part of the bion-anti-bion in the compactified theory and the imaginary part of the IR renormalon in the original theory cancels each other,

$$\operatorname{Im}\mathbb{B}(g^2)_{\theta^\pm} \text{ on } \mathbb{R}^2 + \operatorname{Im}[B_{ij}\bar{B}_{ij}]_{\theta^\pm} \text{ on } \mathbb{R}^1 \times S^1 = 0, \text{to order of } e^{-\frac{4S_I}{N}}, \tag{4.122}$$

then there must be a deep relation between the CP^{N-1} on $\mathbb{R}^2$ and CP^{N-1} on $\mathbb{R}^1 \times S^1$ with periodic boundary condition on fermions.

For $N_f = 0$, there is an ambiguous imaginary part in bion amplitudes. Therefore, in a pure bosonic CP^{N-1} theory, the leading ambiguity is of order $e^{-\frac{2S_I}{N}}$, but the closest IR renormalon on $\mathbb{R}^2$ is of order $e^{-\frac{4S_I}{N}}$ so there is a mismatch. There are

doi:10.6342/NTU201700825



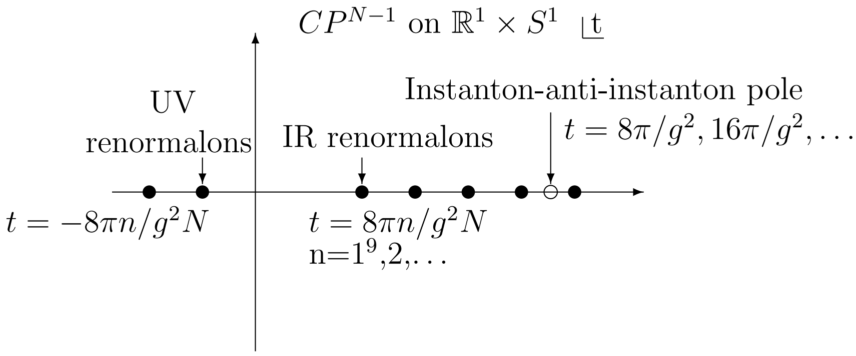


Figure 10: Upper figure: The conjectured pole structure of the Borel plane for CP^{N-1} on $\mathbb{R}^2$. Lower figure: The semiclassical singularities corresponding to the topological molecules in CP^{N-1} on small $\mathbb{R} \times S^1$. There is an extra singularity closer to the origin for $N_f = 0$. For $N_f \geq 1$, the pole structure of the topological molecules and the IR renormalons concide. Although the analysis is done in weakly coupled region and the IR renormalons come from strongly coupled region, we conjectured that the pole structure does not change significantly. This is an evidence to show that bion-anti-bion pairs on $\mathbb{R}^1 \times S^1$ may be the semiclassical realization of the IR renormalons on $\mathbb{R}^2$.

doi:10.6342/NTU201700825

several possibilities. First possibility is that the IR renormalons can not always be realized by semiclassical configurations. Namely, there would be two types of singularities on the Borel plane, one type is the instanton pole which can be realized semiclassically, the other type is the IR renormalon which can not be realized by the semiclassical configurations. Second possibility is that the conjectured Borel plane structure for $N_f = 0$ CP^{N-1} model is wrong, there is also an extra singularity appearing closer to the origin when $N_f = 0$, but we have not found this kind of thing happen yet. Third, some kind of phase transition may have happend when we change the length of the compactified direction from small-L limit to large-L, the analysis we have done in weak coupling region does not work for strong coupling.

Let's see the cancellation to the leading order in the compactified theory, in the small-L limit, the 2-d field theory can be reduced to a 1-d quantum mechanics, the asymptotic perturbative expansion of the ground state energy in units of the natural frequency ω in CP^{N-1} model on $\mathbb{R} \times S^1$ is given by [13, 38],

$$P(g^2) = E_0(g^2)\omega^{-1} = \sum_{n=0}^{\infty} a_n g^{2n}, \quad a_n \sim -\frac{2}{\pi}(\frac{1}{4\omega})^n n!(1 - \frac{5}{2n} + O(n^{-2}). \tag{4.123}$$

This is a non-alternating series and hence non-Borel summable. The Borel transform of it is,

$$B_P(g^2) = -\frac{2}{\pi}\sum_{n=0}^{\infty}(\frac{g^2}{4\omega})^n = -\frac{2}{\pi}\frac{1}{1-\frac{g^2}{4\omega}}. \tag{4.124}$$

The lateral Borel sums are,

$$\mathbb{B}(g^2)_{\theta\pm} = \int_{0,C^\pm}^{\infty} B_P(g^2 t)\ e^{-t} dt = \mathrm{Re}\,\mathbb{B}(g^2) \mp i\frac{8\omega}{g^2}e^{-\frac{4\omega}{g^2}} \tag{4.125}$$

$$= \mathrm{Re}\,\mathbb{B}(g^2) \mp i\frac{16\pi}{g^2 N}e^{-\frac{8\pi}{g^2 N}}. \tag{4.126}$$

On the otherhand, the leading ambiguity comes from the bion amplitude,

$$[B_{ii}]_{\theta\pm} = \mathrm{Re}[B_{ii}] + \mathrm{Im}[B_{ii}]_{\theta\pm} \tag{4.127}$$

$$= (\log\left(\frac{g^2 N}{8\pi}\right) - \gamma)\frac{16}{g^2 N}e^{-\frac{8\pi}{g^2 N}} \pm i\frac{16\pi}{g^2 N}e^{-\frac{8\pi}{g^2 N}}. \tag{4.128}$$

Remarkably, they canel each other exactly,

$$\mathrm{Im}\,\mathbb{B}_{\theta\pm} + \mathrm{Im}[B_{ii}]_{\theta\pm} = 0,\ \text{to the order of } e^{-\frac{2S_I}{N}}. \tag{4.129}$$

doi:10.6342/NTU201700825

This is an evidence that the resurgence relation works in the compactified CP^{N-1} model with periodic fermions. The exact resurgence relation in CP^1 QM has been found by Toshiaki Fujimori,Syo Kamata,Tatsuhiro Misumi,Muneto Nitta and Norisuke Sakai [25, 39], it seems the relation should also work in CP^{N-1} QM.

5 Conclusion and Future directions

Resurgence theory works well in quantum mechanics. We can define observables in quantum mechanics non-perturbatively by the construction of trans-series. By finding all the non-perturbative saddles in path integral, the physical quantities are well-defined. The coefficients of the expansion still need to be taken care of since we do not know how to express all the coefficients. Some evidences of the non-trivial relation between the coefficients of different saddle points have been found in quantum mechanics [10, 11].

In quantum field theories, because the coupling constant runs with the scale, there are renormalons closer to the origin by a factor of order N_c then the instantons. In some theories after compactification, we can find new semiclassical configurations which has the action less then the instantons by a factor of order N_c too. However, only in few theories, the position of the singularity produced by the new semiclassical configuration concide with the IR renormalon. In addition, the origin of the renormalons and the instanon-anti-instanton singularities seems to be so different. One comes from the factorial growth of only one set of Feymann diagrams, another one comes from the factorial growth of various Feymann diagrams. As we have seen in section 2, the renormalon divergence comes from the high momentum and low momentum contribution to the Adler function, which is very different from instanton divergence. We can ask, can renormalon really be realized by semiclassical configurations, just like instanton-anti-instanton singularities? If it can, then there must be a deep connection between the running of the coupling and those semiclassical configurations. Unfortunately we have not seen this kind of relation yet. Perhaps this kind of semiclassical realization can only be used in quantum mechanics case.

Let's summarize the problems about the semiclassical realization to renormalons. The first is why the position of the newly found semiclassical configurations do not

doi:10.6342/NTU201700825

concide with the IR renormalons in some theories. If the position of the renormalons would change with compactified radius, then how to show that? Second, in some theories, the compactification of one direction to small radius results in phase transition. We cannot continuously change the radius from small to infinity, so even though we find new semiclassical configuration, they cannot correspond to the renormalons in the original theory. Third, as mentioned above, the renoamalons are obviously related to the running coupling, so the new semiclassical configuration should also related to it, but we did not find such relation until now.

Acknowledgement

I offer my sincerest gratitude to my academic advisor Kazuo Hosomichi. Thanks for your encouragement, patience and the assistance. This thesis would never have been if without you. I would also like to thank those who have taught me durung the tume that I have stayed in National Taiwan University: Heng-Yu Chen, Pei-Ming Ho,Yu-Tin Huang. Special thanks to my best friend Chia-Wei Chen and those who have accompanied me during the two years: Chih-Kai Chang, Tsung-Hsuan Tsai, Ta-Yu Chiang, En-Jui Kuo and many others.

Appendices

A Ambiguity of Borel resummation

Consider this factorially divergent series,

$$P(g) = \sum_{n=0}^{\infty} n! g^n \tag{A.1}$$

The sign of this series is non-alternating so it is not Borel summable.. After Borel transformed, it becomes

$$B_P(g) = \sum_{n=0}^{\infty} g^n = \frac{1}{1-g} \tag{A.2}$$

Now the original series $P(g)$ can be represented by $B_P(g)$

$$P(g) = \int_0^{\infty} dt\ e^{-t} B_P(gt) = \int_0^{\infty} dt\ e^{-t} \frac{1}{1-gt} \tag{A.3}$$

doi:10.6342/NTU201700825

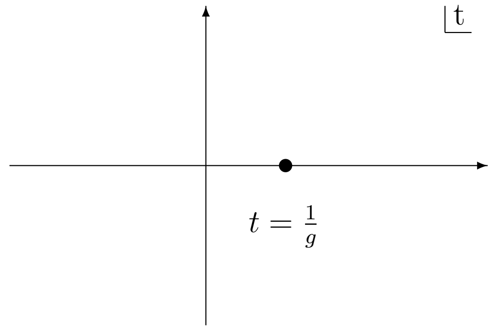


Figure 11: The pole on the Borel complex t plane

It has a pole on the positive real axis. In order to compute the integral, we need to do analytic continuation to avoid the pole. We can choose the path to be C_+ or C_-.

$$P(g + i\epsilon) = \int_{C_+} dt\ e^{-t} \frac{1}{1 - gt} = \operatorname{Re} P(g) + i \operatorname{Im} P(g), \tag{A.4}$$

$$P(g - i\epsilon) = \int_{C_-} dt\ e^{-t} \frac{1}{1 - gt} = \operatorname{Re} P(g) - i \operatorname{Im} P(g). \tag{A.5}$$

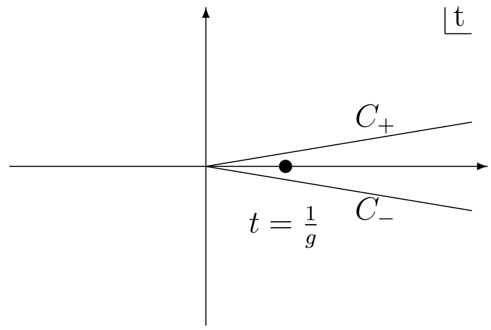


Figure 12: The different choice of paths

Thus the integral is ambiguous, it has two possible different values. The discontinuity is,

$$P(g + i\epsilon) - P(g - i\epsilon) = 2i \operatorname{Im} P(g) = \int_{C_+ - C_-} dt\ e^{-t} \frac{1}{1 - gt}. \tag{A.6}$$

This is a contour integral around the pole at $t = \frac{1}{g}$, the value is given by $2\pi i$ multiplying the residue,

$$\int_{C_+ - C_-} dt\ e^{-t} \frac{1}{1 - gt} = \frac{2\pi i}{g} e^{-\frac{1}{g}}. \tag{A.7}$$

Therefore, the ambiguous imaginary part of the series P(g) is $\pm i \frac{\pi}{g} e^{-1/g}$.

doi:10.6342/NTU201700825

B Computation of the functional determinant in Quantum Mechanics

Consider an operator $M(t_1, t_2)$ given by (3.33) with some boundary conditions which depend on the boundary condition of the path integral. The determinant of M is realized as the product over its eigenvalues. Let q_n be the orthonormal eigenfunctions of M.

$$\int dt_2 M(t_1, t_2) q_n(t_2) = \lambda_n q_n(t_1). \tag{B.1}$$

We can write more precisely,

$$[-\frac{d^2}{dt^2} - V''(q_c(t))]q_n(t) = \lambda_n q_n(t), \tag{B.2}$$

and

$$\int dt\ q_n(t) q_m(t) = \delta_{nm}. \tag{B.3}$$

The determinant of M is,

$$\det M = \prod_n \lambda_n. \tag{B.4}$$

In general, the operator which we often want to compute can have zero modes and negative modes,

$$[-\frac{d^2}{dt^2} - V''(q_c(t))]q_0(t) = 0, \tag{B.5}$$

if $q_0(t)$ is not 0, then $q_0(t)$ is a zero mode of M.

$$[-\frac{d^2}{dt^2} - V''(q_c(t))]q_a(t) = \lambda_a q_a(t), \tag{B.6}$$

if $\lambda_a < 0$ then $q_a(t)$ is an negative mode of M.

We first discuss the issue of zero mode. Naively, the zero mode makes the functional determinant become zero. We need to be careful with this. What we want to compute is the integral,

$$\int D[q_f(t)] \exp\left[-\frac{1}{2}\int dt_1 dt_2 q_f(t_1) M(t_1, t_2) q_f(t_2)\right] = (\det M)^{-\frac{1}{2}}. \tag{B.7}$$

doi:10.6342/NTU201700825

If we expand $q_f(t)$ by its normalized eigenmodes $q_n(t)$ of M and $q_0(t)$ is the zero mode,

$$q_f(t) = \sum_{n\geq 0} c_n q_n(t), \tag{B.8}$$

then the integral become,

$$(\det M)^{-\frac{1}{2}} = \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{n\geq 0}\lambda_n c_n^2} = \int \frac{dc_0}{\sqrt{2\pi}} (\det{}' M)^{-\frac{1}{2}}, \tag{B.9}$$

where

$$\det{}' M = \prod_{n\neq 1} \lambda_n, \tag{B.10}$$

is the determinant of which zero mode is removed. The integral of c_0 results in infinity. In order to compute this, we need to know what is the zero mode $q_0(t)$. Actually, $q_0(t)$ is propotional to $\dot{q}_c(t)$. Recall that $q_c(t)$ is a solution of the EOM of the Euclidean action.

$$\ddot{q}_c(t) + V'(q_c(t)) = 0, \tag{B.11}$$

doing derivative on t, we get,

$$\frac{d^2}{dt^2}\dot{q}_c(t) + V''(q_c(t))\dot{q}_c(t) = 0, \tag{B.12}$$

thus $\dot{q}_c(t)$ is a zero mode of M. The relation between $q_0(t)$ and $\dot{q}_c(t)$ is,

$$q_0(t) = \frac{1}{||\dot{q}_c||}\dot{q}_c(t). \tag{B.13}$$

The norm is given by,

$$||\dot{q}_c(t)||^2 = \int_{-\beta/2}^{\beta/2} dt\ \dot{q}_c(t)^2, \tag{B.14}$$

we can use the energy conservation (3.19),

$$||\dot{q}_c(t)||^2 = \int_{-\beta/2}^{\beta/2} dt\ \frac{1}{2}\dot{q}_c(t)^2 - \int_{-\beta/2}^{\beta/2} dt\ V(q_c(t)) = S_c, \tag{B.15}$$

which is just the action of the instanton trajectory. Note c_0 is the collective parameter of $q_0(t)$, t_0 is the collective parameter of $\dot{q}_c(t)$. We can find the relation between c_0 and t_0. Doing variation to q_0 with respect to c_0,

$$q_0(t)\delta c_0 = \frac{1}{||\dot{q}_c(t)||}\dot{q}_c(t)\delta c_0. \tag{B.16}$$

doi:10.6342/NTU201700825

We can also do variation to $\dot{q}_c$ with respect to t_0,

$$q_0(t)\delta c_0 = \frac{1}{||\dot{q}_c(t)||}\dot{q}_c(t)\delta c_0 = \dot{q}_c(t)\delta t_0. \tag{B.17}$$

Then we can find the Jacobian of changing c_0 to t_0, which is,

$$J = \frac{\delta c_0}{\delta t_0} = S_c^{1/2}. \tag{B.18}$$

At the end, the integral of c_0 becomes,

$$\int \frac{dc_0}{\sqrt{2\pi}} = \frac{S_c^{1/2}}{\sqrt{2\pi}} \int_{-\beta/2}^{\beta/2} dt_0 = \frac{\beta S_c^{1/2}}{\sqrt{2\pi}}. \tag{B.19}$$

This is why we need to multiply this factor at Sec 3.1.1., it comes from the zero mode integration. Therefore, the determinant (B.9) becomes,

$$(\det M)^{-\frac{1}{2}} = \frac{\beta S_c^{1/2}}{\sqrt{2\pi}}(\det' M)^{-\frac{1}{2}}. \tag{B.20}$$

We have extracted the zero mode from the determinant.

The situation of the negative modes are simpler. A negative mode would make the determinant of M become negative and $(\det M)^{\frac{1}{2}}$ would become imaginary. If there is an imaginary part in the ground state energy, it means the potential is unstable. The vacuum which we do expansion is a false vacuum and it would eventually decay. So if the vancum we choose is stable, we don't need to worry about the negative mode, it will not appear in the operator M.

Appendix B.A Gelfand-Yaglom method

There are several ways to compute the functional determinant. The most intuitive way is to find all the spectrum of the operator, then doing regularization. However, it is hard to do this usually. There is one useful method to compute the functional determinant in Quantum mechanics without knowing all the spectrum. It is known as the Gelfand-Yaglom theorem [40].

Consider a second order differential equation with some boundary conditions at the interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$. The eigenfunction equation is given by,

$$M\phi(t) = [-\frac{d^2}{dt^2} + V(t)]\phi(t) = \lambda\phi(t), \tag{B.21}$$

doi:10.6342/NTU201700825

where $M = [\frac{d^2}{dt^2} + V(t)]$ is the operator which we want to compute the determinant, λ is the eigenvalue. We denote $\phi_\lambda^{(1,2)}(t)$ to be the solutions with the initial condition,

$$\begin{aligned} \phi_\lambda^{(1)}(-\beta/2) = 1, \quad \phi_\lambda^{(2)}(-\beta/2) = 0, \\ \dot{\phi}_\lambda^{(1)}(-\beta/2) = 0, \quad \dot{\phi}_\lambda^{(2)}(-\beta/2) = 1. \end{aligned} \tag{B.22}$$

These two solutions do not need to satisfy the boundary condition, since the ODE is second order, we can always find solutions satisfy (B.22). We can use them to construct a matrix,

$$E_\lambda(t) = \begin{pmatrix} \phi_\lambda^{(1)}(t) & \phi_\lambda^{(2)}(t) \\ \dot{\phi}_\lambda^{(1)}(t) & \dot{\phi}_\lambda^{(2)}(t) \end{pmatrix}. \tag{B.23}$$

Every solutions of the eigenfunction equation with arbitrary initial condition can be construct by the matrix $E_\lambda(t)$,

$$\begin{pmatrix} \phi_\lambda(t) \\ \dot{\phi}_\lambda(t) \end{pmatrix} = E_\lambda(t) \begin{pmatrix} \phi_\lambda(-\frac{\beta}{2}) \\ \dot{\phi}_\lambda(-\frac{\beta}{2}) \end{pmatrix}. \tag{B.24}$$

Now we can write down the most general boundary condition for the eigenvalue problem,

$$A \begin{pmatrix} \phi_\lambda(-\frac{\beta}{2}) \\ \dot{\phi}_\lambda(-\frac{\beta}{2}) \end{pmatrix} + B \begin{pmatrix} \phi_\lambda(\frac{\beta}{2}) \\ \dot{\phi}_\lambda(\frac{\beta}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{B.25}$$

where A and B are matrices which depend on the boundary condition. For example,

Drichlet: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Neumann: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Periodic: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Antiperiodic: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We can use (B.24) to rewrite (B.25), then we find,

$$[A + BE_\lambda(\frac{\beta}{2})] \begin{pmatrix} \phi_\lambda(-\frac{\beta}{2}) \\ \dot{\phi}_\lambda(-\frac{\beta}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{B.26}$$

doi:10.6342/NTU201700825

So if this equation is satisfied for some special value of λ, there is a solution of this eigenvalue problem and this λ is the eigenvalue. The condition on λ is,

$$\det\Big[A + BE_\lambda(\frac{\beta}{2})\Big] = 0. \tag{B.27}$$

Since $\det\big[A + BE_\lambda(\frac{\beta}{2})\big]$ equal to 0 only when λ is equal to the eigenvalue, we can write,

$$\det\Big[A + BE_\lambda(\frac{\beta}{2})\Big] = \prod_i(\lambda_i - \lambda) = \det(M - \lambda), \tag{B.28}$$

so if we want to compute the determinant of M, we only need to construct the matrix $E_\lambda(t)$. We do not need to know all the spectrum of M. If there are zero modes in M and we want to remove them, we just write,

$$\det{}' M = -\frac{\partial}{\partial\lambda}\det(M - \lambda)|_{\lambda=0}. \tag{B.29}$$

If there are more than one zero modes, we just need to do more derivatives.

Let's take the periodic boundary condition as an example. First we note $A = \mathbf{1}$ and $B = -\mathbf{1}$ then the determinant become,

$$\det\Big[1 - E_\lambda(\frac{\beta}{2})\Big] = 1 - \mathrm{Tr}\Big(E_\lambda(\frac{\beta}{2})\Big) + \det\Big[E_\lambda(\frac{\beta}{2})\Big]. \tag{B.30}$$

Since the Wronskian is constant,

$$\det\Big[E_\lambda(\frac{\beta}{2})\Big] = 1. \tag{B.31}$$

The condition of λ become,

$$\mathrm{Tr}\Big(1 - E_\lambda(\frac{\beta}{2})\Big) = 2 - \phi_\lambda^{(1)}(\frac{\beta}{2}) - \dot{\phi}_\lambda^{(2)}(\frac{\beta}{2}) = 0, \tag{B.32}$$

and

$$\det M = [2 - \phi_\lambda^{(1)}(\frac{\beta}{2}) - \dot{\phi}_\lambda^{(2)}(\frac{\beta}{2})]|_{\lambda=0}. \tag{B.33}$$

So we only need to find $\phi_\lambda^{(1,2)}(t)$ for the operator M. It is easy to find for λ equal to 0. We need to solve

$$[-\frac{d^2}{dt^2} + V(t)]\phi_0(t) = 0. \tag{B.34}$$

The two linear independent zero modes are given by,

$$\phi_0(t) = A\dot{q}_c(t) + B\dot{q}_c(t)\int_{-\beta/2}^{t} dt' \frac{1}{\dot{q}_c(t')^2}. \tag{B.35}$$

doi:10.6342/NTU201700825

We have known that $\dot{q}_c(t)$ is one of the zero mode solution of M. The other one can be easily found by assuming the solution is $\dot{q}_c(t)a(t)$. Solve $a(t)$ can give us the result (B.35). We can translate t_0 to make $\ddot{q}_c(-\beta/2) = 0$. Then we can find $\phi_\lambda^{(1,2)}(t)$ for $\lambda = 0$ are,

$$\phi_0^{(1)}(t) = \frac{\dot{q}_c(t)}{\dot{q}_c(-\beta/2)}, \quad \phi_0^{(2)}(t) = \dot{q}_c(-\beta/2)\dot{q}_c(t)\int_{-\beta/2}^{t} dt' \frac{1}{\dot{q}_c(t')^2}. \tag{B.36}$$

And we obtain the determinant is,

$$\det M = [2 - \phi_0^{(1)}(\frac{\beta}{2}) - \dot{\phi}_0^{(2)}(\frac{\beta}{2})], \tag{B.37}$$

for periodic boundary condition. For other different boundary conditions, we just need to change the matrices A and B.

When we need to remove the zero mode in $\det M$, we need to compute $\phi_\lambda^{(1,2}(t)$ to first λ order. We can use $\phi_0^{(1,2}$ and the Green's function to construct it. To first order, it is

$$\phi_\lambda^{(1,2)}(t) = \phi_0^{(1,2)}(t) + \lambda[\phi_0^{(1)}(t)\int_{-\beta/2}^{t} dt'\phi_0^{(2)}(t')\phi_0^{(1,2)}(t') - \phi_0^{(2)}(t)\int_{-\beta/2}^{t} dt'\phi_0^{(1)}(t')\phi_0^{(1,2)}(t')] + O(\lambda^2). \tag{B.38}$$

We find,

$$\det{}'M = -\frac{\partial}{\partial\lambda}\det M|_{\lambda=0} = -\frac{\partial}{\partial\lambda}[2 - \phi_\lambda^{(1)}(\frac{\beta}{2}) - \dot{\phi}_\lambda^{(2)}(\frac{\beta}{2})]|_{\lambda=0}. \tag{B.39}$$

Let's use the operator M appear in section (3.1.1) as an example. We want to compute (3.46). Note we need to use anti-periodic boundary condition. The operator is given by,

$$M(t) = -\frac{d^2}{dt^2} + 1 - \frac{3}{2\cosh^2(t/2)}, \tag{B.40}$$

there is one zero mode inside this operator. From (B.28) and (B.29), we find,

$$\det{}'M = -\frac{\partial}{\partial\lambda}[\phi_\lambda^{(1)}(t) + \dot{\phi}_\lambda^{(2)}(t)]|_{\lambda=0}, \tag{B.41}$$

it can be rewritten by $\dot{q}_c(t)$,

$$\det{}'M = \frac{1}{2}\int_{-\beta/2}^{\beta/2} dt\ \dot{q}_c(t)^2 \int_{-\beta/2}^{\beta/2} \frac{dt'}{\dot{q}_c(t')^2}. \tag{B.42}$$

Recall the instanton solution is,

$$q_c(t) = \frac{1}{\sqrt{g}}\frac{1}{1+e^t}, \quad \dot{q}_c(t) = -\frac{e^t}{(1+e^t)^2}, \tag{B.43}$$

doi:10.6342/NTU201700825

then we can compute (B.42). The first integral is just the action of the instanton.

$$S_c = \frac{1}{6g}. \tag{B.44}$$

The second integral is,

$$\int_{-\beta/2}^{\beta/2} \frac{dt'}{\dot{q}_c(t')^2} = g \int_{-\beta/2}^{\beta/2} dt\ [(e^{-\frac{t}{2}} + e^{\frac{t}{2}})^4] = 2g[3\beta + 8\sinh(\beta/2) + \sinh(\beta)], \tag{B.45}$$

so the determinant is

$$\det{}' M = \frac{\beta}{2} + \frac{4}{3}\sinh(\beta/2) + \frac{1}{6}\sinh(\beta). \tag{B.46}$$

We also need to compute the reference determinant, which is given by,

$$M_0(t) = -\frac{d^2}{dt^2} + 1, \tag{B.47}$$

where

$$\phi_0^{(1)}(t) = \cosh(t + \beta/2), \quad \phi_0^{(2)}(t) = \sinh(t + \beta/2), \tag{B.48}$$

for anti-periodic boundary condition,

$$\det M_0 = 2 + 2\cosh(\beta). \tag{B.49}$$

Finally, after takes the limit $\beta \to \infty$,

$$\frac{\det{}' M}{\det M_0} = \frac{1}{12}. \tag{B.50}$$

This is the final result we want to know. Using this method, we can compute the determinant without solving the eigenfunction equation completely. This method can also be generalized to Sturm-Liouville problems. See [23].

Bibliography

[1] Freeman J Dyson. Divergence of perturbation theory in quantum electrodynamics. *Physical Review*, 85(4):631, 1952.

[2] LN Lipatov. Divergence of perturbation theory series and the quasiclassical theory. *Zhurnal Ehksperimental'noj i Teoreticheskoj Fiziki*, 72(2):411–427, 1977.

[3] Jean Zinn-Justin. Quantum field theory and critical phenomena. 2002.

doi:10.6342/NTU201700825

[4] E Brezin, JC Le Guillou, and Jean Zinn-Justin. Perturbation theory at large order. i. the φ 2 n interaction. *Physical Review D*, 15(6):1544, 1977.

[5] Jean-Claude Le Guillou and Jean Zinn-Justin. *Large-order behaviour of perturbation theory*, volume 7. Elsevier, 2012.

[6] E.B. Bogomolny. Calculation of instanton-anti-instanton contributions in quantum mechanics. *Physics Letters B*, 91(3):431 – 435, 1980.

[7] J. Zinn-Justin. Multi-instanton contributions in quantum mechanics. *Nuclear Physics B*, 192(1):125 – 140, 1981.

[8] J Zinn-Justin. Perturbation series at large orders in quantum mechanics and field theories: Application to the problem of resummation. *Physics Reports*, 70(2):109 – 167, 1981.

[9] Gerald V. Dunne and Mithat Ünsal. Generating nonperturbative physics from perturbation theory. *Phys. Rev. D*, 89:041701, Feb 2014.

[10] Gokce Basar, Gerald V. Dunne, and Mithat Unsal. Resurgence theory, ghost-instantons, and analytic continuation of path integrals. *JHEP*, 10:041, 2013.

[11] Gerald V. Dunne and Mithat Unsal. Deconstructing zero: resurgence, supersymmetry and complex saddles. *JHEP*, 12:002, 2016.

[12] Gerald V. Dunne and Mithat Unsal. Uniform WKB, Multi-instantons, and Resurgent Trans-Series. *Phys. Rev.*, D89(10):105009, 2014.

[13] Gerald V. Dunne and Mithat Unsal. Resurgence and Trans-series in Quantum Field Theory: The CP(N-1) Model. *JHEP*, 11:170, 2012.

[14] Philip C. Argyres and Mithat Unsal. The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects. *JHEP*, 08:063, 2012.

[15] M. Shifman. New and Old about Renormalons: in Memoriam Kolya Uraltsev. *Int. J. Mod. Phys.*, A30(10):1543001, 2015.

doi:10.6342/NTU201700825

[16] E.B. Bogomolny. Calculation of instanton-anti-instanton contributions in quantum mechanics. *Physics Letters B*, 91(34):431 – 435, 1980.

[17] J. Koplik, A. Neveu, and S. Nussinov. Some aspects of the planar perturbation series. *Nuclear Physics B*, 123:109–131, May 1977.

[18] M. Beneke. Renormalons. *Phys. Rept.*, 317:1–142, 1999.

[19] V. I. Zakharov. Qcd perturbative expansions in large orders. *Nuclear Physics B*, 385(3):452–480, 10 1992.

[20] Matthias Neubert. Scale setting in QCD and the momentum flow in Feynman diagrams. *Phys. Rev.*, D51:5924–5941, 1995.

[21] Marcos Marino. *Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory.* Cambridge University Press, 2015.

[22] Gerald V. Dunne. Functional determinants in quantum field theory. *J. Phys.*, A41:304006, 2008.

[23] Klaus Kirsten and Alan J. McKane. Functional determinants for general Sturm-Liouville problems. *J. Phys.*, A37:4649–4670, 2004.

[24] Gerald V. Dunne and Mithat Unsal. Deconstructing zero: resurgence, supersymmetry and complex saddles. *JHEP*, 12:002, 2016.

[25] Toshiaki Fujimori, Syo Kamata, Tatsuhiro Misumi, Muneto Nitta, and Norisuke Sakai. Exact Resurgent Trans-series and Multi-Bion Contributions to All Orders. 2017.

[26] I. M. Suslov. Divergent perturbation series. *Zh. Eksp. Teor. Fiz.*, 127:1350, 2005. [J. Exp. Theor. Phys.100,1188(2005)].

[27] I. M. Suslov. Renormalization Group Functions of the ϕ^4 Theory in the Strong Coupling Limit: Analytical Results. *J. Exp. Theor. Phys.*, 107:413–429, 2008. [Zh. Eksp. Teor. Fiz.134,490(2008)].

[28] I. M. Suslov. Exact asymptotics for beta-function in QED. *J. Exp. Theor. Phys.*, 108:980–984, 2009.

doi:10.6342/NTU201700825

[29] Philip Argyres and Mithat Unsal. A semiclassical realization of infrared renormalons. *Phys. Rev. Lett.*, 109:121601, 2012.

[30] Pavel Kovtun, Mithat Unsal, and Laurence G. Yaffe. Volume independence in large N(c) QCD-like gauge theories. *JHEP*, 06:019, 2007.

[31] Alexander M. Polyakov. Quark Confinement and Topology of Gauge Groups. *Nucl. Phys.*, B120:429–458, 1977.

[32] Ian Affleck, Jeffrey A. Harvey, and Edward Witten. Instantons and (Super)Symmetry Breaking in (2+1)-Dimensions. *Nucl. Phys.*, B206:413–439, 1982.

[33] David J. Gross, Robert D. Pisarski, and Laurence G. Yaffe. Qcd and instantons at finite temperature. *Rev. Mod. Phys.*, 53:43–80, Jan 1981.

[34] Tom M. W. Nye and Michael A. Singer. An L**2 index theorem for Dirac operators on S**1 x R**3. *Submitted to: J. Funct. Anal.*, 2000.

[35] Erich Poppitz and Mithat Unsal. Index theorem for topological excitations on R**3 x S**1 and Chern-Simons theory. *JHEP*, 03:027, 2009.

[36] F. David. CANCELLATIONS OF INFRARED DIVERGENCES IN TWO-DIMENSIONAL CHIRAL MODELS. *Phys. Lett.*, B96:371–374, 1980.

[37] Francois David. Instantons and condensates in two-dimensional cpn-1 models. *Physics Letters B*, 138(1):139 – 144, 1984.

[38] Michael Stone and Jeffrey Reeve. Late terms in the asymptotic expansion for the energy levels of a periodic potential. *Phys. Rev. D*, 18:4746–4751, Dec 1978.

[39] Toshiaki Fujimori, Syo Kamata, Tatsuhiro Misumi, Muneto Nitta, and Norisuke Sakai. Nonperturbative contributions from complexified solutions in $\mathbb{C}P^{N-1}$models. *Phys. Rev.*, D94(10):105002, 2016.

[40] I. M. Gelfand and A. M. Yaglom. Integration in functional spaces and it applications in quantum physics. *J. Math. Phys.*, 1:48, 1960.

doi:10.6342/NTU201700825