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博士論文

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Doctoral Dissertation

檢測有殘餘應力的彈性物體中內含物大小

Detecting an Inclusion in an Elastic Body with Residual Stress

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中文摘要

我們考慮估計內含物大小(D)的反問題，在有殘餘應力的彈性系統中 $(\Omega,\ D \subset \Omega)$，由於存在殘餘應力，所以該彈性系統的結構方程式不是各方向同性的，我們證明，透過量測Ω邊界之應力與位移量，可得內含物尺寸上下界的估計。

關鍵詞：反問題、檢測內在異質物體、彈性系統、殘餘應力、卡爾曼估計、三球不等式、利普希茨之小的傳播

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Abstract

We only consider the inverse problem for estimating the size of an inclusion D, $D \subset \Omega$, in an elastic body with residual stress. The constitutive equation of this elasticity system is not isotropic, due to the presence of residual stresses. We prove that the size of the inclusion can be estimated both from above and below by using only one pair of traction-displacement measurement on the boundary of Ω.

Keywords: Inverse Problem, Detecting inclusions, Elasticity system, Residual stress, Carleman estimate, Three-Sphere inequality, Lipschitz propagation of smallness

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1 Introduction

1.1 Elasticity system with residual stress

We consider the three dimension problem, so we assume $n = 3$. In linear elastic systems, the general equation for linear elasticity with residual stress is given by

$$\sigma = T + (\nabla u)T + L(\widehat{\nabla} u), \tag{1.1}$$

where $L(\widehat{\nabla} u)$ is the incremental elasticity tensor and

$$\widehat{\nabla} u = \frac{1}{2}(\nabla u + (\nabla u)^t) \tag{1.2}$$

the residual stress T should satisfy

$$\nabla T = 0, \quad T = T^t \text{ [9]}.$$

Applying Hartig's law in three dimensions we can write

$$L(\widehat{\nabla} u) = H(\widehat{\nabla} u) + D(T, \widehat{\nabla} u). \tag{1.3}$$

Then using the result in [12], we can express $L(\widehat{\nabla} u)$ as

$$\begin{aligned} L(\varepsilon) \;=\; & \bar{\lambda}(\mathrm{tr}\widehat{\nabla} u)I + 2\bar{\mu}\widehat{\nabla} u + \beta_1(\mathrm{tr}\widehat{\nabla} u)(\mathrm{tr}T) + \beta_2(\mathrm{tr}T)\widehat{\nabla} u \\ & + \beta_3((\mathrm{tr}\widehat{\nabla} u)T + \mathrm{tr}(\widehat{\nabla} uT)I) + \beta_4(\widehat{\nabla} uT + T\widehat{\nabla} u), \end{aligned} \tag{1.4}$$

where $\bar{\lambda}, \bar{\nu}$ are Lamé parameters and $\beta_1, \beta_2, \beta_3, \beta_4$ are material parameters.

Now we have the equation

$$\begin{aligned} \sigma \;& = T + (\nabla u)T + \bar{\lambda}(\mathrm{tr}\widehat{\nabla} u)I + 2\bar{\mu}\widehat{\nabla} u \\ & + \beta_1(\mathrm{tr}\widehat{\nabla} u)(\mathrm{tr}T) + \beta_2(\mathrm{tr}T)\widehat{\nabla} u \\ & + \beta_3((\mathrm{tr}\widehat{\nabla} u)T + \mathrm{tr}(\widehat{\nabla} uT)I) + \beta_4(\widehat{\nabla} uT + T\widehat{\nabla} u) \end{aligned} \tag{1.5}$$

In this thesis, we will mainly focus on Equation (1.5). This equation is much closer to the real elastic system than Lamé System. The results we derive can be applied to a wider range.

1.2 Inverse problem

Now we begin by defining our mathematical model and present the results of this paper from the point of view of Inverse Problem. We let $\lambda = \bar{\lambda} + \beta_1(\mathrm{tr}T)$ and $\mu = \bar{\mu} + \frac{1}{2}\beta_2(\mathrm{tr}T)$. We can rewrite (1.5) as

$$\begin{aligned} \sigma \;& = T + (\nabla u)T + \lambda(\mathrm{tr}\widehat{\nabla} u)I + 2\mu\widehat{\nabla} u \\ & + \beta_3((\mathrm{tr}\widehat{\nabla} u)T + \mathrm{tr}(\widehat{\nabla} uT)I) + \beta_4(\widehat{\nabla} uT + T\widehat{\nabla} u) \end{aligned} \tag{1.6}$$

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To simplify our derivation, we take $\beta_3 = \beta_4 = 0$. We consider the equation

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\mathrm{tr}\widehat{\nabla}u)I + 2\mu(x)\widehat{\nabla}u. \tag{1.7}$$

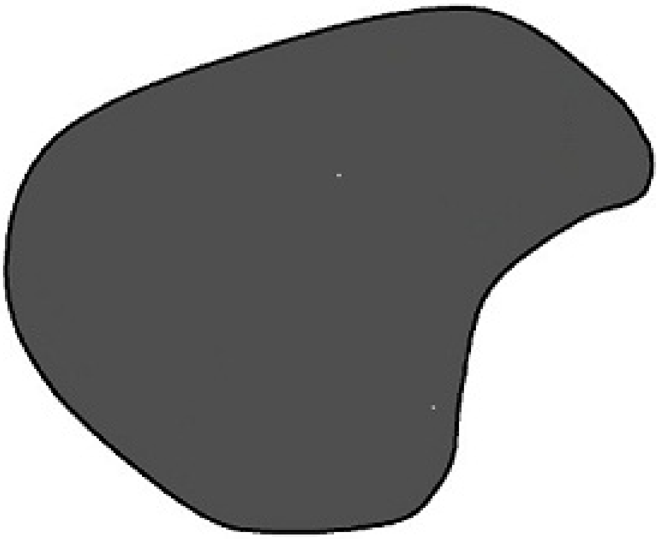

Figure 1: Ω

Let u be the displacement in elasticity system with general residual stress

$$0 = \nabla\cdot\left[\lambda(x)tr(\widehat{\nabla}u)I + 2\mu(x)\widehat{\nabla}u + T(x) + (\nabla u)T(x)\right]. \tag{1.8}$$

We can re-express (1.8) in another format. If we define the elasticity tensor $\mathbb{C} = (\mathbf{C}_{ijkl})^3_{i,j,k,l=1}$ by

$$\mathbf{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + t_{jl}\delta_{ik}, \tag{1.9}$$

then (1.8)) is equivalent to

$$\nabla\cdot(\mathbf{C}\nabla u) = \partial_{x_j}(\mathbf{C}_{ijkl}\partial_{x_l}u_k) = 0 \text{ in } \Omega. \tag{1.10}$$

However $\mathbb{C}$ loses some symmetry properties, so that it maybe

$$C_{ijkl} \neq C_{jikl}, C_{ijkl} \neq C_{ijlk}. \tag{1.11}$$

Let $\tilde{\mathbb{C}}$ be the elasticity tensor field of D. First we use (1.8) as our model and introduce Neumann boundary conditions so that we have

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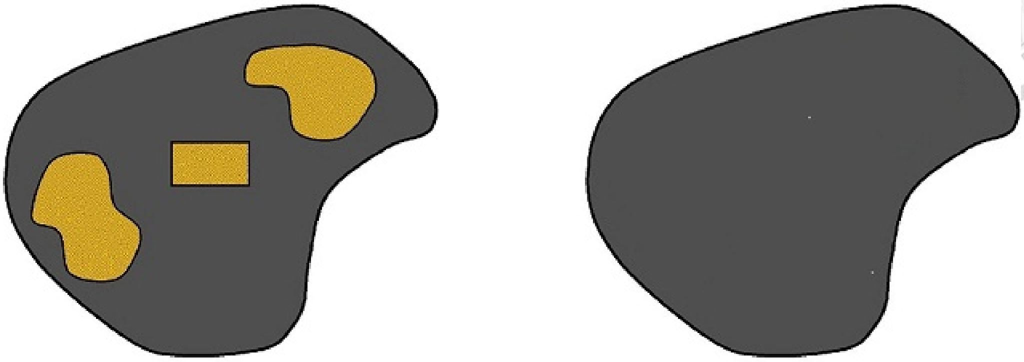

Figure 2: two condition

$$\begin{array}{rlcrl} \nabla \cdot ((\chi_{\Omega\setminus D}\mathbb{C} + \chi_D\tilde{\mathbb{C}})\nabla u) & = 0 & \Omega & \nabla \cdot (\chi_\Omega \mathbb{C}\nabla u_0) & = 0 \\ (\mathbb{C}\nabla u)\nu & = \varphi & \partial\Omega & (\mathbb{C}\nabla u_0)\nu & = \varphi \end{array} \quad (1.12)$$

here we set $\int_\Omega u = \int_\Omega u_0 = 0$ for uniqueness.

For inverse problem, we assume we don't know whether D really exists, and the position and size of D are unknown. Consequently we assume the real elastic coefficient of the known substance Ω is $\mathbb{C}$ and the elastic coefficient of the substance D is $\tilde{\mathbb{C}}$.

Our inverse problem is that: Can we estimate the size of the unknown inclusion D without breaking Ω?

In order to achieve this goal, we can try to measure the stress, φ, and deformation, g, of the surface after giving appropriate external force without destroying the material. We infer the size of D by the measurement results.

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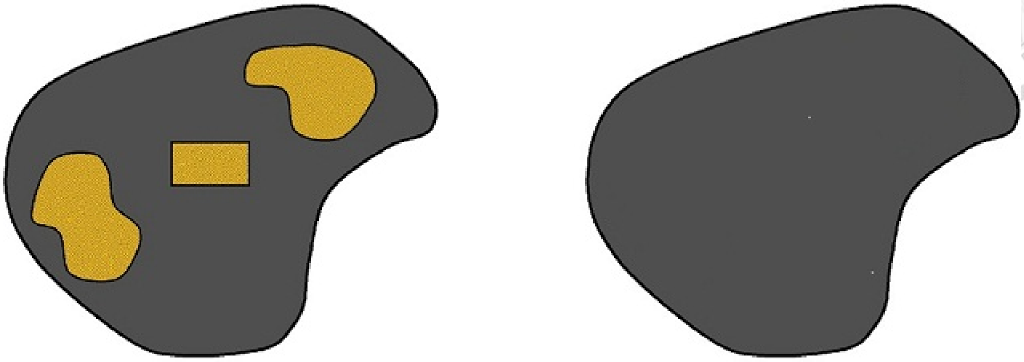

$$\begin{aligned} \nabla\cdot((\chi_{\Omega/\,D}\mathbb{C}+\chi_D\tilde{\mathbb{C}})\nabla u) &= 0 && \Omega \\ (\mathbb{C}\nabla u)\nu &= \varphi && \partial\Omega \\ u &= g && \partial\Omega \end{aligned} \qquad \begin{aligned} \nabla\cdot(\chi_\Omega\mathbb{C}\nabla u_0) &= 0 \\ (\mathbb{C}\nabla u_0)\nu &= \varphi \\ g_0 &:= u_0 \end{aligned}$$

where $\int_\Omega u_0 = 0$. This is our inverse system.

1.3 Estimate size of an inclusion in an elastic body with residual stress

In this research, the focus is on an inverse problem for the elasticity with residual stresses (1.14). The main purpose is to estimate the size of an unknown embedded domain in an elastic body. This embedded domain could represent the region in which the defect occurs. In order to better define the problem, we consider an elastic body with residual stresses. The residual stresses are the remainder after the original cause of the stresses, e.g. thermal treatment, has been removed. The existence of residual stresses may cause premature failure of a structure. For the development of detecting inclusion of elasticity system issue for this kind of inverse problems, we refer to [4], [3] and [5].

To define our problem more precisely, let Ω be a connected open set in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assuming that $u(x) = (u_i(x))_{i=1}^3$ is a three-dimensional vector field. We consider the following equilibrium equation for u:

$$\nabla\cdot\sigma = 0 \ \text{ in } \ \Omega, \tag{1.13}$$

where $\sigma = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor field given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\mathrm{tr}\widehat{\nabla}u)I + 2\mu(x)\widehat{\nabla}u, \tag{1.14}$$

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where $\widehat{\nabla} u(x) = (\nabla u + \nabla u^t)/2$ is the infinitesimal strain and λ, μ are Lamé parameters. The tensor $T(x) = (t_{jl}(x))_{j,l=1}^3$ represents the residual stress, which satisfies $\nabla \cdot T = 0$ and $t_{jl} = t_{lj}$ for all $1 \leq j, l \leq 3$.

The expression (1.14) is a simple constitutive equation modeling the linear elasticity with residual stress, which has been considered in existing literature [18], [8], [15] and [16]. We consider (1.14) because for (1.13) with (1.14) we have the three spheres inequalities, which are an essential tool in this research.

We already know we can express (1.14) into the new form

$$\nabla \cdot (\mathbf{C} \nabla u) = \partial_{x_j}(\mathbf{C}_{ijkl} \partial_{x_l} u_k) = 0 \text{ in } \Omega. \tag{1.15}$$

It is rather important to notice that, for this elasticity system, the minor symmetry properties, i.e., $\mathbf{C}_{ijkl} = \mathbf{C}_{jikl}$ and $\mathbf{C}_{ijkl} = \mathbf{C}_{ijlk}$, may not hold. However, it still satisfies the major symmetry property, $\mathbf{C}_{ijkl} = \mathbf{C}_{klij}$, meaning that (1.13) is a hyper elasticity system.

Now let $D \subset \Omega$ represent an unknown domain embedded in Ω. Let $\tilde{\mathbf{C}}$ denote the elasticity tensor in D. We consider the equilibrium system

$$\nabla \cdot ((\chi_{\Omega \setminus D} \mathbf{C} + \chi_D \tilde{\mathbf{C}}) \nabla u) = 0 \text{ in } \Omega, \tag{1.16}$$

where χ_E denotes the characteristic function of domain E. Let u be the solution for (1.16) satisfying the Neumann condition

$$(\mathbf{C} \nabla u)\nu = \varphi \text{ on } \partial\Omega, \tag{1.17}$$

where ν is the unit exterior normal to $\partial\Omega$. Here we investigate the following inverse problem: *assuming that the background media* $\mathbf{C}$ *is known, we would like to estimate the size of* D *using the knowledge of* $\{\varphi, u|_{\partial\Omega}\}$ only.

The ultimate goal for this inverse problem is to retrieve all geometric information of D by one pair of $\{\varphi, u|_{\partial\Omega}\}$ only. Detecting size of an inclusion has been studied using various models but yields similar results. We give three significant examples: modelling electrically conducting body [17], modelling the Lamé system of elasticity [4] and modelling the elastic plates [13].

In existing literature, the proof of important result is often based on three spheres inequalities for (1.13), (1.14). The qualitative unique continuation property (UCP) for (1.13), (1.14) has been proved in [18]. Our task here is to derive a quantitative estimate of the UCP and three-sphere inequality for (1.13) and (1.14). The main tool for deriving such quantitative estimate is the Carleman estimate. Unfortunately, we can not apply the Carleman estimate in [18] directly to our problem here. To overcome this difficulty, we borrow some ideas in [14] to derive the estimates we need. The estimate

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of $|D|$ is described in Theorem 3.1, which shows that $|D|$ can be bounded both from above and below by the difference of power for the unperturbed system (without D) and the perturbed system (with D) under the fatness condition (Assumption 4 of section 2). Of course, it is more informative to study the problem without the fatness condition. To do this, we need the quantitative form of the strong unique continuation property (SUCP) for (1.13) and (1.14), i.e. doubling inequalities. However, whether the SUCP holds for (1.13) and (1.14) or not is still an unsolved problem.

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2 Elementary concepts and notations

If you are already familiar with most PDE notations, you may skip this section.

The next two sections will supplement some basic theories and their proofs. The basic knowledge required for the whole article is as follows: 1. Basic measure theory 2. Basic theorem of calculus 3. Basic inequalities, such as $ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$, etc. If you are already familiar with the above mentioned mathematical skills, you may skip this section.

We will first define the notations we use, then briefly introduce the concept of Sobolev Space and Weak Solution, and finally use Fourier transform and dual space to generalize the differential concept to any real number.

2.1 Notations

Let $U \in \mathbb{R}^n$ be open.

Definition 2.1. We define the following notations:
1. If $f : U \to \mathbb{R}^m$,

$$f(U) := \{f(x)|x \in U\}$$

2. Let $v, u \in \mathbb{R}^n$,

$$v_i u^i := \sum_{i=1}^{n} v_i u_i = v \cdot u.$$

3. If $u \in \mathbb{R}^n$,

$$|u| := (u_i u^i)^{\frac{1}{2}}$$

4.

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

5. If $1 \leq p < \infty$,

$$L^p(U) := \{f|f \text{ is measureable and satisfy } \int_U |f|^p < \infty\}$$

and

$$\|f\|_{L^p(U)} := (\int_U |f|^p)^{\frac{1}{p}}$$

6.

$$L^\infty(U) := \{f|f \text{ is measureable and exists a constant } K \text{ s.t. } |f| \leq K \text{ a.e. on } U\}$$

and

$$\|f\|_{L^\infty(U)} := \operatorname{ess\,sup} |f|.$$

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7.
$$L^1_{loc}(U) := \{f | f \in L^1(V) \text{ for all } V \Subset U\}.$$

8. If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a n-tuple of non-negative integer α_i and $x \in \mathbb{R}^n$,
$$x^\alpha := \Pi_{i=1}^n x_i^{\alpha_i}$$
and
$$|\alpha| := \sum_{i=1}^n \alpha_i$$

9. Denote
$$D_i := \partial / \partial x_i$$
and
$$D^\alpha := D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n}$$

First, we introduce the concept of weak differential.

Definition 2.2 (Weak partial derivative)**.** Suppose $f \in L^1_{loc}(U)$ and α is a multi-index. We say f is α weak partial derivative if there exist $g \in L^1_{loc}(U)$ such that
$$\int_U f D^\alpha \eta dx = (-1)^{|\alpha|} \int_U g \eta dx \tag{2.1}$$
$\forall \eta \in C_0^\infty(U)$. We denote
$$D^\alpha f := g. \tag{2.2}$$

It is easy to check weak-derivative is unique. Now we can define Sobolev space.

2.2 Sobolev spaces

Definition 2.3 (Sobolev Spac)**.** For any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote
$$W^{k,p}(U) := \{f \in L^1_{loc}(U) | \text{ weak derivative} D^\alpha f \in L_P(U) \forall |\alpha| \leq k\} \tag{2.3}$$
We also define its norm as
$$\|f\|_{W^{k,p}} := \begin{cases} (\sum_{|\alpha| \leq k} \int_U |D^\alpha f|^p)^{\frac{1}{p}} & 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_U |D^\alpha f| & p = \infty. \end{cases} \tag{2.4}$$

We write $H^k(U) = W^{k,2}(U)$.

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Then we have the property

$$C^\infty(U) \cap W^{k,p}(U) \text{ is dense in } W^{k,p}(u). \tag{2.5}$$

For any $k \in \mathbb{N}$, apply Fourier transform we know $f \in L^2(\mathbb{R}^)$ belongs to $H^k(\mathbb{R}^n)$ if and only if

$$(1 + |y|^k)\hat{f} \in L^2(\mathbb{R}^n). \tag{2.6}$$

Moreover, there exists a constant c such that

$$\frac{1}{c}\|f\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{f})\|_{L^2(\mathbb{R}^n)} \leq c\|f\|_{H^k(\mathbb{R}^n)} \tag{2.7}$$

for all $f \in H^k(\mathbb{R}^n)$. So we extend k to real number.

Definition 2.4. For any $0 \leq s < \infty$ we define

$$H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \quad | \quad (1 + |y|^k)\hat{f} \in L^2(\mathbb{R}^n)\}. \tag{2.8}$$

Although we can define $W^{s,p}$, we choose not to do that because this article does not need to use its properties. Finally to extend s to negative part by dual space, we need the following notation.

Definition 2.5. We denote

$$W_0^{k,p}(U) := \{f \in W^{k,p}(U) \quad | \quad \exists\{f_k\} \subset C_0^\infty(U) \text{ s.t. } f_k \to f \text{ in } W^{k,p}\} \tag{2.9}$$

Definition 2.6. If $s > 0$ we denote

$$H^{-s}(U) := \text{ dual space to } H_0^s(U). \tag{2.10}$$

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3 Assumptions and main result

3.1 Assumptions

First, we introduce some assumptions used in this paper. Our attention is restricted to the dimension $n = 3$, which is physically relevant to elasticity.

Let Ω be a bound domain in $\mathbb{R}^3$, and unknown $D \subset \Omega$. For convenience, we order that $\mathbf{C}$ is the elasticity tensor if $\mathbf{C}$ satisfies the following conditions:

$$\mathbf{C} = (\mathbf{C}_{ijkl})_{i,j,k,l=1}^{3} \in L^\infty(\Omega),$$

$$\mathbf{C}_{ijkl} = \mathbf{C}_{klij} \text{ for all } i,j,k,l = 1,2,3.$$

Let $\mathbf{C}$ and $\tilde{\mathbf{C}}$ be elasticity tensor relevant to Ω and D, respectively. We assume that $\mathbf{C}$, which will be explained in detail in Assumption 1, satisfies the Legendre condition(strongly convex), which guarantees the existence of the direct Neumann problem.

We measure the traction φ and displacement $u|_{\partial\Omega} = g$ from the boundary of Ω. Here we assume that $\varphi, g \in L^2(\partial\Omega, \mathbb{R}^3)$ and φ satisfy the compatibility conditions. Let u be the displacement and satisfies the following elasticity system

$$\begin{cases} \nabla \cdot ((\chi_{\Omega\setminus D}\mathbf{C} + \chi_D\tilde{\mathbf{C}})\nabla u) = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \\ (\mathbf{C}\nabla u)\cdot\nu = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

Let u_0 be the displacement with the same traction φ on the boundary and satisfies

$$\begin{cases} \nabla \cdot (\mathbf{C}\nabla u_0) = 0 & \text{in} \quad \Omega, \\ (\mathbf{C}\nabla u_0)\cdot\nu = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

For any $p \in \mathbb{R}^3$, it can be easily shown that $u_0 + p$ also satisfies (3.2). Therefore, we choose u_0 such that $\int_\Omega u_0 = 0$. Set $g_0 := u_0|_{\partial\Omega}$. Then we can estimate the size of D from g, φ and g_0. It is important that we only need the measurements g and φ while the value of g_0 is derived from the system (3.1).

In order to obtain the estimation of inclusion, the following assumptions are necessary.

Assumption 1. (Strongly convex with constant θ)
We assume that $\mathbf{C}$ is strongly convex and T is positive definite in Ω, meaning that a positive constant θ exists such that

$$\theta|A|^2 \le \mathbf{C}(x)A\cdot A \text{ for a.e. } x \in \Omega$$

and

$$\theta|\eta|^2 \le T(x)\eta\cdot\eta \text{ for a.e. } x \in \Omega,$$

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for any 3×3 matrix A and $\eta \in \mathbb{R}^3$.
Assumption 2.($\partial\Omega \in C^{1,1}$ with constants r_0 and M_0)
For every $x \in \mathbb{R}^3$, we set $x = (x', x_3)$, where $x' \in \mathbb{R}^2$. We assume that $\partial\Omega$ belongs to $C^{1,1}$, with constants r_0 and M_0. In other word, for any $x_0 \in \partial\Omega$, a rigid transformation of coordinates exists such that $x_0 = 0$ and

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) | x_3 > \phi(\acute{x})\},$$

where ϕ is $C^{1,1}$ function on $B_{r_0}(0) \subset \mathbb{R}^2$ satisfying

$$\phi(0) = |\nabla\phi(0)| = 0$$

and

$$\|\phi\|_{C^{1,1}(B_{r_0}(0))} \leq r_0 M_0.$$

Assumption 3.(Strictly contained with constant d_0)
A positive constant d_0 exists such that $\text{dist}(D, \partial\Omega) \geq d_0$.
Assumption 4.(Fatness-condition with constant h_1)

$$|\{x \in D | \text{dist}(x, \partial D) > h_1\}| \geq \frac{1}{2}|D|,$$

for a given positive constant h_1.
Assumption 5.(Bounds on the jump and uniform strong convexity for $\tilde{\mathbf{C}}$ with constants δ and η)
We also need the relation between $\tilde{\mathbf{C}}$ and $\mathbf{C}$:
either there exist $\eta > 0$ and $\delta > 1$ such that

$$\eta\mathbf{C} \leq \tilde{\mathbf{C}} - \mathbf{C} \leq (\delta - 1)\mathbf{C} \text{ a.e. in } \Omega, \tag{3.3}$$

or there exists $\eta > 0$ and $0 < \delta < 1$ such that

$$-(1-\delta)\mathbf{C} \leq \tilde{\mathbf{C}} - \mathbf{C} \leq -\eta\mathbf{C} \text{ a.e. in } \Omega. \tag{3.4}$$

Here we denote that $\tilde{\mathbf{C}} \leq \mathbf{C}$ if $\tilde{\mathbf{C}}A \cdot A \leq \mathbf{C}A \cdot A$ for every 3×3 matrix A.
Assumption 6.($\mathbf{C} \in C^3 \cap W^{4,\infty}$ with constant M.)
Let X be a norm space. We say that $\mathbf{C} \in X$ if $\lambda, \mu, t_{jl} \in X$ for all $j, l = 1, 2, 3$, and let

$$\|\mathbf{C}\|_X := \|\lambda\|_X + 2\|\mu\|_X + \sum_{j,l=1}^{3} \|t_{jl}\|_X.$$

We assume $\mathbf{C} \in C^3 \cap W^{4,\infty}$. For convenience, denote $M > 0$ such that

$$\|\mathbf{C}\|_{W^{4,\infty}} \leq M.$$

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Remarks. 1). In this paper, the only assumption of $\tilde{\mathbf{C}}$ is the elasticity tensor which satisfies Assumption 5. It is a very mild assumption for an unknown inclusion since the inclusion D may consist of any anisotropic material which is either harder (case (3.3)) or softer (case (3.4)) than the surrounding material in Ω, and no additional regularity assumption is required on the elasticity tensor inside D. 2). D can be disconnected. 3). The existence of residual stresses may cause the loss of several symmetry properties. To overcome this difficulty, we make an additional the Assumption 6 for regularity.

3.2 Main result(theorem)

Now we have

Theorem 3.1. *Let Ω be bounded domain in $\mathbb{R}^3$ and D be any measurable subset of Ω. Let $\mathbf{C}$ and $\tilde{\mathbf{C}}$ be elasticity tensors to Ω and D, respectively. If Assumptions 1-6 hold and $\mathbf{C}$ satisfies (1.9):*

if (3.3) holds, then we have

$$\frac{1}{\delta-1}C_1^+\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi} \le |D| \le \frac{\delta}{\eta}C_2^+\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi},$$

if (3.4) holds, then we have

$$\frac{\delta}{1-\delta}C_1^-\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi} \le |D| \le \frac{1}{\eta}C_2^-\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi},$$

where C_1^+, C_1^- only depend on d_0, $|\Omega|$, θ, M, r_0 and M_0, and C_2^+, C_2^- only depend on d_0, $|\Omega|$, θ, M, r_0, M_0, h_1 and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$.

3.3 Strategy

We will introduce our basic analysis tools in section 4. Starting from section 5, we will derive the main result according to the strategy listed in the

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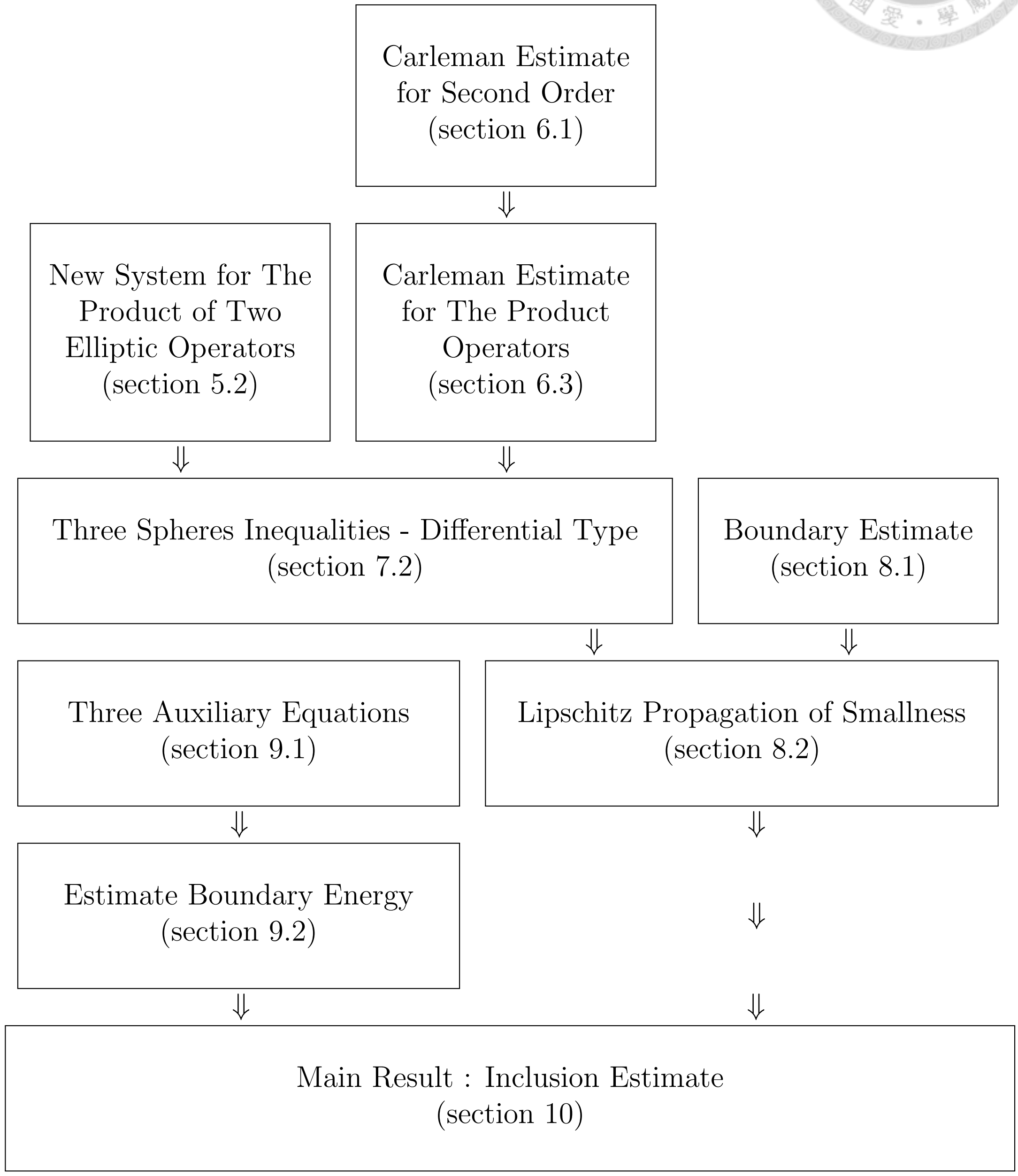
following chart.



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4 Standard estimate tools

The main purpose of this section is to try to make this article as self-contained as possible. This section contains some inequalities that we will use in our derivation of our new systems. You may skip this section at first reading.

4.1 Interior estimate

First, we recall the following standard regularity proposition which can be found in [6]. First, we recall the following standard regularity proposition which can be found in [6].

Proposition 4.1. *Let $u \in W^{1,2}(\Omega;\mathbb{R}^3)$ be a weak solution of (1.15). Assume that $\mathbf{C} \in C^{1,1}(\Omega)$ satisfies strong convexity condition, then $u \in W^{4,2}_{loc}(\Omega;\mathbb{R}^3)$.*

Proposition 4.2. *Let $\mathbf{C} \in L^\infty(\Omega)$ be strongly convex and $F = (F^i_j)^3_{i,j=1} \in L^2(\Omega, \mathbb{R}^{3\times 3})$. If $V \in W^{1,2}_{loc}(\Omega;\mathbb{R}^3)$ satisfies*

$$\int (\mathbf{C}_{ijkl}\partial_l V_k)\partial_j \varphi_i = \int F^i_j \partial_j \varphi_i$$

for all $\varphi = (\varphi_i)^3_{i=1} \in C^\infty_0(\Omega, \mathbb{R}^3)$, then for any $r > 0$ we have

$$\int_{a_1 r\le|x|\le a_2 r} |\nabla V|^2 \le C\int_{a_3 r\le|x|\le a_4 r} |x|^{-2}|V|^2 + C\int_{a_3 r\le|x|\le a_4 r} |F|^2, \tag{4.1}$$

where $0 < a_3 < a_1 < a_2 < a_4 < \frac{R}{r}$ and $C = C(\theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{L^\infty})$.

Proof of Proposition 4.2. Let $\eta \in C^\infty_0(\mathbb{R}^3)$ satisfy $0 \le \eta \le 1$,

$$\eta(x) = \begin{cases} 0 & |x| \le a_3 r, \\ 1 & a_1 r \le |x| \le a_2 r, \\ 0 & a_4 r \le |x|, \end{cases}$$

and $|\nabla^\alpha \eta| \le C|x|^{-|\alpha|}$ for any multi-index α, where C is independent of r. From the strong convexity condition, we obtain

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$$\lambda \int \eta^2 |\nabla V|^2 \leq \int \eta^2 (\mathbf{C}_{ijkl} \partial_l V_k) \partial_j V_i \tag{4.2}$$

$$= \int (\mathbf{C}_{ijkl} \partial_l V_k) \partial_j (\eta^2 V_i) - 2\eta (\mathbf{C}_{ijkl} \partial_l V_k) V_j \partial_j \eta \tag{4.3}$$

$$= \int F^i_j \partial_j (\eta^2 V_i) - 2\eta (\mathbf{C}_{ijkl} \partial_l V_k) V_j \partial_j \eta \tag{4.4}$$

$$= \int \eta^2 F^i_j \partial_j V_i + \int 2\eta F^i_j \partial_j \eta V_i - 2\eta (\mathbf{C}_{ijkl} \partial_l V_k) V_j \partial_j \eta \tag{4.5}$$

$$\leq \epsilon \int \eta^2 |\nabla V|^2 + \frac{C}{\epsilon} \int \eta^2 |F|^2 + \int_{a_3 r \leq |x| \leq a_4 r} \frac{C}{\epsilon} \frac{|V|^2}{|x|^2}. \tag{4.6}$$

When ϵ is small, we obtain (4.1). □

Corollary 4.3 (Interior estimate)**.** *Let $u \in W^{1,2}(\Omega; \mathbb{R}^3)$ be a weak solution of (1.15) . Assume that $\mathbf{C} \in C^3(\Omega)$ satisfies strong convexity condition, then $u \in W^{4,2}_{loc}(\Omega; \mathbb{R}^3)$ and*

$$\sum_{k=1}^{4} \int_{a_1 r \leq |x| \leq a_2 r} |x|^{l+2k} |\nabla^k u|^2 \leq C \int_{a_3 r \leq |x| \leq a_4 r} |x|^l |u|^2, \tag{4.7}$$

for all $r > 0$, where $0 < a_3 < a_1 < a_2 < a_4 < \frac{R}{r}$ and
$C = C(l, \theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{W^{3,\infty}})$.

Proof of Corollary 4.3. 1. By Proposition 4.2, we have

$$\int_{a_1 r \leq |x| \leq a_2 r} |\nabla u|^2 \leq C \int_{a_3 r \leq |x| \leq a_4 r} |x|^{-2} |u|^2.$$

For any $\varphi_i \in C_0^\infty(\Omega)$ and $t \in \{1, 2, 3\}$, we have

$$0 = \int (\mathbf{C}_{ijkl} \partial_l u_k) \partial_j (\partial_t \varphi_i) \tag{4.8}$$

$$= -\int \partial_t \mathbf{C}_{ijkl} \partial_l u_k \partial_j \varphi_i - \int (\mathbf{C}_{ijkl} \partial_l (\partial_t u_k)) \partial_j \varphi_i. \tag{4.9}$$

(4.10)

Apply Proposition 4.2 with $F^i_j = F^i_{j,t} = -\partial_t \mathbf{C}_{ijkl} \partial_l u_k$, we have

$$\int_{a_1 r \leq |x| \leq a_2 r} |\nabla (\partial_t u)|^2 \leq C \int_{a_3 r \leq |x| \leq a_4 r} |x|^{-2} |\partial_t u|^2 + C \int_{a_3 r \leq |x| \leq a_4 r} |F_t|^2,$$

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where $F_t = (F^i_{j,t})^3_{i,j=1}$. Sum up with respect to $t = 1, 2, 3$, we have

$$\int_{a_1 r \le |x| \le a_2 r} |\nabla^2 u|^2 \le C \int_{a_3 r \le |x| \le a_4 r} |x|^{-2} |\nabla u|^2,$$

where $C = C(\theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{W^{1,\infty}})$.

2. Similarly, we obtain

$$\int_{a_1 r \le |x| \le a_2 r} |\nabla^3 u|^2 \le C \int_{a_3 r \le |x| \le a_4 r} (|x|^{-2} |\nabla^2 u|^2 + |x|^{-4} |\nabla u|^2)$$

and

$$\int_{a_1 r \le |x| \le a_2 r} |\nabla^4 u|^2 \le C \int_{a_3 r \le |x| \le a_4 r} (|x|^{-2} |\nabla^3 u|^2 + |x|^{-4} |\nabla^2 u|^2 + |x|^{-6} |\nabla u|^2),$$

where $C = C(\theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{W^{3,\infty}})$.

3. By Proposition 4.2, we have

$$\int_{a_1 r \le |x| \le a_2 r} |\nabla u|^2 \le C \int_{a_3 r \le |x| \le a_4 r} |x|^{-2} |u|^2.$$

If $l \ge 0$, then

$$\int_{a_1 r \le |x| \le a_2 r} |x|^l |\nabla u|^2 \tag{4.11}$$

$$\le \int_{a_1 r \le |x| \le a_2 r} (a_2 r)^l |\nabla u|^2 \tag{4.12}$$

$$\le C (a_2 r)^l \int_{a_3 r \le |x| \le a_4 r} |x|^{-2} |u|^2 \tag{4.13}$$

$$\le C (a_2 r)^l \int_{a_3 r \le |x| \le a_4 r} (\frac{|x|}{a_3 r})^l |x|^{-2} |u|^2 \tag{4.14}$$

$$\le C (\frac{a_2}{a_3})^l \int_{a_3 r \le |x| \le a_4 r} |x|^{l-2} |u|^2. \tag{4.15}$$

Similarly, if $l < 0$, then

$$\int_{a_1 r \le |x| \le a_2 r} |x|^l |\nabla u|^2 \le C (\frac{a_4}{a_2})^l \int_{a_3 r \le |x| \le a_4 r} |x|^{l-2} |u|^2,$$

where $C = C(l, \theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{W^{1,\infty}})$.

4. We redo steps 1-3 with suitable range, then we have

$$\int_{a_1 r \le |x| \le a_2 r} |x|^l |\nabla^k u|^2 \le C \int_{a_3 r \le |x| \le a_4 r} |x|^{l-2k} |u|^2$$

for $k = 2, 3, 4$, where $C = C(l, \theta, a_1, a_2, a_3, a_4, \|\mathbf{C}\|_{W^{3,\infty}})$. □

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4.2 Caccioppoli-type inequality

We introduce the following notation and definition.

Definition 4.4. Let

$$\{C_{ij}^{\alpha\beta}(x)\}_{1\le\imath,\jmath\le m}^{1\le\alpha,\beta\le n} \in L^\infty(\Omega) \tag{4.16}$$

which is said to satisfy **Legendre condition**(strongly convex), if there exists a $\Lambda > 0$ such that

$$C_{ij}^{\alpha\beta} A\cdot A \ge \Lambda |A|^2, \quad \forall A \in \mathbb{R}^{m\times n} \tag{4.17}$$

Theorem 4.5 (General type of Caccioppoli's inequality for Elliptic Systems)**.** *Let $u \in W^{1,2}(\Omega;\ \mathbb{R}^m)$ be a solution of*

$$D_\alpha(C_{ij}^{\alpha\beta} D_\beta u^j) = D_\alpha f_i^\alpha - f_i \tag{4.18}$$

where $D_\alpha f_i^\alpha, f_i in L^2$, and $C_{ij}^{\alpha\beta} D_\beta u^j$ satisfy the Legendre condition. Then for any ball $\beta_{r(x_0)} \subset \Omega$ and $0 < r < R$ we have

$$\int_{\beta_{r(x_0)}} |Du|^2 dx \le \frac{c}{(R-r)^2}\int_{\beta_{R(x_0)}} |u-\eta|^2 dx \tag{4.19}$$

$$+cR^2\int_{B_{R(x_0)}} \sum_{1\le i\le m} f_i^2 dx + c\int_{\beta_{r(x_0)}} \sum_{1\le i\le m, 1\le\alpha\le n} (f_i^\alpha)^2 dx \tag{4.20}$$

$\forall \eta \in \mathbb{R}^m$*, where* $c = c(n, m, \Lambda, \sup|A|)$

Proof. For convenience let $x_0 = 0$. First we construct a cut-off function $\xi \in C_0^\infty(\mathbb{R}^n)$ where ξ satisfy $0 \le \xi \le 1$ and

$$\xi(x) = \begin{cases} 1, & |x| \le r, \\ H(|x|), & r \le |x| \le R, \\ 0, & R \le |x| \end{cases} \tag{4.21}$$

where $0 \le H \le 1$ and $|\nabla^\alpha H| \le C|x|^{-|\alpha|}$ for any multi-index α. From

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Legendre condition we have,

$$\Lambda\int_{B_R}\xi^2|Du|^2dx \le \int_{B_R}\xi^2 C_{ij}^{\alpha\beta}D_\alpha u^i D_\beta u^j dx$$
$$= \int_{B_R}\xi^2 C_{ij}^{\alpha\beta}D_\alpha(u^i-\eta^i)D_\beta u^j dx$$
$$= -\int_{B_R}2\xi(u^i-\eta^i)C_{ij}^{\alpha\beta}D_\beta u^j D_\alpha\xi dx$$
$$+\int_{B_R}\xi^2(u^u-\eta^i)D_\alpha(C_{ij}^{\alpha\beta}D_\beta u^j)dx$$
$$= -\int_{B_R}2\xi(u^i-\eta^i)C_{ij}^{\alpha\beta}D_\beta u^j D_\alpha\xi dx$$
$$+\int_{B_R}\xi^2(u^u-\eta^i)D_\alpha f_i^\alpha dx - \int_{B_R}\xi^2(u^u-\eta^i)f_i dx$$
$$= -\int_{B_R}2\xi(u^i-\eta^i)C_{ij}^{\alpha\beta}D_\beta u^j D_\alpha\xi dx$$
$$+\int_{B_R}\xi^2(u^u-\eta^i)D_\alpha f_i^\alpha dx - \int_{B_R}\xi^2\frac{1}{R}(u^u-\eta^i)Rf_i dx$$
$$= -\int_{B_R}2\xi(u^i-\eta^i)C_{ij}^{\alpha\beta}D_\beta u^j D_\alpha\xi dx$$
$$-\int_{B_R}2\xi f_i^\alpha(u^u-\eta^i)D_\alpha\xi dx - \int_{B_R}2\xi^2 f_i^\alpha D_\alpha u^u dx$$
$$-\int_{B_R}\xi^2\frac{1}{R}(u^u-\eta^i)Rf_i dx$$
$$\le \epsilon\int_{B_R}\xi^2|Du|^2dx + \frac{c}{R^2}\int_{B_R}|u-\eta|^2dx + c\int_{B_R}|f|^2$$
$$\le \epsilon\int_{B_R}\xi^2|Du|^2dx + \frac{c}{(R-r)^2}\int_{B_R}|u-\eta|^2dx + c\int_{B_R}|f|^2$$

where $c = c(n, m, \epsilon, \sup|C|)$. If ϵ is small enough we can remove $\int_{B_R}\xi^2|Du|^2dx$ into

$$\Lambda\int_{B_r}\xi^2|Du|^2dx \le \Lambda\int_{B_R}\xi^2|Du|^2dx \tag{4.22}$$
$$\le \frac{c}{(R-r)^2}\int_{B_R}|u-\eta|^2dx + c\int_{B_R}|f|^2. \tag{4.23}$$

where $c = c(n, m, \Lambda, \sup|C|)$. □

Let $n = m = 3$ and $f_i^\alpha = f_i = 0$, we can reduce it to the simple form.

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Lemma 4.6. *(Caccioppoli-type inequality) If* $\mathbf{C}$ *is strongly convex with form (1.9) and* $u \in W^{1,2}(\Omega, \mathbb{R}^3)$ *is a solution to (1.15), then for any ball* $B_R \subset \Omega$ *and* $0 < r < R$ *the the following Caccioppoli inequality holds*

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(\hat{R} - r)^2} \int_{B_R} |u|^2 \tag{4.24}$$

where $C = C(\theta, ||\mathbf{C}||_{L^\infty})$.

Given $u \in W^{1,p}(\Omega)$ and S be any measurable subset of Ω, set $u_S := 1/|S| \int_S u$.

4.3 Poincaré inequality

In mathematics, Poincaré inequality allows us to get bounds only using its derivatives and the geometry domain. This inequality is very important in modern analysis. In general, there are two versions of poincare inequalities, one is the compact support version, and the other is the subtracted average version.

Theorem 4.7 (Poincaré inequality - boundary support)**.** *If domain* U *has finite width, there exists a constant* $Q = Q(p, diam(U))$ *such that for all* $f \in C_0^\infty(U)$*, we have*

$$\|f\|_{L^p(U)} \leq c\|Df\|_{L^p(U)}. \tag{4.25}$$

Proof. For convenience we assume that U lies between hyperplanes $x_n = 0$ and $x_n = c > 0$. Given $f \in C_0^\infty$. Let $(x', x_n) = x \in U$, we have

$$f(x) = \int_0^{x_n} D_n f(x', t) dt. \tag{4.26}$$

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So that

$$\begin{aligned}\|f\|^p_{L^p(U)} &= \int_{\mathbb{R}^{n-1}}\int_0^c |f(x',x_n)|^p dx_n dx' \\ &= \int_{\mathbb{R}^{n-1}}\int_0^c |\int_0^{x_n} D_n f(x',t)dt|^p dx_n dx' \\ &\leq \int_{\mathbb{R}^{n-1}}\int_0^c |(\int_0^{x_n} |D_n f(x',t)|^p dt)^{\frac{1}{p}}(x_n)^{\frac{p-1}{p}}|^p dx_n dx' \\ &= \int_{\mathbb{R}^{n-1}}\int_0^c (\int_0^{x_n} |D_n f(x',t)|^p dt)(x_n)^{p-1} dx_n dx' \\ &\leq \int_{\mathbb{R}^{n-1}}\int_0^c |(\int_0^{x_n} |D_n f(x',t)|^p dt)^{\frac{1}{p}}(x_n)^{\frac{p-1}{p}}|^p dx_n dx' \\ &\leq \int_{\mathbb{R}^{n-1}}\int_0^c (x_n)^{p-1}\int_0^c |D_n f(x',t)|^p dt dx_n dx' \\ &\leq \frac{c^p}{p}\int_U |D_n f(x)|^p dx \\ &\leq Q(p, diam(U))^p \|D_n f\|^p_{L^p(U)}\end{aligned}$$

where $Q(p, diam(U)) = \dfrac{diam(U)}{p^{\frac{1}{p}}}$. □

The following standard inequality can be found in [7].

Lemma 4.8 (Poincaré inequality - subtract mean)**.** *If U is convex and $u \in W^{1,2}(U)$, then we have*

$$||u - u_S||_{L^2(U)} \leq (\frac{\omega_3}{|S|})^{1-1/3} d^3 ||\nabla V||_{L^2(U)}, \tag{4.27}$$

where $d = diam(U)$ and $u_S = \frac{\int_S u}{|S|}$ for any measurable $S \subset U$.

Proof. Since we know $C^1(U)$ is dense in $W^{1,2}(U)$, it is enough to show $u \in C^1(U)$. We have

$$u(x) - u(y) = -\int_0^{|x-y|} D_r u(x + r\eta)dr \tag{4.28}$$

where $\eta = \frac{y-x}{|y-x|}$. Then integrate both sides of (4.28) over S, we obtain

$$|S|(u(x) - u_S) = \int_S (u(x) - u(y))dy = -\int_S \int_0^{|x-y|} D_r u(x + r\eta)drdy. \tag{4.29}$$

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Let

$$V(x) = \begin{cases} |D_r u(x)|, & x \in U \\ 0, & otherwise \end{cases}$$

we have

$$\begin{aligned}
&|u(x) - u_s| \\
&\le \frac{1}{S} \int_S \int_0^{|x-y|} V(x + r\eta) dr dy \\
&\le \frac{1}{S} \int_{|x-y|<d} \int_0^{\infty} V(x + r\eta) dr dy \\
&= \frac{1}{|S|} \int_0^{\infty} \int_{|x-y|<d} V(x + r\eta) dy dr \\
&= \frac{1}{|S|} \int_0^{\infty} \int_{|\eta|=1} \int_0^{d} V(x + r\eta) \rho^{n-1} d\rho d\eta dr \\
&= \frac{d^n}{n|S|} \int_0^{\infty} \int_{|\eta|=1} V(x + r\eta) d\eta dr \\
&= \frac{d^n}{n|S|} \int_{\mathbb{R}^n} |x - y|^{1-n} V(y) dy \\
&= \frac{d^n}{n|S|} \int_U |x - y|^{1-n} |D_r u(y)| dy \qquad (4.30)
\end{aligned}$$

Let $\mu \in (0, 1]$, we have $n(\mu - 1) \le 0$. Let $R > 0$ such that $|U| = |B_R(x)| = w_n R^n$. It is easy to find

$$\begin{aligned}
&\int_U |x - y|^{n(\mu-1)} dy \\
&\le \int_{B_R(x)} |x - y|^{n(\mu-1)} dy \\
&= \int_0^R \int_{\partial B(x,r)} r^{n(\mu-1)} ds dr \\
&= \int_0^R r^{n(\mu-1)} n w_n r^{n-1} dr \\
&= \int_0^R r^{n\mu-1} n w_n r^{n-1} dr \\
&= \frac{1}{\mu} R^{n\mu} w_n \\
&= \frac{1}{\mu} w_n^{1-\mu} |U|^{\mu}.
\end{aligned}$$

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Let $\mu = \frac{1}{n}$, then

$$\int_U |x-y|^{1-n} dy \leq n w_n^{1-\frac{1}{n}} |U|^{\frac{1}{n}}. \tag{4.31}$$

Since $|x-y|^{1-n}|D_r u(y)| = |x-y|^{\frac{1-n}{2}}|x-y|^{\frac{1-n}{2}}|D_r u(y)|$, then apply Holder inequality to obtain

$$\int_U |x-y|^{1-n}|D_r u(y)| dy \leq (\int_U |x-y|^{1-n}|D_r u(y)|^2 dy)^{\frac{1}{2}} (\int_U |x-y|^{1-n} dy)^{\frac{1}{2}} \tag{4.32}$$

So that

$$\begin{aligned}
&\int_U (\int_U |x-y|^{1-n}|D_r u(y)| dy)^2 dx \\
&\leq \int_U (\int_U |x-y|^{1-n}|D_r u(y)|^2 dy)(\int_U |x-y|^{1-n} dy) dx \\
&\leq n w_n^{1-\frac{1}{n}} |U|^{\frac{1}{n}} \int_U \int_U |x-y|^{1-n}|D_r u(y)|^2 dy dx \\
&\leq n w_n^{1-\frac{1}{n}} |U|^{\frac{1}{n}} \int_U |D_r u(y)|^2 \int_U |x-y|^{1-n} dx dy \\
&\leq n^2 w_n^{2-\frac{2}{n}} |U|^{\frac{2}{n}} \int_U |D_r u(y)|^2.
\end{aligned}$$

Combine the last inequality and (4.30) with $n = 3$, we complete the proof. □

Now, if $u \in W^{1,2}(\Omega, \mathbb{R}^3)$, $\hat{R} < 1$, $S = B_r$ and $E = B_{\hat{R}}$, we obtain

$$\int_{B_{\hat{R}}} |u - u_r|^2 \leq C(\frac{\hat{R}}{r})^{6-2} R^2 \int_{B_{\hat{R}}} |\nabla u|^2, \tag{4.33}$$

where $u_r = \frac{1}{|B_r|}\int_{B_r} u$.

4.4 Sobolev inequality

In mathematics, there is a class of Sobolev inequalities for analysis of norms in Sobolev spaces. These inequalities can be used to prove the Sobolev embedding theorem, giving the inclusion relations of some Sobolev spaces. Further, the Rellich-Kondrachov theorem states that under slightly stronger conditions, some Sobolev spaces can be tightly embedded into another space.

Sobolev inequalities is really a big and important class of tools. Here we only present the Theorem and reference.

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Theorem 4.9 (Sobolev inequality)**.** *When $mp < n$, there exists a finite constant K such that for every $u \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |u(x)|^q dx \leq K^q (\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |D^\alpha u(x))^p dx)^{\frac{q}{p}} \tag{4.34}$$

if and only if $q = \frac{np}{n-mp}$.

The detailed proof can be found in ([1]).

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5 Transformation of the original system into two new systems

In order to obtain the three spheres inequalities for solution u to system (1.15), we need a suitable form of Carleman estimate. For this purpose, we transform system (1.15) with $\mathbf{C}$ satisfying (1.9) into a new system with the uncoupled principal part. To begin with, we recall a standard property.

5.1 Auxiliary new system

Proposition 5.1. *Let* $\mathbf{C}$ *be of the form (1.9) and satisfies Assumption 1. We have*

$$\begin{cases} \Sigma_{jl} t_{jl}\xi_j\xi_l + \mu|\xi|^2 \geq \theta|\xi|^2, \\ \Sigma_{jl} t_{jl}\xi_j\xi_l + (\lambda+2\mu)|\xi|^2 \geq \theta|\xi|^2, \end{cases} \tag{5.1}$$

which means that

$$A_1(x,D) := \Sigma_{jl}(\mu\delta_{jl} + t_{jl})\partial^2_{x_j x_l}$$

and

$$A_2(x,D) := \Sigma_{jl}((\lambda+2\mu)\delta_{jl} + t_{jl})\partial^2_{x_j x_l}$$

are both uniform elliptic operators.

We assume that $\lambda, \mu, t_{jk} \in W^{2,\infty}(\Omega)$. Then we can rewrite (1.15) in the form

$$A_1(x,D)u + (\lambda+\mu)\nabla(\nabla\cdot u) = \tilde{P}_1(x,D)(u), \tag{5.2}$$

where $\tilde{P}_1$ is the first order differential operator with $W^{1,\infty}(\Omega)$ coefficients. We denote two auxiliary functions $v(x) := \nabla\cdot u(x)$ and $w(x) := \nabla\times u(x)$. The equation becomes

$$A_1(x,D)u = P_1(x,D)(u,v). \tag{5.3}$$

Take the divergence on (5.3), we derive the equation

$$A_2(x,D)v = Q_2(x,D)(u) + Q_1(x,D)(u,v), \tag{5.4}$$

where $Q_2(x,D)(u) = -2(\partial_i\mu)\triangle u_i - (\partial_i t_{jl})\partial_{jl}u_i$ and Q_1 is first order differential operator with $L^\infty(\Omega)$ coefficients. Take the curl on (5.3), and we have

$$A_1(x,D)w = R_2(x,D)(u) + R_1(x,D)(u,v,w), \tag{5.5}$$

where $R_2(x,D)(u) = -(\nabla t_{jl})\times(\partial_{jl}u) - \nabla\mu\times\triangle u$ and R_1 is the first order differential operator with $L^\infty(\Omega)$ coefficients. Now, we have the following property.

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Proposition 5.2. *If u satisfies (1.15) and $\lambda, \mu, t_{jl} \in W^{2,\infty}(\Omega)$ for all $j, l = 1, 2, 3$, then u also satisfies*

$$\begin{cases} A_1 u = P_1(u, v), \\ A_2 v = Q_1(u, v) + Q_2(u), \\ A_1 w = R_1(u, v, w) + R_2(u) \end{cases} \tag{5.6}$$

where $v := \nabla \cdot u$ and $w := \nabla \times u$.

5.2 New system for the product of two elliptic operators

We assume that $\lambda, \mu, t_{jl} \in W^{4,\infty}(\Omega)$ for all $j, l = 1, 2, 3$. Take $\triangle$ on system (5.6). Since $\triangle u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = \nabla v - \nabla \times w$, we have the following proposition.

Proposition 5.3 (New system for the product of two elliptic operators)**.** *If u satisfies (1.15) and $\lambda, \mu, t_{jl} \in W^{4,\infty}(\Omega)$ for all $j, l = 1, 2, 3$, then u also satisfies*

$$\begin{cases} \triangle(A_1(x, D)u) = Q_1^3(x, D)(u, v, w), \\ \triangle(A_2(x, D)v) = Q_2^3(x, D)(u, v, w), \\ \triangle(A_1(x, D)w) = Q_3^3(x, D)(u, v, w), \end{cases} \tag{5.7}$$

where $v := \nabla \cdot u$, $w := \nabla \times u$ and Q_j^3 is third order differential operator with $L^\infty(\Omega)$.

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6 Carleman estimates

In general, we need a suitable Carleman estimate to derive Three sphere inequality. In order to make a long story short, we directly use Theorem of [14] to obtain the suitable Carleman estimate we need. Because the theorem involves many symbols, so we first explain the symbols used in it.

Let $g = \{g_{ij}(x)\}_{i,j=1}^3$ be positive definite. So there exists $g^{-1}(x) = \{g^{ij}(x)\}_{i,j=1}^3$, which is the inverse matrix of $g(x)$. We know $g^{-1}(x)$ is positive definite.

For convenience, we shall use the following notations. Let $g_1(x) = \{g_1^{ij}(x)\}_{i,j=1}^3$ and $g_2(x) = \{g_2^{ij}(x)\}_{i,j=1}^3$ be two symmetric matrix real value functions which satisfy:

1).Let $a, b \in \mathbb{R}^n$ we denote

$$(a,b) := \sum_{i=1}^{n} a_i b_i, \quad |a|^2 := (a,a) \tag{6.1}$$

2).

$$\lambda|\xi|^2 \leq g_k^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \tag{6.2}$$

for every $x, \xi \in \mathbb{R}^3$;

3).

$$\sum_{i,j=1}^{3} |g_k^{ij}(x) - g_k^{ij}(y)| \leq \Lambda|x-y| \tag{6.3}$$

for every $x, y \in \mathbb{R}^3$.

Set $\Lambda_1 := \max_{k\in\{1,2\}} \sum_{i,j=1}^3 \|g_k^{ij}\|_{W^{2,\infty}(\mathbb{R}^3)}$. Let $L_k := \sum_{i,j=1}^3 g_k^{ij}\partial_i\partial_j$ be the second order differential operator for k =1,2 and set $\mathcal{L} := L_2(L_1)$.

4).

$$|g| := (\sum_{i,j=1}^{3} (g^{ij})^2)^{\frac{1}{2}}$$

5).Let $\Gamma = \{\gamma_{ij}\}_{i,j=1}^n$ be a matrix. Let m_* and m^* be the minimum and the maximum eigenvalue of Γ such that

$$m_*|x|^2 \leq (\Gamma x, x) \leq m^*|x|^2 \quad \text{for every} x \in \mathbb{R}^n. \tag{6.4}$$

Sometimes we omit the lower index and obtain the following notations

$$\nabla_g u(x) = g^{-1}\nabla_g u(x), \tag{6.5}$$

$$\triangle_g u = \operatorname{div}(\nabla_g u(x)). \tag{6.6}$$

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Note that $\triangle_{g_k} \neq L_k$. Let $f \in C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$ but we have

$$|\triangle_{g_k} f| \leq |L_k f| + c|\nabla f| \tag{6.7}$$

where $c = c(\Lambda)$.

6.1 Second order type Carleman estimate

The following Carleman estimate is from Theorem 4.5 of [14].

Theorem 6.1 (Original second order type Carleman estimate)**.** *Let β be a number such that $\beta > w_0$, let*

$$\varphi(s) = \exp^{-s^{-\tau}} \tag{6.8}$$

and let $w(x) = \varphi(\sigma(x))$ and $\sigma(x) = (\Gamma x, x)^{\frac{1}{2}}_{n}$. There exist constant C, τ_1 and r_0, ($C \geq 1, \tau_1 \geq 1, 0 < r_0 \leq 1$) depending only on $\lambda, \Lambda, m_*, m^*$ and β such that for every $u \in C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$ and for every $\tau \geq \tau_1$ the following inequality holds true

$$\beta^3 \int \sigma^{-\tau-2} w^{-2\beta} u^2 + \beta \int \sigma^\tau w^{-2\beta} |\nabla_g u|^2 \leq C \int \sigma^{2\tau+2} w^{-2\beta} (\triangle_g u)^2. \tag{6.9}$$

Then using our notations, we deduce for this inequality the following lemma.

Lemma 6.2 (Carleman estimate for second order elliptic operator)**.** *There exist C, β_0 and r_0 ($C \geq 1, \beta_0 \geq 1, 0 < r_0 \leq 1$) depending only on λ and Λ such that for every $u \in C_0^\infty(B_{r_0} \setminus \{0\})$ and for every $\beta > \beta_0$ we have*

$$\beta^3 \int r^{-\tau-2} \varphi_\beta^2 |u|^2 + \beta \int r^\tau \varphi_\beta^2 |\nabla u|^2 \leq C \int r^{2\tau+2} \varphi_\beta^2 |L_i u|^2 \tag{6.10}$$

where $\varphi_\beta = \varphi_\beta(|x|) = \exp(\beta |x|^{-\tau})$.

Proof. Let $\Gamma := I$, so that $m_* = m^* = 1$ and $\sigma(x) = |x|^{\frac{1}{2}}$. In addition, we have

$$\varphi_\beta^2(x) = \exp(2\beta |x|^{-\tau}) = \varphi^{-2\beta}(|x|) = w^{-2\beta}(x).$$

So that (6.9) reduces to

$$\beta^3 \int r^{-\tau-2} \varphi_\beta^2 u^2 + \beta \int r^\tau \varphi_\beta^2 |\nabla_g u|^2 \leq C \int r^{2\tau+2} \varphi_\beta^2 (\triangle_g u)^2. \tag{6.11}$$

Applying (6.2) and (6.3), we have

$$|\nabla u| \leq c |\nabla_g u| \tag{6.12}$$

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and use (6.7) we obtain

$$\beta^3 \int r^{-\tau-2}\varphi_\beta^2|u|^2 + \beta \int r^\tau \varphi_\beta^2 |\nabla u|^2 \tag{6.13}$$

$$\leq \beta^3 \int r^{-\tau-2}\varphi_\beta^2 u^2 + c\beta \int r^\tau \varphi_\beta^2 |\nabla_g u|^2 \tag{6.14}$$

$$\leq C \int r^{2\tau+2}\varphi_\beta^2 (\triangle_g u)^2 \tag{6.15}$$

$$\leq C \int r^{2\tau+2}\varphi_\beta^2 |L_k u|^2 + C \int r^{2\tau+2}\varphi_\beta^2 |\nabla u|^2 \tag{6.16}$$

We cancel the last term of (6.16) and completes the proof. □

6.2 Auxiliary Carleman estimate form

We need the following standard proposition to derive a new Carleman estimate (Theorem 6.4). This proposition can be found in [14], and we include the proof in Appendix A for reader's convenience.

Proposition 6.3. *Given $a \in C^1(\mathbb{R}^3 \setminus \{0\})$ and $u \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, we have the following inequalities*

$$\int a^2|\nabla^2 u|^2 \leq C(\int a^2|L_i u|^2 + \int (a^2 + |\nabla a|^2)|\nabla u|^2), \ i = 1, 2, \tag{6.17}$$

$$\int a^2|\nabla^3 u|^2 \leq C(\int a^2|\mathcal{L}u||\nabla^2 u| + \int (a^2 + |\nabla a|^2)|\nabla^2 u|^2), \tag{6.18}$$

where $C = C(\lambda, \Lambda)$.

Proof of Proposition 6.3. To simplify the notation, we omit the index k in L_k. For any $l \in \{1, 2, 3\}$, we have

$$\int Lu\partial_{ll}^2 u a^2 = -\int \partial_l(a^2 g^{ij}\partial_{ij}^2 u)\partial_l u$$

$$= -\int a^2 g^{ij}\partial_{ijl}^3 u \partial_l u - 2\int a\partial_l a g^{ij}\partial_{ij}^2 u\partial_l u - \int a^2(\partial_l g^{ij})\partial_{ij}^2 u \partial_l u$$

$$= \int a^2 g^{ij}\partial_{il}^2 u \partial_{jl}^2 u + \partial_j(a^2 g^{ij})\partial_{il}^2 u\partial_l u - 2\int a\partial_l a g^{ij}\partial_{ij}^2 u\partial_l u$$

$$- \int a^2(\partial_l g^{ij})\partial_{ij}^2 u\partial_l u$$

$$\geq \lambda \sum_l \int a^2|\nabla\partial_l u|^2 - C\int (|a| + |\nabla a|)|a||\nabla u||\nabla^2 u|,$$

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where $C = C(\lambda, \Lambda)$. Summing up the last inequality with respect to l and applying the inequality $2xy \leq \epsilon x^2 + \frac{1}{\epsilon}y^2$ yield (6.17).

Let $v \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Observe that

$$\lambda \int a^2|\nabla v|^2 \leq \int a^2 g^{ij}\partial_i v \partial_j v$$
$$= -\int a^2 g^{ij}\partial_{ij}^2 vv - 2\int a\partial_j a g^{ij}\partial_i vv - \int a^2\partial_j g^{ij}\partial_i vv,$$

we have

$$\int a^2|\nabla v|^2 \leq C\left(\int a^2|L_2 v||v| + \int (a^2 + |\nabla a|^2)v^2\right), \tag{6.19}$$

where $C = C(\lambda, \Lambda)$. Apply the inequality,

$$|L_1(\partial_l u)| \leq |\partial_l(L_1 u)| + C|\nabla^2 u|,$$

and take $v = L_1 u$ to the inequality (6.19), we have

$$\int a^2|L_1(\partial_l u)|^2 \leq C\left(\int a^2|\mathcal{L}u||\nabla^2 u| + \int (a^2 + |\nabla a|^2)|\nabla^2 u|^2\right),$$

where $C = C(\lambda, \Lambda)$. Summing up with respect to l and applying inequality (6.17) yield inequality (6.18). □

6.3 Production of two second order type Carleman estimate

By Lemma 6.2 and Proposition 6.3, we can derive a new Carleman estimate. We include the proof of the following Theorem in Appendix A.

Theorem 6.4 (Carleman estimate for the product of two elliptic operators)**.** *There exist C, β_* and r_* ($C \geq 1, \beta_* \geq 1, 0 < r_* \leq 1$) depending only on λ, Λ and Λ_1 such that for every $V \in C_0^\infty(B_{r_*}\setminus\{0\})$ and for every $\beta > \beta_*$, we have*

$$\sum_{k=0}^{3} \beta^{6-2k}\int r^{-\tau-2+k(2\tau+2)}\varphi_\beta^2|\nabla^k V|^2 \leq C\int r^{5\tau+6}\varphi_\beta^2|\mathcal{L}V|^2. \tag{6.20}$$

Proof of Theorem 6.4. We will prove this theorem using arguments similar to [14]. We recall the constant $\tau = \lambda^{-2}$ and the function $\varphi_\beta = \exp^{\beta|x|^{-\tau}}$ as in

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section 4. By applying inequality (6.10) with the function $V = r^{\frac{3}{2}\tau+2}v$, we have

$$\beta^3 \int r^{2\tau+2}\varphi_\beta^2|v|^2 = \beta^3 \int r^{-\tau-2}\varphi_\beta^2|r^{\frac{3}{2}\tau+2}v|^2 \le C \int r^{2\tau+2}\varphi_\beta^2|L_2(r^{\frac{3}{2}\tau+2}v)|^2 \tag{6.21}$$

for every $\beta \ge \beta_0$. Since

$$|L_2(r^{\frac{3}{2}\tau+2}v)| \le r^{\frac{3}{2}\tau+2}|L_2 v| + Cr^{\frac{3}{2}\tau+1}|\nabla v| + Cr^{\frac{3}{2}\tau}|v|, \tag{6.22}$$

we have

$$\beta^3 \int r^{2\tau+2}\varphi_\beta^2|v|^2 \le C \int r^{5\tau+6}\varphi_\beta^2|L_2 v|^2 + C \int r^{5\tau+4}\varphi_\beta^2|\nabla v|^2 \tag{6.23}$$

for every $\beta \ge \beta_1$, where C only depends on λ and Λ. Apply inequality (6.10) again with $V = r^{2\tau+2}v$, we have

$$\beta \int r^{\tau}\varphi_\beta^2|\nabla(r^{2\tau+2}v)|^2 \le C \int r^{2\tau+2}\varphi_\beta^2|L_2(r^{2\tau+2}v)|^2. \tag{6.24}$$

By

$$|\nabla(r^{2\tau+2}v)|^2 + Cr^{4\tau+2}v|v|^2 \ge \frac{1}{2}r^{4\tau+4}v|\nabla v|^2,$$

(6.24) and

$$|L_2(r^{2\tau+2}v)| \le r^{2\tau+2}|L_2 v| + Cr^{2\tau+1}|\nabla v| + Cr^{2\tau}|v|,$$

we have

$$\begin{aligned}
&\frac{\beta}{2}\int r^{5\tau+4}\varphi_\beta^2|\nabla v|^2 \\
&\le \beta \int r^{\tau}\varphi_\beta^2|\nabla(r^{2\tau+2}v)|^2 + C\beta \int r^{5\tau+2}\varphi_\beta^2|v|^2 \\
&\le C \int r^{2\tau+2}\varphi_\beta^2|L_2(r^{2\tau+2}v)|^2 + C\beta \int r^{5\tau+2}\varphi_\beta^2|v|^2 \\
&\le C \int r^{6\tau+6}\varphi_\beta^2|L_2 v|^2 + C \int r^{6\tau+4}\varphi_\beta^2|\nabla v|^2 + C\beta \int r^{5\tau+2}\varphi_\beta^2|v|^2.
\end{aligned}$$

Reduce the last inequality to

$$\beta \int r^{5\tau+4}\varphi_\beta^2|\nabla v|^2 \le C \int r^{6\tau+6}\varphi_\beta^2|L_2 v|^2 + C\beta \int r^{5\tau+2}\varphi_\beta^2|v|^2 \tag{6.25}$$

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for every $\beta \geq \beta_2$. Combining (6.23) and (6.25) gives

$$\beta^3 \int r^{2\tau+2}\varphi_\beta^2|v|^2 \leq C \int r^{5\tau+6}\varphi_\beta^2|L_2 v|^2 \tag{6.26}$$

for every $\beta \geq \beta_3$. Let $v = L_1 u$, then

$$\beta^3 \int r^{2\tau+2}\varphi_\beta^2|L_1 u|^2 \leq C \int r^{5\tau+6}\varphi_\beta^2|L_2(L_1 u)|^2. \tag{6.27}$$

Apply (6.10) with $V = u$ and (6.27), we have

$$\beta^6 \int r^{-\tau-2}\varphi_\beta^2|u|^2 + \beta^4 \int r^{\tau}\varphi_\beta^2|\nabla u|^2 \leq C \int r^{5\tau+6}\varphi_\beta^2|L_2(L_1 u)|^2. \tag{6.28}$$

Apply inequality (6.17) with $a = r^{\frac{3}{2}\tau+1}\varphi_\beta$, then

$$\int r^{3\tau+2}\varphi_\beta^2|\nabla^2 u|^2 \leq C(\int r^{3\tau+2}\varphi_\beta^2|L_1 u|^2 + \beta^2 \int r^{\tau}\varphi_\beta^2|\nabla u|^2). \tag{6.29}$$

Combine (6.27), (6.28) and (6.29), we have

$$\beta^2 \int r^{3\tau+2}\varphi_\beta^2|\nabla^2 u|^2 \leq C \int r^{5\tau+6}\varphi_\beta^2|L_2(L_1 u)|^2 \tag{6.30}$$

for every $\beta \geq \beta_4$.

Apply inequality (6.18) with $a = r^{-\frac{5}{2}\tau+2}\varphi_\beta$, then

$$\int r^{5\tau+4}\varphi_\beta^2|\nabla^3 u|^2 \leq C \int r^{5\tau+4}\varphi_\beta^2|L_2(L_1 u)||\nabla^2 u| + C\beta^2 \int r^{3\tau+2}\varphi_\beta^2|\nabla^2 u|^2. \tag{6.31}$$

Since

$$\begin{aligned} & r^{5\tau+4}|L_2(L_1 u)||\nabla^2 u| \\ & = r^{\frac{3}{2}\tau+1}|\nabla^2 u| r^{\frac{7}{2}\tau+3}|L_2(L_1 u)| \\ & \leq \frac{1}{2} r^{3\tau+2}|\nabla^2 u|^2 + \frac{1}{2} r^{7\tau+6}|L_2(L_1 u)|^2, \end{aligned}$$

we have

$$\int r^{5\tau+4}\varphi_\beta^2|\nabla^3 u|^2 \leq C \int r^{7\tau+6}\varphi_\beta^2|L_2(L_1 u)|^2 + C\beta^2 \int r^{3\tau+2}\varphi_\beta^2|\nabla^2 u|^2. \tag{6.32}$$

Combining (6.30) and (6.32) gives

$$\int r^{5\tau+4}\varphi_\beta^2|\nabla^3 u|^2 + \beta^2 \int r^{3\tau+2}\varphi_\beta^2|\nabla^2 u|^2 \leq C \int r^{5\tau+6}\varphi_\beta^2|L_2(L_1 u)|^2 \tag{6.33}$$

for every $\beta \geq \beta_5$. By (6.28) and (6.33), we obtain the claimed result. □

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7 Three spheres inequalities

In this section, we derive the main tool: three spheres inequalities for solution u to system (1.15), the elasticity system with residual stress. The idea used in [10] plays a key role in our arguments here. According to [10], we shall need two suitable auxiliary tools interior estimate (Corollary 4.3) and the Carleman estimate (Theorem 6.4) for our system.

In order to simplify the derivations and notations in this section, we only consider $\Omega = B_R := \{x \in \mathbb{R}^3 : |x| < R\}$. Moreover, if X is a norm space and $\mathbf{C}$ is an elasticity tensor, we shall denote $\mathbf{C} \in X$ if $\lambda, \mu, t_{jl} \in X$ for all $j, l = 1, 2, 3$, and let

$$\|\mathbf{C}\|_X := \|\lambda\|_X + 2\|\mu\|_X + \sum_{j,l=1}^{3} \|t_{jl}\|_X.$$

7.1 Three spheres inequality - normal type

Now we have all the tools to obtain three spheres inequalities. By system (5.7) with $w = \nabla \times u$ and $v = \nabla \cdot u$, we rewrite

$$U(x) := \begin{pmatrix} u(x) \\ w(x) \\ v(x) \end{pmatrix} : \Omega \to \mathbb{R}^7,$$

$$R(U(x)) := \begin{pmatrix} L_3^1(u, v, w) \\ L_3^3(u, v, w) \\ L_3^2(u, v, w) \end{pmatrix} : \Omega \to \mathbb{R}^7.$$

Let $\mathcal{L}_i := \triangle(A_1)$ for $i = 1, \cdots, 6$ and $\mathcal{L}_7 := \triangle(A_2)$, (5.7) can be rewritten in the form

$$\mathcal{L}_i U_i = R_i(U) \tag{7.1}$$

for $i = 1, \cdots, 7$, where R_i is the third order differential operator with $L^\infty(\Omega)$ coefficients. Now, we have the following inequality

$$|L_i U_i| \le C \sum_{k=0}^{3} |\nabla^k U| \tag{7.2}$$

for every $i = 1, 2, \cdots, 7$, where $C = C(M)$.

Theorem 7.1 (Three spheres inequalities). *If $\mathbf{C}$ is a elasticity tensor satisfying (1.9) and Assumption 1 and 6, there exists a positive number $r_* < 1$*

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depending only on θ, M, such that for every $0 < r_1 < r_2 < r_3 = 2r_2 < \min\{R, r_*\}$ and for every u satisfying (1.15) in B_R, we have

$$\int_{B_{r_2}} |u|^2 \leq C \left(\int_{B_{r_1}} |u|^2 \right)^{\delta} \left(\int_{B_{r_3}} |u|^2 \right)^{1-\delta}, \tag{7.3}$$

where the constants $C = C(\theta, M, r_1, r_2)$, $\delta = \delta(\theta, M, r_1, r_2)$.

Proof of Theorem 7.1. From Assumption 1 and 6, we know that the constant λ, Λ and Λ_1 in Theorem 6.4 depend only on θ and M.

Let $\xi \in C_0^\infty(\mathbb{R}^3)$ satisfy $0 \leq \xi \leq 1$

$$\xi(x) = \begin{cases} 0, & |x| \leq \frac{r_1}{4}, \\ 1, & \frac{r_1}{2} \leq |x| \leq \frac{3r_2}{2}, \\ 0, & 2r_2 \leq |x| \end{cases}$$

and $|\nabla^\alpha \xi| \leq C|x|^{-|\alpha|}$ for any multi-index α. Apply Theorem 6.4 with $u = \xi^4 U_i$, then

$$\begin{aligned} &\sum_{k=0}^{3} \beta^{6-2k} \int_{\{\frac{r_1}{2} \leq |x| \leq \frac{3r_2}{2}\}} r^{-\tau-2+k(2\tau+2)} \varphi_\beta^2 |\nabla^k U_i|^2 \\ &\leq C \int r^{5\tau+6} \varphi_\beta^2 |\mathcal{L}_i(\xi^4 U_i)|^2 \\ &= C \int_{\{\frac{r_1}{4} \leq |x| \leq \frac{r_1}{2}\} \cup \{\frac{3r_2}{2} \leq |x| \leq 2r_2\}} r^{5\tau+6} \varphi_\beta^2 |\mathcal{L}_i(\xi^4 U_i)|^2 + C \int_{\{\frac{r_1}{2} \leq |x| \leq \frac{3r_2}{2}\}} r^{5\tau+6} \varphi_\beta^2 |\mathcal{L}_i U_i|^2 \end{aligned}$$

for $i = 1, \cdots, 7$. Sum up with respect to $i = 1, \cdots, 7$, then

$$\begin{aligned} &\sum_{k=0}^{3} \beta^{6-2k} \int_{\{\frac{r_1}{2} \leq |x| \leq \frac{3r_2}{2}\}} r^{-\tau-2+k(2\tau+2)} \varphi_\beta^2 |\nabla^k U|^2 \\ &\leq C \sum_{i=1}^{7} \int_{\{\frac{r_1}{4} \leq |x| \leq \frac{r_1}{2}\} \cup \{\frac{3r_2}{2} \leq |x| \leq 2r_2\}} r^{5\tau+6} \varphi_\beta^2 |\mathcal{L}_i(\xi^4 U_i)|^2 \\ &\quad + C \sum_{i=1}^{7} \int_{\{\frac{r_1}{2} \leq |x| \leq \frac{3r_2}{2}\}} r^{5\tau+6} \varphi_\beta^2 |\mathcal{L}_i U_i|^2. \end{aligned}$$

Let β be large enough and r_2 be small enough ($\beta > \beta^*$, $r_2 < \frac{r_*}{2}$, where β_* and r_* are the constants in Theorem 6.4) and apply the inequality (7.2), and we

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can remove the second term on the right hand side and obtain the following simpler inequality:

$$\sum_{k=0}^{3}\beta^{6-2k}\int_{\{\frac{r_1}{2}\le|x|\le\frac{3r_2}{2}\}} r^{-\tau-2+k(2\tau+2)}\varphi_\beta^2|\nabla^k U|^2$$
$$\le C\sum_{i=1}^{7}\int_{\{\frac{r_1}{4}\le|x|\le\frac{r_1}{2}\}\cup\{\frac{3r_2}{2}\le|x|\le 2r_2\}} r^{5\tau+6}\varphi_\beta^2|\mathcal{L}_i(\xi^4U_i)|^2.$$

Since $\varphi_\beta(r)$ is decreasing with r, the last inequality yields

$$\beta^6\int_{\{\frac{r_1}{2}\le|x|\le r_2\}} r^{-\tau-2}\varphi_\beta^2(r_2)|u|^2$$
$$\le C\sum_{i=1}^{7}\int_{\{\frac{r_1}{4}\le|x|\le\frac{r_1}{2}\}} r^{5\tau+6}\varphi_\beta^2(\frac{r_1}{4})|\mathcal{L}_i(\xi^4U_i)|^2$$
$$+ C\sum_{i=1}^{7}\int_{\{\frac{3r_2}{2}\le|x|\le 2r_2\}} r^{5\tau+6}\varphi_\beta^2(\frac{3r_2}{2})|\mathcal{L}_i(\xi^4U_i)|^2.$$

We reduce the last inequality to

$$\int_{\{\frac{r_1}{2}\le|x|\le r_2\}} \varphi_\beta^2(r_2)|u|^2$$
$$\le C\sum_{k=0}^{3}\int_{\{\frac{r_1}{4}\le|x|\le\frac{r_1}{2}\}} r^{5\tau-2+2k}\varphi_\beta^2(\frac{r_1}{4})|\nabla^k U|^2$$
$$+ C\sum_{k=0}^{3}\int_{\{\frac{3r_2}{2}\le|x|\le 2r_2\}} r^{5\tau-2+2k}\varphi_\beta^2(\frac{3r_2}{2})|\nabla^k U|^2.$$

Apply Corollary 4.3(Interior estimate) to last inequality, and we get

$$\int_{\{\frac{r_1}{2}\le|x|\le r_2\}} \varphi_\beta^2(r_2)|u|^2$$
$$\le C\int_{\{\frac{r_1}{8}\le|x|\le r_1\}} r^{5\tau-4}\varphi_\beta^2(\frac{r_1}{4})|u|^2$$
$$+ C\int_{\{r_2\le|x|\le 3r_2\}} r^{5\tau-4}\varphi_\beta^2(\frac{3r_2}{2})|u|^2.$$

We reduce it to

$$\int_{\{\frac{r_1}{2}\le|x|\le r_2\}} |u|^2$$
$$\le C\int_{\{\frac{r_1}{4}\le|x|\le\frac{r_1}{2}\}} \phi_1(\beta)|u|^2 + C\int_{\{\frac{3r_2}{2}\le|x|\le 2r_2\}} \phi_2(\beta)|u|^2$$

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for all $\beta > \beta_*$, where $\phi_1(\beta) = \frac{\varphi_\beta^2(r_1/4)}{\varphi_\beta^2(r_2)} > 1$, $\phi_2(\beta) = \frac{\varphi_\beta^2(3r_2/2)}{\varphi_\beta^2(r_2)}$ and $C = C(\theta, M, r_1, r_2)$. Adding $\int_{|x|<\frac{r_1}{2}} |u|^2$ to both sides leads to

$$\int_{|x|<r_2} |u|^2 \leq C\phi_1(\beta) \int_{|x|<r_1} |u|^2 + C\phi_2(\beta) \int_{|x|<2r_2} |u|^2. \tag{7.4}$$

We observe that ϕ_1 is increasing with β and that ϕ_2 is decreasing with β. If $\int_{|x|<r_1} |u|^2 = 0$, then $\int_{|x|<r_2} |u|^2 = 0$ as $\beta \to \infty$. If $\int_{|x|<r_1} |u|^2 \neq 0$ and $\phi_1(\beta^*) \int_{|x|<r_1} |u|^2 < \phi_2(\beta^*) \int_{|x|<2r_2} |u|^2$, there exists $\beta_1 > \beta^*$ such that

$$\phi_1(\beta_1) \int_{|x|<r_1} |u|^2 = \phi_2(\beta_1) \int_{|x|<1} |u|^2.$$

Set $\beta = \beta_1$ in (7.4), then we obtain

$$\begin{aligned}
&\int_{|x|<r_2} |u|^2 \\
&\leq 2C\phi_1(\beta_1) \int_{|x|<r_1} |u|^2 \\
&= 2C(\int_{|x|<r_1} |u|^2)^{1-\delta} (\int_{|x|<2r_2} |u|^2)^{\delta},
\end{aligned}$$

where

$$\delta = \frac{(\frac{r_1}{4})^{-\tau} - (r_2)^{-\tau}}{(\frac{r_1}{4})^{-\tau} - (\frac{3r_2}{2})^{-\tau}} \in (0, 1).$$

If $\int_{|x|<r_1} |u|^2 \neq 0$ and $\phi_1(\beta^*) \int_{|x|<r_1} |u|^2 \geq \phi_2(\beta^*) \int_{|x|<2r_2} |u|^2$, we have

$$\begin{aligned}
&\int_{|x|<r_2} |u|^2 \\
\leq &\int_{|x|<2r_2} |u|^2 \\
= &(\int_{|x|<2r_2} |u|^2)^{1-\delta} (\int_{|x|<2r_2} |u|^2)^{\delta} \\
\leq &(\frac{\phi_1(\beta^*)}{\phi_2(\beta^*)})^{1-\delta} (\int_{|x|<r_1} |u|^2)^{1-\delta} (\int_{|x|<2r_2} |u|^2)^{\delta} \\
\leq &C(\int_{|x|<r_1} |u|^2)^{1-\delta} (\int_{|x|<2r_2} |u|^2)^{\delta},
\end{aligned}$$

where δ can be chosen as above. This completes the proof. □

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7.2 Three spheres inequality - differential type

Corollary 7.2 (Three spheres inequalities - differential type)**.** *If* $\mathbf{C}$ *is an elasticity tensor satisfying (1.9) and Assumptions 1 and 6, there exists a positive number* $r_* < 1$ *depending only on* θ, M*, such that, for every* $0 < r_1 < r_2 < r_3 = 3r_2 < \min\{R, r_*\}$ *and for every* u *satisfying (1.15) in* B_R*, we have*

$$\int_{B_{r_2}} |\nabla u|^2 \leq C \left(\int_{B_{r_1}} |\nabla u|^2 \right)^{\delta} \left(\int_{B_{r_3}} |\nabla u|^2 \right)^{1-\delta}, \tag{7.5}$$

where the constants $C = C(\theta, M, r_1, r_2)$*,* $\delta = \delta(\theta, M, r_1, r_2)$*.*

Proof of Corollary 7.2. Let $u_r := \frac{1}{|B_r|}\int_{B_r} u$ and $v := u - u_r$, then v satisfies the hypothesis of Theorem 7.1 and $\nabla v = \nabla u$. We apply Caccioppoli's inequality with $r = r_2$, $\hat{R} = \frac{3r_2}{2}$, Theorem 7.1 and Poincaré inequality twice with $r = \hat{R} = r_1$ and $r = r_1$, $\hat{R} = 2r_2$, respectively, then

$$\begin{aligned}
\int_{B_{r_2}} |\nabla u|^2 &= \int_{B_{r_2}} |\nabla v|^2 \\
&\leq \frac{C}{(r_2)^2} \int_{B_{\frac{3r_2}{2}}} |v|^2 \\
&\leq \frac{C}{(r_2)^2} (\int_{B_{r_1}} |v|^2)^{1-\delta} (\int_{B_{3r_2}} |v|^2)^{\delta} \\
&\leq \frac{C}{(r_2)^2} (r_1^2 \int_{B_{r_1}} |\nabla u|^2)^{1-\delta} ((\frac{3r_2}{r_1})^{6-2} (3r_2)^2 \int_{B_{3r_2}} |\nabla u|^2)^{\delta} \\
&\leq C (\frac{3r_2}{r_1})^{2\delta} (\frac{r_3}{r_1})^{6\delta} (\int_{B_{r_1}} |\nabla u|^2)^{1-\delta} (\int_{B_{3r_2}} |\nabla u|^2)^{\delta}.
\end{aligned}$$

□

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8 Lipschitz propagation of smallness

8.1 Boundary estimate

We prove the main theorem(Theorem 3.1) with the following auxiliary lemmas, which are the analogues of the lemmas in [4]. The proofs of the following lemmas are shown in Appendix B.

Lemma 8.1 (Boundary estimate)**.** *Let* $\mathbf{C}$ *be a elasticity tensor satisfying (1.9) and Assumptions 1, 2 and 6. For any positive integer* m*. If* $u_0 \in H^1(\Omega, \mathbb{R}^3)$ *is solution of* (1.15)*, then we have*

$$\int_{\Omega\setminus\Omega_{(3m+1)\rho}} |\nabla u_0|^2 \le C\rho^{1/3}||\varphi||^2_{L^2(\partial\Omega)}, \tag{8.1}$$

where $C = C(\theta, \|\mathbf{C}\|_{W^{2,\infty}}, r_0, M_0, |\Omega|, m)$*.*

Proof of Lemma 8.1. For convenience, we suppress the subscript 0 in u_0. Apply Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega\setminus\Omega_{(3m+1)\rho}} |\nabla u|^2 &\le |\Omega\setminus\Omega_{(3m+1)\rho}|^{\frac{1}{3}}(\int_{\Omega\setminus\Omega_{(3m+1)\rho}} |\nabla u|^3)^{\frac{2}{3}} \\ &= |\Omega\setminus\Omega_{(3m+1)\rho}|^{\frac{1}{3}}\|\nabla u\|^2_{L^3(\Omega\setminus\Omega_{(3m+1)\rho})}.\end{aligned}$$

Apply Sobolev inequality (see [1]), we have

$$\|\nabla u\|^2_{L^3(\Omega)} \le C\|\nabla u\|^2_{H^{\frac{1}{2}}(\Omega)} \le C\|u\|^2_{H^{\frac{3}{2}}(\Omega)}.$$

Combine the last two inequalities, we obtain

$$\|\nabla u\|^2_{L^2(\Omega\setminus\Omega_{(3m+1)\rho})} \le C|\Omega\setminus\Omega_{(3m+1)\rho}|^{\frac{1}{3}}\|u\|^2_{H^{\frac{3}{2}}(\Omega)}, \tag{8.2}$$

where $C = C(r_0, M_0, |\Omega|)$. By the global estimates for the Neumann problem (see [2]), we have

$$\|u\|_{H^1(\Omega)} \le C_1\|\varphi\|_{H^{\frac{-1}{2}}(\partial\Omega)}$$

and

$$\|u\|_{H^2(\Omega)} \le C_2\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

By interpolation (see [11]), we have

$$\|u\|_{H^{\frac{3}{2}}(\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}, \tag{8.3}$$

where $C = C(r_0, M_0, |\Omega|, \mathbf{C})$. By (A.3) of [17], we obtain the inequality

$$|\Omega\setminus\Omega_{(3m+1)\rho}| \le C\rho, \tag{8.4}$$

where $C = C(r_0, M_0, |\Omega|, m)$. Combining (8.2), (8.3) and(8.4) yields the result. □

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8.2 Theorem of Lipschitz propagation of smallness

Lemma 8.2. *(Lipschitz propagation of smallness) Let* $\mathbf{C}$ *be an elasticity tensor satisfying (1.9) and Assumptions 1, 2 and 6. If* $u_0 \in H^1(\Omega, \mathbb{R}^3)$ *is solution of* (1.15)*, we have*

$$\int_{B_\rho(x)} |\nabla u_0|^2 \geq C_\rho \int_\Omega |\nabla u_0|^2 \tag{8.5}$$

for any $\rho > 0$ *and for every* $x \in \Omega_{9\rho}$*, where*
$C_\rho = C_\rho(\theta, M, |\Omega|, r_0, M_0, \|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}, \rho)$.

Proof of Lemma 8.2. For convenience, we suppress the subscript 0 in u_0. By Assumption 2, there exists ρ_0 such that $\Omega_{9\rho}$ is connected for every $\rho \leq \rho_0$. Without loss of generality, we may assume, for this proof, $\rho \leq \rho_0$. Given any $y \in \Omega_{9\rho}$, let γ be an arc in $\Omega_{9\rho}$ joining x and y. We define $\{x_i\}_{i=1}^L$ as follows: Set $x_1 = x$. If $|x_i - y| > 2\rho$, we set $x_{i+1} = \gamma(t_i)$, where $t_i = max\{t : |\gamma(t) - x_i| = 2\rho\}$. Otherwise let $i = L$ and stop the process. Then, by construction, the balls $B_\rho(x_i)$ are pairwise disjoint, $|x_{i+1} - x_i| = 2\rho$ for $i = 1, \cdots, L-1$, $|x_L - y| \leq 2\rho$.

By Corollary 7.2, we have $\int_{B_{r_2}} |\nabla u|^2 \leq C(\int_{B_{r_1}} |\nabla u|^2)^\delta (\int_{B_{r_3}} |\nabla u|^2)^{1-\delta}$ with x_i, $r_1 = \rho$, $r_2 = 3\rho$, $r_3 = 9\rho$, $C = C(\theta, M, \rho)$ and $\delta = \delta(\theta, M, \rho)$. Since

$$||\nabla u||_{L^2(B_\rho(x_{i+1}))} \leq ||\nabla u||_{L^2(B_{3\rho}(x_i))} \leq C||\nabla u||^\delta_{L^2(B_\rho(x_i))} ||\nabla u||^{1-\delta}_{L^2(\Omega)},$$

we have

$$\frac{||\nabla u||_{L^2(B_\rho(x_{i+1}))}}{||\nabla u||_{L^2(\Omega)}} \leq C(\frac{||\nabla u||_{L^2(B_\rho(x_i))}}{||\nabla u||_{L^2(\Omega)}})^\delta.$$

Sum up with respect to i, then we derive

$$\frac{||\nabla u||_{L^2(B_\rho(y))}}{||\nabla u||_{L^2(\Omega)}} \leq C(C(\frac{||\nabla u||_{L^2(B_\rho(x_{L-2}))}}{||\nabla u||_{L^2(\Omega)}})^\delta)^\delta \tag{8.6}$$

$$\leq C^{1+\delta+\delta^2+\cdots}(\frac{||\nabla u||_{L^2(B_\rho(x))}}{||\nabla u||_{L^2(\Omega)}})^{\delta L}. \tag{8.7}$$

Since $B_\rho(x_i) \cap B_\rho(x_j) = \emptyset$ if $i \neq j$, we have $L \leq \frac{|\Omega|}{w_3 \rho^3}$. Let us cover $\Omega_{10\rho}$ with non-overlapping closed cubes of side $l = \frac{2\rho}{\sqrt{3}}$. The number of the cubes is controlled by $N = \frac{|\Omega| 3^{\frac{3}{2}}}{2^3 \rho^3}$. Clearly, any such cube is contained in $B_\rho(y)$ for some $y \in \Omega_{9\rho}$. Therefore, from (8.7) we have

$$\frac{||\nabla u||_{L^2(\Omega_{10\rho})}}{||\nabla u||_{L^2(\Omega)}} \leq \frac{C}{\rho^{\frac{3}{2}}}(\frac{||\nabla u||_{L^2(B_\rho(x))}}{||\nabla u||_{L^2(\Omega)}})^{\delta L}. \tag{8.8}$$

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Clearly, we have the following identity

$$\frac{||\nabla u||^2_{L^2(\Omega_{10\rho})}}{||\nabla u||^2_{L^2(\Omega)}} = 1 - \frac{||\nabla u||^2_{L^2(\Omega/\ \Omega_{10\rho})}}{||\nabla u||^2_{L^2(\Omega)}}. \tag{8.9}$$

By trace inequality, then

$$||\varphi||_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C||\nabla u||_{L^2(\Omega)}.$$

By Lemma 8.1 and te last inequality, we have

$$\begin{aligned} ||\nabla u||^2_{L^2(\Omega/\ \Omega_{10\rho})} &\leq C\rho^{\frac{1}{3}}||\varphi||^2_{L^2(\partial\Omega)} && (8.10) \\ &\leq C\rho^{\frac{1}{3}}||\varphi||^2_{L^2(\partial\Omega)}\frac{||\varphi||^2_{H^{-\frac{1}{2}}(\partial\Omega)}}{||\varphi||^2_{H^{-\frac{1}{2}}(\partial\Omega)}} && (8.11) \\ &\leq C\rho^{\frac{1}{3}}||\nabla u||^2_{L^2(\Omega)}. && (8.12) \end{aligned}$$

From (8.9) and (8.12), there exists $\bar{\rho} > 0$ ($\bar{\rho} \leq \rho_0$) such that

$$\frac{||\nabla u||^2_{L^2(\Omega_{10\rho})}}{||\nabla u||^2_{L^2(\Omega)}} \geq \frac{1}{2} \tag{8.13}$$

for every $0 < \rho \leq \bar{\rho}$.

From (8.8) and (8.13), we have

$$C_\rho||\nabla u||^2_{L^2(\Omega)} \leq ||\nabla u||^2_{L^2(B_\rho(x))}$$

for every $0 < \rho \leq \bar{\rho}$ and for every $x \in \Omega_{9\rho}$. If $\rho > \bar{\rho}$, we also have

$$\int_{B_\rho(x)} |\nabla u|^2 \geq \int_{B_{\bar{\rho}}(x)} |\nabla u|^2 \geq C_{\bar{\rho}} \int_\Omega |\nabla u|^2$$

for every $x \in \Omega_{9\rho}$. This completes the proof. □

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9 Auxiliary lemmas

Below we will introduce two simple Lemma. With the help of these two Lemma, we can prove that the main results are more concise and clear.

9.1 Three auxiliary equations

Lemma 9.1. *[Three auxiliary equations] Let* $\mathbf{C}$ *and* $\tilde{\mathbf{C}}$ *be elasticity tensors and* $\mathbf{C}$ *also satisfy Assumption 1. If* u *,* $u_0 \in H^1(\Omega, \mathbb{R}^3)$ *are weak solutions for the traction problems (3.1), (3.2), respectively, then we have the following identities:*

$$\int_\Omega (\chi_{\Omega\setminus D}\mathbf{C}+\chi_D\tilde{\mathbf{C}})\nabla(u-u_0)\cdot\nabla(u-u_0)-\int_D(\tilde{\mathbf{C}}-\mathbf{C})\nabla u_0\cdot\nabla u_0=\int_{\partial\Omega}(g-g_0)\cdot\varphi, \tag{9.1}$$

$$\int_\Omega \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0)+\int_D(\tilde{\mathbf{C}}-\mathbf{C})\nabla u\cdot\nabla u=\int_{\partial\Omega}(g_0-g)\cdot\varphi, \tag{9.2}$$

$$\int_D(\tilde{\mathbf{C}}-\mathbf{C})\nabla u\cdot\nabla u_0=\int_{\partial\Omega}(g_0-g)\cdot\varphi, \tag{9.3}$$

where $g, g_0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ *are the displacement of* u *and* u_0*, respectively, on* $\partial\Omega$*.*

Proof of Lemma 9.1. Set $H := \tilde{\mathbf{C}} - \mathbf{C}$. Let D_1 and D_2 be two subsets of Ω. Let u_1 and u_2 be functions such that

$$\begin{cases} \operatorname{div}((\chi_{\Omega\setminus D_i}\mathbf{C}+\chi_{D_i}\tilde{\mathbf{C}})\nabla u_i)=0 & \text{in} \quad \Omega \\ (\mathbf{C}\nabla u_i)\cdot\nu=\varphi & \text{on} \quad \partial\Omega \end{cases} \tag{9.4}$$

with $g_i := u_i|_{\partial\Omega}$ for $i = 1, 2$. For any $w \in H^1(\Omega, \mathbb{R}^3)$, we have

$$\begin{aligned}
&\int_\Omega (\mathbf{C}+\chi_{D_1}H)\nabla u_1\cdot\nabla w \\
&= -\int_\Omega \nabla((\mathbf{C}+\chi_{D_1}H)\nabla u_1)\cdot w+\int_{\partial\Omega}(\mathbf{C}\nabla u_1)\nu\cdot w \\
&= 0+\int_{\partial\Omega}\varphi\cdot w \\
&= -\int_\Omega \nabla((\mathbf{C}+\chi_{D_2}H)\nabla u_2)\cdot w+\int_{\partial\Omega}(\mathbf{C}\nabla u_2)\nu\cdot w \\
&= \int_\Omega (\mathbf{C}+\chi_{D_2}H)\nabla u_2\cdot\nabla w.
\end{aligned}$$

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Subtract $\int_\Omega (\mathbf{C} + \chi_{D_1} H)\nabla u_2 \cdot \nabla w$ from both sides of the last equation, then we have

$$\int_\Omega (\mathbf{C} + \chi_{D_1} H)\nabla(u_1 - u_2) \cdot \nabla w = \int_\Omega (\chi_{D_2} - \chi_{D_1}) H \nabla u_2 \cdot \nabla w. \tag{9.5}$$

Since $\mathbf{C}_{ijkl} = \mathbf{C}_{klij}$, taking $w = u_1$ in (9.5) yields

$$\int_\Omega (\chi_{D_2} - \chi_{D_1}) H \nabla u_2 \cdot \nabla u_1 = \int_\Omega (\mathbf{C} + \chi_{D_1} H)\nabla(u_1 - u_2) \cdot \nabla u_1 = \int_{\partial\Omega} \varphi \cdot (g_1 - g_2).$$

Combine the last identity and (9.5) with $w = u_1 - u_2$, we have

$$\begin{aligned}
&\int_\Omega (\mathbf{C} + \chi_{D_1} H)\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2) \\
&= \int_\Omega (\chi_{D_2} - \chi_{D_1}) H \nabla u_2 \cdot \nabla(u_1 - u_2) \\
&= \int_\Omega (\chi_{D_1} - \chi_{D_2}) H \nabla u_2 \cdot \nabla u_2 + \int_{\partial\Omega} \varphi \cdot (g_1 - g_2).
\end{aligned}$$

The last identity implies

$$\begin{aligned}
\int_\Omega (\mathbf{C} + \chi_{D_1} H)\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2) + \int_{D_2 \setminus D_1} H \nabla u_2 \cdot \nabla u_2 \\
= \int_{\partial\Omega} \varphi \cdot (g_1 - g_2) + \int_{D_1 \setminus D_2} H \nabla u_2 \cdot \nabla u_2.
\end{aligned} \tag{9.6}$$

1. We choose $D_1 = D$ and $D_2 = \emptyset$, hence $u_1 = u$ and $u_2 = u_0$. Substitute them into (9.6), then

$$\int_\Omega (\mathbf{C} + \chi_D H)\nabla(u - u_0) \cdot \nabla(u - u_0) + 0 = \int_{\partial\Omega} \varphi \cdot (g - g_0) + \int_D (\tilde{\mathbf{C}} - \mathbf{C})\nabla u_0 \cdot \nabla u_0.$$

This is identity (9.1).

2. We choose $D_1 = \emptyset$ and $D_2 = D$, hence $u_1 = u_0$ and $u_2 = u$. Substitute them into (9.6), then

$$\int_\Omega \mathbf{C}\nabla(u_0 - u) \cdot \nabla(u_0 - u) + \int_D (\tilde{\mathbf{C}} - \mathbf{C})\nabla u \cdot \nabla u = \int_{\partial\Omega} \varphi \cdot (g_0 - g) + 0.$$

This is identity (9.2).

3. We choose $w = u_0$ and $w = u$ in the weak formulation of the traction problems (3.2) and (3.1), respectively, then we have

$$\int_\Omega (\mathbf{C} + \chi_D H)\nabla u \cdot \nabla u_0 = \int_{\partial\Omega} g_0 \cdot \varphi \tag{9.7}$$

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and

$$\int_\Omega \mathbf{C}\nabla u_0 \cdot \nabla u = \int_{\partial\Omega} g \cdot \varphi. \tag{9.8}$$

By subtracting (9.8) from (9.7) we obtain identity (9.3). □

9.2 Estimate boundary energy of strongly elliptic system

Lemma 9.2. *[Estimate boundary energy] Let* $\mathbf{C}$ *and* $\tilde{\mathbf{C}}$ *be elasticity tensors. Let* ξ_l *and* ξ_u*,* $0 < \xi_l < \xi_u$*, such that*

$$\xi_l|A| \leq \mathbf{C}(x)A \cdot A \leq \xi_u|A| \text{ for a.e. } x \in \Omega, \tag{9.9}$$

for any 3×3 *matrix* A*, and let the jump* $\tilde{\mathbf{C}} - \mathbf{C}$ *satisfies either (3.3) or (3.4). Suppose that* u*,* $u_0 \in H^1(\Omega, \mathbb{R}^3)$ *are weak solutions to the traction problems (3.1) and (3.2), respectively. If (3.3) holds, then we have*

$$\frac{\eta\xi_l}{\delta}\int_D |\nabla u_0|^2 \leq \int_{\partial\Omega}(g_0 - g)\cdot\varphi \leq (\delta - 1)\xi_u\int_D |\nabla u_0|^2; \tag{9.10}$$

if (3.4) holds, then we have

$$\eta\xi_l\int_D |\nabla u_0|^2 \leq \int_{\partial\Omega}(g - g_0)\cdot\varphi \leq \frac{1-\delta}{\delta}\xi_u\int_D |\nabla u_0|^2. \tag{9.11}$$

Proof of Lemma 9.2. Set $H = \tilde{\mathbf{C}} - \mathbf{C}$.

1. If (3.3) holds, from identity (9.1), we have

$$\int_{\partial\Omega}\varphi\cdot(g_0 - g) \leq \int_D H\nabla u_0\cdot\nabla u_0 \leq (\delta - 1)\int_D \mathbf{C}\nabla u_0\cdot\nabla u_0 \leq (\delta-1)\xi_u\int_D |\nabla u_0|^2.$$

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For the middle term, we observe

$$
\begin{aligned}
&\int_D H\nabla u_0 \nabla u_0 \\
=&\int H(\nabla u_0 - \nabla u + \nabla u)\cdot(\nabla u_0 - \nabla u + \nabla u) \\
=&\int_D H\nabla(u-u_0)\cdot\nabla(u-u_0) + \int_D H\nabla u\cdot\nabla u + \int_D H\nabla(u_0-u)\cdot\nabla u \\
&+ \int_D H\nabla u\cdot\nabla(u_0-u) \\
\leq&(1+\epsilon)\int_D H\nabla(u-u_0)\cdot\nabla(u-u_0) + (1+\frac{1}{\epsilon})\int_D H\nabla u\cdot\nabla u \\
\leq&(1+\epsilon)(\delta-1)\int_D \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0) + \frac{\epsilon+1}{\epsilon}\int_D H\nabla u\cdot\nabla u \\
=&(1+\epsilon)(\delta-1)\left[\int_D \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0) + \frac{1}{\epsilon(\delta-1)}\int_D H\nabla u\cdot\nabla u\right]
\end{aligned}
$$

for every $\epsilon > 0$. By $\epsilon = \frac{1}{\delta-1} > 0$ and identity (9.2), then we have

$$
\begin{aligned}
&\int_D H\nabla u_0\cdot\nabla u_0 \\
\leq&\delta\left[\int_D \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0) + \int_D H\nabla u\cdot\nabla u\right] \\
\leq&\delta\left[\int_\Omega \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0) + \int_D H\nabla u\cdot\nabla u\right] \\
=&\ \delta\int_{\partial\Omega}(g_0-g)\cdot\varphi. \qquad (9.12)
\end{aligned}
$$

From (3.3), we also have

$$
\int_D H\nabla u_0\cdot\nabla u_0 \geq \eta\int_D \mathbf{C}\nabla u_0\cdot\nabla u_0 \geq \epsilon_l\eta\int_D |\nabla u_0|^2. \qquad (9.13)
$$

Combine (9.12) and (9.13), we complete the proof of (9.10).

2. If (3.4) holds from identity (9.3), we have

$$
\begin{aligned}
&\int_{\partial\Omega}(g-g_0)\cdot\varphi \\
=&\int_D (-H)\nabla u\cdot\nabla u_0
\end{aligned}
$$

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$$\le \frac{\epsilon}{2}\int_D (-H)\nabla u\cdot\nabla u + \frac{1}{2\epsilon}\int_D (-H)\nabla u_0\cdot\nabla u_0 \tag{9.14}$$

for every $\epsilon > 0$. By identity (9.1), (9.2) and inequality (3.4), observing that

$$\tilde{\mathbf{C}} = \chi_D\tilde{\mathbf{C}} + \chi_{\Omega\setminus D}\tilde{\mathbf{C}} + \chi_{\Omega\setminus D}\mathbf{C} - \chi_{\Omega\setminus D}\mathbf{C} = (\chi_{\Omega\setminus D}\mathbf{C} + \chi_D\tilde{\mathbf{C}}) + \chi_{\Omega\setminus D}(\tilde{\mathbf{C}} - \mathbf{C}),$$

we have

$$\begin{aligned}
&\int_D (-H)\nabla u\cdot\nabla u\\
=&\int_{\partial\Omega}(g-g_0)\cdot\varphi + \int_\Omega \mathbf{C}\nabla(u-u_0)\cdot\nabla(u-u_0)\\
\le&\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{1}{\delta}\int_\Omega \tilde{\mathbf{C}}\nabla(u-u_0)\cdot\nabla(u-u_0)\\
\le&\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{1}{\delta}\int_\Omega (\chi_{\Omega\setminus D}\mathbf{C} + \chi_D\tilde{\mathbf{C}})\nabla(u-u_0)\cdot\nabla(u-u_0)\\
=&\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{1}{\delta}\left[\int_{\partial\Omega}(g-g_0)\cdot\varphi + \int_D H\nabla u_0\cdot\nabla u_0\right]\\
=&\frac{\delta+1}{\delta}\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{1}{\delta}\int_D H\nabla u_0\cdot\nabla u_0.
\end{aligned}$$

So we have

$$\int_D (-H)\nabla u\cdot\nabla u$$

$$\le \frac{\delta+1}{\delta}\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{1}{\delta}\int_D H\nabla u_0\cdot\nabla u_0. \tag{9.15}$$

From (9.14) and (9.15), we obtain

$$\begin{aligned}
&\int_{\partial\Omega}(g-g_0)\cdot\varphi\\
\le&\frac{\epsilon}{2}\frac{\delta+1}{\delta}\int_{\partial\Omega}(g-g_0)\cdot\varphi + \frac{\epsilon}{2}\frac{1}{\delta}\int_D H\nabla u_0\cdot\nabla u_0 + \frac{1}{2\epsilon}\int_D (-H)\nabla u_0\cdot\nabla u_0
\end{aligned}$$

for every $\epsilon > 0$. We take $\epsilon = \delta > 0$, then

$$\frac{1-\delta}{2}\int_{\partial\Omega}(g-g_0)\cdot\varphi \le \frac{1-\delta}{2\delta}\int_D (-H)\nabla u_0\cdot\nabla u_0.$$

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Hence

$$\begin{aligned}
&\int_{\partial\Omega}(g-g_0)\cdot\varphi\\
&\le \frac{1}{\delta}\int_D(-H)\nabla u_0\cdot\nabla u_0\\
&\le \frac{1}{\delta}(1-\delta)\int_D \mathbf{C}\nabla u_0\cdot\nabla u_0\\
&\le \frac{1-\delta}{\delta}\xi_u\int_D|\nabla u_0|^2.
\end{aligned}$$

From identity (9.1) and (3.4), we get

$$\begin{aligned}
&\int_{\partial\Omega}(g-g_0)\cdot\varphi\\
&\ge \int_D(-H)\nabla u_0\cdot\nabla u_0\\
&\ge \eta\int_D \mathbf{C}\nabla u_0\cdot\nabla u_0\\
&\ge \eta\xi_l\int_D|\nabla u_0|^2.
\end{aligned}$$

□

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10 Proof of main result

Proof of Theorem 3.1. From Assumption 1 and 6, we choose $\xi_l = \theta$ and $\xi_u = 3M$, such that Lemma 9.2 holds.

1. By standard regularity estimates and Poincaré inequality, we have

$$\|\nabla u_0\|_{L^\infty(D)} \leq C\|u_0\|_{H^1(\Omega_{d_0/2})} \leq C\|\nabla u_0\|_{L^2(\Omega)},$$

where $C = C(d_0, |\Omega|)$. Since u_0 is the solution for (3.2), we also have

$$\|\nabla u_0\|^2_{L^2(\Omega)} \leq C\int_\Omega \mathbf{C}\nabla u_0 \cdot \nabla u_0 = C\int_{\partial\Omega} (\mathbf{C}\nabla u_0)\nu \cdot \nabla u_0 = C\int_{\partial\Omega} \varphi \cdot g_0,$$

where $C = C(d_0, \theta, |\Omega|)$. Apply Lemma (9.2) and the last two inequalities, and we obtain the lower bound for $|D|$ with $C = C(d_0, \theta, |\Omega|)$.

2. Let $D_{h_1} = \{x \in D | \text{dist}(x, \partial D) \geq h_1\}$ and $\epsilon = \min\{\frac{h_1}{\sqrt{3}}, \frac{2d_0}{9}\}$. Because of Assumption 4, there exists $\{Q_l\}_{l=1}^L$, which are the non-overlapping closed cubes with side ϵ that cover D_{h_1}, and which are contained in D. Now we have the estimate

$$\int_D |\nabla u_0|^2 \geq \int_{\bigcup_{l=1}^L} |\nabla u_0|^2 \geq L\int_{Q^*} |\nabla u_0|^2,$$

where Q^* is the cube in $\{Q_l\}_{l=1}^L$ such that

$$\int_{Q^*} |\nabla u_0|^2 = \min_l \int_{Q_l} |\nabla u_0|^2.$$

Since $L\epsilon^3 \geq |D_{h_1}|$, we have

$$\int_D |\nabla u_0|^2 \geq \frac{|D_{h_1}|}{\epsilon^3}\int_{Q^*} |\nabla u_0|^2 \geq \frac{|D_{h_1}|}{\epsilon^3}\int_{B(\bar{x},\epsilon/2)} |\nabla u_0|^2, \tag{10.1}$$

where $\bar{x}$ is the center of Q^*. By Lemma 8.2 and Assumption 3, we obtain

$$\begin{aligned}
&\int_D |\nabla u_0|^2 \geq \\
&\frac{|D_{h_1}|}{\epsilon^3}\int_{B(\bar{x},\epsilon/2)} |\nabla u_0|^2 \\
&\geq \frac{|D_{h_1}|}{\epsilon^3} C_{\frac{\epsilon}{2}} \int_\Omega |\nabla u_0|^2 \\
&\geq C|D_{h_1}|\int_\Omega \mathbf{C}\nabla u_0 \cdot \nabla u_0 \\
&= C|D_{h_1}|\int_\Omega \varphi \cdot g_0 \\
&\geq C\frac{|D|}{2}\int_\Omega \varphi \cdot g_0,
\end{aligned}$$

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where $C = C(\theta, d_0, |\Omega|, r_0, M_0, M, h_1, \|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}}(\partial\Omega))$. By Lemma 9.2, we obtain the upper bound of D. $\square$

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11 Further work

The main contribution of our work is to overcome the need for loses some symmetry properties(1.11) under elasticity system with residual stress. The system we discussed has a wider range of applications and a closer physical reality than Lamé system.

In general, we can complete our work, because we can derive the Lipschitz propagation of smallness(8.2) with our Assumptions 1-6. And the key of this derivation is that we obtain Three-Spheres Inequality(7.2), which is based on we can transform the original elasticity system with residual stress(1.8) into the product of two elliptic operators.

The work we have done is based on the assumption that $\beta_3 = \beta_4 = 0$ for (1.6). In our future work, we can try to get rid of this hypothesis or assume that some β_3 or β_4 are a very small value to get a similar estimate. The biggest difficulty will be how to convert the equation into new system for the product of two elliptic operators.

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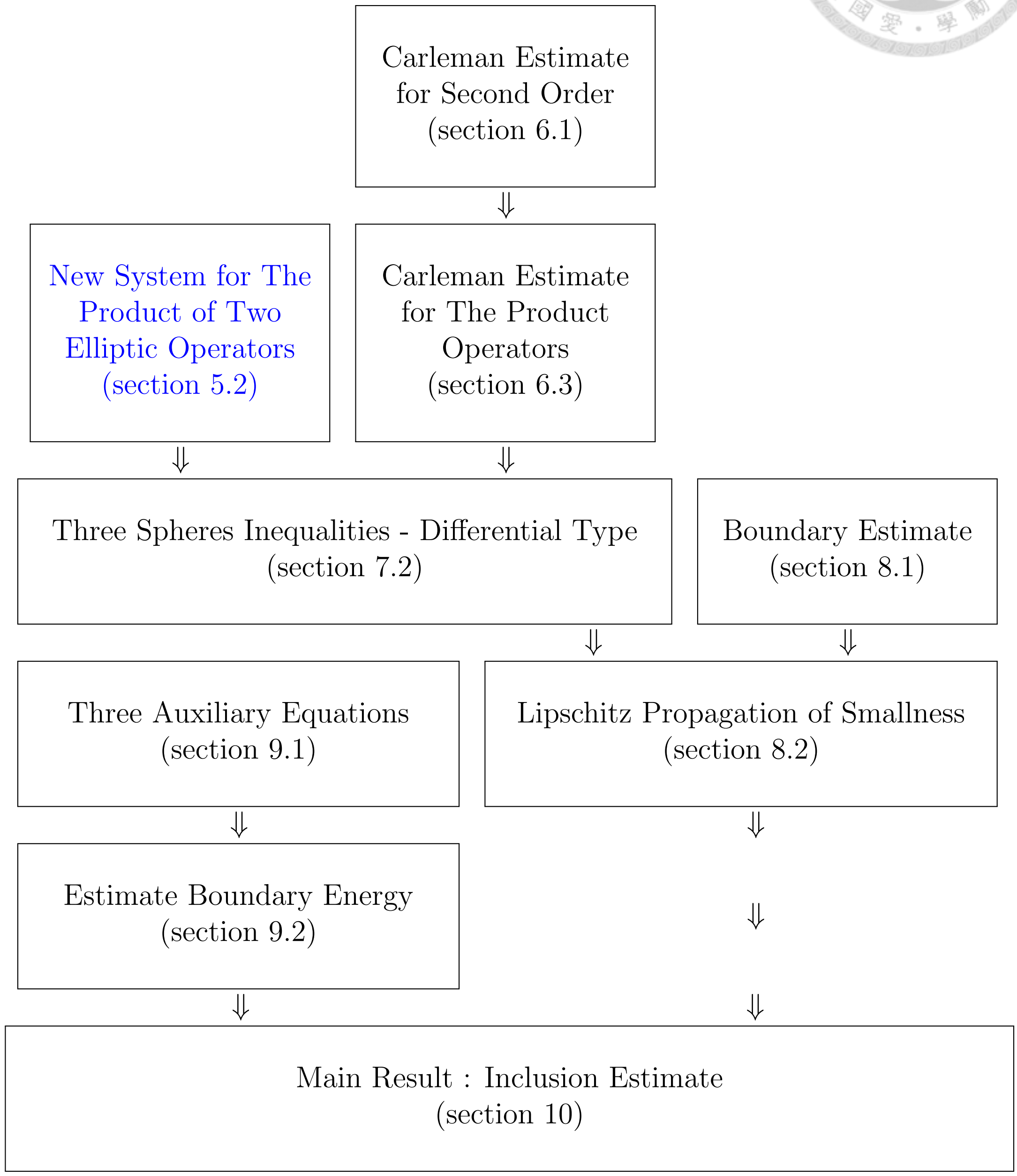

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