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在對稱群上的特徵標

On Characters of Symmetric Groups

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國立臺灣大學碩士學位論文
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在對稱群上的特徵標
On Characters of Symmetric Groups

本論文係 黃子豪 君 （R04221002） 在國立臺灣大學 數學 系完成之碩士學位論文，於民國 107 年 01 月 23 日承下列考試委員審查通過及口試及格，特此證明

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一開始剛入碩士班，我是個完全沒寫過論文，完全無法理解何謂學術研究的人。很感謝于靖教授願意收這樣的我做學生，讓我能一窺學術研究的面貌。也要謝謝于靖教授的指導，指導我說何謂一個數學家應該要有的能力。這篇論文得以完成，碩士班得以順利畢業，最要感謝的就是于靖教授。

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中文摘要

令 p 爲一個質數，G 爲一個有限群，且 $G^{(p)}$ 爲收集 G 的元素中所有滿足元素的階 (order) 和 p 互質的元素。如果對於所有的 G 的不可約特徵標 (irreducible character) χ，都沒辦法找到一個大於 1 的自然數 a 和一個 G 的不可約 p-模特徵標 ϕ (irreducible p-modular character) 使得 $\chi|_{G^{(p)}} = a\phi$，那我們就會說 G 有 (L', p)-性質 ((L', p)-property)。如果對於所有的 G 的不可約特徵標 χ，都能找到一個 G 的不可約 p-模特徵標 ϕ 使得 $\chi|_{G^{(p)}} \geq \phi$ with multiplicity 1，那我們就會說 G 有 (L'', p)-性質 ((L'', p)-property)。又如果對於所有的質數 p，G 恆有 (L'', p)-性質，那我們就會說 G 有 L''-性質 (L''-property)。

在這篇碩士論文中，我們想要證明所有的對稱群 (symmetric groups) 都有 L''-性質；所有的交錯群 (alternating groups) 都有 $(L'', 2)$-性質；且對於所有比 2 大的質數 p，所有的交錯群都有 (L', p)-性質。

關鍵詞：對稱群、交錯群、特徵標、p-模特徵標、(L', p)-性質、(L'', p)-性質、L''-性質。

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Abstract

Let p be a prime number, G be a finite group, and $G^{(p)}$ be the set of all $g \in G$ such that $p \nmid \mathrm{ord}(g)$. We say G has the (L', p)-property if for any irreducible character χ of G, $\chi|_{G^{(p)}} \neq a\phi$ for any irreducible p-modular character ϕ of G and any $a \in \mathbb{N}$ with $a > 1$. We say G has the (L'', p)-property if for any irreducible character χ of G, there exists an irreducible p-modular character ϕ of G such that $\chi|_{G^{(p)}} \geq \phi$ with multiplicity 1. We say G has the L''-property if G has the (L'', p)-property for all p.

In this thesis, we want to show that all symmetric groups have the L''-property, all alternating groups have the $(L'', 2)$-property, and all alternating groups have the (L', p)-property for all prime $p > 2$.

Keywords: symmetric groups, alternating groups, character, modular character, (L', p)-property, (L'', p)-property, L''-property.

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0 Notations and Preliminaries

In this thesis, we denote

$$
\begin{aligned}
n &: \text{a positive number},\\
p &: \text{a prime number},\\
S_n &: \text{the symmetric group of degree } n,\\
A_n &: \text{the alternating group of degree } n,\\
G &: \text{a finite group},\\
G^{(p)} &: \text{the set of all } g \in G \text{ such that } p \nmid \operatorname{ord}(g),\\
H &: \text{a subgroup of } G,\\
F &: \text{an arbitrary field},\\
F_0 &: \text{a field with characteristic } 0,\\
F_p &: \text{a field with characteristic } p.
\end{aligned}
$$

Definition 0.1.

(1) We say a simple FG-module V is absolutely simple if for any field extension L of F, the scalar extension $L \otimes_F V$ is a simple LG-module. We say F is a s-splitting field for G if $\operatorname{char}(F) = s$ and every simple FG-module are absolutely simple.

(2) We say (A, B, C) is a p-modular system if

A : a field complete with respect to a discrete valuation v with characteristic 0

B : the valuation ring of A with respect to v

C : the residue field of A with respect to v with characteristic p.

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Note that if we denote D as the maximal ideal of B, then we have $C = B/D$.

(3) We say (A,B,C) is a splitting p-modular system for G if (A,B,C) is a p-modular system, A is a 0-splitting field for G, and C is a p-splitting field for G.

(4) In the following, we always assume (K_0,R,K_p) is a splitting p-modular system for G, and we denote $\mathfrak{m}$ as the maximal ideal of R.

Remark.

If A contains $\mathrm{lcm}(\mathrm{ord}(g) \mid g \in G)$-th root of unity, then (A,B,C) is a splitting p-modular system for G (cf., for example, [S1], theorem 24, page 94). In the case $G = S_n$, if we denote Q as the p-adic completion of $\mathbb{Q}$, then by theorem 2.10 and theorem 2.17, we can choose $K_0 = Q$ and $v =$ the p-adic valuation of Q.

Definition 0.2.

(1) Let V be an FG-module and let $e \in \mathbb{N}$. We denote eV as the e-fold direct sum of V.

(2) Let M be an FG-module and let $V = \mathrm{Hom}_F(M,F)$. For any $g \in G$, $m \in M$, and $\phi \in V$, define $g \cdot \phi(m) = \phi(g^{-1}m)$. Then V is an FG-module via this definition and we say V is the dual of M. We usually denote the dual of M as M^*.

Definition 0.3.

(1) Let M be an FG-module. Then we can regard M as an FH-module by restricting FG to FH directly, and we denote it as $\mathrm{Res}_H^G(M)$.

(2) Let V be an FH-module. We denote $\mathrm{Ind}_H^G(V) = FG \otimes_{FH} V$.

Definition 0.4.

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Let M be a K_0G-module and let $\theta_g : M \to M$ given by $m \mapsto g \cdot m$. Let S be a K_0-basis of M and let $|S| = a$. Then we can write θ_g as the matrix $\rho_g \in \mathrm{GL}_a(K_0)$ with respect to S. Now define

$$\chi_M : G \to K_0,\ g \mapsto \mathrm{Tr}(\rho_g).$$

Then χ_M is called the character of M (or of G). This definition is independend of the choice of S, and also independend of the choice of K_0. Note that if C is a conjugacy class of G and $g_1, g_2 \in C$, then $\chi_M(g_1) = \chi_M(g_2)$.

Moreover, we say χ_M is irreducible if M is a simple K_0G-module. There are only finitely many irreducible characters.

Proposition 0.5.

Let M be a K_0G-module and let $e_1E_1, \cdots, e_bE_b$ be all composition factors of M, where e_iE_i means there are e_i simple K_0G-modules E_i for $e_i \in \mathbb{N}$, and each E_i is distinct. Then

$$\chi_M = \sum_{i=1}^{b} e_i\chi_{E_i}, \text{ where } \chi_M, \chi_{E_i} \text{ are characters of } M, E_i \text{ respectively,}$$

and the sum is unique, i.e. if $\chi_M = \sum_{j=1}^{b'} d_j\chi_j$, where d_j are integers, $\chi_1, \cdots, \chi_{b'}$ are all distinct irreducible characters with $\chi_j = \chi_{E_j}$ for $j = 1, \cdots, b$, then $d_j = e_j$ for $j = 1, \cdots, b$ and $d_j = 0$ for $j = b+1, \cdots, b'$.

Proof.

Cf., for example, [S1] proposition 32, page 91.

□

Definition 0.6.

Let Δ be the $m := \mathrm{lcm}(\mathrm{ord}(g) \mid g \in G)$-th root of unity in the algebraic clorsure of K_0. Then there is a valuation $\tilde{v}$ of $\tilde{K}_0 := K_0(\Delta)$ such that $\tilde{v}|_{K_0} = v$, where v is a given valuation

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of K_0 (cf., for example, [S2], proposition 3, page 28). The residue field $\tilde{K}_p$ of $\tilde{K}_0$ can be regarded as a field extension of K_p. If $m = p^k m'$ for some $m' \in \mathbb{N}$ with $\gcd(p, m') = 1$, then $\tilde{K}_p$ contains the m'-th root of unity.

Let M be a K_pG-module and let S be a K_p-basis of M. Similary as definition 0.4, for any $g \in G$, we can define $\rho_g \in \mathrm{GL}_a(K_p)$ with respect to S. Assume $g \in G^{(p)}$. Since $\rho_g^{m'}$ is the identity matrix, we can find $\lambda_1, \cdots, \lambda_a \in \tilde{K}_p$ such that they are all eigenvalues of ρ_g. Moreover, there exist roots of unity Λ_i in $\tilde{K}_0$ such that $\Lambda_i + \tilde{\mathfrak{m}} = \lambda_i$, where $\tilde{\mathfrak{m}}$ is the maximal ideal of the valuation ring of $\tilde{K}_0$. Define

$$\phi_M : G^{(p)} \to \tilde{K}_0,\ g \mapsto \sum_{i=1}^{a} \Lambda_i.$$

Then ϕ_M is called the p-modular character of M (or of G). This definition is independent of the choice of S, and independent of the choice of the p-modular system (K_0, R, K_p). Note that if C is a conjugacy class of $G^{(p)}$ and $g_1, g_2 \in C$, then $\phi_M(g_1) = \phi_M(g_2)$.

Moreover, we say ϕ_M is irreducible if M is a simple K_pG-module. There are only finitely many irreducible p-modular characters.

Proposition 0.7.

Let M be a K_pG-module and let $e_1E_1, \cdots, e_bE_b$ be all composition factors of M, where e_iE_i means there are e_i simple K_pG-modules E_i for $e_i \in \mathbb{N}$, and each E_i is distinct. Then

$$\phi_M = \sum_{i=1}^{b} e_i \phi_{E_i}, \text{ where } \phi_M,\ \phi_{E_i} \text{ are } p\text{-modular characters of } M,\ E_i \text{ respectively}$$

and the sum is unique, i.e. if $\phi_M = \sum_{j=1}^{b'} d_j \phi_j$, where d_j are integers and $\phi_1, \cdots, \phi_{b'}$ are all distinct irreducible p-modular characters with $\phi_j = \phi_{E_j}$ for $j = 1, \cdots, b$, then $d_j = e_j$ for $j = 1, \cdots, b$ and $d_j = 0$ for $j = b+1, \cdots, b'$.

Proof.

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Cf., for example, [S1] proposition 40, page 115.

$\square$

Definition 0.8.

(1) If χ is the character of a K_0G-module M, then we denote $\mathrm{Res}_H^G(\chi)$ as the character of the K_0H-module $\mathrm{Res}_H^G(M)$.

(2) If ϕ is the p-modular character of a K_pG-module M, then we denote $\mathrm{Res}_H^G(\phi)$ as the p-modular character of the K_pH-module $\mathrm{Res}_H^G(M)$.

Remark.

Note that $\mathrm{Res}_H^G(\chi) = \chi|_H$ and $\mathrm{Res}_H^G(\phi) = \phi|_{H^{(p)}}$.

Definition 0.9.

Let M be a K_0G-module. We say X is a RG-lattice of M if X is a finite generated R-free RG-module such that $K_0 \otimes_R X = M$. Note that $X/\mathfrak{m}X$ is a K_pG-module. We say $X/\mathfrak{m}X$ is the reduction mod $\mathfrak{m}$ of X, and say $X/\mathfrak{m}X$ is a reduction mod $\mathfrak{m}$ of M.

Proposition 0.10.

Let M be a K_0G-module, χ be the character of M, and X be a RG-lattice of M. Then the p-modular character of $X/\mathfrak{m}X$ is $\chi|_{G^{(p)}}$. Note that it is independent of the choice of X.

Proof.

To see this, it suffices to show that if X, Y are RG-lattices of M, then the K_pG-modules $X/\mathfrak{m}X$ and $Y/\mathfrak{m}Y$ have the same composition factors. For its proof, one may see, for example, [S1] theorem 32, page 125.

$\square$

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Definition 0.11.

Let $\chi_1, \cdots, \chi_m$ be all distinct irreducible characters of G, and let χ be a character of G. Then by proposition 0.5, $\chi = \sum_{i=1}^{m} a_i \chi_i$ for some non-negative integer $a_i \in \mathbb{Z}$. If $a_i \neq 0$, then we denote $\chi \geq \chi_i$ (with multiplicity a_i).

Definition 0.12.

Let $\phi_1, \cdots, \phi_m$ be all distinct irreducible p-modular characters of G, and let ϕ be a p-modular character of G. Then by proposition 0.7, $\phi = \sum_{i=1}^{m} a_i \phi_i$ for some non-negative integer $a_i \in \mathbb{Z}$. If $a_i \neq 0$, then we denote $\phi \geq \phi_i$ (with multiplicity a_i).

Let ϕ' be a p-modular character of G and $\phi' = \sum_{i=1}^{m} b_i \phi_i$ for some non-negative integer $b_i \in \mathbb{Z}$. If $a_i \geq b_i$ for all i, then we denote $\phi \supseteq \phi'$.

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1 Introduction

In this paper, we study (L,p), (L',p), (L'',p)-properties of S_n and A_n for a prime p. With (K_0,R,K_p) fixed, their definitions are:

Definition 1.1.

Let $\phi_1,\cdots,\phi_k$ be all distinct irreducible p-modular characters of G. Then we say G has the (L,p)-property if one of the following happens:

(1) for any $i=1,\cdots,k$, ϕ_i is liftable (we say ϕ_i is liftable if we can find an irreducible character χ_i of G such that $\chi_i|_{G^{(p)}}=\phi_i$),

(2) for some $i=1,\cdots,k$, ϕ_i is not almost liftable. (we say ϕ_i is almost liftable if we can find an irreducible character χ_i of G such that $\chi_i|_{G^{(p)}}=a\phi_i$ for some $a\in\mathbb{N}$).

(In other words, if ϕ is almost liftable for any irreducible p-module characters ϕ, then ϕ is liftable for any ϕ.)

We say G has the L-property if G has the (L,p)-property for all p.

Definition 1.2.

We say G has the (L',p)-property if all irreducible (ordinary) characters χ of G satisfy one of the following:

(1) $\chi|_{G^{(p)}}$ is an irreducible p-modular character of G,

(2) $\chi|_{G^{(p)}}\geq\phi_1$ and $\chi|_{G^{(p)}}\geq\phi_2$, where ϕ_1, ϕ_2 are two distinct irreducible p-modular characters of G.

(In other words, $\chi|_{G^{(p)}}\neq a\phi$ for any irreducible p-modular character ϕ of G and any $a\in\mathbb{N}$ with $a>1$)

We say G has the L'-property if G has the (L',p)-property for all p.

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Definition 1.3.

We say G has the (L'', p)-property if for any irreducible characters χ of G, there exists an irreducible p-modular character ϕ of G such that $\chi|_{G^{(p)}} \geq \phi$ with multiplicity 1.

We say G has the L''-property if G has the (L'', p)-property for all p.

There are three main results in this thesis:

(1) The group S_n has the L''-property for all n.

(2) The group A_n has the (L'', p)-property for $p = 2$, and all n.

(3) The group A_n has the (L', p)-property for $p > 2$, and all n.

The first is described in corollary 3.17.1, the second is described in corollary 4.15.1, and the third is described in theorem 4.21.

Our motivation for studying (L, p)-property, (L', p)-property and (L'', p)-property came from a fall 2016 course on finite group representations which was taught by professor Jing Yu at National Taiwan University. In that course, we have completed all exercises except the exercise 16.6 in the book "Linear Representations of Finite Groups" by Jean-Pierre Serre. This exercise 16.6 is, with (K_0, R, K_p) fixed:

$$e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_p}(G) \text{ if and only if } d(R^+_{K_0}(G)) = R^+_{K_p}(G).$$

The notations d, e mean the d, e of the cde-triangle (we put the definition of it on Appendix B). The notations $R_{K_0}(G)$, $R_{K_p}(G)$, and $P_{K_p}(G)$ denote the Grothendieck groups which are generated by K_0G-modules, K_pG-modules, and projective K_pG-modules respectively. The notations $R^+_{K_0}(G)$, $R^+_{K_p}(G)$, and $P^+_{K_p}(G)$ denote sets which collect the image of K_0G-modules, K_pG-modules, and projective K_pG-modules in $R_{K_0}(G)$, $R_{K_p}(G)$, and $P_{K_p}(G)$ respectively.

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We had no idea how to work out this exercise. So we wrote a letter to Serre, and he wrote back a reply which we put here as Appendix A. In his reply letter, he said that the exercise should be modified as the following:

$$e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G) \text{ if and only if } NR^+_{K_p}(G) \subset d(R^+_{K_0}(G)) \text{ for some } N \in \mathbb{N},$$

and he gave us a proof of this statement (For a proof of this modified exercise, see Appendix B). However, is the question of the original exercise can be answered? According to Serre they did not know the answer yet. This leads us to study a property on given groups, which we called the (L, p)-property. It is easy to see that a given group has the (L, p)-property if and only if the claim of the original exercise holds on the group.

To study the (L, p)-property, we then introduce the (L', p)-property, and the (L'', p)-property. All these properties are properties about the decomposition matrices for the group in questions. It is clear that the (L'', p)-property implies the (L', p)-property, and the (L', p)-property implies the (L, p)-property. In this thesis, we will see that from previous work of James [J2], it follows easily that all the symmetric groups S_n have the L''-property, and we will prove that the groups A_n have the $(L'', 2)$-property and (L', p)-property for $p > 2$. In the Master's thesis [L], Liu proves that $\mathrm{GL}(2, q)$, $\mathrm{SL}(2, q)$, $\mathrm{GL}(3, q)$, and $\mathrm{SL}(3, q)$ have the (L', p)-property for any p, where $q \in \mathbb{N}$ is an arbitrary prime power. On the other hand, by Fong-Swan theorem (cf., for example, [S1], theorem 38, page 135), all p-solvable groups have the (L, p)-property.

Note that there are some groups which do not have the (L', p)-property, but they have the (L, p)-property. A non-abelian group H with order p^k for some $k \in \mathbb{N}$ is an example. Since the order of H is p^k, the only irreducible p-modular character of H is the trivial p-modular character ϕ, and it is liftable clearly. So H has the (L, p)-property. But since H is non-abelian, there is a character χ of H such that $\chi(e) > 1$, where e is the identity of

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H. So $\chi|_{H^{(p)}} = \chi(e)\phi$, and hence H does not have the (L', p)-property.

It is important to find irreducible p-modular characters which are almost liftable but not liftable. Because if all irreducible p-modular characters of a given group G are almost liftable, and one of them is not liftable, then it implies that G does not have the (L, p)-property. The group $O'N \rtimes C_2$, where $O'N$ is the O'Nan group and C_2 is the cyclic group of order 2, give an example that there exists an irreducible 2-modular character of it which is almost liftable but not liftable (communicated to us by Professor Hiss). Unfortunately, not all irreducible 2-modular characters of $O'N \rtimes C_2$ are almost liftable, and hence it has the $(L, 2)$-property. But at least, $O'N \rtimes C_2$ tell us that such an irreducible p-modular character really exists. For more information about $O'N \rtimes C_2$, one can see [web], which contains the decomposition matrix of $O'N \rtimes C_2$.

In conclusion, we still can not prove that all groups have the (L, p)-property. We also can not find any counter example to this.

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2 Specht modules, F_0S_n-Modules, and F_pS_n-modules

2.1 The Definition of the Specht Module

In this subsection, we will introduce the definition of the Specht module. It is described in definition 2.6. The Specht module plays an important role in the group representation theory of S_n.

Definition 2.1.

Let $\mu = (\mu_1, \cdots, \mu_m)$ be a sequence of positive integers. We say μ is a partition of n if $\mu_i \geq \mu_{i+1}$ for all i and $\sum_{i=1}^{m} \mu_i = n$. If there are the same terms in a partition, we usually abbreviate it as in the following example:

$$(5,4,4,2,1,1) = (5,4^{(2)},2,1^{(2)})$$

We usually use a graph to represent a partition of n, i.e. a graph with m rows such that the i-th row fills in μ_i marks lined up on the left. For example, $(3,2)$ can be represented by this graph:

$$\begin{matrix} \times & \times & \times \\ \times & \times & \end{matrix}$$

Definition 2.2.

Let $\mu = (\mu_1, \cdots, \mu_m)$ be a partition of n. We say t is a μ-tableau if t is a graph with m rows such that the i-th row fills in μ_i numbers lined up on the left; moreover, each number in t is distinct and belongs to $\{1, \cdots, n\}$. For example, let $\mu = (3,2,1)$ be a partition of 6

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and let

$$t_1 = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array}, \quad t_2 = \begin{array}{ccc} 4 & 6 & 2 \\ 1 & 5 & \\ 3 & & \end{array},$$

then t_1, t_2 are μ-tableaux.

Note that S_n can act on tableaux in an intuitive way. So we can define two subgroup C_t, R_t of S_n as follows

$$\begin{aligned} C_t =& \{\pi \in S_n \mid \pi \text{ perserves each column of } t\}, \\ R_t =& \{\pi \in S_n \mid \pi \text{ perserves each row of } t\}. \end{aligned}$$

For example, C_{t_2} is the subgroup of S_6 which is generated by (41), (43), (13) and (65).

Definition 2.3.

Let μ be a partition of n and let t_1, t_2 be μ-tableaux. We say t_1, t_2 are equivalent if the numbers in each rows of t_1, t_2 are the same up to order. It is easy to see that it is an equivalent relation. We denote the equivalence class of t_1 by $\{t_1\}$, and call $\{t_1\}$ as a μ-tabloid. For example,

$$t_1 = \begin{array}{ccc} 2 & 1 & 3 \\ 4 & 6 & \\ 5 & & \end{array}, \quad t_2 = \begin{array}{ccc} 3 & 2 & 1 \\ 6 & 4 & \\ 5 & & \end{array}$$

are equivalent, and $\{t_1\}$ is a $(3,2,1)$-tabloid.

Proposition 2.4.

Let μ be a partition of n and let t_1, t_2 be two μ-tableaux. If $t_1 \sim t_2$, then $\pi t_1 \sim \pi t_2$ for

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all $\pi \in S_n$.

Proof.

It is easy to see by observing that $R_{\pi t_2} = \pi R_{t_2} \pi^{-1}$.

$\square$

Definition 2.5.

Let μ be a partition of n. Define

$$M_F^\mu\text{: the vector space over } F \text{ whose basis is } \{\{t\} \mid \{t\} : \mu\text{-tabloid}\}.$$

Note that M_F^μ is an FS_n-module by defining $\pi\{t\} = \{\pi t\}$ for $\pi \in S_n$. It is well-defined by the above proposition.

Definition 2.6.

For $\pi \in S_n$, define

$$\operatorname{sgn}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is an even permutation} \\ -1 & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

For any μ-tableau t, define

$$b_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi \in FS_n,$$

$$e_t = b_t\{t\} = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\{\pi t\} \in M_F^\mu.$$

Moreover, define

$$S_F^\mu\text{: the subspace of } M_F^\mu \text{ which is generated by all } e_t.$$

The element e_t is called the polytabloid and S_F^μ is called the Specht module (with respect to μ).

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Proposition 2.7.

For any μ-tableau t and any $\pi \in S_n$, we have $\pi e_t = e_{\pi t}$. This means S_F^μ is an FS_n-module.

Proof.

Observe that $\pi e_t = \pi b_t \pi^{-1}\{\pi t\}$. So it suffices to show $\pi b_t \pi^{-1} = b_{\pi t}$, which is easy to see by observing $\pi C_t \pi^{-1} = C_{\pi t}$.

□

2.2 Simple F_0S_n-Module and Simple F_pS_n-Module

In this subsection, we will introduce that all Specht F_0S_n-modules form a complete set of isomorphic classes of all simple F_0S_n-modules. Note that Specht F_pS_n-modules may not simple. We will also introduce a complete set of isomorphic classes of all simple F_pS_n-modules.

Definition 2.8.

Let $\mu = (\mu_1, \cdots, \mu_{m_1})$, $\lambda = (\lambda_1, \cdots, \lambda_{m_2})$ be two partitions of n. We say $\mu \trianglerighteq \lambda$ if

$$\sum_{i=1}^{r} \mu_i \geq \sum_{i=1}^{r} \lambda_i \text{ for all } r = 1, \cdots, m_1$$

(note that m_1 must be $\leq m_2$ in this case).

It is called the dominance order. Moreover, we say $\mu \triangleright \lambda$ if $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$. Note that it is a partial order but not a totally order. For example,

$$(5,1,1) \triangleright (4,2,1),\ (5,1,1) \triangleright (4,1,1,1),$$

and $(5,1,1)$, $(4,3)$ has no such relation.

Lemma 2.9.

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Let μ, λ be two partitions of n, and let Θ be an FS_n-homomorphism from M_F^μ to M_F^λ. Then:

(1) If $S_F^\mu \not\subset \ker(\Theta)$, then $\mu \trianglerighteq \lambda$.

(2) If $\mu = \lambda$, then $\Theta|_{S_F^\mu} = f\,\mathrm{Id}_{S_F^\mu}$ for some $f \in F$.

Proof.

Cf., for example, [J1] lemma 4.10, page 16.

□

Theorem 2.10.

(1) Let μ, λ be two partitions of n. Then $S_{F_0}^\mu \cong_{F_0S_n} S_{F_0}^\lambda$ if and only if $\mu = \lambda$.

(2) The set $\{S_{F_0}^\mu \mid \mu:$ partition of $n\}$ is a complete set of isomorphic classes of all simple F_0S_n-modules.

Proof.

(1) The if part is trivial. For the only if part, assume $S_{F_0}^\mu \cong_{F_0S_n} S_{F_0}^\lambda$. Since $\mathrm{char}(F_0) = 0$, there is an F_0S_n-module V such that $M_{F_0}^\mu = S_{F_0}^\mu \oplus V$. Let θ be an F_0S_n-isomorphism from $S_{F_0}^\mu$ to $S_{F_0}^\lambda$. Then define

$$\Theta : M_{F_0}^\mu = S_{F_0}^\mu \oplus V \to S_{F_0}^\lambda \subset M_{F_0}^\lambda \text{ given by } (x,y) \mapsto \theta(x)$$

Since $S_{F_0}^\mu \not\subset \ker(\Theta)$, by lemma 2.9 (1), we obtain $\mu \trianglerighteq \lambda$. So by symmetry, we conclude that $\mu = \lambda$.

(2) By (1), each $S_{F_0}^\mu$ is different when μ varys. So $|\{S_{F_0}^\mu \mid \mu:$ partition of $n\}|$ is equal to the number of conjugacy classes of S_n. Therefore it remains to show each $S_{F_0}^\mu$ is simple.

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Let $U \subset S^{\mu}_{F_0}$ be a nonzero F_0S_n-submodule. Since $\operatorname{char}(F_0) = 0$, we can write $S^{\mu}_{F_0} = U \oplus U'$ for some F_0S_n-module U'. So

$$M^{\mu}_{F_0} = S^{\mu}_{F_0} \oplus V = U \oplus U' \oplus V,$$

where V is as in (1). Now define

$$\Theta : M^{\mu}_{F_0} = U \oplus U' \oplus V \to M^{\mu}_{F_0} \text{ given by } (x,y,z) \mapsto (x,0,0)$$

Then by lemma 2.9 (2), $\Theta|_{S^{\mu}_{F_0}} = f\,\mathrm{Id}_{S^{\mu}_{F_0}}$ for some $f \in F_0$, i.e.

$$(x,0,0) = \Theta|_{S^{\mu}_{F_0}}(x,y,0) = f\,\mathrm{Id}_{S^{\mu}_{F_0}}(x,y,0) = (fx, fy, 0)$$

for any $x \in U$ and $y \in U'$, i.e. $x = fx$ and $0 = fy$. Hence $U' = 0$ since $U \neq 0$. So $U = S^{\mu}_{F_0}$, i.e. $S^{\mu}_{F_0}$ is simple.

$\square$

Definition 2.11.

Define a bilinear form $\langle *,* \rangle : M^{\mu}_{F} \times M^{\mu}_{F} \to F$ given by

$$\langle \{a\}, \{b\} \rangle_F = \delta_{\{a\}\{b\}} \text{ for any } \mu\text{-tabloids } \{a\}, \{b\}$$

So for any subspace V of M^{μ}_{F}, we can define

$$V^{\perp} = \{x \in M^{\mu}_{F} \mid \langle x, v \rangle_F = 0 \text{ for all } v \in V\}.$$

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Note that by definition, it is easy to see that for any x, $y \in M_F^\mu$ and any $\pi \in S_n$,

$$\langle \pi x, \pi y \rangle_F = \langle x, y \rangle_F.$$

Thus if V is an FS_n-module, then so is $V^\perp$.

Definition 2.12.

Let $\mu = (\mu_1, \cdots, \mu_m)$ be a partition of n. We say μ is p-regular if

$$p > |\{k \mid \mu_k = i\}| \text{ for all } i = 1, \cdots, n.$$

Otherwise we say μ is p-singular. For example, $\mu = (3,2,1,1)$ is 2-singular, and it is p-regular for any prime $p > 2$.

Definition 2.13.

For any p-regular partition μ of n. Define

$$D_{F_p}^\mu = S_{F_p}^\mu / (S_{F_p}^\mu \cap (S_{F_p}^\mu)^\perp).$$

Note that $S_{F_p}^\mu = S_{F_p}^\mu \cap (S_{F_p}^\mu)^\perp$ if and only if μ is p-singular. So we define $D_{F_p}^\mu$ only when μ is p-regular (cf., for example, [J1], theorem 11.1, page 39).

Lemma 2.14.

Let μ, λ be partitions of n with μ is p-regular. Let U be an F_pS_n-submodule of $M_{F_p}^\lambda$. If Θ is a non-zero F_pS_n-homomorphism from $S_{F_p}^\mu$ to $M_{F_p}^\lambda / U$, then $\mu \trianglerighteq \lambda$.

Proof.

Cf., for example, [J1] lemma 11.3, page 39.

$\square$

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Lemma 2.15.

The number of p-regular conjugacy classes of S_n is equal to the number of p-regular partitions of n (We say a conjugacy class of a group G is p-regular if the order of elements in the class is prime to p.).

Proof.

Cf., for example, [J1] lemma 10.2, page 36.

$\square$

Lemma 2.16. (Submodule lemma)

If U is an FS_n-submodule of M_F^{μ}, then either $S_F^{\mu} \subset U$ or $U \subset (S_F^{\mu})^{\perp}$.

Proof.

Cf., for example, [J1] theorem 4.8, page 15.

$\square$

Theorem 2.17.

(1) For any two p-regular partitions μ, λ of n, $D_{F_p}^{\mu} \cong_{F_pS_n} D_{F_p}^{\lambda}$ if and only if $\mu = \lambda$.

(2) The set $\{D_{F_p}^{\mu} \mid \mu : p\text{-regular partition of } n\}$ is a complete set of isomorphic classes of all simple F_pS_n-modules.

Proof.

(1) The if part is trivial. For the only if part, assume $D_{F_p}^{\mu} \cong_{F_pS_n} D_{F_p}^{\lambda}$. Let θ be an F_pS_n isomorphism from $D_{F_p}^{\mu}$ to $D_{F_p}^{\lambda}$. Define a map Θ from $S_{F_p}^{\mu}$ to $M_{F_p}^{\lambda}/(S_{F_p}^{\lambda} \cap (S_{F_p}^{\lambda})^{\perp})$ given by

$$\Theta : S_{F_p}^{\mu} \xrightarrow{\alpha} S_{F_p}^{\mu}/(S_{F_p}^{\mu} \cap (S_{F_p}^{\mu})^{\perp}) = D_{F_p}^{\mu} \xrightarrow{\theta} D_{F_p}^{\lambda} \subset M_{F_p}^{\lambda}/(S_{F_p}^{\lambda} \cap (S_{F_p}^{\lambda})^{\perp}),$$

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where α is the canonical map from $S^{\mu}_{F_p}$ to $S^{\mu}_{F_p}/(S^{\mu}_{F_p}\cap(S^{\mu}_{F_p})^{\perp})$. Then clearly Θ is nonzero. So by lemma 2.14, we obtain $\mu \trianglerighteq \lambda$, and by symmetry, we conclude that $\mu=\lambda$.

(2) By (1), each $D^{\mu}_{F_p}$ is distinct, and by lemma 2.15, the number of p-regular conjugacy classes of S_n is equal to the number of all p-regular partitions of n. So it suffices to show each $D^{\mu}_{F_p}$ is a simple F_pS_n-module.

Let $U \subsetneq S^{\mu}_{F_p}$ be a maximal submodule. Then by lemma 2.16, we have $U \subset (S^{\mu}_{F_p})^{\perp}$. So $U \subset S^{\mu}_{F_p}\cap(S^{\mu}_{F_p})^{\perp} \subset S^{\mu}_{F_p}$. Since μ is p-regular, we have $S^{\mu}_{F_p}\cap(S^{\mu}_{F_p})^{\perp} \neq S^{\mu}_{F_p}$. So we conclude that $U = S^{\mu}_{F_p}\cap(S^{\mu}_{F_p})^{\perp}$ since U is maximal, which meams $D^{\mu}_{F_p}$ is simple.

□

2.3 Facts about F_0S_n-Module and F_pS_n-Module

In this subsection, we will introduce some facts about F_0S_n-module and F_pS_n-module. They will be used in following sections.

Definition 2.18.

Let μ be a partition of n, and let t be a μ-tableau. We say t is standard if the numbers in t are increasing along the rows and down the columns. For example, let $\mu=(3,2,1)$ be a partition of 6 and let

$$t = \begin{matrix} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{matrix} ,$$

then t is a standard μ-tableau.

Proposition 2.19.

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Let μ be a partition of n and let $t_1, \cdots, t_m$ be all distinct standard μ-tableaux of n. Then $\{e_{t_1}, \cdots, e_{t_m}\}$ is a basis of S_F^{μ}.

Proposition 2.20.

Let μ be a partition of n. Then $S_{F_0}^{\mu}$ is self dual, i.e

$$(S_{F_0}^{\mu})^* \cong_{F_0 S_n} S_{F_0}^{\mu}.$$

Proof.

Cf., for example, [J1], theorem 1.5, page 3.

□

Proposition 2.21.

Let μ be a p-regular partition of n. Then $D_{F_p}^{\mu}$ is self dual, i.e

$$(D_{F_p}^{\mu})^* \cong_{F_p S_n} D_{F_p}^{\mu}.$$

Proof.

Cf., for example, [J1], theorem 1.5, page 3.

□

Proposition 2.22.

Let λ be a p-regular partition of n and let $t_1, \cdots, t_m$ be all distinct standard λ-tableaux. Denote $M = \left(\langle e_{t_i}, e_{t_j} \rangle_{F_p} \right) \in \mathrm{M}_{m \times m}(F_p)$. Then $\dim_{F_p}(D_{F_p}^{\lambda}) = \mathrm{rank}_{F_p}(M)$.

Proof.

Cf., for example, [J1], theorem 1.6, page 3.

□

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Definition 2.23.

Let λ, λ' be partitions of n. We say λ' is the conjugate partition of λ if the graph of λ' is obtained by interchanging the rows and columns of the graph of λ. For example, the conjugate partition of $(3,2)$ is

$$\begin{array}{cc} \times & \times \\ \times & \times \\ \times & \end{array}$$

i.e. $(2,2,1)$ is the conjugate partition of $(3,2)$. We usually use the notation λ' to denote the conjugate partition of λ.

Proposition 2.24.

Let λ be a partition of n.

(1) If λ is p-singular, then all composition factors of $S^{\lambda}_{F_p}$ have the form $D^{\mu}_{F_p}$ with $\mu \rhd \lambda$,

(2) If λ is p-regular, then $D^{\lambda}_{F_p}$ occurs precisely once in the composition factors of $S^{\lambda}_{F_p}$, and the others (if exist) have the form $D^{\mu}_{F_p}$ with $\mu \rhd \lambda$.

Proof.

Cf., for example, [J1], corollary 12.2, page 42.

□

Proposition 2.25.

Let λ be a partition of n. Then $(S^{\lambda'}_F)^* \cong_{FS_n} S^{\lambda}_F \otimes_F S^{(1^{(n)})}_F$.

Proof.

Cf., for example, [J1], theorem 8.15, page 33.

□

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3 About S_n

3.1 Groups S_5 and S_6 have the (L', p)-property

As we have mentioned in the introduction, from the work of James [J2], we can see that S_n has the L''-property. However, in this subsection, we are going to use our method to show that S_5 and S_6 have the (L', p)-property for all p. This method is different from James, but it gives us an idea that how to show a given group has (L', p)-property.

Lemma 3.1.

Let V, W be simple FG-modules with $\dim_F(V) = 1$. Then $V \otimes_F W$ is also a simple FG-module.

Lemma 3.2.

Let V, W be two K_0G-modules (or K_pG-modules) which afford character (or p-modular character) χ_V, χ_W respectively. Then the character (or p-modular character) of $V \otimes_{K_0} W$ (or $V \otimes_{K_p} W$) is $\chi_V \times \chi_W$.

Lemma 3.3.

Let μ be a partition of n. Then $S^{\mu}_{K_p}$ is a reduction mod $\mathfrak{m}$ of $S^{\mu}_{K_0}$.

Proof.

Let $t_1, \cdots, t_m$ be all distinct standard μ-tableaux. Then by proposition 2.19,

$$S^{\mu}_{K_0} = K_0 e_{t_1} \oplus \cdots \oplus K_0 e_{t_m}.$$

Consider the RS_n-submodule $X = Re_{t_1} \oplus \cdots \oplus Re_{t_m}$. Then clearly X is a RS_n-lattice of $S^{\mu}_{K_0}$,

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and its reduction mod $\mathfrak{m}$ is

$$X/\mathfrak{m}X \cong_{K_pS_n} K_p \otimes_R X \cong_{K_pS_n} K_p e_{t_1} \oplus \cdots \oplus K_p e_{t_m} = S^{\mu}_{K_p}.$$

So $S^{\mu}_{K_p}$ is a reduction mod $\mathfrak{m}$ of $S^{\mu}_{K_0}$.

□

Proposition 3.4.

(1) If λ is a p-regular partition of n, then the character of $S^{\lambda}_{K_0}$ has the (L'', p)-property.

(2) Let μ be a partition of n. If the character of $S^{\mu}_{K_0}$ has the (L', p)-property, then so does $S^{\mu'}_{K_0}$.

Remark.

By revising the proof of the proposition 3.4 (2), it is easy to see we can replace the term "(L', p)-property" to "(L'', p)-property" in (2).

Proof.

(1) Let χ be the character of $S^{\lambda}_{K_0}$, and let ϕ_μ be the p-modular character of $D^{\mu}_{K_p}$ for any p-regular partition μ of n. Then by proposition 0.10 and lemma 3.3, $\chi|_{S_n^{(p)}}$ is the p-modular character of $S^{\mu}_{K_p}$. Now by proposition 2.24, since λ is p-regular, $D^{\lambda}_{K_p}$ occurs precisely once in the composition factors of $S^{\lambda}_{K_p}$, and the others (if exists) have the form $D^{\mu}_{K_p}$ with $\mu \rhd \lambda$. Thus

$$\chi|_{S_n^{(p)}} \geq \phi_\lambda \text{ with multiplicity } 1,$$

i.e. χ has the (L'', p)-property.

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(2) Let χ_λ, $\chi_{\lambda'}$, and χ_1 be characters of $S^{\lambda}_{K_0}$, $S^{\lambda'}_{K_0}$, and $S^{(1^{(n)})}_{K_0}$ respectively. Then by proposition 0.10 and lemma 3.3, $\chi_\lambda|_{S_n^{(p)}}$, $\chi_{\lambda'}|_{S_n^{(p)}}$, and $\chi_1|_{S_n^{(p)}}$ are p-modualr characters of $S^{\lambda}_{K_p}$, $S^{\lambda'}_{K_p}$, and $S^{(1^{(n)})}_{K_p}$ respectively. Moreover, let ϕ^* be the p-modualr character of $(S^{\lambda'}_{K_p})^*$. Then by lemma 2.21, since every simple K_pS_n-modules are self dual, all composition factors of $S^{\lambda'}_{K_p}$ and $(S^{\lambda'}_{K_p})^*$ are equal. So by proposition 0.7, we have

$$\chi_{\lambda'}|_{S_n^{(p)}} = \phi^*.$$

On the other hand, since $(S^{\lambda'}_{K_p})^* \cong_{K_pS_n} S^{\lambda}_{K_p} \otimes_{K_p} S^{(1^{(n)})}_{K_p}$ by proposition 2.25, we obtain $\phi^* = \chi_\lambda|_{S_n^{(p)}} \times \chi_1|_{S_n^{(p)}}$ by lemma 3.2. So

$$\chi_{\lambda'}|_{S_n^{(p)}} = \phi^* = \chi_\lambda|_{S_n^{(p)}} \times \chi_1|_{S_n^{(p)}}.$$

Now by the assumption, since the character of $S^{\lambda}_{K_0}$ has the (L', p)-property,

$$\text{either } \chi_\lambda|_{S_n^{(p)}} = \phi_0 \text{ or } a_1\phi_1 + \cdots + a_z\phi_z$$

for some $z \in \mathbb{N}$, $z > 1$, $a_1, \cdots, a_z \in \mathbb{N}$, and distinct irreducble p-modular characters ϕ_i of S_n. So

$$\text{either } \chi_{\lambda'}|_{S_n^{(p)}} = \phi_0 \times \chi_1|_{S_n^{(p)}} \text{ or } a_1\left(\phi_1 \times \chi_1|_{S_n^{(p)}}\right) + \cdots + a_z\left(\phi_z \times \chi_1|_{S_n^{(p)}}\right).$$

Clearly, each $\phi_1 \times \chi_1|_{S_n^{(p)}}, \cdots, \phi_z \times \chi_1|_{S_n^{(p)}}$ is distinct. So to show $\chi_{\lambda'}$ has the (L', p)-property, it suffices to show $\phi_i \times \chi_1|_{S_n^{(p)}}$ are irreducible p-modular characters for all i.

To show it, observe that since $S^{(1^{(n)})}_{K_p} \cong_{K_pS_n} D^{(1^{(n)})}_{K_p}$, $\chi_1|_{S_n^{(p)}}$ is the p-modular character of $D^{(1^{(n)})}_{K_p}$. Now let D_i be a simple K_pG-module which affords ϕ_i. Then by lemma 3.2,

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the p-modular character of $D_i \otimes_{K_p} D_{K_p}^{(1^{(n)})}$ is

$$\phi_i \times \chi_1|_{S_n^{(p)}}.$$

Since the dimension of $D_{K_p}^{(1^{(n)})}$ is 1, the K_pS_n-module $D_i \otimes_{K_p} D_{K_p}^{(1^{(n)})}$ is simple by lemma 3.1, and hence $\phi_i \times \chi_1|_{S_n^{(p)}}$ is an irreducible p-modular character as we desired.

□

Proposition 3.5.

Let M be a simple K_0G-module and let X be a RG-lattice. If the dimension of M is divisible by the largest power of p dividing the order of G, then $X/\mathfrak{m}X$ is a simple K_pG-module.

Proof.

Cf., for example, [S1] proposition 46, page 136.

□

Example 3.6.

We are going to show that S_5 has the (L', p)-property. First observe that all partitions of S_5 are:

$$(5), \quad (4,1), \quad (3,2), \quad (3,1,1),$$
$$(1,1,1,1,1), \quad (2,1,1,1), \quad (2,2,1).$$

For the case $p > 2$, since (5), $(4,1)$, $(3,2)$, and $(3,1,1)$ are p-regular, by proposition 3.4, we conclude S_5 has the (L', p)-property for $p > 2$.

For the case $p = 2$, observe that all 2-regular partitions are

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$$(5),\ (4,1),\ (3,2),$$

and their conjugate partitions are

$$(1,1,1,1,1),\ (2,1,1,1),\ (2,2,1)$$

respectively. So again by proposition 3.4, all characters of their corresponding Specht modules have the $(L',2)$-property. So it remains to show the character of $S_{K_0}^{(3,1,1)}$ has the $(L',2)$-property.

To show this, we want to find dimensions of $S_{K_0}^{(3,1,1)}$, $D_{K_2}^{(5)}$, $D_{K_2}^{(4,1)}$, and $D_{K_2}^{(3,2)}$. Observe that the number of all standard $(3,1,1)$-tableaux is $\dfrac{4!}{2!\,2!} = 6$. So by proposition 2.19,

$$\dim_{K_0}(S_{K_0}^{(3,1,1)}) = 6.$$

On the other hand, since $D_{K_2}^{(5)}$ is the trivial K_2S_5-module, its dimension is 1, and by using proposition 2.22, we can calculate the dimensions of $D_{K_2}^{(4,1)}$ and $D_{K_2}^{(3,2)}$. The conclusion is that

$$\dim_{K_2}(D_{K_2}^{(4,1)}) = 4 \text{ and } \dim_{K_2}(D_{K_2}^{(3,2)}) = 4.$$

Now we show the character of $S_{K_0}^{(3,1,1)}$ has the $(L',2)$-property. Let χ be the character of $S_{K_0}^{(3,1,1)}$, and let $\phi^{(5)}$, $\phi^{(4,1)}$, $\phi^{(3,2)}$ be 2-modular characters of $D_{K_2}^{(5)}$, $D_{K_2}^{(4,1)}$, $D_{K_2}^{(3,2)}$ respectively. Then we can write

$$\chi|_{S_5^{(2)}} = a\phi^{(5)} + b\phi^{(4,1)} + c\phi^{(3,2)}$$

for some a, b, $c \in \mathbb{N} \cup \{0\}$. Let e be the identity of S_5. Then since

$$\dim_{K_0}(S_{K_0}^{(3,1,1)}) = 6,\ \dim_{K_2}(D_{K_2}^{(5)}) = 1,\ \dim_{K_2}(D_{K_2}^{(4,1)}) = 4, \text{ and } \dim_{K_2}(D_{K_2}^{(3,2)}) = 4,$$

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we have

$$6 = \chi|_{S_5^{(2)}}(e) = a\phi^{(5)}(e) + b\phi^{(4,1)}(e) + c\phi^{(3,2)}(e) = a + 4b + 4c.$$

Moreover, since a, b, $c \in \mathbb{N} \cup \{0\}$, the equation $6 = a + 4b + 4c$ implies that

$$a = 2 \text{ or } 6.$$

Therefore, to show the character χ of $S_{K_0}^{(3,1,1)}$ has the $(L',2)$-property, it suffices to show $a \neq 6$.

Assume $a = 6$. Denote $g = (123)(4)(5) \in S_5$, then

$$\chi(g) = \chi|_{S_5^{(2)}}(g) = 6\phi^{(5)}(g) = 6.$$

Let $r_1, \cdots, r_6 \in K_0$ be all eigenvalues of the matrix representation ρ_g of g with respect to any fixed basis of $S_{K_0}^{(3,1,1)}$. Then since ρ_g^3 is the identity matrix, every r_i are roots of unity in K_0. Since $\chi(g) = 6$, we have

$$r_1 + r_2 + \cdots + r_6 = 6,$$

and hence $r_i = 1$ for all i. This means ρ_g is the identity matrix, i.e. $g \cdot e_t = e_t$ for all $(3,1,1)$-tableaux t. But this is impossible clearly. Therefore $a \neq 6$, and hence χ has the $(L',2)$-property.

We therefore conclude that S_5 has the $(L',2)$-property, and hence S_5 has the (L',p)-property for all prime p.

Remark.

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In fact, this method can show that S_5 has the (L'', p)-property. Indeed, we can use proposition 3.4 to show that all irreducible characters of S_5 have (L'', p)-property for all p, except the case that the irreducible character χ of $S_{K_0}^{(3,1,1)}$ has $(L'', 2)$-property. However, if we use same notations and the same method as in the above example, we will obtain $6 = \chi|_{S_5^{(2)}}(e) = a + 4b + 4c$, and we have shown that $a = 2$. This means $b = 1$ or $c = 1$, and hence χ has $(L'', 2)$-property.

Example 3.7.

We are going to show that S_6 has the (L', p)-property. Observe that all partitions of S_6 are

$$\begin{array}{llllll} (6), & (5,1), & (4,2), & (4,1,1), & (3,3), & (3,2,1), \\ (1,1,1,1,1,1), & (2,1,1,1,1), & (2,2,1,1), & (3,1,1,1), & (2,2,2). & \end{array}$$

For the case $p > 2$, since (6), $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,2,1)$ are p-regular, we conclude that S_6 has (L', p)-property for $p > 2$ by proposition 3.4. It is the same method as in example 3.6.

For the case $p = 2$, observe that all 2-regular partitions are

$$(6), (5,1), (4,2), (3,2,1),$$

and their conjugate partitions are

$$(1,1,1,1,1,1), (2,1,1,1,1), (2,2,1,1), (3,2,1)$$

respectively. Again by proposition 3.4, all characters of their corresponding Specht modules have the $(L', 2)$-property. To show the other characters have the $(L', 2)$-property, by corollary 3.4 (2), it suffices to show characters of $S_{K_0}^{(4,1,1)}$ and $S_{K_0}^{(3,3)}$ have the $(L', 2)$-property.

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Again, we calculate dimensions of $S^{(4,1,1)}_{K_0}$, $S^{(3,3)}_{K_0}$, and all simple K_pS_6-modules. Observe that the number of all standard $(4,1,1)$-tableaux is $\dfrac{5!}{3!\,2!} = 10$, and all standard $(3,3)$-tableaux is $\dfrac{4!}{2!\,2!} - 1 = 5$. So by proposition 2.19,

$$\dim_{K_0}(S^{(4,1,1)}_{K_0}) = 10 \text{ and } \dim_{K_0}(S^{(3,3)}_{K_0}) = 5.$$

On the other hand, since the dimension of $D^{(6)}_{K_2}$ is 1 since it is the trivial K_2S_5-module, and by using proposition 2.22, we can calculate dimensios of $D^{(5,1)}_{K_2}$ and $D^{(4,2)}_{K_2}$. The conclusion is that

$$\dim_{K_2}(D^{(5,1)}_{K_2}) = 4 \text{ and } \dim_{K_2}(D^{(5,1)}_{K_2}) = 4.$$

To see the dimension of $D^{(3,2,1)}_{K_2}$, of course we can use proposition 2.22 to calculate. However, it is a little complicate. So we want to use another way to find its dimension. Observe that the number of all standard $(3,2,1)$-tableaux is 16. So the dimension of $S^{(3,2,1)}_{K_0}$ is 16. Since $|S_6| = 16 \times 45$, by proposition 3.5, any reduction mod $\mathfrak{m}$ of $S^{(3,2,1)}_{K_0}$ is a simple K_2S_6-module. Moreover, by lemma 3.3, we know $S^{(3,2,1)}_{K_2}$ is a reduction mod $\mathfrak{m}$ of $S^{(3,2,1)}_{K_0}$. So by proposition 2.24 (2), we obtain $S^{(3,2,1)}_{K_2} \cong_{K_2S_6} D^{(3,2,1)}_{K_2}$, and hence

$$\dim_{K_2}(D^{(3,2,1)}_{K_2}) = 16.$$

Now we show that characters of $S^{(4,1,1)}_{K_0}$ and $S^{(3,3)}_{K_0}$ have the $(L',2)$-property. Let $\chi^{(4,1,1)}$, $\chi^{(3,3)}$ be characters of $S^{(4,1,1)}_{K_0}$, $S^{(3,3)}_{K_0}$ respectively, and let $\phi^{(6)}$, $\phi^{(5,1)}$, $\phi^{(4,2)}$, $\phi^{(3,2,1)}$ be 2-modular characters of $D^{(6)}_{K_2}$, $D^{(5,1)}_{K_2}$, $D^{(4,2)}_{K_2}$, $D^{(3,2,1)}_{K_2}$ respectively. Then we

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can write

$$\chi^{(4,1,1)}|_{S_6^{(2)}} = a_1\phi^{(6)} + a_2\phi^{(5,1)} + a_3\phi^{(4,2)} + a_4\phi^{(3,2,1)}$$
$$\chi^{(3,3)}|_{S_6^{(2)}} = b_1\phi^{(6)} + b_2\phi^{(5,1)} + b_3\phi^{(4,2)} + b_4\phi^{(3,2,1)}$$

for some a_i, $b_i \in \mathbb{N}\cup\{0\}$. Let e be the identity of S_5. Then since

$$\dim_{K_0}(S_{K_0}^{(4,1,1)}) = 10, \qquad \dim_{K_0}(S_{K_0}^{(3,3)}) = 5, \qquad \dim_{K_2}(D_{K_2}^{(6)}) = 1,$$
$$\dim_{K_2}(D_{K_2}^{(5,1)}) = 4, \qquad \dim_{K_2}(D_{K_2}^{(4,2)}) = 4, \qquad \dim_{K_2}(D_{K_2}^{(3,2,1)}) = 16,$$

we have

$$\begin{aligned} 10 = \chi^{(4,1,1)}|_{S_6^{(2)}}(e) &= a_1\phi^{(6)}(e) + a_2\phi^{(5,1)}(e) + a_3\phi^{(4,2)}(e) + a_4\phi^{(3,2,1)}(e) \\ &= a_1 + 4a_2 + 4a_3 + 16a_4, \end{aligned}$$

and

$$\begin{aligned} 5 = \chi^{(3,3)}|_{S_6^{(2)}}(e) &= b_1\phi^{(6)}(e) + b_2\phi^{(5,1)}(e) + b_3\phi^{(4,2)}(e) + b_4\phi^{(3,2,1)}(e) \\ &= b_1 + 4b_2 + 4b_3 + 16b_4. \end{aligned}$$

Since a_i, $b_i \in \mathbb{N}\cup\{0\}$, the above equations imply that

$$a_1 = 2,\ 6, \text{ or } 10, \text{ and } b_1 = 1 \text{ or } 5$$

Therefore, to show $\chi^{(4,1,1)}$ and $\chi^{(3,3)}$ have the $(L',2)$-property, it suffices to show $a_1 \neq 10$ and $b_1 \neq 5$.

Assume $a_1 = 10$. Denote $g = (123)(4)(5)(6) \in S_6$. Then $\chi^{(4,1,1)}(g) = 10$. Similary as

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in example 3.6, $\chi^{(4,1,1)}(g) = 10$ will imply that $g \cdot e_t = e_t$ for all $(4,1,1)$-tableaux t. But clearly this is impossible. So $a_1 \neq 10$. Using the same method, we can also conclude that $b_1 \neq 5$. Hence characters of $S_{K_0}^{(4,1,1)}$ and $S_{K_0}^{(3,3)}$ have the $(L',2)$-property.

We therefore concldue that S_6 has the $(L',2)$-property, and hence S_6 has the (L',p)-property for all prime p.

Remark.

Let λ be a partition of n such that it is not 2-regular or their conjugate partitions. Let χ be an irreducible character of $S_{K_0}^{\lambda}$, and ϕ be the trivial 2-modular character of S_n. In the above two examples and in the case $p = 2$, we calculate the dimension of $S_{K_0}^{\lambda}$ and dimension of all distinct simple K_2S_n-modules, and using them to show $\chi|_{S_n^{(2)}} \geq \phi$. Then by a little work, we conclude that

$$\chi|_{S_n^{(2)}} \neq \chi(e)\phi,$$

where e is the identity of S_n, and this implies that χ has the $(L',2)$-property.

However, this method does not always work. For example, consider $n = 7$. The dimension of $S_{K_0}^{(4,1,1,1)}$ is 20, and dimensions of all distinct 2-modular characters of S_7 are 1, 6, 8, 14, 20. The equation

$$20 = a_1 + 6a_2 + 8a_3 + 14a_4 + 20a_5,\ a_i \in \mathbb{N} \cup \{0\}$$

can not guarantee that there exists i such that $a_i > 0$.

Even though the method does not always work, it induce we think about a question, that is, for any character χ of G, can we find a p-modular character ϕ of G such that either $\chi|_{G^{(p)}} = \phi$ if $\chi(e) = \phi(e)$, or $\chi|_{G^{(p)}} \geq \phi$ with multiplicity smaller than $\chi(e)/\phi(e)$ if $\chi(e) \neq \phi(e)$? In theorem 3.17, we will see that for any prime p and for any character χ of

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S_n, there always exists a p-modular character ϕ of G such that $\chi|_{S_n^{(p)}} \geq \phi$ with multiplicity 1.

3.2 Diagrams

In this subsection, we will introduce some tools which will be used to show theorem 3.17, that is, for any prime p and for any irreducible character χ of S_n, there always exists an irreducible p-modular character ϕ of G such that $\chi|_{S_n^{(p)}} \geq \phi$ with multiplicity 1.

Definition 3.8.

(1) Consider a fixed origin, and a first axis pointing south and a second axis pointing east. These axes construct a coordinate system, and we define vertices to be elements of $\{(i,j) \mid i,j \in \mathbb{N}\}$.

We say a vertex (i,j) is higher than (k,l) if $i<k$. Similary define "lower than", "to the right of", and "to the left of".

(2) We say $\mathfrak{D}$ is a diagram (for S_n) if $\mathfrak{D}$ is a set which collects n vertices such that if $(i,j) \in \mathfrak{D}$, then $(i,j-1)$, $(i-1,j) \in D$. The vertices which belong to $\mathfrak{D}$ is called the nodes of $\mathfrak{D}$.

Let $\mu = (\mu_1, \cdots, \mu_m)$ be a partition of n and denote

$$\mathfrak{D}_\mu = \{(i,j) : \text{vertex} \mid 1 \leq i \leq m,\ 1 \leq j \leq \mu_i\}.$$

Then it is easy to see $\mathfrak{D}_\mu$ is a diagram, and for any diagram $\mathfrak{D}$, there is an unique partition μ such that $\mathfrak{D} = \mathfrak{D}_\mu$. In this case, we say the diagram $\mathfrak{D}$ corresponds to μ, or μ corresponds to $\mathfrak{D}$.

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Let a be a fixed number such that the set $\{(a,j) \mid \text{if } (a,j) \in \mathfrak{D}\}$ is not empty. Then the set is called the a-th row of $\mathfrak{D}$, and its cardinality is called the length of the a-th row.

Let a be a fixed number such that the set $\{(j,a) \mid \text{if } (j,a) \in \mathfrak{D}\}$ is not empty. Then the set is called the a-th column of $\mathfrak{D}$, and its cardinality is called the length of the a-th column.

We say a diagram is p-regular if no p rows of it have the same length; otherwise the diagram is called p-singular.

For two diagrams $\mathfrak{D}_1$, $\mathfrak{D}_2$, we say $\mathfrak{D}_1 \trianglerighteq \mathfrak{D}_2$ if for every j,

$$\sum_{i \le j} (\text{length of the } i\text{-th row of } \mathfrak{D}_1) \ge \sum_{i \le j} (\text{length of the } i\text{-th row of } \mathfrak{D}_2)$$

We say $\mathfrak{D}_1 \triangleright \mathfrak{D}_2$ if $\mathfrak{D}_1 \trianglerighteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \neq \mathfrak{D}_2$.

(3) If $\mathfrak{D}$ is a diagram for S_n which corresponds to the partition μ of n, then we denote $\chi_{\mathfrak{D}}$ as the character of $S^{\mu}_{K_0}$.

If $\mathfrak{D}$ is a p-regular diagram for S_n which corresponding to the p-regular partition μ of n, then we denote $\phi_{\mathfrak{D}}$ as the p-modular character of $D^{\mu}_{K_p}$.

Definition 3.9.

(1) A ladder is a straight line joining the vertex $(i,1)$ to the point $(1, \dfrac{i-1}{p-1}+1)$. The vertices which a ladder passes through will be called the rungs of the ladder.

Note that the rungs of a ladder are $(i,1)$, $(i-(p-1),2)$, $(i-2(p-1),3)$, and so on. For example, let $p=3$ and consider the ladder passing through $(6,1)$. Then the rungs of the ladder are $(6,1)$, $(4,2)$, $(2,3)$.

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(2) A subset of the rungs of a ladder is called a complete k subset if it consists of the top k rungs of the ladder. We say a vertex x is the 1-st rung of a ladder l if x belongs to the complete 1 subset, and x is the k-th ($k > 1$) rung of l if x belongs to the complete k subset of l but not in the complete $k-1$ subset of l.

For example, let $p = 2$ and consider the ladder passing through $(3,1)$, $(2,2)$, $(1,3)$. Then

$$\begin{aligned} \{(1,3)\} &\text{ is the complete 1 subset,} \\ \{(2,2),(1,3)\} &\text{ is the complete 2 subset,} \\ \{(3,1),(2,2),(1,3)\} &\text{ is the complete 3 subset, and} \\ \{(3,1),(2,2)\} &\text{ is not a complete } k \text{ subset for any } k \in \mathbb{N}. \end{aligned}$$

The vertex $(1,3)$ is the 1-st rung of l, $(2,2)$ is the 2-nd, and $(3,1)$ is the 3-rd.

(3) Suppose we have p colors, which we shall call $0,1,\cdots,p-1$, Color the vertices by letting (i,j) have the color which is the smallest non-negative residue of $j-i$ mod p.

We say two diagrams $\mathfrak{D}_1$, $\mathfrak{D}_2$ belong to the same block if and only if they have the same color content, that is, for every i, the number of nodes of $\mathfrak{D}_1$ colored i is equal to the number of nodes of $\mathfrak{D}_2$ colored i.

Proposition 3.10.

(1) All the rungs of a ladder have the same color.

(2) A diagram $\mathfrak{D}$ is p-regular if and only if the following happens for each ladder l:

If the k-th rung of l belongs to $\mathfrak{D}$, then so does the $k-1$-th rung of l.

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(3) Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two diagrams corresponding to partitions μ_1 and μ_2 respectively. Then $\mathfrak{D}_1$ and $\mathfrak{D}_2$ belong to the same block if and only if χ_{μ_1} and χ_{μ_2} lies in the same p-block, that is, there exists an irreducible p-modular character ϕ such that

$$\chi_{\mu_1}|_{S_n^{(p)}} \geq \phi \text{ and } \chi_{\mu_2}|_{S_n^{(p)}} \geq \phi.$$

Proof.

(1) let l be a ladder and let (a,b) be a rung of l. Then other rungs of l can be written as the form $(a-c(p-1), b+c)$ for some integer c. Clearly, $(a-c(p-1), b+c)$ and (a,b) have the same color.

(2) Assume $\mathfrak{D}$ is p-singular. Then there are p rows of $\mathfrak{D}$, say $a+1$-th, $a+2$-th,$\cdots$,$a+p$-th rows, such that they have the same length, say m. So

$$(a+p,m) \in \mathfrak{D} \text{ and } (a+1,m+1) \notin \mathfrak{D}.$$

If we say $(a+p,m)$ is the k-th rung of a ladder l, then the $k-1$-th rung of l is $(a+p-(p-1), m+1) = (a+1, m+1)$. So the k-th rung of l belong to $\mathfrak{D}$, but the $k-1$-th does not.

Assume $\mathfrak{D}$ is p-regular. Let l be a ladder and its k-th rung ($k>1$) is (i,j). Then the $k-1$-th rung of l is $(i-(p-1), j+1)$. Let m be the length of the i-th row. Then since $\mathfrak{D}$ is p-regular, the number of rows with length m is $<p$. So the length of the $i-(p-1)$-th row must be $\geq m+1$. Hence $(i-(p-1), j+1) \in \mathfrak{D}$, i.e. the $k-1$-th rung of l belongs to $\mathfrak{D}$.

(3) To show this, we have to introduce "p-hook" and "p-core". A p-hook of a diagram $\mathfrak{D}$ is a connected part with p-nodes of the rim of $\mathfrak{D}$ beginning with the last node of

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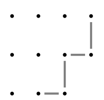
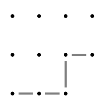
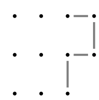
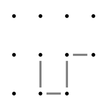
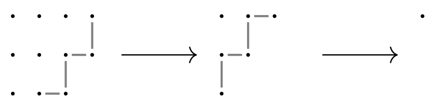
any row and ending with the last node of an earlier column, and these nodes can be removed to leave a diagram. For example, consider following four diagrams. In (a) and (b), the nodes which are connected by lines can be 5-hooks of $\mathfrak{D}_{(4,4,3)}$. In (c) and (d), the nodes which are connected by lines can not be 5-hooks of $\mathfrak{D}_{(4,4,3)}$

(a)  (b)  (c)  (d) 

Moreover, we say a diagram is a p-core of $\mathfrak{D}$ if we remove p-hooks in succession from $\mathfrak{D}$ as many as we can. For example,



The last diagram is the 5-core of $\mathfrak{D}_{(4,4,3)}$. Note that a p-core of a diagram is unique (cf., for example, [R] 4.53, page 85).

Now to see (3), we have to introduce two theorems:

Theorem. (Nakayama's Conjecture)

Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be two diagrams for S_n. Then $\chi_{\mathfrak{D}_1}$, $\chi_{\mathfrak{D}_2}$ are in the same p-block if and only if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ have the same p-core.

Proof. Cf., for example, [R], 5.36, page 98.

Theorem.

The p-cores of two diagrams are equal if and only if they are in the same block.

Proof. Cf., for example, [R], 5.42, page 99.

So we can see that (3) holds by this two theorem.

□

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Definition 3.11.

Let $\mathfrak{D}$ be a diragram. Construct a new set $\mathfrak{D}^r$ from $\mathfrak{D}$ as follows. For each ladder l, if l hits $\mathfrak{D}$ in k nodes, replace these nodes by the complete k subset of l.

For example, let $p = 3$.

$$\text{If } \mathfrak{D} = \begin{array}{lll} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}, \text{ then } \mathfrak{D}^r = \begin{array}{lll} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & & \end{array}.$$

Proposition 3.12.

(1) $\mathfrak{D}^r$ is a p-regular diagram.

(2) $\mathfrak{D}$ and $\mathfrak{D}^r$ belong to the same block.

Proof.

(1) We are going to show $\mathfrak{D}^r$ is a diagram. Assume $x = (i, j) \in \mathfrak{D}^r$. Let l be the ladder in which x lies, and say x is the k-th rung of l. First we claim that $(i-1, j)$ (if $i > 1$) belong to $\mathfrak{D}^r$. Let l_1 be the ladder in which $(i-1, j)$ lie. Observe that the 1-st rung of l is

$$(i-(k-1)(p-1), j+(k-1)).$$

Assume

$$i-(k-1)(p-1) = 1.$$

Then the 1-st rung of l_1 is

$$(i-(k-2)(p-1)-1, j+(k-2)).$$

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So $(i-1,j)$ is the $k-1$-th rung of l_1. Therefore, to show $(i-1,j)\in\mathfrak{D}^r$ in this case, we have to find $k-1$ nodes of $\mathfrak{D}$ lie in l_1.

Since x is the k-th rung of l, we can find k rungs of l belong to $\mathfrak{D}$, say

$$(a,b),\ (a-c_2(p-1),b+c_2),\cdots,(a-c_k(p-1),b+c_k)$$

with $c_i\in\mathbb{N}$ and $c_2<\cdots<c_k$. Note that for convinence, we denote $c_1=0$. So for any $z=1,\cdots,k-1$,

$$a-c_z(p-1)>a-c_k(p-1)\geq i-(k-1)(p-1)=1.$$

Now since $\mathfrak{D}$ is a diagram, by the above inequality, we can guarantee that

$$(a-1,b),\ (a-c_2(p-1)-1,b+c_2),\cdots,(a-c_{k-1}(p-1)-1,b+c_{k-1})\in\mathfrak{D},$$

and since these nodes lie in l_1, we conclude that l_1 hits $\mathfrak{D}$ in at least $k-1$ nodes, and hence $(i-1,j)\in\mathfrak{D}^r$.

On the other hand, assume

$$i-(k-1)(p-1)>1.$$

Then the 1-st rung of l_1 is

$$(i-(k-1)(p-1)-1,j+(k-1)).$$

So $(i-1,j)$ is the k-th rung of l_1. Hence in this case, we have to find k nodes of $\mathfrak{D}$ lie

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in l_1. Observe that for any $z = 1, \cdots, k$,

$$a - c_z(p-1) \geq i - (k-1)(p-1) > 1.$$

So we can guarantee that

$$(a-1,b),\ (a-c_2(p-1)-1, b+c_2), \cdots, (a-c_k(p-1)-1, b+c_k) \in \mathfrak{D},$$

and hence l_1 hits $\mathfrak{D}$ in at least k nodes, i.e. $(i-1, j) \in \mathfrak{D}^r$.

Next we claim that $(i, j-1)$ (if $j > 1$) belong to $\mathfrak{D}^r$. Let l_2 be the ladder in which $(i, j-1)$ lies. Since

$$(i-(k-1)(p-1), j+(k-1))$$

is the 1-st rung of l, the 1-st rung of l_2 is

$$(i-(k-1)(p-1), j+(k-1)-1).$$

So $(i, j-1)$ is the k-th rung of l_2. Hence to show $(i, j-1) \in \mathfrak{D}^r$, we have to find k nodes of $\mathfrak{D}$ lies in l_2.

Assume $b > 1$. Then since

$$(a, b-1),\ (a-c_2(p-1), b+c_2-1), \cdots, (a-c_k(p-1), b+c_k-1) \in \mathfrak{D}$$

and they are all lie in l_2, we can see that l_2 hits $\mathfrak{D}$ in at least k nodes, i.e. $(i, j-1) \in \mathfrak{D}^r$.

Assume $b = 1$. Observe that since $1 + c_z \leq c_k$ for all $z = 1, \cdots, k-1$, we can guarantee

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that the following $k-1$ rungs of l_2 exist and belong to $\mathfrak{D}$:

$$(a-(p-1),b),\ (a-(1+c_2)(p-1),b+c_2),\cdots,(a-(1+c_{k-1})(p-1),b+c_{k-1}).$$

If there is an i between 2 and k such that $1+c_{i-1}<c_i$, then the rung of l_2

$$(a-c_i(p-1),(b+c_i)-1)\in\mathfrak{D}$$

is distinct to the above $k-1$ rungs. So l_2 hits $\mathfrak{D}$ in at least k nodes, i.e. $(i,j-1)\in\mathfrak{D}^r$. If we can not find such i, then it means

$$1+c_{i-1}=c_i \text{ for all } i=2,\cdots,k.$$

Note that if $(a-c_k(p-1),b+c_k)$ is not the 1-st rung of l, then the rung of l_2

$$(a-(1+c_k)(p-1),b+c_k)$$

exists and belongs to $\mathfrak{D}$. Hence l_2 hits $\mathfrak{D}$ in at least k nodes, i.e. $(i,j-1)\in\mathfrak{D}^r$. So it remains to consider the case

$$1+c_{i-1}=c_i \text{ for all } i=2,\cdots,k \text{ and } (a-c_k(p-1),b+c_k) \text{ is the 1-st rung of } l.$$

However, in this case, we have $c_2=1,\ c_3=2,\cdots,c_k=k-1$. So the 1-st rung of l is

$$(a-c_k(p-1),b+c_k)=(a-(k-1)(p-1),1+(k-1))=(a-(k-1)(p-1),k),$$

and hence the k-th rung of l is $(a,1)$, which means $j=1$, which is impossible. Now

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we complete the proof of $\mathfrak{D}^r$ is a diagram.

Finally, to see $\mathfrak{D}^r$ is p-regular, it is clearly by proposition 3.10 (2).

(2) It is clearly by proposition 3.10 (1).

□

Definition 3.13.

(1) Let $\mathfrak{D}$ be a diagram for S_n and $x \in \mathfrak{D}$. If $\mathfrak{D} \setminus \{x\}$ is a diagram for S_{n-1}, then x is called a removable node of $\mathfrak{D}$, and write $\mathfrak{D} - x$ for $\mathfrak{D} \setminus \{x\}$. The node x is called regular-removable if $\mathfrak{D} - x$ is a p-regular diagram for S_{n-1}.

(2) The node x of $\mathfrak{D}$ is called a shadow node if x is regular-removable and no node higher than or equal to x can be raised, retaining its color (the term "raised" means if y is removable in $\mathfrak{D}$ and there is a vertex z which is higher than y and it can be added to $\mathfrak{D} - y$ to give a diagram, then we say y is raised to z).

We say $\mathfrak{D} - x$ is a shadow of $\mathfrak{D}$ if x is a shadow node of $\mathfrak{D}$.

Proposition 3.14.

(1) A diagram $\mathfrak{D}$ has a shadow node if and only if $\mathfrak{D}$ is p-regular.

(2) Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are p-regular diagrams in the same block and x, y are shadow nodes of $\mathfrak{D}_1$, $\mathfrak{D}_2$ respectively, If $\mathfrak{D}_1$, $\mathfrak{D}_2$ are distinct, then $\mathfrak{D}_1 - x \neq \mathfrak{D}_2 - y$.

Proof.

(1) Assume $\mathfrak{D}$ is p-regular. Consider the longest ladder l which hits $\mathfrak{D}$. Then since no nodes of $\mathfrak{D}$ are to the right of this ladder or lower than this ladder, all rungs of l in $\mathfrak{D}$ are removable. Let x be the lowest rung of l in $\mathfrak{D}$. Denote m as the length of the row

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in which x lies. Since x is the lowest rung of l, the number of rows of $\mathfrak{D}$ with length $= m-1$ is $< p-1$. So the number of rows of $\mathfrak{D}-x$ with length $= m-1$ is $< p$. Therefore $\mathfrak{D}-x$ is p-regular since $\mathfrak{D}$ is p-regular, and hence x is regular-removable. Now we claim that x is a shadow node of $\mathfrak{D}$.

Observe that all removable nodes higher than or equal to x are rungs of l higher than or equal to x, and so have the same color as x. Let y be one rung of them and we want to raise y. Then positions which y can be raised to are only such vertices which higher than y and below a rung of l, or to the right of the most right node in the first row of $\mathfrak{D}$. If we raise y to z which below a rung t of l, then z can not have the same color as y since the color of t is the same as y. In the other case, we raise y to z which is to the right of the most right node in the first row of $\mathfrak{D}$. Denote (i,j) as the highest rung of l. Then no any node in $\mathfrak{D}$ can righter than (i,j), and since $\mathfrak{D}$ is p-regular, we have $i<p$. Since all rungs of l have the same color, the color of y is $j-i$ (mod p), which is not equal to j (mod p) because $i<p$. Now since no any node in $\mathfrak{D}$ can righter than (i,j), the coordinate of z is $(1,j+1)$, and hence its color is j (mod p), which is different than y. Hence y can not be raised, rataining its color, i.e. x is a shadow node of $\mathfrak{D}$.

Assume $\mathfrak{D}$ is p-singular, and x is a regular-removable node of $\mathfrak{D}$. Let (i,j) be the coordinate of x. Then it means x lies in the i-th row of $\mathfrak{D}$ whose length is j. Since $\mathfrak{D}$ is p-singular and x is regular-removable, the number of rows of $\mathfrak{D}$ with length j is p. So $i \geq p$ and lengths of i-th, $i-1$-th,$\cdots$, $i-(p-1)$-th rows are all j. If $i=p$, then x can be raised to $(1,j+1)=(i-(p-1),j+1)$. If $i>p$, then since the number of rows of $\mathfrak{D}$ with length j is p, the length of the $i-p$-th row is $>j$, and hence x can be raised to $(i-(p-1),j+1)$. In both cases, we all conclude that x can be raised

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to $(i-(p-1), j+1)$, whose color is $j-i \pmod p$. So x can be raised, retaining its color, i.e. x is not a shadow node of $\mathfrak{D}$.

(2) If $\mathfrak{D}_1 - x = \mathfrak{D}_2 - y$, then x, y have the same color since $\mathfrak{D}_1$, $\mathfrak{D}_2$ are in the same block. Since $\mathfrak{D}_1 \neq \mathfrak{D}_2$ and $\mathfrak{D}_1 - x = \mathfrak{D}_2 - y$, x can not as high as y. Without loss of generality, we may assume y is highter than x. Then it means $x \in \mathfrak{D}_1$ can be raised to y retaining its color, which is impossible since x is a shadow node. Therefore $\mathfrak{D}_1 - x \neq \mathfrak{D}_2 - y$.

□

Definition 3.15.

Let $\mathfrak{D}$ be a p-regular diagram, and let l be the longest ladder which hits $\mathfrak{D}$. Let x be the lowest rung of l in $\mathfrak{D}$. In the proof of proposition 3.14 (1), we see that x is a shadow node of $\mathfrak{D}$. The shadow node x is called the first shadow node of $\mathfrak{D}$, and $\mathfrak{D} - x$ is called the first shadow of $\mathfrak{D}$.

3.3 The L''-Property of S_n

In this subsection, we will see that S_n has the L''-property. but the following lemma should be introduced first. It will be used in the proof of 3.17.

Lemma 3.16. (The branching theorem)

Let μ be a partition of n, and let $x_1, \cdots, x_a$ be all removable nodes of $\mathfrak{D}_\mu$. For all i, denote λ_i as the partition of $n-1$ which corresponds to $\mathfrak{D}_\mu - x_i$. Then

$$\mathrm{Res}^{S_n}_{S_{n-1}}(S^{\mu}_{F_0}) \cong (S^{\lambda_1}_{F_0}) \oplus \cdots \oplus (S^{\lambda_a}_{F_0})$$

Theorem 3.17. (cf. [J2], theorem A)

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Let $\mathfrak{D}$ be a diagram for S_n. Then

$$\chi_{\mathfrak{D}}\big|_{S_n^{(p)}} \geq \phi_{\mathfrak{D}^r} \text{ with multiplicity 1.}$$

Moreover, if $\chi_{\mathfrak{D}}\big|_{S_n^{(p)}} \geq \phi$ for some irreducible p-modular character ϕ with $\phi \neq \phi_{\mathfrak{D}^r}$, then $\phi = \phi_D$ for some p-regular diagram $D \triangleright \mathfrak{D}^r$.

Proof.

For convenience, if E is a diagram for S_n and E_1 is a p-regular diagram for S_n, the notation

$$\chi_E\big|_{S_n^{(p)}} = e\phi_{E_1} + \cdots$$

menas $\chi_E\big|_{S_n^{(p)}} \geq \phi_{E_1}$ with multiplicity e and all its other constituents belong to

$$\{\phi_{E_2} \mid E_2 \triangleright E_1\}.$$

We will prove this theorem by induction. Assume that if E is a diagram for S_{n-1}, then

$$\chi_E\big|_{S_{n-1}^{(p)}} = \phi_{E^r} + \cdots .$$

We claim that it will also hold on an arbitrary diagram for S_n. Note that it holds trivially on S_1.

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Let $\mathfrak{D}$ be an arbitrary diagram for S_n. Let

$$X_{\mathfrak{D}} = \{x_1, \cdots, x_a\} \text{ be the set of removable nodes of } \mathfrak{D}^r,$$

$$Y_{\mathfrak{D}} = \{y_1, \cdots, y_b\} \text{ be the set of regular-removable nodes of } \mathfrak{D}^r,$$

$$Z_{\mathfrak{D}} = \{z_1, \cdots, z_c\} \text{ be the set of removable nodes of } \mathfrak{D}.$$

Put an equivalence relation $\sim$ on $X_{\mathfrak{D}}$ by $x_i \sim x_j$ if and only if they are in the same ladder. Then $Y_{\mathfrak{D}}$ provides $\sim$-class representatives (observe that each y_i must lie in distinct ladders since a node is regular-removable only if it is the lowest rung in $\mathfrak{D}$ of a ladder; on the other hand, if $\{x_{a_1}, \cdots x_{a_j}\}$ is a $\sim$-class, then they lie in the same ladder and the lowest x_{a_i} is regular-removable). Note that the diagrams $\mathfrak{D}^r - x_i$ are totally ordered by $\trianglerighteq$ (the higher x_i is, the smaller $\mathfrak{D}^r - x_i$ is by $\trianglerighteq$), so in particular, we may assume that

$$\mathfrak{D}^r - y_b \triangleright \mathfrak{D}^r - y_{b-1} \triangleright \cdots \triangleright \mathfrak{D}^r - y_1.$$

Also note that under this assumption, y_{i-1} is higher than y_i. Moreover, since $y_1, \cdots, y_b$ are in distinct ladders, if we denote l_i as the ladder in which y_i lies, then y_{i-1} is either lefter than l_i or righter than l_i. However, since $\mathfrak{D}^r$ is p-regular, y_{i-1} can not lefter than l_i. So y_{i-1} is righter than l_i, which means l_{i-1} is longer than l_i. Thus l_1 is the longest ladder between $l_1, \cdots, l_b$, and hence y_1 is the first shadow node of $\mathfrak{D}^r$. In this proof, we continuously adopt the notation l_i.

Denote $r(y_i)$ be the size of the $\sim$-class containing y_i. Then l_1 hits $\mathfrak{D}^r$ in $r(y_1)$ nodes, and so hits $\mathfrak{D}$ in $r(y_1)$ nodes. Since all ladders longer than l_1 miss $\mathfrak{D}^r$, and so miss $\mathfrak{D}$. So the rungs of l_1 in $\mathfrak{D}$ are removable nodes of $\mathfrak{D}$, and we may take them to be z_i for

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$i = 1, \cdots, r(y_1)$. Then for $i = 1, \cdots, r(y_1)$, we have

$$(\mathfrak{D} - z_i)^r = \mathfrak{D}^r - y_1$$

(because y_1 is the lowest rung in $\mathfrak{D}^r$ of l_1), and for $i > r(y_1)$, we have

$$(\mathfrak{D} - z_i)^r = \mathfrak{D}^r - y_j$$

for some $j > 1$ (Note that not every $\mathfrak{D}^r - y_i$ need turn up as a $(\mathfrak{D} - z_i)^r$. For example, $p = 3$, $\mathfrak{D} = \{(1,1),(2,1),(3,1)\}$ has 1 removable node, but $\mathfrak{D}^r = \{(1,1),(2,1),(1,2)\}$ has 2 regular-removable nodes.). So by induction hypothesis, we have

$$\chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} = \phi_{(\mathfrak{D}-z_i)^r} + \cdots = \phi_{\mathfrak{D}^r - y_1} + \cdots, \text{ if } i = 1, \cdots, r(y_1),$$
$$\chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} = \phi_{(\mathfrak{D}-z_i)^r} + \cdots = \phi_{\mathfrak{D}^r - y_j} + \cdots, \text{ if } i > r(y_1)\ (j \neq 1).$$

Now observe that by lemma 3.16, we have $\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}}) = \sum_i (\chi_{\mathfrak{D}-z_i})$. So

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}})|_{S_{n-1}^{(p)}} = \sum_i (\chi_{\mathfrak{D}-z_i})|_{S_{n-1}^{(p)}} = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots$$

(recall that $\mathfrak{D}^r - y_j \triangleright \mathfrak{D}^r - y_1$ for $j > 1$). Moreover, since $\mathfrak{D}^{rr} = \mathfrak{D}^r$ (so $Y_{\mathfrak{D}} = Y_{\mathfrak{D}^r}$) and $\mathfrak{D}$ is arbitrary, we also have

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}^r})|_{S_{n-1}^{(p)}} = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots.$$

On the other hand, if $x_i \sim y_i$, then we have $(\mathfrak{D}^r - x_i)^r = \mathfrak{D}^r - y_i$. Since by lemma 3.16,

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we have $\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_{\mathfrak{D}^r}) = \sum_i(\chi_{\mathfrak{D}^r - x_i})$. So again by induction hypothesis, we have

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_{\mathfrak{D}^r})|_{S^{(p)}_{n-1}} = \sum_i (\chi_{\mathfrak{D}^r - x_i})|_{S^{(p)}_{n-1}} \supseteq r(y_j)\phi_{\mathfrak{D}^r - y_j}.$$

Using these results, we claim the following: (we denote this claim as $(*)$)

If E is a p-regular diagram for S_n (so $E = E^r$), and v is a shadow node of E, then

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\phi_E) \supseteq r(v)\phi_{E-v},$$

where $r(v)$ denotes the number of rungs in E of the ladder in which v lies (it is identical with the above definition). Moreover, if w is the first shadow node of E, then

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\phi_E) = r(w)\phi_{E-w} + \cdots.$$

To prove this, observe that since E is p-regular, by proposition 2.24, we have

$$\chi_E|_{S^{(p)}_n} = \phi_E + \cdots.$$

So $\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_E)|_{S^{(p)}_{n-1}} = \mathrm{Res}^{S_n}_{S_{n-1}}(\chi_E|_{S^{(p)}_n}) \supseteq \mathrm{Res}^{S_n}_{S_{n-1}}(\phi_E)$. Moreover, collecting this result and the previous results, we obtain (recall that a shadow node is regular-removable)

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_E)|_{S^{(p)}_{n-1}} \supseteq \mathrm{Res}^{S_n}_{S_{n-1}}(\phi_E),$$

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_E)|_{S^{(p)}_{n-1}} = r(w)\phi_{E-w} + \cdots,$$

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_E)|_{S^{(p)}_{n-1}} \supseteq r(v)\phi_{E-v}.$$

So, to prove $(*)$, it suffices to prove that there is no p-regular diagram $\widetilde{E}$ for S_n satisfying:

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$$\widetilde{E} \neq E,\ \chi_E|_{S_n^{(p)}} \supseteq \phi_{\widetilde{E}},\text{ and } \mathrm{Res}^{S_n}_{S_{n-1}}(\phi_{\widetilde{E}}) \supseteq \phi_{E-v}.$$

Assume there is a such $\widetilde{E}$. First observe that since $\chi_E|_{S_n^{(p)}} \supseteq \phi_{\widetilde{E}}$, by proposition 2.24, we obtain

$$\widetilde{E} \rhd E.$$

Moreover, since $\widetilde{E}$ is p-regular, we have $\chi_{\widetilde{E}}|_{S_n^{(p)}} = \phi_{\widetilde{E}} + \cdots$. So

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_{\widetilde{E}})|_{S_{n-1}^{(p)}} = \mathrm{Res}^{S_n}_{S_{n-1}}(\chi_{\widetilde{E}}|_{S_n^{(p)}}) \supseteq \mathrm{Res}^{S_n}_{S_{n-1}}(\phi_{\widetilde{E}}) \supseteq \phi_{E-v}.$$

Thus by induction hypothesis, $\widetilde{E}$ has a removable node t such that

$$E - v \trianglerighteq (\widetilde{E} - t)^r.$$

Note that since $\widetilde{E}$ is p-regular,

$$(\widetilde{E} - t)^r = \widetilde{E} - s$$

for some removable node s of $\widetilde{E}$. This two conditions

$$\widetilde{E} \rhd E \text{ and } E - v \trianglerighteq \widetilde{E} - s$$

imply clearly that s is higher than v, and it is not hard to see that E and $\widetilde{E}$ must be of the form:

(1) There exist $R_{a,1}, R_{b,1}, R_{a,2}, R_{b,2}, \cdots, R_{a,m}, R_{b,m} \in \mathbb{N}$ with

$$R_{a,1} < R_{b,1} < R_{a,2} < R_{b,2} < \cdots < R_{a,m-1} < R_{b,m-1} < R_{a,m} < R_{b,m}$$

such that

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$$\text{the 1-st row, the 2-nd row, } \cdots, R_{a,1}-1\text{-th row of } E \text{ and } \widetilde{E}$$

are equal, and the length of the $R_{a,1}$-th row of $\widetilde{E}$ is one more than the length of the $R_{a,1}$-th row of E. Moreover, for $i = 1, \cdots, m-1$,

$$\text{the } R_{a,i}+1\text{-th row, } \cdots, R_{b,i}-1\text{-th row of } E \text{ and } \widetilde{E}$$

are equal, and the length of the $R_{b,i}$-th row of E is one more than the length of the $R_{b,i}$-th row of $\widetilde{E}$, and

$$\text{the } R_{b,i}+1\text{-th row, } \cdots, R_{a,i+1}-1\text{-th row of } E \text{ and } \widetilde{E}$$

are equal, and the length of the $R_{a,i+1}$-th row of $\widetilde{E}$ is one more than the length of the $R_{a,i+1}$-th row of E. Finally,

$$\text{the } R_{a,m}+1\text{-th row , } \cdots, \text{ the } R_{b,m}-1\text{-th row of } E \text{ and } \widetilde{E}$$

are equal, and either

- The last row of E is exactly the $R_{b,m}$-th row and its length is 1, and the last row of $\widetilde{E}$ is exactly the $R_{b,m}-1$-th row.

or

- The number of rows of E and $\widetilde{E}$ are equal. The length of the $R_{b,m}$-th row of E is one more than the length of the $R_{b,m}$-th row of $\widetilde{E}$, and rows of E and $\widetilde{E}$ which below to the $R_{b,m}$-th row (if exist) are all equal.

(2) The node s is the rightest node of the $R_{a,1}-1$-th row, or the $R_{a,1}$-th row of $\widetilde{E}$.

The node v is the rightest node of the $R_{b,m}$-th row of E.

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(3) Note that for each i, the rightest node in the $R_{b,i}$-th row of E is removable.

For example,

Row	E		Row	$\widetilde{E}$
1	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$		1	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \boxed{\cdot}$
$2 = R_{a,1}$	$\cdot\ \cdot\ \cdot\ \cdot$		$2 = R_{a,1}$	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$
3	$\cdot\ \cdot\ \cdot\ \cdot$		3	$\cdot\ \cdot\ \cdot\ \cdot$
4	$\cdot\ \cdot\ \cdot\ \cdot$	and	4	$\cdot\ \cdot\ \cdot\ \cdot$
$5 = R_{b,1}$	$\cdot\ \cdot\ \cdot\ \cdot$		$5 = R_{b,1}$	$\cdot\ \cdot\ \cdot$
6	$\cdot\ \cdot\ \cdot$		6	$\cdot\ \cdot\ \cdot$
$7 = R_{a,2}$	$\cdot\ \cdot$		$7 = R_{a,2}$	$\cdot\ \cdot\ \cdot$
$8 = R_{b,2}$	$\boxed{\cdot}$			

The nodes which are marked in E, $\widetilde{E}$ are v, s respectively.

Now since $\chi_E|_{S_n^{(p)}} \geq \phi_{\widetilde{E}}$ and $\chi_{\widetilde{E}}|_{S_n^{(p)}} \geq \phi_{\widetilde{E}}$, the diagrams E and $\widetilde{E}$ are in the same p-block, and hence by proposition 3.10 (3), they are in the same block. Moreover, if we denote a_i as the rightest nodes in the $R_{a,i}$-th row of $\widetilde{E}$, and b_i as the rightest nodes in the $R_{b,i}$-th row of E for all i, then by using the above observation, we can see that if we raise b_i into the $R_{a,i}$-th row of E for all i, then we can obtain $\widetilde{E}$ from E. So

$$\text{the set of the colors of } b_1, \cdots, b_m$$

is equal to

$$\text{the set of the colors of } a_1, \cdots, a_m.$$

This implies that there exists one of $b_1, \cdots, b_m$ has the same color as a_1, which contradicts to the assumption that v is a shadow node. Thus such $\widetilde{E}$ does not exist, and hence we complete this claim.

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Go back to this theorem. Recall that we have shown

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}})|_{S_{n-1}^{(p)}} = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots$$

and we want to show

$$\chi_{\mathfrak{D}}|_{S_n^{(p)}} = \phi_{\mathfrak{D}^r} + \cdots .$$

Let E be a p-regular diagram with $\chi_{\mathfrak{D}}|_{S_n^{(p)}} \geq \phi_E$ and let w be the first shadow node of E. Then by $(*)$, we have

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}})|_{S_{n-1}^{(p)}} = \mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}}|_{S_n^{(p)}}) \supseteq \mathrm{Res}_{S_{n-1}}^{S_n}(\phi_E) = r(w)\phi_{E-w} + \cdots .$$

So we obtain $E - w \trianglerighteq \mathfrak{D}^r - y_1$. On the other hand, since

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}}|_{S_n^{(p)}}) \supseteq \phi_{\mathfrak{D}^r - y_1},$$

there is some such E, say E_1, such that

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\phi_{E_1}) \supseteq \phi_{\mathfrak{D}^r - y_1}.$$

Moreover, by $(*)$, since (here w_1 denotes the first shadow node of E_1)

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\phi_{E_1}) = r(w_1)\phi_{E_1 - w_1} + \cdots ,$$

we obtain $\mathfrak{D}^r - y_1 \trianglerighteq E_1 - w_1$, and hence

$$\mathfrak{D}^r - y_1 = E_1 - w_1.$$

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Now observe that since $\chi_{\mathfrak{D}}|_{S_n^{(p)}} \geq \phi_{E_1}$ and $\chi_{E_1}|_{S_n^{(p)}} \geq \phi_{E_1}$, by proposition 3.10 (3), $\mathfrak{D}$ and E_1 are in the same block, and since $\mathfrak{D}$ and $\mathfrak{D}^r$ are in the same block by proposition 3.12 (2), $\mathfrak{D}^r$ and E_1 are in the same block. Thus by proposition 3.14 (2), since $\mathfrak{D}^r$ and E_1 are p-reuglar and they are in the same block, the fact $\mathfrak{D}^r - y_1 = E_1 - w_1$ implies

$$\mathfrak{D}^r = E_1,$$

and hence $\chi_{\mathfrak{D}}|_{S_n^{(p)}} \geq \phi_{\mathfrak{D}^r}$. Moreover, observe that $(*)$ gives us

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\phi_{\mathfrak{D}^r}) = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots$$

and recall again that we have shown

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}}|_{S_n^{(p)}}) = \mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}})|_{S_{n-1}^{(p)}} = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots.$$

So these imply that the multiplicity of $\phi_{\mathfrak{D}^r}$ in $\chi_{\mathfrak{D}}|_{S_n^{(p)}}$ must be 1.

It remains to show that if E is a p-regular diagram with $E \neq \mathfrak{D}^r$ and $\chi_{\mathfrak{D}}|_{S_n^{(p)}} \geq \phi_E$, then $E \rhd \mathfrak{D}^r$. First observe that since

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}}|_{S_n^{(p)}}) = r(y_1)\phi_{\mathfrak{D}^r - y_1} + \cdots$$
$$\mathrm{Res}_{S_{n-1}}^{S_n}(\phi_E) \geq \phi_{E-w}$$
$$\mathrm{Res}_{S_{n-1}}^{S_n}(\phi_E) \ngeq \phi_{\mathfrak{D}^r - y_1},$$

we have

$$E - w \rhd \mathfrak{D}^r - y_1,$$

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where w is the first sahdow node of E. Let B be the set of nodes of E not lower than w. Because $\mathfrak{D}^r - y_1$ is p-regular and $E - w \rhd \mathfrak{D}^r - y_1$,

$(**)$ the set of nodes of $\mathfrak{D}^r - y_1$ not lower than w are contained in B.

Indeed, if it is false, then there is a node x, which is not lower than w, of $\mathfrak{D}^r - y_1$ such that it is to the right of the ladder in which w lies. Note that there is no node of E to the right of the ladder of w since w is the first shadow node of E. Say x lies in the i-th row of $\mathfrak{D}^r - y_1$. Then the fact $\mathfrak{D}^r - y_1$ is p-regular implies that for all $j \leq i$, the length of the j-th row of $\mathfrak{D}^r - y_1$ is greater than the length of the j-th row of E (because by proposition 3.10 (2), if the k-th rung of a ladder l belongs to $\mathfrak{D}^r - y_1$, then so does the $k-1$-th rung of l), and this contradicts to $E - w \rhd \mathfrak{D}^r - y_1$.

Observe that there are only three possibility that y_1 may lie in $\mathfrak{D}^r$:

(i) y_1 belongs to B. (ii) y_1 is lower than w. (iii) not (i) and (ii).

In the first case, by $(**)$ and $E - w \rhd \mathfrak{D}^r - y_1$, we can see that $E \rhd \mathfrak{D}^r$. In the second case, by $E - w \rhd \mathfrak{D}^r - y_1$, we also can see that $E \rhd \mathfrak{D}^r$. Now we show that the third case is impossible.

Assume the third case holds, then it means y_1 is not lower than w, and y_1 does not belong to B. Since $\mathfrak{D}^r$ is p-regular, $(**)$, and no node in E is to the right of the ladder in which w lies, the node y_1 is either in the rightest node in the first row of $\mathfrak{D}^r$, or below the 1-st rung of the ladder in which w lies. In both cases, w and y_1 have distinct colors. Moreover, since y_1 belongs to $\mathfrak{D}^r - y_j$ for all $j > 1$, the diagram $E - w \not\trianglerighteq \mathfrak{D}^r - y_j$. Recall

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that we have shown

$$\sum_i \chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} = \mathrm{Res}_{S_{n-1}}^{S_n}(\chi_{\mathfrak{D}})|_{S_{n-1}^{(p)}} \supseteq \phi_{E-w}$$

$$\chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} = \phi_{\mathfrak{D}^r-y_1} + \cdots, \text{ if } i = 1, \cdots, r(y_1),$$

$$\chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} = \phi_{\mathfrak{D}^r-y_j} + \cdots, \text{ if } i > r(y_1)\ (j \neq 1).$$

So the fact $E - w \not\trianglerighteq \mathfrak{D}^r - y_j$ implies that there is a $i \in \{1, \cdots, r(y_1)\}$ such that

$$\chi_{\mathfrak{D}-z_i}|_{S_{n-1}^{(p)}} \supseteq \phi_{E-w}.$$

Moreover, since $E - w$ is p-regular, we also have

$$\chi_{E-w}|_{S_{n-1}^{(p)}} \supseteq \phi_{E-w}.$$

So $\mathfrak{D} - z_i$ and $E - w$ are in the same block by proposition 3.10 (3), and hence

$$(\mathfrak{D} - z_i)^r \text{ and } E - w$$

are in the same block. Note that $(\mathfrak{D} - z_i)^r = \mathfrak{D}^r - y_1$. Now the fact y_1 and w have distinct colors means $\mathfrak{D}^r$ and E are not in the same block, and hence $\mathfrak{D}$ and E are not in the same block. However, recall that

$$\chi_{\mathfrak{D}}|_{S_n^{(p)}} \geq \phi_E,$$

and since E is p-regular, we also have

$$\chi_E|_{S_n^{(p)}} \geq \phi_E.$$

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So $\mathfrak{D}$ and E are in the same block by proposition 3.12. This gets a contradiction.

Now we completes the proof of the theorem.

□

Corollary 3.17.1.

The group S_n has the L''-property.

Proof.

It is obvious by theorem 3.17.

□

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4 About A_n

Recall that (K_0, R, K_p) is a splitting p-modular system for G. But (K_0, R, K_p) may not be a splitting p-modular system for H. So in this section, we assume that (K_0, R, K_p) is a splitting p-modular system for H and for G. For example, if $m = \mathrm{lcm}(\mathrm{ord}(g) \mid g \in G)$ and K_0 contains the m-th root of unity, then it is a splitting p-modular system for any subgroup H of G (cf., for example, [S1], theorem 24, page 94). In the case $H = A_n$ and $G = S_n$, if (K_0, R, K_p) is a splitting p-modular system for A_n, then it is also for S_n.

To investigate a minimal p-splitting field for G, let $\mathrm{char}(F) = p$, $\bar{F}$ be the algebraic closure of F, M be a $\bar{F}G$-module, S be a $\bar{F}$-basis of M, $\theta_g^M : M \to M$ given by $m \mapsto g \cdot m$, and ρ_g^M be the matrix representation with respect to S. Then by [I], theorem 9.14, page 150, if F contains $\mathrm{Tr}(\rho_g^M)$ for all non-isomorphic simple $\bar{F}G$-module M and all $g \in G$, then F is the minimal p-splitting field for G. But it may not be ture when $\mathrm{char}(F) = 0$.

To be an example, we list minimal s-splitting fields of A_5, where $s = 0$, 2, 3, and 5. The field $\mathbb{Q}(\sqrt{5})$ is a minimal 0-splitting field of A_5. The field $\mathbb{F}_4$, $\mathbb{F}_9$, and $\mathbb{F}_5$ are minimal 2, 3, and 5-splitting fields of A_5 respectively, where $\mathbb{F}_q$ denotes the finite field of order q. For more information about the p-splitting field of A_n, one may see [W], theorem 4.1.3. It tells us a p-splitting field of A_n should contain a special element.

Let f be an irreducible character of A_n. To show that f has the the $(L'', 2)$-property, or (L', p)-property for $p > 2$, observe that there exists an irreducible character χ of S_n such that $\mathrm{Res}_{A_n}^{S_n}(\chi) \geq f$, and hence $\mathrm{Res}_{A_n}^{S_n}(\chi|_{S_n^{(p)}}) = \mathrm{Res}_{A_n}^{S_n}(\chi)|_{A_n^{(p)}} \supseteq f|_{A_n^{(p)}}$. So we will discuss about $\mathrm{Res}_{A_n}^{S_n}(\chi)$ and $\mathrm{Res}_{A_n}^{S_n}(\phi)$ for any irreducible character χ and any irreducible p-modular character ϕ of S_n, using Clifford's Theorem. Also theorem 3.17 gives us important informations about $\chi|_{S_n^{(p)}}$ when showing that f has the (L', p)-property for $p > 2$.

Note that by using Frobenius Formula (cf. for example, [F], 4.10, page 49), we can

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obtain the character table of S_n for all n. Moreover, if χ is an irreducible character of S_n, then lemma 4.18 tell us when $\mathrm{Res}_{A_n}^{S_n}(\chi)$ splits, or be an irreducible character of A_n. So by investigating the character value of $\mathrm{Res}_{A_n}^{S_n}(\chi)$ when it splits, we can obtain the character table of A_n by the character table of S_n (cf. for example, [F], proposition 5.3, page 66).

For more informations about irreducible characters of A_n, one may see [F1] and [F2].

4.1 Tools

Proposition 4.1.

Let $\chi_1, \cdots, \chi_m$ be all distinct irreducible characters of G and let M_i be a K_0G-module which affords χ_i. Then

$$\dim_{K_0} \mathrm{Hom}_{K_0G}(M_i, M_j) = \delta_{ij}.$$

Proposition 4.2.

If V is a projective K_pG-module, then there is a unique (up to isomorphism) RG-lattice M such that its reduction mod $\mathfrak{m}$ is V.

Proof.

Cf., for example, [S1], proposition 42, page 119. Note that the completion of K_0 is necessary.

□

Definition 4.3.

A FG-module homomorphism $f : A \to B$ is called essential if $f(A) = B$ and $f(A') \neq B$ for all proper FG-submodule A' of A.

Let P be a projective FG-module. We say P is a projective envelope of an FG-module M if there is an essential FG-homomorphism from P to M.

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Proposition 4.4.

(1) Every FG-module M has a projective envelope which is unique up to isomorphism.

(2) Let $M_1, \cdots, M_k$ be FG-modules and let P_i be the projective envelope of M_i. Then $\bigoplus_{i=1}^{k} P_i$ is the projective envelope of $\bigoplus_{i=1}^{k} M_i$.

(3) Let E_1, E_2 be simple K_pG-modules, and let P_i be the projective envelope of E_i for $i = 1, 2$. Then $P_1 \cong_{K_pG} P_2$ if and only if $E_1 \cong_{K_pG} E_2$.

Proof.

Cf., for example, [S1] proposition 14.1, page 117.

□

Proposition 4.5.

Let $E_1, \cdots, E_k$ be all distinct simple K_pG-modules, and let P_i be the projective envelope of E_i. Then each P_i is indecomposable among projective K_pG-modules. Moreover, if V is a projective K_pG-module, then

$$V = e_1P_1 \oplus \cdots \oplus e_kP_k$$

for some $e_i \in \mathbb{N} \cup \{0\}$.

Proof.

Cf., for example, [S1], corollary 1, page 140.

□

Proposition 4.6.

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Let $E_1, \cdots, E_k$ be all distinct simple K_pG-modules, and let P_i be the projective envelope of E_i. Then

$$\dim_{K_p} \mathrm{Hom}_{K_pG}(P_i, E_j) = \delta_{ij}.$$

Proof.

Cf., for example, [S1], page 121.

$\square$

Proposition 4.7. (Frobenius Reciprocity)

Let M be an FH-module, and N be an FG-module. Then

$$\mathrm{Hom}_{FH}(M, \mathrm{Res}_H^G(N)) \cong_F \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(M), N).$$

Proof.

Cf., for example, [CR], theorem 10.8, page 232.

$\square$

Definition 4.8.

Let M be a semisimple FG-module and write $M = e_1M_1 \oplus \cdots \oplus e_kM_k$ for some pairwisely non-isomorphic FG-module $M_1, \cdots, M_k$, and some $e_1, \cdots, e_k \in \mathbb{N}$. Then e_iM_i is called an FG-homomgeneous component of M.

Definition 4.9.

(1) Assume $H \trianglelefteq G$. Let M be an FH-module. For any $g \in G$, define $M^{(g)}$ as an FH-module which $M^{(g)}$ agrees with M as abelian group, and for any $h \in H$, $m \in M^{(g)}$,

$$h \cdot m := (g^{-1}hg)m.$$

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We say two FH-modules M, N are conjugate (under G) if $N \cong_{FH} M^{(g)}$ for some $g \in G$.

(2) If $F = K_0$ and χ_M is the characters of M, then we denote

$$\chi_M^{(g)}(h) = \chi_M(g^{-1}hg).$$

It is the character of the K_0H-module $M^{(g)}$. We say χ_M and a character χ are conjugate (under G) if $\chi = \chi_M^{(g)}$ for some $g \in G$.

(3) If $F = K_p$ and ϕ_M is the p-modular characters of M, then we denote

$$\phi_M^{(g)}(h) = \phi_M(g^{-1}hg).$$

It is the p-modular character of the K_pH-module $M^{(g)}$. We say ϕ_M and a p-modular character ϕ are conjugate (under G) if $\phi = \phi_M^{(g)}$ for some $g \in G$.

Theorem 4.10. (Clifford's Theorem)

Assume $H \trianglelefteq G$. Let V be a simple FG-module, and M be a simple FH-submodule of $\mathrm{Res}_H^G(V)$. Then $\mathrm{Res}_H^G(V)$ is a semisimple FH-module. Moreover,

(1) Let

$\widetilde{M}$ be the FH-homogeneous components of $\mathrm{Res}_H^G(V)$ containing M,

$\widetilde{H} = \{x \in G \mid x\widetilde{M} = \widetilde{M}\}$ be the stabilizer of $\widetilde{M}$ (note that $\widetilde{H} \trianglelefteq G$),

$g_1, \cdots, g_k$ be a set of representative of $G/\widetilde{H}$.

Then $\{M^{(g_i)} \mid i = 1, \cdots, k\}$ is a complete set of non-isomorphic conjugates of M, and

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each appears with the same multiplicity $t \in \mathbb{N}$ in $\mathrm{Res}_H^G(V)$, i.e.

$$\mathrm{Res}_H^G(V) \cong_{FH} \overset{k}{\underset{i=1}{\oplus}} tM^{(g_i)}.$$

(2) The module $\widetilde{M}$ is a simple $F\widetilde{H}$-module and we have

$$V \cong_{FG} \mathrm{Ind}_{\widetilde{H}}^G(\widetilde{M}).$$

(3) If $F = K_0$ or K_p, then $t^2 \leq |\widetilde{H} : H|$.

Proof.

For (1) and (2), cf., for example, [CR], theorem 11.1, page 259. Now we are going to show (3).

Assume $F = K_0$. Since $\mathrm{Res}_H^G(V) \cong_{K_0H} \overset{k}{\underset{i=1}{\oplus}} tM^{(g_i)}$, we have

$$\dim_{K_0} \mathrm{Hom}_{K_0H}(\mathrm{Res}_H^G(V), \mathrm{Res}_H^G(V)) = \sum_{i=1}^{k} t^2 \dim_{K_0} \mathrm{Hom}_{K_0H}(M^{(g_i)}, M^{(g_i)}) = |G : \widetilde{H}| t^2.$$

On the other hand, by proposition 4.7, we have

$$\dim_{K_0} \mathrm{Hom}_{K_0H}(\mathrm{Res}_H^G(V), \mathrm{Res}_H^G(V)) = \dim_{K_0} \mathrm{Hom}_{K_0G}(\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V)), V).$$

Note that since V occurs in composition factors of $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V))$ at most $|G : H|$ times, we have

$$\dim_{K_0} \mathrm{Hom}_{K_0G}(\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V)), V) \leq |G : H|.$$

Therefore, we obtain $|G : \widetilde{H}| t^2 \leq |G : H|$, and hence $t^2 \leq |\widetilde{H} : H|$.

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Assume $F = K_p$. Denote

$$P_i : \text{the } K_pH \text{ projective envelope of the } K_pH\text{-module } M^{(g_i)},$$

$$P_{\mathrm{Res}_H^G(V)} : \text{the } K_pH \text{ projective envelope of the } K_pH\text{-module } \mathrm{Res}_H^G(V),$$

$$P_V : \text{the } K_pG \text{ projective envelope of the } K_pG\text{-module } V.$$

First observe that by proposition 4.4 (2), we have

$$P_{\mathrm{Res}_H^G(V)} = \bigoplus_{i=1}^{k} tP_i.$$

On the other hand, let f be an essential K_pG-homomorphism from P_V to V. Then f is a surjective K_pH-homomorphism from $\mathrm{Res}_H^G(P_V)$ to $\mathrm{Res}_H^G(V)$. Let h be an essential K_pH-homomorphism from $P_{\mathrm{Res}_H^G(V)}$ to $\mathrm{Res}_H^G(V)$. Then since $\mathrm{Res}_H^G(P_V)$ is a projective K_pH-module, we have the following commutative diagram

$$\begin{tikzcd}
 & P_{\mathrm{Res}_H^G(V)} \arrow[d, two heads, "h"] \\
\mathrm{Res}_H^G(P_V) \arrow[ur, dashed, "\exists\, g"] \arrow[r, two heads, "f"] & \mathrm{Res}_H^G(V)
\end{tikzcd},$$

where g is a K_pH-homomorphism from $\mathrm{Res}_H^G(P_V)$ to $P_{\mathrm{Res}_H^G(V)}$. Moreover, since h is essential and f is onto, g must be onto, and since $P_{\mathrm{Res}_H^G(V)}$ is projective, the fact g is onto implies

$$\mathrm{Res}_H^G(P_V) \cong_{K_pH} P_{\mathrm{Res}_H^G(V)} \oplus \ker(g)$$

(Note that $\ker(g)$ is also a projective K_pH-module). So

$$\mathrm{Res}_H^G(P_V) \cong_{K_pH} \left(\bigoplus_{i=1}^{k} tP_i \right) \oplus \ker(g),$$

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and hence by proposition 4.6, we obtain

$$\dim_{K_p} \operatorname{Hom}_{K_pH}(\operatorname{Res}^G_H(P_V), \operatorname{Res}^G_H(V)) \geq kt^2 = |G : \widetilde{H}|t^2.$$

On the other hand, by proposition 4.7, we have

$$\dim_{K_p} \operatorname{Hom}_{K_pG}(\operatorname{Ind}^G_H(\operatorname{Res}^G_H(P_V)), V) = \dim_{K_p} \operatorname{Hom}_{K_pH}(\operatorname{Res}^G_H(P_V), \operatorname{Res}^G_H(V)).$$

Since $\operatorname{Ind}^G_H(\operatorname{Res}^G_H(P_V)) = K_pG \otimes_{K_pH} \operatorname{Res}^G_H(P_V)$, it is a projective K_pG-module, and its dimension is

$$|G : H| \dim_{K_p}(P_V).$$

So if we write $\operatorname{Ind}^G_H(\operatorname{Res}^G_H(P_V))$ as the direct sum of some indecomposable projective K_pG-modules, then P_V occured in it at most $|G : H|$ times (recall that P_V is an indecomposable projective K_pG-modules since V is a simple K_pG-module). So

$$|G : H| \geq \dim_{K_p} \operatorname{Hom}_{K_pG}(\operatorname{Ind}^G_H(\operatorname{Res}^G_H(P_V)), V).$$

Now combine these inequalities, we obtain $|G : H| \geq |G : \widetilde{H}|t^2$, and hence $|\widetilde{H} : H| \geq t^2$.

□

By theorem 4.10 (3), we immediately have the following two lemmas.

Lemma 4.11.

Assume $H \trianglelefteq G$ with $|G : H| = 2$. Let χ be an irreducible character of G. Then

$$\text{either } \operatorname{Res}^G_H(\chi) \text{ is irreducible or } \operatorname{Res}^G_H(\chi) = \chi_1 + \chi_2$$

for some distinct irreducible characters χ_1, χ_2 of H, where χ_1, χ_2 are conjugate.

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Lemma 4.12.

Assume $H \trianglelefteq G$ with $|G:H| = 2$. Let ϕ be an irreducible p-modular character of G. Then

$$\text{either } \mathrm{Res}_H^G(\phi) \text{ is irreducible or } \mathrm{Res}_H^G(\phi) = \phi_1 + \phi_2$$

for some distinct irreducible p-modular characters ϕ_1, ϕ_2 of H, where ϕ_1, ϕ_2 are conjugate.

Lemma 4.13.

Let f be an irreducible character of H. Then there is an irreducible character χ of G such that $\mathrm{Res}_H^G(\chi) \geq f$.

Proof.

Let M be a simple K_0H-module which affords f. Since K_0H is a K_0H-submodule of $\mathrm{Res}_H^G(K_0G)$, we have

$$\dim_{K_0} \mathrm{Hom}_{K_0H}(\mathrm{Res}_H^G(K_0G), M) \neq 0.$$

On the other hand, let $V_1, \cdots, V_m$ be all non-isomorphic simple K_0G-modules of G. Observe that

$$K_0G \cong_{K_0G} \dim_{K_0}(V_1)V_1 \oplus \cdots \oplus \dim_{K_0}(V_m)V_m.$$

So

$$\mathrm{Res}_H^G(K_0G) \cong_{K_0G} \dim_{K_0}(V_1)\,\mathrm{Res}_H^G(V_1) \oplus \cdots \oplus \dim_{K_0}(V_m)\,\mathrm{Res}_H^G(V_m).$$

Hence

$$0 \neq \dim_{K_0} \mathrm{Hom}_{K_0H}(\mathrm{Res}_H^G(K_0G), M) = \sum_{i=1}^{m} \dim_{K_0}(V_i) \dim_{K_0} \mathrm{Hom}_{K_0H}(\mathrm{Res}_H^G(V_i), M).$$

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This means there exists a $j \in \{1, 2, \cdots, m\}$ such that $\dim_{K_0} \mathrm{Hom}_{K_0 H}(\mathrm{Res}_H^G(V_j), M) \neq 0$, i.e. $\mathrm{Res}_H^G(\chi_j) \geq f$.

□

4.2 The $(L'', 2)$-property of A_n

When we restrict irreducible p-modular characters of S_n from S_n to A_n, the two cases $p = 2$ and $p > 2$ present different phenomenons. So we have to separate our discussion into these two cases. In this subsection, we focus on the case $p = 2$, and in the next subsection we focus on the case $p > 2$.

Lemma 4.14.

Let ϕ_1, ϕ_2 be two distinct irreducible 2-modular characters of S_n. If σ_1, σ_2 are two irreducible 2-modular characters of A_n with

$$\mathrm{Res}_{A_n}^{S_n}(\phi_1) \geq \sigma_1 \text{ and } \mathrm{Res}_{A_n}^{S_n}(\phi_2) \geq \sigma_2$$

then $\sigma_1 \neq \sigma_2$.

Proof.

If $\sigma_1 = \sigma_2$, then by theorem 4.10 (1), we have $\mathrm{Res}_{A_n}^{S_n}(\phi_1) = \mathrm{Res}_{A_n}^{S_n}(\phi_2)$. However, since $S_n^{(2)} = A_n^{(2)}$, the above equality means $\phi_1 = \phi_2$, which gets a contradiction. Thus $\sigma_1 \neq \sigma_2$.

□

Theorem 4.15.

Let f be an irreducible character of A_n. Then there exists an irreducible 2-modular character σ of A_n such that $f|_{A_n^{(2)}} \geq \sigma$ with multiplicitly 1.

Proof.

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Let f be an irreducible character of A_n. Then by lemma 4.13, there is an irreducible character χ of S_n such that $\mathrm{Res}_{A_n}^{S_n}(\chi) \geq f$. Note that by theorem 3.17,

$$\chi|_{S_n^{(2)}} = \phi_{\mu_1} + a_2\phi_{\mu_2} + \cdots + a_m\phi_{\mu_m},$$

where $m \in \mathbb{N}$, $a_2, \cdots, a_m \in \mathbb{N}$, and $\mu_1, \cdots, \mu_m$ are 2-regular partition of n such that $\mu_i \rhd \mu_1$ for all $i = 2, \cdots, m$ (note that if $m = 1$, then it means $\chi|_{S_n^{(2)}} = \phi_{\mu_1}$).

If $f^{(g)} = f$ for all $g \in S_n$, then by lemma 4.11, $\mathrm{Res}_{A_n}^{S_n}(\chi) = f$. So

$$f|_{A_n^{(2)}} = \mathrm{Res}_{A_n}^{S_n}(\chi)|_{A_n^{(2)}} = \mathrm{Res}_{A_n}^{S_n}(\chi|_{S_n^{(2)}}) = \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) + a_2 \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_2}) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_m}).$$

Now by lemma 4.12, there is an irreducible 2-modular character σ of A_n such that

$$\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) \geq \sigma \text{ with multiplicity } 1,$$

and by lemma 4.14, for $i = 2, \cdots, m$, $\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_i}) \geq \sigma$ is impossible. So

$$f|_{A_n^{(2)}} \geq \sigma \text{ with multiplicity } 1.$$

If there exists $g \in S_n$ such that $f^{(g)} \neq f$. Then by lemma 4.11, $\mathrm{Res}_{A_n}^{S_n}(\chi) = f + f^{(g)}$. So

$$f|_{A_n^{(2)}} + f^{(g)}|_{A_n^{(2)}} = \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) + a_2 \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_2}) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_m}).$$

Let σ be as above. Then

$$\text{either } f|_{A_n^{(2)}} \geq \sigma \text{ or } f^{(g)}|_{A_n^{(2)}} \geq \sigma,$$

or equivalently,

$$f|_{A_n^{(2)}} \geq \sigma \text{ or } \sigma^{(g^{-1})}.$$

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Since $\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) \geq \sigma$ and $\sigma^{(g^{-1})}$ by Clifford theorem, $\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_i}) \geq \sigma$ or $\sigma^{(g^{-1})}$ is impossible for $i = 2, \cdots, m$ by lemma 4.14. Hence we conclude that

$$f|_{A_n^{(2)}} \geq \sigma \text{ or } \sigma^{(g^{-1})} \text{ with multiplicity 1.}$$

□

Corollary 4.15.1.

The group A_n has the $(L'', 2)$-property.

Remark.

In the case $p = 2$, bceause $S_n^{(2)} = A_n^{(2)}$, we can prove theorem 4.15 without introducing many informations about irreducible 2-modular characters of S_n. But in the case $p > 2$, we will introduce four lemmas to show that A_n has the (L', p)-property.

For more informations about irreducible 2-modular characters of S_n, one may see [B].

4.3 The (L', p)-property of A_n for $p > 2$

In the subsection, we will show that A_n has the (L', p)-property for $p > 2$. To show this, the following lemmas are necessary. Note that the notation p do not assume > 2 unless we explicitly mention it.

Lemma 4.16.

Assume $H \trianglelefteq G$ with $|G : H| = 2$. Let $s = 0$ or p and M_1, M_2 be two non-isomorphic simple K_sG-modules such that $\mathrm{Res}_H^G(M_1) \cong_{K_sH} V_1 \oplus V_2$ for some non-isomorphic simple K_sH-modules V_1, V_2 which are conjugate (it may happen by lemma 4.11 and lemma 4.12).

(1) If $\mathrm{Res}_H^G(M_2) \cong_{K_sH} V_3 \oplus V_4$for some non-isomorphic simple K_sH-modules V_3, V_4 which are conjugate. Then V_1, V_2, V_3, V_4 are all non-isomorphic as K_sH-modules.

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(2) If $\mathrm{Res}_H^G(M_2) := V_5$ for some simple K_sH-modules V_5, then V_1, V_2, V_5 are all non-isomorphic as K_sH-modules.

Proof.

(1) Observe that by theorem 4.10 (2), we have

$$M_1 \cong_{K_sG} \mathrm{Ind}_H^G(V_1) \cong_{K_sG} \mathrm{Ind}_H^G(V_2),\ M_2 \cong_{K_sG} \mathrm{Ind}_H^G(V_3) \cong_{K_sG} \mathrm{Ind}_H^G(V_4).$$

If $V_i \cong_{K_sH} V_j$ for some $i = 1$ or 2 and $j = 3$ or 4, then

$$M_1 \cong_{K_sG} \mathrm{Ind}_H^G(V_i) \cong_{K_sG} \mathrm{Ind}_H^G(V_j) \cong_{K_sG} M_2,$$

which gets a contradiction. Thus $V_1, \cdots, V_4$ are all distinct.

(2) It is clearly by theorem 4.10 (1).

□

Lemma 4.17. (Schur's Lemma)

Let $s = 0$ or p. Let X and Y be simple K_sG-modules such that there is a K_sG-isomorphism z from X to Y. Let $x_1, \cdots, x_t$ be an any fixed K_s-basis of X. If f is a K_sG-isomorphism from X to Y, then there is an $l \in K_s$ such that $f(x_i) = l \times z(x_i)$ for all i.

Proof.

Let L be the algebraic closure of K_s. Then by the assumption of K_s, $L \otimes_{K_s} X$ and $L \otimes_{K_s} Y$ are simple LS_n-modules.

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Let $A \in \mathrm{GL}_t(K_s)$ be the matrix representation of f by $x_1, \cdots, x_t$ and $z(x_1), \cdots, z(x_t)$. Consider the map

$$1 \otimes f : L \otimes_{K_s} X \to L \otimes_{K_s} Y.$$

Observe $1 \otimes x_1, \cdots, 1 \otimes x_t$ and $1 \otimes z(x_1), \cdots, 1 \otimes z(x_t)$ are L-bases of $L \otimes_{K_s} X$ and $L \otimes_{K_s} Y$ respectively, and the matrix representation of $1 \otimes f$ by these two bases is A. Let $l \in L$ be an eigenvalue of A. Consider the map

$$g : L \otimes_{K_s} X \to L \otimes_{K_s} Y,\ 1 \otimes x_i \mapsto l \otimes z(x_i).$$

Then the matrix representation of $1 \otimes f - g$ by $1 \otimes x_1, \cdots, 1 \otimes x_t$ and $1 \otimes z(x_1), \cdots, 1 \otimes z(x_t)$ is $A - l\,\mathrm{Id}$, where Id is the identity matrix in $\mathrm{GL}_t(L)$.

Now since l is the eigenvalue of A, the kernel of $1 \otimes f - g$ is non-trivial. So the fact $L \otimes_{K_s} X$ is simple LS_n-module implies $1 \otimes f - g$ is a zero map, i.e.

$$A = l\,\mathrm{Id}.$$

Hence $l \in K_s$, and $f(x_i) = l \times z(x_i)$ for all i.

□

Lemma 4.18.

Let μ be a partition of n. Then

(1) If $\mu \neq \mu'$, then $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu})$ is a simple K_0A_n module.

(2) If $\mu = \mu'$, then $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu}) \cong_{K_0A_n} V_1 \oplus V_2$ for some non-isomorphic simple K_0A_n-modules V_1, V_2 which are conjugate.

Proof.

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First observe that by proposition 2.25 and proposition 2.20,

$$S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})} \cong_{K_0 S_n} (S_{K_0}^{\mu'})^* \cong_{K_0 S_n} S_{K_0}^{\mu'}.$$

Also observe that since $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{(1^{(n)})})$ is the trivial K_0A_n-module, we have

$$\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu}) \cong_{K_0 A_n} \mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu'}).$$

Assume $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu})$ is simple. Then $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})})$ is also a simple K_0A_n-modules. If $u = u'$, then there is a K_0S_n-isomorphism

$$f : S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})} \to S_{K_0}^{\mu}.$$

Note that f is also a K_0A_n-isomorphism from $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})})$ to $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu})$. Let $z_1, \cdots, z_t$ be a basis of $S_{K_0}^{\mu}$, and let u be a basis of $S_{K_0}^{(1^{(n)})}$. Then $z_1 \otimes u, \cdots, z_t \otimes u$ is a basis of $S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})}$. Note that for $\pi \in S_n$,

$$\pi \cdot u = \begin{cases} u & \text{if } \pi \text{ is an even permutation} \\ -u & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

Now by lemma 4.17, since the map

$$z_i \otimes u \mapsto z_i$$

is a K_0A_n-isomorphism from $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu} \otimes_{K_0} S_{K_0}^{(1^{(n)})})$ to $\mathrm{Res}_{A_n}^{S_n}(S_{K_0}^{\mu})$, there is a $l \in K_0$ such

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that

$$f(z_i \otimes u) = lz_i.$$

However, since f is a K_0S_n-homomorphism, if $\pi \in S_n$ is an odd permutation, then

$$l(-\pi \cdot z_i) = f((-\pi z_i) \otimes u) = f(\pi(z_i \otimes u)) = \pi f(z_i \otimes u) = \pi \cdot (lz_i),$$

which gets a contradiction. Thus we obtain $u \neq u'$.

Assume $\mathrm{Res}^{S_n}_{A_n}(S^{\mu}_{K_0}) \cong_{K_0A_n} V_1 \oplus V_2$. Then since $\mathrm{Res}^{S_n}_{A_n}(S^{\mu}_{K_0}) \cong_{K_0A_n} \mathrm{Res}^{S_n}_{A_n}(S^{\mu'}_{K_0})$, by lemma 4.16 (1), we obtain $S^{\mu}_{K_0} \cong_{K_0A_n} S^{\mu'}_{K_0}$, and hence $\mu = \mu'$.

Now we completes this lemma.

□

Lemma 4.19.

Assume $p > 2$. Let ψ be an irreducible p-modular character of A_n such that there is an irreducible character ϕ of S_n with $\mathrm{Res}^{S_n}_{A_n}(\phi) = \psi$. Then ϕ and $\phi \cdot \mathrm{sgn}$ are distinct, and they are the only two p-modular characters of S_n such that its restriction to A_n is ψ.

Note that sgn is the p-modular character of $D^{(1^{(n)})}_{K_p}$, and for $\pi \in S^{(p)}_n$,

$$\mathrm{sgn}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is an even permutation} \\ -1 & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

Proof.

Let μ be a p-regular partition of n such that $D^{\mu}_{K_p}$ affords ϕ. Then $\phi \cdot \mathrm{sgn}$ is the character of

$$D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p}.$$

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Note that $D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p}$ is a simple K_pS_n-module by lemma 3.1 and

$$\mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p}) \cong_{K_pA_n} \mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p}) \cong_{K_pA_n} V,$$

where V is a simple K_pA_n-module which affords ψ.

If $\phi = \phi \cdot \mathrm{sgn}$, then there is a K_pS_n-isomorphism

$$f : D^{\mu}_{K_p} \to D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p}.$$

Note that f is also a K_pA_n-isomorphism from $\mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p})$ to $\mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p})$. Let $x_1, \cdots, x_t$ be a K_p-basis of $D^{\mu}_{K_p}$ and u be a K_p-basis of $D^{(1^{(n)})}_{K_p}$. Then $x_1 \otimes u, \cdots, x_t \otimes u$ is a K_p-basis of $D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p}$, and the map

$$x_i \mapsto x_i \otimes u$$

is a K_pA_n-isomorphism from $\mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p})$ to $\mathrm{Res}^{S_n}_{A_n}(D^{\mu}_{K_p} \otimes_{K_p} D^{(1^{(n)})}_{K_p})$. So by lemma 4.17, there is a $l \in K_p$ such that

$$f(x_i) = l(x_i \otimes u).$$

However, since f is a K_pS_n-homomorphism, if $\pi \in S_n$ is an odd permutation, then we have

$$l((\pi x_i) \otimes u) = f(\pi x_i) = \pi f(x_i) = \pi l(x_i \otimes u) = -l((\pi x_i) \otimes u),$$

which gets a contradiction since $p > 2$. Therefore

$$\phi \neq \phi \cdot \mathrm{sgn}.$$

For the second part, if W is a simple K_pS_n-module such that its irreducible p-modular

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character w of S_n satisfies $\mathrm{Res}_{A_n}^{S_n}(w) = \psi$ (i.e. $\mathrm{Res}_{A_n}^{S_n}(W) \cong_{K_pA_n} V$), then by proposition 4.7,

$$1 = \dim_{K_p} \mathrm{Hom}_{K_pA_n}(P_V, \mathrm{Res}_{A_n}^{S_n}(W)) = \dim_{K_p} \mathrm{Hom}_{K_pS_n}(\mathrm{Ind}_{A_n}^{S_n}(P_V), W),$$

where P_V is the K_pA_n-projective envelope of V. Note that $\mathrm{Ind}_{A_n}^{S_n}(P_V)$ is a projective K_pS_n-module. So by proposition 4.5, if we write $\mathrm{Ind}_{A_n}^{S_n}(P_V)$ as the direct sum of indecomposable projective K_pS_n-modules, then P_W occured in it with multiplicity 1, where P_W is the K_pS_n-projective envelope of W.

From now on, we know $W_1 := D_{K_p}^{\mu}$ and $W_2 := D_{K_p}^{\mu} \otimes_{K_p} D_{K_p}^{(1^{(n)})}$ satisfy

$$\mathrm{Res}_{A_n}^{S_n}(W_i) \cong_{K_pA_n} V.$$

Hence P_{W_1}, P_{W_2} occured in the direct sum of indecomposable projective K_pS_n-modules of $\mathrm{Ind}_{A_n}^{S_n}(P_V)$ with multiplicity 1. Note that by proposition 4.4 (3), P_{W_1} and P_{W_2} are non-isomorphic since W_1 and W_2 are non-isomorphic. Hence

$$2\dim_{K_p}(P_V) = \dim_{K_p}(\mathrm{Ind}_{A_n}^{S_n}(P_V)) \geq \dim_{K_p}(P_{W_1}) + \dim_{K_p}(P_{W_2}).$$

Also note that in the proof of theorem 4.10 (3), we have seen that $\mathrm{Res}_{A_n}^{S_n}(P_{W_i})$ is a projective K_pA_n-module, and

$$\mathrm{Res}_{A_n}^{S_n}(P_{W_i}) \cong_{K_pA_n} P_{\mathrm{Res}_{A_n}^{S_n}(W_i)} \oplus Z \cong_{K_pA_n} P_V \oplus Z$$

for some projective K_pA_n-module Z. Hence

$$\dim_{K_p}(P_{W_i}) = \dim_{K_p}(\mathrm{Res}_{A_n}^{S_n}(P_{W_i})) \geq \dim_{K_p}(P_V).$$

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Therefore, $2\dim_{K_p}(P_V) = \dim_{K_p}(\mathrm{Ind}_{A_n}^{S_n}(P_V)) = \dim_{K_p}(P_{W_1}) + \dim_{K_p}(P_{W_2})$ and hence

$$\mathrm{Ind}_{A_n}^{S_n}(P_V) \cong_{K_pS_n} P_{W_1} \oplus P_{W_1}.$$

Therefore, we cannot find another simple K_pS_n-module W_3 which is not isomorphic to W_1 and W_2 such that $\mathrm{Res}_{A_n}^{S_n}(W_3) \cong_{K_pA_n} V$, and hence ϕ and $\phi \cdot \mathrm{sgn}$ are the only two p-modular characters of S_n such that its restriction to A_n is ψ.

□

Lemma 4.20.

Assume $p > 2$. Let λ be a partition of n, and let χ_λ be the irreducible character of $S_{K_0}^\lambda$. If $\chi_\lambda(\pi) = 0$ for all odd permutation $\pi \in S_n^{(p)}$, then $\lambda = \lambda'$

Proof.

By [W], corollary 2.1.2 (iii).

□

Theorem 4.21.

The group A_n has the (L', p)-property for any prime $p > 2$.

Proof.

Let f be an irreducible character of A_n. Then by lemma 4.13, there is a partition λ of n such that the irreducible character χ_λ of the Specht K_0S_n-module $S_{K_0}^\lambda$ satisfies $\mathrm{Res}_{A_n}^{S_n}(\chi_\lambda) \geq f$. Note that since S_n has (L', p)-property,

$$\text{either } \chi_\lambda|_{S_n^{(p)}} = \phi_0 \text{ or } \chi_\lambda|_{S_n^{(p)}} = a_1\phi_1 + \cdots + a_m\phi_m,$$

where m, $a_i \in \mathbb{N}$, $m > 1$, and ϕ_i are irreducible p-modular characters of S_n.

Assume f does not have the (L', p)-property, then $f|_{A_n^{(p)}} = r\sigma$ for some $r \in \mathbb{N}$, $r > 1$, and irreducible p-modular character σ of A_n. We want to get a contradiction.

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If $\lambda \neq \lambda'$, then by lemma 4.18, we have $\mathrm{Res}_{A_n}^{S_n}(\chi_\lambda) = f$. So

$$r\sigma = f|_{A_n^{(p)}} = \mathrm{Res}_{A_n}^{S_n}(\chi_\lambda)|_{A_n^{(p)}} = \mathrm{Res}_{A_n}^{S_n}(\chi_\lambda|_{S_n^{(p)}}) = \begin{cases} \text{either} & \mathrm{Res}_{A_n}^{S_n}(\phi_0) \\ \text{or} & a_1 \mathrm{Res}_{A_n}^{S_n}(\phi_1) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_m). \end{cases}$$

In the first case, by lemma 4.12, it is clearly that it implies $r = 1$, which gets a contradiction. In the second case, by lemma 4.12, $\mathrm{Res}_{A_n}^{S_n}(\phi_1), \cdots, \mathrm{Res}_{A_n}^{S_n}(\phi_m)$ must be irreducible, and by lemma 4.19, m must be 2, ϕ_1 and ϕ_2 are distinct, and $\phi_2 = \phi_1 \cdot \mathrm{sgn}$. So we obtain

$$\chi_\lambda|_{S_n^{(p)}} = a_1\phi_1 + a_2\phi_1 \cdot \mathrm{sgn}\,.$$

Now by the first part of theorem 3.17, there exists i such that $a_i = 1$, say $a_1 = 1$. On the other hand, observe that

$$\chi_{\lambda'}|_{S_n^{(p)}} = (\chi_\lambda \cdot \mathrm{sgn})|_{S_n^{(p)}} = a_2\phi_1 + \phi_1 \cdot \mathrm{sgn}\,.$$

So again by theorem 3.17, a_2 must be also 1, i.e.

$$\chi_\lambda|_{S_n^{(p)}} = \phi_1 + \phi_1 \cdot \mathrm{sgn}\,.$$

This means $\chi_\lambda(\pi) = 0$ for all odd permutation $\pi \in S_n^{(p)}$. Hence by lemma 4.20, it implies that $\lambda = \lambda'$, which gets a contradiction.

If $\lambda = \lambda'$, then by lemma 4.18, there exists a conjugate chatacters f' under S_n which

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is not equal to f such that $\mathrm{Res}_{A_n}^{S_n}(\chi_\lambda) = f + f'$. So

$$r\sigma + r\sigma' = f|_{A_n^{(p)}} + f'|_{A_n^{(p)}} = \mathrm{Res}_{A_n}^{S_n}(\chi_\lambda)|_{A_n^{(p)}} = \begin{cases} \text{either} & \mathrm{Res}_{A_n}^{S_n}(\phi_0) \\ \text{or} & a_1 \mathrm{Res}_{A_n}^{S_n}(\phi_1) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_m), \end{cases}$$

where σ, σ' are conjugate under S_n.

Assume $\sigma \neq \sigma'$. Then

(i) If $r\sigma + r\sigma' = \mathrm{Res}_{A_n}^{S_n}(\phi_0)$, then by lemma 4.12, we have $r = 1$, which is impossible.

(ii) If $r\sigma + r\sigma' = a_1 \mathrm{Res}_{A_n}^{S_n}(\phi_1) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_m)$, then by lemma 4.16, since $m \geq 2$, each $\mathrm{Res}_{A_n}^{S_n}(\phi_i)$ is irreducible. Moreover, since $\sigma \neq \sigma'$, there exists i, j such that

$$\mathrm{Res}_{A_n}^{S_n}(\phi_i) = \sigma \text{ and } \mathrm{Res}_{A_n}^{S_n}(\phi_j) = \sigma'.$$

But by theorem 4.10 (1), $\mathrm{Res}_{A_n}^{S_n}(\phi_i) = \sigma$ and $\sigma \neq \sigma'$ can not hold simultaneously. So this case is impossible.

On the other hand, assume $\sigma = \sigma'$. Then

(i) If $r\sigma + r\sigma' = 2r\sigma = \mathrm{Res}_{A_n}^{S_n}(\phi_0)$, then by lemma 4.12, the case is impossible.

(ii) If $r\sigma + r\sigma' = 2r\sigma = a_1 \mathrm{Res}_{A_n}^{S_n}(\phi_1) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_m)$, then by lemma 4.12, each $\mathrm{Res}_{A_n}^{S_n}(\phi_i)$ is irreducible, and by lemma 4.19, m must be 2, ϕ_1 and ϕ_2 are distinct, and $\phi_2 = \phi_1 \cdot \mathrm{sgn}$. So

$$\chi_\lambda|_{S_n^{(p)}} = a_1\phi_1 + a_2\phi_1 \cdot \mathrm{sgn}\,.$$

Same reason as above, theorem 3.17 gives $a_1 = a_2 = 1$. So

$$2r\sigma = a_1 \mathrm{Res}_{A_n}^{S_n}(\phi_1) + a_2 \mathrm{Res}_{A_n}^{S_n}(\phi_2) = \mathrm{Res}_{A_n}^{S_n}(\phi_1) + \mathrm{Res}_{A_n}^{S_n}(\phi_2) = 2\sigma,$$

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which means $r = 1$, and we gets a contradiction.

Therefore, we get contradictions in all cases, and hence f has the (L', p)-property. This means A_n has the (L', p)-property since f is arbitrary.

$\square$

Remark.

In the proof of theorem 4.15, we show that A_n has $(L'', 2)$-property for all n by using the fact that S_n has $(L'', 2)$-property. However, in the case $p > 2$, proceed as we do in the proof of theorem 4.21, we can only show that A_n has (L', p)-property, not the (L'', p)-property.

Let f be an irreducible character of A_n. Assume $f = \mathrm{Res}_{A_n}^{S_n}(\chi_\lambda)$ for some partition λ of n with $\lambda \neq \lambda'$. Write $\chi_\lambda|_{S_n^{(p)}} = \phi_{\mu_1} + a_2\phi_{\mu_2} + \cdots + a_m\phi_{\mu_m}$ for some $m \geq 1$, $a_i \in \mathbb{N}$, and p-regular partition μ_i of n with $\mu_i \rhd \mu_1$ for all $i \neq 1$. Then

$$f|_{A_n^{(p)}} = \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) + a_2 \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_2}) + \cdots + a_m \mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_m}).$$

If $\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) = \sigma_1 + \sigma_2$ for distinct irreducible p-modular character σ_i of A_n, then

$$f|_{A_n^{(p)}} \geq \sigma_1 \text{ and } \sigma_2 \text{ with multiplicity } 1.$$

However, if $\mathrm{Res}_{A_n}^{S_n}(\phi_{\mu_1}) = \sigma$ for some irreducible p-modular character σ of A_n, $m > 2$, and $\phi_{\mu_2} = \phi_{\mu_1} \cdot \mathrm{sgn}$, then

$$f|_{A_n^{(p)}} \geq \sigma \text{ with multiplicity } 1 + a_2, \text{ which is greater than } 1.$$

Hence if $\chi_\lambda|_{S_n^{(p)}} = \phi_{\mu_1} + a_2\phi_{\mu_1} \cdot \mathrm{sgn} + a_3\phi_{\mu_3} + \cdots + a_m\phi_{\mu_m}$ and $\lambda \neq \lambda'$, then the method used in the proof of theorem 4.21 can not guarantee that there exists an irreducible p-modular character ψ of A_n such that

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$$f \geq \psi \text{ with multiplicity } 1.$$

This case indeed happens. For example, $p = 3$, $n = 9$, and $\lambda = (4,3,2)$. However, A_9 has $(L'', 3)$-property by checking its decomposition matrix (cf., for example, [web2]). It seems that if we want to show A_n has (L'', p)-property for $p > 2$, we need more informations about $\phi_{\mu_2}, \cdots, \phi_{\mu_m}$ and $a_2, \cdots, a_m$, or we have to understand more about p-modular character theory of A_n.

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5 Appendix A: Letter from Jean-Pierre Serre

Paris, 3/8/2017

Dear M. Hao,

I have looked into the exercise you mention (the second of §16.3 of my Representations book), and I am afraid that the proof I had in mind when I wrote it (roughly 40 years ago) is wrong. The correct statement should involve a weaker version of condition (R), namely :

(QR) There exists an integer $N > 0$ *such that* $d(R_K^+) \supset N.R_k^+$.

[In what follows I abbreviate the notation by not writing the group G. Hence R_K means $R_K(G)$; same for R_k, etc.]

Condition (QR) is equivalent to asking that, for every simple $k[G]$-module E, there exists a simple $K[G]$-module F, whose mod p reduction $d([F])$ is a nonzero multiple of $[E]$ (instead of being $[E]$ itself, as in condition (R)).

The correct form of the exercise should have been :

Show that, if K is large enough, then (QR) *is equivalent to* $e(P_A^+) = e(P_A) \cap R_K^+$.

The proof that $(QR) \quad \Rightarrow \quad e(P_A^+) = e(P_A) \cap R_K^+$ is the same as the one in prop.45. The only difference is that, at the end, the formula $d(z_E) = [E]$ should be replaced by $d(z_E) = N.[E]$, for some $N > 0$; one then gets that $N.n_E \geqslant 0$, which means $n_E \geqslant 0$, and we are done.

To prove the converse implication, I find convenient to state a general criterion for surjectivity :

Let V and W be two finite dimensional vector spaces over $\mathbf{Q}$. Let S (resp. T) be a basis for V (resp. W). Define V^+ as the set of linear combinations, with coefficients $\geqslant 0$ of S ; define similarly W^+. Let $d : V \to W$ be a linear map such that $d(V^+) \subset W^+$.

Claim - *The following are equivalent* :
(i) $d(V^+) \;=\; W^+$;
(ii) *Every linear form* ℓ *on* W*, such that* $\ell \circ d$ *is* $\geqslant 0$ *on* V^+*, is* $\geqslant 0$ *on* W^+.

Proof of the claim. Clearly, (i) $\Rightarrow$ (ii). To prove the converse, note that $d(V^+)$ is a closed convex cone in W, hence is an intersection of open half-spaces ; the same is true for W^+ ; if these convex cones were not equal, then one of the half-spaces containing $d(V^+)$ would not contain W^+, and that would give a linear form ℓ contradicting (ii).

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[Note that, in the literature, the basic facts on convex cones are usually proved for vector spaces over **R**, not over **Q**, cf. e.g. Bourbaki EVT II, §5, n°3, cor.5. One needs to use the fact that **Q** is dense in **R** to get them over **Q**.]

Let us apply this to the revised form of the exercise given above. We take $V = \mathbf{Q} \otimes R_K, W = \mathbf{Q} \otimes R_k$, and we choose for S, T their obvious bases (i.e. the classes of simple modules) ; let $d : V \to W$ be the decomposition map, extended by **Q**-linearity.

Remember that we are assuming :

(iii) $e(P_A^+) = e(P_A) \cap R_K^+$,

and we want to prove :

(iv) $d(R_K^+) \supset N.R_k^+$ for a suitable $N > 0$.

We are going to do that by showing that (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv).

The implication (i) $\Rightarrow$ (iv) is almost obvious. If we assume (i), and if z is any element of R_k^+, then there exists an integer $N > 0$ such that Nz is in $d(R_K^+)$, and this gives (iv) (the same N can work for all z).

Let us prove (iii) $\Rightarrow$ (ii). Let ℓ be as in (ii). We want to show that it is $\geqslant 0$ on V^+. By replacing ℓ by a multiple, we may assume that $\ell(T) \subset \mathbf{Z}$. This means that ℓ belongs to the **Z**-dual of R_k, hence can be written as $\ell(x) = x \mapsto \langle p, x\rangle_k$ for some $p \in P_k$. Let $p' = e(p) \in R_K$. For every $z \in R_K^+$, we have :

$$\langle p', z\rangle_K = \langle p, dz\rangle_k = \ell(dz) \geqslant 0,$$

since ℓ is $\geqslant 0$ on R_k^+.

This shows that$\langle p', z\rangle \geqslant 0$ for every $z \in R_K^+$, which is equivalent to $p' \in R_K^+$. Hence $p' \in e(P_A) \cap R_K^+$, which , by (iii), is $e(P_A^+)$. Hence p belongs to P_k^+ and we have $l(z) = \langle p, z\rangle_k \geqslant 0$, which proves (ii).

A curious question arises : is there an example of a group G for which (QR) is true, but not (R) ? We need this to be sure that the original version of the exercise is false ! Unfortunately, it does not seem easy to make such an example. I have asked one or two specialists : no answer yet.

Best wishes,

Yours

J-P. Serre

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6 Appendix B: Proof of Serre's Exercise in Appendix A

In this appendix, we are going to show the modified exercise 16.6, that is,

$$e(P_{K_p}^+(G)) = e(P_{K_p}(G)) \cap R_{K_0}^+(G) \text{ if and only if } NR_{K_p}^+(G) \subset d(R_{K_0}^+(G)) \text{ for some } N \in \mathbb{N},$$

where all notations are as in the introduction.

First we recall the definition of the cde-triangle. If P is a projective K_pG-module, we denote $[P]_{\text{proj}}$ as the image of P in $P_{K_p}(G)$. If E is a K_pG-module, we denote $[E]_p$ as the image of E in $R_{K_p}(G)$. If M is a K_0G-module, we denote $[M]_0$ as the image of M in $R_{K_0}(G)$. The $\mathbb{Z}$-homomorphism $c : P_{K_p}(G) \to R_{K_p}(G)$ is given by

$$[P]_{\text{proj}} \mapsto [P]_p$$

for any projective K_pG-module P. The $\mathbb{Z}$-homomorphism $d : R_{K_0}(G) \to R_{K_p}(G)$ is given by

$$[M]_0 \mapsto [\bar{M}]_p$$

for any K_0G-module M, where $\bar{M}$ is a reduction mod $\mathfrak{m}$ of M (we define it in definition 0.9). Note that $[\bar{M}_1]_p = [\bar{M}_2]_p$ for any two reduction mod $\mathfrak{m}$ $\bar{M}_1, \bar{M}_2$ of M (cf., for example, [S1], theorem 32, page 125). So d is well-defined. Finally, the $\mathbb{Z}$-homomorphism $e : P_{K_p}(G) \to R_{K_0}(G)$ is given by

$$[P]_{\text{proj}} \mapsto [K_0 \otimes_R P']_0$$

for any projective K_pG-module P, where P' is a RG-lattice such that its reduction mod $\mathfrak{m}$ is P. Note that P' must exist and unique up to isomorphism by proposition 4.2. So e is

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well-defined. We can see that c, d and e form the following commutative diagram.

$$\begin{tikzcd} P_{K_p}(G) \arrow[rr, "c"] \arrow[dr, "e"'] & & R_{K_p}(G) \\ & R_{K_0}(G) \arrow[ur, "d"'] & \end{tikzcd}$$

Hence it is called the cde-triangle.

Moreover, let $M_1, \cdots, M_a$ be all distinct simple K_0G-modules, $E_1, \cdots, E_b$ be all distinct simple K_pG-modules, and P_i be the projective envelope of E_i (we define it in definition 4.3) for $i = 1, \cdots, b$. Then

$$[P_1]_{\text{proj}}, \cdots, [P_b]_{\text{proj}}$$

is a $\mathbb{Z}$-basis of $P_{K_p}(G)$,

$$[E_1]_p, \cdots, [E_b]_p$$

is a $\mathbb{Z}$-basis of $R_{K_p}(G)$, and

$$[M_1]_0, \cdots, [M_a]_0$$

is a $\mathbb{Z}$-basis of $R_{K_0}(G)$. We write

$$\begin{aligned} c([P_j]_{\text{proj}}) &= [P_j]_p = \sum_{i=1}^{b} c_{ij}[E_i]_p \text{ for } j = 1, \cdots, b \\ d([M_j]_0) &= [\bar{M}_j]_p = \sum_{i=1}^{b} d_{ij}[E_i]_p \text{ for } j = 1, \cdots, a \\ e([P_j]_{\text{proj}}) &= [K_0 \otimes_R P'_j]_0 = \sum_{i=1}^{a} e_{ij}[M_i]_0 \text{ for } j = 1, \cdots, b, \end{aligned}$$

and put

$$C = (c_{ij}) \in M_{b\times b}((\mathbb{Z}_{\geq 0}), D = (d_{ij}) \in M_{b\times a}(\mathbb{Z}_{\geq 0}), \text{ and } E = (e_{ij}) \in M_{a\times b}(\mathbb{Z}_{\geq 0}).$$

Then $C = DE$ (because $c = d \circ e$) and $E = D^t$ (cf., for example, [S1], 15.4 (c), page

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127), where t denotes the matrix transpose. The matrix C is a Cartan matrix, and D is a decomposition matrix.

Note that for $i = 1, \cdots, b$, if we denote e_i as the column vector of dimension b which is 1 in the i-th coordinate, and 0 in other coordinates, then

(1) We say G has the (L, p)-property is equivalent to say that if we can find the following vectors in columns of D:

$$x_1 e_1, \cdots, x_b e_b \text{ for some } x_i \in \mathbb{N},$$

then we can also find the following vectors in columns of D:

$$e_1, \cdots, e_b.$$

(2) We say G has the (L', p)-property is equivalent to say that each column of D can not be one of the following forms:

$$x_1 e_1, \cdots, x_b e_b \text{ for some } x_i \in \mathbb{N} \text{ with } x_i > 1.$$

(3) We say G has the (L'', p)-property is equivalent to say that each column of D exists an entry 1.

Now we show the modified exercise 16.6:

$e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G)$ if and only if $NR^+_{K_p}(G) \subset d(R^+_{K_0}(G))$ for some $N \in \mathbb{N}$.

Proof.

Denote a and b as numbers of simple K_0G-modules and simple K_pG-modules respectively. Assume $e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G)$, we claim that $NR^+_{K_p}(G) \subset d(R^+_{K_0}(G))$

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for some $N \in \mathbb{N}$. To show this, first note that the condition $e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G)$ holds if and only if the matrix $E = (e_{ij}) \in M_{a\times b}(\mathbb{Z}_{\geq 0})$ with respect to e satisfies the property $(*)$:

$(*)$: all entries of $x \in \mathbb{Z}^b$ are non-negative if and only if so are Ex.

Suppose $e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G)$. Fix any $s \in \{1, \cdots, a\}$. Denote u_s as the row vector which is 1 in the s-th coordinate, and 0 in other coordinates. Assume E has no row of the form βu_s ($\beta \neq 0$). Then for any $i = 1, \cdots, a$, there exists $j \neq s$ such that

$$e_{ij} > 0.$$

So there exists $x_{ij} \in \mathbb{Z}_{\geq 0}$ such that

$$-e_{is} + x_{ij}e_{ij} > 0.$$

Moreover, for other $j \neq s$ such that $e_{ij} = 0$, define $x_{ij} = 1$. Let $\alpha_j = \max(x_{1j}, \cdots, x_{aj})$ for $j = 1, 2, \cdots, b$ except s, and let (t denotes the tranpose)

$$x^t = (\alpha_1, \cdots, \alpha_{s-1}, -1, \alpha_{s+1}, \cdots, \alpha_b),$$

then all entries of Ex are positive, but the s-th coordinate of x is not. This gets a contradiction. It follows that E must have a row of the form βu_s for all $s = 1, \cdots, a$, i.e. E can

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be written in the form (rearrange rows if necessary):

$$\begin{bmatrix} e_{11} & 0 & 0 & \cdots & 0 \\ 0 & e_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e_{bb} \\ e_{b+1,1} & & \cdots & & e_{b+1,b} \\ \vdots & & \cdots & & \vdots \\ e_{a1} & & \cdots & & e_{ab} \end{bmatrix}, \text{ where } e_{ii} \neq 0 \text{ for all } i.$$

Now, since the decomposition matrix $D = E^t$, we complete our claim by taking N be the least common multiple of $e_{11}, ..., e_{bb}$.

On the other hand, we assume $NR^+_{K_p}(G) \subset d(R^+_{K_0}(G))$ for some $N \in \mathbb{N}$. Fix any $s \in \{1, \cdots, a\}$. Let u_s be as above and let $D = (d_{ij}) \in M_{b\times a}(\mathbb{Z}_{\geq 0})$ be the decomposition matrix. Then our assumption means there exist integers $v_1, \cdots, v_a \geq 0$ such that

$$v_1 \begin{pmatrix} d_{11} \\ \vdots \\ d_{b1} \end{pmatrix} + \cdots + v_a \begin{pmatrix} d_{1a} \\ \vdots \\ d_{ba} \end{pmatrix} = N{u_s}^t.$$

Assume D has no column of the form $\beta {u_s}^t$ ($\beta \neq 0$). Then for any j, there exists $i \neq s$ such that

$$d_{ij} > 0.$$

If $v_j > 0$ for some j, then $v_j d_{ij} > 0$ for i as above. This gets a contradiction since

$$v_1 d_{i1} + \cdots + v_a d_{ia} = 0$$

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for all $i \neq s$. So $v_j = 0$ for all j. But this also gets a contradiction since

$$v_1 d_{s1} + \cdots + v_a d_{sa} = N.$$

Therefore D must have a column of the form $\beta {u_s}^t$. This means E has a row of the form βu_s for all $s = 1, \cdots, a$ since $E = D^t$. This implies E satisfies $(*)$, i.e. $e(P^+_{K_p}(G)) = e(P_{K_p}(G)) \cap R^+_{K_0}(G)$ as desired.

$\square$

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