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星星森林的邊拉姆西數

Size Ramsey Numbers of Star Forests

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中文摘要

對於圖 $G_1, G_2, \ldots, G_r$和 F，如果當 F的邊被著上 $1, 2, \ldots, r$這些顏色時，總是存在 i 使得著顏色 i的邊中包含圖 G_i的話，則記作 $F \to (G_1, G_2, \ldots, G_r)$。在所有滿足 $F \to (G_1, G_2, \ldots, G_r)$的 F中，所含邊數的最小值稱為邊拉姆西數，記作 $\hat{r}(G_1, G_2, \ldots, G_r)$。

假設 $G_1 = \cup_{i=1}^{m} K_{1,a_i}$，$G_2 = \cup_{i=1}^{n} K_{1,b_i}$且 $a_1 \geq a_2 \geq \ldots \geq a_m$，$b_1 \geq b_2 \geq \ldots \geq b_n$，令 $\ell_s = \max_{i+j=s+1} (a_i + b_j - 1)$，Burr, Erdős, Faudree, Rousseau and Schelp [4]猜測 $\hat{r}(G_1, G_2) = \sum_{s=1}^{m+n-1} \ell_s$。這篇論文的目的是研究這個猜想在 a_1, b_1以外的數都等於1時的情形。

Abstract

For graphs $G_1, G_2, \ldots, G_r$ and F, we write $F \to (G_1, G_2, \ldots, G_r)$ to mean that if the edges of F are colored by $1, 2, \ldots, r$ then there exists some i such that the edges of color i contains a copy of G_i. The *size Ramsey number* $\hat{r}(G_1, G_2, \ldots, G_r)$ is the least number of edges of a graph F for which $F \to (G_1, G_2, \ldots, G_r)$.

Suppose $G_1 = \cup_{i=1}^{m} K_{1,a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_m$ and $G_2 = \cup_{i=1}^{n} K_{1,b_i}$ with $b_1 \geq b_2 \geq \ldots \geq b_n$. Let $\ell_s = \max_{i+j=s+1}(a_i + b_j - 1)$. Burr, Erdős, Faudree, Rousseau and Schelp [4] conjectured that $\hat{r}(G_1, G_2) = \sum_{s=1}^{m+n-1} \ell_s$. The purpose of this thesis is to study the conjecture for the case when $a_i = b_j = 1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$.

Contents

1 Introduction

For graphs $G_1, G_2, \ldots, G_r$ and F, we write $F \to (G_1, G_2, \ldots, G_r)$ to mean that if the edges of F are colored by $1, 2, \ldots, r$ then there exists some i such that the edges of color i contains a copy of G_i. The classical *Ramsey number* $R(G_1, G_2, \ldots, G_r)$ is the least number of vertices in a graph F for which $F \to (G_1, G_2, \ldots, G_r)$. For the case when all G_i are isomorphic, we write $(G; r)$ as a short notation for $(G_1, G_2, \ldots, G_r)$.

In general, for any graph parameter ρ, the *ρ-Ramsey number* $R_\rho(G_1, G_2, \ldots, G_r)$, which was introduced by West [7], is the minimum value of $\rho(F)$ for a graph F with $F \to (G_1, G_2, ..., G_r)$. This concept was introduced by Burr, Erdős and Lovász [12]. A graph parameter ρ is *monotone* if $\rho(G) \leq \rho(H)$ for any two graphs $G \subseteq H$. It is easy to see that if ρ is monotone, then so is R_ρ.

In the literature, there are many studies in ρ-Ramsey numbers for various graph parameters ρ. For instance, Folkman [10] proved that the clique Ramsey number $R_\omega(G, G)$ equals the clique number $\omega(G)$. Nešetřil and Rödl [13] extended the result to $R_\omega(G; r) = \omega(G)$ for every r. Let $\mathcal{H}om(G)$ denote the set of all homomorphism images of G. For any set $\mathcal{A}$ of graphs, $R(\mathcal{A})$ denotes the minimal number n such that for every 2-edge-coloring of K_n there exists a monochromatic subgraph isomorphic to one in $\mathcal{A}$. When the parameter is the chromatic number $\chi(G)$, Burr, Erdős and Lovász [6] showed that $R_\chi(G, G) = R(\mathcal{H}om(G))$. They proved that $\min\{R_\chi(G; r) : \chi(G) = k\} \leq k^r$ and conjectured that $\min\{R_\chi(G; r) : \chi(G) = k\} = k^r + 1$. Zhu [16] proved the conjecture for $k \leq 5$ and $s = 2$. The conjecture remains open in general.

Erdős, Faudree, Rousseau and Schelp [9] were the first to consider the *size Ramsey number* $\hat{r}(G_1, G_2, \ldots, G_r)$, which is the least number of edges of a graph F for which $F \to (G_1, G_2, \ldots, G_r)$. Notice that $\hat{r} = R_m$, where $m(G)$ is the number of edges of G.

For more study on the size Ramsey number, please see [1, 2, 8, 14]. It is easy to see that $\hat{r}(K_{1,a}, K_{1,b}) = a + b - 1$ and $\hat{r}(mK_2, nK_2) = m + n - 1$. More generally, for star forests $\cup_{i=1}^{m} K_{1,a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_m$ and $\cup_{i=1}^{n} K_{1,b_i}$ with $b_1 \geq b_2 \geq \ldots \geq b_n$, let

$$\ell_s = \max_{i+j=s+1} (a_i + b_j - 1)$$

for $1 \leq s \leq m + n - 1$. There is a famous conjecture about the size Ramsey number of star forests by Burr, Erdős, Faudree, Rousseau and Schelp [4].

Conjecture 1. (Burr, Erdős, Faudree, Rousseau and Schelp [4])

$$\hat{r}(\cup_{i=1}^{m} K_{1,a_i}, \cup_{i=1}^{n} K_{1,b_i}) = \sum_{s=1}^{m+n-1} \ell_s.$$

In the same paper, they confirmed the conjecture for $a_1 = a_2 = \ldots = a_m = a$ and $b_1 = b_2 = \ldots = b_n = b$ by proving that

$$\hat{r}(mK_{1,a}, nK_{1,b}) = (m + n - 1)(a + b - 1).$$

Min [12] generalize the result to

$$\hat{r}(n_1 K_{1,a_1}, n_2 K_{1,a_2}, \ldots, n_r K_{1,a_r}) = \left(\sum_{i=1}^{r} n_i - r + 1 \right) \left(\sum_{i=1}^{r} a_i - r + 1 \right).$$

The purpose of this thesis is to study the conjecture for the case when $a_i = b_j = 1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$.

2 Main Result

In this thesis, we also write down the proofs for known results for completeness.

It's easy to see the following proposition.

Proposition 2. If $F \to (G_1, G_2)$ and $H_1 \subseteq G_1, H_2 \subseteq G_2$, then $F \to (H_1, H_2)$ and F contains both G_1 and G_2.

We first notice that $\hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i}) \leq \sum_{s=1}^{m+n-1} \ell_s$, which gives the upper bound in Conjecture 1.

For convenience, by a 2-edge-coloring of a graph F, we always mean a coloring of the edges by red and blue. We then use $(F)_R$ and $(F)_B$ to denote the subgraph of F induced by the set of red edges and the set of blue edges, respectively.

Theorem 3. **([5])** $\cup_{s=1}^{m+n-1} K_{1,\ell_s} \to (\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i})$ and so $\hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i}) \leq \sum_{s=1}^{m+n-1} \ell_s$.

Proof. Let $F = \cup_{s=1}^{m+n-1} K_{1,\ell_s}$ and consider any 2-edge-coloring of F. We claim that for every t with $0 \leq t \leq m+n-1$ there exist i and j with $i+j=t$ such that in the subgraph $\cup_{s=1}^t K_{1,\ell_s}$, $(F)_R$ contains a copy of $\cup_{s=1}^i K_{1,a_s}$ and $(F)_B$ contains a copy of $\cup_{s=1}^j K_{1,b_s}$. We shall prove the claim by induction on t. The case of $t=0$ is clear. Suppose the claim holds for t. Since $\ell_{t+1} = \max_{i'+j'=t+2}(a_{i'} + b_{j'} - 1) \geq a_{i+1} + b_{j+1} - 1$, we have $K_{1,\ell_{t+1}} \to (K_{1,a_{i+1}}, K_{1,b_{j+1}})$ and so there is a red $K_{1,a_{i+1}}$ or a blue $K_{1,b_{j+1}}$ in $K_{1,\ell_{t+1}}$. Therefore, in the subgraph $\cup_{s=1}^{t+1} K_{1,\ell_s}$, either "$(F)_R$ contains a copy of $\cup_{s=1}^{i+1} K_{1,a_s}$ and $(F)_B$ contains a copy of $\cup_{s=1}^{j} K_{1,b_s}$" or "$(F)_R$ contains a copy of $\cup_{s=1}^{i} K_{1,a_s}$ and $(F)_B$ contains a copy of $\cup_{s=1}^{j+1} K_{1,b_s}$". So the claim holds by induction. For the case of $t = m+n-1$, by the pigeonhole principle, $i \geq m$ or $j \geq n$, which gives $\hat{r}(F) \to (\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i})$.

It then follows that $\hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i}) \leq m(F) = \sum_{s=1}^{m+n-1} \ell_s$. $\square$

A graph F is (G, H)-*minimal* if $F \to (G, H)$ but $F' \nrightarrow (G, H)$ for every $F' \subsetneq F$.

Proposition 4. The graph $F = \cup_{s=1}^{m+n-1} K_{1,\ell_s}$ is $(\cup_{i=1}^m K_{1,a_i}, \cup_{i=1}^n K_{1,b_i})$-minimal.

Proof. For any $F' \subsetneq F$, let $F' = \cup_{s=1}^{m+n-1} K_{1,\ell'_s}$, where $\ell'_1 \geq \ell'_2 \geq \ldots \geq \ell'_{m+n-1} \geq 0$. Since $m(F') < m(F)$, there exists some s_0 such that $\ell'_{s_0} < \ell_{s_0} = a_i + b_j - 1$ with $i + j = s_0 + 1$. For any $s \geq s_0$, we have $\ell_s \leq (a_i - 1) + (b_j - 1)$ and so $K_{1,\ell_s} \nrightarrow (K_{1,a_i}, K_{1,b_j})$. If we color $\cup_{s=1}^{i-1} K_{1,\ell_s}$ red and $\cup_{s=i}^{s_0} K_{1,\ell_s}$ blue. Every component of $\cup_{s=s_0}^{m+n-1} K_{1,\ell_s}$ contains neither K_{1,a_i} nor K_{1,b_j}. So $(F')_R$ contains no $\cup_{s=1}^{i} K_{1,a_s}$ and $(F')_B$ contains no $\cup_{s=1}^{j} K_{1,b_s}$, $F' \nrightarrow (\cup_{i=1}^{m} K_{1,a_i}, \cup_{i=1}^{n} K_{1,b_i})$. □

Proposition 4 tells us that if there is a counter example to the conjecture, then it is not a subgraph of $\cup_{s=1}^{m+n-1} K_{1,\ell_s}$.

A proper *k-edge-coloring* is a coloring on edges with k colors so that incident edges have different colors. The *edge-chromatic number* $\chi'(G)$ is the least k such that G has a proper k-edge-coloring.

The following two famous theorems can be found in [15].

Theorem 5. (Vizing) $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

A *k-factor* of a graph G is a k-regular subgraph of G.

Theorem 6. (Peterson) If G is $2k$-regular, then G can be decomposed into k 2-factors.

The following lemma is Theorem 1.3 in [11].

Lemma 7. Any connected graph G with $\Delta(G) \leq k$ can be embedded into a k-regular connected graph G'. When k is even and G is not k-regular, G can be embedded in a connected graph G' in which all vertices are of degree k except exactly two vertices have odd degree less then k.

Proof. For odd k, let H_k be the graph constructed by adding a vertex x adjacent to all vertices of one part in $K_{k-1,k-1}$ and adding a perfect matching on the other part.

Then H_k has x with degree $k-1$ and all other vertices with degree k. For even k, let H_k be the graph constructed by adding a vertex x_1 adjacent to all vertices in one part of $K_{k-1,k-1}$ and another vertex x_2 adjacent to all vertices in the other part. Then H_k has exactly two vertex with degree $k-1$ and all other vertices with degree k.

Add edges from vertices of degree less than k in G to distinct new vertices until the degree of all vertices in G become k. If k is odd, for each new vertex we add a copy of H_k and identify the new vertex with x in H_k. Then the resulting graph is k-regular. If k is even, the number of new vertices must be even since the sum of degrees is even. We can partition them into pairs of two vertices. For each pair we add a copy of H_k and identify x_1 and x_2 with these two vertices. Then the resulting graph is k-regular. For the second statement, we just discard the the final copy of H_k. $\square$

The following three lemmas consider lower bounds of the maximum degree for a graph F such that $F \to (K_{1,a}, K_{1,b})$.

Lemma 8. If $F \to (K_{1,a}, K_{1,b})$, then $\Delta(F) \geq a+b-2$.

Proof. Suppose to the contrary that $\Delta(F) \leq a+b-3$. By Vizing's theorem, we may properly color the edges of F by using colors $1, 2, \ldots, a+b-2$. Recolor the edges with colors $1, 2, \ldots, a-1$ by red and the edges with colors $a, a+1, \ldots, a+b-2$ by blue. Then $(F)_R$ contains no $K_{1,a}$ and $(F)_B$ contains no $K_{1,b}$. So $F \nrightarrow (K_{1,a}, K_{1,b})$, a contradiction. $\square$

Lemma 9. ([5]) If a and b are odd and $F \to (K_{1,a}, K_{1,b})$, then $\Delta(F) \geq a+b-1$.

Proof. Suppose to the contrary that $\Delta(F) \leq a+b-2$. By Lemma 7, F can be embedded into an $(a+b-2)$-regular graph F'. Since $a+b-2$ is even, by Peterson's theorem, F'

can be decomposed into $(a+b-2)/2$ 2-factors. Color $(a-1)/2$ of these 2-factors red and other $(b-1)/2$ 2-factors blue. Then $(F')_R$ is $(a-1)$-regular and so contains no $K_{1,a}$, and $(F')_B$ is $(b-1)$-regular and so contains no $K_{1,b}$. Therefore, $F' \nrightarrow (K_{1,a}, K_{1,b})$ and then $F \nrightarrow (K_{1,a}, K_{1,b})$, a contradiction. $\square$

Lemma 10. ([6]) If a is even and $F \to (K_{1,a}, K_{1,a})$, then either $\Delta(F) \geq 2a-1$ or F contains a $(2a-2)$-regular odd component.

Proof. For every $F \to (K_{1,a}, K_{1,a})$, there is a component C of F such that $C \to (K_{1,a}, K_{1,a})$, for otherwise we can color every component such that there is no monochromatic $K_{1,a}$. So we may assume that F is connected. Suppose $\Delta(F) \leq 2a-2$ but F is not a $(2a-2)$-regular graph of odd order.

We first consider the case when F is $(2a-2)$-regular and has an even number of vertices. Then F has an eulerian tour with an even number of edges since $(2a-2)|V(F)|/2$ is even. Color the edges of this eulerian tour alternately by red and blue. This yields a 2-coloring of F for which both $(F)_R$ and $(F)_B$ are $(a-1)$-regular. So $F \nrightarrow (K_{1,a}, K_{1,a})$.

Next, consider the case when F is not $(2a-2)$-regular. By Lemma 7, F can be embedded into a graph F', where $\Delta(F') = 2a-2$ and F' has exactly two vertices of odd degree. There is an eulerian trail between these two odd vertices. Color the edges of the eulerian trail alternately by red and blue. Then each of $(F')_R$ and $(F')_B$ has maximum degree at most $a-1$. So $F' \nrightarrow (K_{1,a}, K_{1,a})$ and hance $F \nrightarrow (K_{1,a}, K_{1,a})$. $\square$

In the rest of this section, we need the following notation. For a graph F, let $F_1 = F$. Having defined F_i, let v_i be a vertex of degree $d_i = \Delta(F_i)$ in F_i and $F_{i+1} = F_i - v_i$.

Lemma 11. If $F \to (\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j})$ and $p+q = s+1$, then $F_s \to (\cup_{i=p}^m K_{1,a_i}, \cup_{j=q}^n K_{1,b_j})$.

Proof. We shall prove the lemma by induction on s. For the case of $s = 1$, we have $p = q = 1$ and so the lemma is clear. Suppose $s \geq 2$ and the lemma holds for $s-1$. For $p+q = s+1$, without loss of generality we may assume that $p \geq 2$. As $(p-1)+q = (s-1)+1$, by the induction hypothesis, $F_{s-1} \to (\cup_{i=p-1}^m K_{1,a_i}, \cup_{j=q}^n K_{1,b_j})$. For every coloring of F_s, we color edges adjacent to v_{s-1} in F_{s-1} by red. These give a coloring of F_{s-1}. By the induction hypothesis, F_{s-1} has a red $\cup_{i=p-1}^m K_{1,a_i}$ or a blue $\cup_{j=q}^n K_{1,b_j}$. For the later case, the blue $\cup_{j=q}^n K_{1,b_j}$ is also in F_s. For the former case, suppose v_{s-1} appears in the k-th component K_{1,a_k} in $\cup_{i=q-1}^m K_{1,a_i}$. Then, F_s contains a red $(\cup_{i=q-1}^{k-1} K_{1,a_i}) \cup (\cup_{i=k+1}^m K_{1,a_i})$. Since each $a_i \geq a_{i+1}$, $(\cup_{i=p-1}^{k-1} K_{1,a_i}) \cup (\cup_{i=k+1}^m K_{1,a_i})$ contains $\cup_{i=p}^m K_{1,a_i}$. Hence, $F_s \to (\cup_{i=p}^m K_{1,a_i}, \cup_{j=q}^n K_{1,b_j})$. The lemma then follows from induction. $\square$

For $\cup_{i=1}^m K_{1,a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_m$ and $\cup_{i=1}^n K_{1,b_i}$ with $b_1 \geq b_2 \geq \ldots \geq b_n$, let

$$u_{i,j} = \begin{cases} a_i + b_j - 1, & \text{if } a_i \text{ and } b_j \text{ are odd, or one of } a_i \text{ and } b_j \text{ is } 1; \\ a_i + b_j - 2, & \text{otherwise.} \end{cases}$$

If $F \to (K_{1,a_i}, K_{1,b_j})$, then $\Delta(F) \geq u_{i,j}$ by Lemmas 8 and 9 and Proposition 2.

Lemma 12. If $F \to (\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j})$, then $m(F) \geq \sum_{s=1}^{m+n-1} \max_{i+j=s+1} u_{i,j}$.

Proof. For all i and j with $i+j = s+1$, we have $F_s \to (K_{1,a_i}, K_{1,b_j})$ by Lemma 11 and Proposition 2 and so $d_s \geq \max_{i+j=s+1} u_{i,j}$, which gives $m(F) \geq \sum_{s=1}^{m+n-1} d_s \geq \sum_{s=1}^{m+n-1} \max_{i+j=s+1} u_{i,j}$. $\square$

Corollary 13. $\hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j})) \geq \sum_{s=1}^{m+n-1} \ell_s - (m+n-1)$.

Proof. The corollary follows from Lemma 12 and that $\max_{i+j=s+1} u_{i,j} \geq \ell_s - 1$. $\square$

Theorem 14. Conjecture 1 holds if either (1) all a_i and b_j are odd, or (2) all $b_j = 1$.

Proof. For either case $u_{i,j} = a_i + b_j - 1$ and so

$$m(F) \geq \sum_{s=1}^{m+n-1} \max_{i+j=s+1} u_{i,j} = \sum_{s=1}^{m+n-1} \max_{i+j=s+1}(a_i + b_j - 1) = \sum_{s=1}^{m+n-1} \ell_s,$$

which gives the lower bound of Conjecture 1. $\square$

In the rest of this section, we consider the case of $G_1 = \cup_{i=1}^{m} K_{1,a_i}$ with $a_i = 1$ for $2 \leq i \leq m$ and $G_2 = \cup_{j=1}^{n} K_{1,b_j}$ with $b_j = 1$ for $2 \leq j \leq n$.

Proposition 15. Suppose $a_1 \geq b_1$ and $a_i = b_j = 1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$.
(1) If $m \leq n$, then $\sum_{s=1}^{m+n-1} \ell_s = na_1 + b_1 + m - 2$.
(2) If $m > n$, then $\sum_{s=1}^{m+n-1} \ell_s = na_1 + (m-n+1)b_1 + n - 2$.

Proof. If $m \leq n$ and $a_1 \geq b_1$, then $\ell_1 = a_1 + b_1 - 1$, $\ell_2 = \ell_3 = \ldots = \ell_n = a_1$ and $\ell_{n+1} = \ell_{n+2} = \ldots = \ell_{m+n-1} = 1$, which gives $\sum_{s=1}^{m+n-1} \ell_s = (a_1 + b_1 - 1) + (n-1)a_1 + (m-1) = na_1 + b_1 + m - 2$.

If $m < n$ and $a_1 \geq b_1$, then $\ell_1 = a_1 + b_1 - 1$, $\ell_2 = \ell_3 = \ldots = \ell_n = a_1$, $\ell_{n+1} = \ell_{n+2} = \ldots = \ell_m = b_1$ and $\ell_{m+1} = \ell_{m+2} = \ldots = \ell_{m+n-1} = 1$, which gives $\sum_{s=1}^{m+n-1} \ell_s = (a_1+b_1-1)+(n-1)a_1+(m-n)b_1+(n-1) = na_1+(m-n+1)b_1+n-2$. $\square$

Lemma 16. If Conjecture 1 is false for $(\cup_{i=1}^{m} K_{1,a_i}, \cup_{j=1}^{n} K_{1,b_j})$ with $m \leq n$, $a_1 \geq b_1$, $a_i = b_j = 1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$, then there exists $F \to (\cup_{i=1}^{m} K_{1,a_i}, \cup_{j=1}^{n} K_{1,b_j})$ such that $\hat{r} = (\cup_{i=1}^{m} K_{1,a_i}, \cup_{j=1}^{n} K_{1,b_j}) = m(F) = \sum_{s=1}^{m+n-1} \ell_s - 1$ and the following hold.
(1) $d_1 = a_1 + b_1 - 2$ and $d_i = a_1$ for $2 \leq i \leq n$.

(2) $F_n = K_{1,a_1} \cup (m-1)K_2$ and $F_{n+1} = (m-1)K_2$.

(3) v_i is not adjacent to v_j for $2 \le i < j \le n$.

(4) For $2 \le i \le n$, $N[v_i] = \{v_i\} \cup N(v_i)$ is apart from F_{n+1}.

Proof. Choose a graph $F \to (\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j})$ with $m(F) = \hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j})$. Since $u_{1,1} \ge a_1 + b_1 - 2$ and $u_{i,j} = a_i + b_j - 1$ for $(i,j) \ne (1,1)$, we have $d_1 \ge \ell_1 - 1$ and $d_s \ge \ell_s$ for $s \ge 2$. Since Conjecture 1 is false, we have

$$\textstyle\sum_{s=1}^{m+n-1} \ell_s > \hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j}) = m(F) \ge \sum_{s=1}^{m+n-1} d_s \ge \sum_{s=1}^{m+n-1} \ell_s - 1.$$

Therefore, $\hat{r}(\cup_{i=1}^m K_{1,a_i}, \cup_{j=1}^n K_{1,b_j}) = m(F) = \sum_{s=1}^{m+n-1} \ell_s - 1$. And, (1) also follows.

(2) By Lemma 11, $F_n \to (\cup_{i=1}^m K_{1,a_i}, \cup_{j=n}^n K_{1,b_j}) = (K_{1,a_i} \cup (m-1)K_2, K_2)$. So F_n contains $K_{1,a_1} \cup (m-1)K_2$. And by (1), $m(F_n) = \sum_{s=n}^{m+n-1} \ell_s = a_1 + m = m(K_{1,a_i} \cup (m-1)K_2)$. Therefore, $F_n = K_{1,a_1} \cup K_2$. So (2) also follows.

(3) If v_i is adjacent to v_j, then $d_{F_i}(v_j) > d_{F_j}(v_j) = a_1$, which is impossible as $d_i = \Delta(F_i) \ge d_{F_i}(v_j)$.

(4) For any permutation of $(2, 3, \ldots, n)$, say $(s_2, s_3, \ldots, s_{n1})$, we can choose this as a new sequence $(v'_2, v'_3, \ldots, v'_n) = (v_{s_2}, v_{s_3}, \ldots, v_{s_n})$ and relate $F'_2 = F_2, F'_3, \ldots, F'_n, F'_{n+1} = F_{n+1}$. Since $d_{F'_i}(v'_i) = \Delta(F'_i)$, we have $F'_n = K_{1,a_1} \cup (m-1)K_2$. So $N[v_{s_n}]$ is apart from F_{n+1}. We can choose any v_i as v_{s_n}, so every $N[v_i]$ is apart from F_{n+1}. $\square$

By Lemma 16 (3) and (4), we know that F_2 is a bipartite graph.

Lemma 17. Suppose G is a bipartite graph with part sets X and Y. If $\Delta(G) = k \ge 2$ and all vertices in X have degree k except possibly one is of degree at least $k-1$, then there exist a matching M saturate X such that $\Delta(G - M) \le k - 1$.

Proof. We first claim that there is a matching M' saturate X by checking Hall's condition. For any vertex set $I \subseteq X$ of size m, the degree sum of vertices in I is at least $km-1$. If $|N(I)| < m$, then there exists a vertex y in $N(I)$ such that

$$d_G(y) \geq (km-1)/(m-1) \geq (km-k)/(m-1) > k,$$

a contradiction. So the claim holds. Let $Y' = \{y \in Y : d_G(y) = k\}$. By using the same argument to Y' and $N(Y')$, there is a matching saturate Y'. Using properties of matroids, see Section 8.2 in [15], Y is an independent set in the transversal matroid formed from the bipartite graph. This independent set can be extended to a basis Y'', which has size equal to $|X| = |M'|$. Let M be the matching between X and Y'' correspondent to Y''. Then M meets every vertex of degree k and so $\Delta(G-M) \leq k-1$. □

Theorem 18. Conjecture 1 holds for the case when $m \leq n$ and $a_1 \geq b_1$, $a_i = b_j = 1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$.

Proof. Suppose to the contrary that Conjecture 1 is false for the specified conditions. Choose graph F satisfying conditions in Lemma 16. So $d_F(v_1) = a_1 + b_1 - 2$ and $T = F_2 - F_n$ is a bipartite graph with partite sets X and Y, where $X = \{v_2, v_3, \ldots, v_n\}$. Notice that $d_{F_2}(v_2) = d_{F_2}(v_3) = \ldots = d_{F_2}(v_n) = a_1$.

Case 1: $b_1 \geq 4$. Since F_2 is bipartite and $\Delta(F_2) = a_1$, there is a proper edge a_1-coloring of F_2, see Theorem 7.1.7 in [15]. Recolor the edges of colors $1, 2, \ldots, a_1 - 2$ by red and edges of colors $a_1 - 1, a_1$ by blue. For edges adjacent to v_1, color $a_1 - 1$ edges red and $b_1 - 1$ edges blue. Then $\Delta((F)_R) < a_1$ and $\Delta((F)_B) \leq \max\{b_1 - 1, 3\} < b_1$. So $F \nrightarrow (\cup_{i=1}^{m} K_{1,a_i}, \cup_{j=1}^{n} K_{1,b_j})$.

Case 2: $b_1 = 3$. $\Delta(F) = a_1 + b_1 - 2 = a_1 + 1$. Every vertex of degree $a_1 + 1$ are pairwise adjacent since $d_2 = a_1 < a_1 + 1$. There is at most 3 such vertices since F_2

is bipartite. Delete edges from v_1 to other such vertex forms F'. Then $F' - F_n$ is a bipartite graph satisfies the condition in Lemma 17. There is a matching M such that $\Delta(F' - M) < a_1$. Color edges in M blue, edges in $F' - M$ red and remaining two edges blue. Then $\Delta((F)_R) < a_1$ and $\Delta((F)_B) < 3$.

Case 3: $b_1 = 2$. $\Delta(F) = a_1 + b_1 - 2 = a - 1$, $F - F_n$ is bipartite and satisfies conditions in Lemma 16. The argument is similar to Case 2. $\square$

3 Structure of Ramsey graphs with minimum number of edges

It was proved in [4] that if $F \to (mK_{1,a}, nK_{1,b})$ with minimum number of edges, then every component of F is a triangle or a star. In general, this is not true.

For example, $C_5 \cup K_2 \to (K_{1,2} \cup K_2, K_{1,2} \cup K_2)$ and $C_7 \cup 2K_2 \to (K_{1,2} + \cup 2K_2, K_{1,2} \cup K_2)$, each of them has a minimum number of edges.

Lemmas 20 and 21 below give some property of F with $F \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$.

Lemma 19. If $G' \cup F \to (G' \cup G_1, G_2)$, then $F \to (G_1, G_2)$.

Proof. Suppose to the contrary that $F \nrightarrow (G_1, G_2)$. Color the edges of F such that $(F)_R$ contains no G_1 and $(F)_B$ contains no G_2. Then further color G' red. This results a coloring of $G' \cup F$ which contains no red $G' \cup G_1$ or blue G_2, a contradiction. $\square$

Lemma 20. Suppose $m \leq n$. If $n \geq 2$, then $C_5 \cup (n-2)K_{1,2} \cup (m-1)K_2 \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$. If $n \geq 3$, then $C_7 \cup (n-3)K_{1,2} \cup (m-1)K_2 \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$.

Proof. First, consider the case of $m = n$. For any 2-edge-coloring of $C_5 \cup (n-2)K_{1,2} \cup (n-1)K_2$, there must be a monochromatic $K_{1,2}$ in C_5, say red. If there is a red $K_{1,2} \cup K_2$ in C_5, then for any coloring of the following $(n-2)K_{1,2} \cup (n-1)K_2$, red edges appear in $n-2$ components (so there is a red $K_{1,2} \cup (n-1)K_2$ and we are done), or there are n blue components, one of them is $K_{1,2}$. So the blue subgraph contains $K_{1,2} + (n-1)K_2$. If there is no red $K_{1,2} \cup K_2$ in C_5, then there is also a blue $K_{1,2}$ in C_5. In the remaining $2n-3$ components, by the pigeonhole principle, there exists a monochromatic $(n-1)K_2$, and we are done.

For any 2-edge-coloring of $C_7 \cup (n-3)K_{1,2} \cup (n-1)K_2$, there must be a monochromatic $K_{1,2}$ in C_7, say red. If there is a red $K_{1,2} \cup 2K_2$ in C_7, then the lemma follows the claim in the previous case. Otherwise there are both red and blue $K_{1,2}$ and they appear in consecutive edges in C_7. Consider the graph with $2n-1$ components formed by the union of the edge in C_7 which dos not touch the two $K_{1,2}$ and remaining $(n-3)K_{1,2} \cup (n-1)K_2$. Then the lemma follows from the claim in the case $C_5 \cup (n-2)K_{1,2} \cup (n-1)K_2$.

Let $G' = (n-m)K_2$. Then the result of the above case can be written as $(C_5 \cup (n-2)K_{1,2} \cup (m-1)K_2) \cup G' \to ((K_{1,2} \cup (m-1)K_2) \cup G', K_{1,2} \cup (n-1)K_2)$ and $(C_7 \cup (n-3)K_{1,2} \cup (m-1)K_2) \cup G' \to ((K_{1,2} \cup (m-1)K_2) \cup G', K_{1,2} \cup (n-1)K_2)$. By Lemma 19, $C_5 \cup (n-2)K_{1,2} \cup (m-1)K_2 \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$ and $C_7 \cup (n-3)K_{1,2} \cup (m-1)K_2 \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$. $\square$

Lemma 21. If $F \to (K_{1,2} \cup (m-1)K_2, K_{1,2} \cup (n-1)K_2)$ with a minimum number of edges and $\Delta(F) < 3$, then the length of the maximum cycle of $F \leq 7$.

Proof. Without lose of generality, we may assume that $m \leq n$. By Lemma 10, there is a component C of F which is a cycle with size $2t+3$.

Case 1: $t > n-1$. Color a path with length $2n-1$ blue in C and others red. Then $(F)_B$ contains no $K_{1,2}+(n-1)K_2$ and $m((F)_R) = 3+2(n-1)+(m-1)-(2n-1) = m+1$. Since there is at least 4 red edges in C, which contains at most two components, $(F)_R$ contains at most $2+(m+1-4) = m-1$ components and then contains no $K_{1,2}+(m-1)K_2$.

Case 2: $2 < t \leq n-1$. Let $F' = F - C$. Color C and edges adjacent to v_i' red for $i \leq n-t-2$ and all other edges blue. Then $(F)_R$ contains no $K_{1,2} \cup (n-1)K_2$ since C contains at most $t+1$ component of $K_{1,2} \cup (n-1)K_2$. Therefore, $(F)_B$ contains $K_{1,2} \cup (n-1)K_2$. Since $(F)_B = F'_{n-t-1}$, $\Delta(F_i') = 2$ for $1 \leq i \leq n-t-1$ and then $m(F'_{n-t-1}) = m(F) - m((F)_R) = 3+2(n-1)+(m-1)-(2t+3)-2(n-t-2) = m+1$. So $F'_{n-t-1} = K_{1,2}+(m-1)K_2$. Label the edges of C clockwise by $1, 2, \ldots, 2t+3$. Color $1, 4, 7$ in C and $F'_{n-t} = F_{n+1} = (m-1)K_2$ red and others blue. Then $(F)_R$ contains no $K_{1,2}$ and the number of components of $(F)_B$ is at most t(in C)$+(n-t-1) = n-1$. So $(F)_B$ contains no $K_{1,2} \cup (n-1)K_2$. □

We close the thesis by posting two problems.

Problem 22. Is the diameter of a minimum graph bounded?

Problem 23. When is the minimum graph unique?

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