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金融相依性之風險測量

Measuring Risk on Financial Interdependence

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金融相依性之風險測量
Measuring Risk on Financial Interdependence

本論文係趙士綱君（R98723073）在國立臺灣大學財務金融學系完成之碩士學位論文，於民國 100 年 5 月 19 日承下列考試委員審查通過及口試及格，特此證明

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中 文 摘 要

近年來金融危機的發生有越來越頻繁和嚴重的趨勢。其中有個明顯的現象，一家金融機構的危機似乎會連帶影響到其他的金融機構。許多財務學家開始研究如何評估來自於連帶影響造成的系統性風險。在本篇論文中，我們根據 Adrian and Brunnermeier (2010)提出的廣義 CoVaR，提出了一個具體的可操作定義。在此定義下，我們以多元更新理論提出解析近似公式。根據此公式，利用常態和雙指數跳躍過程，以數值法計算近似 CoVaR，與利用蒙地卡羅模擬法所計算的 CoVaR 做比較。此外，我們也利用常態近似 CoVaR 與 t 分配蒙地卡羅模擬的結果作比較，因為在 t 分配下無法計算近似 CoVaR。結果顯示不同模型的確會影響計算的準確度，而蒙地卡羅模擬法需要較長的計算時間，近似法在某些狀況下可以提供較準確的值並更有效率。最後我們亦提供未來研究的方向。

關鍵字：相依風險值、風險值、溢出效果、財務相依性、多元更新理論、首度通過時間、隨機漫步

Abstract

Financial crisis seems to come more regularly in recent years. A prominent phenomenon is the spillover effect shown in the time of crisis. Many researchers begin to find a simple measure to characterize the risk of dependence in financial market. In this study, we propose a special case of CoVaR, which is a measure of dependence risk proposed by Adrian and Brunnermeier (2010). The asymptotic conditional distribution is derived from multivariate renewal theory under normal distribution and DEJP process in discrete time setting. The CoVaR's are computed numerically and are compared with the benchmarks from Monte Carlo simulation. We also compare the normal asymptotic CoVaR with the t distribution Monte Carlo simulated CoVaR since it is hard to get the asymptotic CoVaR under t distribution. We find that model assumption is likely to affect the CoVaR values and that the Monte Carlo simulation is computationally demanding. The asymptotic CoVaR's are suitably accurate in some most needed situations with higher time-efficiency. Possibilities for further researches are also suggested in the conclusion.

Key words：CoVaR, Value-at-Risk, Spillover Effect, Financial Interdependence, Multivariate Renewal Theory, First-Passage Time, Random Walk

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1. Introduction

In the last two decades, the world has gone through several severe financial crises, such as the Black Monday on October 19 in 1987, the market turmoil in 1998 and the collapse of CDO (Collateral Debt Obligation) and the later financial tsunami in 2008 and 2009. These events have urged the financial academics as well as the practitioners to think how to appropriately measure the market risk. There seems to be a tendency that financial system as a whole can be brought down by the bad news of a few financial firms.

Brady (1988) provide a thorough review of 1987 crisis and it mentions that the crisis start from the great loss of pension fund using portfolio insurance trading strategy, and then leads to the loss of investment banks and stock market. Rubin et al. (1999) review the crisis in 1998 and suggests the default of LTCM has not only results in the loss of its counterparties but also undermines the market confidence, so that increasing the market risk. In the most recent financial crises of 2008 to 2009, taking for an example, the bankrupcy of Lehman Brothers and financial distress of AIG has caused a tremendous effect on global financial market because the counterparty risk and the fear they aroused.

There are several kinds of systemic risk which have been identified in the literature. Bandt et al. (2009) review the related topics on systemetic risk. They argue that the systemic risk is driven by systemic events, which is composed by two components: shocks and propagation mechanisms. Shocks can be idiosyncratic or systemic, if the shock is systemic, such as crash of stock market or tightening liquidity, all assets are affected based on their market exposure. If the shock is idiosyncratic, the shock by definition will not directly endanger the financial system. However, idiosyncratic shock can be transmitted by propagation mechanism, and this creates systemic risk.

The propogation mechanism is very complicated, which may involve many interlocking events. For example, the fail of one company may not only impair its counterparties, but the market liquidity also tightens, thus reducing the funding resource for its peers, and possibly leading to more default events. There are many other phenomenon being identified and modeled in financial literature, such as herding, can attributes to the propogation of idiosyncratic risk.

Some authors borrow the term "contagion" from epidemiology to refer to the passing on of negative information in the financial market. However, the using of this term has not achieved a concensus among researchers. The terminology "interdependence" is used by Forbes and Rigobon (2002) to describe the comovement of international stock market, they argue that the concept of "contagion" is related to the increase of correlation in financial crisis. However, the correlation is conditioning on volatility. The increase of correlation can be due to the heteroskedasticity. "Interdependence" is used to describe the transmission of market turmoil without the rise of correlation. This terminology is more specific than the vague and controversial terminology like "contagion" in describing the approach we use in which the correlation between assets stay constant over time. In this study, we use the "interdependence risk" to refer to the systemic risk imposed by the fail of one financial institution in the system without the rising of true correlation.

A simple measure of financial independence is desired by market regulators and policy makers to define a clear-cut criterion for the financial institutions to follow. The most common risk measure is the value at risk (VaR), which bas been anacted in Basel II and III as a standard approach to measure the market risk. Nontheless, traditional VaR measures may not be able to reflect the spillover risk because it only looks at the individual asset price distribution and does not consider the increase of systemic risk which arises from the crash of financial institutions.

To properly measure the systemic risk, there are a number of researches proposing different risk measures. This area is relatively new in financial academics and the optimal systemic risk measure is still under debating. We follow the line of CoVaR which has been discussed by many researchers. Following Adrian and Brunnermeier (2010), if $C(X)$ is some event of X, define $\mathrm{CoVaR}_q^{W|X}$ the q-quantile of the conditional probability distribution:

$$P(W \leq \mathrm{CoVaR}_q^{W|X} | C(X)) = q. \tag{1}$$

Many researchers of central banks have started studying this quantity.

Adrian and Brunnermeier (2010) computes a special case of weekly CoVar where $C(X) = \{X = VaR_X\}$ by using linear quantile regression on a vast data set compriced of all publicly traded commercial banks, brokerdealers, insurance companies, and real estate companies. They estimate the asset i VaR and the market return conditional on asset i by running linear quantile regression on one-week lag of state variable,

$$X_t^i = \alpha^i + \gamma^i M_{t-1} + \epsilon_t^i,$$
$$X_t^{system} = \alpha^{system|i} + \beta^{system|i} X_t^i + \gamma^{system|i} M_{t-1} + \epsilon_t^{system|i}$$

where M_t is a state variable, reflecting the common risk factor. The predicted values can be generated by

$$VaR_t^i = \alpha^i + \gamma^i M_{t-1},$$
$$CoVaR_t^i = \alpha^{system|i} + \beta^{system|i} VaR_t^i + \gamma^{system|i} M_{t-1}.$$

With a different data set but under the same framework, Wong and Fong (2010) estimate the CoVaR on the CDS of Asia-Pacific banks.

Adams et al. (2010) estimate the state-dependent sensitivity VaR (SDSVaR) by using two-step estimation: first estimating VaR and then doing quantile regression to estimate SDSVaR of i:

$$VaR_{m,t} = \mu_{m,t} + z_\alpha \sigma_{m,t};$$
$$SDSVaR_{\{i;j,k,l\},t,\theta} = \alpha_\theta + \beta_{1,\theta} VaR_{j,t} + \beta_{2,\theta} VaR_{k,t} + \beta_{3,\theta} VaR_{l,t} + \beta_{4,\theta} VaR_{i,t-1} + u_{i,t}.$$

and $SDSVaR_{\{i;j,k,l\},t,\theta}$ satisfies

$$P(X_i \leq SDSVaR_{\{i;j,k,l\},t,\theta} | X_{m,t} = VaR_{j,t}, m = j,k,l, X_{i,t-1} = VaR_{i,t-1}) = q$$

where $m = i,j,k,l$ represent four financial indexes: insurance companies, commercial banks, investment banks and hedge funds. $\mu_{m,t}$ is the mean of index $m = i, j, k, l$ and $\sigma_{m,t}$ follows a volatiltiy process extracting from GARCH(1,1). They investigate how $(\alpha_\theta, \beta_{1,\theta}, \beta_{2,\theta}, \beta_{3,\theta}, \beta_{4,\theta})$ change when substituting different regressand and θ. They argue that the VaR as dependent variables can reflect the state or condition of financial market, so that the estimated quantile will account for the condition of financial market.

Under similar setting as (1), Zhou (2009) proposes different methodology to estimate CoVaR based on multivariate extreme value theory. Gauthier et al. (2009) also compute the CoVaR satisfying

$$P(W < CoVaR_i | X_i \in [VaR_i(1-\epsilon), VaR_i(1+\epsilon)]) = 0.05$$

where VaR_i is the value of risk of asset X_i and $\epsilon = 0.1$.

In this study, we place the problem of CoVaR under sequential setting. We consider two assets log return S_t and W_t, which are stochastic processes varying with time t. Moreover, they are assumed to be random walks with i.i.d. increments $\{(X_i, Y_i)\}_{i=1}^t$ and $(X_i, Y_i) \sim P$ where P is a joint probability measure with finite third moment. In particular, X_i and Y_i are linearly correlated with correlation coefficient ρ. The goal is finding the α-quantile of the distribution of W_T given the event $\min_{0 \le t \le T} S_t \le a$, $a < 0$.

The CoVaR in this study is defined by

$$P(W_T \le CoVaR_q^{W_T|S_t} | \tau < T) = \alpha, \tag{2}$$

where $\tau = \inf\{t > 0 : S_t \le a\}$ and $a < 0$ is a negative return level. For the ease of developing method in computing (2), we consider the reverse probability measure $\tilde{P} = -P$. Under the measure $\tilde{P}$ Equ. (2) is equivalent to

$$P(W_T \leq -CoVaR_q^{W_T|S_t} | \tau < T) = 1 - \alpha, \tag{3}$$

where $\tau = \inf\{t > 0 : S_t \geq a\}$ and $a > 0$. The correlation and variance of S_t and W_t do not change. Henceforth in this study whenever the CoVaR is mentioned, we mean the CoVaR satisfies Equ. (3).

(3) is similar to the general CoVaR definition (1) of Adrian and Brunnermeier (2010). However, former researchers focus on the one period return distribution, in our definition, T can be any time point in the future. We do not care about the behavior of S_t before time T if it does not fall below the lower level a. However, once it hits a, it triggers market fear and information spillover effect, and the whole system is affected. This corresponds to our observation of the market that asset prices fluctuate all the time but it does not cause market instability; on the other hand, large price drop imposes large negative effect on the market. The choice of a may be arbitrary. We set a minimum of $a = 3.0$ which means the actual accumulated return $-S_t$ falls to $e^{-3} \doteqdot 0.05$ of its beginning value.

We place our model under discrete time setting because it accomodates the actual data collection process. The continuous time model can be taken as an approximation of the discrete time model. The drawback of continuous time model is that it can induces large bias from the discrete time model, as suggested in the chapter 3 of Siegmund (1988). Therefore, a discrete adjustment is needed to get a more solid estimation.

In order to compute the CoVaR of (3), we propose an asymptotic approach based on multivariate renewal theory. The simulated CoVaRs are benchmarks. The asymptotic CoVaR can be computed for normal distribution and DEJP, even stable distribution, but not for t distribution, because it is not stable and its characteristic function has no closed form. We set normal, t and double exponential jump process (DEJP) for simulation. t distribution and DEJP are possibly more realistic because they can generate heavy tail and asymmetric jump.

In some cases the asymptotic CoVaR can be sufficiently accurate. By "accurate" we mean the asympotic values are close to the benchmarks computed by simulation. In our study, we find that when the correlation of the system and the conditioned asset process is positive, the values are more accurate. In practice, our care in the positive correlated case is greater than that in the negative correlated case. Positive correlation means the bad event from one asset can impose negative effect on the system. This can be seen in the time of financial crisis that all assets seem to be positively correlated.

Simulating rare event can be very computationally demanding. The rare event here refers to the case in which a is negatively far from zero and T is small. Sometimes it even takes a few weeks to simulate one or two values. Table 1 shows that the simulation based approach are much more time consuming than asymptotic based approach. The computer we use is with CPU Intel(R) Duo core T8100 2.10GHz and 2.09 GHz and 2.00 GB RAM. The normal Monte Carlo simulation is more time consuming than the heavy tail t distribution and DEJP process because normal random variable is less likely to generate extreme values.

Table 1: Computational time (seconds) of each panel in Section 3 and Section 4.

	Monte Carlo Simulation Based				Equ. (7) Based	
	Normal	DEJP	t(8)	t(6)	Normal	DEJP
$T = 90$	263373.2	150862.85	29407.02	20493.18	57.64	6032.86
$T = 120$	57353.95	111444.36	9762.05	8070.23	83.44	7975.17
$T = 150$	24771.37	78314.02	5295.16	4842.25	104.39	10094.46
$T = 180$	14813.12	68487.09	3733.53	3960.16	123.54	11716.73
$T = 210$	10823.53	62083.58	3077.57	3243.49	146.75	15190.09

This study is organized as follows: Section 2 introduces the methodology of computing the asymptotic CoVaR. Section 3 presents the CoVaRs computed by asymptotic method under normal assumptions and are compared with the benchmarks from Monte Carlo simulation. Section 4 shows the asymptotic CoVaRs under DEJP model and simulated CoVaRs under DEJP and t distribution. We also illustrate how to find asymptotic CoVaR under stable distributions.

2. Asymptotic CoVaR

The goal of this section is to find the way to compute the $1-\alpha$-quantile of the distribution

$$P(W_T \leq w | \tau < T),$$

under given fixed time T and τ is defined by

$$\tau = \inf\{t : S_t \geq a\}.$$

This quantity is compatible with our definition of CoVaR given in the Equ. (3) in the introduction.

As suggested in the introduction, modeling the negative accumulated return of a stock from time 0 to time t by:

$$S_t = X_1 + ... + X_t, \quad S_0 = 0,$$

where $X_1, X_2, ...$ are i.i.d. random variables denoting the stock log return innovations.

Modeling the market index (or system) accumulated log return process as univariate random walk $W_t = \sum_{j=1}^{t} Y_j$, $Y_j \in \mathbb{R}$ are i.i.d. Notice that W_t can be a linear combination of many log return processes from individual asset. To keep our model simple, here we assume that the weights of these assets in W_t do not change in time. Moreover, the mean, variance and the correlation between assets are constant over time.

In particular, under the Black-Sholes economy in discrete version, suppose we have m assets $V = (V_i)_{i=1}^{m}$, the i-th asset log return follows

$$V_t^i = \sum_{j=1}^{t} \epsilon_j^i,$$

where $\epsilon \sim N(\mu_i, \sigma_i^2)$. Suppose the shares invested $\pi = (\pi_i)_{i=1}^{m}$ are real numbers, so

$$Y_j = \sum_{i=1}^{m} \pi_i \epsilon_j^i \sim N\left(\sum_{i=1}^{m} \pi_i \mu_i, \sum_{i=1}^{m} \pi_i^2 \sigma_i^2 + \sum_{i,k}^{m} \pi_i \pi_k \rho_{ik} \sigma_i \sigma_k\right).$$

If the assets do not follow normal distribution, we directly assume the stock index follows a stochastic process itself, and we look at the relationship between this process and the stock process S_t.

For the validity of the renewal theory (see Appendix A), we further assume that the third moment of X_i and Y_i exist and the probability measure $\tilde{P}$

$$(X_1, Y_1) \sim \tilde{P}$$

where $\tilde{P}$ is defined in the introduction and satisfies Cramer's condition (Definition A.1 in Appendix). Denoting $\mathrm{E}X_1 = \nu$ and $\mathrm{E}Y_1 = \mu$ under $\tilde{P}$.

By strong Markov property, decomposing the conditional probability to get (please see Appendix B for details)

$$\begin{aligned} & P(W_T \leq w|\tau < T) \\ & = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(\tau = t) \int_{-\infty}^{\infty} P(W_\tau \leq w - x|\tau = t) P\left(\sum_{j=t+1}^{T} Y_j = x\right) dx \end{aligned} \tag{4}$$

The last equality follows from the fact that the random variables $Y_{t+1}, ..., Y_T$ are independent with the event $\{\tau = t\} \in \mathcal{F}_t$. One thing to notice is that the value of the last expression is between 0 and 1.

This quantity has three part to compute. The first part is

$$P\left(\sum_{j=t+1}^{T} Y_j = x\right) = \phi\left(\frac{x - (T-t)\mu}{\sigma_Y \sqrt{T-t}}\right),$$

if $Y_i \sim N(\mu, \sigma_Y^2)$. This normal assumption is employed in Section 3. However, in Section 4, other processes are applied. Notice that this is the only part which depends on model assumption, the other parts introduced below based on renewal theory are model free.

The tasks left are the stopping time distribution: $P(\tau < T)$ and the joint probability density function of W_τ and τ: $P(W_\tau \leq w - x, \tau = T)$.

The stopping time distribution $P(\tau < T)$ as $a \to \infty$ can be approximated using the result from Siegmund (1988). This formula is constructed by first deriving the stopping time distribution of Brownian motion and then make discrete adjustment.

Lemma 2.1. Let τ and S_t be defined as above. For fixed $T \in \mathbb{N}$ large,

$$P(\tau \leq T) \cong 1 - \Phi\left[\left(\frac{a}{\sigma_X} + \rho\right) T^{-1/2} - \frac{\nu}{\sigma_X} T^{1/2}\right] + e^{2(a/\sigma_X + \rho)(\nu/\sigma_X)} \Phi\left[-\left(\frac{a}{\sigma_X} + \rho\right) T^{-1/2} - \frac{\nu}{\sigma_X} T^{1/2}\right], \quad (5)$$

where

$$\rho = -\pi^{-1} \int_0^\infty t^{-2} \log\{2(1 - e^{-t^2/2})/t^2\} dt \cong 0.583.$$

up to terms of order $o(T^{-1/2})$.

The probability density $P(\tau = t)$ can be easily derived by differentiating the cumulative distribution given in the Lemma 2.1.

The final part is to estimate the probability $P(W_\tau \leq w - x | \tau < T)$. It is given by Corollary A.4:

$$P(W_\tau < w) = \Phi(\hat{w}) + \sqrt{\nu/a}\phi(\hat{w})H_1(\hat{w}) + o\{a^{(-1-\delta)/2}\sqrt{\log a}\}$$

as $a \to \infty$, uniformly in w, where $\hat{w} = (w - \gamma a)/(\sigma\sqrt{a/\nu})$, $\sigma^2 = \mathrm{var}(Y_1 - \gamma X_1) = \mathrm{E}Y_1^2 + \gamma^2\mathrm{E}X_1^2 - 2\gamma\mathrm{E}Y_1X_1$, $\mathrm{E}Z_1^3 = \sigma^{-3}(\mathrm{E}Y_1^3 - 3\gamma\mathrm{E}Y_1^2X_1 + 3\gamma^2\mathrm{E}Y_1X_1^2 - \gamma^3\mathrm{E}X_1^3)$ and $\mathrm{E}X_1Z_1 = \sigma^{-1}(\mathrm{E}Y_1X_1 - \gamma\mathrm{E}X_1^2)$

$$H_1(\hat{w}) = (\hat{w}^2 - 1)\left\{-\frac{1}{6}\mathrm{E}Z_1^3 + \frac{\mathrm{E}X_1Y_1}{2\nu}\right\} - \frac{\gamma\mathrm{E}S_\tau^2}{2\nu\sigma\mathrm{E}\tau} \tag{6}$$

Finally, the conditional distribution formula is given by

$$P(W_T \le w|\tau < T) \approx \frac{1}{F_\tau(T)}\sum_{t=1}^{T-1} f_\tau(t)\int_{-\infty}^{\infty}(\Phi(\hat{w}_x) + \sqrt{\nu/a}\phi(\hat{w}_x)H_1(\hat{w}_x))f_t(x)dx \tag{7}$$

where $a > 0$, $\nu > 0$ and $f_t(x)$ is the probability density function of $\sum_{j=t+1}^{T} Y_j$, $\hat{w}_x = (w - x - \gamma a)/(\sigma\sqrt{a/\nu})$,

$$F_\tau(T) = 1 - \Phi(A(T)) + e^{2(a/\sigma_X+\rho)(\nu/\sigma_X)}\Phi(B(T))$$
$$A(T) = \left(\frac{a}{\sigma_X} + \rho\right)T^{-1/2} - \frac{\nu}{\sigma_X}T^{1/2}$$
$$B(T) = A - 2\left(\frac{a}{\sigma_X} + \rho\right)T^{-1/2}$$
$$\hat{w} = (w - \gamma a)/(\sigma\sqrt{a/\nu})$$
$$\sigma^2 = \mathrm{var}(Y_1 - \gamma X_1) = \mathrm{E}Y_1^2 + \gamma^2\mathrm{E}X_1^2 - 2\gamma\mathrm{E}Y_1X_1$$
$$\mathrm{E}Z_1^3 = \sigma^{-3}(\mathrm{E}Y_1^3 - 3\gamma\mathrm{E}Y_1^2X_1 + 3\gamma^2\mathrm{E}Y_1X_1^2 - \gamma^3\mathrm{E}X_1^3)$$
$$\mathrm{E}X_1Z_1 = \sigma^{-1}(\mathrm{E}Y_1X_1 - \gamma\mathrm{E}X_1^2)$$
$$H_1(\hat{w}) = (\hat{w}^2 - 1)\left\{-\frac{1}{6}\mathrm{E}Z_1^3 + \frac{\mathrm{E}X_1Y_1}{2\nu}\right\} - \frac{\gamma\mathrm{E}S_\tau^2}{2\nu\sigma\mathrm{E}\tau}$$

and

$$
\begin{aligned}
f_\tau(t) &= \frac{d}{dt}F_\tau(t) \\
&= \frac{t^{-1.5}}{2}\phi\left(\left(\frac{a}{\sigma_X}+\rho\right)t^{-1/2} - \frac{\nu}{\sigma_X}t^{1/2}\right)\left(\frac{a}{\sigma_X}+\rho\right) - \frac{\nu}{2\sigma_X}t^{-0.5} + \\
& e^{2(a/\sigma_X+\rho)(\nu/\sigma_X)}\phi\left(-\left(\frac{a}{\sigma_X}+\rho\right)\sigma_X t^{-0.5} - \frac{\nu}{\sigma_X}t^{0.5}\right)\left(\frac{a}{\sigma_X}+\rho\right)\frac{t^{-1.5}}{2} - \frac{\nu}{2\sigma_X}t^{-0.5}
\end{aligned}
$$

Equ. (7) is the weighted sum of the convolution of asymptotic distribution (before hitting boundary) and idiosyncratic distribution (after hitting boundary). It just combines the two distribution of W_t before (using asymptotic distribution) and after (using assumed distribution) S_t hits a, and weighted by the stopping time distribution of the event $\{\min_{t\in[0,T]} S_t \geq a\}$.

Given $0 \leq \alpha \leq 1$ and $\epsilon > 0$ small, numerically searching for $CoVaR^{W_T|S_t}$ until

$$\left|\hat{P}(W_T \leq -CoVaR_q^{W_T|S_t}|\tau < T) - (1-\alpha)\right| \leq \epsilon$$

holds, where $\hat{P}(W_T \leq CoVaR^{W_T|S_t}|\tau < T)$ is the asymptotic probability of $P(W_T \leq CoVaR^{W_T|S_t}|\tau < T)$ given by Equ. (7). In this study, we set $\alpha = 0.05$.

As suggested by the multirenewal theory, this approximation is accurate for a far from zero. a also defines the event in CoVaR. In section 3 we will show CoVaR for different levels of a. In practice, the choice of a depends on the purpose of user.

The CoVaR estimation for large a is troublesome if the Monte Carlo simulation is applied. As shown in Table 1 that the computational time increases when a gets smaller. By contract, the proposed method of computing

CoVaR by inverting (7) gives accurate value in some cases and the computational time is much shorter than the Monte Carlo simulation. Therefore, this method can be more favorable than Monte Carlo simulation in CoVaR computation.

3. Numerical Computation of CoVaR

3.1. Choice of Parameters

In this section, the quantiles of the conditional marginal probability

$$P(W_T \leq w | \tau < T)$$

are computed by the normal Monte Carlo simulation and by inverting (7). The model is as described in Section 2. The purpose of this study is to see how well the asymptotic conditional quantiles approximate the exact simulated conditional quantiles.

The time step in this study is daily. Suppose the observation dates are $T = 30, 60, 90, 120, 150, 180$ and 210 days. We consider stock log return threshold $a = 0.2, 0.5, 1.5, 3.0$. The largest threshold is 3.0 because $e^{-3} = 0.04979$. That is, as log return S_t hits $a = 3.0$, the asset price has plunged over 95%.

To get practical parameters, a daily log returns data set of Citigroup and S&P500 from 1986Q1 to 2010Q1 is obtained.

Let S_t be the log return process of a financial institution and W_t be the log return process of the stock market under the reverse probabiltiy measure

$\tilde{P}$. The Citigroup is chosen as the representative of S_t. Citigroup Inc. underwent severe financial distress in the financial tsunami of 2008 and 2009. As Citigroup is once the largest commercial bank in the U.S. by market value, how the stock market responds to its crash is worth exploring.

Be noted that our purpose is to acquire parameters with reasonable scale to carry out the simulation, a complete empirical study is not our goal. The parameter estimations may not be the most accurate ones. Estimating the first and second moment of the increment X_i of S_t by taking simple mean and standard deviation daily returns of Citigroup Inc. from June 2008 to June 2009. During this period Citigroup Inc. drop 64.65% in market value (from our data set). We get X_i mean $\nu = 0.004$ and second moment 0.008. The first and second moment of W are estimated from the weekly log return of S&P500. We set Y_i mean $\mu = -0.0006$ and second moment $\mathrm{E}Y_1^2 = 0.0014$.

3.2. Normal Monte Carlo Simulation

For computing the conditional quantiles, our benchmark is the simulation value. In this section, the exact CoVaRs are based on great amount of simulations using normal distribution. The simulation is very computational demanding. All parameters are daily as described in Section 3.1. We run the program on freeware R (version 2.12.1), and the computer we use is equipped with Intel(R) Duo core T8100 2.10GHz and 2.09 GHz and 2.00 GB RAM.

First the Monte Carlo simulation procedure is as follows:

Algorithm 3.1. Monte Carlo Simulation of the Conditional Quantiles Using Normal Distribution.

1. Generating T 2-dimensional $\{(X_t, Y_t)\}_{t=1}^{T}$ from bivariate normal distribution

$$N\left(\begin{pmatrix} \nu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_W \\ \rho\sigma_X\sigma_W & \sigma_W^2 \end{pmatrix}\right)$$

with the command "rmvnorm" from the multivariate statistical package "mvtnorm".

2. Computing the marginal log return at time t: $S_t = X_1 + ... + X_t$, and recording S_t for each $t = 0, 1, 2, ..., T$.

3. If $S_t \geq a$ for some t ,(in other words, $\max_{1\leq t\leq T} S_t \geq a$) then computing $W_T = Y_1 + ...Y_T$ and recording W_T. If $S_t < a$ for all t, going back to step 1.

4. Repeating step 1 to 3 until $N = 10,000$ values of W_T are collected. Therefore, we obtain a distribution of W_T conditional on the event $S_t \geq a$ for some t based on these N samples.

5. Computing the $1 - \alpha$-th quantile out of this empirical distribution, where $\alpha = 0.05$.

The simulation can be very time-consuming, because $P(\tau < T)$ is small. As shown in Table 3.2.1, the probablities are very small in scale when $a = 3.0$ and T small. This shows that it need to take great amount of computation in order to get one trajectory of S_t which $\max_{0\leq t\leq T} \geq 3.0$.

Table 3.2.1: The probability $P(\tau \leq T)$ for different a by using Lemma 2.1.

a	$T=30$	$T=60$	$T=90$	$T=120$	$T=150$	$T=180$	$T=210$
1.5	0.00325	0.05199	0.1372	0.2268	0.3091	0.3816	0.4447
3.0	2.0829×10^{-9}	4.5975×10^{-5}	1.3676×10^{-3}	7.633×10^{-3}	0.02165	0.04365	0.07224

Table 3.2.2: Normal Monte Carlo simulation of the CoVaR.

	a	$\rho=-0.9$	$\rho=-0.5$	$\rho=0$	$\rho=0.5$	$\rho=0.9$
	0.2	-0.1876	-0.2580	-0.3276	-0.3552	-0.3550
$T=30$ days	0.5	-0.0445	-0.1868	-0.3176	-0.3926	-0.4157
	1.5	0.3801	0.0179	-0.3146	-0.5851	-0.7141
	0.2	-0.3122	-0.3796	-0.4359	-0.4697	-0.4668
$T=60$ days	0.5	-0.1687	-0.3167	-0.4372	-0.5185	-0.5181
	1.5	0.2598	-0.1085	-0.4478	-0.6744	-0.7636
	0.2	-0.4070	-0.4722	-0.5293	-0.5551	-0.5572
$T=90$ days	0.5	-0.2636	-0.4155	-0.5292	-0.5976	-0.5873
	1.5	0.1690	-0.2130	-0.5212	-0.7541	-0.8351
	3.0	0.7696	0.1109	-0.5368	-1.0335	-1.2883
	0.2	-0.4797	-0.5535	-0.6164	-0.6116	-0.6285
$T=120$ days	0.5	-0.3364	-0.4913	-0.5953	-0.6407	-0.6678
	1.5	0.0872	-0.2766	-0.6163	-0.7926	-0.8524
	3.0	0.7069	0.0353	-0.5998	-1.0788	-1.3008
	0.2	-0.5569	-0.5966	-0.6546	-0.6729	-0.6980
$T=150$ days	0.5	-0.4163	-0.5553	-0.6625	-0.6999	-0.7099
	1.5	0.0137	-0.3530	-0.6801	-0.8446	-0.9138
	3.0	0.6145	-0.0521	-0.6524	-1.1200	-1.3143
	0.2	-0.6201	-0.6736	-0.7017	-0.7222	-0.7344
$T=180$ days	0.5	-0.4638	-0.6189	-0.7222	-0.7707	-0.7684
	1.5	-0.0406	-0.4253	-0.7186	-0.8985	-0.9168
	3.0	0.5790	-0.1033	-0.7226	-1.1392	-1.3067
	0.2	-0.6454	-0.6876	-0.7692	-0.7875	-0.8051
$T=210$ days	0.5	-0.5096	-0.6765	-0.7508	-0.7980	-0.8335
	1.5	-0.1044	-0.4904	-0.7805	-0.9289	-0.9503
	3.0	0.4906	-0.1732	-0.7639	-1.1864	-1.3261

Table 3.2.2 presents the exact conditional quantiles from simulated distribution. In each panel, the quantiles are decreasing with rising correlations.

This is intuitive because W_t is more likely to be dragged down as S_t hits a lower boundary when their correlation is high. In particular, the aforementioned correlation effect is more significant when a is large, for example, in the panel $T = 150$ days, at the line $a = 0.2$, the quantiles decrease from -0.5569 of $\rho = -0.9$ to -0.6980 of $\rho = 0.9$, but at the line $a = 3.0$, the quantiles decrease from a positive 0.6145 of $\rho = -0.9$ to -1.3143 of $\rho = 0.9$. This shows that the behavior of W_T is closely linked to the behavior of S_t. Another interesting feature is that the quantiles are increasing in a for negative correlation but decreasing for positive correlation. The quantiles does not change much in a for zero correlation. For negative correlation, S_t and W_t tend to go opposite way. If S_t goes lower, W_t tends to go higher. The quantiles are even positive at $a = 3.0$ when $\rho = -0.9$ for all T. For positive correlation, the pattern reverses. S_t and W_t go lower together.

If we turn our attention to changes between panels, we can see that the quantiles are decreasing as T getting longer. The reason is that W_T is the sum of T i.i.d. random variables Y_t. When T is large, the variance of W_T is also large because $\mathrm{var} W_T = T\sigma_Y^2$, so that the left α quantile is smaller.

3.3. Asymptotic Conditional Quantiles

Recall from (7) that the conditional distribution decomposition is

$$P(W_T \le w|\tau < T) = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(\tau = t) \int_{-\infty}^{\infty} P(W_\tau \le w - x|\tau = t) P\left(\sum_{j=t+1}^{T} Y_j = x\right) dx.$$

This probability is approximated by

$$P(\tau \le T) \cong 1-\Phi\left[\left(\frac{a}{\sigma_X}+\rho\right)T^{-1/2}-\frac{\nu}{\sigma_X}T^{1/2}\right]+ e^{2(a/\sigma_X+\rho)(\nu/\sigma_X)}\Phi\left[-\left(\frac{a}{\sigma_X}+\rho\right)T^{-1/2}-\frac{\nu}{\sigma_X}T^{1/2}\right],$$

up to terms of order $o(T^{-1/2})$, where $\rho = 0.583$, and

$$P(W_\tau < w|\tau < T) \cong \Phi(\hat{w}) + \sqrt{\nu/a}\phi(\hat{w})H_1(\hat{w})$$
$$H_1(\hat{w}) = (\hat{w}^2-1)\left\{-\frac{1}{6}\mathrm{E}Z^3+\frac{\mathrm{E}XZ}{2\nu}\right\}-\frac{\gamma \mathrm{E}S_\tau^2}{2\nu\sigma\mathrm{E}\tau}$$

up to terms of order $o\{a^{(-1-\delta)/2}\sqrt{\log a}\}$.

Given $0 \le \alpha \le 1$ and $\epsilon > 0$ small, numerically searching for $CoVaR^{W_T|S_t}$ until

$$\left|\hat{P}(W_T \le -CoVaR_q^{W_T|S_t}|\tau < T) - (1-\alpha)\right| \le \epsilon$$

holds, where $\hat{P}(W_T \le CoVaR^{W_T|S_t}|\tau < T)$ is the asymptotic probability of $P(W_T \le CoVaR^{W_T|S_t}|\tau < T)$ given by Equ. (7). In this study, we set $\alpha = 0.05$. The indefinite integration involved in (7) is computed by the "integrate" command in R.

Table 3.3.1 shows the asymtotic conditional quantiles by inverting (7) using numerical method. [1] The daily parameters are $\nu = 0.004$, $\mu = -0.0006$, $\mathrm{E}X^2 = 0.008$, $\mathrm{E}W^2 = 0.0014$ as described in Section 3.1. These quantiles represent the conditional log returns of asset W_T, for different cases of T.

The CoVaR's in Table 3.3.1 are all negative and the quantiles are in-

[1]Since $P(W_T \le w|\tau < T)$ is conditional distribution function, it is an increasing function of w. We recursively search the value of w, until the difference of $P(W_T \le w|\tau < T)$ and α is less than 10^{-6}.

creasing with correlations at $a = 0.2, 0.5, 1.5$ and decreasing at $a = 3.0$. By contrary, Table 3.2.2 has both positive and negative values and its value are all decreasing with rising correlation. In essence, the asymptotic value is accurate for a larger. Therefore, it is reasonable that at $a = 3.0$ the asymptotic quantiles have similar pattern to the exact quantiles at $a = 3.0$. Another feature is that the asymptotic conditional quantiles are decreasing with T at $a = 0.2, 0.5, 1.5$ but increasing at $a = 3.0$, but this feature is not very obvious. Lastly, The change of CoVaR's in T is not so obvious as that in a.

Table 3.3.1: The asymptotic CoVaR computed by inverting the decomposition (7) under normal distribution.

	a	$\rho=-0.9$	$\rho=-0.5$	$\rho=0$	$\rho=0.5$	$\rho=0.9$
	0.2	-0.5854	-0.5028	-0.3916	-0.3941	-0.4091
$T=30$ days	0.5	-0.7468	-0.6350	-0.5212	-0.5494	-0.5831
	1.5	-0.9924	-0.8074	-0.7755	-0.8548	-0.9246
	0.2	-0.6142	-0.5554	-0.4692	-0.4547	-0.4603
$T=60$ days	0.5	-0.7661	-0.6606	-0.5624	-0.5803	-0.6087
	1.5	-0.9933	-0.8147	-0.7851	-0.8617	-0.9301
	0.2	-0.6428	-0.6095	-0.5393	-0.5129	-0.5103
$T=90$ days	0.5	-0.7770	-0.6910	-0.6043	-0.6125	-0.6356
	1.5	-0.9929	-0.8244	-0.7971	-0.8704	-0.9370
	3.0	-1.0585	-0.8459	-0.9393	-1.0805	-1.1905
	0.2	-0.6784	-0.6604	-0.6021	-0.5682	-0.5585
$T=120$ days	0.5	-0.7867	-0.7230	-0.6453	-0.6449	-0.6627
	1.5	-0.9910	-0.8357	-0.8105	-0.8803	-0.9448
	3.0	-1.0531	-0.8500	-0.9429	-1.0829	-1.1922
	0.2	-0.7166	-0.7074	-0.6584	-0.6203	-0.6050
$T=150$ days	0.5	-0.7994	-0.7550	-0.6848	-0.6770	-0.6898
	1.5	-0.9880	-0.8482	-0.8248	-0.8909	-0.9532
	3.0	-1.0469	-0.8547	-0.9470	-1.0857	-1.1941
	0.2	-0.7545	-0.7507	-0.7091	-0.6692	-0.6495
$T=180$ days	0.5	-0.8156	-0.7863	-0.7227	-0.7085	-0.7166
	1.5	-0.9845	-0.8614	-0.8397	-0.9019	-0.9620
	3.0	-1.0398	-0.8599	-0.9516	-1.0887	-1.1963
	0.2	-0.7910	-0.7907	-0.7551	-0.7150	-0.6920
$T=210$ days	0.5	-0.8345	-0.8166	-0.7588	-0.7393	-0.7431
	1.5	-0.9811	-0.8751	-0.8548	-0.9133	-0.9711
	3.0	-1.0321	-0.8656	-0.9566	-1.0921	-1.1987

Table 3.3.2 shows the absolute errors between the asymptotic conditional quantile and normal simulation quantiles. Each number in the table is computed by substracting normal Monte Carlo simulated value (Table 3.2.2) from theoretical value (Table 3.3.1) and then taking absolute value.

In general, the asymptotic conditional quantiles perform better for positive and zero correlation. Most of the absolute difference value are less than 2 digits under the decimal point. The worst performance happens for negative correlations. However, except a little anomaly [2], the absolute errors reduce as T is getting larger.

[2]For $a = 0.2$, there are some disorder that the absolute errors increase form $T = 120$ to $T = 150$. This may be due to the fact that $a = 0.2$ is not negative enough to get a good asymptotic result.

Table 3.3.2: The absolute errors between conditional marginal quantiles and normal Monte Carlo simulated quantiles.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	0.3978	0.2448	0.0640	0.0389	0.0541
$T = 30$ days	0.5	0.7023	0.4482	0.2036	0.1568	0.1674
	1.5	1.3725	0.8253	0.4609	0.2697	0.2105
	0.2	0.3020	0.1758	0.0333	0.0150	0.0065
$T = 60$ days	0.5	0.5974	0.3439	0.1252	0.0618	0.0906
	1.5	1.2531	0.7062	0.3373	0.1873	0.1665
	0.2	0.2358	0.1373	0.0100	0.0422	0.0469
$T = 90$ days	0.5	0.5134	0.2755	0.0751	0.0149	0.0483
	1.5	1.1619	0.6114	0.2759	0.1163	0.1019
	3.0	1.8281	0.9568	0.4025	0.0470	0.0978
	0.2	0.1987	0.1069	0.0143	0.0434	0.0700
$T = 120$ days	0.5	0.4503	0.2317	0.0500	0.0042	0.0051
	1.5	1.0782	0.5591	0.1942	0.0877	0.0924
	3.0	1.7600	0.8853	0.3431	0.0041	0.1086
	0.2	0.1597	0.1108	0.0038	0.0526	0.0930
$T = 150$ days	0.5	0.3831	0.1997	0.0223	0.0229	0.0201
	1.5	1.0017	0.4952	0.1447	0.0463	0.0394
	3.0	1.6614	0.8026	0.2946	0.0343	0.1202
	0.2	0.1344	0.0771	0.0074	0.0530	0.0849
$T = 180$ days	0.5	0.3518	0.1674	0.0005	0.0622	0.0518
	1.5	0.9439	0.4361	0.1211	0.0034	0.0452
	3.0	1.6188	0.7566	0.2290	0.0505	0.1104
	0.2	0.1456	0.1031	0.0141	0.0725	0.1131
$T = 210$ days	0.5	0.3249	0.1401	0.0080	0.0587	0.0904
	1.5	0.8767	0.3847	0.0743	0.0156	0.0208
	3.0	1.5227	0.6924	0.1927	0.0943	0.1274

4. CoVaR: Other Stock Return Models

The results in Section 3.1 is based on the assumption that the increments Y_i follow normal distribution. In fact, our model can be applied to any stochastic process for which the probability density function

$$P\left(\sum_{j=t+1}^{T} Y_j = x\right)$$

exists in (7). By the Fourier inversion theorem (see Durret (2005)), this is equivalent to the condition that its characteristic exists. Therefore, we can in fact extend our result into Lévy process. In this section, we discuss DEJP porcess and stable distribution whose characteristic functions are known in closed form.

The characteristic function of t is in generally not known in closed form (Cont and Tankov (2004)), the asymptotic conditional quantiles by (7) is hard to compute by the Fourier inversion formula. Hurst (1995) derive a general formula for the characteristic function of univariate t distribution with positive degree of freedom by the subordinate method introduced in Feller (1966). If $V \sim t(\nu)$, its characteristic function is given by

$$\mathrm{E}(e^{i\theta X}) = \frac{K_{\frac{1}{2}\nu}(\sqrt{\nu}|\theta|)(\sqrt{\nu}|\theta|)^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu)2^{\frac{1}{2}\nu-1}} \quad \theta \in \mathbb{R},$$

where ν is the degree of freedom, $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ and $\Gamma(\cdot)$ is the Gamma function. Dreier and Kotz (2002) obtains another expression of the characteristic function of t-distribution involving an indefinite integral by the theory of positive definite densities, which is

$$\mathrm{E}(e^{i\theta X}) = \frac{2^{\nu}\nu^{\nu/2}}{\Gamma(\nu)} \int_0^{\infty} e^{-\sqrt{\nu}(2x+|\theta|)}(x(x+|\theta|))^{(\nu-1)/2}dx, \quad \theta \in \mathbb{R}.$$

where ν is the degree of freedom and $\Gamma(\cdot)$ is the Gamma function. As can be seen from the above discussion, Fourier inversion for the characteristic function of t distribution is generally not feasible.

4.1. Double Exponential Jump Process (DEJP)

If the stock log return S_t follows DEJP, the dynamics is given by the stochastic differential equation

$$dS_t = \mu dt + \sigma dB_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right).$$

Where B_t is a standard Brownian motion, N_t a Poisson process with rate λ. $\{V_i\}$ is i.i.d. nonnegative random variables so that $\log(V)$ has an asymmetric double exponential distribution:

$$f(z) = p \cdot \eta_1 e^{-\eta_1 z}\mathbf{1}_{\{z\geq 0\}} + q \cdot \eta_2 e^{\eta_2 z}\mathbf{1}_{\{z<0\}}, \quad \eta_1 > 1, \eta_2 > 0,$$

where $p, q \geq 0$, $p + q = 1$. DEJP seperately models the positive and negative jumps, which gives more freedom in modeling the jump dynamics of asset returns.

Double Exponential Jump Process is a special case of Lévy process, which admits the unique Lévy-Khinchin representation, see Kou and Wang (2004). By Kou and Wang (2003), the characteristic function of the double exponential jump process $Z(t)$ is $\mathrm{E}[e^{i\theta Z(t)}] = \exp\{G(i\theta)t\}$, where

$$G(x) = x\mu + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right). \tag{8}$$

Under the assumption that the increments Y_i follow DEJP, in order to use (7), we need to find the probability density function $P\left(\sum_{j=t+1}^{T} Y_j = x\right)$. Fourier inversion theorem can be applied. The pdf is given by (please see Appendix C for details)

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\exp(i\theta x)\exp(G(i\theta)t)]d\theta \qquad (9)\\ &= \frac{1}{\pi}\int_0^\infty \exp(A(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2))\cos(B(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2))d\theta \end{aligned}$$

where

$$\begin{aligned} A(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2) &= -\frac{\theta^2\sigma^2 t}{2} - \lambda t + \lambda\left(p\frac{\eta_1^2 t}{\eta_1^2 + \theta^2} + q\frac{\eta_2^2 t}{\eta_2^2 + \theta^2}\right);\\ B(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2) &= \theta\left(-x + t\mu + p\frac{\lambda\eta_1 t}{\eta_1^2 + \theta^2} - q\frac{\lambda\eta_2 t}{\eta_2^2 + \theta^2}\right). \end{aligned}$$

In the following numerical experiments, the parameters are retrieved from the empirical study of DEJP by Ramezani and Zeng (2006). They use maximum likelihood mathod to estimate the parameters $(\lambda_u, \lambda_d, p, \eta_1, \eta_2)$ in the Pareto-Beta Jump-Diffusion (PBJD) model. This model is proposed by Ramezani and Zeng (1998). In this model the positive and negative jumps are generated by two independent Poisson processes with rates λ_1 and λ_2. They argued that the connection between DEJP and PBJD is that $\lambda = \lambda_1 + \lambda_2$ and $p = \lambda_1/\lambda$. They estimate the parameters for PBJD, and in our study we use the relation they argued to compute the DEJP parameters. The data is daily returns for 100 firms in CRSP, starting from October 31st 1996 through December 31st, 1998, 547 observations in sum. Their resulting daily parameters are $\lambda_1 = 0.49$, $\lambda_2 = 0.3$, $\eta_1 = 53.82$, and $\eta_2 = 48.98$.

In order to make this setting comparable with the results under normal distribution, we need to adjust the first and second moment of DEJP so that they have the same standpoint. If $W(t)$ follows DEJP, the first and second moments of DEJP given elapsed time t are:

$$\mathrm{E}[W_t] = t\left(\mu + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)\right) = -0.0006; \tag{10}$$

$$\mathrm{E}[W_t^2] = t\left(\left(\mu + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)\right)^2 + \left(\sigma^2 + 2\lambda\left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right)\right)\right) = 0.0014. \tag{11}$$

Taking (μ, λ) as variables and $\sigma^2, \eta_1, \eta_2, p$ as given, [3] solving the two equations above gives $\mu = -0.006477$ and $\lambda = 1.87647$. To summarize, all parameters for process W_t used in this section are $\mu = 0.006477$, $\sigma_Y = 0.0014$, $\lambda_Y = 1.87647$, $p_Y = 0.3/0.79 = 0.3797$, $\eta_2^Y = 53.82$ and $\eta_1^Y = 48.98$.

In the Monte Carlo simulation study, as in the case of normal distribution in section 3.2, two correlated double exponential jump diffusion processes are simulated simultaneously. In order to simplify the procedure, we assume that the correlation exists only in the Brownian motion part, and the jump part

[3] We tried setting η_1, η_2 as variables and solve for (10) and (11). However, the solutions are not reasonable. Setting (μ, λ) as variables gives more reasonable result. The setting of σ is somewhat arbitrary, but it does not hurt a lot.

of the two processes are independent. Similar to what we do in Section 3, the daily return of Citigroup is taken as our representative for process S_t. Again from the paper of Ramezani and Zeng (1998), using the daily return of the Citigroup from October, 1996 through December, 1998, MLE gives the log double exponential diffusion parameters $\lambda_1 = 0.2$, $\lambda_2 = 0.41$, $\eta_1 = 40.91$ and $\eta_2 = 63.42$. To keep the first and second moment of the DEJP increments to be the same as the normal increments in section 3, we also change the values of (ν, λ_X) just like what we do for W_t. In sum, the parameters for S_t are $\nu = 0.0404$, $\sigma_X = 0.008$, $\lambda_X = 9.4554$, $p_X = 0.41/0.61 = 0.6721$, $\eta_2^X = 40.91$ and $\eta_1^X = 63.42$.

First we do a Monte Carlo simulation study of the conditional quantiles. The algorithm for the simulation of jump part is from Cont and Tankov (2004). The algorithm used here is very similar to Algorithm 3.1. Given the parameters for the increments of S_t and W_t, the whole trajectory of S_t is generated in each iteration, and we check if S_t ever hit the lower boundary a. If it does, we compute and record W_T. Note that for W_t we only need to know its realization at time T. Repeating this process until N samples of W_T are collected. The precise algorithm is described as follows.

Algorithm 4.1. Monte Carlo Simulation of the Conditional Qantiles Using DEJP.

1. Drawing T pairs of (X_i, Y_i) from normal distribution

$$(X_i, Y_i) \sim N\left(\begin{pmatrix} \nu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_W \\ \rho\sigma_X\sigma_W & \sigma_W^2 \end{pmatrix}\right),$$

with R command "rmvnorm" in the multivariate statistical package "mvtnorm".

2. Generating the total number of jumps of S_t in $[0, T]$ using Poisson distribution with rate $T\lambda_X$ (R command: "rpois"). Denote the total number of jumps of S_t as M_X.

3. Generating the jump times by simulating M_X uniformly distributed random variables in range $[0, T]$. (R command: "runif")

4. Generating M_X bernoulli random numbers B_i, $1 \le i \le M_X$, taking value in $\{0, 1\}$ with probability p_X. This tells us whether the jump is positive or negative. The jump size J_i, $1 \le i \le M_X$, is randomly drawn from an exponential distribution with rate η_1^X if the jump is positive ($B_i = 1$), or from an exponential distribution with rate η_2^X if the jump is negative ($B_i = 0$). (R command: "rexp")

5. The trajectory of S_t is given by

$$S_t = \sum_{i=1}^{t} (X_i + e^{J_i} - 1).$$

6. If S_t does not hit the boundary a for all $t \in [0, T]$, going back to Step 1.

7. If S_t hits the boundary a for $t \in [0, T]$, then computing W_T as follows: Generating the total number of jumps of W_t in $[0, T]$ using Poisson distribution with rate $T\lambda_Y$. Denote the total number of jumps of W_t as M_Y. Generating M_Y bernoulli random numbers C_i, $1 \le i \le M_Y$, taking

value in $\{0, 1\}$ with probability p_Y. The jump size L_i is randomly drawn from an exponential distribution with rate η_1^Y if the jump is positive ($C_i = 1$), or from an exponential distribution with rate η_2^Y if the jump is negative ($C_i = 0$). Then

$$W_T = \sum_{i=1}^{T}(Y_i + e^{L_i} - 1).$$

8. Repeat Step 1-7 until $N = 10000$ samples of W_T are collected, where N is a positive integer. Computing the $1 - \alpha = 0.95$ quantile from the empirical distribution generated by these N samples.

Table 4.1.1 shows the exact conditional quantiles by the Monte Carlo simulation using DEJP. Comparing to the exact conditional quantiles by normal simultion of Table 3.2.2, the quantiles are much smaller, regardless we have rendered the first and second moment of the two process to be equal to that of the normal simulation. This may be due to the fact that DEJP generates leptokurtic effect. DEJP has heavier tail than Black-Scholes model which is basically sum of normal random variables; therefore, the distribution generated from DEJP process is more disperse and leads to smaller quantiles.

Table 4.1.2 shows the conditional quantiles under DEJP model using (7). The patterns is very similar to that under normal distribution in Table 3.3.1. In the case of $a = 0.2, 0.5, 1.5$ the values are increasing with rising correlation, while in the case of $a = 3.0$, the values are decreasing.

However, the values in Table 4.1.2 seem to be smaller (more negative) than those in Table 3.3.1. This may be due to the fact that DEJP shows asymmetrically leptokurtic (Kou and Wang (2004)) effect, so extreme negative returns are more likely to appear in case of DEJP than in the case of normal distribution. This results in the smaller quantile values.

Table 4.1.3 presents the errors computed by substracting Table 4.1.1 DEJP exact conditional quantiles from asymptotic conditional quantiles Table 4.1.2 and then taking absolute value.

Comparing Table 4.1.3 to Table 3.3.2, the errors in both tables are likely to be smaller for a close to zero and larger for a far from zero. Generally speaking, the largest errors occur at $a = 3.0$. The errors seems to be smaller for zero and positive correlation, and larger for negative correlation. The errors tend to reduce as T increase.

Table 4.1.4 shows the absolute difference between DEJP exact quantiles and the normal asymptotic quantiles. We see that the differences are greater than in Table 4.1.3. This suggests that the choice of model may be important when computing the CoVaR. If using normal distribution to approximate non-normal models may lead to large bias.

Table 4.1.1: DEJP Monte Carlo simulation of CoVaR.

	a	$\rho=-0.9$	$\rho=-0.5$	$\rho=0$	$\rho=0.5$	$\rho=0.9$
	0.2	-0.3701	-0.3990	-0.4430	-0.4548	-0.4784
$T = 30$ days	0.5	-0.3084	-0.3628	-0.4280	-0.4810	-0.5191
	1.5	-0.1005	-0.2555	-0.4403	-0.5870	-0.6953
	0.2	-0.5319	-0.5448	-0.5884	-0.6192	-0.6388
$T = 60$ days	0.5	-0.4657	-0.5525	-0.5993	-0.6245	-0.6816
	1.5	-0.2566	-0.4008	-0.6026	-0.7460	-0.8318
	0.2	-0.6418	-0.6446	-0.7104	-0.7206	-0.7500
$T = 90$ days	0.5	-0.5649	-0.6384	-0.6983	-0.7416	-0.7892
	1.5	-0.3820	-0.5333	-0.7098	-0.8561	-0.9335
	3.0	-0.0461	-0.3584	-0.6971	-0.9979	-1.2137
	0.2	-0.7075	-0.7521	-0.7789	-0.7975	-0.8300
$T = 120$ days	0.5	-0.6569	-0.7240	-0.7868	-0.8255	-0.8680
	1.5	-0.4531	-0.6260	-0.7744	-0.9266	-1.0248
	3.0	-0.1546	-0.4602	-0.7947	-1.0719	-1.2907
	0.2	-0.7867	-0.8054	-0.8847	-0.8847	-0.9005
$T = 150$ days	0.5	-0.7258	-0.8051	-0.8652	-0.9084	-0.9517
	1.5	-0.5416	-0.6871	-0.8805	-1.0138	-1.1268
	3.0	-0.1171	-0.5340	-0.8599	-1.1613	-1.3504
	0.2	-0.8751	-0.9153	-0.9590	-0.9385	-0.9651
$T = 180$ days	0.5	-0.7963	-0.8594	-0.9374	-0.9704	-0.9851
	1.5	-0.5817	-0.7418	-0.9042	-1.0434	-1.1720
	3.0	-0.2698	-0.5798	-0.9226	-1.1906	-1.4171
	0.2	-0.8925	-0.9663	-0.9520	-0.9975	-1.0081
$T = 210$ days	0.5	-0.8486	-0.8798	-0.9821	-1.0272	-1.0517
	1.5	-0.6384	-0.8007	-0.9697	-1.1241	-1.2035
	3.0	-0.3277	-0.6542	-0.9875	-1.2619	-1.4351

Table 4.1.2: The CoVaR computed by inverting the decomposition (7) under DEJP model.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	-0.6242	-0.5634	-0.4728	-0.4600	-0.4670
$T = 30$ days	0.5	-0.7697	-0.6625	-0.5595	-0.5803	-0.6103
	1.5	-0.9957	-0.8139	-0.7833	-0.8613	-0.9304
	0.2	-0.7024	-0.6826	-0.6207	-0.5864	-0.5777
$T = 60$ days	0.5	-0.8025	-0.7319	-0.6478	-0.6509	-0.6712
	1.5	-1.0011	-0.8353	-0.8079	-0.8807	-0.9470
	0.2	-0.7943	-0.7896	-0.7453	-0.7037	-0.6840
$T = 90$ days	0.5	-0.8417	-0.8062	-0.7354	-0.7235	-0.7343
	1.5	-1.0043	-0.8632	-0.8383	-0.9050	-0.9678
	3.0	-1.0565	-0.8593	-0.9518	-1.0912	-1.2001
	0.2	-0.8828	-0.8842	-0.8516	-0.8102	-0.7841
$T = 120$ days	0.5	-0.8926	-0.8788	-0.8189	-0.7958	-0.7979
	1.5	-1.0072	-0.8950	-0.8720	-0.9323	-0.9912
	3.0	-1.0502	-0.8717	-0.9631	-1.1001	-1.2077
	0.2	-0.9646	-0.9691	-0.9443	-0.9060	-0.8771
$T = 150$ days	0.5	-0.9494	-0.9480	-0.8976	-0.8667	-0.8611
	1.5	-1.0128	-0.9293	-0.9076	-0.9612	-1.0162
	3.0	-1.0431	-0.8864	-0.9762	-1.1105	-1.2165
	0.2	-1.0400	-1.0462	-1.0270	-0.9923	-0.9630
$T = 180$ days	0.5	-1.0075	-1.0136	-0.9713	-0.9356	-0.9235
	1.5	-1.0228	-0.9650	-0.9443	-0.9915	-1.0425
	3.0	-1.0363	-0.9025	-0.9906	-1.1220	-1.2263
	0.2	-1.1098	-1.1168	-1.1018	-1.0708	-1.0422
$T = 210$ days	0.5	-1.0647	-1.0754	-1.0403	-1.0021	-0.9846
	1.5	-1.0377	-1.0014	-0.9817	-1.0225	-1.0695
	3.0	-1.0304	-0.9198	-1.0061	-1.1345	-1.2369

Table 4.1.3: The absolute errors between DEJP asymptotic conditional marginal quantiles and DEJP Monte Carlo simulated quantiles.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	0.2541	0.1644	0.0298	0.0052	0.0114
$T = 30$ days	0.5	0.4613	0.2997	0.1315	0.0993	0.0912
	1.5	0.8952	0.5584	0.3430	0.2743	0.2351
	0.2	0.1705	0.1378	0.0323	0.0328	0.0611
$T = 60$ days	0.5	0.3368	0.1794	0.0485	0.0264	0.0104
	1.5	0.7445	0.4345	0.2053	0.1347	0.1152
	0.2	0.1525	0.1450	0.0349	0.0169	0.0660
$T = 90$ days	0.5	0.2768	0.1678	0.0371	0.0181	0.0549
	1.5	0.6223	0.3299	0.1285	0.0489	0.0343
	3.0	1.0104	0.5009	0.2547	0.0933	0.0136
	0.2	0.1753	0.1321	0.0727	0.0127	0.0459
$T = 120$ days	0.5	0.2357	0.1548	0.0321	0.0297	0.0701
	1.5	0.5541	0.2690	0.0976	0.0057	0.0336
	3.0	0.8956	0.4115	0.1684	0.0282	0.0830
	0.2	0.1779	0.1637	0.0596	0.0213	0.0234
$T = 150$ days	0.5	0.2236	0.1429	0.0324	0.0417	0.0906
	1.5	0.4712	0.2422	0.0271	0.0526	0.1106
	3.0	0.9260	0.3524	0.1163	0.0508	0.1339
	0.2	0.1649	0.1309	0.0680	0.0538	0.0021
$T = 180$ days	0.5	0.2112	0.1542	0.0339	0.0348	0.0616
	1.5	0.4411	0.2232	0.0401	0.0519	0.1295
	3.0	0.7665	0.3227	0.0680	0.0686	0.1908
	0.2	0.2173	0.1505	0.1498	0.0733	0.0341
$T = 210$ days	0.5	0.2161	0.1956	0.0582	0.0251	0.0671
	1.5	0.3993	0.2007	0.0120	0.1016	0.1340
	3.0	0.7027	0.2656	0.0186	0.1274	0.1982

Table 4.1.4: The absolute errors between normal asymptotic conditional marginal quantiles and DEJP Monte Carlo simulated quantiles.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	0.2153	0.1038	0.0514	0.0607	0.0693
$T = 30$ days	0.5	0.4384	0.2722	0.0932	0.0684	0.0640
	1.5	0.8919	0.5519	0.3352	0.2678	0.2293
	0.2	0.0823	0.0106	0.1192	0.1645	0.1785
$T = 60$ days	0.5	0.3004	0.1081	0.0369	0.0442	0.0729
	1.5	0.7367	0.4139	0.1825	0.1157	0.0983
	0.2	0.0010	0.0351	0.1711	0.2077	0.2397
$T = 90$ days	0.5	0.2121	0.0526	0.0940	0.1291	0.1536
	1.5	0.6109	0.2911	0.0873	0.0143	0.0035
	3.0	1.0124	0.4875	0.2422	0.0826	0.0232
	0.2	0.0291	0.0917	0.1768	0.2293	0.2715
$T = 120$ days	0.5	0.1298	0.0010	0.1415	0.1806	0.2053
	1.5	0.5379	0.2097	0.0361	0.0463	0.0800
	3.0	0.8985	0.3898	0.1482	0.0110	0.0985
	0.2	0.0701	0.0980	0.2263	0.2644	0.2955
$T = 150$ days	0.5	0.0736	0.0501	0.1804	0.2314	0.2619
	1.5	0.4464	0.1611	0.0557	0.1229	0.1736
	3.0	0.9298	0.3207	0.0871	0.0756	0.1563
	0.2	0.1206	0.1646	0.2499	0.2693	0.3156
$T = 180$ days	0.5	0.0193	0.0731	0.2147	0.2619	0.2685
	1.5	0.4028	0.1196	0.0645	0.1415	0.2100
	3.0	0.7700	0.2801	0.0290	0.1019	0.2208
	0.2	0.1015	0.1756	0.1969	0.2825	0.3161
$T = 210$ days	0.5	0.0141	0.0632	0.2233	0.2879	0.3086
	1.5	0.3427	0.0744	0.1149	0.2108	0.2324
	3.0	0.7044	0.2114	0.0309	0.1698	0.2364

4.2. Monte Carlo Simulation with t distribution

There is an increasing belief that the joint asset returns do not follow normal distribution. Kan and Zhou (2006) run a multivariate kurtosis test on the Fama and French (1993) 25 assets returns and from January 1963 to December 2002 and rejected the null hypothesis of multivariate normality with a p-value less than 0.01. However, the multivariate kurtosis tests do not reject the null that the data are from t-distribution with degree of freedom 6 and 8, neither do the multivariate skewness tests. They further estimate the degree of freedom by the maximum likelihood method and obtain the point estimate value 8.66. Nevertheless, for robustness, they report most of their result based on degree of freedom 6,8 and 10.

In our simulation study, we follow Kan and Zhou (2006) and use degree of freedom 6 and 8 to reflect possible thick tail phenomenon in assets returns during the period of financial crisis. The Simulation again run on R. The computer we use quipped with Intel(R) Duo core T8100 2.10GHz and 2.09 GHz and 2.00 GB RAM. The simulation procedure is similar to Algorithm 3.1:

Algorithm 4.2. Monte Carlo Simulation of the Conditional Qantiles Using t Distribution.

1. Generating T 2-dimensional $\{(X_t, Y_t)\}_{t=1}^{T}$ from bivariate t distribution

$$t\left(\begin{pmatrix} \nu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_W \\ \rho\sigma_X\sigma_W & \sigma_W^2 \end{pmatrix}, df\right),$$

   where df=6,8. The command "rmvt" from the multivariate statistical package "mvtnorm" is used.

2. Computing the marginal log return at time t: $S_t = X_1 + ... + X_t$, and recording S_t for each $t = 0, 1, 2, ..., T$.

3. If $S_t \geq a$ for some t ,(in other words, $\max_{1 \leq t \leq T} S_t \geq a$) then computing $W_T = Y_1 + ...Y_T$ and recording W_T. If $S_t < a$ for all t, going back to step 1.

4. Repeating step 1 to 3 until $N = 10,000$ values of W_T are collected. Therefore, we obtain a distribution of W_T conditional on the event $S_t \geq a$ for some t based on these N samples.

5. Computing the $1 - \alpha$-th quantile out of this empirical distribution, where $\alpha = 0.05$.

The results are shown in table 4.2.1 for df=8 and table 4.2.2 for df=6. They show similar pattern to table 3.2.2, but here the quantiles are smaller, because the leptokurtic property of t distribution makes the simulated distribution more disperse. If we compare table 4.2.1 to table 4.2.2, we can also find that the quantiles simulated from $t(6)$ is smaller than that simulated from $t(8)$. Therefore, we conclude that the log return distribution with heavier tail leads to smaller exact conditional quantiles.

Table 4.2.3 and table 4.2.4 presents the numbers which are computed by substracting table 4.2.1 and table 4.2.2 DEJP exact conditional quantiles from asymptotic conditional quantiles table 3.3.1 and then taking absolute value. Since the characteristic function of t distribution has no closed form as explained in the beginning of this section, it is difficult to compute the asymptotic CoVaR. Therefore, we compare the t simulation results with the normal asymptotic CoVaR.

The two tables present that the errors do occur when using normal assumption in asymptotic CoVaR computation. This corresponds to the results of Table 4.1.4 where we compare the normal asymptotic CoVaR with the DEJP exact CoVaR. The asymptotic CoVaR underestimate the risk of heavy tail by overestimating the values of CoVaR. We can see that in table 3.3.1 the CoVaR's in positive correlation group are often higher than the corresponding CoVaR in Table 4.2.1 and Table 4.2.2. On the other hand, the CoVaR's in negative correlation group are often underestimated. This overestimates the risk if the stock log return follows t distribution.

Table 4.2.1: $t(8)$ Monte Carlo simulation of the CoVaR.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	-0.2402	-0.2999	-0.3683	-0.4046	-0.4088
$T = 30$ days	0.5	-0.0826	-0.2389	-0.3767	-0.4594	-0.4723
	1.5	0.3459	-0.0249	-0.3815	-0.6445	-0.7638
	0.2	-0.3676	-0.4580	-0.5068	-0.5306	-0.5441
$T = 60$ days	0.5	-0.2404	-0.3901	-0.5278	-0.5771	-0.5960
	1.5	0.2160	-0.1770	-0.5217	-0.7619	-0.8421
	0.2	-0.4750	-0.5535	-0.6231	-0.6420	-0.6511
$T = 90$ days	0.5	-0.3399	-0.4820	-0.6191	-0.6780	-0.7041
	1.5	0.1062	-0.2875	-0.6298	-0.8332	-0.8994
	3.0	0.7120	0.0302	-0.6130	-1.1137	-1.3732
	0.2	-0.5894	-0.6549	-0.7262	-0.7444	-0.7332
$T = 120$ days	0.5	-0.4362	-0.5876	-0.7058	-0.7745	-0.7604
	1.5	0.0125	-0.3788	-0.7001	-0.8957	-0.9791
	3.0	0.6389	-0.0683	-0.7046	-1.1854	-1.3803
	0.2	-0.6568	-0.7318	-0.7898	-0.7928	-0.8110
$T = 150$ days	0.5	-0.5292	-0.6505	-0.7796	-0.8545	-0.8553
	1.5	-0.0585	-0.4714	-0.7733	-0.9815	-1.0135
	3.0	0.5585	-0.1674	-0.7623	-1.2471	-1.4231
	0.2	-0.7343	-0.7851	-0.8743	-0.8678	-0.8611
$T = 180$ days	0.5	-0.5832	-0.7128	-0.8399	-0.8997	-0.8943
	1.5	-0.1490	-0.5372	-0.8450	-1.0366	-1.0671
	3.0	0.4817	-0.2289	-0.8411	-1.2732	-1.4383
	0.2	-0.7921	-0.8449	-0.9244	-0.9399	-0.9122
$T = 210$ days	0.5	-0.6345	-0.7971	-0.8768	-0.9628	-0.9402
	1.5	-0.2054	-0.6061	-0.9170	-1.0779	-1.0758
	3.0	0.4174	-0.2896	-0.8939	-1.2997	-1.4622

Table 4.2.2: $t(6)$ Monte Carlo simulation of the CoVaR.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	-0.2513	-0.3291	-0.3848	-0.4258	-0.4288
$T = 30$ days	0.5	-0.1069	-0.2630	-0.4011	-0.4808	-0.5135
	1.5	0.3262	-0.0575	-0.4233	-0.6802	-0.7900
	0.2	-0.4129	-0.4708	-0.5549	-0.5714	-0.5841
$T = 60$ days	0.5	-0.2583	-0.4157	-0.5574	-0.6253	-0.6259
	1.5	0.1856	-0.1981	-0.5592	-0.7921	-0.8818
	0.2	-0.5313	-0.6124	-0.6554	-0.6864	-0.6879
$T = 90$ days	0.5	-0.3896	-0.5437	-0.6560	-0.7160	-0.7228
	1.5	0.0768	-0.3307	-0.6702	-0.8706	-0.9563
	3.0	0.6908	-0.0154	-0.6716	-1.1802	-1.4030
	0.2	-0.6069	-0.6885	-0.7384	-0.7763	-0.7798
$T = 120$ days	0.5	-0.4826	-0.6332	-0.7525	-0.8178	-0.8308
	1.5	-0.0254	-0.4279	-0.7604	-0.9693	-1.0337
	3.0	0.6060	-0.1113	-0.7632	-1.2397	-1.4331
	0.2	-0.7002	-0.7705	-0.8328	-0.8369	-0.8553
$T = 150$ days	0.5	-0.5576	-0.6951	-0.8336	-0.8826	-0.8868
	1.5	-0.1196	-0.5147	-0.8444	-1.0335	-1.0759
	3.0	0.5020	-0.2088	-0.8273	-1.2934	-1.4687
	0.2	-0.7890	-0.8514	-0.9042	-0.9325	-0.9390
$T = 180$ days	0.5	-0.6332	-0.7717	-0.8996	-0.9404	-0.9614
	1.5	-0.1925	-0.5721	-0.9062	-1.0750	-1.1165
	3.0	0.4503	-0.2736	-0.8915	-1.3491	-1.4924
	0.2	-0.8598	-0.9154	-0.9717	-1.0089	-0.9949
$T = 210$ days	0.5	-0.7040	-0.8431	-0.9777	-1.0093	-1.0174
	1.5	-0.2681	-0.6431	-0.9770	-1.1400	-1.1855
	3.0	0.3663	-0.3399	-0.9758	-1.3907	-1.5345

Table 4.2.3: The absolute errors between asymptotic conditional quantiles and $t(8)$ Monte Carlo simulated exact conditional quantiles.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	0.3452	0.2029	0.0233	0.0105	0.0003
$T = 30$ days	0.5	0.6642	0.3961	0.1445	0.0900	0.1108
	1.5	1.3383	0.7825	0.3940	0.2103	0.1608
	0.2	0.2466	0.0974	0.0376	0.0759	0.0838
$T = 60$ days	0.5	0.5257	0.2705	0.0346	0.0032	0.0127
	1.5	1.2093	0.6377	0.2634	0.0998	0.0880
	0.2	0.1678	0.0560	0.0838	0.1291	0.1408
$T = 90$ days	0.5	0.4371	0.2090	0.0148	0.0655	0.0685
	1.5	1.0991	0.5369	0.1673	0.0372	0.0376
	3.0	1.7705	0.8761	0.3263	0.0332	0.1827
	0.2	0.0890	0.0055	0.1241	0.1762	0.1747
$T = 120$ days	0.5	0.3505	0.1354	0.0605	0.1296	0.0977
	1.5	1.0035	0.4569	0.1104	0.0154	0.0343
	3.0	1.6920	0.7817	0.2383	0.1025	0.1881
	0.2	0.0598	0.0244	0.1314	0.1725	0.2060
$T = 150$ days	0.5	0.2702	0.1045	0.0948	0.1775	0.1655
	1.5	0.9295	0.3768	0.0515	0.0906	0.0603
	3.0	1.6054	0.6873	0.1847	0.1614	0.2290
	0.2	0.0202	0.0344	0.1652	0.1986	0.2116
$T = 180$ days	0.5	0.2324	0.0735	0.1172	0.1912	0.1777
	1.5	0.8355	0.3242	0.0053	0.1347	0.1051
	3.0	1.5215	0.6310	0.1105	0.1845	0.2420
	0.2	0.0011	0.0542	0.1693	0.2249	0.2202
$T = 210$ days	0.5	0.2000	0.0195	0.1180	0.2235	0.1971
	1.5	0.7757	0.2690	0.0622	0.1646	0.1047
	3.0	1.4495	0.5760	0.0627	0.2076	0.2635

Table 4.2.4: The absolute errors between asymptotic conditional quantiles and $t(6)$ Monte Carlo simulated exact conditional quantiles.

	a	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	0.2	0.3341	0.1737	0.0068	0.0317	0.0197
$T = 30$ days	0.5	0.6399	0.3720	0.1201	0.0686	0.0696
	1.5	1.3186	0.7499	0.3522	0.1746	0.1346
	0.2	0.2013	0.0846	0.0857	0.1167	0.1238
$T = 60$ days	0.5	0.5078	0.2449	0.0050	0.0450	0.0172
	1.5	1.1789	0.6166	0.2259	0.0696	0.0483
	0.2	0.1115	0.0029	0.1161	0.1735	0.1776
$T = 90$ days	0.5	0.3874	0.1473	0.0517	0.1035	0.0872
	1.5	1.0697	0.4937	0.1269	0.0002	0.0193
	3.0	1.7493	0.8305	0.2677	0.0997	0.2125
	0.2	0.0715	0.0281	0.1363	0.2081	0.2213
$T = 120$ days	0.5	0.3041	0.0898	0.1072	0.1729	0.1681
	1.5	0.9656	0.4078	0.0501	0.0890	0.0889
	3.0	1.6591	0.7387	0.1797	0.1568	0.2409
	0.2	0.0164	0.0631	0.1744	0.2166	0.2503
$T = 150$ days	0.5	0.2418	0.0599	0.1488	0.2056	0.1970
	1.5	0.8684	0.3335	0.0196	0.1426	0.1227
	3.0	1.5489	0.6459	0.1197	0.2077	0.2746
	0.2	0.0345	0.1007	0.1951	0.2633	0.2895
$T = 180$ days	0.5	0.1824	0.0146	0.1769	0.2319	0.2448
	1.5	0.7920	0.2893	0.0665	0.1731	0.1545
	3.0	1.4901	0.5863	0.0601	0.2604	0.2961
	0.2	0.0688	0.1247	0.2166	0.2939	0.3029
$T = 210$ days	0.5	0.1305	0.0265	0.2189	0.2700	0.2743
	1.5	0.7130	0.2320	0.1222	0.2267	0.2144
	3.0	1.3984	0.5257	0.0192	0.2986	0.3358

4.3. Stable Distributions

Stable distributions can also generate heavy tail log returns, which is more close to the stylized facts of stock returns. The use of stable distributions to model the asset return can be trace back to Mandelbrot (1960) and Fama (1963). Mandelbrot (1960) suggests that the Gaussian distribution is incapable of capturing the extreme tails of the empirical distribution of asset returns. Fama (1963) empirically and theoretically examine Mandelbrot's stable distribution suggestion. Since then, tons of researches have been dedicated to the application of stable distribution in finance. For a thorough discussion and references about the stable distribution, Nolan (2011) is a good source.

We can use stable distribution to compute the asymptotic conditional quantiles. The stable distributions are defined by the characteristic function of the form

$$\varphi(\theta; \mu, \alpha, \beta, c) = \exp[i\theta\mu - |c\theta|^{\alpha}(1 - i\beta sgn(\theta)\Phi)], \tag{12}$$

where

$$\Phi = \begin{cases} \tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ -\frac{2}{\pi}\log|\theta|, & \alpha = 1. \end{cases}$$

By the Fourier inversion formula, the pdf of stable distribution is (Appendix D)

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \exp(-|c\theta|^{\alpha})\cos(\theta\mu - \theta x + |c\theta|^{\alpha}\beta\Phi)d\theta, \tag{13}$$

where Φ is given as above. From this formula we are able to numerically compute the asymptotic conditional quantiles.

5. Conclusion

In this study we show how to compute the CoVaR in (3) in sequential setting. We provide an asymptotic method to compute the CoVaR and compare with the Monte Carlo simulation. We do it under normal, t distribution and double exponential jump process. We can get sufficient accurate result to the CoVaR in positively correlated case. It is also shown that there is a difference in the computational time. The asymptotic method is more time-efficient.

Some topics are worth further research. First, in this study except for the normal case, we only do univariate CoVaR rather than portfolio CoVaR. However, in practice portfolio CoVaR may be of more concerns. The portfolio CoVaR may need higher dimensional renewal theory. Second, correlation may not be a sufficient tool to describe the dependence between assets. It is suggested that copula allow a more general and flexible framework in dealing with dependence. Third, we see that simulation is very time-consuming. This is due to the fact that rare event has very low probability of occuring. Multivariate efficient simulation theory can possibly provide a solution to this problem. For the Monte Carlo simulation with multivariate normal and t distribution, Glasserman et al. (2000) and Glasserman et al. (2002) discuss the importance sampling under large deviation theory. Fuh and Hu (2004) and Fuh et al. (2011) show how to do efficient simulation under moderate deviation. For general Markov process such as stochastic volatility model, Fuh and Hu (2007) also provide a efficient simulation method under moderate deviation. Finally, empirical study can be done based on our study.

Appendices

A. Multivariate Renewal Theory

The theoretical background of this study is the multivariate renewal theory developed by Keener (2006). The theory is developed in a general multivariate framework. Suppose $X_1, X_2, ...$ i.i.d. random variables, the sum

$$S_t = X_1 + ... + X_t, \quad S_0 = 0,$$

is an univariate random walk. Define

$$\tau = \inf\{t : S_t \geq a\}.$$

Suppose P is a probability measure and

$$(X_1, \mathbf{Y}_1) \sim P,$$

and the sum of $\{\mathbf{Y}_i\}_{i=1}^{\infty}$, $\mathbf{Y}_i \in \mathbb{R}^m$ forms multivariate random walk $W_t = \sum_{i=1}^{t} \mathbf{Y}_i$. There is possibly nonzero correlation between the two increments X_1 and $\mathbf{Y}_1$. $\{(X_i, \mathbf{Y}_i)\}_i$ is i.i.d. $(S_t, \mathbf{W}_t)$ forms a multivariate dimensional random walk. Suppose $\mathrm{E}X_i = \nu$ and $\mathrm{E}Y_i = \mu$, $\nu > 0$, $\mu \in \mathbb{R}^m$.

In our study, we assume that $\nu < 0$ and $a < 0$, so that the stopping time τ is well-defined. Keener's theory can still applied.

Keener (1990) and Keener (2006) derive the second order expansion with respect to the Lebesque measure for the multivariate renewal measure, which is defined by

$$R = \sum_{n=0}^{\infty} P^{*n}.$$

Where P is defined above and $*$ is the convolution operator. This problem is crucial in sequential analysis where the distribution of τ is of great concern. Sequential statistical inference application calls for the need for the asymptotic distribution of the likelihood process when the process hits a given threshold. More details can be found in Siegmund (1988).

Keener's technique is based on the Edgewarth expansion introduced in Feller (1966), which is different from Carlsson (1983) and Carlsson and Wainger (1983) who use distribution theory. Keener gets a more explicit result and its error term converge to zero with a faster rate than Carlsson and Wainger (1983). However, the price he pays is that the algebra involved is more difficult.

Before introducing Keener's theory, some conditions and notations must be introduced first.

Definition A.1 (Cramer's Condition)**.** The random vector $(X_1, \mathbf{Y}_1) \in \mathbb{R}^{1+m}$, $m \in \mathbb{N}$ satisfies Cramer's condition if

$$\limsup_{|(\xi_1,\xi_2)|\to\infty} |\mathrm{E}e^{\xi_1 X_1 + \xi_2 \cdot \mathbf{Y}_1}| < 1.$$

An immediate consequence of this definition is

$$\inf_{\mathbb{R}^{m+1}-N_0} |\mathrm{E}e^{\xi_1 X_1 + \xi_2 \cdot \mathbf{Y}_1}| < 1.$$

where N_0 is any neighborhood of 0.

Definition A.2 (Oscillation Function)**.** For any function f defined on $\mathbb{R}^n$, the oscillation function ω_f is given by

$$\omega_f(\mathbf{x};\epsilon) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \|\mathbf{x}-\mathbf{y}\| \leq \epsilon\}$$

Some notations:

$$\phi : \text{ The standard normal pdf}$$

$$\mathbf{W}_t^* = \begin{pmatrix} \mathbf{W}_t \\ t \end{pmatrix}$$

$$\gamma = \frac{\mathrm{E}\mathbf{Y}_1}{\mathrm{E}X_1} = \frac{\mu}{\nu} \tag{14}$$

$$\gamma^* = \frac{1}{\nu}\begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

$$\sigma^2 = \mathrm{E}(\mathbf{Y}_1 - \gamma X_1)^2 \tag{15}$$

$$\Sigma_* = \mathrm{E}\begin{pmatrix} \mathbf{Y}_1 - \frac{\mu}{\nu}X_1 \\ 1 - \frac{1}{\nu}X \end{pmatrix}\begin{pmatrix} \mathbf{Y}_1 - \frac{\mu}{\nu}X_1 \\ 1 - \frac{1}{\nu}X \end{pmatrix}'$$

$$\mathbf{Z}_n = \Sigma^{-1/2}(\mathbf{W}_n - \gamma S_n) \tag{16}$$

$$\mathbf{Z}^* = Z_1^* = \Sigma_*^{-1/2}\left(\begin{pmatrix} \mathbf{Y}_1 \\ 1 \end{pmatrix} - \gamma^* a\right)\sqrt{\frac{\nu}{a}}$$

$$\tilde{\mathbf{q}} = \Sigma^{-1/2}((\mathbf{w} - \gamma a)\sqrt{\nu/a}$$

$$\tilde{\mathbf{q}}^*(\mathbf{w}, t) = \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \Sigma_*^{-1/2}\left(\begin{pmatrix} \mathbf{w} \\ t \end{pmatrix} - \gamma^* a\right)\sqrt{\nu/a}$$

$$H_0(\tilde{\mathbf{q}}) = \frac{1}{6}\mathrm{E}(\tilde{\mathbf{q}}\cdot\mathbf{Z}_1)^3 - \frac{1}{2}\mathrm{E}\mathbf{Z}_1^2\tilde{\mathbf{q}}\cdot\mathbf{Z}_1 + \frac{(m+2-|\tilde{\mathbf{q}}|^2)\mathrm{E}X_1\tilde{\mathbf{q}}\cdot\mathbf{Z}_1}{2\nu} + \frac{\tilde{\mathbf{q}}\cdot\Sigma^{-1/2}\gamma}{2\nu\mathrm{E}\tau}\mathrm{E}S_\tau^2$$

$$H_0^*(\tilde{\mathbf{q}}^*) = \frac{1}{6}\mathrm{E}(\tilde{\mathbf{q}}^*\cdot\mathbf{Z}_1^*)^3 - \frac{1}{2}\mathrm{E}\mathbf{Z}_1^{*^2}\tilde{\mathbf{q}}_1^*\cdot\mathbf{Z}_1^* + \frac{(m+3-|\tilde{\mathbf{q}}^*|^2)\mathrm{E}X_1\tilde{\mathbf{q}}^*\cdot\mathbf{Z}_1^*}{2\nu} + \frac{\tilde{\mathbf{q}}^*\cdot\Sigma_*^{-1/2}\gamma^*}{2\nu\mathrm{E}\tau}\mathrm{E}S_\tau^2$$

$$\frac{dQ_0}{d\lambda_0} = \frac{\phi(\tilde{\mathbf{q}})}{\sqrt{|\Sigma|}(a/\nu)^{m/2}}\left\{1 + \sqrt{\frac{\nu}{a}}H_0(\tilde{\mathbf{q}})\right\}$$

$$\frac{dQ_0^*}{d\lambda_0^*} = \frac{\phi(\tilde{\mathbf{q}}^*)}{\sqrt{|\Sigma_*|}(a/\nu)^{(m+1)/2}}\left\{1 + \sqrt{\frac{\nu}{a}}H_0^*(\tilde{\mathbf{q}}^*)\right\}$$

Where λ_0 is the Lebesque measure on $\mathbb{R}^m$ and λ_0^* is the product measure of Lebesque measure on $\mathbb{R}^m$ and counting measure on $\mathbb{Z}$. It is obvious that $\mathbf{Z}_n$ is also a random walk with i.i.d. increments. One of the main results in this paper is given in the next theorem.

Theorem A.3 (Keener (2006), Lemma 5.1)**.** Suppose for $m \in \mathbb{N}$, $(X_1, \mathbf{Y}_1) \in \mathbb{R}^{1+m}$ satisfies Cramer's condition, $\nu > 0$, $\mathrm{E}|X_1|^{2+\delta} < \infty$ and $\mathrm{E}|\mathbf{Z}_1|^{3+\delta} < \infty$, where $\delta \in (0,1)$. Then for some $\eta > 0$,

$$\mathrm{E}f(\mathbf{W}_\tau) = \int f d\hat{Q}_0 + O(1)\int \omega_f(\cdot; e^{-\eta a})d\hat{Q}_0 + o\{a^{(-1-\delta)/2}(\log a)^{m/2}\}$$

as $a \to \infty$, uniformly for nonnegative measurable f bounded above by one. If the moment condition for X is strengthened to $\mathrm{E}X^{3+\delta}$, then for some $\eta > 0$,

$$\mathrm{E}f(\mathbf{W}_\tau^*) = \int f d\hat{Q}_0^* + O(1) \int \omega_f(\cdot; e^{-\eta a}) d\hat{Q}_0^* + o\{a^{(-1-\delta)/2}(\log a)^{(m+1)/2}\}$$

as $a \to \infty$, uniformly for nonnegative measurable f bounded above by 1.

By the theorem above, the following corollary can be derived from computation.

Corollary A.4. [Keener (2006), Corollary 5.2] Suppose $\dim(W_t) = 1$, $0 < \delta < 1$, (X_1, Y_1) satisfies Cramér's condition, $\mathrm{E}X_1 = \nu > 0$, $\mathrm{E}Y_1 = \mu \in \mathbb{R}$, $E|X_1|^{2+\delta} < \infty$ and $E|Z_1|^{3+\delta} < \infty$, $\gamma = \mathrm{E}Y_1/\nu = \mu/\nu$, then

$$P(W_\tau < w) = \Phi(\hat{w}) + \sqrt{\nu/a}\phi(\hat{w})H_1(\hat{w}) + o\{a^{(-1-\delta)/2}\sqrt{\log a}\} \tag{17}$$

as $a \to \infty$, uniformly in w, where $\hat{w} = (w - \gamma a)/(\sigma\sqrt{a/\nu})$, $\sigma^2 = \mathrm{var}(Y_1 - \gamma X_1) = \mathrm{E}Y_1^2 + \gamma^2 \mathrm{E}X_1^2 - 2\gamma \mathrm{E}Y_1 X_1$, $\mathrm{E}Z_1^3 = \sigma^{-3}(\mathrm{E}Y_1^3 - 3\gamma \mathrm{E}Y_1^2 X_1 + 3\gamma^2 \mathrm{E}Y_1 X_1^2 - \gamma^3 \mathrm{E}X_1^3)$ and $\mathrm{E}X_1 Z_1 = \sigma^{-1}(\mathrm{E}Y_1 X_1 - \gamma \mathrm{E}X_1^2)$

$$H_1(\hat{w}) = (\hat{w}^2 - 1)\left\{-\frac{1}{6}\mathrm{E}Z_1^3 + \frac{\mathrm{E}X_1 Z_1}{2\nu}\right\} - \frac{\gamma \mathrm{E}S_\tau^2}{2\nu\sigma \mathrm{E}\tau} \tag{18}$$

B. Derivation of (7)

By strong Markov property, decomposing the conditional probability

$$
\begin{aligned}
&P(W_T \le w|\tau < T) \\
&\quad = \frac{P(W_T \le w, \tau < T)}{P(\tau < T)} \\
&\quad = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(W_T \le w, \tau = t) \\
&\quad = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(W_T \le w|\tau = t) P(\tau = t) \\
&\quad = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(\tau = t) \int_{-\infty}^{\infty} P\left(W_\tau \le w - x, \sum_{j=t+1}^{T} Y_j = x|\tau = t\right) dx \\
&\quad = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(\tau = t) \int_{-\infty}^{\infty} P(W_\tau \le w - x|\tau = t) P\left(\sum_{j=t+1}^{T} Y_j = x|\tau = t\right) dx \\
&\quad = \frac{1}{P(\tau < T)} \sum_{t=1}^{T-1} P(\tau = t) \int_{-\infty}^{\infty} P(W_\tau \le w - x|\tau = t) P\left(\sum_{j=t+1}^{T} Y_j = x\right) dx
\end{aligned}
$$

C. Probability Density Function of DEJP

Using (8), since sin is odd function, the Fourier inversion formula gives that

$$
\begin{aligned}
f(x) &= \frac{1}{\pi} \int_0^\infty \mathrm{Re}[\exp(i\theta x)\exp(G(i\theta)t)]d\theta \\
&= \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\exp\left(-i\theta x + it\theta\mu - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{q\eta_2}{\eta_2 + i\theta} - 1\right)\right)\right] d\theta \\
&= \frac{1}{\pi} \int_0^\infty \exp\left(-\frac{\theta^2\sigma^2 t}{2} - \lambda t\right) \mathrm{Re}\left[\exp\left(-i\theta x + it\theta\mu + \lambda\left(\frac{p\eta_1 t(\eta_1 + i\theta)}{\eta_1^2 + \theta^2} + \frac{q\eta_2 t(\eta_2 - i\theta)}{\eta_2^2 + \theta^2}\right)\right)\right] d\theta \\
&= \frac{1}{\pi} \int_0^\infty \exp(A(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2)) \cos(B(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2)) d\theta,
\end{aligned}
$$

where

$$A(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2) = -\frac{\theta^2\sigma^2 t}{2} - \lambda t + \lambda\left(p\frac{\eta_1^2 t}{\eta_1^2+\theta^2} + q\frac{\eta_2^2 t}{\eta_2^2+\theta^2}\right);$$
$$B(\theta, t; \sigma, \lambda, p, q, \eta_1, \eta_2) = \theta\left(-x + t\mu + p\frac{\lambda\eta_1 t}{\eta_1^2+\theta^2} - q\frac{\lambda\eta_2 t}{\eta_2^2+\theta^2}\right).$$

D. Probability Density Function of Stable Distribution

By (12), since sin is odd function, the Fourier inversion formula gives that

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\exp(-i\theta x + i\theta\mu - |c\theta|^\alpha(1 - i\beta sgn(t)\Phi))]d\theta \\ &= \frac{1}{\pi}\int_0^\infty \exp(-|c\theta|^\alpha)\mathrm{Re}[\exp(i(-\theta x + \theta\mu - |c\theta|^\alpha\beta\Phi))]d\theta \\ &= \frac{1}{\pi}\int_0^\infty \exp(-|c\theta|^\alpha)\cos(-\theta x + \theta\mu - |c\theta|^\alpha\beta\Phi)d\theta \end{aligned}$$

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