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關於 Köthe Conjecture 與其相關結果

A Survey on Köthe Conjecture and related topics

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中文摘要

文章分成三個部分，第一部分簡單介紹一下 Köthe Conjecture，並簡述其與 Jacobson radical 的關係。
第二部分參考 Krempa 的文章，給出一些等價敘述。
最後的部分有三個 topics，第一個我們指出:在我們的 ring 滿足特殊的條件下，Koethe Conjecture 成立。
第二個，我們介紹 E.R. Puczylowski 與 Agata Smoktunowicz 的結果:當 R 是一個 nil ring 時，佈於其上的多項式環 R[x]是所謂的 Brown-McCoy radical ring，即，R[x]無法同態映滿一個非零的 ring with identity。
最後，我們介紹 Agata Smoktunowicz 的一個例子:存在一個 nil ring R，佈於其上的多項式環 R[x]不是 nil ring。

而在本文章中，ring(環)一詞總是代表結合環，但不一定有乘法單位元素。

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Abstract

There are three parts of this paper.
The first part we give an introduction of Köthe Conjecture, and list some properties of Jacobson radical.
The second part we go through some results of Krempa, to see some equivalent statements of our conjecture.
Finally, we will see some related topics of Köthe Conjecture.

In this paper, the word “ring” always means an associative ring, but may not have an identity.

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(一)Introduction

(1.1) Köthe Conjecture

Köthe Conjecture 是在環論中相當著名的猜想之一，至今仍未解決，它在 1930 年被奧地利數學家 Gottfried Köthe 提出，其有很多等價敘述，我們採用的是以下的敘述:

Köthe Conjecture [方便起見，以下稱之為 K .C]:

In any ring, the sum of two nil left ideal is nil.

初看之下，這個問題似乎是簡單的，而且若我們將敘述做一些自然地加強，則這些結果均會成立，而且證明是相當容易的，底下指出兩個:

1. In any ring, the sum of two nil two-sided ideal is nil.

證明:

[Claim]: Let R be a ring, and I be a two sided ideal of R, then R is nil if and only if both R/I and I are nil.

"only if" part is obvious, it suffices to show the remaining part.

Suppose that both R/I and I are nil, then given any $x \in R$, there is a positive integer n such that $x^n \in I$.

Since I is also nil, there is a positive integer m such that $(x^n)^m = x^{nm} = 0$, that is, x is a nilpotent element in R. That is, R is nil.

Now return to our goal, given nil two sided ideal I_1, I_2, we have, by Isomorphism Theorem, that

$$\frac{I_1 + I_2}{I_1} \cong \frac{I_2}{I_1 \cap I_2} \dots (*)$$

The right hand side of $(*)$ is nil by the claim above, and hence $\frac{I_1 + I_2}{I_1}$ is nil, again by the claim, $I_1 + I_2$ is nil.

■

2. In any ring, the sum of two nilpotent left ideal is nilpotent, and hence, is nil.

證明:

Let A and B be two nilpotent left ideals of a ring R, then there are positive

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integers m and n such that

$$A^m = B^n = 0$$

Now consider

$$(A+B)^{m+n-1} \dots (*)$$

Expand each term of $(*)$, each term is of the form $X_1X_2\dots X_{n+m-1}$, where $X_i \in \{A,B\}$, for all $i = 1,2,\dots,n+m-1$.
在每個展開項中，至少有 m 個 A 或 n 個 B(否則長度不足 n+m-1)
Since A and B are left ideal, $X_1X_2\dots X_{n+m-1} \subseteq A^m$, or A^mR, or B^n, or B^nR, each of them are zero. So we get that $(A+B)^{m+n-1} = 0$.

■

而(K.C.)似乎只是將吸收律去掉一邊，從這個觀點看來，問題似乎並不困難，然而，我們將在第二與第三部份看到，他的等價敘述"剛好卡在中間"，而從這個角度可以看出，這個問題其實遠非容易。

(1.2) Jacobson radical
由於第二部分的等價敘述會用到 Jacobson radical 來刻劃，我們在此簡述一下 Jacobson radical 的基本性質，但略去證明，底下參考[1]第一章的內容:

Definition: Let R be a ring, the Jacobson radical of R, written as J(R), is defined by

$$J(R):=\cap \mathrm{Ann}(M)$$

, where the intersection runs over all irreducible left R-module M. If R has no irreducible left R-module, then we put J(R)=R.
(Note: Ann(M):={a∈R: am=0, for all m∈M.})

Definition: Let R be a ring, a left(right) ideal λ of R is said to be modular if there is an a∈R such that x-xa∈λ(x-ax∈λ) for all x∈R.

底下我們介紹一些可能會用到的性質:
首先，Jacobson radical 無分左右，即:
Proposition1.2.1: J(R):=∩Ann(M)
, where the intersection runs over all irreducible right R-module M.

再來，Jacobson radical 可以用 maximal ideal 來刻劃，而且無分左右，即:
Proposition1.2.2:

1. J(R)= ∩λ, where the intersection runs over all maximal left ideals of R which are modular.
2. J(R)= ∩ρ, where the intersection runs over all maximal right ideals of R

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which are modular.

有時候用元素來刻劃 Jacobson radical 是有益的，以下是一個定義:

Definition:

1. An element a∈R is said to be left-quasi-regular (l.q.r) if there is an b∈R such that a+b+ba=0.
2. An element a∈R is said to be right-quasi-regular (r.q.r) if there is an b∈R such that a+b+ab=0.
3. An element a∈R is said to be quasi-regular (q.r.) if it is both left and right quasi-regular.
4. A left ideal of R is left-quasi-regular if each of its element is.
5. A right ideal of R is right-quasi-regular if each of its element is.

而 Jacobson radical 有以下的性質，此性質亦無分左右:

Proposition1.2.3:

1. J(R) is a left-quasi-regular left ideal of R and contains all the left-quasi-regular left ideals of R.
2. J(R) is a left-quasi-regular right ideal of R and contains all the right-quasi-regular right ideals of R.

底下做一個簡單的觀察:

If a∈R is nilpotent, then $a^m = 0, for \ \ some \ \ m \in \mathbb{N}$. Put $\mathrm{b} = -\mathrm{a} + a^2 - a^3 + \cdots + (-1)^{m-1}a^{m-1}$. Then by a simple calculation, we have a+b+ab=0=a+b+ba.

從這與 Proposition1.2.3 我們可以得到一個推論:

Corollary1.2.1:

If I is a nil left(right) ideal of R, then I⊆J(R).

最後我們列舉兩個性質:

Proposition1.2.4:

If A is an ideal of R, then J(A)=J(R)∩A.

Proposition1.2.5:

$\mathrm{J}\big(\mathrm{M_n(R)}\big) = \mathrm{M_n}\big(\mathrm{J(R)}\big)$.

(1.3) Köthe Conjecture and Jacobson radical.

我們可以從兩個面向來看 Köthe Conjecture，第一個由(1.1)的討論可以看出，由於兩個 nilpotent left ideal 加起來還是 nilpotent，(K.C.)可以視為邏輯上的推

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廣。

第二個可以從 Jacobson radical 的性質看出，由 Corollary1.2.1，我們知道若 I 是一個 nil left ideal, 則 I⊆J(R)。

現在假設有兩個 nil left ideal A，B，則由 Corollary1.2.1，我們知道 A⊆J(R)與 B⊆J(R)，所以 A+B⊆J(R)，這表明 Jacobson radical 對 nil left ideal 似乎有某種程度上的"封閉性"，從這個面向來看，討論 A+B 本身的"封閉性"也是相當自然的。

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(二)Main Theorem

這部份主要參考[2]，證明一些關於 Köthe Conjecture 的等價敘述。

(2.1)

首先我們先岔開主題，觀察一個自然的問題:

(K.C.)中我們問:對任意一個 ring R 中的兩個 nil left ideal 的和是否為 nil，注意到這個問題我們並沒有對 R 作限制，一個自然的想法是這個問題應該不分左右，即以下敘述應當等價:

In any ring, the sum of two nil left ideals is nil.

In any ring, the sum of two nil right ideals is nil.

我們先證明這個簡單的事實，再回過頭看我們的主要定理

首先幾個簡單的引理:

Lemma2.1.1

Let R be a ring, then R has no nonzero nil left ideal if and only if R has no nonzero nil right ideal.

證明: Suppose that R has a nil left ideal λ, pick x≠0 in λ, and consider the right ideal xR+ℤx, xR+ℤx≠{0} since x∈ xR+ℤx

Let mx+xr∈ xR+ℤx ,the element mx+rx∈λ, so ∃n∈ℕ such that $(\mathrm{mx}+\mathrm{rx})^{n}=0$, then $(\mathrm{mx}+\mathrm{xr})^{n+1}=x(\mathrm{mx}+\mathrm{rx})^{n}(m+r)=0$. This implies that xR+ℤx is a nonzero nil right ideal of R.

The other part is similar.

■

Lemma2.1.2

Let R be a ring, the following statements are equivalent:

(1) Every nil left ideal of R is contained in a nil two-sided ideal of R.

(2) Sum of two nil left ideal of R is nil.

證明:

(1)⟹(2) is trivial.

(2)⟹(1) Consider

$$I=\sum \lambda$$

,where the sum is taken all over the nil left ideal of R.

By assumption, I is a nil left ideal, we claim that I is a two-sided ideal of R.

Let r∈R ,then Ir is a nil left ideal of R and hence Ir⊆I, for all r∈R, i.e. I is a two-

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sided nil ideal of R, and λ⊆I, for all nil left ideal λ of R.

■

Lemma2.1.3

Let R be a ring, λ be a nil left (resp. right) ideal of R, and I be a nil two-sided left ideal of R, then λ+I is a nil left (resp. right) ideal of R.

證明:

Consider the ring $\overline{R}=\frac{R}{I}$, and let $x+r\in\lambda+I$, $x\in\lambda$, $r\in I$.

Then $\overline{x+r}=\overline{x}$ in $\overline{R}$, since λ is nil, so $\exists n\in\mathbb{N}$ such that $\overline{x+r}^n=\overline{x}^n=\overline{x^n}=0$ in $\overline{R}$. i.e. $(x+r)^n\in I$, but since I is nil, so $\exists m\in\mathbb{N}$ such that $[(x+r)^n]^m=0$, i.e. $x+r$ is a nilpotent. i.e. $\lambda+I$ is nil.

■

Proposition2.1.1

The following statements are equivalent:

(1) In any ring, every nil left ideal is contained in a nil two sided ideal.

(2) If R is a ring without nonzero nil two-sided ideal, then R has no nil left ideal.

證明:

(1)⟹(2) is trivial.

(2)⟹(1)

Let R be a ring, and let λ be a nil left ideal of R, consider

$$I=\sum J$$

,where the sum is taken all over the nil two-sided ideal of R.

Then λ+I is also a nil left ideal of R by Lemma2.1.3, hence

$\overline{\lambda}=\frac{\lambda+I}{I}$ is a nil left ideal of the ring $\overline{R}=\frac{R}{I}$.

Note that $\overline{R}$ has no nonzero nil two − sided ideal and hence $\overline{\lambda}=0$ by assumption.

i.e. λ+I⊆I, so λ⊆I.

■

注意到上述論證均為左右對稱，故結合上述結果，我們有

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Proposition2.1.2:
The following statements are equivalent:
(1) In any ring, the sum of two nil left ideals is nil.
(2) In any ring, the sum of two nil right ideals is nil.
證明:
If (1) is true, then by Lemma2.1.2, in any ring, every nil left ideal is contained in a nil two sided ideal. So by Proposition2.1.1, if R is a ring without nonzero nil two-sided ideal, then R has no nil left ideal, so by Lemma2.1.1, if R is a ring without nonzero nil two-sided ideal, then R has no nil right ideal, then by Proposition2.1.1, in any ring, every nil right ideal is contained in a nil two sided ideal. Hence by Lemma 2.1.2, in any ring, the sum of two nil right ideals is nil.

■

這便表明了(K.C)的敘述是左右對稱的。

(2.2)
接下來我們進入主要的內容，我們將證明下面的敘述彼此等價，即:
Theorem2.2.1:
The following statements are equivalent:
(1) If R is a nil ring, then the polynomial ring R[x] is a radical ring. (i.e. J(R[x])=R[x]).
(2) If a ring R contains a one-sided nil ideal A, then A is contained in a two-sided nil ideal of R.
(3) If R is a nil ring, then $R_2 = M_2(R)$ is also a nil ring.

我們先證明一個 Lemma，實際上即為 Theorem2.2.1 的(1)⟺(3)

Lemma2.2.1
Let R be a ring, then the polynomial ring R[x] is radical if and only if the matrix ring R_n is nil, for all n∈ℕ.

這個證明需要用到 Amitsur 的結果，這個結論的證明並非容易，但離我們的目標稍遠，我們在此敘述，證明放在 Appendix 中

Lemma2.2.2:
Let N=J(R[x])∩R ,then J(R[x])≠0 implies N≠0.

Lemma2.2.3:
J(R[x])=N[x] ,where N=J(R[x])∩R.

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Lemma2.2.4:
N is a nil ideal in R.

Proposition2.2.1
If R[x] is Jacobson radical, then R is nil.

Proof of Lemma2.2.1:
"Only if"
Suppose that R[x] is radical, then by Proposition1.2.5, $J(R[x]_n)=J(R[x])_n=R[x]_n$, for all $n\in\mathbb{N}$.
But we have that $R[x]_n=R_n[x]$, so we have that $R_n[x]$ is radical, by Proposition2.2.1, R_n is nil, for all $n\in\mathbb{N}$.
"If"
Suppose that R_n is nil, for all $n\in\mathbb{N}$, our goal is to show that all elements in xR[x] is right quasi regular.
If this is true, then $xR[x]\subseteq J(R[x])$, and for any $a\in R$, $ax\in J(R[x])$, by Lemma2.2.3, $ax\in N[x]$, hence $a\in N=J(R[x])\cap R$, i.e. $a\in J(R[x])$, for all $a\in R$.
Hence $R[x]=R+xR[x]\subseteq J(R[x])\Longrightarrow J(R[x])=R[x]$.
The main idea as follows:
Let $p(x)=a_1x+a_2x^2+\cdots+a_mx^m$, it may be difficult to find $q(x)\in R[x]$ directly, but if we allow $q(x)\in R[[x]]$, then the coefficient of q(x) can be defined inductively, say $q(x)= b_1x+b_2x^2+\cdots+b_nx^n+\cdots$ our goal is to show that for k is sufficient large, $b_l=0$, for all $l\geq k$. And hence $q(x)\in R[x]$

Suppose that there is $q(x)\in R[[x]]$ such that $q(x)=p(x)+p(x)q(x)$
Let $q(x)= b_1x+b_2x^2+\cdots+b_nx^n+\cdots$,then
$p(x)q(x)=a_1b_1x^2+(a_1b_2+a_2b_1)x^3+\cdots+(a_mb_1+\cdots a_1b_m)x^m+\cdots+(a_mb_{n-m-1}+\cdots a_1b_{n-1})x^n+\cdots$
$\Longrightarrow (b_1-a_1)x+\cdots+(b_m-a_m)x^m+(b_{m+1})x^{m+1}+\cdots=p(x)q(x)\ \ldots(1)$

Compare each coefficient of (1), we can define b_n inductively as follows:
$b_1=a_1$
$b_2=a_2+a_1b_1\ldots$

$$b_n=\begin{cases} a_n+\sum_{j=1}^{n-1}a_{n-j}b_j\,,\text{if } n\leq m \\ \sum_{j=1}^{m}a_{m-j+1}b_{n-m+j-1}\ ,\text{if } n>m \end{cases}\quad\ldots(2)$$

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It is easy to check that q(x)=p(x)+p(x)q(x).

Our next goal is to find a m×m matrix (c_{ij}) such that for all k≥m, we have

$$b_{k+i} = \sum_{j=1}^{m} c_{ij} b_{k-m+j}\,,\text{for } i = 1, 2, \ldots, m.$$

We define c_{ij} inductively, suppose that there is (c_{ij}) such that

$$b_{k+i} = \sum_{j=1}^{m} c_{ij} b_{k-m+j}\,,\text{for } i = 1, 2, \ldots, m, \text{for all } k \geq m.$$

Then $b_{k+1} = \sum_{j=1}^{m} c_{1j} b_{k-m+j}$, from (2), we may choose $c_{1j} = a_{m-j+1}$,

for $j = 1, 2, \ldots m$.

Next, $b_{k+2} = \sum_{j=1}^{m} c_{2j} b_{k-m+j} = c_{21} b_{k+1-m} + c_{22} b_{k+2-m} + \cdots + c_{2m} b_k$.

On the other hand, $b_{k+2} = \sum_{j=1}^{m} a_{m-j+1} b_{k+2-m+j-1}$

$= a_m b_{k+2-m} + a_{m-1} b_{k+3-m} + \cdots + a_1 b_{k+1}$

Since $b_{k+1} = \sum_{j=1}^{m} c_{1j} b_{k-m+j}$, so

$$b_{k+2} = a_m b_{k+2-m} + a_{m-1} b_{k+3-m} + \cdots + a_1 \sum_{j=1}^{m} c_{1j} b_{k-m+j}$$

$$= a_m b_{k+2-m} + a_{m-1} b_{k+3-m} + \cdots + a_1 \sum_{j=1}^{m} c_{1j} b_{k-m+j}$$

$= a_1 c_{11} b_{k-m+1} + (a_m + a_1 c_{12}) b_{k+2-m} + \cdots + (a_2 + a_1 c_{1m}) b_k$.

Put $c_{21} = a_1 c_{11}, c_{22} = a_m + a_1 c_{12}, \ldots, c_{2m} = a_2 + a_1 c_{1m}$,

$$b_{k+3} = \sum_{j=1}^{m} a_{m-j+1} b_{k+2-m+j} = a_m b_{k+3-m} + a_{m-1} b_{k+4-m} + \cdots + a_1 b_{k+2}$$

$$= a_m b_{k+3-m} + a_{m-1} b_{k+4-m} + \cdots + a_2 \sum_{j=1}^{m} c_{1j} b_{k-m+j} + a_1 \sum_{j=1}^{m} c_{2j} b_{k-m+j}$$

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$= (a_2c_{11} + a_1c_{21})b_{k+1-m} + (a_2c_{12} + a_1c_{22})b_{k+2-m}$
$+(a_m + a_2c_{13} + a_1c_{23})b_{k+3-m} + \cdots + (a_3 + a_2c_{1m} + a_1c_{2m})b_k$
Put $c_{31} = a_2c_{11} + a_1c_{12}, c_{32} = a_2c_{12} + a_1c_{22}, c_{33} = a_m + a_2c_{13} + a_1c_{23}, \dots$
$c_{3m} = a_3 + a_2c_{1m} + a_1c_{2m}$.
More generally, suppose we have defined c_{ij}, for 1≤i<l≤m, j=1, 2, …, m.

$$\text{Define } c_{lj} = \begin{cases} \displaystyle\sum_{i=1}^{l-1} a_{l-i}c_{ij}, \text{if } 1 \le j < l. \\ a_{m+l-j} + \displaystyle\sum_{i=1}^{l-1} a_{l-i}c_{ij}, \text{if } l \le j \le m. \end{cases} \quad \dots (3)$$

Then by the argument above, we have

$$b_{k+i} = \sum_{j=1}^{m} c_{ij}b_{k-m+j}, \text{for } i = 1, 2, \dots, m, \text{for all } k \ge m. \dots (4)$$

Next, consider the matrix $D=(d_{ij}) \in R_{m+1}$ defined by $d_{ij} = c_{ij}$,$1 \le i$,$j \le m$ $d_{i(m+1)} = b_i, i = 1, 2, \dots, m$. And $d_{(m+1)i} = 0, i = 1, 2, \dots, m+1$.

We claim that for any $k \in \mathbb{N}, (D^k)_{i(m+1)} = b_{(k-1)m+i}$, for $i = 1, 2, \dots, m$,

and $(D^k)_{(m+1)(m+1)} = 0$.

For $k = 1$, the statement holds obviously, assume that

$(D^k)_{i(m+1)} = b_{(k-1)m+i}$, for $i = 1, 2, \dots, m, (D^k)_{(m+1)(m+1)} = 0$,

for some $k \in \mathbb{N}$, then

$$D^{k+1} = DD^k, \text{so } (D^{k+1})_{i(m+1)} = \sum_{l=1}^{m+1} d_{il}(D^k)_{l(m+1)} = \sum_{l=1}^{m+1} c_{il}b_{(k-1)m+l}$$

$$= \sum_{l=1}^{m} c_{il}b_{(k-1)m+l}, \text{hence } (D^{k+1})_{i(m+1)} = \sum_{l=1}^{m} c_{il}b_{km-m+l}, \text{by (4)}, (D^{k+1})_{i(m+1)}$$

$$= b_{km+i}, i = 1, 2, \dots, m. \text{ And } (D^{k+1})_{(m+1)(m+1)} = \sum_{l=1}^{m+1} c_{(m+1)l}b_{(k-1)m+l} = 0.$$

Hence $(D^k)_{i(m+1)} = b_{(k-1)m+i}$, for $i = 1, 2, \dots, m, (D^k)_{(m+1)(m+1)} =$

0 , for all $k \in \mathbb{N}$.
By our assumption, R_{m+1} is nil, so there is $k \in \mathbb{N}$ such that $D^k = 0$

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$\Rightarrow D^l = 0, \text{for all } l \geq k.$

This implies that $b_{(l-1)m+1} = b_{(l-1)m+2} = \cdots = b_{(l-1)m+m} = 0, \text{for all } l \geq k,$

hence $b_p = 0, \text{for all } p \geq (l-1)m + 1.$ i.e. $q(x) \in R[x].$

$p(x) - q(x) + p(x)q(x) = 0.$

So q(x)+(-p(x))+(-p(x))q(x)=0

So $-p(x)$ is quasi regular, for all $p(x) \in xR[x],$

So $p(x)$ is quasi regular, for all $p(x) \in xR[x],$

So $xR[x] \subseteq J(R[x]).$ Hence $J(R[x]) \subseteq R[x] = R + xR[x] \subseteq J(R[x]),$

i.e. $J(R[x]) = R[x].$

■

Now we are ready to prove our main theorem.

Proof of Theorem2.2.1:

(1)⟹(2)

Let A be an nil left ideal of R, then A is itself a nil ring, so by our assumption, J(A[x])=A[x], i.e. every element in A[x] is left quasi regular, and clearly A[x] is a nil left ideal of R[x], so A[x]⊆J(R[x]), hence J(R[x])≠0, so by Lemma2.2.2 N=J(R[x])∩R≠0, by Lemma 2.2.3, J(R[x])=N[x], Lemma 2.2.4 tell us that N is nil two-sided ideal.

Now A[x]⊆J(R[x])=N[x], hence A⊆N, note that this argument is also true for right ideals, hence (2) holds.

(2)⟹(3)

Let R be a nil ring, and let A'=$\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\right\} = R_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$

A''=$\left\{\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in R\right\} = R_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, C'=$\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in R\}$, C''=$\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in R\}$, it is not hard to check that C' is an ideal of A', and C'' is an ideal of A''. Furthermore, A'/C' and A''/C'' are both isomorphic to R, and hence is nil. But $C'^2 = 0,$ and $C''$is clearly nil. So C' and C'' are nil ideal of A' and A'', respectively, this implies that A' and A'' are nil left ideals of R_2. By our assumption, A' and A'' are contained in a nil two-sided ideal of R_2, say A'⊆B', A'' ⊆B'', where B' and B'' are nil two-sided ideal of R_2.

$R_2 = A' + A'' \subseteq B' + B'' \subseteq R_2,$ hence $R_2 = B' + B'',$ which is nil.

(3)⟹(1)

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By Lemma2.2.1, we only need to show that R_n is nil, for all $n \in \mathbb{N}$.
Since $R_{2^{n+1}} \cong (R_{2^n})_2$, for all $n = 0, 1, 2, \ldots$
So by induction, R_{2^n} is nil, for all $n \geq 0$, but for arbitrarily $n \in \mathbb{N}$, pick $m \in \mathbb{N}$ so that $2^m > n$,

then R_n is isomorphic to (written as block form) $\begin{pmatrix} R_n & 0 \\ 0 & 0 \end{pmatrix}_{2^m \times 2^m} \subseteq R_{2^m}$,

and R_{2^m} is nil, hence R_n is nil, for all $n \in \mathbb{N}$, and (1) follows from Lemma2.2.1.

■

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(三) Related topics.

(3.1) Some special class of rings.

我們一開始即指出，(K.C)在最一般的情形下仍未解決，一個自然的問題是:(K.C)是否會在某些特殊的環上成立，倘若這個 class 足夠大，或者說條件是自然的，那麼這些跡象或許可以表明此猜想為正確的可能性

在這個小節，我們考慮三個特殊的 class，這三個情形均在某種程度上描述了”有限性”條件。

前兩個是較為明顯的結論，這兩個描述了類似有限維的條件，參考[1]的證明:

1. Artinian rings.

Theorem3.1.1: Köthe Conjecture holds for any left(right) artinian rings.

利用下面這個定理即可立刻推出 Theorem3.1.1 的證明:

Theorem3.1.2: Let R be a left(right) artinian ring, then J(R) is nilpotent.

Proof:

Assume that R is right artinian.

Put J=J(R), consider the descending chain of ideals: $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$

Since R is right artinian, so there is an integer n such that $J^n = J^{n+1} = \cdots$

Our goal is to show that $J^n = 0$, assume that $J^n \neq 0$, consider $W=\{x \in J : xJ^n = 0\}$, then W is an ideal of R.

If $W \supseteq J^n$, then $0 = WJ^n \supseteq J^nJ^n = J^{2n} = J^n$, which is a contradiction. So $W \nsupseteq J^n$.

Consider $\overline{R} = {}^{R}\!/_{W}$, then by our assumption, $\overline{J^n} = {}^{J^n}\!/_{W} \neq 0$, since $\overline{R}$ is also right artinian, so there is a minimal right ideal $\overline{\rho} \subseteq \overline{J^n}$.

Claim: $\overline{\rho}$ is irreducible, if not, then $\overline{\rho}\overline{R} = 0$, so $\overline{\rho}\ \overline{J^n} = 0$, that is, $\rho J^n \subseteq W$, i.e. $\rho J^n J^n = 0$, but by $J^{2n} = J^n$, we have $.\rho J^n = 0$, hence $\rho \subseteq W$, a contradiction!

So $\overline{\rho}$ is irreducible, hence $\overline{\rho} J(\overline{R}) = 0$, but note that $\overline{J^n} \subseteq J(\overline{R})$ ($\overline{J^n}$ is a quasi regular ideal of $\overline{R}$).

So we have $\overline{\rho}\ \overline{J^n}$, which is impossible! Hence $J^n = 0$, the similar argument holds for R is left artinian.

■

Now we prove Theorem3.1.1

Proof of Theorem3.1.1:

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Let R be a left(right) artinian ring.
By Corollary1.2.1, any nil left(right) ideal of R is contained in J(R), by Theorem3.1.2, J(R) is nilpotent, so any nil left(right) ideal of R is nilpotent, hence by (1.1) 2, sum of any two nil ideals of R is nil.

■

2. Noetherian rings.

Theorem3.1.3: Köthe Conjecture holds for any left(right) noetherian rings.

這個證明可由下面的定理推出

Theorem3.1.4: If R is a left(right) noetherian ring, then any nil one-sided ideal is nilpotent.

這裡我們先證明一個 Lemma:
Lemma3.1.1: If R is a ring without nonzero nilpotent ideal, then R has no nonzero nilpotent one-sided ideal.
Proof: Suppose ρ is a nonzero nilpotent right ideal of R, if Rρ=0, then ρ is itself a two-sided ideal which is nilpotent.
If Rρ≠0, then Rρ is a nonzero two-sided ideal of R, and $(R\rho)^n = (R\rho)(R\rho)\dots(R\rho) \subseteq R(\rho R)(\rho R)\dots = 0$, if n is large enough since ρ is nilpotent.
A similar argument shows that R cannot have a nonzero nilpotent left ideal.

■

Proof of Theorem3.1.4:
Claim: If R is a left(right) Noetherian ring without any nilpotent ideal, then R has no nil one-sided ideal.
Assume that R is right Noetherian and A is a nonzero nil one-sided ideal of R.
Pick an 0≠a∈A so that Ra≠0(This can be done, for if Ra=0, for all a∈A, in particular, AA=0, then A is a nilpotent one-sided ideal,so by Lemma3.1.1, R has a nilpotent two-sided ideal, which contradicts with our assumption.) and consider U=Ra, U is a left ideal of R.
Claim: U is nil.
If A is a left ideal, then U⊆A and hence A is nil, if A is a left ideal, let ra∈U, then ar∈U, hence there is n∈ℕ such that $(ar)^n = 0$, hence, $(ra)^{n+1} = r(ar)^n a = 0$.
So U is nil.
Now for u∈U, let r(u)={x∈R:ux=0}, then $\{r(u)\}_{u\in U-\{0\}}$is a collection of right ideals of R. Since R is right Noetherian, there is $u_0 \in U - \{0\}$ such that

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$r(u_0)$ is maximal.
Now for $x \in R$, $xu \in U$ and $r(xu) \supseteq r(u)$, so if $x \in R$ with $xu_0 \neq 0$, then $r(xu_0) = r(u_0)$ by maximality of $r(u_0)$.
Now for $y \in R$ with $yu_0 \neq 0$, since $yu_0 \in U$ and U is nil, so there is $k \geq 2$ such that $(yu_0)^k = 0$, but $(yu_0)^{k-1} \neq 0$
Since $(yu_0)^{k-1} \in U$, and $(yu_0)^{k-1} = au_0$, for some $a \in R$, so $r\left((yu_0)^{k-1}\right) = r(u_0)$. And since $(yu_0)^k = 0$, $yu_0 \in r\left((yu_0)^{k-1}\right) = r(u_0)$, hence $u_0yu_0 = 0$.
For $yu_0 = 0$, it is obvious that $u_0yu_0 = 0$. So we get $u_0yu_0 = 0$, for all $y \in R$.
Finally, for $xu_0, yu_0 \in U$, $(xu_0)(yu_0) = 0$,
so U is a nonzero nilpotent left ideal of R, by Lemma3.1.1, this cannot happen! So the claim holds.
For the general case, assume that R is a right Noetherian ring, then R has a maximal nilpotent ideal N, if ρ is a nil one-sided ideal of R such that $\rho \not\subseteq N$, then

$$\bar{\rho} = \frac{\rho + N}{N} \text{ is a nonzero nil one-sided ideal of } \bar{R} = \frac{R}{N},$$

but $\bar{R}$ is a right noetherian ring which has no nonzero nilpotent ideal. So by Lemma 3.1.1, this cannot happen, hence $\rho \subseteq N$.
That is, ρ is nilpotent.

■

Proof of Theorem3.1.3:
This result follows from Theorem3.1.4 and (1.1) 2.

■

3. Polynomial identity rings.(PI-rings)

這個情形相對特殊一些，但一方面來說，這還是滿足某種條件下的有限性，在這小節最後我們指出一個特殊例子來說明這個 class 夠大(至少，包含矩陣環)。
We follow the proof given in [4]

我們先介紹一些定義:
Definition3.1.1: A ring R is said to be satisfies a polynomial identity if there is $n \in \mathbb{N}$, and a nonzero $f(x_1, \dots, x_n) \in \mathbb{Z}\langle x_1, \dots, x_n\rangle$(Free $\mathbb{Z}$ − algebra) such that $f(r_1, \dots, r_n) = 0$, for all $r_i \in R$.
Here, we say that f is a polynomial identity of R, the degree of f is defined as natural way, f is monic if at least one of the highest degree term has coefficient 1.

我們由上面的定義看出，R 滿足一個有限條件的等式，但以下我們給一個例子:
Example3.1.1:
$R = \mathbb{Z}_2[x]$,這個環顯然滿足 $f = 2x_1 \in \mathbb{Z}\langle x_1\rangle$，但這個作用顯然是無趣的。

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由上面的例子，為了規避上述情形的發生，我們對 PI-ring 有一個更加嚴格的定義。

Definition3.1.2: A polynomial identity ring(PI-ring) is a ring satisfies some monic polynomial identity.

Remark:在上述定義中，我們並沒有假設其常數項為 0，實際上，若常數項不為 0，例如 $f(x_1,\dots,x_n)\in\mathbb{Z}\langle x_1,\dots,x_n\rangle$，其常數項不為 0，如果這個環 R 沒有乘法單位元，那麼$f(0,\dots,0)$這個元素是沒有定義的，而即便有乘法單位元素，由於 $f(0,\dots,0)=m=0$ in R，代表這個環滿足 mx=0, for all x∈R，於是我們顯然可以把$f(x_1,\dots,x_n)$用 $f(x_1,\dots,x_n)-m$取代，而這個多項式常數項為 0。
因此，以下如果我們沒有特地指出，均設$f(x_1,\dots,x_n)$的常數項為 0。

下面這個定理指出，如果環滿足一個 polynomial identity，那麼我們可以針對這個多項式作調整使其整齊一點，首先我們有以下的定義:

Definition3.1.3: A multilinear polynomial of degree n is a nonzero element $f\in\mathbb{Z}\langle x_1,\dots,x_n\rangle$ taking the form:

$$\sum_{\sigma\in S_n} a_\sigma x_{\sigma(1)}\dots x_{\sigma(n)}$$

with each $a_\sigma\in\mathbb{Z}$.

所謂的比較整齊即下面的性質:

Proposition3.1.1: If R satisfies an identity f of degree d then R also satisfies a multilinear identity g of degree d. Furthermore, each coefficient in g is also a coefficient in f; and if f is monic, so too is g.

Proof: Each term of f can be partitioned into those in which x_1 occurs and the others. Say

$$f(x_1,\dots,x_n)=f_1(x_1,\dots,x_n)+f_2(x_2,\dots,x_n)$$

Set $x_1=0$, we get f_2, and so, f_1 are identities of R.
Since f is monic, one of f_1, f_2 are monic. So we pick the monic one.
Continue this process, we can assume that each x_i occurs in every term of the expansion of f. And this new identity has degree at most deg(f).

If f is not multilinear, then the term of highest degree cannot be multilinear, so at least one x_i, say x_1, appears in this term twice.
Consider the polynomial:

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$$g(x_1,\dots,x_{n+1}) = f(x_1 + x_{n+1},\dots,x_n) - f(x_1,\dots,x_n) - f(x_{n+1},\dots,x_n)$$

A simple calculation shows that each term of g all come from those terms w in f which have degree at least 2 in x_1; they are the terms obtained by replacing some, but not all, of the $x_1{}'s$ w by x_{n+1}. They retain the same coefficient as w; and if f is monic, then g is monic.

Continue this process, we may reduce f to be multilinear.

■

Next we note a trivial fact:

Lemma3.1.2: Let $g(x_1,\dots,x_m)$ be a monic multilinear identity whose monic term is $x_1 \dots x_m$. Then

$$g= g_1(x_1,\dots,x_{m-1})x_m + g_2(x_1,\dots,x_m),$$

where g_1 is monic multilinear and g_2 is multilinear with no term ending in x_m.

Now we can prove:

Theorem3.1.5:

A nonzero nil PI ring has a nonzero nilpotent ideal.

Proof:

Called a PI ring R has minimal degree m if m is the least possible degree of a monic polynomial identity of R. As above, we can assume that this identity is monic.

Our proof based on induction on the minimal degree m of R.

First, by Lemma3.1.1, it is sufficient to show that R has a nonzero one-sided nilpotent ideal.

Let R be a nonzero nil PI ring with minimal degree m, then m>1.

If m=2, then this identity must of the form $x_1x_2 - nx_2x_1$, for some $n \in \mathbb{Z}$.

Choose an nonzero b∈R such that $b^2 = 0$

Then bRb=nb^2R=0. So if B=ℤb+bR, then $B^2 = 0$.

Next, suppose the result holds for all smaller degree, choose an nonzero b∈R such that $b^2 = 0$.

If bR=0, then B=bℤ is a nilpotent right ideal of R.

If bR≠0 and R satisfies $g(x_1,\dots,x_m)$, write

$$g= g_1(x_1,\dots,x_{m-1})x_m + g_2(x_1,\dots,x_m)$$

as in Lemma3.1.2.

Note that for all $r_1,\dots,r_{m-1} \in R, g_2(br_1,\dots,br_{m-1},b) = 0$

So $g_1(br_1,\dots,br_{m-1})b = 0$.

Let W={r∈bR:rbR=0}. Then W⊴bR; and g_1 is monic, multilinear of degree less than m satisfied by bR/W.

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If W=bR, then bR is a nilpotent right ideal of R, otherwise, bR/W contains a nonzero nilpotent right ideal.
So there is a right ideal I of bR, I⊇W, I≠W, with $I^2 \subseteq W$.
Now IbR is a nonzero right ideal of bR(Since I≠W). And

$$(IbR)^2 \subseteq I^2bR \subseteq WbR = 0.$$

■

Corollary3.1.1: If R is a nonzero PI ring has a nonzero nil right ideal, then R has a nilpotent ideal.
Proof: It is sufficient to show that R has a nilpotent one-sided ideal.
Let ρ be a nonzero nil right ideal of R, then ∃0≠I⊴ρ which is nilpotent.
If IR=0, then I is itself a right ideal of R which is nilpotent.
If IR≠0, then there is x in I such that xR≠0, let W={r∈ xR:rxR=0}, then W is an ideal of xR.
If W=xR, then xRxR=0, and hence R has a nonzero nilpotent right ideal.
Otherwise, xR/W contains a nonzero nilpotent right ideal, so there is a right ideal J of xR, I⊇W, I≠W, with $I^2 \subseteq W$.
Now IxR is a nonzero right ideal of R(Since I≠W). And

$$(IxR)^2 \subseteq I^2xR \subseteq WbR = 0.$$

■

Corollary3.1.2: If R is a PI ring, then every nil one-sided ideal is contained in a nil two-sided ideal
Proof:
Let R be a PI ring and N be its nilradical, then R/N has no nonzero nilpotent ideal, so R/N has no nonzero one-sided ideal by Corollary3.1.1.
Now, if ρ is a nil one-sided ideal of R, then $\overline{\rho}$ is a nil one-sided ideal of R/N, so $\overline{\rho}$=0, hence ρ⊆N.

■

By Corollary3.1.2 and (1.1)1. Köthe Conjecture holds for any PI rings.

Application:
(3.1)剩下的部分，我們證明:PI ring 這個 class 包含了我們常見的一些環，並舉出一個小應用
一個顯然的事實是 PI ring 包含了所有的交換環(滿足$x_1x_2 - x_2x_1$)
實際上，我們要證明:

Theorem3.1.6: Let R be a commutative ring, then $M_n(R)$ is PI, for all n∈ℕ.
Proof:
It is sufficient to show that R is a commutative ring with identity, if this is true, let

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R^1 be the usual extension of R, then $M_n(R)$ is a subring of $M_n(R^1)$, and hence, is PI.
So let us assume that R has an identity, and let $e_{ij} \in M_n(R)$ defined by

$$(e_{ij})_{mk} = \begin{cases} 1, \text{if } m = i \text{ and } k = j \\ 0, \text{otherwise} \end{cases}, 1 \leq i, j \leq n.$$

Note that every element of $M_n(R)$ can be written as a linear combination of e_{ij}'s over R.
Moreover, take $f(x_1 \dots x_{n^2+1}) \in \mathbb{Z}\langle x_1, \dots, x_{n^2+1}\rangle$

$$f(x_1, \dots, x_{n^2+1}) = \sum_{\sigma \in S_{n^2+1}} (\text{sgn}\sigma) x_{\sigma(1)} \dots x_{\sigma(n^2+1)}.$$

Then $f(r_1, \dots, r_{n^2+1}) = 0$, for all $r_1, \dots, r_{n^2+1} \in \{e_{ij}: 1 \leq i, j \leq n\}$.
Since f is multilinear and very element of $M_n(R)$ can be written as a linear combination of $e_{ij}'s$ over R, so $M_n(R)$ satisfies the identity f. Hence $M_n(R)$ is PI, for all $n \in \mathbb{N}$.

■

從 Theorem3.1.6 有一個簡單的推論:可以拿它來計算$M_n(R)$的 nilradical.

Let R be a ring, denote the nilradical of R by

$$N(R) := \sum_{I \trianglelefteq R, I \text{ is nil.}} I$$

Theorem3.1.7:
Let R be a commutative ring with identity, then $N(M_n(R)) = M_n(N(R))$
Proof:
Since R has identity, so $N(M_n(R)) = M_n(I)$, for some $I \trianglelefteq R$, so we must have $I \subseteq N(R)$. So $N(M_n(R)) \subseteq M_n(N(R))$
Conversely, since N(R) is nil, $M_n(R)$ is PI.
Consider the right ideals $e_{ii}M_n(N(R))$, $i = 1, 2, \dots n$. A simple calculation shows that each $e_{ii}M_n(N(R))$ is a nil right ideal of $M_n(R)$. Since $M_n(R)$ is PI, so $M_n(N(R)) = e_{11}M_n(N(R)) + \dots + e_{nn}M_n(N(R))$ is nil, hence $M_n(N(R)) \subseteq N(M_n(R))$.
So $N(M_n(R)) = M_n(N(R))$.

■

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(3.2) A Theorem of E.R Puczylowski and Agata Smoktunowicz ([5]).

在 Theorem2.2.1 中，我們指出 Köthe Conjecture 等價於下面的敘述:
If R is a nil ring, then R[x] is Jacobson radical.

作為這個的推論，我們有:
Proposition3.2.1: Assume K.C. holds, then R[x] cannot be homomorphically mapped onto a nonzero ring with identity.
Proof:
Since R[x] is Jacobson radical, so is every homomorphic image of R[x], i.e. J(S)=S, where S is a homomorphic image of R[x].
But if a nonzero ring S has identity, then $J(S)\neq S$ since S has a maximal left ideal ρ, and M=S/ρ is an irreducible S-module such that $1\notin Ann(M)$.

■

也就是說，Proposition3.2.1 為 K.C.的必要條件，而本節我們將證明這個是正確的，即我們要證明:

Theorem3.2.1(E.R. Puczylowski and Agata Smoktunowicz):
If R is a nil ring, then R[x] cannot be homomorphically mapped onto a nonzero ring with identity.

底下我們先給幾個定義:
Definition3.2.1:
1. Let R be a ring, and let $f(x)=a_mx^m+\cdots+a_nx^n\in R[x]$, $m\leq n$, $a_m\neq 0$, and $a_n\neq 0$, call a_n the leading coefficient of f(x) and put minf(x)=m, degf(x)=n, l(f(x))=n-m+1. So minf(x)+l(f(x))=degf(x)+1.
2. We call a ring homomorphism $f:R[x]\longrightarrow P$ of R[x] onto a simple ring P proper if f(xI[x])=P, for every nonzero ideal I of R.

Lemma3.2.1:
Let $f:R[x]\longrightarrow P$ be a proper homomorphism of R[x] onto a simple ring P with identity, and let $w(x)\in xR[x]$ be such that f(w(x))=1.
Then for every non-zero ideal I of R there exists $t(x)\in xI[x]$ with f(t(x))=1, $mint(x)\geq degw(x)$ and $l(t(x))\leq deg(w(x))$.
Proof:
Given a polynomial $p(x)= a_mx^m+\cdots+a_nx^n\in xI[x]$ such that $a_m\neq$

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0, and $f\big(p(x)\big) = 1$, then consider the polynomial

$$q(x) \coloneqq w(x)a_m x^m + \cdots + a_n x^n.$$

Clearly $q(x) \in xI[x]$, and $f\big(q(x)\big) = f\big(w(x)\big)f(a_m x^m) + \cdots + f(a_n x^n)$

$= 1 \times f(a_m x^m) + \cdots + f(a_n x^n) = f(a_m x^m + \cdots + a_n x^n) = 1.$

Next we look at $\min q(x)$ and $l(q(x))$, first, since $\min w(x) \geq 1$, we have $\min q(x) \geq \min p(x)+1$.

If $\deg w(x)a_m x^m \leq n$, then $l(q(x)) \leq l(p(x))-1$.

If $\deg w(x)a_m x^m > n$, then $l(q(x)) \leq [\deg w(x)+m]-(m+1)+1=\deg w(x)$.

So we have $l(q(x)) \leq \max\{\deg w(x), l(p(x))-1\}$.

So if we have $l(q(x)) > \deg w(x)$, then replace $p(x)$ by $q(x)$ and continue this way, we can find a $t(x) \in xI[x]$ with the properties that $f(t(x))=1$, $\min t(x) \geq \deg w(x)$ and $l(t(x)) \leq \deg(w(x))$.

■

Proposition3.2.1:

Suppose that $f: R[x] \longrightarrow P$ is a proper homomorphism of $R[x]$ onto a simple ring P with identity 1, and $w(x)= a_1 x + \cdots + a_n x^n$ is a polynomial of minimal length in $xR[x]$ such that $f(w(x))=1$. Then $a_i \in Z(R)$, for all $i = 1, 2, \dots n$, here $Z(R) = \{x \in R: ax = xa, \text{for all } a \in R\}$.

Proof:

Suppose not, then there is k of maximal number such that $a_k \notin Z(R)$, i.e. $ra_k - a_k r \neq 0$, for some $r \in R$, say $c = ra_k - a_k r$.

Let $I = \mathbb{Z}c + Rc + cR + RcR \lhd R$ (i.e. I is the ideal generated by c).

Since f is proper and P is simple, we have $f(xI[x]) = P$, and for $m \geq 1$, $f(xI^m[x]) \supseteq [f(xI[x])]^m = P$, so $f(xI^m[x]) = P$.

By Lemma3.2.1, for each m, there exists $t_m(x) \in xI^m[x]$ such that $f\big(t_m(x)\big) = 1$, $\min t_m(x) \geq \deg w(x)$ and $l\big(t_m(x)\big) \leq \deg w(x)$.

We may choose $t_m(x)$ with these properties with minimal length l_m.

By our choice of $w(x)$, $l\big(w(x)\big) \leq l_m$, for all $m \geq 1$.

For any $m \geq 2$, the leading coefficient a of $t_m(x) \in I^m$. Hence there are $c_1, \dots c_l \in I^{m-1}$ and $j_1, \dots, j_l \in I$ such that $a = c_1 j_1 + \cdots + c_l j_l$, since $j_1, \dots, j_l \in I$.

, there are $m_k \in \mathbb{Z}, n_k, p_k, q_k, s_k \in R$ such that $j_k = m_k c + n_k c + c p_k + q_k c s_k$. $k = 1, 2 \dots, l$.

So $a = c_1(m_1 c + n_1 c + c p_1 + q_1 c s_1) + \cdots + c_l(m_l c + n_l c + c p_l + q_l c s_l)$, put $i_k = c_k m_k + c_k n_k + c_k q_k \in I^{m-1}$ and $r_k = 2 + p_k + s_k \in R^1$, then $a = i_1 c r_1 + \cdots + i_l c r_l$, here R^1 denote the usual extension with an identity of R.

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Next, put g(x)=rw(x)-w(x)r, the leading coefficient of g(x) is c, degg(x)≤degw(x) and l(g(x)) ≤l(w(x)).

$\deg t_m(x) \geq \min t_m(x) \geq \deg w(x)$. Now

$\bar{t}_m(x) = t_m(x) - [i_1 g(x) r_1 + \cdots + i_l g(x) r_l] x^{\deg t_m(x) - \deg g(x)} \in I^{m-1}[x]$.

$f\left(\bar{t}_m(x)\right) = f\left(t_m(x)\right) - f\left(i_1 g(x)\right) f\left(r_1 x^{\deg t_m(x) - \deg g(x)}\right) - \cdots$

$-f\left(i_l g(x)\right) f\left(r_l x^{\deg t_m(x) - \deg g(x)}\right) = 1 - 0 = 1.$

Now we compute $l\left(\bar{t}_m(x)\right)$, since $l\left(t_m(x)\right) \geq l\left(w(x)\right) \geq l\left(g(x)\right) \geq$

$l\left((i_1 g(x) r_1 + \cdots + i_l g(x) r_l) x^{\deg t_m(x) - \deg g(x)}\right)$,

and the leading coefficient of $t_m(x)$ and

$(i_1 g(x) r_1 + \cdots + i_l g(x) r_l) x^{\deg t_m(x) - \deg g(x)}$ are equal, so $l\left(\bar{t}_m(x)\right) < l\left(t_m(x)\right)$

and $\min \bar{t}_m(x) \geq \deg w(x)$ $\left(\text{since} \min g(x) \geq w(x)\right)$.

So by our choice of $t_{m-1}(x)$, $l_m > l\left(\bar{t}_m(x)\right) \geq l_{m-1}$, so $l_m > l_{m-1}$, for all $m \geq 2$, which is impossible since $l_m \leq \deg w(x)$ for all m.

■

The next lemma is the key of our proof of Theorem3.2.1:

Lemma3.2.2:

(1) If f:R[x]→P is a ring homomorphism of R[x] onto a simple ring P with identity, then (R∩kerf)[x]⊆kerf, and either A:=R/(R∩kerf)≅P or the ring homomorphism g:A[x]→P induced by f is proper.

(2) If f:R[x]→P is a ring homomorphism of R[x] onto a simple ring P with identity, then R/(R∩kerf) has an non-nilpotent element.

Proof:

(1) Note that if I is an ideal of R such that f(I)=0, then f(I[x])=0, if not, then f(I[x])=P, but 0=f(I)f(R[x])=f((IR)[x])=f(I[x])f(R[x])=P, which is a contradiction. In particular, (R∩kerf)[x]⊆kerf.

Now, put A=R/ R∩kerf and define g:A[x]→P by $g(\bar{a}_0 + \cdots + \bar{a}_n x^n) = f(a_0 + \cdots + a_n x^n)$. Since (R∩kerf)[x]⊆kerf, g is well-defined, g is clearly a ring homomorphism form A[x] onto P.

If $g(xA[x]) = 0$, then $A \cong P$.

If $g(xA[x]) = P$, then for every nonzero ideal I of A, $g(I) \neq 0$ since $A = R \cap \ker f$, $I \neq 0$ means that $I \not\subseteq \ker f$ and hence $g(I) \neq 0$.

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Now, if $g(xI[x]) = 0$, then $P = g(I[x]) = g(I)$, so $0 = g(xI[x]) \supseteq g(I)g(xA[x]) = P$. This is impossible, so $g(xI[x]) = P$, i.e., g is proper.

(2) If A:=R/(R∩kerf)≅P, then the result holds obviously, suppose not, then the ring homomorphism g:A[x]→P induced by f is proper.

So by Proposition3.2.1, there is a polynomial t(x)∈Z(A)[x] such that f(t(x))=1, so at least one coefficient of t(x) is not a nilpotent element.(If not, note that Z(A)[x] is a commutative ring, so sum of two nilpotent elements is again a nilpotent elements.)

So A contatins a non-nilpotent element.

■

Proof of Theorem3.2.1:

If R is a nil ring, so every element in R is nilpotent, by Lemma3.2.2 (2), R[x] cannot be homomorphically mapped onto a nonzero simple ring with identity.

This implies that R[x] cannot be homomorphically mapped onto a nonzero ring with identity. If not, suppose f:R[x]⟶P is a ring homomorphism from R[x] onto P, where P is a nonzero ring with identity, then P has a maximal ideal I, denote g:P⟶P/I the canonical projection of P onto P/I, then g∘f is a ring homomorphism from R[x] onto a simple ring with identity, which is a contradiction.

■

這就證明了我們所要的結果。

更進一步的，M.Chebotar, W.-F. Ke, P.-H. Lee 與 E.R. Puczylowski 四人將此結果推廣，證明了下述定理[7]:

Theorem: If R is a nil ring, then the polynomial ring R$[x_1, \dots x_n]$ cannot be homomorphically mapped onto a nonzero simple ring with identity.

但在此我們不談這個定理。

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(3.3) An example of Agata Smoktunowicz ([6]).

在 Theorem2.2.1 中，我們指出 Köthe Conjecture 等價於下面的敘述:
If R is a nil ring, then R[x] is Jacobson radical.

那麼，一個可行的做法是考慮下列的問題:
If R is a nil ring, then R[x] is also a nil ring.
若這個問題是正確的，那麼 Köthe Conjecture 自然成立。
但在 2000 年，Agata Smoktunowicz 構造了一個反例，即存在一個 nil ring R，佈於其上的多項式環 R[x]不是 nil，這代表試圖以這個方向證明是不可行的，但很可惜的是，這個反例的構造超出我的理解，因此我僅在此列出這個反例，而不做證明，但我仍試圖給一個直覺上的觀察，希望能說明這個可能會出現問題(這當然，不是一個詳細的證明)。

Let K be a countably infinite field, and let K{x，y，z} denote the polynomial ring of three noncommutative indeterminate, take A be the subring of K{x, y, z} with zero constant term.
Since K is countable, so is A, write A={f_i: $i = 1, 2, \dots$}.
Consider the ideal I generated by $\{{f_i}^{10m_{i+1}}: i \in \mathbb{N}\}$, where m_i satisfying:
(i) $m_1 > 10^8, and\ \ m_{i+1} > m_i \times 2^{i+101}\ \ for\ \ i \geq 1,$
(ii) $each\ \ m_i\ \ divides\ \ m_{i+1},$
(iii) $m_i > 3^{2\deg(f_i)}(\deg(f_i))^2 40^2, for\ \ i \geq 1.$
$Put\ \ \overline{A} = \mathrm{A/I}$

Theorem3.3.1:
The polynomial ring $\overline{A}[X, Y]$ with two commutative indeterminate is not nil.

Corollary3.3.1:
There exists a nil ring N such that the polynomial ring N[x] is not nil.

Proof: If N[x] is nil for all nil ring N, then N[x，y]=N[x][y] is nil, for all nil ring N, but we have an nil ring $\overline{A}$ such that $\overline{A}[X，Y]$ is not nil, so there is a nil ring N such that N[x] is not nil.

■

這裡可以給一個直覺上的說明，首先，$\overline{A}$的交換性相當差，當我們取一個 f∈$\overline{A}[X, Y]$，並計算他的次方時，可以想像他的項數會越來越多，而由我們 I 的

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構造，每個項的係數 0 掉的速度越來越慢(指數部份成長相當快速)，可以直覺想像，f 次方的成長追不上這些m_i的成長，那 f 可能就不會是 nilpotent。
總結:Köthe Conjecture 可以從很多面向來看這個問題，首先當然是把 nilpotent ideal 對 nil ideal 做推廣，第二個我們可以從 radical 來看，在環論中，我們可以把 radical 裡面的元素想像成不好的元素，像是 nilpotent 這種自己自乘某個次方後會等於 0 的元素通常不是我們喜歡的，就如同考慮可測函數一樣，一般來說一個環內當然有可能有一些不好的點(例如可測函數有些點會取值正無窮或負無窮)，關鍵在於這些點是否會影響到環的結構，所謂影響結構，一個觀點就是這些點多到可以形成 ideal，從而就可以做 quotient，例如我們看矩陣環$M_n(\mathbb{R})$，這矩陣內當然有一些 nilpotent 的元素(例如嚴格上三角或嚴格下三角矩陣等等)，但這個矩陣的 nilradical 是 0，直覺上看來就相當於說明這些 nilpotent 元素數量不多，不至於影響環的結構。而 Köthe Conjecture 可以這樣看:單邊 ideal 所影響的是 left 或 right module 的結構，我們想問的是:如果這些 nilpotent 多到能影響 module 的結構，那是否他就能影響到整個環的結構?
這個問題看來單純，但從 Theorem2.2.1 中我們可以發覺，定理中第三個敘述是自然的，第二個則是可以從 Köthe Conjecture 中直接得到，但第一個敘述相當不自然，一般來說，會問:”R 是 nil，則 R[x]是 nil.”這樣的問題顯然比較自然，但(3.3)卻指出這是錯的，(3.2)與(3.3)說明了 Köthe Conjecture 剛好"卡在中間”，可見這個問題並不是如他的外表看來那麼容易解決，乃至於自 1930 年開始至今均懸而未決。

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Appendix:

Here we shall prove Lemma 2.2.2, Lemma 2.2.3, and Lemma 2.2.4. The proof of these lemma due to Amitsur([3]).

We first prove some elementary properties of Jacobson radical.

Lemma1:
Let R and S be rings and φ:R→S be a ring homomorphism from R onto S, then $\varphi(J(R))\subseteq J(S)$.
Proof:
Since φ is onto, $\varphi(J(R))$ is an two-sided ideal of S, it is clear that every element in $\varphi(J(R))$ is quasi regular, hence $\varphi(J(R))\subseteq J(S)$.

■

This implies an easy corollary:
Corollary1:
Let N be an ideal of a ring R such that $N\subseteq J(R)$, then $J(R/N) =J(R)/N=\{a+N: a\in J(R)\}$
Proof:
Consider the canonical projection φ:R→R/N, then by Lemma1, $\varphi(J(R))=J(R)/N\subseteq J(R/N)$.
Conversely, note that $J(R/N)=I/N$, for some ideal I of R that containing N.(Correspondence theorem)
Let $a\in I$, then there is $b\in I$ such that $a+b+ab\in N\subseteq J(R)$, so there is $c\in R$ such that $a+b+ab+c+ac+bc+abc=0=a+(b+c+bc)+a(b+c+bc)$, hence a is right quasi regular, so $I\subseteq J(R)$, hence $J(R/N) =J(R)/N$

■

Lemma2: Let R be a ring and $a\in R$. If $aR\subseteq J(R)$, then $a\in J(R)$.
Proof:
If $a\notin J(R)$, then by the definition of Jacobson radical, there is an irreducible left R-module M such that $aM\neq 0$, but M is irreducible implies that M=RM, so $aRM \neq 0$, i.e. there is an $r\in R$ such that $arM\neq 0$, i.e. $ar\notin J(R)$, which is a contradiction.

■

Now we can prove
Lemma 2.2.2:
Let $N=J(R[x])\cap R$,then $J(R[x])\neq 0$ implies $N\neq 0$.

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Proof:

Let J=J(R[x]). Suppose that the lemma is not true, then there is a ring R with J≠0, but N=0.

Our main ideal as follows:

Let f(x) be a nonzero polynomial of minimum degree such that f(x)∈J.

Then consider φ:R[x]→R[x] defined by φ(f(x))=f(x+1) (Note that R may not have identity, but f(x+1) is well-defined as usual way.)

So by our corollary1, f(x+1) ∈J, hence f(x+1)-f(x) ∈J, but the degree of f(x+1)-f(x) is smaller than f(x), hence f(x+1)-f(x)=0, i.e. f(x+1)=f(x), in the special case that R is a field, then f(x)≡a≠0, which is a contradiction, our goal is to show that this is also true for any ring.

Step1:

For a prime integer p, let $R_p = \{a \in R: pa = 0\}$. It is clear that R_p is an ideal of R, and hence $R_p[x]$ is an ideal of R[x].

We now show that we can assume that f(x)∈$R_p[x]$.

Let f(x)=$a_n x^n + \cdots + a_0$ with $n \geq 1 (\text{i.e. } a_n \neq 0)$, then f(x+1)-f(x)=$na_n x^{n-1} + \cdots = 0$, so na_n=0, let m be the minimal integer such that ma_n=0 (Note that m≥2) and let p be a prime integer dividing m. Then we have

$\frac{m}{p} a_n \neq 0$, and clearly $\frac{m}{p} a_n \in R_p$, now, replace f(x) by $\frac{m}{p} f(x) \in J$, and

$p\left[\frac{m}{p} f(x)\right]$ is a polynomial in J with degree less than n, so $p\left[\frac{m}{p} f(x)\right] = 0$

, and hence $\frac{m}{p} f(x) \in R_p[x]$.

Step2:

We claim that if f(x) ∈$R_p[x]$ satisfies that f(x+1)=f(x), then f must of the form f(x)=$h(x^p - x)$, for some $h(x) \in R_p[x]$.

First, note that R_p can be viewed as an algebra over Z_p.

Next, consider the ring

$R_p{}^1 = R_p \times Z_p$, with addition defined componentwise, and

$(a, m)(b, n) \coloneqq (ab + na + mb, mn)$

Since R_p is an algebra over Z_p, the operation of $R_p{}^1$ is well − defined, R_p can be viewed as a subring of $R_p{}^1$, and $R_p{}^1$ is an algebra over Z_p with identity.

Now, we use induction on the degree of f(x), note that f(x)∈ $R_p[x] \subseteq R_p{}^1[x]$.

If deg(f(x))<p, since f(x+1)=f(x), so f(m)=a, for some a∈ $R_p \subseteq R_p{}^1$,m=0, 1,..., p-

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1, so by division algorithm and factor theorem, we have f(x)≡a∈ R_p.
Next, if degree f(x)≥p, then by division algorithm in $R_p{}^1$, we have $f(x)=(x^p-x)h(x)+k(x)$, where deg(k(x))<p, so by our assumption, f(x+1)=f(x), this implies that

$$(x^p-x)[h(x+1)\text{-}f(x)]=k(x)\text{-}k(x+1)\ldots(1)$$

compare the degree of both side, note that the degree of right hand side of (1) is less than p, and if h(x+1)-h(x) is not zero, then the degree of left hand side of (1) must ≥p, so h(x+1)=h(x), and k(x)=k(x+1), where deg(h(x))<deg(f(x)), and deg(k(x))<p, so k(x)≡a∈$R_p{}^1$, and k(x)∈$R_p{}^1[x^p-x]$, so $f(x)=(x^p-x)h(x)+a\in R_p{}^1[x^p-x]$, and hence f(x)∈$R_p[x]$(The final conclusion holds by view every coefficient when we use division algorithm).

Step3:
Claim: If $h(x^p-x)\in J(R_p[x])$, then $h(x^p-x)\in J(R_p[x^p-x])$.
By Lemma2, it is sufficient to show that $h(x^p-x)R_p[x^p-x]\subseteq J(R_p[x^p-x])$.
Let g(x)∈ $h(x^p-x)R_p[x^p-x]$, then g(x) ∈$J(R_p[x])$, so it is quasi regular, say g(x)+k(x)+g(x)k(x)=g(x)+k(x)+k(x)g(x)=0, since the quasi inverse of g(x+1) is k(x+1) and g(x)=g(x+1), so k(x) is also the quasi inverse of g(x+1), then by the uniqueness of quasi inverse, we have k(x)=k(x+1), then by Step2, we have k(x) ∈$R_p[x^p-x]$, hence $h(x^p-x)R_p[x^p-x]$ is a quasi-regular right ideal of $R_p[x^p-x]$. So $h(x^p-x)\in J(R_p[x^p-x])$ by Lemma2.

Step4:
Now return to our goal, let f(x) be a nonzero polynomial of minimum degree such that f(x)∈J. By step1, we may assume f(x) ∈$J\cap R_p[x]=J(R_p[x])$(Proposition1.2.4)
Next, since f(x)=f(x+1), so by Step2 and Step3, $f(x)=g(x^p-x)\in J(R_p[x^p-x])$.
Consider the map φ: $R_p[x^p-x]\rightarrow R_p[x]$ defined by $\varphi[g(x^p-x)]=g(x)$, this is, clearly, a ring homomorphism. So by Lemma1, $\varphi(J(R_p[x^p-x]))\subseteq J(R_p[x])$, hence g(x)∈$J(R_p[x])$. But g has degree less than f(Since f is nonconstant by our assumption.), this leads to a contradiction. Hence N≠0.

■

Lemma2.2.3 J(R[x])=N[x].
Proof:
Since N⊆J=J(R[x]), so N[x]R[x]⊆NR[x]⊆J. That is f(x)R[x] ⊆J, for all f(x)∈N[x], so by Lemma2, N[x] ⊆J.
Next, consider the canonical projection from R[x] to R[x]/N[x], since N[x] ⊆J. By Corollary1, we have J(R[x]/N[x])=J(R[x])/N[x].

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Put $\overline{R} = R/N$, then $R[x]/N[x] \cong \overline{R}[x]$, $J(\overline{R}[x]) \cap \overline{R} \cong (J/N[x]) \cap \{(R + N[x])/N[x]\}$
$=[J \cap (R+N[x])]/N[x]) = [J \cap R + N[x]]/N[x]$(Dedekind modular law.)
$=[N+N[x]]/N[x]=0$, so
$J(\overline{R}[x]) \cap \overline{R}=0$, by Lemma2.2.2, $J(\overline{R}[x]) = 0$.
That is, $J/N[x]=0$, so $J \subseteq N[x]$, hence $J=N[x]$.

■

Lemma2.2.4:
N is a nil ideal of R.
Proof:
It is clear that N is an ideal of R.
Now let $r \in N$, we have $r^2 x \in J$, so there is a $q(x) \in R[x]$ such that
$q(x) + r^2 x + q(x) r^2 x = 0$.
So $q(x) = -r^2 x - q(x) r^2 x = -r^2 x + (r^2 x)^3 + \cdots + (-1)^{n+1} (r^2 x)^{n+1} +$
$(-1)^{n+1} (r^2 x)^{n+1} q(x)$.
Choose $n > \deg(q(x))$, and compare the coefficient of x^n, we have $r^{2n} = 0$
, hence N is nil.

■

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