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博士論文

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三維代數多樣體
Geometry of Algebraic Threefolds

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中文摘要

這篇論文包含兩個部份。於第一部份我們證明了一個平滑三維多樣體和其極小模型的貝堤數的差可被該平滑三維多樣體的皮喀數所限制。於第二部份我們證明了任一小平維度為一的平滑三維多樣體的第九十六個複正則系統會決定其飯高纖維。

關鍵詞：

複三體，極小模型計劃，貝堤數，複正則系統，有效飯高猜想，黑肯-瑪柯能猜想。

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Abstract

This thesis consists of two parts. In the first part we prove that the difference of the Betti numbers of a smooth threefold and its minimal model can be bounded by a constant depending only on the Picard number of the smooth threefold. In the second part we prove that the 96-th pluricanonical system of a smooth threefold of Kodaira dimension one defines the Iitaka fibration.

Keywords:

Complex threefolds, minimal model program, Betti numbers, pluricanonical systems, effective Iitaka conjecture, Hacon-M^{c}Kernan conjecture.

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Chapter 1

Introduction

In birational geometry, people study algebraic varieties in birational ways. That is, comparing varieties which are isomorphic on a dense open set. If two varieties are birational, then they have the same function field. Hence the category of birational classes is a reasonable category and the theoreies in birational geometry could have many applications in other branch of algebraic geometry. There is a fundamental approach in birational geometry, so called the minimal model program, concerning with how to find a good representative in each birational class. In the end of 20 century, the minimal model program for varieties with mild singularities and dimension less than or equal to three was established. Over the past twenty years, the three-dimensional minimal model program becomes a foundation of three-dimensional birational geometry.

After the minimal model theorem being proved, there are two basic approaches in the three-dimensional birational geometry: study minimal threefolds, and compare the given smooth threefold with its minimal model. A minimal threefold has several good property. For example, the abundance conjecture is known to be true in dimension three, so that a sufficiently divisible pluricanonical system of a minimal threefold is base point free and defines the Iitaka fibration. If additionally we assume that the given minimal threefold is of general type, then the plurigenura can be directly computed and one can estimate some important geometric invariants such as the volume or the self-intersection of the canonical divisor. J. A. Chen and M. Chen have work on minimal threefolds of general type for many years. They have good estimate for the volume of threefolds of general type, and they can precisely describe those minimal threefolds of general type with small volume.

The second approach is to explicitly describe each step of the minimal model program. Three-dimensional minimal model program consists divisorial contrations to points or curves

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and flips. Hayakawa, Kawakita and Yamamoto had classified three-dimensional divisorial contractions to points. J. A. Chen and Hacon proved that one can factorize divisorial contractions to curves and flips into (inverse of) divisorial contractions to points, blowing-up local complete intersection curves and flops. Hence the comparison of a smooth threefold and its minimal model is possible nowadays.

In this paper we discuss two different problems, which correspond to the two approaches we mentioned above. The first problem is the comparison of Betti numbers between a smooth threefold and its minimal model. This problem is motivated by the estimate of the self-intersection of the canonical divisor. The self-intersection of the canonical divisor is an important geometric quantity. For smooth varieties, this number is exactly the first Chern number (up to a sign), which could be linked to the theories of algebraic topology. Since this number is not a birational invariant, it is an important issue that how dose this number change under birational morphisms. In dimension three, one can prove that under elementary birational morphisms the change of this number can be bounded by Betti numbers. Hence it becomes an interesting problem that how does Betti numbers change under elementary birational maps. We will prove that among steps of minimal model program begin with a smooth threefold, the change of Betti numbers could be bounded by some constant depending on the Picard number of the original threefold.

Another problem we are going to discuss is to study threefolds via its Iitaka fibration. Given a smooth threefold, the Kodaira dimension could be $-\infty$, 0, 1, 2 or 3. Threefolds of Kodaira dimension three, so call threefolds of general type, were recently studied in detail by J. A. Chen and M. Chen. They can estimate the volume of threefolds of general type and they proved that the 61-th pluricanonical system defines the Iitaka fibration. They can also describe the extreme cases in detail. In the Kodaira dimension two case, since the Iitaka fibration gives an elliptic fibration structure, threefolds of Kodaira dimension two can be studied simply using Fujino-Mori's canonical bundle formula. Ringler have proved that the 48-th pluri canonical system defines the Iitaka fibration. For threefolds of Kodaira dimension zero, Kawamata and Morrison proved that the m_0-pluricanonical system is non-empty where $m_0 = 2^5 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19$.

We will study threefolds of Kodaira dimension one here. The main difficulty is that the technique used to study threefolds of Kodaira dimension 2 or 3 do not behave well in Kodaira dimension one case. In the research of threefolds of general type people use Reid's singular Riemann-Roch formula to compute the dimension of the pluricanonical system. This technique

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do not work in non-general type cases. When dealing with threefolds of Kodaira dimension two the Iitaka fibration is an elliptic fibration, so the canonical bundle formula were simple. However in our situation the Iitaka fibration is a two-dimensional fibration and the canonical bundle formula has very large denominators, so we can not get any reasonable bound from it. The solution is to apply the theories of terminal threefolds and study the structure of Iitaka fibration at the same time. We will prove that the 96-th pluricanonical system defines the Iitaka fibration and we can give the geometric description in the extreme cases.

This thesis is organized as follows. Chapter 2 is a preliminary section. We will introduce of minimal model program and several basic theories for terminal threefolds. We will discuss the change of Betti numbers in the minimal model program in Chapter 3 and threefolds of Kodaira dimension one in Chapter 4.

1.1 Convention and notation

Through this paper a variety is always projective and over complex numbers.

The word "divisor" always means a Weil divisor. A $\mathbb{Q}$-divisor means a finite sum of prime divisors $D = \sum_{i=1}^{n} a_i D_i$ such that $a_i \in \mathbb{Q}$. The round down of D is the integral divisor $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$, where $\lfloor a_i \rfloor$ is the largest integer smaller than or equal to a_i. Similarly the round up of D is the integral divisor $\lceil D \rceil = \sum \lceil a_i \rceil D_i$ with $\lceil a_i \rceil$ being the smallest integer greater than or equal to a_i.

For any divisor D on an variety X, $\mathcal{O}_X(D)$ will denote the sheaf associated to D. Given two divisor D and D', the notation $D \sim D'$ means D is linear equivalent to D'. Assume that both D and D' are $\mathbb{Q}$-Cartier divisors, then the notation $D \equiv D'$ means D is numerically equivalent to D', i.e. $D.C = D'.C$ for any curve C.

1.2 Algebraic geometric background

1.2.1 Ample, nef and big divisors

Recall that a Cartier divisor D is said to be ample if nD is very ample for some integer n, that is, $|nD|$ defines an embedding to a projective space.

Theorem 1.2.1 (Nakai-Moishezon criterion, cf. [KM92] Theorem 1.37)**.** *Let X be a proper scheme over a field and D be a Cartier divisor on X. Then D is ample if and only if* $D^{\dim Z}.Z > 0$

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for any closed integral subscheme $Z \subset X$.

Definition. A Cartier divisor D is said to be *nef* if $D.C \geq 0$ for any curve C.

Definition. A Cartier divisor D on an n-dimensional projective variety X is called *big* if

$$h^0(X, \mathcal{O}_X(kD)) > ck^n$$

for some $c > 0$ and and for all $k \gg 1$.

Lemma 1.2.2 ([KM98], Lemma 2.60)**.** *Let X be an n-dimensional projective variety and D be a Cartier divisor. The following are equivalent:*

1. *D is big.*
2. *$mD \sim A + E$ where A is ample and E is effective for some $m > 0$.*
3. *For some $m > 0$ the rational map $\phi_{|mD|}$ is birational.*

1.2.2 Iitaka fibration and the Kodaira dimension

Let X be a normal algebraic variety and L be a line bundle on X such that $H^0(X, L) \neq 0$. Then exists a rational map $\phi_{|L|} X \dashrightarrow \mathbb{P}H^0(X, L)$.

Definition. Assume that there exists $n \in \mathbb{N}$ such that $H^0(X, L^n) \neq 0$. We define the Iitaka dimension of L to be

$$\kappa(X, L) = \max \left\{ \dim im(\phi_{|L^n|}) \,\middle|\, n \in \mathbb{N} \right\}.$$

If $H^0(X, nL) = 0$ for all $n \in \mathbb{N}$, then we define $\kappa(X, L) = -\infty$.

Now assume that X is smooth, then we define the Kodaira dimension of X to be $\kappa(X, K_X)$ and we will denote it by $\kappa(X)$. When X is singular, we define $\kappa(X) = \kappa(\tilde{X})$ for any smooth model $\tilde{X}$ of X.

Note that if X and X' are two smooth varieties birational to each other, then there exists a smooth variety $\tilde{X}$ such that there exists birational morphisms $f : \tilde{X} \to X$ and $f' : \tilde{X} \to X'$. One has $K_{\tilde{X}} = f^* K_X + E = f'^* K_X + E'$ where E and E' are effective exceptional divisors. It follows that

$$\dim im(\phi_{|nK_X|}) = \dim im(\phi_{|nK_U|}) = \dim im(\phi_{|nK_{f^{-1}U}|}) = \dim im(\phi_{|nK_{\tilde{X}}|})$$

where U is the open set on X such that $f|_U$ is isomorphic. This tell us that $\kappa(X) = \kappa(\tilde{X})$ and similarly $\kappa(X') = \kappa(\tilde{X})$. Thus the definition of Kodaira dimension is well-defined.

Theorem 1.2.3 (Iitaka fibration theorem, [Laz04] Theorem 2.1.19)**.** *Let X be a normal projective variety and L be a line bundle on X such that $\kappa(X, L) > 0$. Then there exists an algebraic fiber space (i.e. surjective morphism with connected fiber) $\phi_\infty : X_\infty \to Y_\infty$ such that for all sufficiently large and divisible integer k, we have the following commutative diagram*

$$\begin{tikzcd}
X \arrow[d, dashed, "\phi_k"'] & X_\infty \arrow[l, "u_\infty"'] \arrow[d, "\phi_\infty"] \\
Y_k & Y_\infty \arrow[l, dashed]
\end{tikzcd},$$

where $\phi_k = \phi_{|L^k|}$, Y_k is the closure of $im(\phi_{|L^k|})$, the horizontal maps are birational and $(u_\infty^* L)|_F$ has Kodaira dimension zero for a very general fiber F of $X_\infty \to Y_\infty$.

Chapter 2

Minimal Model Program and Terminal Threefolds

The goal of the minimal model program is to find a good representative in each birational class. Assume that X is a smooth projective curve, then any smooth curve Y birational to X is in fact isomorphic to X. Hence any birational class for algebraic curves has a unique smooth element. However, when the dimension is greater than one, things become complicated.

Assume X is a smooth variety of dimension greater than or equal to two. One can always blow-up a point on X, and obtained another smooth variety which is birational to X. The original idea of the minimal model program is to find a variety which is not the blow-up of other smooth variety. This approach works in dimension two. Assume that X is a projective smooth surface, then one can always contract a minus-one curve (a rational curve which has self-intersection -1) on it and get another smooth surface which contains no -1-curves (so that it is not a blow-up of any other smooth surface). This surface is called the minimal model of X. The word minimal means that it can not map to any other smooth surface birationally.

In higher dimensional cases things become more and more complicated. Instead of working on the category of smooth varieties, one should study varieties with mild singularities, or even more generally, study a pair. That is, a variety plus a boundary divisor. The approach of higher dimensional minimal model program leads to lots of interesting problems and there are rich theories associated to it.

The category of terminal varieties is the smallest category such that the minimal model program works. In dimension less than three being terminal is equivalent to being smooth. However, there exist non-smooth terminal threefolds. Fortunately, three-dimenisonal terminal sin-

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gularities are not hard to study: their canonical cover has only compound Du Val singularities. Reid and Mori give a complete classification of three-dimenisonal terminal singularities. With the help of the classification one can study terminal threefolds in detail.

This chapter contains two parts. In the first section we will introduce the notion of minimal model program. In the second section we will quickly review some known results about terminal threefolds.

2.1 Minimal model program

2.1.1 Minimal model program for surfaces

Through this subsection S will denote a complex projective smooth surface.

Castelnuovo's contraction theorem

Definition. An irreducible curve $C \subset S$ is called a -1-curve if C is a smooth rational curve and $C^2 = -1$.

Theorem 2.1.1 (Castelnuovo's contraction theorem, [Bea78] Theorem II.17)**.** *Assume that C is a -1-curve on S. Then there is a birational morphism $f : S \to S'$ to a smooth surface S', such that $f(C)$ is a point and $S - C$ is isomorphic to $S' - f(C)$.*

Note that the Picard number decreased by one after constracting a -1-curve. Since the Picard number is always a positive integer, there are only finitely many -1-curve on a smooth surface. Thus Castelnuovo's contraction theorem implies that there exists a birational morphism $S \to S_{min}$ for some smooth surface S_{min}, such that S_{min} do not contain any -1-curve. Such kind of surfaces are called minimal surfaces.

If we blow-up a point on a smooth surface, then the exceptional divisor is a -1-curve. Hence if S is minimal, then S cannot be the blowing-up of other smooth surfaces. In fact, by Theorem 2.1.2, there is no birational morphism $S \to S'$ to any smooth surface S'. Thus S is a minimal element in the smooth birational class, where the partial order in defined by $S \geq S'$ if there exists a morpshim $S \to S'$.

Theorem 2.1.2 ([Bea78], Theorem II.11)**.** *Let $f : S \to S_0$ be a birational morphism. Then*

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there exists a sequence of blow-ups

$$S \cong S_k \to S_{k-1} \to \dots \to S_1 \to S_0$$

such that f is the composition of above morphisms.

Classification of algebraic surfaces

Definition. A surface is said to be *ruled* if it is birational to $C \times \mathbb{P}^1$. If $C \cong \mathbb{P}^1$, then the surface is said to be rational.

The Hirzebruch surfaces $\mathbb{F}_n$ is defined by $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, for all integer $n \geq 0$.

Theorem 2.1.3 ([Bea78], Theorem V.10)**.** *Let S be a minimal rational surface. Then S is isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \neq 1$.*

We remark that $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blowing-up a point, which is not minimal.

Theorem 2.1.4 (Enriques' Theorem, [Bea78] Theorem VI.17)**.** *Assume that S is a smooth surface and $H^0(S, 12K_S) = 0$, then S is ruled. In particular, any minimal surface with negative Kodaira dimension is ruled.*

Theorem 2.1.5 ([Bea78], Theorem VIII.2)**.** *Assume that S is minimal of Kodaira dimension zero, then S belongs to one of the four following cases:*

1. *$p_g = q = 0$ and $2K \sim 0$. We say S is an Enriques surface.*
2. *$p_g = 0$, $q = 1$ and S is a bielliptic surface.*
3. *$p_g = 1$, $q = 0$ and $K \sim 0$. We say S is a K3 surface.*
4. *$p_q = 1$, $q = 2$ and S is an abelian surface.*

Proposition 2.1.6 ([Bea78], Proposition IX.2 (b))**.** *Assume that S is a minimal surface of Kodaira dimension one. Then there is a smooth curve B and a surjective morphism $S \to B$ whose generic fiber is an elliptic curve.*

Theorem 2.1.7 ([Iit70], Corollary after Proposition 8)**.** *Assume that S is an surface of Kodaira dimension one. Then $|86K_S|$ defines the Iitaka fibration.*

Theorem 2.1.8 ([Bom70])**.** *Assume that S is a surface of general type, then $|5K_S|$ defines a birational map.*

2.1.2 Cone theorem

Definition. Let $K = \mathbb{Q}$ or $\mathbb{R}$ and V be a K-vector space. A subset $N \subset V$ is called a cone if $0 \in N$ and N is closed under multiplication by positive scalars.

A subcone $M \subset N$ is called *extremal* if $v, w \in N$ and $v + w \in M$ implies v and $w \in M$. An one dimensional extremal subcone is called an extremal ray.

Definition. Let X be a proper variety. A 1-cycle is a formal linear combination of irreducible, reduced and proper curves $C = \sum_i a_i C_i$. We say C is effective if $a_i \geq 0$ for all i. Given two 1-cycles C and C', we say that C is numerically equivalent to C', denoted by $C \equiv C'$, if $D.C = D.C'$ for all Cartier divisor D on X. The space of 1-cycles on X with real coefficient module numerical equivalence is denoted by $N_1(X)$, which is a finite dimensional vector space.

We will denote $NE(X) = \{\text{effective 1-cycles}\} \subset N_1(X)$ and $\overline{NE(X)}$ is the closure of $NE(X)$ in $N_1(X)$.

Theorem 2.1.9 (Cone theorem, [KM98], Theorem 1.24)**.** *Let X be a smooth projective variety.*

1. *There are countably many rational curves $C_j \subset X$ such that $0 < -K_X.C_j \leq \dim X + 1$, and*

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

2. *For any $\epsilon > 0$ and ample divisor H,*

$$\overline{NE(X)} = \overline{NE(X)}_{K_X + \epsilon H \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_j].$$

Theorem 2.1.10 ([Mor82], Theorem 3.1)**.** *Let X be a smooth projective threefold and $R \subset \overline{NE(X)}$ be an extremal ray, then there exists a morphism $\phi : X \to Y$ such that $\phi_* \mathcal{O}_X = \mathcal{O}_Y$ and for any irreducible curve C in X, $\phi(C)$ is a point if and only if $[C] \in R$.*

Theorem 2.1.11 ([Mor82], Theorem 3.3, Corollary 3.4)**.** *Notation as in the previous theorem. Assume that R is not numerically effective (that is, there exists an effective divisor D such that $D.C < 0$ for some curve C with $[C] \in R$), then there is a divisor D on X such that ϕ is isomorphism on $X - D$ and $\dim \phi(D) \leq 1$. One of the following holds.*

1. *Y is smooth and $X = Bl_{\phi(D)} Y$, $\phi(D)$ is either a point or a smooth curve.*

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2. *$\phi(D) = Q$ is a point. The completion $\mathcal{O}_{\hat{Y},Q}$ of $\mathcal{O}_{Y,Q}$ is given by*

$$k[[x,y,z,u]]/(x^2+y^2+z^2+u^2), \quad k[[x,y,z,u]]/(x^2+y^2+z^2+u^3) \quad or \quad k[[x,y,z]]^{\mathbb{Z}_2},$$

where the $\mathbb{Z}_2$ action is given by $(x,y,z) \mapsto (-x,-y,-z)$.

Theorem 2.1.12 ([Mor82], Corollary 3.5)**.** *If R is numerically effective, then Y is smooth and we have*

1. $\dim Y = 2$, *$X \to Y$ is a conic bundle (i.e., the generic fiber X_η is a conic in $\mathbb{P}^2$).*

2. $\dim Y = 1$, *$X \to Y$ is a del Pezzo fibration (i.e. the generic fiber X_η is an irreducible reduced surface such that $-K_{X_\eta}$ is ample).*

3. $\dim Y = 0$, *$-K_X$ is ample.*

2.1.3 Singularities in minimal model program

Definition. A pair (X, Δ) is a normal projective variety X and an (effective) $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Definition. Let (X, Δ) be a pair over complex numbers. Let E be an exceptional divisor over X. We define the discrepancy of E over (X, Δ), denoted by $a(X, \Delta, E)$, to be the coefficient of $K_Y - f^*(K_X + \Delta)$ along E.

We say that (X, Δ) is terminal (resp. canonical, log terminal, kawamata log terminal, log canonical) if $a(X, \Delta, E) > 0$ (resp. ≥ 0, > -1, > -1 and $\lfloor \Delta \rfloor = 0$, ≥ -1).

Those singularities has their geometric meaning. The category of terminal varieties is the smallest category which is closed under minimal model program, and the category of varieties of kawamata log terminal (klt for short) singularities is the largest category such that the minimal model program works. Canonical singularities coming from the pluricanonical maps of smooth varieties of general type. Log canonical singularities are the worst singularities which can be described using the language of the discrepancy.

2.1.4 Higher dimensional minimal model program

Theorem 2.1.13 (Relative cone theorem, [KM98] Theorem 3.25)**.** *Let (X, Δ) be a klt pair and assume that Δ is effective. Let $g : X \to Z$ be a projective morphism. Then*

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1. *There are countably many rational curves* $C_j \subset X$ *such that* $g(C_j)$ *is a point,* $0 < -K_X.C_j \leq 2\dim X$ *and*

$$\overline{NE(X/Z)} = \overline{NE(X/Z)}_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

2. *For any* $\epsilon > 0$ *and* f*-ample divisor* H*,*

$$\overline{NE(X/Z)} = \overline{NE(X/Z)}_{K_X+\Delta+\epsilon H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_j].$$

3. *Let* $F \subset \overline{NE(X/Z)}$ *be a* K_X*-negative extremal subcone. Then there exists a unique morphism* $f_F : X \to Y$ *over* Z*, such that* $(f_F)_*\mathcal{O}_X = \mathcal{O}_Y$ *and an irreducible curve* C *maps to a point if and only if* $[C] \in F$*.*

4. *Assume that L is a line bundle such that* $L.C = 0$ *for all curve* C *such that* $[C] \in F$*. Then* $L \cong f_F^* L_Y$ *for some line bundle* L_Y *on* Y*.*

Definition. Let (X, Δ) be a pair. A $K_X+\Delta$-flipping contraction is a proper birational morphism $f : X \to Y$ such that $Exc(f)$ has codimension greater than or equal to two and $-(K_X + \Delta)$ is f-ample.

A normal variety X^+ together with a proper birational morphism $f_+ : X^+ \to Y$ is called a $K_X + \Delta$-flip of f if $K_{X^+} + \Delta_+$ is $\mathbb{Q}$-Cartier and f_+-ample and $Exc(f_+)$ has codimension at least two, where Δ_+ is the birational transform of Δ.

Definition. Let (X, Δ) be a pair. A proper morphism $f : X \to S$ is called a *Mori fiber space* if $\dim S < \dim X$, $\rho(X/S) = 1$ and $-(K_X + \Delta)$ is f- ample.

Now let (X, Δ) be a $\mathbb{Q}$-factorial projective klt pair. The ideal minimal model program goes as follows: Assume that $K_X + \Delta$ is nef, then we have done. If it is not nef, then there is an $K_X + \Delta$-negative extremal ray. By Theorem 2.1.13 one can contract the extremal ray and get a morphism $f : X \to Y$ such that $-(K_X + \Delta)$ is f-ample. We have the following three possible situaiton.

1. $X \to Y$ is a divisorial contraction. Note that Y is $\mathbb{Q}$-factorial and $(Y, f_*\Delta)$ is klt by Proposition 2.1.14 and Lemma 2.1.16. In this case we replace (X, Δ) by $(Y, f_*\Delta)$ and continue the process. We have $\rho(Y) = \rho(X) - 1$, hence there are only finitely many divisorial contractions could occur in this process.

2. $X \to Y$ is a flipping contraction. We have to construct a $K_X + \Delta$-flip $X \dashrightarrow X^+$. By Proposition 2.1.15 and Lemma 2.1.16 X^+ is $\mathbb{Q}$-factorial and (X^+, Δ^+) is klt provided that X^+ exists, so one can keep doing this process with (X^+, Δ^+).

3. $\dim Y < \dim X$. By Proposition 2.1.14 we have $\rho(X/Y) = 1$ and we get a Mori fiber space.

Proposition 2.1.14 ([KM98], Proposition 3.36)**.** *Let (X, Δ) be a porjective $\mathbb{Q}$-factorial klt pair and $g : X \to Y$ is a contraction of $K_X + \Delta$-negative extremal ray. Assume that either g is divisorial or* $\dim Y < \dim X$*, then Y is also $\mathbb{Q}$-factorial and $\rho(Y) = \rho(X) - 1$.*

Proposition 2.1.15 ([KM98], Proposition 3.37)**.** *Let (X, Δ) be a porjective $\mathbb{Q}$-factorial klt pair and $X \dashrightarrow X^+$ is a $K_X + \Delta$-flip. Then X^+ is $\mathbb{Q}$-factorial and $\rho(X^+) = \rho(X)$.*

Lemma 2.1.16 ([KM98], Lemma 3.38)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& Y &
\end{tikzcd}$$

where X, X' and Y are normal varieties and f, f' are proper and biraitonal. Let Δ (resp. Δ') be a $\mathbb{Q}$-divisor on X (resp. X') such that $f_*\Delta = f'_*\Delta'$, and both $-(K_X + \Delta)$ and $K_{X'} + \Delta'$ are $\mathbb{Q}$-Cartier and f-nef. Then for any exceptional divisor E over Y, we have

$$a(E, X, \Delta) \leq a(E, X', \Delta').$$

Strict inequality holds if either $-(K_X + \Delta)$ is f-ample and f is not an isomorphism above the generic point of $Center_Y E$, or $K_{X'} + \Delta'$ is f'-ample and f is not an isomorphism above the generic point of $Center_Y E$.

To successfully run the minimal model program there are two problems should be solved: the existence of flips and the termination of flips. In dimension three it had been already solved.

Theorem 2.1.17 ([Mor88], Theorem 0.2.5)**.** *Let $X \to Y$ be a three-dimensional flipping contraction, then the K_X-flip exists.*

Theorem 2.1.18 (Shokurov, cf. [Mor88] Theorem 0.2.7)**.** *To each algebraic threefold with only terminal singularities there is a well-defined non-negative integer $d(X)$ called the difficulty, such that $d(X^+) < d(X)$ if $X \dashrightarrow X^+$ is a K_X-flip.*

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Corollary 2.1.19 (Minimal model program for terminal threefolds)**.** *Let X be a $\mathbb{Q}$-factorial projective terminal threefold. Then there exists a sequence of birational morphism between $\mathbb{Q}$-factorial terminal threefolds*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_{k-1} \dashrightarrow X_k$$

such that $X_i \dashrightarrow X_{i+1}$ is either a divisorial contraction or a flip, and X_k is either minimal (that is, K_{X_k} is nef), or has a Mori fiber space structure.

For dimension greater than three the existence of flips and termination of flips are still open. In [BCHM10], Birkar, Cascini, Hacon and M$^{\text{c}}$Kernan proved that the existence and termination of some special flips, and proved that in very general case the minimal model program still works.

Theorem 2.1.20 ([BCHM10], Theorem 1.2)**.** *Let (X, Δ) be a klt pair. Let $\pi : X \to U$ be a projective morphism of quasi-projective varieties.*

If either Δ is π-big and $K_X + \Delta$ is π-pseudo-effective or $K_X + \Delta$ is π-big, then

1. *$K_X + \Delta$ has a minimal model over U.*
2. *The $\mathcal{O}_U$-algebra $\bigoplus_{m\in\mathbb{N}} \pi^*\mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor)$ is finitely generated.*

Corollary 2.1.21 ([BCHM10], Corollary 1.1.1)**.** *Let X be a smooth projective variety of general type, then*

1. *X has a minimal model.*
2. *X has a canonical model.*
3. *The ring $\bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor)$ is finitely generated.*

2.1.5 Abundance conjecture

Assume that (X, Δ) is a pair such that its minimal model (X_{min}, Δ_{min}) exists. People expect that the minimal model satisfied the following good property.

Conjecture 2.1.22 (Abundance conjecture)**.** Assume that (X_{min}, Δ_{min}) is minimal, then $K_{X_{min}} + \Delta_{min}$ is semi-ample. That is, $|m(K_{X_{min}} + \Delta_{min})|$ is basepoint-free for some $m \in \mathbb{N}$.

The abundance conjecture is known for lower-dimensional varieties.

Theorem 2.1.23 (Abundance theorem for log surfaces, cf. [Kol92] Theorem 11.1.3)**.** *Let* (X, Δ) *be a two-dimenisonal minimal log canonical pair. Then* $K_X + \Delta$ *is semi-ample.*

Theorem 2.1.24 (Abundance theorem for terminal threefolds, cf. [Kol92] Theorem 11.1.1)**.** *Let* X *be a minimal terminal threefold. Then* K_X *is semi-ample.*

2.2 Terminal threefolds

Through this section X will be a terminal threefold.

2.2.1 Classification of terminal threefolds

The local classification of terminal threefolds were done by Reid [Rei83] for Gorenstein case and Mori [Mor85] for non-Gorenstein case.

Definition. A compound Du Val point $P \in X$ is a hypersurface singularity locally analytically defined by $f(x, y, z) + tg(x, y, z, t) = 0$, where $f(x, y, z)$ defines a Du Val singularity.

Theorem 2.2.1 ([Rei83], Theorem 1.1)**.** *Let* $P \in X$ *be a point of threefold. Then* $P \in X$ *is an isolated compound Du Val point if and only if* $P \in X$ *is terminal of index one.*

Let G be a cyclic group of order r and assume that G acts on $\mathbb{A}^n_{(x_1,...,x_n)}$ by $x_i \mapsto \xi_r^{a_i} x_i$, where ξ_r is a fixed r-th roots of unity and $a_i \in \mathbb{Z}$. We will denote

$$\mathbb{A}^n/G = \mathbb{A}^n/\frac{1}{r}(a_1, ..., a_n).$$

Theorem 2.2.2 ([Mor85], cf. [Rei87] Theorem 6.1)**.** *Let* $P \in X$ *be a germ of three-dimensional terminal singularity of index* $r > 1$*. Then*

$$X \cong (f(x, y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(a_1, ..., a_4)$$

such that $f(x, y, z, u)$*,* r *and* a_i *is given by Table 2.1.*

By [Rei87, (6.4)] any three-dimensional terminal singularity of indice greater than one could be deformed to cyclic quotient singularities. These data is called the *basket* of the singularity.

Type	$f(x,y,z,u)$	r	a_i	condition
cA/r	$xy + g(z^r, u)$	any	$(\alpha, -\alpha, 1, r)$	$g \subset m_P^2$ α and r are coprime
$cAx/4$	$xy + z^2 + g(u)$ $x^2 + z^2 + g(y,u)$	4	$(1,1,3,2)$	$g \in m_P^3$
$cAx/2$	$xy + g(z,u)$	2	$(0,1,1,1)$	$g \in m_P^4$
$cD/3$	$x^2 + y^3 + z^3 + u^3$ $x^2 + y^3 + z^2u + yg(z,u) + h(z,u)$ $x^2 + y^3 + z^3 + yg(z,u) + h(z,u)$	3	$(0,2,1,1)$	$g \in m_P^4$ $h \in m_P^6$
$cD/2$	$x^2 + y^3 + yzu + g(z,u)$ $x^2 + yzu + y^n + g(z,u)$ $x^2 + yz^2 + y^n + g(z,u)$	2	$(1,0,1,1)$	$g \in m_P^4, \quad n \geq 4$ $n \geq 3$
$cE/2$	$x^2 + y^3 + yg(z,u) + h(z,u)$	2	$(1,0,1,1)$	$g, h \in m_P^4$ $h_4 \neq 0$

Table 2.1: Classification of terminal threefolds

Type	deformation	general elephant	basket
cA/r	$f(x,y,z,u) + tu$	$z = 0$	$k \times (r,b)$
$cAx/2$	$f(x,y,z,u) + tx$	$\lambda z + \mu u = 0$	$2 \times (2,1)$
$cAx/4$	$f(x,y,z,u) + tu$	$x - y = 0$	$(4,1), k-1 \times (2,1)$
$cD/2$	$f(x,y,z,u) + ty$	$\lambda z + \mu u = 0$	$k \times (2,1)$
$cD/3$	$f(x,y,z,u) + tx$	$\lambda z + \mu u = 0$	$2 \times (3,1)$
$cE/2$	$f(x,y,z,u) + ty$	$\lambda z + \mu u = 0$	$3 \times (2,1)$

Table 2.2: Basket for three-dimensional terminal singularities

Uasally we denote (r,b) for the cyclic quotient $\frac{1}{r}(1,-1,b)$. The number of the cyclic quotient points is called *axial weight*. Please see Table 2.2 (cf. [Rei87, (6.4)], [CH11, Remark 2.1]) for the explicit basket for each case.

2.2.2 Singular Riemann-Roch formula

An basic tool to study terminal threefolds is Reid's singular Riemann-Roch formula [Rei87]:

$$\begin{aligned}\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) &+ \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}D.c_2(X) \\ &+ \sum_{P \in \mathcal{B}(X)} \left(-i_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_P - 1} \frac{\overline{jb_P}(r_P - \overline{jb_P})}{2r_P} \right),\end{aligned}$$

where $\mathcal{B}(X) = \{(r_P, b_P)\}$ is the basket data of X and i_P is the integer such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(i_P K_X)$ near P.

Take $D = K_X$, one have

$$K_X.c_2(X) = -24\chi(\mathcal{O}_X) + \sum_{P \in \mathcal{B}(X)} \left(r_P - \frac{1}{r_P}. \right)$$

Now take $D = mK_X$ and replace $K_X.c_2(X)$ by $\chi(\mathcal{O}_X)$ and the contribution of singularities, we get the following plurigenus formula [CH09, Section 2]:

$$\chi(mK_X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 + (1-2m)\chi(\mathcal{O}_X) + l(m),$$

here

$$l(m) = \sum_{P \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{jb_P}(r_P - \overline{jb_P})}{2r_P}.$$

If one assumes X is minimal and of Kodaira dimension one, then $K_X^3 = 0$ and one has

$$\chi(mK_X) = (1-2m)\chi(\mathcal{O}_X) + l(m).$$

2.2.3 Weighted blow-up

Let $X \cong \mathbb{A}^n/\frac{1}{r}(a_1, ..., a_n)$ be a cyclic quotient singularity. There is an elementary way to construct a birational morphism $Y \to X$, so called the weighted blow-up, defined as follows.

We write everything in the language of toric varieties. Let N be the lattice $\langle e_1, ..., e_n, v\rangle_{\mathbb{Z}}$, where e_1, ..., e_n is the standard basic of $\mathbb{R}^n$ and $v = \frac{1}{r}(a_1, ..., a_n)$. Let $\sigma = \langle e_1, ..., e_n\rangle_{\mathbb{R}_{\geq 0}}$. We have $X \cong$ Spec $\mathbb{C}[N^\vee \cap \sigma^\vee]$.

Let $w = \frac{1}{r}(b_1, ..., b_n)$ be a vector such that $b_i = \lambda a_i + k_i r$ for $\lambda \in \mathbb{N}$ and $k_i \in \mathbb{Z}$. We define a weighted blow-up of X with weight w to be the toric variety defined by the fan consists of those cones

$$\sigma_i = \langle e_1, ..., e_{i-1}, w, e_{i+1}, ..., e_n\rangle.$$

Let U_i be the toric variety defined by the cone σ_i and lattice N.

Lemma 2.2.3. *Let*

$$v' = \frac{1}{b_i}(-b_1, ..., -b_{i-1}, r, -b_{i+1}, ..., -b_n)$$

and

$$w' = \frac{1}{rb_i}(a_1b_i - a_ib_1, ..., a_{i-1}b_i - a_ib_{i-1}, ra_i, a_{i+1}b_i - a_ib_{i+1}, ..., a_nb_i - a_ib_n).$$

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Assume that $u = \frac{1}{r'}(a'_1, ..., a'_n)$ *is a vector such that* $\langle e_1, ..., e_n, v', w'\rangle_{\mathbb{Z}} = \langle e_1, .., e_n, u\rangle_{\mathbb{Z}}$*, then*

$$U_i \cong \mathbb{A}^n / \frac{1}{r'}(a'_1, ..., a'_n).$$

In particular, if $\lambda = 1$*, then* $U_i \cong \frac{1}{b_i}(-b_1, ..., -b_{i-1}, r, -b_{i+1}, ..., -b_n)$.

Proof. Let T_i be a linear transformation such that $T_i e_j = e_j$ if $j \neq i$ and $T_i w = e_i$. One can see that

$$T_i e_i = \frac{r}{b_i}(e_i - \sum_{j \neq i} \frac{b_j}{r} e_j) = v'$$

and

$$T_i v = \sum_{j \neq i} \frac{a_j}{r} e_j + \frac{a_i}{r}\frac{r}{b_i}(e_i - \sum_{j \neq i} \frac{b_j}{r} e_j) = \frac{a_i}{b_i} e_i + \sum_{j \neq i} \frac{a_j b_i - a_i b_j}{r b_j} e_j = w'.$$

Under this linear transformation σ_i becomes the standard cone $\langle e_1, ..., e_n\rangle_{\mathbb{R}_{\geq 0}}$. Note that

$$\begin{aligned} k_i v' + \lambda w' &= \frac{k_i r + \lambda a_i}{b_i} e_i + \sum_{j \neq i} \frac{\lambda(a_j b_i - a_i b_j) - k_i b_j r}{r b_i} e_j \\ &= e_i + \sum_{j \neq i} \frac{\lambda a_j b_i - b_i b_j}{r b_i} e_j = e_i - \sum_{j \neq i} k_j e_j. \end{aligned}$$

Hence $e_i \in T_i N$ and $T_i N = \langle e_1, ..., e_n, u\rangle_{\mathbb{Z}}$. This implies U_i has cyclic quotient singularity which is defined by the vector u.

Now assume that $\lambda = 1$, then one can see that

$$w' = e_i - \sum_{j \neq i} k_j e_j - k_i v',$$

so one can take $u = v'$. ☐

Corollary 2.2.4. *Let* x_1*, ...,* x_n *be the local coordinates of* X *and* y_1*, ...,* y_n *be the local coordinates of* U_i*. The change of coordinates of* $U_i \to X$ *are given by* $x_j = y_j y_i^{\frac{b_j}{r}}$ *and* $x_i = y_i^{\frac{b_i}{r}}$*.*

Proof. The change of coordinate is defined by T_i^t, where T_i is defined as in Lemma 2.2.3. ☐

Corollary 2.2.5. *Assume that*

$$S = (f_1(x_1, ..., x_n) = ... = f_k(x_1, ..., x_n) = 0) \subset X$$

is a complete intersection and S' *is the proper transform of* S *on* Y*. Assume that the exceptional* *locus E of $S' \to S$ is irreducible and reduced. Then*

$$a(S,E) = \frac{b_1 + ... + b_n}{r} - \sum_{i=1}^{k} wt_w f_k(x_1, ..., x_n) - 1.$$

Proof. Assume first that $k = 0$. Denote $\phi : Y \to X$. Then on U_i we have

$$\phi^* dx_1 \wedge ... \wedge dx_n = \frac{b_i}{r} y_i^{\frac{b_i}{r}-1} \left(\prod_{j \neq i} y_i^{\frac{b_j}{r}} \right) dy_1 \wedge ... \wedge dy_n,$$

hence $K_Y = \phi^* K_X + (\frac{b_1+...+b_n}{r} - 1)E$.

Now the statement follows from adjunction formula. □

2.2.4 Divisorial contraction to points

We briefly introduce the classification of three-dimensional extremal divisorial contraction to points between terminal threefolds. These result were done by Hayakawa, Kawakita and Yamamoto [Hay99, Hay00, Hay05, Hay1, Hay2, Kawak01, Kawak05, Kawak12, Y].

Assume that $Y \to X$ be an extremal divisorial contraction to point between terminal threefolds and let E be the exception divisor which maps to a point $P \in X$. Let r_P be the index of P, that is, the smallest integer such that $r_P K_X$ is Cartier at P.

Theorem 2.2.6. *There exists an local embedding*

$$X \cong (f(x,y,z,u) = 0) \hookrightarrow \mathbb{A}^4_{(x,y,z,u)} / \frac{1}{r}(a_1, ..., a_4) \text{ or}$$

$$X \cong (f_1(x,y,z,u,t) = f_2(x,y,z,u,t) = 0) \hookrightarrow \mathbb{A}^5_{(x,y,z,u,t)} / \frac{1}{r}(a_1, ..., a_5)$$

and Y is obtained by weighted blow-up of weight w and one of the following holds.

(i) $r_P > 1$ and $a(X,E) = 1/r_P$. In this case a_i, f_j and w is given by Table 2.3 and Table 2.4. This kind of divisorial contraction to point is called a w-morphism.

(ii) $r_P = a(X,E) = 1$. In this case f_j and w is given by Table 2.5.

(iii) $a(X,E) > 1/r_P$. In this case a_i, f_j and w is given by Table 2.6.

We use the notation that p_k denote a polynomial which is weighted homogeneous of weight k and $g_{\geq k}$ denote a polynomial such that $wt_w g \geq k$.

Proof. Please see Table 2.7 for the reference of (1), Table 2.8 for (2) and Table 2.9 for (3). □

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No.	defining equations	$\frac{(r;a_i)}{\text{weight}}$	type	condition
1	$xy+g_{\geq m}(z,u)$	$\frac{(r;\alpha,-\alpha,1,r)}{\frac{1}{r}(a,b,1,r)}$	cA/r	$a\equiv\alpha(\mod r)$ $a+b=rm$
2	$x^2+y^2+g_{\geq\frac{2k+1}{2}}(z,u)$	$\frac{(4;1,3,1,2)}{\frac{1}{4}(2k+a,2k+b,1,2)}$	$cAx/4$	k even, $(a,b)=(1,3)$, or k odd, $(a,b)=(3,1)$
3	$x^2+y^2\pm(a-3)xp_{\frac{2k+1}{4}}(z,u)\pm$ $(b-3)yp_{\frac{2k+1}{4}}(z,u)+g_{\geq\frac{2k+3}{2}}(z,u)$	$\frac{(4;1,3,1,2)}{\frac{1}{4}(2k+a,2k+b,1,2)}$	$cAx/4$	k even, $(a,b)=(5,3)$, or k odd, $(a,b)=(3,5)$
4	$x^2+y^2+g_{\geq k}(z,u)$	$\frac{(2;0,1,1,1)}{\frac{1}{2}(k+a,k+b,1,1)}$	$cAx/2$	k even, $(a,b)=(0,1)$, or k odd, $(a,b)=(1,0)$
5	$x^2+y^2\pm(a-1)xp_{\frac{k}{2}}(z,u)\pm$ $(b-1)yp_{\frac{k}{2}}(z,u)+g_{\geq k+1}(z,u)$	$\frac{(2;0,1,1,1)}{\frac{1}{2}(k+a,k+b,1,2)}$	$cAx/2$	k even, $(a,b)=(2,1)$, or k odd, $(a,b)=(1,2)$
6	$x^2+y^3+zu(z\pm su)$, or $x^2+y^3+z^2u+g_{\geq 2}(y,z,u)$	$\frac{(3;0,2,1,1)}{\frac{1}{3}(3,2,a,b)}$	$cD/3$	$(a,b)=(1,4)$ or $(4,1)$
7	$x^2+y^3+z^3+g_{\geq 2}(y,z,u)$	$\frac{(3;0,2,1,1)}{\frac{1}{3}(3,2,4,1)}$	$cD/3$	
8	$x^2+y^3+\lambda y^2u^2+$ $z^3+z^3+g_{\geq 4}(y,z,u)$	$\frac{(3;0,2,1,1)}{\frac{1}{3}(6,5,4,1)}$	$cD/3$	
9	$x^2+y^3+g_{\geq 3}(y,z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(3,2,a,b)}$	$cE/2$	$(a,b)=(1,3)$ or $(3,1)$
10	$x^2+y^3+\lambda y^2u^2+g_{\geq 3}(y,z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(3,2,1,3)}$	$cE/2$	
11	$x^2+y^3+\lambda y^2u^2+g_{\geq 5}(y,z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(5,4,3,1)}$	$cE/2$	
12	$x^2\pm(\lambda yu+\mu zu^2+\nu u^5)x+$ $y^3+g_{\geq 6}(y,z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(7,4,3,1)}$	$cE/2$	
13	$x^2+y^3+g_{\geq 9}(y,z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(9,6,5,1)}$	$cE/2$	
14	$x^2+yzu+y^r+z^s+u^w$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(a,b,c,d)}$	$cD/2$	$(a,b,c,d)=\begin{matrix}(3,1,1,2)\\(3,1,3,2)\\(3,3,1,2)\\(3,1,1,4)\end{matrix}$
15	$\begin{cases}x^2+yt+z^r+u^s\\ zu+y^3+t\end{cases}$	$\frac{(2;1,1,1,0,1)}{\frac{1}{2}(3,1,1,2,5)}$	$cD/2$	
16	$x^2+y^2u+\lambda yz^k+g_{\geq l}(z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(b+2,b,1,4)}$	$cD/2$	$b\leq\min\{k-2,l-2\}$ and b is odd
17	$x^2+y^2u+\lambda yz^k+g_{\geq l}(z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(b,b,1,2)}$	$cD/2$	b is odd, $\begin{matrix}k=b\leq l\text{, or}\\ l=b\leq k\end{matrix}$
18	$\begin{cases}x^2+yt+g_{\geq 2b}(z,u)\\ yu+\lambda z^b+t\end{cases}$	$\frac{(2;1,1,1,0,1)}{\frac{1}{2}(b,b-2,1,2,b+2)}$	$cD/2$	b is odd
19	$x^2\pm xp_b(z,u)+y^2u+$ $\lambda yz^k+g_{\geq b+1}(z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(b+2,b,1,2)}$	$cD/2$	$k\geq b$, b is odd.
20	$\begin{cases}x^2+ut+\lambda yz^k+g_{\geq b+2}(z,u)\\ y^2\pm xp_{\frac{b}{2}-1}(z,u)+h_{\geq b}(z,u)+t\end{cases}$	$\frac{(2;1,1,1,0,0)}{\frac{1}{2}(b+2,b,1,2,2b+2)}$	$cD/2$	k is odd, $k\geq b+2$
21	$x^2+y^2u+\lambda yz^k+g_{\geq b}(z,u)$	$\frac{(2;1,1,1,0)}{\frac{1}{2}(b+1,b-1,1,2)}$	$cD/2$	b is even, $k\geq b+1$

Table 2.3: Classfication of divisorial contraction to points: w-morphism cases

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No.	defining equations	$\frac{(r; a_i)}{\text{weight}}$	type	condition
22	$\begin{cases} x^2 + ut + \lambda yz^k + g_{\geq b+1}(z, u) \\ y^2 + p_{b-1}(z, u) + t \end{cases}$	$\frac{(2; 1, 1, 1, 0, 0)}{\frac{1}{2}(b+1, b-1, 1, 2, 2b)}$	$cD/2$	b is even, $k \geq b+3$
23	$x^2 \pm yup_{\frac{b-1}{2}}(z, u) + y^2u +$ $\lambda yz^k + g_{\geq b+1}(z, u)$	$\frac{(2; 1, 1, 1, 0)}{\frac{1}{2}(b+1, b+1, 1, 2)}$	$cD/2$	$k \geq b+1$, b is even.
24	$x^2 + g_{\geq b}(z, u) +$ $(y - p_{\frac{b}{2}-1}(z, u))(yu + \lambda z^b + up_{\frac{b}{2}-1}(z, u))$	$\frac{(2; 1, 1, 1, 0)}{\frac{1}{2}(b, b, 1, 2)}$	$cD/2$	b is odd.
25	$\begin{cases} x^2 + yt + g_{\geq b}(z, u) \\ yu + p_b(z, u) + \lambda z^b + t \end{cases}$	$\frac{(2; 1, 1, 1, 0, 0)}{\frac{1}{2}(b, b-2, 1, 2, b+2)}$	$cD/2$	b is odd

Table 2.4: Classfication of divisorial contraction to points: w-morphism cases, continued

No.	defining equations	weight	type	condition
1	$x^2 + y^2u + \lambda yz^k + g_{\geq 2b}(z, u)$	$(b, b-1, 1, 2)$	cD	$k \geq b+1$
2	$x^2 + y^2u + yp_l(z, u) + \lambda yz^k +$ $g_{\geq 2l}(z, u)$	$(b, b, 1, 1)$	cD	$b = k \leq l$, or $b = l \leq k$
3	$\begin{cases} x^2 + ut + \lambda yz^k + g_{\geq 2b+2}(z, u) \\ y^2 + p_{2b}(x, z, u) + t \end{cases}$	$(b+1, b, 1, 1, 2b+1)$	cD	$k \geq b+2$
4	$x^2 + y^2u + \lambda yz^k + g_{\geq 2b+1}(x, z, u)$	$(b+1, b, 1, 1)$	cD	$k \geq b+1$
5	$\begin{cases} x^2 + yt + g_{\geq 2b}(z, u) \\ yu + z^b + p_p(z, u) + t \end{cases}$	$(b, b-1, 1, 1, b+1)$	cD	
6	$x^2 + y^3 + g_{\geq 4}(y, z, u)$	$(2, 2, 1, 1)$	cE	
7	$x^2 + 2xp_3(y, z) + y^3 + g_{\geq 6}(y, z, u)$	$(3, 2, 1, 1)$	cE	
8	$x^2 + y^3 + g_{\geq 6}(y, z, u)$	$(3, 2, 2, 1)$	cE	
9	$x^2 + y^3 + 3\lambda y^2u^2 + g_{\geq 8}(y, z, u)$	$(4, 3, 2, 1)$	cE	
10	$x^2 \pm xp_4(y, z, u) + y^3 + g_{\geq 9}(y, z, u)$	$(5, 3, 2, 1)$	cE	
11	$x^2 + y^3 + y^2p_3(z, u) + g_{\geq 9}(y, z, u)$	$(5, 4, 2, 1)$	cE	
12	$x^2 + y^3 + g_{\geq 12}(y, z, u)$	$(6, 4, 3, 1)$	cE	
13	$x^2 + y^3 + y^2p_4(z, u) + g_{\geq 14}(y, z, u)$	$(7, 5, 3, 1)$	cE	
14	$x^2 \pm xp_7(y, z, u) + y^3 + g_{\geq 15}(y, z, u)$	$(8, 5, 3, 1)$	cE	
15	$x^2 + y^3 + g_{\geq 18}(y, z, u)$	$(9, 6, 4, 1)$	cE	
16	$x^2 + y^3 + y^2p_6(z, u) + g_{\geq 20}(y, z, u)$	$(10, 7, 4, 1)$	cE	
17	$x^2 + y^3 + g_{\geq 24}(y, z, u)$	$(12, 8, 5, 1)$	cE	
18	$x^2 + y^3 + g_{\geq 30}(y, z, u)$	$(15, 10, 6, 1)$	cE	
19	$\begin{cases} x^2 + y^3 + tu + g_{\geq 6}(y, z, u) \\ p_4(x, y, z, u) + t \end{cases}$	$(3, 2, 1, 1, 5)$	cE	
20	$x^2 + xp_2(y, z, u) + y^3 + g_{\geq 6}(y, z, u)$	$(4, 2, 1, 1)$	cE_6	
21	$\begin{cases} x^2 + y^3 + tp_2(z, u) + g_{\geq (}(y, z, u) \\ q_3(y, z, u) + t \end{cases}$	$(3, 2, 1, 1, 4)$	cE_7	
22	$x^2 + y^3 + y^2p_2(z, u) + yq_3(z, u) +$ $g_{\geq 6}(y, z, u)$	$(3, 3, 1, 1)$	cE_7	
23	$\begin{cases} x^2 + yt + g_{\geq 10}(y, z, u) \\ y^2 + p_6(y, z, u) + t \end{cases}$	$(5, 3, 2, 1, 7)$	cE	

Table 2.5: Classfication of divisorial contraction to points: Gorenstein and discrepancy one cases

No.	defining equations	$\frac{(r;a_i)}{\text{weight}}$	$\frac{\text{type}}{a(X,E)}$	condition
1	$xy + z^{rm} + g_{\geq ma}(z,u)$	$\frac{(r;\alpha,-\alpha,1,r)}{\frac{1}{r}(b,c,a,r)}$	$\frac{cA/r}{a/r}$	$b \equiv \alpha,$ $b+c=rma$
2	$x^2+y^2+z^3+xu^2+g_{\geq 6}(x,y,z,u)$	$\frac{(1;-)}{(4,3,2,1)}$	$\frac{cA_2}{3}$	
3	$x^2+y^2u+z^m+g_{\geq 2b+1}(x,y,z,u)$	$\frac{(1;-)}{(b+1,b,a,1)}$	$\frac{cD}{a}$	$ma=2b+1$
4	$\begin{cases} x^2+yt+g_{\geq 2b+2}(y,z,u) \\ yu+z^m+up_b(z,u)+t \end{cases}$	$\frac{(1,-)}{(b+1,b,a,1,b+2)}$	$\frac{cD}{a}$	$ma=b+1$
5	$\begin{cases} x^2+\lambda yz^m+ut+g_{\geq b+1}(z,u) \\ y^2+2xp_{\frac{b-3}{2}}(z,u)+q_{b-1}(z,u)+t \end{cases}$	$\frac{(1;-)}{(\frac{b+1}{2},\frac{b-1}{2},4,1,b)}$	$\frac{cD}{4}$	$8m \geq b+3$
6	$\begin{cases} x^2+\lambda yz^m+ut+g_{\geq b+1}(z,u) \\ y^2+2xp_{\frac{b-3}{2}}(z,u)+q_{b-1}(z,u)+t \end{cases}$	$\frac{(1;-)}{(\frac{b+1}{2},\frac{b-1}{2},2,1,b)}$	$\frac{cD}{2}$	$4m \geq b+3$
7	$x^2+y^2u+2yup_{b-1}(z,u)+$ $\lambda yz^m+z^b+g_{\geq 2b}(z,u)$	$\frac{(1;-)}{(b,b,2,1)}$	$\frac{cD}{2}$	$m \geq \frac{b}{2}$
8	$x^2+y^2u+2yup_2(z,u)+$ $u^3+g_{\geq 6}(z,u)$	$\frac{(1;-)}{(3,3,1,2)}$	$\frac{cD_4}{2}$	
9	$x^2+y^2u+z^3+\mu yu^2+$ $2yup_{2,3}(z,u)+g_{\geq 6}(z,u)$	$\frac{(1;-)}{(3,4,2,1)}$	$\frac{cD_4}{3}$	
10	$x^2+y^2u+z^m+g_{\geq b+1}(x,y,z,u)$	$\frac{(2;1,1,1,0)}{(\frac{b}{2}+1,\frac{b}{2},\frac{a}{2},1)}$	$\frac{cD/2}{a/2}$	$g(x,y,0,0)=0$ $\frac{\partial^2}{\partial x^2}g(x,y,z,u)=0$ $\frac{\partial^2}{\partial y^2}g(x,y,z,u)=0$ $ma=2b+2$ a and b are odd
11	$\begin{cases} x^2+yt+g_{\geq b+2}(z,u) \\ yu+z^m+p_{\frac{b}{2}+1}(z,u)+t \end{cases}$	$\frac{(2;1,1,1,0,1)}{(\frac{b}{2}+1,\frac{b}{2},\frac{a}{2},1,\frac{b}{2}+2)}$	$\frac{cD/2}{a/2}$	$ma=b+2$ a and b are odd
12	$x^2+y^2u+z^{4b}+g_{\geq 4b}(y,z,u)$	$\frac{(2;1,1,1,0)}{(2b,2b,1,1)}$	$\frac{cD/2}{1}$	
13	$x^2+yzu+y^4+z^b+u^c$	$\frac{(2;1,1,1,0)}{(2,2,1,1)}$	$\frac{cD/2}{1}$	$b,c \geq 4$ b is even
14	$\begin{cases} x^2+ut+\lambda yz^{b+2}+\alpha z^{2b+2}+g_{\geq 2b+2}(y,z,u) \\ y^2+\mu xz^{b-1}+\beta z^{2b}+p_{2b}(x,z,u)+t \end{cases}$	$\frac{(2;1,1,1,0,0)}{(b+1,b,1,1,2b+1)}$	$\frac{cD/2}{1}$	$\frac{\partial^2}{\partial y^2}g(y,z,u)=0$ $\frac{\partial^2}{\partial x^2}p(x,z,u)=0$ b odd and $\alpha^2+\beta\lambda^2 \neq 0$, or b even and $\beta^2+\alpha\mu^2 \neq 0$
15	$\begin{cases} x^2+ut+y^4+z^4 \\ yz+u^2+t \end{cases}$	$\frac{(2;1,1,1,0,0)}{(2,1,1,1,3)}$	$\frac{cD/2}{1}$	
16	$\begin{cases} x^2+yt+g_{\geq 4b+2}(z,u) \\ yu+z^{2b+1}+p_{2b+1}(z,u)+t \end{cases}$	$\frac{(2;1,1,1,0,1)}{(2b+1,2b,1,1,2b+2)}$	$\frac{cD/2}{1}$	
17	$\begin{cases} x^2+ut+\lambda yz^{\frac{b+3}{4}}+\mu' z^{\frac{b+1}{2}}+g_{\geq b+1}(y,z,u) \\ y^2+\lambda' xz^{\frac{b-3}{4}}+\mu z^{\frac{b-1}{2}}+p_{b-1}(x,z,t)+t \end{cases}$	$\frac{(2;1,1,1,0,0)}{(\frac{b+1}{2},\frac{b-1}{2},2,1,b)}$	$\frac{cD/2}{2}$	$b=8k+1$ and $\lambda\mu \neq 0$, or $b=8k+7, \lambda'\mu' \neq 0$
18	$x^2+(y-p_2(z,u))^3+$ $yu^3+g_{\geq 6}(y,z,u)$	$\frac{(1;-)}{(3,3,2,1)}$	$\frac{cE_6}{2}$	
19	$\begin{cases} x^2+yt+g_{\geq 10}(y,z,u) \\ y^2+p_6(z,u)+t \end{cases}$	$\frac{(1;-)}{(5,3,2,2,7)}$	$\frac{cE_7}{2}$	
20	$x^2+y^3+u^7+g_{\geq 14}(z,u)$	$\frac{(1;-)}{(7,5,3,2)}$	$\frac{cE_{7,8}}{2}$	
21	$x^2+y^3+z^4+u^8+$ $\lambda y^2u^2+g_{\geq 8}(y,z,u)$	$\frac{(2;1,0,1,1)}{(4,3,2,1)}$	$\frac{cE/2}{1}$	$\frac{\partial^2}{\partial y^2}g(y,z,u)=0$

Table 2.6: Classfication of divisorial contraction to points: large disprepancy cases

No.	Reference
1	[Hay99, Theorem 6.4]
2	[Hay99, Theorem 7.4]
3	[Hay99, Theorem 7.9]
4	[Hay99, Theorem 8.4]
5	[Hay99, Theorem 8.8]
6	[Hay99, Theorem 9.9, Theorem 9.14]
7	[Hay99, Theorem 9.20]
8	[Hay99, Theorem 9.25]
9	[Hay99, Theorem 10.11, Theorem 10.17, Theorem 10.22, Theorem 10.28, Theorem 10.41]
10, 11	[Hay99, Theorem 10.33, Theorem 10.47]
12	[Hay99, Theorem 10.54, Theorem 10.61]
13	[Hay99, Theorem 10.67]
14	[Hay00, Proposition 4.4, Proposition 4.7, Proposition 4.12]
15	[Hay00, Proposition 4.9]
16	[Hay00, Proposition 5.4]
17	[Hay00, Proposition 5.8, Proposition 5.13]
18	[Hay00, Proposition 5.9]
19	[Hay00, Proposition 5.16]
20	[Hay00, Proposition 5.18]
21	[Hay00, Proposition 5.22, Proposition 5.32]
22	[Hay00, Proposition 5.25]
23	[Hay00, Proposition 5.28]
24	[Hay00, Proposition 5.35]
25	[Hay00, Proposition 5.36]

Table 2.7: Reference for Table 2.3, Table 2.4

No.	Reference
1-5	[Hay1, Theorem 2.1-2.5]
6-23	[Hay2, Theorem 1.1]

Table 2.8: Reference for Table 2.5

No.	Reference
1	[Kawak05, Theorem 1.2 (i)]
2	[Y, Theorem 2.6]
3,4	[Kawak05, Theorem 1.2 (ii)]
5	[Y, Theorem 2.2]
6	[Y, Theorem 2.3]
7,8	[Y, Theorem 2.4]
9	[Y, Theorem 2.7]
10,11	[Kawak05, Theorem 1.2 (ii)]
12-16	[Hay05, Theorem 1.1]
17	[Kawak12, Theorem 2]
18	[Y, Theorem 2.5]
19	[Y, Theorem 2.9]
20	[Y, Theorem 2.10]
21	[Hay05, Theorem 1.2]

Table 2.9: Reference for Table 2.6

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Chapter 3

Betti numbers in the three dimensional minimal model program

The relations between topology and geometry are the research topic of many studies. A typical question, which was originally asked by Hirzebruch and modified by Kotschick (cf. [Kot08]) states, if one fixes the topology of a smooth algebraic variety, can its Chern numbers only assume finitely many values? This question is trivial in dimension one and has been proved in dimension two (please see [Kot08]). Cascini and Tasin [CT17] have proved the statement in some special cases in dimension three, but in general this question is still open for dimension greater than two.

We will briefly introduce the result of three dimensional case [CT17]. One needs to show that c_1^3 and $c_1.c_2$ of a smooth threefold is bounded by a constant depending only on the topological type of the threefold. The Riemann-Roch formula asserts that $c_1.c_2$ can be bounded by a combination of Betti numbers. If our variety is minimal, then the Miyaoka-Yau inequality says that one can use $c_1.c_2$ to bound c_1^3. Assume the variety is not minimal, then a natural approach is to run the minimal model program.

One has to estimate the change of c_1^3 under the minimal model program. In the case of divisorial contractions to points, this quantity can be bounded by b_2 and in the case of blowing-up smooth curves, it can be bounded by b_3 and the cubic form. Hence for those smooth threefolds with the property that the process of the minimal model program consists of only divisorial contractions to points and blowing-up smooth curves, their c_1^3 will be bounded by topology. This is the main theorem of [CT17].

One can see that how to estimate the change of Betti numbers under the minimal model

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program becomes an important issue if we want to generalize the result of [CT17] into general situations.

In the process of the three-dimensional minimal model program, one can prove that b_0, b_1, b_5 and b_6 won't change and b_2 and b_4 change regularly (cf. Proposition 3.1.12). However, the change of b_3 could be arbitrary positive or arbitrary negative (please see Section 3.4). The point is that because other Betti numbers change regularly, computing the change of b_3 is equivalent to computing the change of topological Euler characteristic, which can be compute directly thanks to the classification of three-dimensional divisorial contractions and the Chen-Hacon factorization. The goal of this chapter is to prove Theorem 3.3.6, which states that the difference of b_3 under steps of minimal model program begin with a smooth threefold could be bounded by some constant depends only on Picard number of the smooth threefold.

3.1 Preliminary

3.1.1 Biraitonal maps between terminal threefolds

In this subsection we introduce the Chen-Hacon factorization, which factorize a step of the three-dimensional minimal model program into simple birational maps. Let $X \dashrightarrow X'$ be a step of the minimal model program. If X is Gorenstein, then this birational map is well studied in [Cut88] (please see Remark 3.1.4). To study birational maps begin with a non-Gorenstein terminal threefolds, we first recall the definition of depth, which is a quantity measures the complexity of non-Gorenstein singularities.

Definition. Let X be a terminal threefold. A *w-morphism* is a extremal divisorial contraction which contract exceptional divisor to a point of index $r > 1$, such that the discrepancy of the exceptional divisor is $1/r$.

The depth of X, denoted by $dep(X)$, is the minimal length of sequence of w-morphisms $X_n \to X_{n-1} \to ... \to X_1 \to X$, such that X_n is Gorenstein. Note that by [Hay00, Theorem 1.2], for any terminal threefold X, $dep(X)$ exists and is finite.

In the world of birational geometry, depth satisfied some good property as follows.

Proposition 3.1.1 ([CH11], Proposition 2.15)**.** *If $f : Y \supset E \to X \ni P$ be (the germ of) a divisorial contraction to a point. Then $dep(Y) \geq dep(X) - 1$.*

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Proposition 3.1.2 ([CH11], Proposition 3.8)**.** *Let $X \to W$ be a flipping contraction and $X \dashrightarrow X'$ be the flip, then $dep(X) > dep(X')$.*

Remark 3.1.3. If $X \dashrightarrow X'$ is a flop. Then by [Kol89] the singularities of X and X' are locally isomorphic, hence $dep(X) = dep(X')$.

Remark 3.1.4. Let X be a terminal threefold. Then $dep(X) = 0$ if and only if X is Gorenstein. In this case, by [Ben85, Corollary 0.1], there is no flipping contraction. Also, if $X \to W$ is a divisorial contraction to a curve, then X is obtained by blowing-up an LCI curve on W (cf. [Cut88, Theorem 4]).

Now let X be a smooth threefold. Then the singularities appear in the minimal model program of X can be bounded by the Picard number of X.

Proposition 3.1.5 (Cascini, D.-Q. Zhang, in the proof of [CZ14] Proposition 3.3)**.** *Let X be a smooth projective threefold and assume that*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow ... \dashrightarrow X_k = Z$$

is a sequence of steps for the K_X-minimal model program of X. Then $dep(Z) \leq \rho(X)$.

Remark 3.1.6. In fact, the argument in [CZ14] Proposition 3.3 implies $dep(Z) \leq \rho(X/Z)$.

Now we are ready to state the Chen-Hacon factorization.

Theorem 3.1.7 ([CH11], Theorem 3.3)**.** *Let $g : X \supset C \to W \ni P$ be an extremal neighborhood which is isolated (resp. divisorial). If X is not Gorenstein, then we have a diagram*

$$\begin{tikzcd}
Y \arrow[rr, dashed] \arrow[d, "f"] & & Y' \arrow[d, "f'"'] \\
X \arrow[dr, "g"] & & X' \arrow[dl, "g'"'] \\
& W &
\end{tikzcd}$$

where $Y \dashrightarrow Y'$ consists of flips and flops over W, f is a w-morphism, f' is a divisorial contraction (resp. a divisorial contraction to a curve) and $g' : X' \to W$ is the flip of g (resp. g' is divisorial contraction to a point).

Remark 3.1.8. The diagram above satisfies more properties.

(i) $dep(Y) = dep(X) - 1$. This is by the construction of Y in [CH11].

(ii) $Y \dashrightarrow Y'$ can be decomposed into $Y = Y_0 \dashrightarrow Y_1 \dashrightarrow ... \dashrightarrow Y_l = Y'$, such that $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i > 0$ and $Y_0 \dashrightarrow Y_1$ is either a flip or a flop. This is the step 4 in the proof of [CH11, Theorem 3.3].

3.1.2 Topology of terminal threefolds

In this subsection, we will compute the change of all Betti numbers except for b_3 under three dimensional birational maps. For the divisorial contraction cases it is known to experts.

Lemma 3.1.9 ([CT17], Lemma 2.17)**.** *Let $Y \to X$ be an elementary divisorial contraction within $\mathbb{Q}$-factorial projective threefolds with terminal singularities. Then $b_i(Y) = b_i(X)$ if $i = 0, 1, 5, 6$, and $b_i(Y) = b_i(X) + 1$ if $i = 2, 4$.*

Corollary 3.1.10. *If $X \to W$ is extremal divisorial contraction, then*

$$b_3(W) - b_3(X) = \chi_{top}(X) - \chi_{top}(W) - 2.$$

The following statement is well-known to experts. However, we are unable to find appropriate reference hence we provide a proof here.

Lemma 3.1.11. *Assume that $X \dashrightarrow X'$ is a three-dimensional terminal flop, then $b_i(X) = b_i(X')$ for all i.*

Proof. By [Che11] (please see Theorem 3.4.4 below) there exists $f : Y \to X$ (resp. $f' : Y' \to X'$) such that Y (resp. Y') is smooth and f (resp. f') is a combination of divisorial contractions to points. By Lemma 3.1.9, we have $b_i(Y) = b_i(X)$ (resp. $b_i(Y') = b_i(X')$) for $i = 0, 1, 5, 6$, and $b_j(Y) = b_j(X) + \rho(Y/X)$ (resp. $b_j(Y') = b_j(X') + \rho(Y'/X')$) for $j = 2, 4$.

Claim. $b_i(Y) = b_i(Y')$ for $i = 0$, 1, 2, 4, 5 and 6.

Note that X and X' has the same singularities by [Kol89, Theorem 2.4], hence $\rho(Y/X) = \rho(Y'/X')$. Thus the above claim implies

$$b_i(X) = b_i(X') \text{ for } i = 0, 1, 2, 4, 5, 6.$$

To prove the claim, one only need to prove that $b_5(Y) = b_5(Y')$ and $b_2(Y) = b_2(Y')$ because $b_i(Y) = b_i(Y') = 1$ for $i = 0$, 6 and $b_1(Y) = b_5(Y)$ (resp. $b_1(Y') = b_5(Y')$), $b_2(Y) = b_4(Y)$

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(resp. $b_2(Y') = b_4(Y')$) since both Y and Y' are smooth. Let $\phi : Z \to Y$ be the resolution of the indeterminacy of $Y \dashrightarrow Y'$ which is obtained by a sequence of blowing-up smooth centers on Y. Let $\phi' : Z \to Y'$ be the induced morphism. Then $Z \to Y$ is a composition of elementary divisorial contractions, hence $b_5(Y) = b_5(Z)$ and $b_2(Y) = b_2(Z) - \rho(Z/Y)$. By [CT17, Lemma 2.15] we have that

$$0 \to H^5(Y', \mathbb{Q}) \to H^5(Z, \mathbb{Q}) \oplus H^5(\phi(E'), \mathbb{Q}) \to H^5(E', \mathbb{Q}) \to 0$$

is exact, where $E' = Exc(\phi')$. This implies $b_5(Y') = b_5(Z) = b_5(Y)$.

On the other hand, in the proof of [CT17, Lemma 2.16] one can see that

$$0 \to H_2(Z/Y', \mathbb{C}) \to H_2(Z, \mathbb{C}) \to H_2(Y', \mathbb{C}) \to 0$$

is exact, where $H_2(Z/Y', \mathbb{C}) \subset H_2(Z, \mathbb{C})$ is the subspace generated by the image of $H_2(E', \mathbb{C})$ in $H_2(Z, \mathbb{C})$. Hence

$$b_2(Z) = b_2(Y') + \dim H_2(Z/Y', \mathbb{C}).$$

As mentioned in [CT17, Lemma 2.16] we have $H_2(Z/Y')$ is generated by algebraic cycles, hence $\rho(Z/Y') \leq \dim H_2(Z/Y', \mathbb{C})$. Also we have

$$\rho(Z/Y') = \rho(Z/X') - \rho(Y'/X') = \rho(Z/X) - \rho(Y/X) = \rho(Z/Y).$$

The conclusion is

$$b_2(Y') = b_2(Z) - \dim H_2(Z/Y', \mathbb{C}) \leq b_2(Z) - \rho(Z/Y') = b_2(Z) - \rho(Z/Y) = b_2(Y).$$

However by the symmetry of Y and Y' one can also show that $b_2(Y) \leq b_2(Y')$, hence $b_2(Y) = b_2(Y')$.

Now we have proved that $b_i(X) = b_i(X')$ for $i \neq 3$. Since $\chi_{top}(X) = \chi_{top}(X')$ by the construction in [Kol89, Theorem 2.4], we have $b_3(X) = b_3(X')$. $\square$

After applying the Chen-Hacon factorization, one can deal with the flip case.

Proposition 3.1.12. *Let X be a smooth threefold and $X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_m$ be the process of minimal model program. Then b_0, b_1, b_5 and b_6 are constant and both b_2 and b_4*

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decrease. Moreover, b_2 *and* b_4 *strictly decrease by one if* $X_i \to X_{i+1}$ *is a divisorial contraction, and remain unchanged if* $X_i \dashrightarrow X_{i+1}$ *is a flip.*

Proof. If $X_i \to X_{i+1}$ is a divisorial contraction we can use Lemma 3.1.9. Assume that $X_i \dashrightarrow X_{i+1}$ is a flip. We will apply Theorem 3.1.7 and induction on $dep(X_i)$. One has the diagram

$$\begin{tikzcd}
Y \arrow[rr, dashed] \arrow[d, "f"] & & Y' \arrow[d, "f'"'] \\
X_i \arrow[dr, "g"] & & X_{i+1} \arrow[dl, "g'"'] \\
& W &
\end{tikzcd} .$$

Note that by Remark 3.1.8 we have $dep(Y) = dep(X_i) - 1$. One can write

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow ... \dashrightarrow Y_l = Y'$$

and $Y_j \dashrightarrow Y_{j+1}$ is a flip or flop for all j, hence $dep(Y_j) \geq dep(Y_{j+1})$ by Proposition 3.1.2 and Remark 3.1.3. By the induction hypothesis and Lemma 3.1.11, we have

$$b_i(Y) = b_i(Y') \text{ for } i \neq 3.$$

Hence

$$b_i(X_i) = b_i(Y) = b_i(Y') = b_i(X_{i+1}) \text{ for } i = 0, 1, 5, 6$$

and

$$b_i(X_i) = b_i(Y) - 1 = b_i(Y') - 1 = b_i(X_{i+1}) \text{ for } i = 2, 4.$$

□

3.2 The estimate on topology

The purpose of this section is to estimate the topological Euler characteristic of the exceptional divisor of a birational morphism which is contracted to a point. We know that any such kind of divisorial contractions is obtained by weighted blow-up. In most common situation the exceptional divisor is contained in a weighted projective space. However in some special cases the exceptional divisor is contained in a cyclic quotient of a weighted projective space. To deal

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with such kind of special cases we have to introduce the following generalization of weighted projective spaces.

Definition. An n-dimensional variety X is a *generically cyclic quotient space* if there exists a Zariski open set $U \subset X$ such that

1. $U \cong \mathbb{A}^n/\frac{1}{r}(a_1, ..., a_n)$ for some natural numbers r and a_1, ..., a_n.
2. $X - U$ is also a generically cyclic quotient space.

Let D be an integral Weil divisor on X. We define

$$\deg D = \max\{\deg \phi_U^{-1}(D|_U), \deg D|_{X-U}\},$$

where $\phi_U : \mathbb{A}^n \to U$ is the natural quotient map and we define the degree of a divisor in $\mathbb{A}^n$ to be the degree of the defining equation of this divisor.

Remark 3.2.1. It is clear that a weighted projective space is a generically cyclic quotient space. Moreover, let $W \cong \mathbb{A}^n/\frac{1}{r}(a_1, ..., a_n)$ be a cyclic quotient singularity and let $W' \to W$ be the weighted blow-up the origin of W with a weight w so that $w(x, y, z, u) = \frac{1}{r}(b_1, ..., b_n)$. Assume that $b_i \equiv \lambda a_i (\text{mod } r)$ with $\lambda = 1$ or r. One can see that the exceptional divisor E of $W' \to W$ is a generically cyclic quotient space. In fact, E is a weighted projective space if $\lambda = 1$ and a cyclic quotient of a weighted projective space if $\lambda = r$. Assume that D is a Weil divisor on W such that $\text{wt}_w(D) = \frac{m}{r}$, then $\deg D \leq m$.

By the classification of divisorial contractions to points, one can check that if $Y \to X$ is a divisorial contraction to a point, then the exceptional divisor is an LCI locus in a generically cyclic quotient space of dimension four or five.

Theorem 3.2.2. *Fix three positive integers n, k and d.*

(i) There is an integer $N^n_{d,k}$ such that for any algebraic set $X_I \subset \mathbb{A}^n$ defined by an ideal $I = (f_1, ..., f_k)$ with $\deg f_i \leq d$ for all i, we have $|\chi_{top}(X_I)| \leq N^n_{d,k}$.

(ii) There is an integer $M^n_{d,k}$ satisfying the following property. Let Y be an n-dimensional generically cyclic quotient space and let $W = D_1 \cap ... D_k$ be a finite intersection of reduced prime Weil divisors such that $\deg D_i \leq d$ for all $i = 1, ..., k$. Then

$$|\chi_{top}(W)| \leq M^n_{d,k}.$$

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Note that Theorem 3.2.2 (i) can be easily proved using Milnor-Thom Theorem which states as follows.

Theorem 3.2.3 ([Mil64], Theorem 2, cf. also [Tho07])**.** *Let* $V \subset \mathbb{R}^m$ *is defined by polynomials* $f_1, ..., f_k$. *If* $\deg f_i \leq d$, *then the sum of Betti numbers of* V *is bounded by* $d(2d-1)^{m-1}$.

However we will write a pure algebraic proof here. To prove the existence of N-constants and M-constants, the basic idea is to reduce the question into lower dimensional cases. We will prove:

Proposition 3.2.4. *Assume that* $N^{n-1}_{e,l}$ *exists for all* e, $l \in \mathbb{N}$, *then* $N^n_{d,k}$ *exists.*

Proposition 3.2.5. *If* $M^m_{d,k}$ *exists for all* $m < n$ *and* $N^n_{d,l}$ *exists for all* $l \geq k$, *then* $M^n_{d,k}$ *exists.*

3.2.1 The existence of N-constant

In this subsection we prove Proposition 3.2.4. Given $X_I \subset \mathbb{A}^n$, where $I = (f_1, ..., f_k)$ satisfying $\deg f_i \leq d$. Consider the natural map

$$K[x_1, ..., x_{n-1}] \hookrightarrow K[x_1, ..., x_n] \to K[x_1, ..., x_n]/I,$$

here K is the ground field. This gives a morphism ϕ from X_I to $\{x_n = 0\} \cong \mathbb{A}^{n-1}$. Fix $p = (a_1, ..., a_{n-1}) \in \mathbb{A}^{n-1}$, then

$$\phi^{-1}(p) = \{(a_1, ..., a_{n-1}, x_n) \in \mathbb{A}^n \,|\, f_1(a_1, ..., a_{n-1}, x_n) = ... = f_k(a_1, ..., a_{n-1}, x_n) = 0\}.$$

Thus $\phi^{-1}(p)$ can be studied via the equations f_1, ..., f_k. Now assume that the topology of the image is known, then since the fibres can be studied, the topology of the original space X_I could be computed. This is the reason why one can reduce the problem to the lower dimensional case.

For the induction reason, we will prove a stronger statement.

Proposition 3.2.6. *Assume* $N^{n-1}_{c,m}$ *exists for all integers* c *and* m. *Let* Z *be an algebraic subset in* $\mathbb{A}^{n-1}$ *which is defined by an ideal* $J = (g_1, ..., g_l)$ *and assuming* $\deg g_j \leq e$ *for some constant* e. *Then there is an fixed integer* $L^n_{d,k,e,l}$ *such that* $|\chi_{top}(\phi^{-1}Z)| \leq L^n_{d,k,e,l}$.

We divide this subsection into four parts. In the first part we study the common roots of a collection of polynomials, which is the main tool we will use to study the fibre of the projection

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ϕ. After the tool is developed, we could get much information between the points in H and its fibre in $\mathbb{A}^n$, provided that the degree of f_1, ... f_k do not be too small. This is the second part of this subsection. In the third part we deal with the case when the degree of f_i is too small for some i so that above technique does not work. Finally in the last part we run a complicated induction and prove Proposition 3.2.6.

The generalized resultant

We generalize the idea of the *resultant* in classical algebra to describe the condition that a collection of polynomials has a common zero.

Let g_1, ..., $g_k \in K[x]$ be one variable polynomials with $\deg g_i = d_i > 0$. One write $g_i = \sum_j a_{i,j}x^j$ and we will denote

$$A^i_{g_1,\ldots,g_k} = \begin{pmatrix} a_{i,d_i} & & & \\ a_{i,d_i-1} & a_{i,d_i} & & \\ \vdots & \vdots & \ddots & a_{i,d_i} \\ a_{i,0} & \vdots & \ddots & \vdots \\ & a_{i,0} & & \vdots \\ & & & a_{i,0} \end{pmatrix}$$

which is a $(d_i + d_k) \times d_k$ matrix satisfying

$$(A^i_{g_1,\ldots,g_k})_{pq} = \begin{cases} a_{i,d_i-q+p} & 0 \le p-q \le d_i \\ 0 & \text{otherwise} \end{cases}.$$

Also define

$$B^i_{g_1,\ldots,g_k} = \begin{pmatrix} a_{k,d_k} & & & \\ a_{k,d_k-1} & a_{k,d_k} & & \\ \vdots & \vdots & \ddots & a_{k,d_k} \\ a_{k,0} & \vdots & \ddots & \vdots \\ & a_{k,0} & & \vdots \\ & & & a_{k,0} \end{pmatrix}$$

be a $(d_i + d_k) \times d_i$ matrix such that

$$(B^i_{g_1,...,g_k})_{pq} = \begin{cases} a_{k,d_k-q+p} & 0 \leq p - q \leq d_k \\ 0 & \text{otherwise} \end{cases}.$$

Consider

$$T_{g_1,...,g_k} = \begin{pmatrix} A^1_{g_1,...,g_k} & B^1_{g_1,...,g_k} & 0 & \cdots & 0 \\ A^2_{g_1,...,g_k} & 0 & B^2_{g_1,...,g_k} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ A^{k-1}_{g_1,...,g_k} & 0 & \cdots & 0 & B^{k-1}_{g_1,...,g_k} \end{pmatrix},$$

which is a $(d_1 + ... + d_{k-1} + (k-1)d_k) \times (d_1 + ... + d_k)$ matrix.

Lemma 3.2.7. *The polynomials g_1, ..., g_k have common zeros if and only if the matrix $T_{g_1,...,g_k}$ is not full rank. Moreover, the number of the common zeros is exactly the nullity of $T_{g_1,...,g_k}$, counted with multiplicity.*

Proof.

Claim. g_1, ..., g_k has common zero if and only if there is polynomials h_1, ..., h_k such that deg $h_i <$ deg g_i and $h_i g_k = h_k g_i$ for all $i < k$.

Indeed, if the polynomials has common zeros, then they have a common factor in the polynomial ring $K[x]$. So we may write $g_i = bh_i$, where $b = \text{gcd}(g_1, ..., g_k)$ and then deg $h_i <$ deg g_i and $h_i g_k = h_k g_i$. Conversely, assume $h_i g_k = h_k g_i$ for some h_1, ..., h_k with deg $h_i <$ deg g_i. If g_k and h_k has no common root, then every root of g_k is a root of g_i for all i thanks to the relation $h_i g_k = h_k g_i$. Otherwise let $l = \text{gcd}(g_k, h_k)$ and define $\bar{g}_k = g_k/l$, $\bar{h}_k = h_k/l$. Then deg $\bar{g}_k > 0$. We still have the relation $h_i \bar{g}_k = \bar{h}_k g_i$ and $\text{gcd}(\bar{g}_k, \bar{h}_k) = 1$. As the previous discussion the root of $\bar{g}_k$ will be a root of g_i for all i.

Thus to prove the lemma, one only need to find h_i satisfied the condition above. Let

$$v = \left(r_{k,d_k-1}, ..., r_{k,0}, -r_{1,d_1-1}, ..., -r_{1,0}, -r_{2,d_2-1}, ..., -r_{k-1,d_{k-1}-1}, ..., -r_{k-1,0}\right)^t$$

be a column vector in $K^{d_1+...+d_k}$, and let $h_i = \sum_j r_{i,j} x^j$, then one can check that the condition $h_i g_k = h_k g_i$ is exactly the linear condition $T_{g_1,...,g_k} v = 0$. Hence g_1, ..., g_k has common zeros if and only if $T_{g_1,...,g_k}$ is not full rank.

Now notice that if $b = \text{gcd}(g_1, ..., g_k)$ and let $\alpha_i = g_i/b$, then the number of common zeros of g_1, ..., g_k is exactly deg b. For $1 \leq j \leq$ deg b, Let v_j be the vector in $K^{d_1+...+d_k}$ corresponds to the collection of polynomials $\{x^{j-1}\alpha_i\}_{i=1}^{k}$, then v_j is lying on the null space of M and v_1, ..., $v_{\deg b}$ are linearly independent.

Conversely assume $T_{g_1,...,g_k}w = 0$ for some $w \in K^{d_1+...+d_k}$, then w corresponds to a collection of polynomials h_1, ..., h_k satisfying $h_i g_k = h_k g_i$ and $\deg h_i < \deg g_i$. We claim that α_i divides h_i for all i.

Let $c_i = \gcd(g_i, g_k)$, $g_i = c_i\beta_i$ and $g_k = c_i\gamma_i$. The relation $h_i g_k = h_k g_i$ yields $h_i\gamma_i = h_k\beta_i$. Since $\gcd(\beta_i, \gamma_i) = 1$ we have γ_i divides h_k for all i, hence $l.c.m.(\gamma_1, ..., \gamma_{k-1})$ divides h_k. On the other hand we have the relation $g_k = b\alpha_k = c_i\gamma_i$. Note that $b = \gcd(c_1, ..., c_{k-1})$, hence γ_i divide α_k for all i and so $l.c.m.(\gamma_1, ..., \gamma_{k-1})$ divides α_k. If $\alpha_k \neq l.c.m.(\gamma_1, ..., \gamma_{k-1})$ then $\alpha_k / l.c.m.(\gamma_1, ..., \gamma_{k-1})$ will divide c_i for all i, contradict to $b = \gcd(c_1, ..., c_{k-1})$. Thus $\alpha_k = l.c.m.(\gamma_1, ..., \gamma_{k-1})$ divides h_k. Finally the relation $h_i g_k = h_k g_i$ gives that $h_i\alpha_k = h_k\alpha_i$. Since α_k divide h_k, we have α_i divide h_i for all i.

Now $\deg h_i < \deg g_i = \deg b + \deg \alpha_i$, hence $h_i = h'\alpha_i$ for some polynomial h' and $\deg h' < \deg b$. Thus w is lying on the subspace generated by v_1, ..., $v_{\deg b}$ and then $null(T_{g_1,...,g_k}) = \deg b$ and the last part of the lemma is proved. □

Lemma 3.2.8. *Assume* $\deg g_i > 1$ *for all* i. *Let*

$$s_0 = null(T_{g_1,...,g_k}); \quad s_1 = null(T_{g_1,...,g_k,g_1',...,g_k'}),$$

here g_i' *denotes the formal derivative of polynomials. Then the number of distinct common roots of* g_1, ..., g_k *is exactly* $s_0 - s_1$.

Proof. Let $b = \gcd(g_1, ..., g_k)$. We will show that $g.c.d(b, b') = \gcd(g_1, ..., g_k, g_1', ..., g_k')$. Indeed, if we write $g_i = bh_i$, then $g_i' = b'h_i + bh_i'$, hence $g.c.d(b, b')$ divides g_i and g_i' for all i and then $\gcd(b, b')$ divides $\gcd(g_1, ..., g_k, g_1', ..., g_k')$. Conversely, if p is a polynomial divides g_i and g_i' for all i, then p will divide $\gcd(g_1, ..., g_k) = b$. The condition p divides g_i' implies p divides $b'h_i$ for all i. However, $\gcd(h_1, ..., h_k) = 1$. Thus p divides b' and hence p divides $\gcd(b, b')$. That is, $\gcd(g_1, ..., g_k, g_1', ..., g_k')$ divides $\gcd(b, b')$.

Now write $b = (x - a_1)^{r_1}...(x - a_m)^{r_m}$, then the number of distinct common roots of g_1, ..., g_k is m. On the other hand,

$$b' = \left((x - a_1)^{r_1 - 1}...(x - a_m)^{r_m - 1}\right)\left(\sum_i r_i(x - a_1)...(x - a_{i-1})(x - a_{i+1})...(x - a_m)\right).$$

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Hence gcd$(b, b') = (x - a_1)^{r_1 - 1} ... (x - a_m)^{r_m - 1}$. By Lemma 3.2.7, $s_0 = \deg b = r_1 + ... + r_m$ and $s_1 = \deg(\gcd(b, b')) = (r_1 - 1) + ... + (r_m - 1) = r_1 + ... + r_m - m$. A conclusion is that $s_0 - s_1 = m$, as we want. □

The geometry of the projection map

In this part we study the fibre of $\phi : X_I \to \mathbb{A}^{n-1}$. We will view f_i as a polynomial in x_n and we will denote $f_i' = \frac{\partial}{\partial x_n} f_i$. Let

$$T^0 = \begin{cases} T_{f_1, f_1'} & \text{if } k = 1, \\ T_{f_1, ..., f_k} & \text{if } k > 1. \end{cases} \qquad T^1 = \begin{cases} T_{f_1, f_1', f_1''} & \text{if } k = 1, \\ T_{f_1, ..., f_k, f_1', ..., f_k'} & \text{if } k > 1. \end{cases}$$

provided that all the polynomials are non-constant. Note that T^0 and T^1 are matrices with all entries being a polynomial in $K[x_1, ..., x_{n-1}]$.

Convention. For $j = 0$, 1, we say the *condition* (A^j) *are satisfied* if T^j is defined. That is, $\deg f_i > j$ (resp. $j + 1$) for all i if $k > 1$ (resp. $k = 1$).

When (A^j) is satisfied, one could study the fiber of ϕ via the nullity of T^j. There are three possibility of the fiber of ϕ: empty, finite points or a $\mathbb{A}^1$. The fiber is a $\mathbb{A}^1$ at a point $P \in \mathbb{A}^{n-1}$ if and only if all f_i vanishes at P, which is easy to detect. The main question is that how to find the locus on $\mathbb{A}^{n-1}$ such that the pre-image is finite, and how to find the cardinality of the fiber.

Assume (A^0) one could solve the first question (cf. Lemma 3.2.9, Lemma 3.2.10). If (A^1) holds and assuming more conditions one could count the cardinality of the fiber (cf. Lemma 3.2.11).

Lemma 3.2.9. *Assume (A^0). Fix $p \in \mathbb{A}^{n-1}$ and assume that $T^0(p)$ is full rank. Then*

$$|\phi^{-1}(p)| = \begin{cases} \deg f_1 & \textit{if } k = 1, \\ 0 & \textit{if } k > 1. \end{cases}$$

Proof. Assume $k = 1$. If the leading coefficient vanishes over p, then the first row of T^0 is always zero. Since T^0 is a square matrix, this implies T^0 is not full rank. Hence we may assume the leading coefficient do not vanishing at p, so both f_1 and f_1' are non-constant. Using Lemma 3.2.7, we see that $T^0(p)$ is full rank implies f_1 and f_1' consist no common zero. Hence f_1 consists no multiple roots over p, so $|\phi^{-1}(p)| = \deg f_1$.

Now assume $k > 1$. First assume f_i is constant over p for some i. Then if f_i is identically zero, T^0 can not be full rank. On the other hand, if f_i is a non-zero constant, then $\phi^{-1}(p)$ is always empty so the conclusion is always true. Finally assume f_i is non-constant for all i, then for any $p \in H$, $\phi^{-1}(p)$ is non-empty only if f_1, ..., f_k admit common zeros. By Lemma 3.2.7, this implies the matrix T^0 is not full rank. ☐

Lemma 3.2.10. *Assume (A^0). Given $p \in \mathbb{A}^{n-1}$ and assume that $T^0(p)$ is not full rank. Assume further that the leading coefficient of f_i do not vanish at p for all i. Then if $k > 1$, we have that p is contained in the image of ϕ. For $k = 1$, one can say that ϕ is a finite morphism near p and p is lying on the ramification locus.*

Proof. First assume $k > 1$. The hypothesis implies that f_1, ..., f_k is non-constant polynomial in x_n over p. By Lemma 3.2.7, T^0 is not full rank at p if and only if f_1, ..., f_k admits a common zero, say $\xi \in K$. If we write $p = (a_1, ..., a_{n-1})$, then the point $(a_1, ..., a_{n-1}, \xi)$ is lying on X_I and is mapped to p by ϕ. Hence p is contained in the image of ϕ.

For the $k = 1$ case, note that T^0 is defined implies deg $f_1 > 1$. By assumption, the leading coefficient of f_1 do not vanish at p, hence it do not vanish on a neighborhood U of p. We see that for any point $q \in U$ we have f_1 is a polynomial of positive degree in x_n over q, so the pre-image of ϕ consists only finitely many points and so ϕ is a finite morphism on U. Now the condition that $T^0(p)$ is not full rank implies f_1 consists multiple root over p, hence p is lying in the ramification locus of ϕ. ☐

Now let $Z \subset \mathbb{A}^{n-1}$ be a subset contained in the image of ϕ. For $p \in Z$ we will denote $r(p) = |\phi^{-1}(p)|$ and $r(Z) = \max_{p \in Z}\{r(p)\}$. Also define $s_0(p) = null(T^0(p))$ and $s_1(p) = null(T^1(p))$. What we want to do is to find the locus which consists of the points $p \in Z$ such that $r(p) \neq r(Z)$. Such point could be determined using the number s_0 and s_1, under suitable conditions.

Lemma 3.2.11. *Fix $Z \subset \mathbb{A}^{n-1}$ be any subset. Assume that the leading coefficient of f_i do not vanish over Z for all i. When $k = 1$ (resp. $k > 1$) assume (A^0)(reps. (A^1)). Then for any $p \in Z$ we have*

(i) Assume $k = 1$, then $r(p) = \deg f_1(p) - s_0(p)$.

(ii) Assume $k > 1$, then $r(p) = s_0(p) - s_1(p)$.

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Proof. First assume $k > 1$. By Lemma 3.2.8 we have $r(p) = s_0(p) - s_1(p)$ for all $p \in Z$. Now assume $k = 1$. The assumption that T^0 exists and the leading coefficient of f_1 do not vanish implies that ϕ is a finite morphism over Z. For any p in Z the number $r(p)$ is the number of distinct roots of f_1 over p. Assume $f_1(p) = (x_n - a_1)^{r_1} ... (x_n - a_m)^{r_m}(x_n - b_1)...(x_n - b_l)$ with $r_i > 1$. We have $r(p) = m + l$, $\deg f_1(p) = r_1 + ... + r_m + l$, $s_0(p) = (r_1 - 1) + ... + (r_m - 1) = r_1 + ... + r_m - m$ by Lemma 3.2.7, hence $r(p) = \deg f_1 - s_0(p)$.

□

Corollary 3.2.12. *Fix $Z \subset \mathbb{A}^{n-1}$. Assume that the leading coefficient of f_i do not vanish over Z for all i and one of the following condition holds:*

(i) $k = 1$ and (A^0) holds.

(ii) $k > 1$, (A^1) holds and s_0 is constant over Z.

Then ϕ is a finite morphism over Z. When $k = 1$ (resp. $k > 1$) the ramification locus of ϕ is exactly the locus where the function s_0 (resp. s_1) do not reach its minimum.

The small degree cases

In this section we deal with the cases that the $\deg f_i$ is too small so that (A^0) or (A^1) dose not hold.

Lemma 3.2.13. *Under the assumption and notation in Proposition 3.2.6, if $k = 1$ and (A^0) dose not hold over Z, then the conclusion of Proposition 3.2.6 is true.*

Proof. The assumption says that $\deg f_1 < 2$ over Z. If $\deg f_1 = 0$, then $f_1 \in K[x_1, ..., x_{n-1}]$ is independent of x_n. Let Z' be the zero locus of the ideal $J + (f_1)$, then $|\chi_{top}(Z')| \leq N^{n-1}_{\max\{e,d\},l+1}$. One see that outside Z', the pre-image of ϕ is empty, and $\phi^{-1}Z' \cong Z' \times \mathbb{A}^1$. Hence $|\chi_{top}(\phi^{-1}Z)| = |\chi_{top}(\phi^{-1}Z')| = |\chi_{top}(Z')| \leq N^{n-1}_{\max\{e,d\},l+1}$.

On the other hand, assume $\deg f_1 = 1$. Write $f_1 = a_1 x_n + a_0$. Let Z_0 be the zero locus defined by $J + (a_1)$ and $Z_1 = Z - Z_0$. Then $\chi_{top}(\phi^{-1}Z_0)$ can be computed in the previous case since we can replace f_1 by a_0 and replace Z by Z_0. On the other hand, since f_1 is a degree one polynomial over any points in Z_1, we have $\phi^{-1}Z_1 \cong Z_1$. Now $|\chi_{top}(\phi^{-1}Z_1)| = |\chi_{top}(Z_1)| = |\chi_{top}(Z) - \chi_{top}(Z_0)| \leq N^{n-1}_{e,l} + N^{n-1}_{\max\{e,d\},l+1}$ can be compute. Thus the lemma is proved. □

The other case is that (A^0) holds but (A^1) dose not hold. This happened when $k = 1$ and $\deg f_1 = 2$ or $k > 1$ and $\deg f_i = 1$ for some i.

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Lemma 3.2.14. *Let $Z \subset H$ and assume the following.*

(i) (A^0) holds but (A^1) dose not hold.

(ii) $T^0(p)$ is not full rank for all $p \in Z$.

(iii) The leading coefficient of f_i do not vanishing for all i for any point $p \in Z$.

Then ϕ is one-to-one over Z. In particular, $\chi_{top}(\phi^{-1}Z) = \chi_{top}(Z)$.

Proof. First assume $k > 1$. By Lemma 3.2.10 the assumption yields that Z is contained in the image of ϕ. On the other hand, T^0 is defined but T^1 is not defined implies deg $f_i = 1$ for some i, hence ϕ is one-to-one over Z.

Now assume $k = 1$. Since T^0 is defined but T^1 is not defined, we have deg $f_1 = 2$, hence ϕ is two-to-one over some open neighborhood of Z. However, Lemma 3.2.10 implies that Z is lying on the ramification locus, hence ϕ is one-to-one over Z. □

The main proof

We will need the following lemma.

Lemma 3.2.15. *Let $S = S_1 \cup S_2 \cup ... \cup S_k$ for some algebraic set S_i. For any $I \subset \{1, ..., k\}$, we denote $S_I = \bigcap_{i \in I} S_i$. Assume that $|\chi_{top}(S_I)| \leq M$ for some integer M and for all $I \subset \{1, ..., k\}$. Then*

$$|\chi_{top}(S)| \leq (2^k - 1)M.$$

Proof. We prove by induction on k. The case that $k = 1$ is trivial. In general let $S' = S_1 \cup ... \cup S_{k-1}$, then $S' \cap S_k = (S_1 \cap S_k) \cup ... \cup (S_{k-1} \cap S_k)$, hence

$$|\chi_{top}(S' \cap S_k)| \leq (2^{k-1} - 1)M$$

by the induction hypothesis. We also have $|\chi_{top}(S')| \leq (2^{k-1} - 1)M$. Thus

$$\begin{aligned}
|\chi_{top}(S)| &= |\chi_{top}(S' - (S' \cap S_k)) + \chi_{top}(S_k - (S' \cap S_k)) + \chi_{top}(S' \cap S_k)| \\
&\leq |\chi_{top}(S')| + |\chi_{top}(S_k)| + |\chi_{top}(S' \cap S_k)| \\
&\leq (2(2^{k-1} - 1) + 1)M = (2^k - 1)M.
\end{aligned}$$

□

Proof of Proposition 3.2.6. We will divide Z into many pieces, and treat each piece separately. In each piece, either the topology of the pre-image can be easily computed, or after cut out some closed subset the pre-image can be computed, and there is some quantum which strictly decrease after restrict to the subset above. In the latter case we can use induction on the special quantum and finally the problem could be solved. We will treat the following cases.

Case(I) (A^0) holds.

Let

$$Z' = \left\{p \in Z \,\middle|\, T^0(p) \text{ is not full rank } \right\}$$

and $Z'' = Z - Z'$. By Lemma 3.2.9,

$$\chi_{top}(\phi^{-1}Z'') = \begin{cases} (\deg f_1)\chi_{top}(Z'') & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

We further divide Z' into

$$Z_- = \{p \in Z' \,|\, \text{The leading coefficient of } f_i \text{ vanish over } p \text{ for some } i\}$$

and $Z_+ = Z' - Z_-$. To compute $\chi_{top}(\phi^{-1}Z_-)$, let a_i be the leading coefficient of f_i. For $S \subset \{1, ..., k\}$, let W_S be the zero locus defined by $J + (a_{i_0}...a_{i_p})$ if $S = \{i_0, ..., i_p\}$ and J is the defining ideal of Z. Then W_S is the locus in Z such that the leading coefficient of f_i vanish for all $i \in S$. Hence $Z_- = \bigcup_{1\leq i\leq k} W_i$ and $W_S = \bigcap_{i\in S} W_i$. Furthermore, one may induction on the number deg $f_1 + ... +$ deg f_k so that we may assume $\chi_{top}(\phi^{-1}W_S)$ can be computed. By Lemma 3.2.15, $\chi_{top}(\phi^{-1}Z_-)$ can be bounded.

Now one has to compute $\chi_{top}(\phi^{-1}Z_+)$. If (A^1) is not true, then Lemma 3.2.14 implies that $\chi_{top}(\phi^{-1}Z_+) = \chi_{top}(Z_+)$. Assume (A^1) is true. We divide Z_+ into

$$Z_0 = \{p \in Z_+ \,|\, s_0(p) \text{ reach its minimum in } Z_+\}$$

and $Z'_0 = Z_+ - Z_0$. One may replace Z by Z'_0 and induction on $\min_{p\in Z}\{s_0(p)\}$. This number is increasing and always less or equal than deg f_i for all i, so after finite step, Z'_0 would be empty.

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If $k > 1$ we further divide Z_0 into

$$Z_1 = \{p \in Z_0 \,|\, s_1(p) \text{ reach its minimum in } Z_0\}$$

and $Z_1' = Z_0 - Z_1$. By Corollary 3.2.12, when $k = 1$ (resp. $k > 1$) ϕ is unramified over Z_0 (resp. Z_1). Hence

$$|\chi_{top}(\phi^{-1}Z_i)| = r(Z_i)|\chi_{top}(Z_i)| \leq d|\chi_{top}(Z_i)|,$$

with $i = 0$ (resp. $i = 1$) in $k = 1$ (resp. $k > 1$) case.

When $k > 1$ we have $r(Z_1') < r(Z_0)$. We will replace Z by Z_1' and induction on the number r. When $r(Z_0) = 1$ Z_1' is always empty, so the induction works.

Case(II) (A^0) does not hold. If $k = 1$, this case can be solved by Lemma 3.2.13. Now assume $k > 1$. In this case deg $f_i = 0$ for some i. If f_i is a non-zero constant, then $\phi^{-1}Z$ is empty, so there is nothing to prove. If f_i is identically zero, we can drop out f_i from the generator of I, and goes to the case with smaller k. By induction on k, this situation is solved.

We have to show that Z', Z_-, Z_1' and Z_0' can be defined by algebraic equations, and the total number and the degree of those equations can be bounded by some integer depends on d and k, so the induction could work.

To see this, let c_i and r_i be the number of columns and rows of T^i, respectively, for $i = 0, 1$. Then $c_0 \leq dk$, $r_0 \leq 2(k-1)d$, and $c_1 \leq 2c_0$, $r_1 \leq 2r_0$. Let R be the ideal containing all maximal minors of T^0, then R can be generated by $C_{c_0}^{r_0}$ many generators and each generator is a degree at most dr_0 polynomial. One can see that Z' is generated by $J + R$. Since $Z_- = \bigcup_{1 \leq i \leq k} W_i$ and the defining ideal of W_i are bounded, the defining ideal of Z_- is bounded.

Now let $t_i = \max_{p \in Z} rk(T^i(p))$. $p \in Z$ satisfied $s_i(p)$ do not reach minimum if and only if $s_i(p) + t_i > r_i$. Hence one only need to find those points in Z such that the rank of T^i at that point is less than t_i, or equivalently, all $t_i \times t_i$ minors of T^i vanishes. Let Q_i be the ideal containing all $t_i \times t_i$ minors of T^i, then Q_i is generated by at most $r_i c_i$ many elements and each element is a degree at most $dt_i \leq dr_i$ polynomial in $K[x_1, ..., x_{n-1}]$. One can easily see that Z_i' is defined by $J + Q_i$ for $i = 0, 1$.

The other task is to compute $\chi_{top}(Z'')$, $\chi_{top}(Z_+)$ and $\chi_{top}(Z_i)$ for $i = 0, 1$. Since Z' is

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generated by $J + R$,

$$|\chi_{top}(Z')| \leq N^{n-1}_{e+dr_0,l+C^{r_0}_{c_0}}.$$

We have $\chi_{top}(Z'') = \chi_{top}(Z) - \chi_{top}(Z')$. Thus

$$|\chi_{top}(Z'')| \leq N^{n-1}_{e,l} + N^{n-1}_{e+dr_0,l+C^{r_0}_{c_0}}.$$

Now consider $|\chi_{top}(W_S)| \leq N^{n-1}_{e+d^{|S|},l+1} \leq N^{n-1}_{e+d^k,l+1}$ for all $S \subset \{1, ..., k\}$, hence

$$|\chi_{top}(Z_-)| \leq (2^k - 1)N^{n-1}_{e+d^k,l+1}$$

by Lemma 3.2.15. A conclusion is that $\chi_{top}(Z_+) = \chi_{top}(Z') - \chi_{top}(Z_-)$ can be bounded.

Finally we try to bound $\chi_{top}(Z_i)$. As the argument above Z'_i is defined by the ideal $J + Q_i$ for $i = 0$ and 1, hence $|\chi_{top}(Z'_i)| \leq N^{n-1}_{e+dr_i,l+r_ic_r}$ can be bounded. Thus

$$\chi_{top}(Z_0) = \chi_{top}(Z_+) - \chi_{top}(Z'_0) \text{ and } \chi_{top}(Z_1) = \chi_{top}(Z_0) - \chi_{top}(Z'_1)$$

can be bounded. □

Proof of Proposition 3.2.4. One can take $N^n_{d,k} = L^n_{d,k,0,1}$ by considering J in Proposition 3.2.6 to be the zero ideal. □

3.2.2 The existence of M-constant

We prove Proposition 3.2.5 in the following way: let Y be the given generically cyclic quotient space. Then Y can be decomposed into a cyclic quotient part and a lower-dimensional generically cyclic quotient part. By induction on the dimension we may assume the lower dimensional part can be controlled. The cyclic quotient part can be studied via the natural quotient map. One only need to understand the ramification locus of the natural quotient map. Each irreducible component of the ramification locus is a lower dimensional cyclic quotient, and one can write it as a difference of two lower dimensional weighted projective spaces. Again by induction on the dimension one can estimate the topological Euler characteristic of it. This leads to the proof of Proposition 3.2.5.

Proof of Proposition 3.2.5. We use induction on the dimension n. When $n = 1$, one can take $M_{1,d} = d$. Now assume $n > 1$. We know that there exists an open set $U \subset Y$ such that

$U \cong \mathbb{A}^n/\frac{1}{r}(a_1, ..., a_n)$. Let $W' = W|_U$ and $W'' = W|_{Y-U}$. We have $|\chi_{top}(W'')| \leq M_{n-1,d}$ by the induction hypothesis. One only need to estimate $\chi_{top}(W')$.

Consider the natural quotient map $\phi : \mathbb{A}^n \to U$. Let $\bar{W} = \phi^{-1}W'$. One has $|\chi_{top}(\bar{W})| \leq N^n_{d,k}$ by Theorem 3.2.2 (i). We know that $\bar{W} \to W'$ is a branched covering. Let $\bar{R} \subset \bar{W}$ be the branched locus and $R \subset W'$ be the image of $\bar{R}'$, then

$$\begin{aligned}|\chi_{top}(W')| &= |\frac{1}{r}\chi_{top}(\bar{W} - \bar{R}) + \chi_{top}(R')| \\ &\leq |\chi_{top}(\bar{W} - \bar{R})| + |\chi_{top}(R')| \\ &\leq |\chi_{top}(\bar{W})| + |\chi_{top}(\bar{R})| + |\chi_{top}(R')| \leq N^n_{d,k} + |\chi_{top}(\bar{R})| + |\chi_{top}(R')|.\end{aligned}$$

Hence one only have to estimate $\chi_{top}(\bar{R})$ and $\chi_{top}(R')$.

Note that the morphism $\mathbb{A}^n \to U$ could only ramify at $\{x_{i_1} = ... = x_{i_l} = 0\}$ for some i_1, ..., i_l, where x_1, ..., x_n is the coordinate of $\mathbb{A}^n$. Let Ξ_1, ..., Ξ_j be irreducible components of the ramification locus on $\mathbb{A}^n$ and let $\bar{S_i} = \Xi_i \cap \bar{R}$. One can see that

$$|\chi_{top}(\bar{S_i})| \leq N_{n-l_i,d}$$

if $\Xi_i = \{x_{i_1} = ... = x_{i_{l_i}} = 0\}$ and

$$|\chi_{top}(\bar{S_{i_1}} \cap ... \cap \bar{S_{i_m}})| \leq N_{n-l',d}$$

if $\Xi_{i_1} \cap ... \cap \Xi_{i_m}$ is of codimension l'. Moreover, the number of irreducible components of ramification locus of $\mathbb{A}^n \to \mathbb{A}^n/\frac{1}{r}(a_1, .., a_n)$ is less than 2^n. Hence by Lemma 3.2.15,

$$|\chi_{top}(\bar{R})| \leq \left(2^{2^n} - 1\right) N_{n-1,d}.$$

To compute $\chi_{top}(R)$, we denote by S_i the image of $\bar{S_i}$. Consider $\Xi_i \cong \mathbb{A}^{n_i}$ for some n_i and the morphism $\Xi_i \to im(\Xi_i)$ can be viewed as a cyclic quotient $\mathbb{A}^{n_i} \to \mathbb{A}^{n_i}/\frac{1}{r_i}(b_{i_1}, ..., b_{i_{n_i}})$ for some integers r_i and b_{i_1}, ..., $b_{i_{n_i}}$. Consider

$$S_i \subset im(\Xi_i) \cong \mathbb{A}^{n_i}/\frac{1}{r_i}(b_{i_1}, ..., b_{i_{r_i}}) \subset \mathbb{P}(r_i, b_{i_i}, ..., b_{i_{n_i}})$$

and let $\tilde{S}_i$ be the closure of S_i in $\mathbb{P}(r_i, b_{i_i}, ..., b_{i_{r_i}})$. We have

$$|\chi_{top}(\tilde{S}_i)| \leq M_{n_i,d} \quad \text{and} \quad |\chi_{top}(\tilde{S}_i - S_i)| \leq M_{n_i-1,d},$$

hence

$$|\chi_{top}(S_i)| \leq M_{n_i,d} + M_{n_i-1,d}.$$

Moreover, for any $i_1, ..., i_l \in \{1, ..., j\}$ we have $\Xi_{i_1} \cap ... \cap \Xi_{i_l} \cong \mathbb{A}^{n_{i_1 \cdots i_l}}$ for some integer $n_{i_1 ... i_l}$ and the same argument shows that

$$|\chi_{top}(S_{i_1} \cap ... \cap S_{i_l})| \leq M_{n_{i_1 ... i_l},d} + M_{n_{i_1 ... i_l}-1,d}.$$

One may assume $M^n_{d,k}$ is an increasing function of n, then we have

$$|\chi_{top}(R)| = |\chi_{top}(S_1 \cup ... \cup S_j)| \leq 2\left(2^{2^n} - 1\right) M_{n-1,d}$$

by Lemma 3.2.15.

The conclusion is

$$|\chi_{top}(W')| \leq N^n_{d,k} + |\chi_{top}(\bar{R})| + |\chi_{top}(R')| \leq N^n_{d,k} + \left(2^{2^n} - 1\right) N_{n-1,d} + 2\left(2^{2^n} - 1\right) M_{n-1,d}.$$

Hence

$$\begin{aligned} |\chi_{top}(W)| &\leq |\chi_{top}(W')| + |\chi_{top}(W'')| \\ &\leq M_{n-1,d} + N^n_{d,k} + \left(2^{2^n} - 1\right) N_{n-1,d} + 2\left(2^{2^n} - 1\right) M_{n-1,d} \\ &\leq 2^{2^n}\left(2M_{n-1,d} + d(2d-1)^{2n-1}\right). \end{aligned}$$

□

Proof of Theorem 3.2.2. One can easily see that $M^1_{d,k} = N^1_{d,k} = d$. Hence Proposition 3.2.4 and Proposition 3.2.5 implies the theorem. □

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3.3 The boundedness of Betti numbers

In this section we will bound the variance of b_3. Thanks to Corollary 3.1.10, it is equivalent to bound the variance of the topological Euler characteristic, which is much easier to compute. The following statement is a corollary of Theorem 3.2.2, which could help us to bound the variance of the topological Euler characteristic under divisorial contractions to points.

Corollary 3.3.1. *Assume that X is a cyclic quotient of a local complete intersection locus in $\mathbb{A}^n$ of codimension k, and $Y \to X$ is a weighted blow-up of weight σ. If the σ-weight of the defining equations of X is bounded by a constant d, then $|\chi_{top}(Y) - \chi_{top}(X)| \leq M^n_{d,k} + 1$.*

Proof. Write $\sigma = \frac{1}{m}(a_0, ..., a_n)$. The exceptional locus E of $Y \to X$ is contained in a generically cyclic quotient space and has degree less than d. Hence $|\chi_{top}(E)| \leq M^n_{d,k}$. Now

$$|\chi_{top}(Y) - \chi_{top}(X)| = |\chi_{top}(E) - \chi_{top}(point)| \leq M^n_{d,k} + 1.$$

□

Given a divisorial contraction $Y \to X$ which contracts a divisor to a point, we will show that the difference of the topological Euler characteristic can be bound by a constant depending only on $dep(X)$ if $Y \to X$ is a w-morphism, and on $dep(Y)$ in general. The reason we need the first statement is that the inverse of w-morphisms occur in the Chen-Hacon factorization.

Proposition 3.3.2. *Let $Y \to X$ be a divisorial contraction which contracts a divisor E to a point $P \in X$. Assume the index of P is $r > 1$ and assume $a(E, X) = 1/r$. Then*

$$|\chi_{top}(Y) - \chi_{top}(X)| \leq D_{dep(X)}$$

for some integer $D_{dep(X)}$ depending only on $dep(X)$.

Proof. We already known that $Y \to X$ is a weighted blow-up of a cyclic quotient of local complete intersection locus in $\mathbb{A}^4$ or $\mathbb{A}^5$. We will denote by d the upper bound of the weight of the exceptional locus viewed as a subvariety in the weighted projective space. What we have to do is to show that d can be determined by $dep(X)$ and then $|\chi_{top}(Y) - \chi_{top}(X)| \leq M^n_{d,k} + 1$ for $(n, k) = (4, 1)$ or $(5, 2)$, which is an integer depends only on $dep(X)$. We discuss each case in Table 2.3 and 2.4.

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cA/m. We are in the case No. 1 of Table 2.3. One can check that there are two cyclic quotient points on Y, one is of index a and the other one is of index b. We conclude that $dep(X) \geq a+b-1$ and the exceptional locus can be viewed as a weighted hypersuface in $\mathbb{P}(a, b, 1, m)$ with weight $mk = a+b \leq dep(X)+1$, hence we take $d = dep(X)+1$.

$cAx/4$. In case No. 2 there is a cyclic quotient point on Y of index $2k+3$ which implies hence $dep(X) \geq 2k+3$. In case No. 3 there is a cyclic uotient point on Y of index $2k+5$ and so $dep(X) \geq 2k+5$. In the both cases we take $d = 2dep(X) - 4$.

$cAx/2$. Using similar argument as the previous case, one can take $d = 2dep(X) - 2$.

$cD/3$. We are in the cases No. 6-8. One can check that $d = 12$ satisfied the condition.

$cE/2$. In the cases No. 9-13, one can check that one can take $d = 18$.

$cD/2$. In case No. 14 and 15 one can see that $d = 6$. For case No. 16-23, one can check that Y contains at least one cyclic quotient or cA/r singularity and the lower bound of $dep(X) = dep(Y) + 1$ can be derived. Please see the following table.

No.	d	singular point on Y	lower bound of $dep(X)$
16	$2b+4$	$\frac{1}{b}(2,-2,1)$	b
17	$2b$	cA/b	$2b-1$
18	$2b$	$\frac{1}{b-2}(2,-2,1)$	$2b$
19	$2b+2$	$\frac{1}{b}(2,-2,1)$	b
20	$2b+4$	$\frac{1}{2b+2}(b+2,b,1)$	$2b+2$
21	$2b$	$\frac{1}{b-1}(2,-2,1)$	$b-1$
22	$2b+2$	$\frac{1}{2b}(b+1,b-1,1)$	$2b$
23	$2b+2$	$cA/b+1$	$2b+1$
24	$2b$	cA/b	$2b-1$
25	$2b$	$\frac{1}{b-2}(2,-2,1)$	$b-2$

One can take $d = 2dep(X) + 4$

The conclusion is that one can take

$$D_{dep(X)} = \max_{n=4,5}\{M^n_{2dep(X)+4,n-3}, M^n_{18,n-3}\} + 1.$$

$\square$

Proposition 3.3.3. *Assume that $f : Y \to X$ is a divisorial contraction to a point. Then there is an integer $D'_{dep(Y)}$ depending only on $dep(Y)$ such that*

$$|\chi_{top}(Y) - \chi_{top}(X)| \leq D'_{dep(Y)}.$$

Proof. By Theorem 2.2.6 X is a LCI locus in cyclic quotient of $\mathbb{A}^4$ or $\mathbb{A}^5$ and Y is obtained by weighted blow-up. One only to show that there is an upper bound d of the weight of defining equation of X, which can be bounded by some constant depends only on $dep(Y)$.

(i) $Y \to X$ belongs to Theorem 2.2.6 (i). Then $Y \to X$ is a w-morphism and the statement follows by Proposition 3.3.2.

(ii) $Y \to X$ belongs to Theorem 2.2.6 (ii). We discuss each case in Table 2.5.

No. 1. We have $d = 2b$ and there are a cyclic quotient point of type $\frac{1}{b-1}(1, -1, 1)$ on Y. Hence $d \leq 2dep(Y) + 4$.

No. 2. We have $d = 2b$ and Y contains a cA/b point. Hence $d \leq dep(Y) + 2$.

No. 3. We have $d = 2b + 2$ and Y contains a $\frac{1}{2b+1}(b + 1, b, 1)$ point. Hence $d \leq dep(Y) + 2$.

No. 4. We have $d = 2b+1$ and Y contains a $\frac{1}{b}(1, -1, 1)$ point. Hence $d \leq 2dep(Y)+3$.

No. 5. We have $d = 2b$ and Y contains a $\frac{1}{b+1}(b, 1, 1)$ point. Hence $d \leq 2dep(Y)$.

No. 6-23. One can take $d = 30$.

(iii) $Y \to X$ belongs to Theorem 2.2.6 (iii). We have to check each case in Table 2.6.

No. 1. The are two cyclic quotient points on Y with indices b and c respectively. Hence $dep(Y) \geq b + c - 2 = d - 2$, or $d \leq dep(Y) + 2$.

No. 2. One can take $d = 6$.

No. 3. We have $d = 2b$ and there is a cyclic quotient point of type $\frac{1}{b}(1, -1, a)$ on Y, hence $d \leq 2dep(Y) + 2$.

No. 4. We have $d = 2b+2$ and there is a cyclic quotient point of type $\frac{1}{b+2}(b+1, a, 1)$, hence $d \leq 2dep(Y)$.

No. 5, 6. We have $d = b + 1$ and there is a $\frac{1}{b}(\frac{b+1}{2}, \frac{b-1}{2}, a)$ poiont on Y, for $a = 2$ or 4. Hence $d \leq dep(Y) + 2$.

No. 7. We have $d = 2b$ and there is a cA/b point on Y, hence $d \leq dep(Y) + 2$.

No. 8, 9. We have $d = 6$.

No. 10. We have $d = 2b+2$ and there is a $\frac{1}{b}(2, -2, a)$ point on Y, hence $d \leq 2dep(Y)+4$.

No. 11. We have $d = 2b + 4$ and there is a $\frac{1}{b+4}(b + 2, a, 2)$ point on Y, hence $d \leq 2dep(Y) - 2$.

No. 12. We have $d = 4b$ and there is a $cA/4b$ point on Y, hence $d \leq dep(Y)$.

No. 13. One can take $d = 4$.

No. 14. We have $d = 2b + 2$ and there is a $\frac{1}{4b+2}(-1, 1, 2b - 1)$ point on Y, hence $d \leq dep(Y)$.

No. 15. One can take $d = 4$.

No. 16. We have $d = 4b + 2$ and there is a $\frac{1}{4b+4}(1, 2b + 1, -1)$ point on Y, hence $d \leq dep(Y)$.

No. 17. We have $d = b+1$ and there is a $\frac{1}{2b}(1, -1, b+4)$ point on Y, hence $d \leq dep(Y)$.

No. 18-21. One can take $d = 14$.

We conclude that one can take $D'_{dep(Y)}$ to be

$$\max_{n=4,5}\{D_{dep(Y)+1}, M^n_{2dep(Y)+4,n-3}, M^n_{30,n-3}\} + 1.$$

□

In the case of blowing-up LCI curves, the difference of the topological Euler characteristic is easy to compute.

Lemma 3.3.4. *Assume that $C \subset X$ is a LCI curve and $f : Y = Bl_C X \to X$, then*

$$\chi_{top}(Y) - \chi_{top}(X) = \chi_{top}(C).$$

Proof. Let E be the exceptional divisor of $Y \to X$. At first we show that over any point $P \in C$, the fiber $f^{-1}(P)$ is isomorphic to $\mathbb{P}^1$. Indeed, assume that C is defined by the ideal I which is locally generated by two regular functions α and β. Then Y is isomorphic to $Proj \bigoplus_{n\geq 0} I^n$ and the natural map $\mathcal{O}_X[x,y] \to \bigoplus_{n\geq 0} I^n$ defined by $x \mapsto \alpha$, $y \mapsto \beta$ gives an inclusion $Y \hookrightarrow X \times \mathbb{P}^1$. Hence every fiber over C is a $\mathbb{P}^1$.

Now there exists a open set $U \subset C$ such that $f^{-1}U \cong U \times \mathbb{P}^1$ since geometric ruled surfaces are ruled, hence we have

$$\chi_{top}(E) = \chi_{top}(f^{-1}U) + \chi_{top}(f^{-1}(C-U)) = 2\chi_{top}(U) + 2\chi_{top}(C-U) = 2\chi_{top}(C)$$

and then

$$\chi_{top}(Y) - \chi_{top}(X) = \chi_{top}(E) - \chi_{top}(C) = \chi_{top}(C).$$

□

Now let X be a smooth threefold and consider the process of the minimal model program

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_m.$$

We will use above results to estimate the third Betti number of X_i.

Proposition 3.3.5. *Let $X \to W$ be a divisorial contraction and $X \dashrightarrow X'$ be a flip or a flop. Then there is a constant $\Phi_{dep(X)}$ depending only on $dep(X)$ such that $b_3(W) \leq \Phi_{dep(X)} + b_3(X)$ and $b_3(X') \leq \Phi_{dep(X)} + b_3(X)$*

Proof. Assume that $X \to W$ is a divisorial contraction to point, then by Corollary 3.1.10 and Proposition 3.3.3 we have $|b_3(X) - b_3(W)| = |\chi_{top}(X) - \chi_{top}(W) - 2| \leq D'_{dep(X)} + 2$, hence

$$b_3(W) \leq D'_{dep(X)} + 2 + b_3(X).$$

If $X \to W$ is obtained by blowing-up an LCI curve $C \subset W$, then using Corollary 3.1.10 and Lemma 3.3.4 one has

$$b_3(W) - b_3(X) = \chi_{top}(X) - \chi_{top}(W) - 2 = \chi_{top}(C) - 2 \leq 0,$$

hence

$$b_3(W) \leq b_3(X).$$

If $X \dashrightarrow X'$ is a flop, then $b_3(X) = b_3(X')$ by Lemma 3.1.11. Thus if $dep(X) = 0$, then the

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statement is proved by Remark 3.1.4.

Now assume that $X \to W$ or $X \dashrightarrow X'$ is not the three kinds of elementary maps we mentioned above. We will prove by induction on $dep(X)$. By Theorem 3.1.7 we have the diagram

$$\begin{tikzcd}
Y \arrow[rr, dashed] \arrow[d, "f"] & & Y' . \arrow[d, "f'"'] \\
X \arrow[dr, "g"] & & X' \arrow[dl, "g'"'] \\
& W &
\end{tikzcd}$$

At first note that by Corollary 3.1.10 and Proposition 3.3.2 one has

$$|b_3(Y) - b_3(X)| = |\chi_{top}(Y) - \chi_{top}(X)| + 2 \leq D_{dep(X)} + 2,$$

hence $b_3(Y) \leq D_{dep(X)} + 2 + b_3(X)$. On the other hand, one may write

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow ... \dashrightarrow Y_l = Y',$$

such that $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i > 0$ and $Y_0 \dashrightarrow Y_1$ is either a flip or a flop by Remark 3.1.8. The conclusion is

$$0 \leq dep(Y_l) < dep(Y_{l+1}) < ... < dep(Y_1) \leq dep(Y_0) < dep(X).$$

Thus $l \leq dep(X)$.

By the induction hypothesis we have $b_3(Y_{i+1}) < \Phi_{dep(Y_i)} + b_3(Y_i)$. Define $\Psi^0_{dep(X)} = D_{dep(X)} + 2$ and $\Psi^n_{dep(X)} = \Phi_{dep(X)-1} + \Psi^{n-1}_{dep(X)}$, then we have

$$b_3(Y_0) = b_3(Y) \leq \Psi^0_{dep(X)} + b_3(X)$$

and

$$b_3(Y_{i+1}) \leq \Phi_{dep(Y_i)} + b_3(Y_i) \leq \Phi_{dep(X)-1} + b_3(Y_i) = \Psi^{i+1}_{dep(X)} + b_3(X)$$

by induction on i. We conclude that $b_3(Y') = b_3(Y_l) \leq \Psi^{dep(X)}_{dep(X)} + b_3(X)$. Finally

$$b_3(X') \leq \Phi_{dep(Y')} + b_3(Y') \leq \Phi_{dep(X)-1} + \Psi^{dep(X)}_{dep(X)} + b_3(X)$$

since $dep(Y') < dep(X)$. So we finish the case when $X \dashrightarrow X'$ is a flip.

Now assume that $X \to W$ is a divisorial contraction to a curve. One has to estimate $b_3(W)$. In this case $g' : X' \to W$ is a divisorial contraction to a point, hence one may apply Proposition 3.3.3 to get

$$|b_3(X') - b_3(W)| = |\chi_{top}(X') - \chi_{top}(W)| + 2 \le D'_{dep(X')} + 2$$

and then

$$b_3(W) \le D'_{dep(X')} + 2 + b_3(X') \le D'_{dep(X)} + 2 + \Phi_{dep(X)-1} + \Psi^{dep(X)}_{dep(X)} + b_3(X).$$

□

Theorem 3.3.6. *Let X be a smooth threefold and $X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_m$ be the process of minimal model program. Then*

(i) $b_i(X_j) = b_i(X)$ *for* $i = 0, 1, 5, 6$ *and for all* j.

(ii) *If* $j > k$, *then* $b_i(X_j) \le b_i(X_k)$ *for* $i = 2, 4$. *Equality holds if and only if* X_j *and* X_k *are connected by flips.*

(iii) *There exists an integer* $\bar{\Phi}_{\rho(X)}$ *depending only on the Picard number of* X, *such that* $b_3(X_j) \le \bar{\Phi}_{\rho(X)} + b_3(X)$ *for all* j.

Proof. (i) and (ii) are Proposition 3.1.12. Also as in Remark 3.1.5 we have $dep(X_i) \le \rho(X)$ for all i. So Proposition 3.3.5 implies

$$b_3(X_i) \le \Phi_{\rho(X)} + b_3(X_{i-1}) \le i\Phi_{\rho(X)} + b_3(X).$$

Now $i \le 2\rho(X)$ by [CZ14, Lemma 3.1]. One conclude that one can take $\bar{\Phi}_{\rho(X)} = 2\rho(X)\Phi_{\rho(X)}$.

□

3.4 Examples and applications

Let $Y \to X$ be an extremal divisorial contraction between terminal threefolds. By Lemma 3.1.9 we know that $0 \le b_i(Y) - b_i(X) \le 1$ for $i \neq 3$. In the previous section we have shown that

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$|b_3(Y)-b_3(X)|$ can be bounded by some constant depending only on the depth of X or Y. The following examples assert that the dependence is non-trivial. If X or Y has very large depth, then $|b_3(Y)-b_3(X)|$ could be very large.

Example 3.4.1. Assume that X is isomorphic to

$$(x^2+y^2+z^{4k+2}+u^{2k+1}=0)\subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{4}(1,3,1,2).$$

This is an isolated terminal point of type $cAx/4$. Assume that k is even and let Y be the weighted blow up of the weight $\frac{1}{4}(2k+1,2k+3,1,2)$. Then $Y\to X$ is an extremal divisorial contraction with discrepancy $1/4$. Let E be the exceptional divisor. We have

$$b_3(X)-b_3(Y)=\chi_{top}(Y)-\chi_{top}(X)-2=\chi_{top}(E)-3.$$

Hence to compute $b_3(X)-b_3(Y)$ is equivalent to compute $\chi_{top}(E)$.

Now in this case

$$E\cong(x^2+z^{4k+2}+u^{2k+1}=0)\subset\mathbb{P}(2k+1,2k+3,1,2).$$

On $U_z=\{z\neq 0\}$ we have $E|_{U_z}\cong(x^2+u^{2k+1}+1)\subset\mathbb{A}^3_{(x,y,u)}$. This is a line bundle over a smooth curve $C=(x^2+u^{2k+1}+1)\subset\mathbb{A}^2_{(x,u)}$, which is of degree $2k+1$. Hence

$$\chi_{top}(E|_{U_z})=\chi_{top}(C)=-(2k-2)(2k+1)-(2k+1).$$

On the other hand, one can show that $E|_{\{z=0\}}$ is isomorphic to $\mathbb{P}^1$. Hence

$$\chi_{top}(E)=-(2k-2)(2k+1)-(2k+1)+2$$

tends to $-\infty$ when k tends to ∞. This shows that $b_3(X)-b_3(Y)$ can be arbitrary negative.

Example 3.4.2. Assume that X is isomorphic to

$$(xy+z^{rk}+u^k=0)\subset\mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(\alpha,-\alpha,1,r)$$

with $(\alpha,r)=1$. This is an isolated terminal point of type cA/r. Let Y be the weighted blow-up of the weight $\frac{1}{r}(a,b,1,r)$ with $a\equiv\alpha$ mod r and $a+b=rk$. Then $Y\to X$ is an extremal

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divisorial contraction with discrepancy $1/r$. The exceptional divisor E is isomorphic to

$$(xy + z^{rk} + u^k = 0) \subset \mathbb{P}(a, b, 1, r).$$

On the affine open set $U_y = \{y \neq 0\}$ we have $E|_{U_y} \cong (x + z^{rk} + u^k = 0) \subset \mathbb{A}^3/\frac{1}{b}(a, 1, r)$, which is isomorphic to $\mathbb{A}^2/\frac{1}{b}(1, r)$. One can compute that $\chi_{top}(E|_{U_y}) = 1$.

Now let

$$E' = E|_{\{y=0\}} \cong (z^{rk} + u^k = 0) \subset \mathbb{P}(a, 1, r).$$

We have $E'|_{\{z\neq 0\}} \cong (u^k + 1 = 0) \subset \mathbb{A}^2_{(x,u)}$, which are k lines. Also $E'|_{\{z=0\}}$ is a point, hence

$$\chi_{top}(E') = k + 1.$$

A conclusion is that $\chi_{top}(E) = k + 2$ can be arbitrary large when k growth to infinity, hence $b_3(X) - b_3(Y)$ can be arbitrary positive.

As an application of Theorem 3.3.6, we try to bound the intersection Betti numbers. Intersection homology was developed by Goresky and MacPherson in the eighth decade of the twentieth century, which was defined on singular manifolds and satisfied some nice properties as the original singular homology on smooth manifolds. One may expect that the difference of original Betti numbers and intersection Betti numbers can be controlled by singularities. In this paper we prove a weaker statement. We will denote by $IH^i(X, \mathbb{Q})$ the middle-perversity intersection cohomology group and denote $Ib_i(X)$ by the dimension of $IH^i(X, \mathbb{Q})$.

Theorem 3.4.3. *Let X be a projective $\mathbb{Q}$-factorial terminal threefold over $\mathbb{C}$. Then there is an integer Θ_i depending only on singularities of X, such that*

$$Ib_i(X) \leq b_i(X) + \Theta_i.$$

Let X be a projective $\mathbb{Q}$-factorial terminal threefold over $\mathbb{C}$. For any singular point $P \in X$, we say that there exists a *feasible resolution* for P if there is a sequence

$$X_n \to X_{n-1} \to \ldots \to X_0 = X$$

so that X_n is smooth over P and $X_{i+1} \to X_i$ is an extremal divisorial contraction to a point P_i with discrepancy equals to $1/index(P)$.

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Theorem 3.4.4 ([Che11], Theorem 2)**.** *Given a three-dimensional terminal singularity $P \in X$, there exists a feasible resolution for $P \in X$.*

Corollary 3.4.5. *Let X be a projective $\mathbb{Q}$-factorial terminal threefold over $\mathbb{C}$. There is a smooth variety Y such that $Y \to X$ is a composition of steps of K_Y-minimal model program, and the relative Picard number $\rho(Y/X)$ depending only on the singularities (that is, the local equation near singular points) of X.*

Corollary 3.4.6. *Notation as above. We have $b_i(Y) \leq b_i(X) + \Theta_i$, where Θ_i is a constant depending only on the singularities of X.*

Proof. We apply Theorem 3.3.6. When $i = 0, 1, 5, 6$, one take $\Theta_i = 0$. For $i = 2, 4$ we choose Θ_i to be $\rho(Y/X)$. Now assume $i = 3$ and assume that $Y \to X$ factors through

$$Y = X_n \to X_{n-1} \to ... \to X_0 = X,$$

where $X_i \to X_{i+1}$ is an extremal divisorial contraction to a point. By Proposition 3.3.3 and Corollary 3.1.10 we have

$$|b_3(X_{i+1}) - b_3(X_i)| \leq |\chi_{top}(X_i) - \chi_{top}(X_{i+1})| + 2 \leq D'_{dep(X_{i+1})} + 2.$$

Now n is equals to $\rho(Y/X)$ and $dep(X_{i+1})$ is bounded by $\rho(Y/X)$ by Remark 3.1.6. Hence

$$|b_3(Y) - b_3(X)| \leq n(D'_{\rho(Y/X)} + 2),$$

which is a constant depending only on singularities of X. □

Proof of Theorem 3.4.3. Let

$$Y = X_n \to X_{n-1} \to ... \to X_0 = X$$

be a feasible resolution. By [CT17, Lemma 2.16] we have

$$0 \to IH^i(X_j, \mathbb{Q}) \to IH^i(X_{j+1}, \mathbb{Q}) \oplus IH^i(P_j, \mathbb{Q}) \to IH^i(E_j, \mathbb{Q}) \to 0$$

is exact for $i \geq 1$, here $E_j = exc(X_{j+1} \to X_j)$ and P_j is the image of E_j. Hence $Ib_i(X_{j+1}) \geq Ib_i(X_j)$ for all j and for all $i \geq 1$. Thus $Ib_i(X) \leq Ib_i(Y) = b_i(Y) \leq b_i(X) + \Theta_i$ by Corollary

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3.4.6. □

Chapter 4

Threefolds of Kodaira dimension one

By the result of Hacon-M$^{\text{c}}$Kernan [HM06], Takayama [Tak06] and Tsuji [Tsu07], it is known that for any positive integer n there exists an integer r_n such that if X is an n-dimensional smooth complex projective variety of general type, then $|rK_X|$ defines a birational map for all $r \geq r_n$. It is conjectured in [HM06] that a similar phenomenon occurs for any projective variety of non-negative Kodaira dimension. That is, for any positive integer n there exists a constant s_n such that, if X is an n-dimensional smooth projective variety of non-negative Kodaira dimension and $s \geq s_n$ is sufficiently divisible, then the s-th pluricanonical map of X is birational to the Iitaka fibration.

We list some known results related to this problem. In 1986, Kawamata [Kawak05] proved that there is an integer m_0 such that for any terminal threefold X of Kodaira dimension zero, the m_0-th plurigenera of X is non-zero. Later on, Morrison proved that one can take $m_0 = 2^5 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19$. Please see [Morsn86] for details. In 2000, Fujino and Mori [FM00] proved that if X is a smooth projective variety of Kodaira dimension one and F is a general fiber of the Iitaka fibration of X, then there exists a integer M, which depends on the dimension of X, the middle Betti number of some finite covering of F and the smallest integer so that the pluricanonical system of F is non-trivial, such that the M-th pluricanonical map of X is birational to the Iitaka fibration. Viehweg and D.-Q. Zhang [VZh09] proved the analog result for the Kodaira dimension two case. Recently, Birkar and D.-Q. Zhang [BZ16] proved that Fujino-Mori type statement holds for every variety of non-negative Kodaira dimension. Note that if C is a curve of Kodaira dimension zero, then $|K_C|$ is non-trivial and $b_1(C) = 2$. Also if S is a surface of Kodaira dimension zero, then $|12K_S|$ is non-trivial and $b_2(S) \leq 22$. Thus the Hacon-M$^{\text{c}}$Kernan conjecture holds for varieties of dimension less than or equal to three.

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It is also interesting to find an explicit value to bound the Iitaka fibration. In dimension one, it is well-known that the third-pluricanonical map is the Iitaka fibration. For the surfaces case, Iitaka [Iit70] proved that the m-th pluricanonical system is birational to the Iitaka fibration if $m \geq 86$ and is divisible by 12. For threefolds of general type, J. A. Chen and M. Chen [CM14] proved that the m-th pluricanonical map is birational if $m \geq 61$. For threefolds of Kodaira dimension two, Ringler [Rin07] proved that the m-th pluricanonical map is birational to the Iitaka fibration if $m \geq 48$ and is divisible by 12. In this chapter we will prove that 96 is an effective bound for Iitaka fibration for threefolds of Kodaira dimension one, please see Theorem 4.5.1 for details.

We now give a rough idea of the proof of Theorem 4.5.1. If the Iitaka fibration maps to a non-rational curve, then the boundedness of the Iitaka fibration can be easily derived using weak-positivity. Now assume the Iitaka fibration of X maps to a rational curve. We may assume X is minimal and hence the general fiber of the Iitaka fibration is a K3 surface, an Enriques surface, an abelian surface or a bielliptic surface. If the general fiber has non-zero Euler characteristic, i.e., if the Iitaka fibration is a K3 fibration or an Enriques fibration, we observe the following fact. One may write K_X as a pull-back of an ample $\mathbb{Q}$-divisor. The degree of this $\mathbb{Q}$-divisor is determined by the singularities of X. If the degree is large, then a small multiple of K_X defines the Iitaka fibration. Assume the degree is small, then the singularities of X are bounded: the local index of singular points of X can not be too large, and the total number of singular points is bounded. This implies the degree has an lower bound. With the help of a computer, we get a good estimate of this lower bound and hence a good effective bound for the Iitaka fibration.

If the Iitaka fibration is an abelian fibration or a bielliptic fibration, then above techniques do not work. Instead, we use Fujino-Mori's canonical bundle formula. The main difficulty is to control the moduli part of the canonical bundle formula. If the moduli part is zero, then with the help of the the theory of holomorphic two-forms developed by Campana and Peternell, one can prove that the Iitaka fibration is isotrivial and it is not hard to estimate the degree of K_X over C. If the moduli part is non-zero, then one can show that the degree of the moduli part is large, and hence it is easy to find enough section in the pluricanonical system.

One can always replace our smooth threefold by its minimal model, hence throughout this article, we always assume our threefold is minimal and has terminal singularities. Since the abundance conjecture is known to be true in dimension three, the Iitaka fibration is a morphism. We will denote it by $f : X \to C$ and hence K_X is a pull-back of some ample $\mathbb{Q}$-divisor on C. If

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C is not rational, we will prove the desired boundedness in Section 4.1. In the later sections we will always assume C is a rational curve. We discuss K3/Enriques fibrations, abelian fibrations and bielliptic fibrations in Section 4.2, 4.3 and 4.4 respectively. We will prove Theorem 4.5.1 in Section 4.5, which is a collection of the result in previous sections. We also compute several examples, which are threefolds of Kodaira dimension one such that a small pluricanonical system do not define the Iitaka fibration.

4.1 Preliminary

4.1.1 The canonical bundle formula

Let X be a minimal terminal threefold of Kodaira dimension one. Since the abundance conjecture holds for threefolds, K_X is semi-ample. Hence the Iitaka fibration $f : X \to C$ is a morphism and K_X is a pull-back of an ample divisor on C. We denote a general fiber of $X \to C$ by F. By [FM00] we have the following canonical bundle formula

$$bK_X = f^*(b(K_C + M + B)),$$

here b is the smallest integer such that $|bK_F|$ is non-empty, M is a nef $\mathbb{Q}$-divisor such that bNM is integral for some N which depends on the middle Betti number of the finite covering of F defined by $|bK_F|$, $B = \sum_{i \in I} s_i P_i$ where s_i is of the form $(1 - \frac{v_i}{bNu_i})$ for some integers u_i and v_i, such that $v_i \leq bN$. We will write $A = K_C + M + B$ for convenience.

Lemma 4.1.1. *We have $f_*\mathcal{O}_X(rbK_X) = \mathcal{O}_C(\lfloor rbA \rfloor)$ for all integers $r \geq 0$. In particular, if $C \cong \mathbb{P}^1$ and $h^0(X, rbK_X) \geq 2$, then $|rbK_X|$ defines the Iitaka fibration.*

Proof. By [FM00, Proposition 2.2] and the projection formula we have $f_*\mathcal{O}_X(rbK_X)^{**} = \mathcal{O}_C(\lfloor rbA \rfloor)$. Since $f_*\mathcal{O}_X(rbK_X)$ is torsion free, it is locally free, hence $f_*\mathcal{O}_X(rbK_X)^{**} = f_*\mathcal{O}_X(rbK_X)$ and we have

$$f_*\mathcal{O}_X(rbK_X) = \mathcal{O}_C(\lfloor rbA \rfloor).$$

Now if $C \cong \mathbb{P}^1$ and $H^0(X, rbK_X) \geq 2$, then $\lfloor rbA \rfloor$ has positive degree, hence very ample. Thus

$$H^0(X, \mathcal{O}_X(rbK_X)) = H^0(X, f_*\mathcal{O}_X(rbK_X)) = H^0(C, \mathcal{O}_C(\lfloor rbA \rfloor)$$

defines the morphsim $X \to C$. □

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Lemma 4.1.2. *Assume that $C \cong \mathbb{P}^1$. Then either* $\deg M = 0$*, or* $\lfloor 2B \rfloor \geq 3$*.*

Proof. We first prove that if $\deg M \neq 0$, then there are at least three multiple fibers.

To see it, assume that the number of multiple fibers is less than three, then there exists a finite morphism $\phi : C' \to C$ which is étale over $\mathbb{C}^*$, such that the base-change $X' = X \times_C C'$ has a semistable model $f' : Z' \to X' \to C'$. Now we have

$$\deg \phi^* \mathcal{O}_C(\lfloor rbM \rfloor) = \deg f'_* \mathcal{O}_{Z'}(rbK_{Z'/C'}) = \deg \det(f'_* \mathcal{O}_{Z'}(rbK_{Z'/C'})) = 0$$

for all $r \in \mathbb{N}$, where the first equality follows from [FM00, Corollary 2.5], the second equality follows form the fact that $f'_* \mathcal{O}_{Z'}(rbK_{Z'/C'})$ is a line bundle (cf. Remark 4.1.4 below) and the last equality follows from [VZ01, Proposition 4.2]. This implies $\deg M = 0$.

Now assume that there are at least three multiple fibers. Let P_1, P_2 and P_3 be three points on C such that $f^{-1}P_i$ is not reduced for $i = 1, ..., 3$. Note that for all $P \in C$ we have $\mathrm{coeff}_P B = 1 - lct(X, f^*P)$ (c.f. [FM00, proof of Proposition 4.7] or [Fuj03, Definition 3.4]). Since $f^{-1}P_i$ is not reduced, we have $lct(X, f^*P_i) \leq \frac{1}{2}$ for $i = 1, ..., 3$. Thus the coefficient of B over P_i is greater than or equal to $\frac{1}{2}$ and so $\lfloor 2B \rfloor \geq 3$. ☐

4.1.2 Kollár vanishing theorem

Theorem 4.1.3 ([Kol95], Theorem 10.19)**.** *Let $f : X \to Y$ be a surjective morphism between normal and proper varieties. Let N, N' be rank 1, reflexive, torsion-free sheaves on X. Assume that $N \equiv K_X + \Delta + f^*M$, where M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y and (X, Δ) is klt. Then*

1. *$R^j f_* N$ is torsion free for $j \geq 0$.*
2. *Assume in addition that M is nef and big. Then*

$$H^i(Y, R^j f_* N) = 0 \quad \textit{for } i > 0, j \geq 0.$$

3. *Assume that M is nef and big and let D be any effective Weil divisor on X such that $f(D) \neq Y$. Then*

$$H^j(X, N) \to H^j(X, N(D)) \quad \textit{is injective for } j \geq 0.$$

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4. *If f is generically finite and $N' \equiv K_X + \Delta + F$ where F is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X which is f-nef, then $R^j f_* N' = 0$ for all $j > 0$.*

Remark 4.1.4. Under our assumption that X is a minimal terminal threefold of Kodaira dimension one and $f : X \to C$ is the Iitaka fibration, Theorem 4.1.3 (1) implies that $R^j f_*(mK_X)$ is locally free because any torsion-free sheaf on a curve is locally free.

Proposition 4.1.5 ([Kol86],Proposition 7.6)**.** *Let X, Y be smooth projective varieties,* $\dim X = n$, $\dim Y = k$, *and let $\pi : X \to Y$ be a surjective map with connected fibers. Then $R^{n-k}\pi_*\omega_X = \omega_Y$.*

Corollary 4.1.6. *The same conclusion of above proposition holds if X has canonical singularities.*

Proof. Let $\phi : \tilde{X} \to X$ be a resolution of X and $\tilde{\pi} : \tilde{X} \to Y$ be the composition of π and ϕ. By the Grothendieck spectral sequence, we have

$$E_2^{p,q} = R^p\pi_*(R^q\phi_*\omega_{\tilde{X}}) \Rightarrow R^{p+q}\tilde{\pi}_*\omega_{\tilde{X}}.$$

By Grauert-Riemenschneider vanishing (or Theorem 4.1.3 (4) above) we have $R^q\phi_*\omega_{\tilde{X}} = 0$ for all $q > 0$, hence

$$\omega_Y = R^{n-k}\tilde{\pi}_*\omega_{\tilde{X}} = R^{n-k}\pi_*(\phi_*\omega_{\tilde{X}}) = R^{n-k}\pi_*\omega_X.$$

□

4.1.3 Weak positivity

Theorem 4.1.7 ([Vie83], Theorem III)**.** *Let $g : T \to W$ be any surjective morphism between non-singular projective varieties. Then $g_*\omega^k_{T/W}$ is weakly positive for any $k > 0$.*

Remark 4.1.8. When $T \to W$ is the Iitaka fibration of T and W is a curve, we have $g_*\omega^k_{T/W}$ is a line bundle by Remark 4.1.4. By [Kawak12, Section 5], $g_*\omega^k_{T/W}$ is pseudo-effective, which is equivalent to say that $\deg g_*\omega^k_{T/W} \geq 0$.

Furthermore, the same conclusion holds if T has canonical singularities because if $\phi : \tilde{T} \to T$ is a resolution of singularity, then $\phi_*\omega^k_{\tilde{T}} = \omega^k_T$.

Proposition 4.1.9. *Let X be a minimal terminal threefold with Kodaira dimension one and let $f : X \to C$ be the Iitaka fibration of X. Assume that $g(C) \geq 1$. Let F be a general fiber and let*

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b be an integer such that $|bK_F|$ is non-empty and $b \geq 2$. Then $|bK_X|$ is non-empty and $|3bK_X|$ defines the Iitaka fibration.

In particular, $|mK_X|$ defines the Iitaka fibration for all $m \geq 24$ and divisible by 12.

Proof. First assume $g(C) \geq 2$. By weak-positivity,

$$\deg f_*\mathcal{O}_X(bK_X) \geq \deg \mathcal{O}_C(bK_C) = b(2g(C)-2).$$

We have

$$\begin{aligned} H^0(X, \mathcal{O}_X(bK_X)) = H^0(C, f_*\mathcal{O}_X(bK_X)) &\geq \chi(f_*\mathcal{O}_X(bK_X)) \\ &= \deg f_*\mathcal{O}_X(bK_X) + 1 - g(C) \geq (2b-1)(g(C)-1) > 0. \end{aligned}$$

Moreover, $\deg f_*\mathcal{O}_X(2bK_X) \geq 8g(C) - 8 > 2g(C) + 1$ under our assumption that $b \geq 2$, hence $f_*\mathcal{O}_X(2bK_X)$ is very-ample, which implies $|2bK_X|$ defines the Iitaka fibration. Since $H^0(X, \mathcal{O}_X(bK_X)) \neq 0$, $|3bK_X|$ also defines the Iitaka fibration.

In the case that $g(C) = 1$, one has $\deg f_*\mathcal{O}_X(bK_X) \geq 0$. Moreover

$$H^1(C, f_*\mathcal{O}_X(bK_X) \otimes P) = 0$$

for all $P \in Pic^0(C)$ by Theorem 4.1.3 (2). Hence $\deg f_*\mathcal{O}_X(bK_X)$ should be positive since otherwise after taking $P = (f_*\mathcal{O}_X(bK_X))^*$ we get $H^1(C, \mathcal{O}_C) = 0$, which is a contradiction. Thus $h^0(bK_X) \neq 0$. Now we have $\deg f_*\mathcal{O}_X(3bK_X) \geq 3$ since one has the natural inclusion $f_*\mathcal{O}_X(bK_X)^{\otimes 3} \to f_*\mathcal{O}_X(3bK_X)$. The conclusion is that $f_*\mathcal{O}_X(3bK_X)$ is very-ample, so $|3bK_X|$ defines the Iitaka fibration.

Note that $|mK_X|$ defines the Iitaka fibration if $m \geq 3b$ and m is divisible by b. We know that $b \in \{2, 3, 4, 6\}$. It is easy to see that $|mK_X|$ defines the Iitaka fibration for all $m \geq 24$ and divisible by 12. ☐

4.2 K3 or Enriques fibrations

Let X be a minimal terminal threefold of Kodaira dimension one and

$$f : X \to C \cong \mathbb{P}^1$$

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be the Iitaka fibration. Let F be a general fiber of $X \to C$ and assume that F is a K3 surface or an Enriques surface. We have $H^1(F, K_F) = 0$. Thus $R^1 f_* \mathcal{O}_X(K_X) = 0$ because it is locally free by Remark 4.1.4. We have

$$\begin{aligned} h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X(K_X)) &= h^0(C, R^2 f_* \mathcal{O}_X(K_X)) + h^1(C, R^1 f_* \mathcal{O}_X(K_X)) \\ &= h^0(C, \mathcal{O}_C(K_C)) = 0 \end{aligned}$$

by Corollary 4.1.6 and

$$h^2(X, \mathcal{O}_X) = h^1(X, \mathcal{O}_X(K_X)) = h^0(C, R^1 f_* \mathcal{O}_X(K_X)) + h^1(C, f_* \mathcal{O}_X(K_X)) = h^1(C, f_* \mathcal{O}_X(K_X)).$$

If F is an Enriques surface, then $H^0(F, K_F) = 0$, hence $f_* \mathcal{O}_X(K_X) = 0$ since it is a line bundle. We have $h^2(X, \mathcal{O}_X) = 0$ and $h^3(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X)) = h^0(C, f_* \mathcal{O}_X(K_X)) = 0$. Thus $\chi(\mathcal{O}_X) = 1$. When F is a K3 surface by the weak positivity we have deg $f_* \mathcal{O}_X(K_X) \geq \mathcal{O}_C(K_C)$, hence $f_* \mathcal{O}_X(K_X)$ is of degree greater than or equal to -2 and so $h^1(C, f_* \mathcal{O}_X(K_X)) \leq 1$. Thus $h^2(\mathcal{O}_X) \leq 1$ which implies $\chi(\mathcal{O}_X) \leq 2$. If $\chi(\mathcal{O}_X) < 0$, then $h^0(X, K_X) \geq 2$ and $|K_X|$ defines the Iitaka fibration by Lemma 4.1.1. From now on we assume $0 \leq \chi(\mathcal{O}_X) \leq 2$.

There exists integers m and d such that $\mathcal{O}_X(mK_X) = f^* \mathcal{O}_C(d)$. We write $\lambda = \frac{m}{d}$, so that $F \equiv \lambda K_X$.

Lemma 4.2.1. *$h^0(X, mK_X) \geq r$ if $m > \lambda r + 1$ and $|mK_F|$ is non-empty.*

Proof. Choose r general fibers F_1, ..., F_r. Consider the exact sequence

$$0 \to \mathcal{O}_X(mK_X - F_1 - ... - F_r) \to \mathcal{O}_X(mK_X) \to \bigoplus_{i=1}^{r} \mathcal{O}_{F_i}(mK_X) = \bigoplus_{i=1}^{r} \mathcal{O}_{F_i} \to 0.$$

Note that $mK_X - F_1 - ... - F_r \equiv K_X + (m - 1 - \lambda r) K_X$. By our assumption $m - 1 - \lambda r > 0$, hence

$$H^1(X, \mathcal{O}_X(mK_X - F_1 - ... - F_r)) \to H^1(X, \mathcal{O}_X(mK_X))$$

is injective by Theorem 4.1.3 (3) with $M = \frac{m-1-\lambda r}{\lambda} P$, $D = F_1 + ... + F_r$ and $\Delta = 0$, here P is a general point on C. Hence

$$H^0(X, \mathcal{O}_X(mK_X)) \to \bigoplus_{i=1}^{r} H^0(F_i, \mathcal{O}_{F_i})$$

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is surjective and so $h^0(X, \mathcal{O}_X(mK_X)) \geq r$. $\square$

Proposition 4.2.2. *If $X \to C$ is a K3 fibration, then $h^0(X, mK_X) \geq 2$ for $m \geq 86$. If $X \to C$ is an Enriques fibration, we have $h^0(X, mK_X) \geq 2$ if m is even and greater than or equal to 42. In particular, $|mK_X|$ defines the Iitaka fibration if $m \geq 86$ (resp. m is even and $m \geq 42$) if $X \to C$ is a K3 (resp. an Enriques) fibration.*

Proof. We use the notation as in Lemma 4.2.1, so λ is the rational number such that $F \equiv \lambda K_X$. We only need to say that $\lambda \leq 42$ if F is a K3 surface and $\lambda \leq 20$ if F is an Enriques surface. Then Lemma 4.2.1 implies the statement.

We have

$$\chi(\mathcal{O}_F) = \frac{1}{12}c_2(F) = \frac{1}{12}F.c_2(X) = \frac{\lambda}{12}K_X.c_2(X)$$

by the Noether's equality. Hence

$$\frac{12}{\lambda}\chi(\mathcal{O}_F) = K_X.c_2(X) = -24\chi(\mathcal{O}_X) + \sum_{P \in \mathcal{B}(X)} \left(r_P - \frac{1}{r_P}\right). \tag{4.1}$$

Assume $\lambda > N$ for some integer N, then

$$24\chi(\mathcal{O}_X) < \sum_{P \in \mathcal{B}(X)} \left(r_P - \frac{1}{r_P}\right) < 24\chi(\mathcal{O}_X) + \frac{12}{N}\chi(\mathcal{O}_F).$$

This tells us that $r_P \leq 24\chi(\mathcal{O}_X) + \frac{12}{N}\chi(\mathcal{O}_F)$ for all P and there is at most

$$\frac{2}{3}\left(24\chi(\mathcal{O}_X) + \frac{12}{N}\chi(\mathcal{O}_F)\right)$$

non-Gorenstein points on X since $r - \frac{1}{r} \geq \frac{3}{2}$ for all integer $r > 1$. Note that we assume $0 \leq \chi(\mathcal{O}_X) \leq 2$, hence there are only finitely many possible basket data. By Equation (4.1) a basket data corresponds to a unique λ once $\chi(\mathcal{O}_X)$ is fixed. This tells us that λ has an upper bound.

Note that the basket data should satisfy more conditions. We have $h^1(X, mK_X) = 0$ since $h^1(C, f_*(mK_X)) = 0$ by Theorem 4.1.3 (2) and $R^1f_*(mK_X)$ is a zero sheaf because $h^1(F, mK_F) = 0$ and $R^1f_*(mK_X)$ is locally free by Remark 4.1.4. Also $h^3(X, mK_X) = h^0(X, (1-m)K_X) = 0$ for all $m > 1$ since K_X is psudo-effective and not numerically trivial.

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Hence the plurigenus formula yields

$$0 \leq \chi(mK_X) = (1-2m)\chi(\mathcal{O}_X) + l(m), \tag{4.2}$$

where

$$l(m) = \sum_{P \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{jb_P}(r_P - \overline{jb_P})}{2r_P},$$

for all $m > 1$. As we have seen before, for a fixed integer N and assuming $\lambda > N$, there are only finitely many possible basket data of X. Using a computer, one can write down all possible basket data and check whether such basket satisfies (4.2) or not. In the K3 fibration case if one take $N = 42$, then there is no basket data satisfying (4.2) for all $m > 1$, hence $\lambda \leq 42$. In the Enriques fibration case using the same technique one can prove that $\lambda \leq 20$.

Finally note that if $h^0(X, mK_X) \geq 2$ then $|mK_X|$ defines the Iitaka fibration by Lemma 4.1.1. □

Remark 4.2.3. We remark that the worst possible basket data occurs when $X \to C$ is a K3 fibration, $\chi(\mathcal{O}_X) = 2$ and the basket data is

$$\{(2,1) \times 8, (3,1) \times 6, (7,1), (7,2), (7,3)\}$$

with $\lambda = 42$. This kind of fibration exists. Please see Examle 4.5.6.

4.3 Abelian fibrations

Let X be a terminal minimal threefold of Kodaira dimension one and assume $X \to C \cong \mathbb{P}^1$ is the Iitaka fibration. Let F be the general fiber of $X \to C$ and assume that F is an abelian surface. Let $W \to X$ be a resolution of singularities of X. We will denote by

$$g : W \to X \to C$$

the composition of the two morphisms.

Definition. Let W be a smooth threefold of Kodaira dimension one and let ω be a two-form on W. Let $g : W \to C$ be the Iitaka fibration. We say ω is a vertical two-form (with respect to the

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Iitaka fibration) if ω corresponds to an element s such that

$$s \in H^0(W, T_{W/C} \otimes K_W) \subset H^0(W, T_W \otimes K_W) \cong H^0(W, \Omega_W^2).$$

Here T_W denotes the tangent bundle of W and $T_{W/C}$ is the kernel of $T_W \to g^*T_C$.

Theorem 4.3.1 ([CT00], Theorem 4.2)**.** *Let W be a smooth projective threefold of Kodaira dimension one and let ω be a two-form on W. Assume that ω is not vertical with respect to the Iitaka fibration. Let X be a minimal model with Iitaka fibration $f : X \to C$. Then there is a finite base change $\tilde{C} \to C$ with induced fiber space $\tilde{f} : \tilde{X} \to \tilde{C}$ such that $\tilde{X} \cong F \times \tilde{C}$, where F is abelian or K3.*

Assume that there is a non-vertical two-form on W. We may assume that $\tilde{C} \to C$ is Galois. Let $G = Gal(\tilde{C}/C)$. Note that there is a natural G-action on $\tilde{X}$ such that $X \cong \tilde{X}/G$. We have the induced action on F. Since finite automorphism groups of an abelian surface are discrete, G acts on $\tilde{X} \cong F \times \tilde{C}$ diagonally. We may assume further that G acts on F faithfully since the kernel of G acting on F is a normal subgroup of G and one can replace $\tilde{C}$ by $\tilde{C}$ module the kernel. In particular $\tilde{X} \to X$ is étale in codimension one.

Proposition 4.3.2. *If the smooth model of X admits a non-vertical two-form, then $|mK_X|$ defines the Iitaka fibration if $m \geq 86$.*

Proof. We have the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\
\tilde{C} \arrow[r, "\phi"] & C.
\end{tikzcd}$$

One may write $K_X = f^*A$ for some $\mathbb{Q}$-divisor A on C. We also have $K_{\tilde{C}} = \phi^*(K_C + B)$, here B is a $\mathbb{Q}$-divisor with $\mathrm{coeff}_P(B) = 1 - \frac{1}{m}$ if the stabilizer of pre-image of P is of order m. We have

$$\tilde{\phi}^* f^* A = \tilde{\phi}^* K_X = K_{\tilde{X}} = K_F \boxtimes K_{\tilde{C}} = \tilde{f}^* K_{\tilde{C}} = \tilde{f}^* \phi^* (K_C + B)$$

since $\tilde{\phi}$ is étale in codimension one. Thus $A \sim_{\mathbb{Q}} K_C + B$.

Take an integer m such that both mA and $m(K_C+B)$ are integral and $\mathcal{O}_C(mA) = \mathcal{O}_C(m(K_C+B)) = \mathcal{O}(d)$. Let $\lambda = \frac{m}{d}$ and $\delta = \frac{1}{\lambda} = \frac{d}{m}$. Assume $B = \sum_{i \in I}(1 - \frac{1}{m_i})P_i$, then $\delta = -2 + \sum_{i \in I}(1 - \frac{1}{m_i})$. Note that $\kappa(\tilde{X}) \geq \kappa(X) = 1$, hence $\kappa(\tilde{C}) = 1$ and $\tilde{C}$ is of general type. It is well-known that $\delta \geq \frac{1}{42}$ (by a simple calculation or using the fact that $|Aut(\tilde{C})| \leq 84(g(\tilde{C}) - 1)$),

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hence $\lambda \leq 42$. By Lemma 4.2.1, for all $r \leq 86$, we have $h^0(X, rK_X) \geq 2$ and $|rK_X|$ defines the Iitaka fibration.

□

From now on we assume that the smooth model W of X admits no non-vertical two-form.

Lemma 4.3.3. $h^1(X, \mathcal{O}_X) \leq 1$.

To prove the lemma, we need the following estimate on the irregularity.

Theorem 4.3.4 ([Fuj05], Theorem 1.6)**.** *Let $f : X \to Y$ be surjective morphism between non-singular projective varieties with connected fibers. Then*

$$q(Y) \leq q(X) \leq q(Y) + q(F),$$

where F is a general fiber of f.

Proof of Lemma 4.3.3. Assume that $h^1(X, \mathcal{O}_X) > 1$, then $h^1(X, \mathcal{O}_X) = 2$ by Theorem 4.3.4. Let $a : X \to A = Alb(X)$ be the Albanese map of X and let F be a general fiber of $X \to C$. Note that A is an abelian surface. Assume that $a(F)$ is two-dimensional, then there is a surjective map $F \to A$. The pull-back of the global two-form on A is a non-vertical two-form on X, contradicting our assumption that there is no non-vertical two-form.

Assume that $a(F)$ is one-dimensional. Note that $a(F)$ should be an elliptic curve since otherwise there is a holomorphic map

$$F \to a(F) \to Jac(a(F))$$

and the image should be a (translation of) non-trivial sub-complex torus, which is impossible. Since there are only countably many elliptic curves up to translation contained in a fixed abelian variety, we have $a(F) \cong E$ for some one-dimensional abelian subvariety E of A, for general F. By [BL04], Theorem 5.3.5, There is an isogeny $A \to E \times T$ for some elliptic curve T. Consider the morphism

$$X \to A \to E \times T \to T$$

which contracts the general fiber of $X \to C$. Thus there is an induced morphism $C \to T$. But $C \cong \mathbb{P}^1$ and T is an elliptic curve, which is impossible.

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Finally assume $a(F)$ is a point, then $a(X)$ is a curve of genus ≥ 2. However this induces a morphism $C \to a(X)$, which is again impossible. $\square$

Consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & T_{W/C} \otimes K_W \arrow[r] \arrow[d] & T_W \otimes K_W \arrow[d, "\sim"'] \\
0 \arrow[r] & \Omega_{W/C} \otimes g^*\Omega_C \arrow[r] & \Omega_W^2 \\
 & \pi(\alpha) \otimes \beta \arrow[r, mapsto] & \alpha \wedge \eta(\beta)
\end{tikzcd}$$

where $\eta : g^*\Omega_C \to \Omega_W$ and $\pi : \Omega_W \to \Omega_{W/C}$ denote the natural maps. Note that the bottom map is well-defined and injective since $g^*\Omega_C$ is locally free of rank one.

Under the condition that there is no non-vertical two-form on W, one has

$$\begin{tikzcd}
H^0(W, T_{W/C} \otimes K_W) \arrow[r, "\sim"] \arrow[d] & H^0(W, T_W \otimes K_W) \arrow[d, "\sim"'] \\
H^0(W, \Omega_{W/C} \otimes g^*\Omega_C) \arrow[r, hook] & H^0(W, \Omega_W^2).
\end{tikzcd}$$

Hence $H^0(W, \Omega_{W/C} \otimes g^*\Omega_C) \cong H^0(W, \Omega_W^2)$.

Lemma 4.3.5. *If there is no non-vertical two-form on W, then $h^2(W, \mathcal{O}_W) = 0$. Hence $h^2(X, \mathcal{O}_X) = h^1(X, K_X) = 0$.*

Proof. We compute $h^0(W, \Omega_{W/C} \otimes g^*\Omega_C) = h^0(W, \Omega_W^2) = h^2(W, \mathcal{O}_W)$. Consider the exact sequence

$$0 \to g^*\Omega_C \otimes g^*\Omega_C \to \Omega_W \otimes g^*\Omega_C \to \Omega_{W/C} \otimes g^*\Omega_C \to 0.$$

Applying the push-forward functor we get

$$0 \to g_*(g^*\Omega_C \otimes g^*\Omega_C) \to g_*(\Omega_W \otimes g^*\Omega_C) \to g_*(\Omega_{W/C} \otimes g^*\Omega_C) \to R^1 g_*(g^*\Omega_C \otimes g^*\Omega_C) \to \dots$$

which induces the following exact sequences

$$0 \to g_*(g^*\Omega_C \otimes g^*\Omega_C) \to g_*(\Omega_W \otimes g^*\Omega_C) \to \mathcal{F} \to 0,$$

$$0 \to \mathcal{F} \to g_*(\Omega_{W/C} \otimes g^*\Omega_C) \to \mathcal{G} \to 0,$$

and

$$0 \to \mathcal{G} \to R^1 g_*(g^*\Omega_C \otimes g^*\Omega_C).$$

Note that

$$H^0(C, R^1 g_*(g^*\Omega_C \otimes g^*\Omega_C)) = H^0(C, R^1 g_*(\mathcal{O}_W \otimes g^*\mathcal{O}_C(-4))) = H^0(C, R^1 g_*\mathcal{O}_W \otimes \mathcal{O}_C(-4)).$$

Since $h^0(C, R^1 g_*\mathcal{O}_W) \leq 1$ (otherwise $h^1(W, \mathcal{O}_W) = h^1(X, \mathcal{O}_X) > 1$, contradicting Lemma 4.3.3) and $R^1 g_*\mathcal{O}_W$ is torsion free (cf. [Kol86, Step 6 in the proof of Theorem 2.2]), we have $R^1 g_*\mathcal{O}_W \cong \mathcal{O}_C(\alpha) \oplus \mathcal{O}_C(\beta)$ with both α and $\beta \leq 0$. Hence

$$H^0(C, R^1 g_*\mathcal{O}_W \otimes \mathcal{O}_C(-4)) = 0$$

and so $H^0(C, \mathcal{G}) = 0$. Thus

$$H^0(C, g_*(\Omega_{W/C} \otimes g^*\Omega_C)) \cong H^0(C, \mathcal{F}).$$

Now $g_*\Omega_W$ is locally free of rank two. One may write $g_*\Omega_W \cong \mathcal{O}_C(d_1) \oplus \mathcal{O}_C(d_2)$. Note that either $d_1 = 0$ and $d_2 < 0$, or both d_1 and $d_2 < 0$ since otherwise $h^1(X, \mathcal{O}_X) = h^1(W, \mathcal{O}_W) = h^0(W, \Omega_W) > 1$. In particular $d_1 + d_2 < 0$. Now the third exact sequence becomes

$$0 \to \mathcal{O}_C(-4) \to \mathcal{O}_C(d_1 - 2) \oplus \mathcal{O}_C(d_2 - 2) \to \mathcal{F} \to 0.$$

Since $\mathcal{F}$ is locally free of rank one (use the fact that $g_*\Omega_{W/C}$ is torsion free), $\mathcal{F} \cong \mathcal{O}_C(d_1 + d_2)$. We know that $d_1 + d_2 < 0$, so $h^0(C, \mathcal{F}) = 0$, hence

$$h^2(W, \mathcal{O}_W) = h^0(W, \Omega_{W/C} \otimes g^*\Omega_C) = h^0(C, g_*(\Omega_{W/C} \otimes g^*\Omega_C)) = 0.$$

□

Lemma 4.3.6. *Using the notation as in Section 4.1.1. Assume* $h^1(X, \mathcal{O}_X(K_X)) = 0$, *then we have* $\deg\lfloor M \rfloor \geq 1$.

Proof. We have $h^1(C, f_*\mathcal{O}_X(K_X)) = 0$ since

$$0 = h^1(X, \mathcal{O}_X(K_X)) = h^1(C, f_*\mathcal{O}_X(K_X)) + h^0(C, R^1 f_*\mathcal{O}_X(K_X)).$$

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By Lemma 4.1.1, we have $h^1(C, \mathcal{O}_C(\lfloor A \rfloor)) = 0$. Hence $\deg\lfloor A \rfloor \geq -1$, so $\deg\lfloor M \rfloor \geq 1$. ☐

Proposition 4.3.7. *Assume the smooth model of X admits no non-vertical two-form. Then $|mK_X|$ defines the Iitaka fibration if $m = 2$ or $m \geq 4$.*

Proof. We has to prove that $\deg\lfloor mA \rfloor = \deg\lfloor m(K_C + M + B)\rfloor \geq 1$. By Lemma 4.3.5 and Lemma 4.3.6, we have $\deg\lfloor M \rfloor \geq 1$. In particular $\deg M \neq 0$. Lemma 4.1.2 implies that $\deg\lfloor 2B \rfloor \geq 3$. Now we have

$$\deg\lfloor mA \rfloor = -2m + \deg\lfloor mM \rfloor + \deg\lfloor mB \rfloor \geq -m + 3\lfloor \frac{m}{2} \rfloor \geq 1$$

if $m = 2$ or $m \geq 4$. ☐

4.4 Bielliptic fibrations

Assume X is a minimal terminal threefold of Kodaira dimension one and $X \to C \cong \mathbb{P}^1$ is a bielliptic fibration. We apply the same construction as [FM00, Remark 2.6] to get $g : Z \to X$ such that Z is smooth, the general fiber $\bar{F}$ of $Z \to C$ is an étale covering of F and $|K_{\bar{F}}|$ in non-empty.

Note that we have a natural inclusion $g^*\mathcal{O}_X(K_X) \hookrightarrow \mathcal{O}_Z(K_Z)$ since the pull-back of a top-form on X is a top-form on Z, hence $K_Z = g^*K_X + R$ for some effective divisor R. Since $Z \to X$ is étale over general points on C, R is supported over singular fibers of $Z \to C$. Thus Z is of Kodaira dimension one. Let $h : Z_0 \to C$ be the relative minimal model of Z over C. Then K_Z as well as K_{Z_0} is $\mathbb{Q}$-linearly equivalent to a sum of effective vertical divisors. Thus K_{Z_0} intersects horizontal curves positively. This implies Z_0 is in fact minimal and $Z_0 \to C$ is the Iitaka fibration of Z_0, which is an abelian fibration.

Write

$$K_Z = (g \circ f)^*(K_C + M_Z + B_Z) + E_Z$$

and

$$K_{Z_0} = h^*(K_C + M_0 + B_0)$$

as the canonical bundle formula over C. Recall that in Section 4.1.1 we denote

$$bK_X = f^*(b(K_C + M + \sum_{i \in I} s_i P_i)),$$

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Lemma 4.4.1. *Either* $M = 0$ *or* $\deg\lfloor M \rfloor \geq 1$.

Proof. By [FM00, Lemma 3.4] we have $M_Z = M$. Since the the moduli part is a birational invariant, we have $M_Z = M_0$. Thus $M_0 = M$. Hence we only need to show that either $M_0 = 0$ or $\deg\lfloor M_0 \rfloor \geq 1$. If $Z_0 \to C$ is isotrivial, then $M_0 = 0$. Assume $Z_0 \to C$ is not isotrivial, then there is no non-vertical two-form on Z by Theorem 4.3.1. Lemma 4.3.5 and Lemma 4.3.6 imply our statement.

□

Proposition 4.4.2. $|mK_X|$ *defines the Iitaka fibration if* $m \geq 96$ *and divisible by* 12.

Proof. If $\deg\lfloor M \rfloor \geq 1$, then using the same argument as in the proof of Proposition 4.3.7, one can show that $|rbK_X|$ defines the Iitaka fibration if rb is even or $rb \geq 4$. In particular, $|mK_X|$ defines the Iitaka fibration if m is divisible by 12.

Now assume $M = 0$. In this case Z_0 is isotrivial and we have the following diagram.

$$\begin{tikzcd}
\bar{F} \times C' \arrow[r] \arrow[dd] & Z_0 \arrow[dd] & Z \arrow[l, dashed, "\text{bir.}"'] \arrow[d] & Z' \arrow[l] \arrow[d] \arrow[lll, dashed, bend right, "\text{bir.}"'] \\
 & & X \arrow[d] & X' \arrow[l] \arrow[d] \\
C' \arrow[r] & C \arrow[r, "="] & C & C' \arrow[l]
\end{tikzcd}$$

where $C' \to C$ is a finite Galois covering, the left square is a fiber product, Z' and X' are the normalizations of the base-change of Z and X respectively. Let $U \subset C$ be an open set such that $Z \to X$ is étale over U. Let U' be the pre-image of U on C'. One may assume $U' \to U$ is étale. Let X_U, Z_U and $(Z_0)_U$ be the pre-images of X, Z and Z_0 over U respectively and let X'_U and Z'_U be the pre-images of X' and Z' over U'. We may assume $Z_U \cong (Z_0)_U$ and $Z'_U \cong \bar{F} \times U'$.

Note that $Z_U \to X_U$ is a cyclic cover defined by $|bK_{X_U}|$, hence there exists a cyclic group H such that $X_U = Z_U/H$ and the restriction of the action on fibers over U is the natural action on the abelian surface $\bar{F}$ such that $\bar{F}/H = F$. Hence there exists an H-action on $Z'_U \cong \bar{F} \times U'$ by acting on $\bar{F}$ and fixing U'. Note that for any point $P \in Z'_U$, the H-orbit of P maps to a point through the morphism $Z'_U \to X'_U$. Since the diagram

$$\begin{tikzcd}
Z_U \arrow[d] & Z'_U \arrow[l] \arrow[d] \\
X_U & X'_U \arrow[l]
\end{tikzcd}$$

commutes, the H-orbit on Z'_U maps to a H-orbit on Z_U. Thus we have a natural map

$$F \times U' \cong Z'_U/H \to Z_U/H = X_U.$$

The morphism $F \times U' \to X_U$ factors through the fiber product $X'_U \to X_U$. Since both morphisms are finite with the same degree, we have $X'_U \cong F \times U'$. Let $G = Gal(C'/C) = Gal(U'/U)$, then there exists a G-action on X'_U such that $X'_U/G \cong X_U$. Thus X is birational to $(F \times C')/G$.

Now we have the étale covering $Z_0 \cong (\bar{F} \times C')/G \to (F \times C')/G$. Since Z_0 is terminal, we have $(F \times C')/G$ is terminal. Replacing X by $(F \times C')/G$ one may assume that our given bielliptic fibration is isotrivial. In this case using the same argument as in the proof of Proposition 4.3.2 one can show that $|mK_X|$ is birational to the Iitaka fibration if $m \geq 86$ and is divisible by 12, or equivalently, $m \geq 96$ and divisible by 12. □

4.5 Boundedness of Iitaka fibration for Kodaira dimension one

Theorem 4.5.1. *Let X be a smooth complex projective threefold of Kodaira dimension one. Then $|mK_X|$ defines the Iitaka fibration if $m \geq 96$ and is divisible by 12. More precisely, let F be a general fiber of the Iitaka fibration of X, we have*

1. *If F is birational to a K3 surface, then $|mK_X|$ defines the Iitaka fibration if $m \geq 86$.*
2. *If F is birational to an Enriques surface, then $|mK_X|$ defines the Iitaka fibration if $m \geq 42$ and is even.*
3. *If F is birational to an abelian surface, then $|mK_X|$ defines the Iitaka fibration if $m \geq 86$. Moreover, assume the Iitaka fibration is not isotrivial, then $|mK_X|$ defines the Iitaka fibration if $m = 2$ or $m \geq 4$.*

4. *If F is birational to a bielliptic surface, then $|mK_X|$ defines the Iitaka fibration if $m \geq 96$ and is divisible by* 12.

Proof. If C is not rational, this follows from Proposition 4.1.9. The K3 or Enriques cases follow by Proposition 4.2.2. The isotrivial abelian fibration case follows from Proposition 4.3.2 and the non-isotrivial abelian fibration case follows from Proposition 4.3.7. The bielliptic case follows from Proposition 4.4.2. □

Combining the work of J. A. Chen-M. Chen [CM14] and Ringler [Rin07], we have the following effective bound for threefolds of positive Kodaira dimension.

Corollary 4.5.2. *Let X be a smooth complex projective threefold of positive Kodaira dimension. Then $|mK_X|$ defines the Iitaka fibration if $m \geq 96$ and is divisible by 12.*

We remark that we do not know whether our estimate is optimal or not. However, as in Example 4.5.6, one can construct a threefold of Kodaira dimension one, such that $|iK_X|$ is not birational to the Iitaka fibration for all $i < 42$. Since the optimal value of the Iitaka fibration for threefolds of Kodaira dimension one should be divisible by 12, we have the following estimate.

Corollary 4.5.3. *If m is the smallest integer such that $|mK_X|$ is birational to the Iitaka fibration for all smooth projective threefold of Kodaira dimension one, then* $48 \leq m \leq 96$.

In the remaining part we will compute several examples.

Example 4.5.4. The first example is a trivial example. Let F be a bielliptic curve such that $|6K_F|$ is non-empty but $|iK_F|$ is empty for all $i \leq 5$ and let C be a curve of general type. Then $X = F \times C$ is a smooth threefold of Kodaira dimension one such that $|6K_X|$ defines the Iitaka fibration but $|iK_X|$ is empty for all $i \leq 5$.

Example 4.5.5. Let E be an elliptic curve. Pick two different points P and Q on E. One can find a line bundle L such that $L^2 = \mathcal{O}_E(P+Q)$. Let C be the cyclic cover corresponds to L^2. Then C is a curve of genus two and $\phi : C \to E$ is a double cover ramified at P and Q. Let $G = Aut(C/E)$, which is a cyclic group of order two and let F be an abelian surface. One can define a G-action on F via $-Id$. Let $X = (F \times C)/G$.

The singular points of X are of the type $\frac{1}{2}(1,1,1)$, hence X has terminal singularities. We want to show that $|4K_X|$ defines the Iitaka fibration, and $|iK_X|$ does not define the Iitaka fibration for $i \leq 3$. One has

$$H^0(X, mK_X) = H^0(F \times C, mK_F \boxtimes mK_C)^G = H^0(C, mK_C)^G$$

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since the unique section in $H^0(F, mK_F)$ is fixed by G for all m. To compute $H^0(C, mK_C)^G$, note that $\phi_*\mathcal{O}_C = \mathcal{O}_E \oplus L^{-1}$ and $\mathcal{O}_C(2K_C) = \phi^*\mathcal{O}_E(2K_E + P + Q) = \phi^*L^2$, hence $\phi_*\mathcal{O}_C(2kK_C) = L^{2k} \oplus L^{2k-1}$ by the projection formula. The G-invariant part of $H^0(C, 2kK_C)$ is $H^0(E, L^{2k})$ and L^{2k} is very ample if and only if $k \geq 2$. Hence $|2K_X|$ does not define the Iitaka fibration, but $|4K_X|$ does.

On the other hand, by Grothendieck duality we have

$$\phi_*(2k+1)K_C = \phi_* Hom_{\mathcal{O}_C}(-2kK_C, K_C) \cong Hom_{\mathcal{O}_E}(\phi_*(-2kK_C), K_E) \cong L^{2k} \oplus L^{2k+1}.$$

This shows that $h^0(C, 3K_C)^G = h^0(E, L^2) = 2$ and hence $|3K_X|$ do not define the Iitaka fibration.

We remark that this is the worst example we know for abelian fibrations.

Example 4.5.6. Let C be the Klein quartic

$$(x^3y + y^3z + z^3x = 0) \subset \mathbb{P}^2.$$

It is known that

$$|G| = |Aut(C)| = 168 = 42(2g(C) - 2)$$

(c.f. [Dol12, Section 6.5.3]). Let

$$F = (x^3y + y^3z + z^3x + u^4 = 0) \subset \mathbb{P}^3,$$

which is a K3 surface. Define the G-action on F by $g([x : y : z : u]) = [g([x : y : z]) : u]$. Let $X = (F \times C)/G$. We will prove the following:

(1) X has terminal singularities.

(2) $H^0(X, iK_X) \leq 1$ for $i \leq 41$ and $H^0(X, 42K_X) = 2$.

Hence the smooth model of X is a threefold of Kodaira dimension one, such that $|42K_X|$ defines the Iitaka fibration, but $|iK_X|$ do not define the Iitaka fibration for $i \leq 41$.

First we prove (1). Since $|Aut(C)| = 168 = 42(2g(C) - 2)$, it is well-known that the morphism $C \to C/G$ ramified at three points P_2, P_3 and $P_7 \in C/G$ and the stabilizer of points over P_r is a cyclic group of order r for $r = 2$, 3 and 7 (cf. also [Elk99, Proposition in Section 2.1]). Let $F_r \subset X$ be the fiber over P_r. We need to compute the singularities of F_r.

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(i) $r = 7$. Note that any order 7 subgroup of G is a Sylow-subgroup, which is unique up to conjugation. To compute the singularities we may assume the stabilizer is the cyclic group generated by the element (please see [Elk99, Section 1.1] for the description of elements in G)

$$\sigma = \begin{pmatrix} \xi^4 & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi \end{pmatrix}, \quad \xi = e^{\frac{2\pi i}{7}}.$$

One can compute that $H_7 = \langle \sigma \rangle$ has three fixed point $[1:0:0:0]$, $[0:1:0:0]$ and $[0:0:1:0]$ on F_7. The H_7 action around those fixed points is of the form $\frac{1}{7}(4,3)$, $\frac{1}{7}(2,5)$ and $\frac{1}{7}(1,6)$ respectively. The conclusion is that X has three singular points over P_7 which are cyclic quotient points of the form $\frac{1}{7}(1,6,1)$, $\frac{1}{7}(2,5,1)$ and $\frac{1}{7}(3,4,1)$ respectively.

(ii) $r = 3$. As before any order 3 subgroup of G is a Sylow-subgroup and hence we may assume the stabilizer is generated by

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are six fixed points on F_3, namely $[1:\omega:\omega^2:0]$, $[1:\omega^2:\omega:0]$ and $[1:1:1:u_i]_{i=1,\dots,4}$, where $w = e^{\frac{2\pi i}{3}}$ and u_1, ..., u_4 are four roots of the equation $u^4 + 3 = 0$. Let $P = [1:\omega:\omega^2:0]$. The local coordinates near $P \in \mathbb{A}^3$ are $y' = y/x - \omega$, $z' = z/x - \omega^2$ and $u' = u/x$. Let $\alpha = y' + z'$ and $\beta = \omega y' + \omega^2 z'$. Then α, β and u' are also local coordinates near P and the defining equation of F_3 near P can be written as

$$(1 + 3\omega)\alpha + \text{ higher order terms,}$$

hence the local coordinate of $P \in F_3$ is β and u'.

We have

$$\tau(u') = \tau(\frac{u}{x}) = \frac{u}{z} = \frac{1}{z' + \omega^2} u' = \omega\phi u',$$

where ϕ is a holomorphic function satisfying $\phi(P) = 1$ and $\phi\tau(\phi)\tau^2(\phi) = 1$. Let λ be a

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holomorphic function near P such that $\lambda^3 = \phi$ and $\lambda(P) = 1$. One can check that

$$\tau(\lambda' u') = \omega \lambda' u', \quad \text{where } \lambda' = \lambda^2 \tau(\lambda).$$

On the other hand,

$$\tau(\beta) = \frac{\omega}{z' + \omega^2} \beta = \omega^2 \phi \beta$$

and we also have

$$\tau(\lambda' \beta) = \omega^2 \lambda' \beta.$$

The conclusion is that the singularity of $P \in F_3$ is of the form $\frac{1}{3}(1,2)$, and hence the singularity of $P \in X$ is a terminal cyclic quotient $\frac{1}{3}(1,2,1)$. A similar computation (simply interchange ω and ω^2 in the calculation) shows that $P' = [1 : \omega^2 : \omega : 0] \in F_3 \subset X$ is also a terminal cyclic quotient point.

Finally we compute the singularities of $Q_i = [1 : 1 : 1 : u_i]$ for $i = 1, ..., 4$. Let $y_0 = y/x - 1$ and $z_0 = z/x - 1$. We take $\alpha_0 = \omega y_0 + \omega^2 z_0$ and $\beta_0 = \omega^2 y_0 + \omega z_0$ as local coordinates of $Q_i \in F_3$. One can see that

$$\tau(\alpha_0) = \tau(\omega y_0 + \omega^2 z_0) = \tau(\omega \frac{y}{x} + \omega^2 \frac{z}{x} + 1) = \frac{x}{z}(\omega + \omega^2 \frac{y}{x} + \frac{z}{x}) = \frac{\omega}{z_0 + 1} \alpha_0$$

and

$$\tau(\beta_0) = \frac{\omega^2}{z_0 + 1} \beta_0.$$

Using the same technique above we can say that the singularity of $Q_i \in F_3$ is of the form $\frac{1}{3}(1,2)$, hence $Q_i \in X$ is also a terminal cyclic quotient point for $i = 1, ..., 4$.

(iii) $r = 2$. Let $\mu \in G$ be an order two element. We have to compute the singularities of $F_2/\langle \mu \rangle$. By [Elk99, Proposition in Section 2.1], μ fixes a line and a point in $\mathbb{P}^2$. By the character table of G (cf. [Elk99, Section 1.1]), we know that the three-dimensional character of μ is equal to -1. This implies the fixed line of μ in $\mathbb{P}^2$ corresponds to the two-dimensional eigenspace with eigenvalue -1, and the fixed point of μ in $\mathbb{P}^2$ corresponds to the one-dimensional eigenspace with eigenvalue 1. Assume that L is the fixed line of μ in $\mathbb{P}^2$, $L \cap C = [x_i : y_i : z_i]_{i=1,...,4}$ and the fixed point of μ in $\mathbb{P}^2$ is $[x_5 : y_5 : z_5]$. One can check that the fixed point of μ on F_2 is $[x_i : y_i : z_i : 0]$ for $i = 1, ..., 4$ and $[x_5 : y_5 : z_5 : u_j]_{j=1,...,4}$, where u_j are the roots of the equation $u^4 + x_5^3 y_5 + y_5^3 z_5 + z_5^3 x_5 = 0$. The conclusion is that there are eight cyclic quotient points of indices two on F_2. Since they are isolated, all the singular points should be the from $\frac{1}{2}(1,1)$. The conclusion is that there are eight singular points on X which is of the from $\frac{1}{2}(1,1,1)$.

Note that the Iitaka fibration of X is a K3 fibration, and the basket data of X is

$$\{(2,1)\times 8,(3,1)\times 6,(7,1),(7,2),(7,3)\}.$$

It is the worst case in Section 4.2.

Now we prove (2). We need to compute $H^0(X,mK_X)=H^0(F\times C,mK_F\boxtimes mK_C)^G$. Consider the long exact sequence

$$\begin{aligned}0\to&H^0(\mathbb{P}^3,mK_{\mathbb{P}^3}+(m-1)F)\to H^0(\mathbb{P}^3,m(K_{\mathbb{P}^3}+F))\to H^0(F,mK_F)\to\\&H^1(\mathbb{P}^3,mK_{\mathbb{P}^3}+(m-1)F)\to\cdots.\end{aligned}$$

Since $H^i(\mathbb{P}^3,mK_{\mathbb{P}^3}+(m-1)F)=0$ for $i=0$, 1, we have

$$H^0(F,mK_F)=H^0(\mathbb{P}^3,m(K_{\mathbb{P}^3}+F))=H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}).$$

Thus any section in $H^0(F,mK_F)$ is G-invariant. This tell us that

$$H^0(X,mK_X)=H^0(F\times C,mK_F\boxtimes mK_C)^G=H^0(C,mK_C)^G.$$

One can consider the following long exact sequence

$$\begin{aligned}0\to&H^0(\mathbb{P}^2,mK_{\mathbb{P}^2}+(m-1)C)\to H^0(\mathbb{P}^2,m(K_{\mathbb{P}^2}+C))\to H^0(C,mK_C)\to\\&H^1(\mathbb{P}^2,mK_{\mathbb{P}^2}+(m-1)C)\to\cdots.\end{aligned}$$

Since $H^1(\mathbb{P}^2,mK_{\mathbb{P}^2}+(m-1)C)=0$, the restriction map

$$H^0(\mathbb{P}^2,m(K_{\mathbb{P}^2}+C))=H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m))\to H^0(C,mK_C)$$

is surjective. Thus to find G-invariant sections in $H^0(C,mK_C)$ is equivalent to find G-invariant polynomials of degree m on C. It is known that (c.f. [Elk99, Section 1.2]) the G-invariant polynomials are generated by three elements f_6, f_{14} and f_{21}, where f_d is a polynomial of degree

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d, satisfying $f_{21}^2 = f_{14}^3 - 1728 f_6^7$. Hence $h^0(C, iK_C)^G \leq 1$ for all $i \leq 41$ and $H^0(C, 42K_C)^G$ is spanned by f_6^7 and f_{14}^3. Thus $h^0(X, iK_X) \leq 1$ for $i \leq 41$ and $h^0(X, 42K_X) = 2$.

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