國立臺灣大學理學院數學系
碩士論文

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College of Science
National Taiwan University
Master Thesis

分離原理及其應用
A separation principle and its applications

陸韋翰
Wei-Han Lu

指導教授：劉豐哲 教授
Advisor: Professor , Fon-Che Liu

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摘要

此論文研究一個可推得 Hahn-Banach 等相關定理的分離原理的等價定理。由此，我們為 Agnew-Morse 定理、交換群上的順從性、Haar integral 的存在性提供自然簡潔的證明。可想像的是這個方法也能幫助分析上其他相關的結果。

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Abstract

This thesis studies an analytic variant of a well-known separation principle from which follows Hahn-Banach theorem and many basic theorems in convex analysis. By using this analytic variant, we provide natural and elegant proofs for Agnew-Morse Theorem, the amenability of abelian groups, and the existence of Haar integrals. It is conceivable that the approach we suggest here might lead to clarification of some results in convex analysis.

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1 Introduction and preliminary definitions

There is a well-known separation principle which states as follows(cf.[9, §III.3]).

Theorem 1. *If E is a nonempty linearly open convex cone in a real v.s. X such that $Y \cap E = \emptyset$ where Y is a vector subspace of X, then there is a hyperplane H in X s.t. $Y \subset H$ and $H \cap E = \emptyset$.*

Theorem 1 is equivalent to the following Theorem 2(cf.[13, §5.4]), if we consider the quotient space X/Y instead of X.

Theorem 2. *If E is a nonempty linearly open convex cone in a real v.s. X such that $0 \notin E$, then there is a hyperplane H in X s.t. $H \cap E = \emptyset$.*

Intuitively simple as Theorem 2 is, it is very useful in linear analysis. For examples, the well-known minimax theorem of von Neumann, Hahn-Banach theorem, and Mazur-Olicz's generalization of Hahn-Banach theorem can be derived from Theorem 2 without much effort as shown in [13, §5.4] and [12].

In this thesis, we aim to look more closely at Theorem 2 and derive an analytic variant of Theorem 2(See Theorem 5 below) with applications to Agnew-Morse theorem and to the existence of Haar integrals.

Throughout the thesis, all vector spaces considered are real vector spaces. Let X be a vector space. A subset E of X is said to be **convex** if $\alpha x+\beta y \in E$ whenever x and y are in E and α and β are nonnegative numbers with

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$\alpha + \beta = 1$. E is called a **convex cone** if it is convex and $\gamma E \subset E$ for all $\gamma > 0$. For a set $S \subset X$, the smallest convex cone containing S denotes Con S.

A set $E \subset X$ is said to be **linearly open** if for any $x \in E$ and $y \in X$, $x + ty \in E$ if $|t|$ is small enough; in other words, the intersection of any line in X with E is an open subset of the line. Note that if a linearly open convex cone containing the origin 0, then $E = X$.

For convenience, the fact that a real-valued function f assumes value $\geq \alpha$ on a set A will be expressed by $f(A) \geq \alpha$; the meaning of each of the expression $f(A) > \alpha$, $f(A) \leq \alpha$, and $f(A) < \alpha$ is parallelly given. Moreover, for subsets A, B of vector space X, the subset $\{a + b| \ a \in A \text{ and } b \in B\}$ and $\{-a| \ a \in A\}$ denote $A + B$ and $-A$ respectively. The set of all positive real numbers, $\{r \in \mathbb{R} : r > 0\}$, is denoted by $\mathbb{R}^+$.

We adopt the usual terminology of linear algebra; in particular, X' will denote the algebraic dual of a vector space X, i.e. X' is the space of all linear functional on X. A real-valued function p on X is called a sublinear functional if $p(x + y) \leq p(x) + p(y)$ for x, y in X and if $p(\alpha x) = \alpha p(x)$ for $\alpha > 0$ and $x \in X$. For convenience, given a sublinear functional p on X, $X'(p)$ denotes the set of all linear functionals ℓ in X' such that $\ell \leq p$ on X. From the definition of a sublinear functional p, $p(0) = 0$ and the following

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inequalities hold

$$p(\Sigma_{k=1}^{m} x_k) \leq \Sigma_{k=1}^{m} p(x_k); \tag{1}$$

$$p(y + x) \geq p(y) - p(-x). \tag{2}$$

For a given sublinear functional p on X, the set $Q = \{x \in X : p(x) < 0\}$ is a linearly open cone in X. Q is refered to as negative cone of p. Note that Q might be empty.

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2 The main theorem

In this section, we give a proof of Theorem 2, together with some applications when the concerned convex cone E is the negative cone of a sublinear functional. For the proof of Theorem 2, we first give a criterion of hyperplanes.

Proposition 1. *Let X be a vector space and H a vector subspace of X. H is a hyperplane if and only if there is a linearly open convex cone D such that $D \cup H \cup (-D) = X$ is a disjoint union.*

Proof. Assume that H is a hyperplane. Choose $x \notin H$. Put $D = \{rx + h : r > 0,\ h \in H\}$. It is easy to see that D is a convex cone and $-D = \{-rx + h : r > 0,\ h \in H\}$. Also $D \cap (-D) = \emptyset$, $H \cap (-D) = \emptyset$, and $D \cap H = \emptyset$. Since H is a hyperplane, $< x > +H = X$. Thus, $< x > +H = (\mathbb{R}^+ x + H) \cup H \cup (-\mathbb{R}^+ x + H) = D \cup H \cup (-D) = X$. It suffices to check that D is linearly open. For $z \in D$, and $y \in X$, there are $r_1 > 0, r_2$ in $\mathbb{R}$ and h_1, h_2 in H such that $z = r_1 x + h_1$ and $y = r_2 x + h_2$. If $r_2 = 0$, then $z + ty \in D$ for all t. Otherwise, let $|t| \leq \frac{|r_1|}{2|r_2|}$, then $z + ty = (r_1 + tr_2)x + (h_1 + th_2) \in D$ since $r_1 + tr_2 > 0$.

Assume now that there is a linearly open convex cone D such that $D \cup H \cup (-D) = X$ is a disjoint union. For any $x \notin H$, we claim $< x > +H = X$. WLOG, we let $x \in D$. For $y \in -D$, consider the line segment $[x, y] := \{\alpha x + \beta y : \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\}$, which is a connected set in

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the line L passing through x and y. $D \cap [x, y]$ and $(-D) \cap [x, y]$ are disjoint open sets in $[x, y]$ containing x and y respectively. Since $[x, y]$ is connected, $[x, y] \supsetneq (D \cap [x, y]) \cup ((-D) \cap [x, y])$; thus, there is an h in H such that $h = \alpha x + \beta y \in [x, y]$ i.e. $y = -\frac{\alpha x}{\beta} + \frac{h}{\beta}$, where $\alpha > 0$ and $\beta > 0$ follow from the fact $x,\ y \notin H$. That is, $y \in < x > +H$, or $(-D) \subset < x > +H$, from which $D \subset < x > +H$ follows. Thus, $X = D \cup H \cup (-D) \subset < x > +H$. $\square$

Proof of Theorem 2

Denote by $\mathcal{F}$ the family of all vector subspaces F of X such that $F \cap E = \emptyset$. $\mathcal{F}$ is not empty since $\{0\} \in \mathcal{F}$. Order $\mathcal{F}$ by set-inclusion i.e. $F_1 \leq F_2$ if $F_1 \subset F_2$ for F_1 and F_2 in $\mathcal{F}$. If $\mathcal{T}$ is a totally ordered subfamily of $\mathcal{F}$, then $\bigcup_{F \in \mathcal{T}} F$ is in $\mathcal{F}$ and is an upper bound of $\mathcal{T}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element H.

Let $D = H + E$. Since E is a linearly open convex cone, so is D. It follows from $0 \notin E$ that $D \cap H = \emptyset$. We show that H is a hyperplane to conclude the proof; for this, it suffices to check that $D \cup H \cup -D = X$ by Proposition 1. Let $x \notin H$. Then $< x > +H$ meets E since H is maximal in $\mathcal{F}$. Thus, there is $h \in H$ and nonzero $\lambda \in \mathbb{R}$ such that $h + \lambda x \in E$, as a consequence $\lambda x \in H + E = D$. Then $x \in D$ or $-D$ depending on $\lambda > 0$ or $\lambda < 0$. This shows that $X = D \cup H \cup -D$. $\square$

Theorem 3. *Suppose that E is a nonempty linearly open convex cone in X*

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and C a nonempty convex set in X such that $C \cap E = \emptyset$, then there is $\ell \in X'$ such that $\ell(C) \geq 0$ and $\ell(E) < 0$.

Proof. It is clear that $(\text{Con } C) \cap E = \emptyset$; hence if we put $D = E - (\text{Con } C)$, D is a linearly open convex cone not containing 0. By Theorem 2, there is a hyperplane H in X such that $H \cap D = \emptyset$. Choose $\ell \in X'$ with $\ker \ell = H$ and $\ell(D) < 0$. Now for $x \in$Con C, $y \in E$, and $\gamma > 0$,

$$\begin{cases} \ell(y) < \gamma \ell(x); \\ \gamma \ell(y) < \ell(x). \end{cases}$$

Let $\gamma \to 0$. It follows that $\ell(y) \leq 0$ for $y \in E$ and $\ell(x) \geq 0$ for $x \in$Con C. In particular, $\ell(C) \geq 0$.

It remains to show that $\ell(y) < 0$ for $y \in E$. Choose $x_0 \in X$ with $\ell(x_0) > 0$, then $y + tx_0 \in E$ if $|t|$ is small enough because E is linearly open. Since $y + tx_0 \in E$, $\ell(y + tx_0) \leq 0$; hence $\ell(y) \leq -t\ell(x_0) < 0$. □

Theorem 4. *Suppose that p is a sublinear functional on a vector space X and C a nonempty convex cone in X. Then there is an $\ell \in X'(p)$ such that $\ell(C) \geq 0$ if and only if $p(C) \geq 0$.*

Proof. Assume $p(C) \geq 0$. If the negative cone Q of p is empty i.e. $p(X) \geq 0$, then we choose the zero functional $\ell = 0$. Suppose now that Q is a nonempty linearly open convex cone in X; since $p(C) \geq 0$ and $p(Q) < 0$, $Q \cap C = \emptyset$.,

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by Theorem 3, there is $\hat{\ell} \in X'$, such that $\hat{\ell}(C) \geq 0$ and $\hat{\ell}(Q) < 0$. It will be shown presently that there is $\sigma > 0$ such that $\sigma\hat{\ell} \leq p$.

Define a map f from X into $\mathbb{R}^2$ by

$$f(x) = (p(x), -\hat{\ell}(x)), \quad x \in X,$$

and let D be the smallest convex cone containing $f(X)$. We claim that $D \cap E = \emptyset$, where $E = \{(r_1, r_2) : r_1 < 0 \text{ and } r_2 < 0\}$ is a linearly open convex cone. If $d \in D$, then there are $\alpha_1, ..., \alpha_k$ in $\mathbb{R}^+$ and $x_1, ..., x_k$ in X such that

$$\begin{aligned} d =& \Sigma_{k=1}^m \alpha_k f(x_k) \\ =& (\Sigma_{k=1}^m \alpha_k p(x_k), -\Sigma_{k=1}^m \alpha_k \hat{\ell}(x_k)) \\ =& (\Sigma_{k=1}^m p(\alpha_k x_k), -\hat{\ell}(\Sigma_{k=1}^m \alpha_k x_k)). \end{aligned}$$

Assume $\Sigma_{k=1}^m p(\alpha_k x_k) < 0$. Since $p(\Sigma_{k=1}^m \alpha_k x_k) \leq \Sigma_{k=1}^m p(\alpha_k x_k) < 0$, $\Sigma_{k=1}^m \alpha_k x_k$ is in Q. But the fact that $\hat{\ell}(Q) < 0$ implies $-\hat{\ell}(\Sigma_{k=1}^m \alpha_k x_k)) \geq 0$ i.e. $D \cap E = \emptyset$.

By Theorem 3, there is (α_1, α_2) in $\mathbb{R}^2$ with $\alpha_1^2 + \alpha_2^2 > 0$ such that

$$\begin{cases} \alpha_1 r_1 + \alpha_2 r_2 < 0 & \text{for } (r_1, r_2) \in E; \\ \alpha_1 p(x) - \alpha_2 \hat{\ell}(x) \geq 0 & \text{for } x \in X. \end{cases}$$

The first inequality shows that $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, while the second inequality shows that $\alpha_1 > 0$ and $\alpha_2 > 0$ due to the fact $Q \neq \emptyset$. The second inequality also shows that $\sigma\hat{\ell} \leq p$, for $\sigma = \frac{\alpha_2}{\alpha_1}$. If we put $\ell = \sigma\hat{\ell}$, then $\ell \in X'(p)$ and

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$\ell(C) \geq 0$, this proves one direction of Theorem 4. The other direction is obvious. $\square$

Remark. *If the negative cone Q of the given sublinear functional p is empty, Theorem 4 does not give much information; on the other hand, if $Q \neq \emptyset$, Theorem 4 guarantees the existence of nonzero $\ell \in X'(p)$ with $\ell(C) \geq 0$.*

Observe that if Y is a vector subspace of X and $\ell(Y) \geq 0$ for $\ell \in X'$, then $\ell(Y) = 0$. This fact is a special case of the following proposition.

Proposition 2. *Let Y be a vector subspace of vector space X and C a nonempty convex cone in X. Then it follows from $\ell(C + Y) \geq 0$ for $\ell \in X'$ that $\ell(Y) = 0$.*

Proof. Assume that $\ell(C + Y) \geq 0$. Let $c \in C$ and $y \in Y$. Then $\ell(c + \alpha y) = \ell(c) + \alpha\ell(y) \geq 0$ because $c + \alpha y \in C + Y$ for all $\alpha \in \mathbb{R}$. Suppose that there is a $y \in Y$ such that $\ell(y) \neq 0$; then if we put $\alpha = \frac{-|\ell(c)|-1}{\ell(y)}$, we have $\ell(c) + \alpha\ell(y) < 0$, which contradicts to the fact that $\ell(c) + \alpha\ell(y) \geq 0$ for all $\alpha \in \mathbb{R}$. Therefore, $\ell(y) = 0$ for all $y \in Y$. $\square$

A useful generalization of Theorem 4 is the following main theorem of this thesis:

Theorem 5. *Suppose that p is a sublinear functional on a vector space X, C a nonempty convex cone in X, and Y a vector subspace of X. Then there is an $\ell \in X'(p)$ such that $\ell(C) \geq 0$ and $\ell(Y) = 0$ if and only if $p(C+Y) \geq 0$.*

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Proof. It is clear that $C+Y$ is a convex cone. By Theorem 4, there is an $\ell \in X'(p)$ such that $\ell(C+Y) \geq 0$. By Proposition 2, $\ell(Y)=0$. Thus for all $c \in C$, $\ell(c)=\ell(c)+\ell(y)=\ell(c+y) \geq 0$, which implies $\ell(C) \geq 0$. The other direction is obvious. □

Corollary 1. *Suppose that p is a sublinear functional on a vector space X, and Y a vector subspace of X. Then there is an $\ell \in X'(p)$ such that $\ell(Y)=0$ if and only if $p(Y) \geq 0$.*

Proof. Put $C=\{0\}$ and use Theorem 5. □

Although Corollary 1 is a special case of Theorem 5, we shall see in §3 that a very elegant and important generalization of Hahn-Banach theorem follows from it (see Theorem 8).

Remark. *For a proper linearly open cone Q in a vector space X, one can construct a sublinear functional p such that $Q=\{x \in X : p(x)<0\}$ by the following steps.*

1. Fix $x_0 \in Q$ and consider the family

$$L=\{\ell \in X' : \ell<0 \text{ on } Q \text{ and } \ell(x_0)=-1\}.$$

L is nonempty by Theroem 2.

2. For $x \in X$, there is a $\sigma>0$ such that $x_0+\sigma x \in Q$. For $\ell \in L$, $\ell(x)=\frac{\ell(x_0+\sigma x)-\ell(x_0)}{\sigma} \leq \frac{1}{\sigma}$.

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3. For $z \in Q$, there is a $\sigma > 0$ such that $z - \sigma x_0 \in Q$. For $\ell \in L$, $\ell(z) = \ell(z - \sigma x_0) + \ell(\sigma x_0) \leq -\sigma$.

4. For $z \notin Q$, $\{z\}$ is a convex set disjoint to Q. By Theorem 3, there is an ℓ s.t. $\ell(Q) < 0$ and $\ell(z) \geq 0$, where the linear functional $\hat{\ell}$ with $\hat{\ell}(x) = \frac{\ell(x)}{-\ell(x_0)}$ and $\hat{\ell}(x_0) = -1$ is in L. Hence $p(z) \geq \hat{\ell}(z) \geq 0$.

5. Define $p(x) = \sup_{\ell \in L} \ell(x)$. By (2), p is finite and hence is a sublinear functional. By (3) and (4), $Q = \{x : p(x) < 0\}$.

To conclude this section, we consider an immediate consequence of Theorem 4.

Theorem 6. *Let X and Y be compact Hausdorff spaces, $\{f_i\}$ and $\{g_i\}$ be two families of real-valued continuous functions defined on X and Y respectively and indexed by the same index set I. Then the following two statements are equivalent:*

($\star$) There exist probability measures μ and ν on X and Y respectively such that

$$\int_X f_i d\mu \leq \int_Y g_i d\nu, \quad i \in I.$$

($\star\star$) For any positive integer n we have

$$Min_{x \in X} \Sigma_{k=1}^{n} \lambda_k f_{i_k}(x) \leq Max_{y \in Y} \Sigma_{k=1}^{n} \lambda_k g_{i_k}(y)$$

for all $i_1, ..., i_n$ in I and all $\lambda_1 \geq 0, ..., \lambda_n \geq 0$.

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Proof. $(\star) \Rightarrow (\star\star)$ is obvious. Assume that $(\star\star)$ holds. Let $h_i = -f_i$ for all $i \in I$. Then $(\star\star)$ implies

$$Max_{x\in X}\Sigma_{k=1}^{n}\lambda_k h_{i_k}(x) + Max_{y\in Y}\Sigma_{k=1}^{n}\lambda_k g_{i_k}(y) \geq 0. \tag{3}$$

We then define

$$C = \{(\Sigma_{k=1}^{n}\lambda_k h_{i_k}, \Sigma_{k=1}^{n}\lambda_k g_{i_k}) | \quad n \in \mathbb{N},\ i_1, ..., i_n \text{ in } I \text{ and all } \lambda_1 \geq 0, ..., \lambda_n \geq 0\}.$$

Clearly C is a convex cone in the vector space $V = C(X) \times C(Y)$ where $C(X)$ and $C(Y)$ are vector spaces consisting of all continuous functions on X and Y respectively. Define $p : V \to \mathbb{R}$ by

$$p(f, g) = Max_{x\in X}f(x) + Max_{y\in Y}g(y),$$

for $f \in C(X)$ and $g \in C(Y)$.

Therefore, $p(C) \geq 0$ from (3). By Theorem 4, there is an $\ell \in V'(p)$ such that $\ell(C) \geq 0$. Since $\ell(f, g) = \ell_1(f) + \ell_2(g)$ where $\ell_1 \in C(X)'$ and $\ell_2 \in C(Y)'$, $\ell(f, g) \leq p(f, g)$ implies $\ell_1(f) \leq p(f, 0)$ and $\ell_2(g) \leq p(0, g)$; it follows then that ℓ_1 and ℓ_2 are positive linear functionals on $C(X)$ and $C(Y)$. By Riesz Representation Theorem (cf.for example, [13, §3.10]), $\ell_1(f) = \int_X f d\mu, \ell_2(g) = \int_Y g d\nu$ for some probability measures μ and ν on X and Y respectively. But $\ell(C) \geq 0$ implies

$$\int_X h_i d\mu + \int_Y g_i d\nu \geq 0, \quad i \in I,$$

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or

$$\int_X f_i d\mu \leq \int_Y g_i d\nu, \quad i \in I.$$

□

At this point, we remark that Theorem 6 is proved in [11] by using the following theorem which is derived from Theorem 2.

Theorem 7. *For i=1,2, let q_i be a sublinear functional on a real vector space E_i and let τ_i be a map from a set S into E_i. Then the following two statements are equivalent:*

($\star$) There are ℓ_1 and ℓ_2 in E_1' and E_1' respectively with $\ell_1 \leq q_1$ and $\ell_2 \leq q_2$ such that

$$\ell_1(\tau_1(s)) \leq \ell_2(\tau_2(s)) \quad \forall s \in S.$$

($\star\star$) For any positive number n we have

$$-q_1(-\Sigma_{i=1}^n \lambda_i \tau_1(s_i)) \leq q_2(\Sigma_{i=1}^n \lambda_i \tau_2(s_i))$$

for all $s_1, ..., s_n$ in S and all $\lambda_1 \geq 0, ..., \lambda_n \geq 0$.

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3 Applications

As an application of Corollary 1, we consider the Agnew-Morse generalization of Hahn-Banach theorem(cf. [10], §3).

Theorem 8 (Agnew-Morse)**.** *Let p be a sublinear functional on a vector space X and $f \in Y'(p)$ where Y is a vector subspace of X. Suppose that there is a collection $\mathcal{A}$ of linear maps A from X into itself such that $AY \subset Y$ and for any $A, B \in \mathcal{A}$, the following relations hold:*

$$A \circ B = B \circ A, \quad p \circ A = p, \quad \text{and} \quad f \circ A|_Y = f.$$

Then there is an $F \in X'(p)$ such that $F|_Y = f$ and $F \circ A = F$ for all $A \in \mathcal{A}$.

Observe that if such an F in Theorem 8 exists, then from $F \circ A = F$, $F(x - Ax) = 0$ follows; and if we let $X_{\mathcal{A}}$ collect all elements of the form $x' = \Sigma_{k=1}^{m}(x_k - A_k x_k)$ where $m \in \mathbb{N}$, $x_1, ..., x_m \in X$ and $A_1, ..., A_m \in \mathcal{A}$, then $F(x') = 0$ for all $x \in X_{\mathcal{A}}$, and consequently,

$$p(x' + y) \geq F(x' + y) = F(y) = f(y)$$

for $x' \in X_{\mathcal{A}}$ and for $y \in Y$. But the following key lemma claims the same conclusion without requiring the existence of F.

Lemma 1. *Assume that $X, Y, p, f, \mathcal{A}$ satisfy the conditions in Theorem 8. Let $X_{\mathcal{A}}$ collect all elements of the form $\Sigma_{k=1}^{m}(x_k - A_k x_k)$ where $m \in \mathbb{N}$,*

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$x_1, ..., x_m \in X$ *and* $A_1, ..., A_m \in \mathcal{A}$. *Then*

$$p(x' + y) \geq f(y)$$

for all $x' \in X_{\mathcal{A}}$ *and* $y \in Y$.

Proof. For every $A \in \mathcal{A}$ and $n \in \mathbb{N}$, define $A^{[n]} = \frac{1}{n}\Sigma_{j=0}^{n-1} A^j$. Then

$$p(A^{[n]}x) = p(\frac{1}{n}(\Sigma_{j=0}^{n-1} A^j)(x)) = \frac{1}{n}p(\Sigma_{j=0}^{n-1}(A^j x)) \leq \frac{1}{n}\Sigma_{j=0}^{n-1} p(A^j x);$$

from $p \circ A = p$, we have

$$\frac{1}{n}\Sigma_{j=0}^{n-1} p(A^j x) = \frac{1}{n}\Sigma_{j=0}^{n-1} p(x) = p(x).$$

i.e.

$$p(x) \geq p(A^{[n]}x).$$

Since $A_1, ..., A_m \in \mathcal{A}$ are mutually commutative, $\Pi_{k=1}^m A_k^{[n_k]}$ is meaningful. By iterating,

$$p(x) \geq p(\Pi_{k=1}^m A_k^{[n_k]} x). \tag{4}$$

Let $x' = \Sigma_{k=1}^m (x_k - A_k x_k)$. For a fixed $n \in \mathbb{N}$, there are correspondent linear maps $A_1^{[n]}, ..., A_m^{[n]}$. By (2) and (4), we have

$$p(y + x') \geq p(\Pi_{j=1}^m A_j^{[n]}(y + x')) \geq p(\Pi_{j=1}^m A_j^{[n]} y) - p(-\Pi_{j=1}^m A_j^{[n]} x').$$

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Regarding the term $p(-\Pi_{j=1}^{m} A_j^{[n]} x')$, we have

$$\begin{aligned}
&p(-\Pi_{r=1}^{m} A_j^{[n]} x') \\
=&p(-\Pi_{r=1}^{m} A_j^{[n]} (\Sigma_{k=1}^{m} (x_k - A_k x_k))) \\
=&p(\Sigma_{k=1}^{m} (\Pi_{j=1}^{m} A_j^{[n]} (A_k x_k - x_k))) \textit{ (by linearity)} \\
\leq&\Sigma_{k=1}^{m} p((\Pi_{j=1}^{m} A_j^{[n]} (A_k x_k - x_k))) \textit{ (by (1))} \\
=&\Sigma_{k=1}^{m} p(\Pi_{j=1, j\neq k}^{m} A_j^{[n]} (\frac{1}{n}(A_k^n x_k - x_k))) \\
\leq&\Sigma_{k=1}^{m} p(\frac{1}{n}(A_k^n x_k - x_k)) \textit{ (by (4))} \\
\leq&\Sigma_{k=1}^{m} (\frac{1}{n}(p(A_k^n x_k) + p(-x_k))) \textit{ (by (1))} \\
=&\frac{1}{n}\Sigma_{k=1}^{m} (p(x_k) + p(-x_k)) \textit{ (observe that } p \circ A = p).
\end{aligned}$$

Namely,

$$-p(-\Pi_{j=1}^{m} A_j^{[n]} x') \geq -\frac{1}{n}\Sigma_{k=1}^{m} (p(x_k) + p(-x_k)).$$

Hence if we put c= $\Sigma_{k=1}^{m} (p(x_k) + p(-x_k))$, we obtain

$$p(x' + y) \geq p(\Pi_{j=1}^{m} A_j^{[n]} y) - p(-\Pi_{j=1}^{m} A_j^{[n]} x') \geq f(\Pi_{j=1}^{m} A_j^{[n]} y) - \frac{c}{n} = f(y) - \frac{c}{n}.$$

Then Lemma 1 follows as $n \to \infty$. □

Proof of Theorem 8

Define $\hat{p} : X \oplus Y \to \mathbb{R}$ by

$$\hat{p}(x, y) = p(x) + f(y), \quad x \in X, \quad y \in Y.$$

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It is clear that $X_{\mathcal{A}}$ is a vector subspace of X. Let $\hat{Y} = \{(x' + y, -y) : x \in X_{\mathcal{A}}$ and $y \in Y\}$. $\hat{Y}$ is a vector subspace of $X \oplus Y$. For all $(x' + y, -y) \in \hat{Y}$, $\hat{p}(x' + y, -y) = p(x' + y) + f(-y) \geq 0$ by Lemma 1. Namely, $\hat{p}(\hat{Y}) \geq 0$.

Now we apply Corollary 1; there is then an (F, F_Y) in $(X \oplus Y)'(\hat{p})$ such that $(F, F_Y)(\hat{Y}) = 0$. That (F, F_Y) is in $(X \oplus Y)'(\hat{p})$ implies $F \in X'(p)$ and $F_Y = f$; $F|_Y = F_Y = f$ follows from $(y, -y) \in \hat{Y}$ and $F(y) + F_Y(-y) = 0$; $F \circ A = F$ follows from $(x', 0) \in \hat{Y}$ and $F(x') = 0$. $\Box$.

Remark. *Let $\tilde{\mathcal{A}}$ collect all finite product of elements in $\mathcal{A}$ and identity map I. That is,*

$$\tilde{\mathcal{A}} = \{A_1 \circ A_2, ..., \circ A_n : A_1, ..., A_n \in \mathcal{A}, n \in \mathbb{N}\} \cup \{I\}.$$

Then $\tilde{\mathcal{A}}$ is an abelian semigroup with the operation $\circ$.

If we replace $\mathcal{A}$ by $\tilde{\mathcal{A}}$ in Theorem 8, the arguments in the proof of Lemma 1 still hold. Then it is nature to ask what kind of group would have these conditions as announced in Theorem 8. We might also ask whether a certain class of invariant linear functionals on a vector subspace of a given function space can be extended to the whole space. For the next main application of Theorem 5, the existence of invariant means on an abelian group will be considered(c.f[7, §1.1]).

Let G be a group and denote by $B(G)$ the linear space of all complex-valued bounded functions in G. For $x \in G$ and $f \in B(G)$, $f^x \in B(G)$ where

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$f^x(a) = f(ax^{-1})$ for each $a \in G$. An invariant mean on $B(G)$ is a linear functional $\mu : B(G) \to \mathbb{C}$ such that

(i) $\mu(f) \geq 0$ if $f \geq 0$ and $\mu(1) = 1$;

(ii) $\mu(f^x) = \mu(f)$ for all $f \in B(G)$ and $x \in G$;

(iii) $\mu(\bar{f}) = \overline{\mu(f)}$.

A group G is called an amenable group if there is an invariant mean on $B(G)$.

From (i) and (iii), let $B(G)$ be the space of all real-valued bounded functions instead. The existence of a linear functional μ which satifies

(i) $\mu(f) \geq 0$ if $f \geq 0$ and $\mu(1) = 1$, and

(ii) $\mu(f^x) = \mu(f)$ for all $f \in B(G)$ and $x \in G$

is sufficient to provide an invariant mean. We are ready to prove that there is an invariant mean on every abelian group by using Theorem 5.

Theorem 9. *All abelian groups are amenable.*

Proof. Put $X = B(G)$, $Y = \{\Sigma_{k=1}^m (f_k - f_k^{x_k}) |\ f_1, ..., f_m \in X,\ x_1, ..., x_m \in G\}$, and $C = \{g \geq 0 |\ g \in B(G)\}$. Define $p : X \to \mathbb{R}$ by $p(f) = \sup_{t \in G} f(t)$. Therefore, Y is a vector subspace of X and C a convex cone of X. We claim that $p(C + Y) \geq 0$. Observe first that for $g \in C$ and $y = \Sigma_{k=1}^m (f_k - f_k^{x_k}) \in Y$, we have

$$p(g + y) = \sup(\Sigma_{k=1}^m (f_k - f_k^{x_k} + g) \geq \sup \Sigma_{k=1}^m (f_k - f_k^{x_k}) = p(y).$$

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Thus, it suffices to prove that $p(Y) \geq 0$.

For any $\Sigma_{k=1}^{m}(f_k - f_k^{x_k})(t) = y(t)$ and any $n \in \mathbb{N}$, we put

$$S = \{(\lambda_1, \lambda_2, ..., \lambda_m) | \quad 1 \leq \lambda_k \leq n \text{ with } \lambda_k \in \mathbb{N} \text{ for } k = 1, ..., m\}, \text{ and}$$

$$S_i^k = \{s \in S : \lambda_k = i\}$$

for $k = 1, ..., m$ and $i = 1, ..., n$. For $(\lambda_1, ..., \lambda_m) = s \in S$, we denote by $\bar{s}$ the element $x_1^{\lambda_1} x_2^{\lambda_2} ... x_m^{\lambda_m} \in G$.

Observe that for a fixed k,

$$(I) \qquad S_i^k \cap S_j^k = \emptyset \text{ if } i \neq j.$$

$$(II) \qquad \cup_{i=1}^{n} S_i^k = S.$$

$$(III) \qquad \Sigma_{s \in S_i^k} f_k(\bar{s}) - \Sigma_{s \in S_{i+1}^k} f_k(\bar{s} x_k^{-1}) = 0.$$

From this observation, we have

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$$
\begin{aligned}
\Sigma_{s\in S} y(\bar{s}) &= \Sigma_{s\in S}\Sigma_{k=1}^{m}(f_k(\bar{s}) - f_k^{x_k}(\bar{s})) \\
&= \Sigma_{k=1}^{m}\Sigma_{s\in S}(f_k(\bar{s}) - f_k(\bar{s}x_k^{-1})) \\
&= \Sigma_{k=1}^{m}(\Sigma_{i=1}^{n}\Sigma_{s\in S_i^k}(f_k(\bar{s}) - f_k(\bar{s}x_k^{-1}))) \quad \text{(by I and II)} \\
&= \Sigma_{k=1}^{m}(\Sigma_{s\in S_n^k} f_k(\bar{s}) - \cancel{\Sigma_{s\in S_n^k} f_k(\bar{s}x_k^{-1})} \\
&\qquad + \cancel{\Sigma_{s\in S_{n-1}^k} f_k(\bar{s})} - \cancel{\Sigma_{s\in S_{n-1}^k} f_k(\bar{s}x_k^{-1})} \\
&\qquad \ldots \\
&\qquad + \cancel{\Sigma_{s\in S_1^k} f_k(\bar{s})} - \Sigma_{s\in S_1^k} f_k(\bar{s}x_k^{-1})) \\
&= \Sigma_{k=1}^{m}(\Sigma_{s\in S_n^k} f_k(\bar{s}) - \Sigma_{s\in S_1^k} f_k(\bar{s}x_k^{-1})). \quad \text{(by III)}
\end{aligned}
$$

Note that $|S| = n^m$ and there are only $m \cdot 2n^{m-1}$ terms on RHS. If we put $M = \sup_k \sup_t |f_k(t)|$, we have

$$
\begin{aligned}
n^m \sup_{t\in G} y(t) &\geq \Sigma_{s\in S} y(\bar{s}) \\
&= \Sigma_{k=1}^{m}(\Sigma_{s\in S_n^k} f_k(\bar{s}) - \Sigma_{s\in S_1^k} f_k(\bar{s}x_k^{-1})) \\
&\geq m \cdot 2n^{m-1}(-M),
\end{aligned}
$$

or, $\sup_{t\in G} y(t) \geq -\frac{2mM}{n}$. $p(y) \geq 0$ follows by letting $n \to \infty$.

By Theorem 5, there is an $\mu \in B(G)'(p)$ such that $\mu(C) \geq 0$ and $\mu(Y) = 0$, implying $\mu(f) \geq 0$ whenever $f \geq 0$ and $\mu(f) = \mu(f^x)$ respec-

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tively. Moreover, observe that

$$1 = -(-1) = -\sup(-1) = -p(-1) \leq -\mu(-1) = \mu(1) \leq p(1) = \sup(1) = 1$$

from which $\mu(1) = 1$ follows. □

Note that the existence of invariant means on an abelian semigroup is provided by the same proof. In the two applications above, the commutativity of group operation is essential. When the group considered is not abelian, topologies on the group have to be considered with compactness playing important role. This is what we do next.

Let G be a locally compact Hausdorff topological group and denote by $C_c(G)$ the linear space of all real-valued continuous functions with compact support in G. A right Haar integral on $C_c(G)$ is a linear functional μ in $C_c(G)'$ such that

(*i*) $\mu(f) \geq 0$ when $f \geq 0$;

(*ii*) $\mu(f^x) = \mu(f)$ for all $f \in C_c(G)$ and $x \in G$.

We denote by $C_c^+(G) = \{f \in C_c(G) |\ f \geq 0\}$. To begin with, we need two well-known facts, Urysohn's lemma and uniform continuity, on locally compact Hausdorff topological groups.

Theorem 10 (Urysohn's lemma)**.** *Given a compact set C contained in an*

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open set O, there is an $f \in C_c^+(G)$ such that $0 \le f \le 1$ and

$$(i) f = 1 \text{ on } C;$$

$$(ii) f = 0 \text{ on } O^c.$$

Proof. We claim first that for any compact C and open set U with $C \subset U$, there are compact set C' and open set U' such that $C \subset U' \subset C' \subset U$.

Since G is locally compact, for all $c \in C$, there is a compact set M_c such that $c \in \text{int } M_c \subset M_c \subset U$. Since C is compact, there are $c_1, ..., c_m$ in C such that $C \subset \cup_k^m \text{ int } M_{c_k} \subset \cup_k^m M_{c_k}$. The claim holds by letting $C' = \cup_k^m M_{c_k}$ and $U' = \cup_k^m \text{ int } M_{c_k}$.

If $C \subset U'' \subset C'' \subset U' \subset C' \subset U$ for some C, C', C'' compact and U, U', U'' open, we say that C'' separates C and C'. For given $C \subset O$, there are C' and O' such that $C \subset O' \subset C' \subset O$. Define $C(1) = C$ and $C(0) = C'$. Choose $C(1/2)$ separating $C(0)$ and $C(1)$; then choose $C(1/4)$ separating $C(0)$ and $C(1/2)$ and $C(3/4)$ separating $C(1/2)$ and $C(1)$. Continuing in this process, we get $C(\theta)$ for every θ of the form $\theta = \frac{i}{2^n}$ with $0 \le i \le 2^n$. Then for any real α with $0 \le \alpha \le 1$, define

$$C(\alpha) = \cap_{\theta \le \alpha} C(\theta),$$

where θ is of the form $\frac{i}{2^n}$. Since each $C(\theta)$ is closed, $C(\alpha)$ itself is closed, and so compact. Define $C(\alpha) = X$ if $\alpha < 0$ and $C(\alpha) = \emptyset$ if $\alpha > 1$.

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Then for $\alpha < \beta$, $C(\alpha)$ is a closed neighborhood of $C(\beta)$, compact if $\alpha \geq 0$. Now define $f(x) = \sup\{\alpha : x \in C(\alpha)\}$. Immediately we see that $f(x) = 0$ for all x outside $C(0)$, so that support(f) is compact and contained in O. Furthermore, $f(x) = 1$ for all $x \in C(1)$ and $0 \leq f(x) \leq 1$ everywhere.

To complete this proof, we need to show that f is continuous. For real β and γ we observe that on the one hand

$$f(x) \geq \beta \Leftrightarrow x \in \cap_{\alpha<\beta} C(\alpha) =: D(\beta),$$

and $D(\beta)$ is a closed set, while on the other hand

$$\begin{aligned} f(x) > \gamma &\Leftrightarrow x \in C(\alpha) \text{ for some } \alpha > \gamma \\ &\Leftrightarrow x \in \text{ Int } C(\alpha') \text{ for some } \alpha' > \gamma \\ &\Leftrightarrow x \in \cup_{\alpha'>r} \text{ Int } C(\alpha') =: E(\gamma), \end{aligned}$$

and $E(\gamma)$ is an open set. Thus,

$$\gamma < f(x) < \beta \Leftrightarrow x \in E(\gamma) \cap (D(\beta))^c,$$

which is open; so the preimage of a basic open set in $\mathbb{R}$ is open in X and f is continuous. □

Proposition 3. *For $f \in C_c(G)$, and $\epsilon > 0$, there is an open neighborhood V of e such that $|f(x) - f(y)| < \epsilon$ whenever $yx^{-1} \in V$.*

Proof. For all $x \in G$, $f^{-1}((f(x) - \frac{\epsilon}{2}, f(x) + \frac{\epsilon}{2})) = A$ is an open neighborhood of x; namely, $Ax^{-1} = N(x)$ is an open neighborhood of e such that $|f(y) -$

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$f(x)| < \frac{\epsilon}{2}$ whenever $y \in N(x)x$. Choose a symmetric open neighborhood $M(x)$ of e such that $M^2(x) \subset N(x)$. Let $\text{supp}(f) \subset C$ for some compact set C. Then there are $x_1, ..., x_m \in C$ such that $\cup_{k=1}^m M(x_k)x_k \supset C$. Then $V = \cap_{k=1}^m M(x_k)$ is a symmetric open neighborhood of e. To show that $|f(x) - f(y)| < \epsilon$ when $yx^{-1} \in V$, we may assume that either $x \in C$ or $y \in C$. Note that $yx^{-1} \in V \Leftrightarrow xy^{-1} \in V$ follows from $V = V^{-1}$, so we may assume that $x \in C$, therefore $x \in M(x_k)x_k$ for some k. Then $y \in Vx \subset VM(x_k)x_k \subset N(x_k)x_k \Rightarrow |f(x_k) - f(y)| < \frac{\epsilon}{2}$. Furthermore, $x \in M(x_k)x_k \subset N(x_k)x_k \Rightarrow |f(x_k) - f(x)| < \frac{\epsilon}{2}$. Thus, $|f(x) - f(y)| < \epsilon$. □

To construct an integral on $C_c(G)$, the main concept is to estimate the relative value of any $f, F \in C_c^+(G)$; namely, we are going to construct $(f : F)$ which almost equals $\frac{\mu(f)}{\mu(F)}$.

Observe that for any compact set C and open set O in G, $\{Oc\}_{c \in C}$ is an open cover of C, hence there are $c_1, ..., c_n$ in C such that $\{Oc_k\}_{k=1,..,n}$ is a finite open cover of C. Moreover, there is a smallest n such that $C \subset \cup_{k=1}^n Oc_k$ and it is intuitive to define

$$(C : O) = Min\{n \in \mathbb{N} |\ \exists\ c_1, .., c_n\ s.t.\ C \subset \cup_{k=1}^n Oc_k\}$$

to be the relative value of C and O. The following proposition shows that there is a similar result on continuous functions.

Proposition 4. *Given $f \in C_c^+(G)$, and nonzero $F \in C_c^+(G)$, there are*

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(α_1, x_1), ... ,(α_m, x_m) *in* $\mathbb{R}^+ \times G$ *such that* $f \leq \Sigma_{k=1}^m(\alpha_k F^{x_k})$.

Proof. There are a compact C and a constant $\alpha > 0$ such that $\text{supp}(f) \subset C$ and $f \leq \alpha$. Since F is nonzero, there is a $a \in G$ such that $F(a) = 2\beta$ for some $\beta > 0$. Since F is continuous, there is an open neighborhood O of e such that $F(x) \geq \beta$ for all $x \in Oa$. Since C is compact and $\{Oc\}_{c\in C}$ is an open cover of C, there are $c_1, ..., c_m$ such that $\cup_{k=1}^m Oc_k \supset C$. For $x \in C$, $x \in Oc_k$ for some k; hence $\frac{\alpha}{\beta}F^{a^{-1}c_k}(x) \geq \alpha \geq f(x)$. So $f(x) \leq \Sigma_{k=1}^m \frac{\alpha}{\beta}F^{a^{-1}c_k}(x)$ for all $x \in C$; namely, $f \leq \Sigma_{k=1}^m \frac{\alpha}{\beta}F^{a^{-1}c_k}$. □

Assume that μ is a right Haar integral and $f \leq \Sigma_{k=1}^m(\alpha_k F^{x_k})$. Since μ is positive and invariant, we have $\mu(f) \leq \mu(\Sigma_{k=1}^m(\alpha_k F^{x_k})) = \Sigma_{k=1}^m \alpha_k \mu(F)$, or,$\frac{\mu(f)}{\mu(F)} \leq \Sigma_{k=1}^m \alpha_k$.

We aim to approximate $\frac{\mu(f)}{\mu(F)}$. Thus, it is intuitive to take inf through all possible α_k as shown in the following definition.

Definition 1. *For any* f *in* $C_c^+(G)$ *and nonzero* F *in* $C_c^+(G)$*, we define*

$$(f : F) = \inf\{\Sigma_{k=1}^m \alpha_k | f \leq \Sigma_{k=1}^m(\alpha_k F^{x_k}) \text{ for some } (\alpha_k, x_k)'s \text{ in } \mathbb{R}^+ \times G\}.$$

Note that m is not a fixed arbitrary nature number in the definition.

Remark. *If* G *is compact, then 1 is a continuous function with compact support* G *and* $(f : 1) = \inf\{\alpha : \alpha \text{ is an upper bound of } f\} = \sup(f)$.

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This suggests that we define a sublinear functional on $C_c(G)$ using the concept $(f : F)$ when G is locally compact; it seems that this is not observed in the literature.

We list some simple but useful properties of $(f : F)$ in the following proposition.

Proposition 5. *Given f, f_1, f_2 in $C_c^+(G)$, and nonzero E, F in $C_c^+(G)$, we have:*

(1) For $\alpha > 0$, $(\alpha f : F) = \alpha(f : F)$.

(2) $(f_1 + f_2 : F) \leq (f_1 : F) + (f_2 : F)$.

(3) If $f_1 \leq f_2$, then $(f_1 : F) \leq (f_2 : F)$.

(4) $(E : F) \geq \frac{\sup E}{supF} > 0$; if G is compact, $(1 : F) > 0$.

(5) $(f : F) \leq (f : E)(E : F)$; if G is compact, $(f : F) \leq (f : 1)(1 : F)$.

(6) For $x \in G$, $(f : F) = (f^x : F)$.

Proof.

(1) follows from the fact that $\alpha f \leq \Sigma_{k=1}^{m}(\alpha\alpha_k F^{x_k}) \Leftrightarrow f \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k})$.

Now, if $f_1 \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k})$ and $f_2 \leq \Sigma_{k=1}^{n}(\beta_k F^{y_k})$, then

$$f_1 + f_2 \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k}) + \Sigma_{k=1}^{n}(\beta_k F^{y_k}),$$

or

$$(f_1 + f_2 : F) \leq \Sigma_{k=1}^{m}\alpha_k + \Sigma_{k=1}^{n}\beta_k;$$

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then (2) follows.

If $f_2 \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k})$, then

$$f_1 \leq f_2 \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k}),$$

namely,

$$(f_1 : F) \leq \Sigma_{k=1}^{m}\alpha_k;$$

then (3) follows.

Let $\sup E = r$ and $\sup F = R$. If $E \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k})$, then

$$r = \sup E \leq \sup \Sigma_{k=1}^{m}(\alpha_k F^{x_k}) \leq \Sigma_{k=1}^{m}(\alpha_k \sup F^{x_k}) = \Sigma_{k=1}^{m}(\alpha_k R),$$

namely,

$$\frac{r}{R} \leq \Sigma_{k=1}^{m}\alpha_k;$$

from which (4) follows.

If $f \leq \Sigma_{j=1}^{n}(\beta_j E^{y_j})$ and $E \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k})$, then

$$f \leq \Sigma_{j=1}^{n}(\beta_j(\Sigma_{k=1}^{m}\alpha_k F^{x_k})^{y_j}) = \Sigma_{j=1}^{n}\Sigma_{k=1}^{m}(\beta_j\alpha_k F^{x_k y_j}),$$

namely,

$$(f : F) \leq \Sigma_{j=1}^{n}\Sigma_{k=1}^{m}(\beta_j\alpha_k) = (\Sigma_{j=1}^{n}\beta_j)(\Sigma_{k=1}^{m}\alpha_k);$$

then (5) follows.

(6) follows from the fact that $f \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k}) \Leftrightarrow f^x \leq \Sigma_{k=1}^{m}(\alpha_k F^{x_k x})$. $\square$

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It is a little complicated to prove the existence of right Haar integrals on locally compact Hausdorff group; we shall first proceed to the proof of the existence of right Haar integrals for the case that G is compact.

Lemma 2. *Let G be a compact Hausdorff topological group. Given $\beta > 0$, $f_1, ..., f_n$ in $C_c^+(G)$ and $x_1, ..., x_n$ in G, there is an $\epsilon > 0$ such that*

$$(\Sigma_{j=1}^n f_j^{x_j} : F)(1+\epsilon) < \Sigma_{j=1}^n (f_j : F) + (\beta : F)$$

for any nonzero $F \in C_c^+(G)$.

Proof. By Proposition 5,

$$\begin{aligned}
(\Sigma_{j=1}^n f_j^{x_j} : F)(1+\epsilon) &\leq \Sigma_{j=1}^n (f_j^{x_j} : F)(1+\epsilon) \\
&= \Sigma_{j=1}^n (f_j : F)(1+\epsilon) \\
&= \Sigma_{j=1}^n (f_j : F) + \epsilon \Sigma_{j=1}^n (f_j : F) \\
&\leq \Sigma_{j=1}^n (f_j : F) + \epsilon \Sigma_{j=1}^n (f_j : 1)(1 : F) \\
&= \Sigma_{j=1}^n (f_j : F) + \epsilon \Sigma_{j=1}^n (f_j : 1) \frac{(\beta : F)}{\beta}.
\end{aligned}$$

Lemma 2 follows if we choose $\epsilon < \frac{\beta}{\Sigma_{j=1}^n (f_j : 1)}$. □

One might imagine that $(C : O)$ should be more useful for the purpose of approximation when open set O becomes smaller; the following lemma states an argument realizing such an effect.

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Lemma 3. *Let G be a compact Hausdorff topological group. Given $\beta > 0$ and f_j's$\in C_c^+(G)$, for any $\epsilon > 0$, there is an open set O (small in some sense) and a nonzero $F \in C_c^+(G)$ with supp$(F) \subset O$ such that*

$$\Sigma_{j=1}^n (f_j : F) + (\beta : F) \leq (\Sigma_{j=1}^n f_j + \beta : F)(1+\epsilon).$$

Remark. *Note that $(\Sigma_{j=1}^n f_j + \beta : F) \leq \Sigma_{j=1}^n (f_j : F) + (\beta : F)$ has been established by Proposition 5. But what is the distance between $(\Sigma_{j=1}^n f_j + \beta : F)$ and $\Sigma_{j=1}^n (f_j : F) + (\beta : F)$? Do they close enough to form a right Haar integral? Lemma 3 gives an answer that they can be arbitrary close as long as supp(F) lies in a small open set.*

Proof of Lemma 3

Put $f_{n+1} = \beta$, $p = \Sigma_{j=1}^{n+1} f_j$ and $g_j = f_j/p$ for $j = 1, ..., n+1$. Note that $\{g_j\}_{j=1,..,n+1}$ are continuous on a compact topological group. So there is an open set O such that $|g_j(x) - g_j(y)| < \frac{\epsilon}{n+1}$ if $x^{-1}y \in O$. By Urysohn's lemma, there is a nonzero F such that $supp(F) \subset O$.

Let $p \leq \Sigma_{k=1}^m \alpha_k F^{x_k}$ for some $(\alpha_k, x_k)'s$ in $\mathbb{R}^+ \times G$. Since supp$(F^{x_k} g_j) \subset Ox_k$, $F^{x_k} g_j \leq F^{x_k}(g_j(x_k) + \frac{\epsilon}{n+1})$. Now

$$f_j = pg_j \leq \Sigma_{k=1}^m \alpha_k F^{x_k} g_j \leq \Sigma_{k=1}^m \alpha_k F^{x_k}(g_j(x_k) + \frac{\epsilon}{n+1});$$

thus

$$(f_j : F) \leq \Sigma_{k=1}^m \alpha_k (g_j(x_k) + \frac{\epsilon}{n+1}).$$

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Note that $\Sigma_{j=1}^{n+1}(g_j(x_k) + \frac{\epsilon}{n+1}) = (1+\epsilon)$. We have then (recall that $f_{n+1} = \beta$)

$$\Sigma_{j=1}^{n+1}(f_j : F) \leq \Sigma_{k=1}^{m}\alpha_k(1+\epsilon),$$

and Lemma 3 follows by letting $\Sigma_{k=1}^{m}\alpha_k \to (p : F)$. □

Corollary 2. *Let G be a compact Hausdorff topological group. Given f_j's$\in C_c(G)$ and x_j's$\in G$, then*

$$\sup \Sigma_{j=1}^{n}(f_j - f_j^{x_j}) \geq 0.$$

Proof. For any $f \in C_c(G)$, and any $x \in G$, observe that

$$f - f^x = f^+ - f^- - (f^+)^x + (f^-)^x$$

$$= \{f^+ - (f^+)^x\} + \{(f^-)^x - f^-\};$$

namely,

$$f - f^x = g - g^x + h - h^{x^{-1}}$$

if $g = f^+$ and $h = (f^-)^x$. Thus we may assume that all f_j's are in $C_c^+(G)$. Suppose that $\Sigma_{j=1}^{n}(f_j - f_j^{x_j}) \leq \alpha$, for some $\alpha < 0$. Put $\beta = -\alpha > 0$. We have

$$\Sigma_{j=1}^{n}f_j + \beta \leq \Sigma_{j=1}^{n}f_j^{x_j}. \tag{5}$$

From Lemma 2 and Lemma 3, there is an $\epsilon > 0$ and a nonzero $F \in C_c^+(G)$ such that

$$(\Sigma_{j=1}^{n}f_j^{x_j} : F)(1+\epsilon) < \Sigma_{j=1}^{n}(f_j : F) + (\beta : F) \leq (\Sigma_{j=1}^{n}f_j + \beta : F)(1+\epsilon).$$

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However, (5) and Proposition 5 implies

$$(\Sigma_{j=1}^n f_j + \beta : F)(1+\epsilon) \leq (\Sigma_{j=1}^n f_j^{x_j} : F)(1+\epsilon),$$

which leads to a contradiction. Thus, all upper bound α of $\Sigma_{j=1}^n (f_j - f_j^{x_j})$ are nonnegative, or $\sup \Sigma_{j=1}^n (f_j - f_j^{x_j}) \geq 0$. □

Theorem 11. *There is a nonzero right Haar integral on every compact Hausdorff topological group G.*

Proof. Define sublinear functional $p : C_c(G) \to \mathbb{R}$ by $p(f) = \sup(f)$. Let $C = C_c^+(G)$ and Y a vector subspace spanned by $f - f^x$, for all $f \in C_c(G)$ and $x \in G$. From Corollary 2, $p(c+y) \geq p(y) \geq 0$ for $c \in C$ and $y \in Y$. We conclude that $p(C+Y) \geq 0$. By Theorem 5, there is a linear functional $\mu \in C_c(G)'(p)$ such that $\mu(C) \geq 0$ and $\mu(Y) = 0$; consequently,

(*i*) $\mu(f) \geq 0$ when $f \geq 0$;

(*ii*) $\mu(f^x) = \mu(f)$for all $f \in C_c(G)$ and $x \in G$.

Since $p(-1) = -1$ and $\mu(-1) \leq p(-1)$, μ is nonzero. □

When right Haar integrals on a locally compact Hausdorff group G are concerned, we use similar method. Since 1 is, in general, not a continuous function with compact support, we fix a nonzero continuous function $E \in C_c^+(G)$ which plays the role of 1, where G is compact. Moreover, we construct

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a $q \in C_c^+(G)$ in Lemma 5 such that the quotient of functions still lies in $C_c^+(G)$.

Lemma 4. *Given* $(f_1, x_1), ..., (f_n, x_n)$ *in* $C_c^+(G) \times G$, q, E *in* $C_c^+(G)$ *but nonzero, and* (a_1, c_1),...,(a_ℓ, c_ℓ) *in* $\mathbb{R}^+ \times G$, *then for all* $\epsilon > 0$, *there is a* $\delta > 0$ *such that*

$$(\Sigma_{j=1}^n f_j^{x_j} + \Sigma_{i=1}^\ell a_i E^{c_i} + \delta q : F)(1+\delta) \le \Sigma_{j=1}^n (f_j : F) + (\Sigma_{i=1}^\ell a_i + \epsilon)(E : F)$$

for any nonzero $F \in C_c^+(G)$.

Proof. By Proposition 5,

$$\begin{aligned}
&(\Sigma_{j=1}^n f_j^{x_j} + \Sigma_{i=1}^\ell a_i E^{c_i} + \delta q : F)(1+\delta) \\
\le&[\Sigma_{j=1}^n (f_j^{x_j} : F) + \Sigma_{i=1}^\ell a_i (E^{c_i} : F) + \delta(q : F)](1+\delta) \\
=&[\Sigma_{j=1}^n (f_j : F) + \Sigma_{i=1}^\ell a_i (E : F) + \delta(q : F)](1+\delta) \\
=&\Sigma_{j=1}^n (f_j : F) + (\Sigma_{i=1}^\ell a_i)(E : F) \\
+&\delta[\Sigma_{j=1}^n (f_j : F) + \Sigma_{i=1}^\ell a_i (E : F) + (1+\delta)(q : F)].
\end{aligned}$$

It suffices to choose δ independent of F such that

$$\delta[\Sigma_{j=1}^n (f_j : F) + \Sigma_{i=1}^\ell a_i (E : F) + (1+\delta)(q : F)] \le \epsilon(E : F).$$

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Now,

$$\Sigma_{j=1}^{n}(f_j : F) + \Sigma_{i=1}^{\ell} a_i(E : F) + (1+\delta)(q : F)$$

$$\leq \Sigma_{j=1}^{n}(f_j : E)(E : F) + \Sigma_{i=1}^{\ell} a_i(E : F) + (1+\delta)(q : E)(E : F)$$

$$\leq M(E : F),$$

for some constant $M > 0$ independent of F. Lemma 4 follows if we choose $\delta \leq \frac{\epsilon}{M}$. □

Lemma 5. *Given $f_1, ..., f_n$ in $C_c^+(G)$, nonzero E in $C_c^+(G)$, and (b_1, d_1),...,(b_ℓ, d_ℓ) in $\mathbb{R}^+ \times G$, then for any $\delta > 0$, there is an open set O (small in some sense) and a nonzero $F \in C_c^+(G)$ with supp$(F) \subset O$ such that*

$$\Sigma_{j=1}^{n}(f_j : F) + \Sigma_{i=1}^{\ell} b_i(E : F) \leq (\Sigma_{j=1}^{n} f_j + \Sigma_{i=1}^{\ell} b_i E^{d_i} + \delta q : F)(1+\delta),$$

for some q in $C_c^+(G)$ depending only on f_j's.

Proof. There is a compact set C such that supp$(f_j) \subset C$ for $j = 1, ..., n$. By Urysohn's lemma, there is a $q \in C_c^+(G)$ with $q = 1$ on C. For $i = 1, ..., \ell$ put $f_{n+i} = b_i E^{d_i}$,and $p = \Sigma_{j=1}^{n+\ell} f_j + \delta q$; for $j = 1, ..., n+\ell$, we define

$$g_j(t) = \begin{cases} \frac{f_j}{p}(t) \text{ if } t \in C; \\ 0 \text{ otherwise.} \end{cases}$$

Note that $\{g_j\}_{j=1,..,n+\ell}$ are continuous on G. So there is an open set O such that $|g_j(x) - g_j(y)| < \frac{\delta}{n+\ell}$ if $x^{-1}y \in O$. By Urysohn's lemma, there is a nonzero F such that supp$(F) \subset O$.

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Let $p \le \Sigma_{k=1}^{m} \alpha_k F^{x_k}$, for some $(\alpha_k, x_k)'s$ in $\mathbb{R}^+ \times G$. Since $\text{supp}(F^{x_k} g_j) \subset Ox_k$, $F^{x_k} g_j \le F^{x_k}(g_j(x_k) + \frac{\delta}{n+\ell})$. Now

$$f_j = pg_j \le \Sigma_{k=1}^{m} \alpha_k F^{x_k} g_j \le \Sigma_{k=1}^{m} \alpha_k F^{x_k}(g_j(x_k) + \frac{\delta}{n+\ell});$$

thus

$$(f_j : F) \le \Sigma_{k=1}^{m} \alpha_k (g_j(x_k) + \frac{\delta}{n+\ell}).$$

Note that $\Sigma_{j=1}^{n+\ell}(g_j(x_k) + \frac{\delta}{n+\ell}) \le (1+\delta)$, hence

$$\Sigma_{j=1}^{n+\ell}(f_j : F) \le \Sigma_{k=1}^{m} \alpha_k (1+\delta),$$

and Lemma 5 follows by letting $\Sigma_{k=1}^{m} \alpha_k \to (p : F)$. □

Corollary 3. *Given* $(f_1, x_1), ..., (f_n, x_n)$ *in* $C_c(G) \times G$*, nonzero* E *in* $C_c^+(G)$*, and* (α_1, y_1),...,(α_m, y_m) *in* $\mathbb{R} \times G$*, if*

$$\Sigma_{j=1}^{n}(f_j - f_j^{x_j}) \le \Sigma_{k=1}^{m} \alpha_k E^{y_k},$$

then

$$\Sigma_{k=1}^{m} \alpha_k \ge 0.$$

Proof. By similar argument as in the proof of Corollary 2, we may assume that all f_j's are in $C_c^+(G)$. The equation $\Sigma_{j=1}^{n}(f_j - f_j^{x_j}) \le \Sigma_{k=1}^{m} \alpha_k E^{y_k}$ is equivalent to

$$\Sigma_{j=1}^{n} f_j + \Sigma_{i=1}^{\tilde{\ell}} b_i E^{d_i} \le \Sigma_{j=1}^{n} f_j^{x_j} + \Sigma_{i=1}^{\ell} a_i E^{c_i},$$

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for some (b_1,d_1),...,$(b_{\tilde\ell},d_{\tilde\ell})$, (a_1,c_1),...,(a_ℓ,c_ℓ) in $\mathbb{R}^+\times G$, where $\tilde\ell+\ell=m$, a_i's are the positive terms of α_k's and $-b_i$'s are the negative terms of α_k's. By Proposition 5,

$$(\Sigma_{j=1}^n f_j+\Sigma_{i=1}^{\tilde\ell} b_i E^{d_i}+\delta q:F)(1+\delta)\leq(\Sigma_{j=1}^n f_j^{x_j}+\Sigma_{i=1}^{\ell} a_i E^{c_i}+\delta q:F)(1+\delta).$$

From Lemma 4 and Lemma 5, for all $\epsilon>0$ there is a $\delta>0$ and a nonzero $F\in C_c^+(G)$ such that

$$\begin{aligned}&\Sigma_{j=1}^n(f_j:F)+\Sigma_{i=1}^{\tilde\ell} b_i(E:F)\\ \leq&(\Sigma_{j=1}^n f_j+\Sigma_{i=1}^{\tilde\ell} b_i E^{d_i}+\delta q:F)(1+\delta)\\ \leq&(\Sigma_{j=1}^n f_j^{x_j}+\Sigma_{i=1}^{\ell} a_i E^{c_i}+\delta q:F)(1+\delta)\\ \leq&\Sigma_{j=1}^n(f_j:F)+(\Sigma_{i=1}^{\ell} a_i+\epsilon)(E:F).\end{aligned}$$

Since ϵ can be arbitrarily small, we have $\Sigma_{i=1}^{\ell} a_i-\Sigma_{i=1}^{\tilde\ell} b_i\geq 0$; i.e. $\Sigma_{k=1}^m\alpha_k\geq 0$. □

Theorem 12. *There is a nonzero right Haar integral on every locally compact Hausdorff topological group G.*

Proof. For a fixed nonzero $E\in C_c^+(G)$, define p on $C_c(G)$ by

$$p(f)=\inf\{\Sigma_{k=1}^m\alpha_k|f\leq\Sigma_{k=1}^m(\alpha_k E^{x_k})\text{ for some }(\alpha_k,x_k)'s\text{ in }\mathbb{R}\times G\}.$$

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Clearly, for $\alpha > 0$ and f, f_1, f_2 in $C_c(G)$,

$$(I) \qquad p(\alpha f) = \alpha p(f).$$

$$(II) \qquad p(f_1 + f_2) \leq p(f_1) + p(f_2).$$

$$(III) \qquad f_1 \leq f_2 \Rightarrow p(f_1) \leq p(f_2).$$

By Proposition 4, $p(f) < \infty$. We shall show $p(f) > -\infty$ presently. From Corollary 3,

$$p(\Sigma_{j=1}^{n}(f_j - f_j^{x_j})) = \inf\{\Sigma_{k=1}^{m}\alpha_k | \Sigma_{j=1}^{n}(f_j - f_j^{x_j}) \leq \Sigma_{k=1}^{m}(\alpha_k E^{x_k})\} \geq 0; \quad (6)$$

in particular, $p(0) \geq 0$. Now for any $f \in C_c(G)$, $f = f^+ - f^-$. By (II) and (III),

$$0 \leq p(0) \leq p(f^+) \leq p(f) + p(f^-)$$

from which $p(f) > -\infty$ follows; hence p is a sublinear functional on $C_c(G)$.

Let $C = C_c^+(G)$ and Y a vector subspace spanned by $f - f^x$, for all $f \in C_c(G)$ and $x \in G$. By (III) and Corollary 3, $p(c + y) \geq p(y) \geq 0$ for $c \in C$ and $y \in Y$. We conclude that $p(C + Y) \geq 0$. By Theorem 5, there is a linear functional $\mu \in C_c(G)'(p)$ such that $\mu(C) \geq 0$ and $\mu(Y) = 0$; consequently,

(i) $\quad \mu(f) \geq 0$ when $f \geq 0$;

(ii) $\quad \mu(f^x) = \mu(f)$for all $f \in C_c(G)$ and $x \in G$.

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Since $p(-E) = -1$ and $\mu(-E) \leq p(-E)$, μ is nonzero. $\square$

Remark. *For every nonzero* $E \in C_c^+(G)$*, there is a correspondent sublinear functional* p *and a right Haar integral* μ *such that* $1 = \mu(E) = p(E)$*.*

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4 Miscellaneous remarks

In Theorem 6, assume Y is a compact topological space. If we put

$$\{f_i\}_{i\in I} = \{0\}_{i\in I}$$

and

$$\{g_i\}_{i\in I} = \{h+\Sigma_{k=1}^n(f_k-f_k^{y_k})|\ h,\ f_k\text{'s are continuous with } h\geq 0 \text{ and } y_k's\in Y\},$$

then

$$0 \leq Max_{y\in Y}(h+\Sigma_{k=1}^n f_k - f_k^{y_k}(y)) \tag{7}$$

implies that there is an probability measure ν and

$$0 \leq \int_Y h+\Sigma_{k=1}^n(f_k - f_k^{y_k})d\nu = \int_Y g_i d\nu \quad i\in I,$$

or,

$$\int_Y \Sigma_{k=1}^n f_k^{y_k} d\nu \leq \int_Y h+\Sigma_{k=1}^n f_k d\nu.$$

ν is positive by letting $f_k = 0$ for $k = 1, .., m$; ν is invariant by letting $h = 0$; thus, ν is an invariant measure. Obviously, (7) is equivalent to Corollary 2. That is, Theorem 10 may be derived from Theorem 6 instead of Theorem 5.

With Lemma 1, one may prove Theorem 8 by Hahn-Banach Theorem as following.

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Proof. (Agnew-Morse) Define $\tilde{Y}$ as the subspace of X spanned by Y and $X_{\mathcal{A}}$. Thus for all $z \in \tilde{Y}$, $z = x' + y$ for some $x' \in X_{\mathcal{A}}$ and $y \in Y$. Define $\tilde{f} : \tilde{Y} \to \mathbb{R}$ by

$$\tilde{f}(z) = f(y) \text{ if } z = x' + y \text{ for some } x' \in X_{\mathcal{A}}, y \in Y.$$

We claim $\tilde{f}$ is well-defined. Assume $z = x_1' + y_1 = x_2' + y_2$. Let $x' = x_1' - x_2'$ and $y = y_1 - y_2$. Note that $x' + y = 0$. By Lemma 1, we have

$$0 = p(0) = p(x' + y) \geq f(y) = f(y_1) - f(y_2);$$
$$0 = p(0) = p((-x') + (-y)) \geq f(-y) = f(y_2) - f(y_1),$$

from which $f(y_1) = f(y_2)$ follows. From this definition, for $z = x' + y$ in $\tilde{Y}$, $p(z) = p(x' + y) \geq f(y) = \tilde{f}(z)$. Thus, $\tilde{f}$ is a linear functional in $\tilde{Y}'(p)$. By Hahn-Banach Theorem, there is an F in $X'(p)$ such that $F|_{\tilde{Y}} = \tilde{f}$. Clearly, F satifies conditions required in Theorem 8. □

Historically, Hahn proved the extension theorem in 1927 as follows (cf.[8]).

Theorem. *(Hahn) Let Y be a vector subspace of the real normed vector space X, and f a linear functional of Y. Then there is an $F \in X'$ such that $F|_Y = f$ and $||F|| = ||f||$.*

S. Banach gave a generalization of Hahn's theorem in 1932 which is known as Hahn-Banach Theorem (cf. [3]).

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Theorem. *(Banach) Let* Y *be a vector subspace of the real vector space* X*. Suppose that there is a sublinear functional* p *on* X *and a linear functional* f *on* Y *where* $f(y) \leq p(y)$ *for all* $y \in Y$*. Then there is an* $F \in X'$ *such that* $F|_Y = f$ *and* $F(x) \leq p(x)$ *for all* $x \in X$*.*

However, before Banach proved the Hahn-Banach theorem, he actually have used the idea of Hahn-Banach theorem in 1923 (cf. [2]). In that paper, Banach extended a certain linear functional to give a solution of a problem of measures. Moreover, he proved that the isometry group of $\mathbb{R}^1$ is equipped with the Hahn-Banach extension property, which is later generalized by Agnew and Morse.

Thereafter, von Neumann (cf. [14])developed Banach's idea of the invariant linear functional on topological group. He not only gave the definition of amenable group but also proved the following proposition.

Proposition. *(von Neumann)*

1. Finite groups are amenable.

2. Abelian groups are amenable.

3. If N is a normal subgroup of a group G and both N and G/N are amenable, then G is amenable.

In 1938, Agnew and Morse (cf. [1]), applying von Neumann's results, generalized Hahn-Banach Theorem, which is known as Agnew-Morse Theo-

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rem.

More systematic study on amenable groups and amenable semigroups is carried out by Day in [4], [5] and [6]. Many results and applications of invariant means were exposed systematically by Greenleaf in [7].

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