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局部緊緻群上的權重平移算子

Weighted translation operators on locally compact groups

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本論文係 陳奎佑 君（R04221001）在國立臺灣大學數學系完成之碩士學位論文，於民國 106 年 5 月 15 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

我們主要探討的問題是「局部緊緻群上是否存在超循環權重平移算子？」。首先我們刻劃了非週期性元素的一些等價條件，並給出非週期性元素的超循環權重平移算子存在性。另一方面，我們也給出一些週期性元素的權重平移算子是超循環算子的必要條件。

關鍵詞：非週期性、週期性、超循環算子、權重平移算子、局部緊緻群、L^p空間。

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Abstract

Our main question is ``Does there exist a hypercyclic weighted translation operator on locally compact groups?". In this article, we give several equivalent characterization of aperiodicity of an element on locally compact group. If a is an aperiodic element, we show that there exists a frequently hypercyclic weighted translation $T_{a,w}$. If a is a periodic element, we give necessary conditions of hypercyclic weighted translation $T_{a,w}$.

Keywords: Aperiodicity, periodicity, hypercyclicity, weighted translation operator, locally compact group, L^p-space.

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Chapter 1

Introduction

Let G be a locally compact group and let a be an element of G. The main aim of this article is to investigate the existence of hypercyclic weighted translation operators associated to a and try to construct the hypercyclic weighted translation operators if such operators are known to exist. The goal has been achieved in the case where a is aperiodic. The details in the aspect makes up the content of Part I (Chapter 2) of this article. In Part II (Chapter 3) we prove several necessary conditions for the existence of hypercyclic weighted translation operators associated to a periodic a.

This article is inspired by [5], and also [3, 4], which characterize the chaoticity and hypercyclicity of a weighted translation operator on the L^p space of a locally compact group.

An operator T on a Banach space X is called *hypercyclic* if there exists a vector $x \in X$ such that its orbit

$$\mathrm{orb}(T, x) := \{T^n x | n \in \mathbb{N}\}$$

is dense in the whole space X. In this article, we focus on weighted translation operators. The weighted translation operator $T_{a,w}$ is a bounded linear self-map on the Banach space $L^p(G)$ (by using the right Haar measure on G), for some $p \in [1, \infty)$, defined by

$$T_{a,w}(f)(x) := w(x)f(xa^{-1}),$$

where the weight w is a bounded continuous function from G to $(0, \infty)$. We denote $T_{a,1}$ by T_a so that $T_a w$ is the function w translation by a, while $T_{a,w}$ is a weighted translation operator.

We call an element a of G *torsion* if it has finite order. An element a is *periodic* if the closed subgroup $G(a)$ generated by a (i.e. $G(a) = \overline{< a >}$) is compact in G. An element a is *aperiodic* if it is not periodic. Such classification is meaningful. For example, we have the following important result.

Lemma. [5, Lemma 1.1.] *Let G be a locally compact group and let $a \in G$ be a torsion element. Then any weighted translation $T_{a,w} : L^p(G) \to L^p(G)$ is not hypercyclic, for $1 \leq p < \infty$.*

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[5, Lemma 1.1] gives the non-existence of hypercyclic weighted translations when the element a is torsion. So the question of the existence of hypercyclic weighted translations will be focused on the case in which a is non-torsion, this is, a is either non-torsion periodic or a is aperiodic.

All examples of hypercyclic weighted translation operators in previously known literature are all associated to aperiodic a. One of the most important concrete examples is the weighted backward shift operator on $\ell^p(\mathbb{Z})$ [7, Example 4.15. p.102] and there are also some classical analogous examples relevant to semigroups [12]; in fact, it can correspond to our case for several admissible weights by conjugation. In Chapter 2, we prove the following main theorem of this article, which is also labeled as Theorem 2.3.2.

Theorem 1 (Theorem 2.3.2)**.** *Let G be a second countable locally compact group and let a be an aperiodic element in G, then there exists a weighted translation operator $T_{a,w}$ which is mixing, chaotic and frequently hypercyclic on $L^p(G)$ for all $p \in [1, \infty)$, simultaneously.*

The main theorem is proved by showing that for every aperiodic a, there always exists an weighted translation associated to a, which satisfies Frequent Hypercyclicity Criterion (see Section 2.1). In general, Frequent Hypercyclicity Criterion is a strong condition that implies the hypercyclicity (see Section 2.1).

The existence of hypercyclic weighted translation operators associated to non-torsion periodic element is still an open problem, even no single non-torsion periodic a is known to be have a hypercyclic weighted translation operator.

Note that in Chapter 2 we don't need G to be Hausdorff, while in Chapter 3 we assume G is.

In Section 3.3, we find a necessary condition of hypercyclic weighted translation operators for non-torsion periodic element, which is also labeled as Theorem 3.3.1.

Theorem 2 (Theorem 3.3.1)**.** *Let G be a compact Hausdorff group, and a be an element in G such that $G = \overline{< a >}$. If $T_{a,w}$ is hypercyclic on $L^p(G)$ for some $p \in [1, \infty)$, then*

$$\int_G \ln w = 0.$$

In Section 3.4, we focus on the groups $\mathbb{Z}_p$ and $\mathbb{Q}_p$ (note that any non-zero element in these group are non-torsion periodic). And we also find a necessary condition of hypercyclic weighted translation operators, which is also labeled as Corollary 3.4.2 and Corollary 3.4.3.

Theorem 3 (Corollary 3.4.2 and Corollary 3.4.3)**.** *Let $G = \mathbb{Z}_p$ or $\mathbb{Q}_p$. If w is a locally constant weight for the weighted translation operator $T_{a,w}$, then $T_{a,w}$ is not hypercyclic on $L^{p'}(G)$ for any prime number p, any $p' \in [1, \infty)$ and any $a \in G$.*

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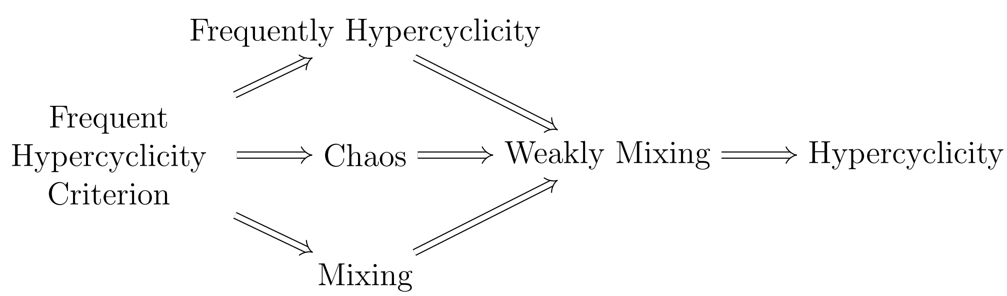
Chapter 2

Weighted translation operators of aperiodic elements

2.1 Notations and Preliminary

An operator T is called *weakly mixing* if $T \oplus T$ defined on $X \times X$ is hypercyclic. An operator T is called *mixing* if for any nonempty open subsets U, V in X, there exists $N \in \mathbb{N}$ such that $T^n U \cap V \neq \varnothing$ for all $n > N$. An operator T is called *chaotic* if it is hypercyclic and the set of periodic points is dense. An operator T is called *frequently hypercyclic* if there is some $x \in X$ such that for any nonempty open subset U of X, $n_k = O(k)$, where n_k is a strictly increasing sequence of integers such that $T^{n_k}x$ is k-th element lying in U (by [7, Proposition 9.3.], this is an equivalent statement of [7, Definition 9.2.]).

The graph below displays some relations between these dynamical properties which have been discussed, see [7]:



The graph above shows the three conditions in the second column which are implied by Frequent Hypercyclicity Criterion (in left) follows from a useful theorem.

Theorem. (Frequent Hypercyclicity Criterion, [7, Theorem 9.9 and Proposition 9.11.]). *Let T be an operator on a separable Fréchet space X. If there is a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$,*

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(1) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(2) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*

(3) $TSx = x$*,*

then T *is frequently hypercyclic. Moreover,* T *is also chaotic and mixing. In particular, it is also weakly mixing and hypercyclic.*

In the paragraphs above, only general conclusions are described; in the following discussions, we will focus on the weighted translation operators.

To analyze the weighted translation operators, we would like to consider more topological properties of the elements of the locally compact group in next section.

Lemma. [5, Lemma 2.1.] *An element* a *in a second countable group* G *is aperiodic if, and only if, for each compact subset* $K \subseteq G$*, there exists* $N \in \mathbb{N}$ *such that* $K \cap Ka^n = \varnothing$ *for* $n > N$*.*

In [5, Lemma 2.1], they gave an equivalence statement of aperiodicity when G is a second countable locally compact Hausdorff group. We give another equivalence statement when G is second countable, which we define in Definition 2.2.8 and named as *terminal pair*. And in section 2.3, we will use it to prove the existence of hypercyclic weighted translation operators.

2.2 Equivalent statements of aperiodicity

Proposition 2.2.1. *Let* G, G' *be locally compact groups, and* $\phi(a)$ *be an aperiodic element of* G'*, where* $\phi : G \to G'$ *is a continuous homomorphism. Then* a *is also an aperiodic element of* G*. In other words, continuous homomorphisms pull back the aperiodicity.*

Proof. If such a is periodic, then $\overline{<a>}$ is a compact group, and so is $\phi(\overline{<a>})$; but $\phi(a) \in \phi(\overline{<a>})$ derives a contradiction, since there is no aperiodic element in compact group. □

Let G be a topological group. we define the *Hausdorffication* of G by the natural continuous quotient map $\pi : G \to \widetilde{G}$, where $\widetilde{G} := G/\overline{\{e\}}$ and e denotes the identity element of G. (This may be related to the "Hausdorffication" which is defined as a left adjoint of forget functor in general topology). The reason that we consider the Hausdorffication is lots of statements in this paper without the assumption that G is Hausdorff. Actually, we first assume G is Hausdorff in proving. Next, we prove the non-Hausdorff case by considering its Hausdorffication and show it can preserve or pull back some conclusion we want.

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The claim form (1) to (4) in next Lemma are basic arguments for topological groups. But it is not so easy to find the reference for the non-Hausdorff group. For the reader's convenience, we will state it below with simple proof for each claim.

Lemma 2.2.2. *Let G be a topological group, and let $\pi : G \to \widetilde{G}$ be its Hausdorffication. Then we have the following statements:*

(1) Each open or closed subset of G is a union of the cosets of $\overline{\{e\}}$.

(2) There's a one to one correspondence between the set of open (closed) of G and $\widetilde{G}$'s. In particular, G is first (second) countable if and only if $\widetilde{G}$ also, and $\widetilde{G}$ is Hausdorff.

(3) π is an open, closed and proper mapping. In particular, G is locally compact iff $\widetilde{G}$ also. On the other hand, the one to one correspondence in (2), not only just closed sets but also closed compact sets (since π is proper).

(4) Let Y be an arbitrary Hausdorff topology space, then $\mathrm{Hom}(G, Y) \cong \mathrm{Hom}(\widetilde{G}, Y)$. (i.e. There's a natural one to one correspondence between the set of continuous functions from G to Y and $\widetilde{G}$'s. Equivalently, one can say that any continuous function from G to Y factor through $\widetilde{G}$. In other words, this is the universal property of Hausdorffication.)

(5) a is an aperiodic in G iff $\pi(a)$ is an aperiodic in $\widetilde{G}$.

proof of (1). Since $\overline{S}x = \overline{Sx}$ for any $S \subseteq G$ and $x \in G$. Choose $S = \{e\}$ and $x \in \overline{\{e\}}$, we have $\overline{\{x\}} = \overline{\{e\}x} = \overline{\{e\}}x = \overline{\{e\}}$ (last equality follows by $x \in \overline{\{e\}}$ and $\overline{\{e\}}$ is a subgroup of G). This shows that $\overline{\{e\}}$ is indiscrete topology space and so is all cosets of $\overline{\{e\}}$. This result implies the statement we are looking for. □

proof of (2). This correspondence is given by $(U \mapsto \pi(U))$ and $(\widetilde{U} \mapsto \pi^{-1}(\widetilde{U}))$ for U is open in G and $\widetilde{U}$ is open in $\widetilde{G}$. To verify that these two maps compose to identity for two sides, we only need to say $\pi^{-1}\pi(U) \subseteq U$ for any U is open in G, since the others are relatively obvious. Assume there exists $x \in \pi^{-1}\pi(U) \setminus U$, that is, $\pi(x) \in \pi(U)$. This means that there is some $y \in U$ such that $\pi(x) = \pi(y)$. Hence $xy^{-1} \in \operatorname{Ker}\pi = \overline{\{e\}}$ and $x \in \overline{\{e\}}y$. But by (1), $\overline{\{e\}}y \subseteq U$, since $y \in U$. A contradiction occurs as this implies $x \in U \cap \pi^{-1}\pi(U) \setminus U = \varnothing$. □

proof of (3). The openness and closedness follow by (1) and (2) immediately. For the properness, let $\widetilde{K}$ be compact in $\widetilde{G}$. We will check that $\pi^{-1}(\widetilde{K})$ is also compact. Let $\{U_\alpha\}$ be a open covering of $\pi^{-1}(\widetilde{K})$. By correspondence, $\{\pi(U_\alpha)\}$ be a open covering of $\widetilde{K}$. Hence it induce a finite subcovering $\{\pi(U_i)\}$. Then we can check the corresponding finite collection $\{U_i\}$ is actually a covering of $\pi^{-1}(\widetilde{K})$. □

proof of (4). Let $w \in \mathrm{Hom}(G, Y)$. We define $\widetilde{w}(\widetilde{x}) := w(x)$ where $\widetilde{x} = \overline{\{e\}}x$, then $\widetilde{w}$ is a well-defined continuous function on $\widetilde{G}$. On the other hand, let $\widetilde{w} \in \mathrm{Hom}(\widetilde{G}, Y)$.

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We have $w := \widetilde{w} \circ \pi$, it is also a well-defined continuous function on G. It is easy to check that these two mappings between $\mathrm{Hom}(G, Y)$ and $\mathrm{Hom}(\widetilde{G}, Y)$ are inverse to each others. □

proof of (5). If $\pi(a)$ is a periodic, then $\overline{<\pi(a)>}$ is compact and so is $\pi^{-1}(\overline{<\pi(a)>})$, since π is proper. However, $a \in \pi^{-1}(\overline{<\pi(a)>})$ which displays a contradiction. The other side follows from Proposition 2.2.1 immediately. □

The idea of the proofs of Proposition 2.2.3, Lemmas 2.2.4 and 2.2.5 and Proposition 2.2.6 follows from [5, Lemma 2.1].

Proposition 2.2.3. *Let G be a locally compact group, and let $a \in G$ satisfy the following properties: for any compact subset K of G, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ for $n > N$. Then a is an aperiodic element in G.*

Proof. Suppose a is a periodic element. Then the closed subgroup $G(a)$ generated by a is compact. Now we set $K = G(a)$, we have $K \cap Ka^n = G(a) \neq \varnothing$ for all $n \in \mathbb{Z}$, which is the negation of sufficient condition. □

Lemma 2.2.4. *Let G be a first countable locally compact Hausdorff group with an aperiodic element a in G. Then $G(a)$ is a second countable compactly generated abelian group.*

Proof. Since a is an aperiodic element, $G(a)$ is a non-compact closed abelian subgroup of G (the commutativity follows from net arguement and Hausdorffness of $G(a)$). Moreover, by [8, Theorem 5.14], there exists a compactly generated subgroup G' of G which contains $G(a)$, since $\{e, a\}$ is a compact subset of G.

Firstly, we will check that it is second countable. Since $G(a)$ is a first countable locally compact Hausdorff group, it is metrizable by [2, Birkhoff-Kakutani metrizable theorem]. On the other hand, since the set $\{a^j\}_{j\in\mathbb{Z}}$ is dense in $G(a)$, $G(a)$ is a separable metrizable space, as a result, it is second countable.

In the last step, we will check that it is compactly generated. Note that not every subgroup of a compactly generated group is compactly generated (even a closed subgroup), but in our case it holds. Let G' be generated by the compact set K_0 where $K_0 := \overline{V_0}$ for some open V_0 containing $\{e, a\}$ (the existence follows from the proof in [8, Theorem 5.14]). We will claim that $G(a)$ is generated by the compact set K_1 where $K_1 := \overline{V_1}$ and $V_1 := V_0 \cap G(a)$.

Since the set $\{a^j\}_{j\in\mathbb{Z}}$ is dense in $G(a)$, this implies that $\{V_1 a^j\}_{j\in\mathbb{Z}}$ is a covering of $G(a)$ (we will explain it in the Remark below). So $G(a) \subseteq \cup_{j\in\mathbb{Z}} V_1 a^j \subseteq \cup_{j\in\mathbb{Z}} K_1 a^j \subseteq G(a)$ (last step follows from $K_1 \subseteq G(a)$). That means that any $x \in G(a)$ can be written as the form ka^j for some $k \in K_1$ and some $j \in \mathbb{Z}$ (that is, $G(a)$ is generated by K_1, since a is also in K_1). □

Remark. To prove that $\{V_1 a^j\}_{j\in\mathbb{Z}}$ is a covering of $G(a)$, it is sufficient to say that every $x \in G(a)$ which is a limit of a subsequence $\{a^{n_k}\}$ has been covered by $\{V_1 a^{n_k}\}$ (we

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view V_1 as the relative open neighborhood of e in $G(a)$ here). Now choose V_2 the symmetric open neighborhood of e in V_1, then $a^{n_k} \in V_2 x$ for k large enough, hence $x \in V_2^{-1} a^{n_k} = V_2 a^{n_k} \subseteq V_1 a^{n_k}$, so the proof is completed.

Lemma 2.2.5. *Let G be a first countable locally compact Hausdorff group with $a \in G$ being an aperiodic element. Then $G(a)$ is topologically isomorphic to $\mathbb{Z}$.*

Proof. According to Lemma 2.2.4 and [8, Theorem 9.8], $G(a) \cong \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{F}$ for some $n, m \in \mathbb{N}$ and $\mathbb{F}$ being a compact group. The element a is identified as an element in $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{F} \setminus (\{0\} \times \{0\} \times \mathbb{F})$. Hence the cyclic subgroup $< a >$ has no accumulation point, which implies $G(a)$ being a discrete group, that is, $G(a) =< a >$, which is also isomorphic to $\mathbb{Z}$. □

Proposition 2.2.6. *Let G be a first countable locally compact group with an aperiodic element a in G. Then, for any compact subset K of G, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ for $n > N$.*

Proof. We first consider the case that G is Hausdorff.

If we assume there exists a compact set K such that $K \cap Ka^n \neq \varnothing$ for infinitely many n's, then for those n's, $a^n \in K^{-1}K$; but it is impossible since there must admit a convergent subsequence in the compact set $K^{-1}K$. This contradicts with the Lemma 2.2.5, so the case of Hausdorff is verified.

Let G be a first countable and not necessary Hausdorff locally compact group and $\pi : G \to \widetilde{G}$ be its Hausdorffication. For any compact set K in G, $\pi(K)$ is compact in $\widetilde{G}$. So there exists N such that

$$\pi(K) \cap \pi(K)\pi(a)^n = \varnothing \text{ for } n > N,$$

then

$$\begin{aligned} K \cap Ka^n &\subseteq (\pi^{-1} \circ \pi)(K) \cap (\pi^{-1} \circ \pi)(K)a^n \\ &= \pi^{-1}(\pi(K)) \cap \pi^{-1}(\pi(K)\pi(a)^n) \\ &= \pi^{-1}(\pi(K) \cap \pi(K)\pi(a)^n) \\ &= \pi^{-1}(\varnothing) \\ &= \varnothing \text{ for } n > N. \end{aligned}$$

□

Combining Proposition 2.2.3 and Proposition 2.2.6, we have the following Theorem.

Theorem 2.2.7. *Let G be a first countable locally compact group, then the following are equivalent:*

(1) $a \in G$ is an aperiodic element.

(2) For any compact subset K of G, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ for $n > N$.

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Definition 2.2.8 (terminal pair)**.** Let G be a locally compact group and $a \in G$. We say G has a terminal pair (A, B) w.r.t. a if there exists a pair of disjoint closed subsets (A, B) of G such that for any given compact subset K in G, we have

$$Ka^n \subseteq A$$

and

$$Ka^{-n} \subseteq B$$

for n large enough.

More intuitively, a shifts any compact subset positively (resp. negatively) into A (resp. B).

Example 2.2.9. One of the simplest cases is $G = \mathbb{Z}$ or $\mathbb{R}$ and $a = 1$, the terminal pair w.r.t. 1 can be given by $(A, B) = ([100, \infty), (-\infty, -100])$.

Example 2.2.10. Let G be a general linear group $GL(n, \mathbb{C})$, $a \in G$ with some eigenvalue λ such that $|\lambda| \neq 1$, then G admits a terminal pair w.r.t. a.

Proof. Without loss of generality, we can assume a itself is a Jordan form,

$$a = \begin{bmatrix} \lambda & * & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda' & * \\ 0 & \cdots & 0 & \lambda'' \end{bmatrix},$$

since we can act a conjugate automorphism on G.

Notation: Let any $x \in G$, we write $x = [x_1|x_2|...|x_n]$, where $x_i \in \mathbb{C}^n$ are the column of x. (Note that x_i will never be zero vector since x is invertible.)

Consider the map $f : G \to \mathbb{R}$, $f(x) = \ln \|x_1\|$. By the calculation, we have $f(xa^n) = f(x) + n * \ln|\lambda|$.

Set $(A, B) = (f^{-1}([1, \infty)), f^{-1}((-\infty, -1]))$, so for any compact subset K in G,

$$\inf_{x \in K} f(xa^n) = (\inf_{x \in K} f(x)) + n * ln|\lambda|$$

and

$$\sup_{x \in K} f(xa^n) = (\sup_{x \in K} f(x)) + n * ln|\lambda|,$$

which imply (A, B) is a terminal pair w.r.t. a. □

Proposition 2.2.11. *Let G be a locally compact group and admit a terminal pair w.r.t. a. Then a is an aperiodic element.*

Proof. Suppose a is a periodic element but G also admits a terminal pair w.r.t. a, then $G(a)a^n = G(a)$ for all $n \in \mathbb{Z}$, which implies that A and B are not disjoint. □

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Proposition 2.2.12. *Let G, G' be locally compact groups, and let G' admit a terminal pair w.r.t. $\phi(a)$, where $\phi : G \to G'$ is a continuous homomorphism. Then G also admits a terminal pair w.r.t. a. In other words, continuous homomorphisms pull back the terminal pair.*

Proof. Let $(A_{G'}, B_{G'})$ be a terminal pair w.r.t. $\phi(a)$. Equivalently, for any given compact subset K' in G', we have

$$K'\phi(a)^n \subseteq A_{G'}$$

and

$$K'\phi(a)^{-n} \subseteq B_{G'}$$

for n large enough.

Set $(A_G, B_G) = (\phi^{-1}(A_{G'}), \phi^{-1}(B_{G'}))$. By continuity of ϕ, (A_G, B_G) is also a pair of disjoint closed sets in G. For any given compact subset K in G, $\phi(K)$ is also compact, we have

$$Ka^n \subseteq \phi^{-1}(\phi(Ka^n)) = \phi^{-1}(\phi(K)\phi(a)^n) \subseteq \phi^{-1}(A_{G'}) = A_G$$

and

$$Ka^{-n} \subseteq \phi^{-1}(\phi(Ka^{-n})) = \phi^{-1}(\phi(K)\phi(a)^{-n}) \subseteq \phi^{-1}(B_{G'}) = B_G$$

for n large enough. □

Example 2.2.13. Let $G = S^1 \times \mathbb{R}$ and $a = (0, 1) \in S^1 \times \mathbb{R}$, where S^1 denotes the circle group. Then the natural quotient map $\phi : G \to \mathbb{R}$ sends a to 1, so terminal pair w.r.t. a can be given by $(A, B) = (\phi^{-1}((-\infty, -100]), \phi^{-1}([100, \infty)))$.

Proposition 2.2.14. *Let G be locally compact group and $\pi : G \to \widetilde{G}$ be the Hausdorff-fication of G. Then (A, B) is a terminal pair w.r.t. a iff $(\widetilde{A}, \widetilde{B}) = (\pi(A), \pi(B))$ is a terminal pair w.r.t. $\pi(a)$ with closed subsets A, B of G.*

Proof. Suppose (A, B) is a terminal pair w.r.t. a. Then $\widetilde{A}$ and $\widetilde{B}$ are also disjoint closed since π is closed and they are union of cosets of $\overline{\{e\}}$. Moreover, $(\widetilde{A}, \widetilde{B})$ is also a terminal pair w.r.t. $\pi(a)$. This follows by the one to one correspondence between the sets of closed compact subsets of G and $\widetilde{G}$ (i.e. $\widetilde{K} = \pi(\pi^{-1}(\widetilde{K}))$). The other side follows from Proposition 2.2.12. □

Recall that any second countable locally compact Hausdorff groups are compatible with a proper "right" invariant metric d [11]. In this metric space the Heine-Borel property holds. Therefore, $B_R(x) := \{y \in G | d(x, y) < R\}$ is a precompact open ball, and $B_R(xa) = B_R(x)a$ by right invariant.

Note that [11] said that there's a proper "left" invariant metric say d_L, but it's easy to induce a proper "right" invariant metric d by set $d(x, y) := d_L(x^{-1}, y^{-1})$.

Theorem 2.2.15. *Let G be a second countable locally compact Hausdorff group. Then the following are equivalent:*

(1) $a \in G$ is an aperiodic element.

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(2) *For any compact subsets K and K' in G, $d(Ka^\ell, K') \to \infty$ as $\ell \to \infty$.*

Proof. (2⇒1). Choose $K = K' = \{e\}$, then $d(e, a^\ell) \to \infty$ as $\ell \to \infty$. But if a is periodic, then $G(a)$ is compact, hence it is bounded. This implies that a^n are uniformly bounded, contradicting to $d(e, a^\ell) \to \infty$ as $\ell \to \infty$.

(1⇒2). Suppose $d(Ka^\ell, K')$ is uniformly bounded by $C > 0$, then for any $\ell \in \mathbb{N}$, there exists $k_\ell \in K$, $k'_\ell \in K'$ such that $d(k_\ell a^\ell, k'_\ell) = d(k_\ell a^\ell k'^{-1}_\ell, e) < C$. Since the closed ball centered at e is compact, so there's a subsequence $\{\ell'\}$, $k^{-1} \in K^{-1}$, $k' \in K'$ and an element b in the ball such that

$$k_{\ell'}^{-1} \to k^{-1},$$

$$k'_{\ell'} \to k',$$

$$k_{\ell'} a^{\ell'} k'^{-1}_{\ell'} \to b \text{ as } \ell' \to \infty.$$

Hence $a^{\ell'} \to k^{-1}bk'$ as $\ell' \to \infty$. This contradicts the Lemma 2.2.5, since $a^{\ell'}$ never converges in $G(a) \cong \mathbb{Z}$. □

Theorem 2.2.16. *Let G be a second countable locally compact group. Then the following are equivalent:*

(1) *$a \in G$ is an aperiodic element.*

(2) *For any compact subset K of G, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ for $n > N$.*

(3) *G has a terminal pair w.r.t. a.*

Proof. According to discussion above, it is sufficient to prove the case (1⇒3).

We first consider the case that G is Hausdorff.

Set

$$J := \left\{ x \in G \mid d(x, e) \le d(xa^{n'}, e) \text{ for all } n' \in \mathbb{Z} \right\}$$

and

$$N_x := min \{ N \in \mathbb{N} \mid 2d(x, e) + 2 < d(xa^n, e) \text{ for all } n \in \mathbb{Z} \text{ with } |n| > N \}.$$

The reason for the definition of J is that we want to simulate the special case $G = \mathbb{R}^2, a = (1, 0)$. In this special case, the terminal pair w.r.t. a can be setting as $(\{(x, y)|x \ge 100\}, \{(x, y)|x \le -100\})$. To deduce this, we need some kind of sense like y-axis which is orthogonal to a, thus, we define J to be the simulation of y-axis.

Note that they are well-defined, since Theorem 2.2.15 implies that both $d(xa^n, e)$ and $d(xa^{-n}, e) \to \infty$ as $n \to \infty$. And for each $x \in G$, there exists $xa^n \in J$ (i.e. $G = \{xa^n | x \in J, n \in \mathbb{Z}\}$), and this also follows from Theorem 2.2.15.

Note that $\mathfrak{B} := \left\{ B_{\frac{1}{4}}(xa^n) \right\}_{x \in J, n \in \mathbb{Z}}$ is a covering of G.

Claim 1: For any $x, y \in J$, $n > N_x$ and $m > N_y$, we have $d(xa^n, ya^{-m}) > 1$.

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Proof of Claim 1: Suppose $1 \geq d(xa^n, ya^{-m}) = d(x, ya^{-n-m}) = d(xa^{n+m}, y)$, then

$$\begin{aligned} 2 + 2d(x,e) &< d(xa^{n+m}, e) \\ &\leq d(xa^{n+m}, y) + d(y,e) \\ &\leq 1 + d(y,e) \\ &\leq 1 + d(ya^{-n-m}, e) \\ &\leq 1 + d(ya^{-n-m}, x) + d(x,e) \\ &\leq 2 + d(x,e), \end{aligned}$$

a contradiction.

Claim 2: For any $x, y \in J$, $n > N_x$ and $m > N_y$, we have

$$d(B_{\frac{1}{4}}(xa^n), B_{\frac{1}{4}}(ya^{-m})) > \frac{1}{4}.$$

Proof of Claim 2: Suppose $d(B_{\frac{1}{4}}(xa^n), B_{\frac{1}{4}}(ya^{-m})) \leq \frac{1}{4} < \frac{1}{3}$, then there exist $x' \in B_{\frac{1}{4}}(xa^n)$ and $y' \in B_{\frac{1}{4}}(ya^{-m})$ such that $d(x', y') < \frac{1}{3}$, so $d(xa^n, ya^{-m}) < \frac{1}{4} + \frac{1}{4} + \frac{1}{3} < 1$, an absurdity.

So far we are ready to set our terminal pair. Set $(A, B) := (\overline{A'}, \overline{B'})$, where

$$A' := \bigcup_{x \in J, n > N_x} B_{\frac{1}{4}}(xa^n)$$

and

$$B' := \bigcup_{x \in J, n > N_x} B_{\frac{1}{4}}(xa^{-n}).$$

By Claim 2, $d(A', B') \geq \frac{1}{4} > 0$, so A and B are disjoint closed set.

Now we are going to show that for each compact set K shifts into A and B. By compactness of K, there exists finitely many balls in $\mathfrak{B}$, $\left\{B_{\frac{1}{4}}(x_i a^{n_i})\right\}_{i=1}^{\ell}$ which covers K. Set
$N := 2\ max_{1 \leq i \leq \ell} \{|n_i - N_{x_i}|, |n_i + N_{x_i}|\}$, so $n_i + n > N_{x_i}$ and $n_i - n < -N_{x_i}$ for $n > N$. Then

$$Ka^n \subseteq \cup_{i=1}^{\ell} B_{\frac{1}{4}}(x_i a^{n_i}) a^n = \cup_{i=1}^{\ell} B_{\frac{1}{4}}(x_i a^{n_i + n}) \subseteq A$$

and

$$Ka^{-n} \subseteq \cup_{i=1}^{\ell} B_{\frac{1}{4}}(x_i a^{n_i}) a^{-n} = \cup_{i=1}^{\ell} B_{\frac{1}{4}}(x_i a^{n_i - n}) \subseteq B$$

for $n > N$, so the case of Hausdorff is verified.

Now we are going to consider the general case. Let G be a second countable locally compact group and $\pi : G \to \widetilde{G}$ be its Hausdorffication, so $\pi(a)$ is also aperiodic, hence $\widetilde{G}$ has a terminal pair w.r.t. $\pi(a)$. The proof is completed by Proposition 2.2.14.

□

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2.3 Existence of hypercyclic weighted translations

In the following section, we would like to discuss how the existence of terminal pair affects the existence of hypercyclic weighted translation operators.

Lemma 2.3.1. *Let G be a second countable locally compact group and admit a terminal pair w.r.t. a. Then there exists a weighted translation operator $T_{a,w}$ which satisfies the frequent hypercyclicity criterion on $L^p(G)$ for all $p \in [1, \infty)$, simultaneously.*

Proof. We are going to construct a weight w by Urysohn's lemma and verify the frequent hypercyclicity criterion [7, Theorem 9.9 and Proposition 9.11.] directly.

Let $\pi : G \to \widetilde{G}$ be the Hausdorffication of G.

Recall that any locally compact Hausdorff group is normal [8, Theorem 8.13]. So $\widetilde{G}$ is normal. Using the notation in Proposition 2.2.14, we set $\widetilde{w}|_{\widetilde{A}} = 2^{-1}$ and $\widetilde{w}|_{\widetilde{B}} = 2$. By Urysohn's lemma $\widetilde{w}$ is a well-defined continuous function on $\widetilde{G}$ with $\widetilde{w}(G) \subset [2^{-1}, 2] \subset (0, \infty)$. Thus we can induce $w := \widetilde{w} \circ \pi$ with $w|_A = 2^{-1}$, $w|_B = 2$ and $w(G) \subset [2^{-1}, 2] \subset (0, \infty)$.

To verify such $T_{a,w}$ satisfying frequent hypercyclicity criterion, we set

$$X_0 := \{\text{bounded compact support functions on } G\}$$

which is a dense subspace in $L^p(G)$ for all $p \in [1, \infty)$, and also set

$$T = T_{a,w} \text{ and } S = T_{a,w}^{-1} = T_{a^{-1},w'},$$

where $w' := (T_{a^{-1}} w)^{-1}$.

Now let $\varphi \in X_0$ and $K := supp(\varphi)$ (note that $supp(\varphi(\cdot\, a^i)) = Ka^{-i}$).
Claim: $\|T^n\varphi\|_\infty$ and $\|S^n\varphi\|_\infty$ decay to zero by exponential type.

That is,

$$\|T^n\varphi\|_\infty \le C\gamma^{-n} \text{ and}$$
$$\|S^n\varphi\|_\infty \le C\gamma^{-n} \text{ for some } C > 0,\ \gamma > 1 \text{ and } n \text{ large enough.}$$

Now we only need to prove the first case, since the second case is symmetric with the first case by replacing A to B, B to A, a to a^{-1} , w to w' and T to S. Since K is compact, there is a large number N such that

$$Ka^n \subseteq A$$

and

$$Ka^{-n} \subseteq B$$

for $n \ge N$, then

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$$
\begin{aligned}
\|T^n\varphi\|_\infty &= \left\|\prod_{i=0}^{n-1} w(xa^{-i})\varphi(xa^{-n})\right\|_\infty \\
&= \left\|\prod_{i=0}^{n-1} w(xa^{n-i})\varphi(x)\right\|_\infty \\
&= \left\|\prod_{j=1}^{n} w(xa^{j})\varphi(x)\right\|_\infty \quad (j = n-i) \\
&\le \left\|\prod_{j=1}^{n} w(xa^{j})|_K\right\|_\infty \|\varphi\|_\infty \\
&\le \left\|\prod_{j=1}^{N} w(xa^{j})\right\|_\infty \|\varphi\|_\infty \prod_{j=N+1}^{n} \left\|w(xa^{j})|_K\right\|_\infty \\
&= \left\|\prod_{j=1}^{N} w(xa^{j})\right\|_\infty \|\varphi\|_\infty \prod_{j=N+1}^{n} \|w(x)|_{Ka^j}\|_\infty \\
&\le \left\|\prod_{j=1}^{N} w(xa^{j})\right\|_\infty \|\varphi\|_\infty \prod_{j=N+1}^{n} 2^{-1} \quad (\text{since } Ka^j \subseteq A).
\end{aligned}
$$

So the claim is verified.

Finally, we will show that

$$\sum_{n=0}^{\infty} T^n\varphi \text{ converges unconditionaly}$$

and

$$\sum_{n=0}^{\infty} S^n\varphi \text{ converges unconditionaly.}$$

Again, we only need to verify the first case, and we will prove that they are actually convergent absolutely. By aperiodicity of a, there is a large number N_0 such that

$$K \cap Ka^{-n} = \varnothing \text{ for } n \ge N_0,$$

then we can easily check $\mathcal{C}_\ell := \{Ka^{-(N_0 j+\ell)}\}_{j\in\mathbb{Z}}$ is a mutually disjoint collection for each $\ell = 0, 1, ..., N_0 - 1$. So

$$\left\|\sum_{n=0}^{\infty} T^n\varphi\right\|_p \le \left\|\sum_{\ell=0}^{N_0-1}\sum_{j=0}^{\infty} \left|T^{N_0 j+\ell}\varphi\right|\right\|_p \le \sum_{\ell=0}^{N_0-1}\left\|\sum_{j=0}^{\infty} \left|T^{N_0 j+\ell}\varphi\right|\right\|_p .$$

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For any given $\ell = 0, 1, ..., N_0 - 1$,

$$\begin{aligned}
\left\| \sum_{j=0}^{\infty} \left| T^{N_0 j+\ell} \varphi \right| \right\|_p^p &= \int \left| \sum_{j=0}^{\infty} T^{N_0 j+\ell} \varphi \right|^p \\
&= \int \sum_{j=0}^{\infty} \left| T^{N_0 j+\ell} \varphi \right|^p \ (C_\ell \text{ are mutually disjoint}) \\
&= \sum_{j=0}^{\infty} \int \left| T^{N_0 j+\ell} \varphi \right|^p \\
&\leq \sum_{n=0}^{\infty} \| T^n \varphi \|_p^p \\
&= \sum_{n=0}^{\infty} \int_{Ka^n} |T^n \varphi|^p \ (supp(T^n \varphi) \subseteq Ka^n) \\
&\leq \sum_{n=0}^{\infty} |Ka^n| \, \| T^n \varphi \|_\infty^p \ \ (|Ka^n| = |K|) \\
&= |K| \sum_{n=0}^{\infty} \| T^n \varphi \|_\infty^p < \infty.
\end{aligned}$$

Notice that the last inequality follows from $\|T^n \varphi\|_\infty$ decaying to zero by exponential type.

Therefore, the condition of frequent hypercyclicity criterion holds. □

Remark. The existence of weighted translation operator $T_{a,w}$ is not unique. In fact, there are uncountable many weighted translations satisfying this lemma by setting $w|_A \equiv \alpha$ and $w|_B \equiv \beta$ for $\alpha \in (0, 1)$ and $\beta \in (1, \infty)$ whatever you like in the proof.

Theorem 2.3.2. *Let G be a second countable locally compact group and let a be an aperiodic element in G. Then there exists a weighted translation operator $T_{a,w}$ which is mixing, chaotic and frequently hypercyclic on $L^p(G)$ for all $p \in [1, \infty)$, simultaneously.*

Proof. By Theorem 2.2.16 and Lemma 2.3.1. □

Example 2.3.3. Let G be an arbitrary Lie group and let a be an aperiodic element in G. Then there exists a weighted translation operator $T_{a,w}$ which is mixing, chaotic and frequently hypercyclic on $L^p(G)$ for all $p \in [1, \infty)$, simultaneously.

Example 2.3.4. Let G be a general linear group $GL(n, \mathbb{C})$. Then a is a periodic element of G iff a is diagonalizable with each eigenvalue has norm 1 (i.e. if λ is a eigenvalue of a, then $|\lambda| = 1$). Without using Theorem 2.2.16, it would be hard to verify G has a terminal pair w.r.t. a when a is aperiodic, for instance, $a = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

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Whatever, if $a \in GL(n, \mathbb{C})$ is not diagonalizable or has an eigenvalue that doesn't has norm 1, then there exist a positive continuous function w such that $T_{a,w}$ is hypercyclic on $L^p(GL(n, \mathbb{C}))$.

Remark. To explain why a is periodic if and only if a is diagonalizable with all eigenvalue has norm 1. We only need to prove the case that if a is nondiagonalizable then a is aperiodic. Since other cases follow by Example 2.2.10 and Proposition 2.2.11. The same as the Example 2.2.10, we can assume a is itself a Jordan form:

$$a = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & \vdots \\ \vdots & \cdots & \ddots & * \\ 0 & \cdots & 0 & \lambda' \end{bmatrix}.$$

Notation: Let any $x \in G$, we write $x = [x_1|x_2|...|x_n]$, where $x_i \in \mathbb{C}^n$ are the column of x. (Note that x_i will never be zero vector since x is invertible.)

Consider the map $f : G \to \mathbb{R}$, $f(x) = |\ln \|x_2\||$, by the calculation, we have $f(a^n) = \left|(n-1)\ln|\lambda| + \ln|n| + \frac{1}{2}\ln|1 + \frac{|\lambda|^2}{n^2}|\right| \to \infty$ as $n \to \infty$ whatever λ might be (λ never be zero since $a \in GL(n, \mathbb{C})$). Suppose a is periodic. Then $f(G(a))$ is compact in $\mathbb{R}$, but $\{f(a^n)\}$ is unbounded in $\mathbb{R}$, an absurdity.

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Chapter 3

Weighted translation operators of periodic elements

3.1 Notations and Preliminary

In this chapter, we only consider that G is a compact Hausdorff group except few examples.

The same as Chapter 2, section 3.2 is prepared for the section 3.3, which argues that the sequence $\{a^{-n}\}_{n\in\mathbb{N}}$ is *homogeneous equidistribution* (Definition 3.2.2). And section 3.3 will use it to prove a necessary condition for some hypercyclic weighted translation operators with non-torsion periodic element. For the section 3.4, we focus on p-adic groups. And we also give a necessary condition for hypercyclic weighted translation for it.

We denote $\widetilde{U}$ to be an arbitrary borel set with U being an open set and $U \subseteq \widetilde{U} \subseteq \overline{U}$. (Obviously, this definition is not well-defined for a given open set U, but we denote those sets by the same symbol). (See also [9, Chapter 3], they call $\widetilde{U}$ a μ-continuity set when $|\partial U| = 0$).

Example. Let $G = S^1 = [0, 1)$ and $U = (0, \frac{1}{2})$, where S^1 denotes the circle group. There are 4 possibilities of the expression $\widetilde{U}$, $(0, \frac{1}{2})$, $[0, \frac{1}{2})$, $(0, \frac{1}{2}]$ and $[0, \frac{1}{2}]$.

We denote $\mathcal{A}$ to be an algebra (closed under finite union and complement) generated by the sets $\{\text{every } \widetilde{U} | \partial U \text{ has measure } 0\}$.

Example. In the case of $G = S^1 = [0, 1)$, these sets $(0, \frac{1}{2})$, $[0, \frac{1}{2})$, $(0, \frac{1}{2}]$ and $[0, \frac{1}{2}]$ belong to $\mathcal{A}$.

There is a simple observation:

Proposition 3.1.1. *The elements in $\mathcal{A}$ are one of the forms below:*

(1) $\widetilde{U}$, where ∂U has measure 0.

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(2) $S = \widetilde{U} \setminus U \subseteq \partial U$*, where* $\widetilde{U}$ *has the form (1).*

(3) $\widetilde{U} \cup \cup_{i=1}^{k} S_i$*, where* $\widetilde{U}$ *has the form (1), and* S_i *have the form (2).*

Note that this classification is not disjoint.

Proof.
1. We first consider the union and complement of form (1). It is not hard to see that the union of form (1) is also form (1). But for complement of form (1) there are two possibilities, form (1) or form (2), determined by whether it is dense in G.
2. For the form (2), let $S \subseteq \partial U$ be an element of the form (2) and K be form (1) or (2) or (3), where $\widetilde{U}$ has the form (1). Then $S \cup K$ has the form (3) and S^c has the form (1), this follows by the facts that $\partial S^c = \partial S \subseteq \overline{S} \subseteq \partial U$ is measure 0 and $S^c = int(S^c) \cup (S^c \setminus int(S^c))$.
3. Finally, we consider the form (3).

It is not hard to see that the union of the elements of the form (3) are also the form (3).

Let $\widetilde{U} \cup \cup_{i=1}^{k} S_i$ be an element of the form (3). Then its complement is $\widetilde{U}^c \cap (\cap_{i=1}^{k} S_i^c)$, which is the finite intersection of the form (1). And the finite intersection of the form (1) is either the form (1) or the form (2).

In conclusion, the sets generated by the form (1) are also of the three forms. □

Recall the representation theory of compact group [6, Chapter 5]. We denote $\hat{G}$ to be the set of unitary equivalence classes of irreducible unitary representations of the compact Hausdorff G. Let $[\pi] \in \hat{G}$, $\pi : G \to U(H_\pi) \subset M(H_\pi)$ be a group homomorphism, where H_π is denoted to be the finite dimension complex Hilbert space w.r.t. π [6, Theorem 5.2]. $M(H_\pi)$ is denoted to be the collection of linear operators on H_π and $U(H_\pi)$ is its subcollection of unitary matries. We denote $\pi_{i,j}(x) :=< \pi(x)e_i, e_j >$ to be the matrix elements, where $\{e_i\}$ is the standard basis of H_π.

3.1.1 Notation for p-adic analysis

This subsection is prepared for the discussions which refer to the p-adic groups.

Let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic field and its ring of integer for some prime number p. Since they are both DVR (discrete valuation ring). We denote their valuation by v_p, (ex: $v_5(100) = v_5(5^2) + v_5(4) = 2 + 0 = 2$). Hence the p-adic norm $|\cdot|_p := p^{v_p(\cdot)}$ defines their topology. (See [10]).

We denote M_a as a multiplication operator which is defined by

$$M_a(f)(x) := f(ax).$$

Remark. In the discussion of the p-adic groups the weighted translation operator will become $T_{a,w}(f)(x) := w(x)f(x-a)$, since we view p-adic groups as additive groups. Both weighted translation operators and multiplication operators act on the $L^{p'}(G)$, where $G = \mathbb{Q}_p$ or $\mathbb{Z}_p$ and $p' \in [1, \infty)$. (Note that the notation for "p" is different from other places in this article.)

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3.2 Homogeneous equidistribution on compact groups

Let $\mathbb{C}(G)$ denote the continuous complex value functions on G.

Lemma 3.2.1. *Let G be a compact Hausdorff group, and let a be an element in G such that $\pi(a)$ do not have nontrivial fixed points (equivalently, $I_{H_\pi} - \pi(a)$ is invertible) for all $[\pi] \in \hat{G} \setminus \{[1]\}$. Then for any $f \in \mathbb{C}(G)$, define $g_{f,N}(x) := \frac{1}{N}\sum_{n=1}^{N-1} f(xa^{-n})$. We have*

$$g_{f,N} \text{ converges uniformly to the constant } \int_G f.$$

Proof. Let S be the collection consisting all f which satisfies this Lemma. It is non-empty, since all constant functions belong to S.

Recall that *span* $\left\{\text{all matrix elements } \pi_{i,j} \text{ of } \pi | [\pi] \in \hat{G}\right\}$ is dense in $\mathbb{C}(G)$ [6, Theorem 5.11]. In particular, the only closed subspace of $\mathbb{C}(G)$ containing all matrix elements is $\mathbb{C}(G)$ itself. So it is sufficient to show S is a closed subspace containing all matrix elements.

Note that $\|g_{h,N}\|_\infty \le \|h\|_\infty$ for any $h \in \mathbb{C}(G)$.
Claim: S is a closed subspace in $\mathbb{C}(G)$ (w.r.t. $\|\cdot\|_\infty$).

S is a subspace, since for any $f, f' \in S$ and $A, B \in \mathbb{C}$ we have $g_{Af+Bf',N} = Ag_{f,N} + Bg_{f',N}$. So $g_{Af+Bf',N}$ also converges uniformly to the constant $\int_G Af + Bf'$.

Let $f_i \in S$ which converges uniformly to f. That means that for any $\varepsilon > 0$, we have $\|f - f_i\|_\infty < \varepsilon/3$ for some i and there is a number N' such that $\left\|g_{f_i,N} - \int_G f_i\right\|_\infty < \varepsilon/3$ for all $N > N'$. Then

$$\begin{aligned}\left\|g_{f,N} - \int_G f\right\|_\infty &\le \left\|g_{f_i,N} - \int_G f_i\right\|_\infty + \|g_{f_i,N} - g_{f,N}\|_\infty + \left\|\int_G f_i - \int_G f\right\|_\infty \\ &\le \left\|g_{f_i,N} - \int_G f_i\right\|_\infty + \left\|g_{(f_i - f),N}\right\|_\infty + \|f_i - f\|_\infty \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Hence S is a closed subspace.
Claim: If $[\pi] \in \hat{G} \setminus \{[1]\}$, then $\pi_{i,j} \in S$.

Firstly, we will show that $\int_G \pi$ is zero matrix. Hence $\int_G \pi_{i,j} = 0$.

By the property of Haar measure on compact groups that it is two-sided invariant, we have

$$\begin{aligned}(\int_G \pi) &= (\int_G \pi(x)dx) \\ &= (\int_G \pi(xy)dx) = (\int_G \pi(x)dx)\pi(y) = (\int_G \pi)\pi(y)\end{aligned}$$

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and

$$\begin{aligned}(\int_G \pi) &= (\int_G \pi(x)dx)\\ &= (\int_G \pi(yx)dx) = \pi(y)(\int_G \pi(x)dx) = \pi(y)(\int_G \pi)\end{aligned}$$

for any $y \in G$. So $\int_G \pi$ commutes with all $\pi(G)$, by Schur's Lemma [6, Lemma 3.5], $\int_G \pi = cI_{H_\pi}$ for some $c \in \mathbb{C}$. If $c \neq 0$, then $(\int_G \pi) = \pi(y)(\int_G \pi)$ implies $I_{H_\pi} = \pi(y)$ for all $y \in G$, which contradicts to the assumption that $[\pi] \neq [1]$. Thus $\int_G \pi$ is zero matrix.

Secondly, let $\|\cdot\|_M$ denote the operator norm. We will show that $\left\|\|g_{\pi,N}\|_M\right\|_\infty \to 0$ as $N \to \infty$. In particular, $\left\|g_{\pi_{i,j},N} - \int_G \pi_{i,j}\right\|_\infty = \left\|g_{\pi_{i,j},N}\right\|_\infty \to 0$ as $N \to \infty$. Hence $\pi_{i,j} \in S$.

$$\begin{aligned}\left\|\|g_{\pi,N}\|_M\right\|_\infty &= \frac{1}{N}\left\|\left\|\sum_{n=1}^{N-1} \pi(xa^{-n})\right\|_M\right\|_\infty\\ &\leq \frac{1}{N}\left\|\|\pi(x)\|_M \left\|\sum_{n=1}^{N-1} \pi(a)^{-n}\right\|_M\right\|_\infty\\ &\leq \frac{1}{N}\left\|\sum_{n=1}^{N-1} \pi(a)^{-n}\right\|_M \quad (\pi(x) \text{ and } \pi(a) \text{ are unitary.})\\ &\leq \frac{1}{N}\left\|(I_{H_\pi} - \pi(a))^{-1}\right\|_M \left\|I_{H_\pi} - \pi(a)^N\right\|_M\\ &\leq \frac{2}{N}\left\|(I_{H_\pi} - \pi(a))^{-1}\right\|_M \to 0 \text{ as } N \to \infty.\end{aligned}$$

□

Definition 3.2.2. Let G be a compact Hausdorff group. We say a sequence $\{x_k\}_{k\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. A, if A is a collection of subsets of G and

$$\sup_{x\in G}\left|Dens_{N,K,\{x_k\}_{k\in\mathbb{N}}}(x) - |K|\right| \to 0 \text{ as } N \to \infty \text{ for any } K \in A,$$

where $Dens_{N,K,\{x_k\}_{k\in\mathbb{N}}}(x) := dens_N(x^{-1}K, \{x_k\}_{k\in\mathbb{N}})$ and

$$dens_N(K, \{x_k\}_{k\in\mathbb{N}}) := \frac{\#\{x_k \in K | k < N\}}{N} = \frac{1}{N}\sum_{k=1}^{N-1} \chi_K(x_k).$$

Lemma 3.2.3. *Let G be a compact Hausdorff group, and a be an element in G such that $\pi(a)$ do not have nontrivial fixed points for all $[\pi] \in \hat{G} \setminus \{[1]\}$. Then $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\mathcal{A}$.*

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Proof. We seperate the proof into two parts. For the first part, we will prove that $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. the form (1) in $\mathcal{A}$. (We have defined $\mathcal{A}$ in Section 3.1.)

For any $\varepsilon > 0$ and U open in G with ∂U being measure 0, define $f^+_{U,\varepsilon}$ and $f^-_{U,\varepsilon}$ to be some continuous functions which satisfy the following:

(1) Both image of $f^+_{U,\varepsilon}$ and $f^-_{U,\varepsilon}$ lie in $[0, 1]$.

(2) $f^+_{U,\varepsilon}|_{\overline{U}} \equiv 1$ and $f^+_{U,\varepsilon}|_{V^c} \equiv 0$, where V is an open neighborhood of $\overline{U}$ such that $|V \setminus U| = |V \setminus \overline{U}| < \varepsilon/2$.

(3) $f^-_{U,\varepsilon}|_K \equiv 1$ and $f^-_{U,\varepsilon}|_{U^c} \equiv 0$, where K is a compact subset of U such that $|U \setminus K| = |\overline{U} \setminus K| < \varepsilon/2$.

The existence of $f^+_{U,\varepsilon}$ and $f^-_{U,\varepsilon}$ follow by the Uryshon's lemma (It works, since G is compact Hausdorff, hence normal.) and the regularity of Haar measure. From the construction, we immediatly have

$$\chi_K \leq f^-_{U,\varepsilon} \leq \chi_{\widetilde{U}} \leq f^+_{U,\varepsilon} \leq \chi_V,$$

hence (use the notation in the proof of Lemma 3.2.1)

$$g_{f^-_{U,\varepsilon},N} \leq dens_N(x^{-1}\widetilde{U}, \{a^{-n}\}_{n\in\mathbb{N}}) \leq g_{f^+_{U,\varepsilon},N} \text{ for any } N.$$

By Lemma 3.2.1, there exists N' independent of x such that

$$\begin{aligned}
|\widetilde{U}| - \varepsilon &= \int_G \chi_{\widetilde{U}} - \varepsilon \\
&\leq \int_G f^-_{U,\varepsilon} - \varepsilon/2 \\
&< g_{f^-_{U,\varepsilon},N} \\
&\leq dens_N(x^{-1}\widetilde{U}, \{a^{-n}\}_{n\in\mathbb{N}}) = Dens_{N,\widetilde{U},\{a^{-n}\}_{n\in\mathbb{N}}}(x) \\
&< g_{f^+_{U,\varepsilon},N} \\
&\leq \int_G f^+_{U,\varepsilon} + \varepsilon/2 \\
&\leq \int_G \chi_{\widetilde{U}} + \varepsilon \\
&= |\widetilde{U}| + \varepsilon
\end{aligned}$$

for all $N > N'$. Then we have

$$\sup_{x\in G} \left| Dens_{N,\widetilde{U},\{a^{-n}\}_{n\in\mathbb{N}}}(x) - |\widetilde{U}| \right| < \varepsilon.$$

For the sencond part, we will consider the additivity of the function $Dens$.

Let X, Y be disjoint subsets of G. If any two of the followings hold, then so does the third:

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(1) $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\{X\}$.

(2) $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\{Y\}$.

(3) $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\{X \cup Y\}$.

This follows by the equations

$$Dens_{N,X\cup Y,\{a^{-n}\}_{n\in\mathbb{N}}} = Dens_{N,X,\{a^{-n}\}_{n\in\mathbb{N}}} + Dens_{N,Y,\{a^{-n}\}_{n\in\mathbb{N}}}$$

and

$$|X \cup Y| = |X| + |Y|.$$

For the form (2) in $\mathcal{A}$. Let $S = \widetilde{U} \setminus U$, where $\widetilde{U}$ has the form (1). We can write $\widetilde{U} = S \cup U$ as a disjoint union. Hence $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\{S\}$ (since U is itself another form (1).)

Similar to the form (3) in $\mathcal{A}$. We can write it as the finite disjoint union of the form (1) and (2). So $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. the form (3) in $\mathcal{A}$.

Hence the lemma has been proved.

□

Corollary 3.2.4. *Let G be a compact Hausdorff group and $a \in G$. Then the following are equivalent:*

(1) $G = \overline{<a>}$.

(2) For any $\pi \in \hat{G}\backslash\{[1]\}$, $\pi(a)$ *does not have nontrivial fixed points for all* $[\pi] \in \hat{G}\backslash\{[1]\}$.

Proof. (1⇒2) Since $G = \overline{<a>}$ is abelian, $\hat{G}$ is its daul group. And we know that $\dim H_\pi = 1$ for all $\pi \in \hat{G}$. So for any $\pi \in \hat{G} \setminus \{1\}$, $1 - \pi(a) \neq 0$ is invertible, since if $\pi(a) = 1$, we have $\pi(a)^n = 1$ for all $n \in \mathbb{Z}$, a contradiction as $\pi \equiv 1$.

(2⇒1) By Lemma 3.2.3, $\{a^{-n}\}_{n\in\mathbb{N}}$ is a homogeneous equidistribution w.r.t. $\mathcal{A}$. This implies that $\{a^{-n}\}_{n\in\mathbb{N}}$ is dense in G (it's sufficient to say any that non-empty open set in G contains an element of form (1) in $\mathcal{A}$, see [9, Chapter 3, example 1.2.]), hence $G = \overline{<a>}$. □

Corollary 3.2.5. *Let G be a finite group. Then the following are equivalent:*

(1) G is non-cyclic.

(2) For any $a \in G$, there exists $\pi \in \hat{G} \setminus \{[1]\}$ such that $\pi(a)$ has an eigenvalue 1.

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3.3 A necessary condition of hypercyclic weighted translation operators

Theorem 3.3.1. *Let G be a compact Hausdorff group, and let a be an element in G such that $G = \overline{<a>}$. If $T_{a,w}$ is hypercyclic on $L^p(G)$ for some $p \in [1,\infty)$, then*

$$\int_G \ln w = 0.$$

Proof. First, we will observe some facts for the step functions:

Let $\phi > 0$ be a strictly positive $\mathcal{A}$-step function (i.e ϕ can write as $\sum_{i=1}^{k} \alpha_i \chi_{E_i}$, where $\alpha_i > 0$ and $E_i \in \mathcal{A}$ are disjoint). Set $m_i := |E_i|$. By Lemma 3.2.3, for any $\varepsilon > 0$, there are N' such that

$$m_i - \varepsilon < \frac{1}{N}\#\{xa^{-n} \in E_i | n < N\} < m_i + \varepsilon$$

for any $x \in G$ and $N > N'$ and $i = 1 \cdots k$. Then

$$\prod_{i=1}^{k} \alpha_i^{m_i - \varepsilon} < \left(\prod_{i=1}^{k} \alpha_i^{\#\{xa^{-n} \in E_i | n < N\}} \right)^{\frac{1}{N}} < \prod_{i=1}^{k} \alpha_i^{m_i + \varepsilon}.$$

Hence

$$\prod_{i=1}^{k} \alpha_i^{m_i - \varepsilon} < (\phi_N)^{\frac{1}{N}} < \prod_{i=1}^{k} \alpha_i^{m_i + \varepsilon},$$

where $\phi_N(x) := \prod_{n=0}^{N-1} \phi(xa^{-n})$, since $\prod_{n=0}^{N-1} \phi(xa^{-n}) = \prod_{i=1}^{k} \alpha_i^{\#\{xa^{-n} \in E_i | n < N\}}$.

Now we are going to claim two things: Let ϕ be a strictly positive $\mathcal{A}$-step function, then

(1) $w \geq \phi$ implies $\int_G \ln \phi \leq 0$.

(2) $w \leq \phi$ implies $\int_G \ln \phi \geq 0$.

Since $\int_G \ln \phi = \sum_{i=1}^{k} m_i \ln \alpha_i = \ln \left(\prod_{i=1}^{k} \alpha_i^{m_i} \right)$, so $\int_G \ln \phi \leq 0 \Leftrightarrow \prod_{i=1}^{k} \alpha_i^{m_i} \leq 1$ and $\int_G \ln \phi \geq 0 \Leftrightarrow \prod_{i=1}^{k} \alpha_i^{m_i} \geq 1$.

For (1). Suppose $\prod_{i=1}^{k} \alpha_i^{m_i} > 1$ but $w \geq \phi$. We can choose ε small enough such that $1 < (\phi_N)^{\frac{1}{N}}$. Then $1 < \phi_N \leq w_N$, where $w_N(x) := \prod_{n=0}^{N-1} w(xa^{-n})$. But $T_{a,w}^N = T_{a^N, w_N}$ is also hypercyclic [1], contradict to the fact that a weighted translation operator with the wieght greater then 1 will never be a hypercyclic operator.

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Simiarly, for (2). Suppose $\prod_{i=1}^{k} \alpha_i^{m_i} < 1$, but $w \leq \phi$. We can choose ε small enough such that $1 > w_N$, contradict to the fact that a weighted translation operator with the wieght smaller then 1 will never be a hypercyclic operator.

Note that we write $\Phi := \ln \phi$ and $W := \ln w$, then $\phi \leq w \Leftrightarrow \Phi \leq W$ and $\phi \geq w \Leftrightarrow \Phi \geq W$. Moreover, Φ is still a $\mathcal{A}$-step function and any $\mathcal{A}$-step function is the form $\ln \phi$.

Finally, we will prove $\int_G \ln w = \int_G W = 0$ by the following:

(i) If the statement "$W \geq \Phi$ implies $\int_G \Phi \leq 0$" holds for all Φ, then $\int_G W \leq 0$.

(ii) If the statement "$W \leq \Phi$ implies $\int_G \Phi \geq 0$" holds for all Φ, then $\int_G W \geq 0$.

To prove this, it is sufficient to show that the $\mathcal{A}$-step functions are enough to approximate the continuous function W from above (from below) in $\|\cdot\|_\infty$ norm.

Consider the sets $B_\beta := \{x | W(x) = \beta\}$. If there are uncountable many β such that B_β has positive measure. Then write G as the disjoint union $\cup_{\beta \in \mathbb{R}} B_\beta$. We get G has infinte measure, which is a contradiction. Hence there are at most countably many β such that B_β has positive measure.

From the consideration of B_β, we see that apart from at most countably many exceptions, the sets $F_\beta := \{x | W(x) \geq \beta\}$ has null measure boundary.

Let $\varepsilon > 0$ and $\inf ImW = \beta_0 < \beta_1 \cdots < \beta_{\ell-1} < \beta_\ell = \sup ImW$ (ImW is means the image of W) such that $E_i := F_{\beta_i}$ has null measure boundary (the existence follows from the discussion above) and $\beta_{i+1} - \beta_i < \varepsilon$ for each $0 \leq i < \ell$. Set $\Phi := \beta_0 + \sum_{i=0}^{\ell-1} (\beta_{i+1} - \beta_i)\chi_{E_i}$. Then for each $x \in X$, there exists an integer j, $0 \leq j < n$ with $\beta_j \leq W(x) < \beta_{j+1}$ such that

$$
\begin{aligned}
0 \leq \Phi(x) - W(x) &= \beta_0 + \sum_{i=0}^{\ell-1} (\beta_{i+1} - \beta_i)\chi_{E_i}(x) - W(x) \\
&= \beta_0 + \sum_{i=0}^{j} (\beta_{i+1} - \beta_i)\chi_{E_i}(x) - W(x) \\
&= \beta_{j+1} - W(x) \\
&\leq \beta_{j+1} - \beta_j \leq \varepsilon.
\end{aligned}
$$

Hence $\|\Phi - W\|_\infty < \varepsilon$ and $W \leq \Phi$, we have

$$
\begin{aligned}
\int_G W &= \int_G (W - \Phi) + \int_G \Phi \\
&\geq -\varepsilon + \int_G \Phi \\
&\geq -\varepsilon.
\end{aligned}
$$

So $\int_G W \geq 0$.

Similarly, we can find $\|\Phi - W\|_\infty < \varepsilon$ and $W \geq \Phi$. Hence $\int_G W \leq 0$. □

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Example 3.3.2. Let $S^1 = [0, 1)$ denote the circle group, $a \in S^1$ be a irrational number. Then the condition of Theorem 3.3.1 holds.

Remark. The necessary condition $\int_G \ln w = 0$ for a hypercyclic weighted translation operator sometimes still holds for some non-compact group G (ex: when G is a p-adic field).

Example 3.3.3. Let $G = \mathbb{Q}_p$ or $\mathbb{Z}_p$ for some prime number p with $a \in G$, where $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic field and its ring of integer. Then if $T_{a,w}$ is hypercyclic on $L^{p'}(G)$ for some $p' \in [1, \infty)$, we also have

$$\int_G \ln w = 0.$$

Proof. Let $k = v_p(a)$. Consider these diagrams. For each coset of $p^k\mathbb{Z}_p$ say $b + p^k\mathbb{Z}_p$

$$\begin{array}{ccc} L^{p'}(G) & \xrightarrow{T_{a,w}} & L^{p'}(G) \\ T_{-b}\downarrow & \circlearrowright & \downarrow T_{-b} \\ L^{p'}(G) & \xrightarrow[T_{a,(T_{-b}w)}]{} & L^{p'}(G) \\ Res\downarrow & \circlearrowright & \downarrow Res \\ L^{p'}(p^k\mathbb{Z}_p) & \xrightarrow[T_{a,\left((T_{-b}w)|_{p^k\mathbb{Z}_p}\right)}]{} & L^{p'}(p^k\mathbb{Z}_p) \end{array},$$

where Res means the restriction map.

If $T_{a,w}$ is hypercyclic on $L^{p'}(G)$, then so is $T_{a,\left((T_{-b}w)|_{p^k\mathbb{Z}_p}\right)}$ on $L^{p'}(p^k\mathbb{Z}_p)$. And since $p^k\mathbb{Z}_p$ is a compact abelian group with $\overline{< a >} = p^k\mathbb{Z}_p$, by Theorem 3.3.1, we have

$$\int_{p^k\mathbb{Z}_p} \ln(T_{-b}w) = 0.$$

But

$$0 = \int_{p^k\mathbb{Z}_p} \ln(T_{-b}w) = \int_{p^k\mathbb{Z}_p} \ln w(x + b)dx = \int_{b+p^k\mathbb{Z}_p} \ln w(x)dx.$$

In short, if $T_{a,w}$ is hypercyclic on $L^{p'}(G)$, then for each coset of $p^k\mathbb{Z}_p$ say $b + p^k\mathbb{Z}_p$, we have

$$\int_{b+p^k\mathbb{Z}_p} \ln w = 0.$$

Hence

$$\int_G \ln w = 0.$$

□

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3.4 Some properties of hypercyclic weighted translations on p-adic

Theorem 3.4.1. *Let $T_{a,w}$ be a weighted translation operator on $L^{p'}(\mathbb{Z}_p)$ for some prime number p and some $p' \in [1, \infty)$. Then for any given $x' \in G$ and $n \in \mathbb{Z} \setminus \{0\}$. Define the sets*

$$U_{w,n,x'} := \{x \in B_{\leq |na|_p}(x') | w_n(x) > 1\}$$

and

$$L_{w,n,x'} := \{x \in B_{\leq |na|_p}(x') | w_n(x) < 1\},$$

where $B_{\leq r}(x') := \{x||x - x'|_p \leq r\}$. If $T_{a,w}$ is hypercyclic, then

$$U_{w,n,x'} \neq \varnothing$$

and

$$L_{w,n,x'} \neq \varnothing.$$

Proof. Let $T_{a,w}$ be hypercyclic. Consider these two diagrams

$$\begin{array}{ccc} L^{p'}(\mathbb{Z}_p) & \xrightarrow{T_{a,w}} & L^{p'}(\mathbb{Z}_p) \\ T_{-x'} \big\downarrow & \circlearrowleft & \big\downarrow T_{-x'} \\ L^{p'}(\mathbb{Z}_p) & \xrightarrow[T_{a,(T_{-x'}w)}]{} & L^{p'}(\mathbb{Z}_p) \end{array}, \qquad \begin{array}{ccc} L^{p'}(\mathbb{Z}_p) & \xrightarrow{T^n_{a,(T_{-x'}w)}} & L^{p'}(\mathbb{Z}_p) \\ M_p^{v_p(na)} \big\downarrow & \circlearrowleft & \big\downarrow M_p^{v_p(na)} \\ L^{p'}(\mathbb{Z}_p) & \xrightarrow[T]{} & L^{p'}(\mathbb{Z}_p) \end{array},$$

where the operator $T := T_{a^{(n)},w^{(n)}}$, $a^{(n)} := \frac{na}{p^{v_p(na)}}$, $w^{(n)}(x) := w_n(p^{v_p(na)}x + x')$.

Since the diagrams commute, so T must be hypercyclic too. Then its weight function must not be identically smaller than 1. Hence

$$\begin{aligned} \varnothing &\neq \{x | w^{(n)}(x) > 1\} \\ &= \{x | w_n(p^{v_p(na)}x + x') > 1\}. \end{aligned}$$

This implies

$$\begin{aligned} \varnothing &\neq \{x \in B_{\leq |na|_p}(0) | w_n(x + x') > 1\} + x' \\ &= \{x + x' \in B_{\leq |na|_p}(x') | w_n(x + x') > 1\} \\ &= \{x \in B_{\leq |na|_p}(x') | w_n(x) > 1\} \\ &= U_{w,n,x'}. \end{aligned}$$

Similar to the case of $L_{w,n,x'}$. □

Corollary 3.4.2. *If w is a locally constant weight for the weighted translation operator $T_{a,w}$, then $T_{a,w}$ is not hypercyclic on $L^{p'}(\mathbb{Z}_p)$ for any prime number p, any $p' \in [1, \infty)$ and any $a \in \mathbb{Z}_p$.*

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Proof. Let w be a locally constant function. Since G is compact, we see that w is a u.l.c.(uniformly locally constant function). (See [10, Lemma 3.1.1, p.178]).

Since any ball in $\mathbb{Z}_p$ is a coset of subgroup $p^k\mathbb{Z}_p$ for some $k \in \mathbb{N}$, so w can be viewed as a composition

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\pi} & \mathbb{Z}_p/p^k\mathbb{Z}_p \\ & \underset{w}{\searrow} \circlearrowright & \downarrow \overline{w} \\ & & (0,\infty) \end{array},$$

where π is the natural quotient map. Then we call w is a k-u.l.c..

Observe that both $T_a w$ and the multiplication of two k-u.l.c. are also k-u.l.c.. These imply that w_n are k-u.l.c. for all $n \in \mathbb{N}$.

But by the previous theorem,

$$U_{w,n,0} = \{x | w_n|_{B_{\leq |na|_p}(0)}(x) > 1\} \neq \varnothing$$

and

$$L_{w,n,0} = \{x | w_n|_{B_{\leq |na|_p}(0)}(x) < 1\} \neq \varnothing$$

for any $n \in \mathbb{N}$.

To make the contradiction, we just choose $n = p^k$, then $w_n|_{B_{\leq |na|_p}(0)}$ is a constant function, which cannot satisfy both of them simultaneously. □

Corollary 3.4.3. *If w is a locally constant weight for the weighted translation operator $T_{a,w}$, then $T_{a,w}$ is not hypercyclic on $L^{p'}(\mathbb{Q}_p)$ for any prime number p, any $p' \in [1, \infty)$ and any $a \in \mathbb{Q}_p$.*

Proof. Consider the diagrams

$$\begin{array}{ccc} L^{p'}(\mathbb{Q}_p) & \xrightarrow{T_{a,w}} & L^{p'}(\mathbb{Q}_p) \\ M_a \downarrow & \circlearrowright & \downarrow M_a \\ L^{p'}(\mathbb{Q}_p) & \xrightarrow[T_{1,(M_a w)}]{} & L^{p'}(\mathbb{Q}_p) \\ Res \downarrow & \circlearrowright & \downarrow Res \\ L^{p'}(\mathbb{Z}_p) & \xrightarrow[T_{1,\left((M_a w)|_{\mathbb{Z}_p}\right)}]{} & L^{p'}(\mathbb{Z}_p) \end{array},$$

where Res means the restriction map.

So if $T_{a,w}$ is hypercyclic on $L^{p'}(\mathbb{Q}_p)$, then $T_{1,\left((M_a w)|_{\mathbb{Z}_p}\right)}$ is also hypercyclic on $L^{p'}(\mathbb{Z}_p)$. But $(M_a w)|_{\mathbb{Z}_p}$ is also a locally constant function, which contradicts to the Corollary 3.4.2. □

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