國立臺灣大學理學院數學系

碩士論文

Department of Mathematics

College of Science

National Taiwan University

Master Thesis

在二項式模型中的高階誤差分析

Analysis of Higher Order Error in the Binomial Model

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中華民國 100 年 6 月

June, 2011

國立臺灣大學數學系

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黃冠閔 撰

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國立臺灣大學碩士學位論文

口試委員會審定書

在二項式模型中的高階誤差分析

Analysis of Higher Order Error in the Binomial Model

本論文係黃冠閔君（R98221005）在國立臺灣大學數學所完成之碩士學位論文，於民國 100 年 06 月 03 日承下列考試委員審查通過及口試及格，特此證明

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謝辭

首先要感謝我的指導教授彭柏堅老師這兩年來的指導，不僅指引我研究的方向並且寫論文的過程中不厭其煩的指導我與幫忙修正。另外要感謝在百忙之中撥空幫我口試的姜祖恕老師與劉淑鶯老師，使我的論文得到更完善的修正與改進。

感謝這兩年來系上各位行政同仁的幫助，使我在很好的環境裡學習，最後我要特別感謝我的家人們，一路走來，給我精神上的支持與鼓勵。

摘要

本文主要在探討在二元樹模型中的歐式買權價格收斂到 Black-Scholes(BS)模型中價格的速度，當每一分割期間的長度愈縮小。在在二元樹模型中，選擇權的價格是由股價的未來變動百分比 u 和 d 及風險中立機率(Risk-Neutral Probability)來決定。文獻一(Chang-Palmer)給出在誤差項中 $1/n$ 的確切係數。在這篇論文中，我們考慮更一般化的 u 和 d 來證明我們的主要定理，應用主要定理加強文獻一所提出的結果，將誤差項提高至 $1/n\sqrt{n}$ 項並給出 $1/n\sqrt{n}$ 的確切係數。我們也利用加強的結果在 Joshi 模型中來說明二元樹模型中的價格與 BS 模型中的價格，兩者誤差是 $O(1/n^2)$。我們也應用主要定理在 Leisen-Reimer 模型中得到一個收斂定理，在 Tian 模型中得到一個新定理。

Abstract

In this paper, we study the rate of convergence of the European call option price by the binomial model to the Black-Scholes price as the number of period n tends to infinity. The binomial option pricing is determined by the jump sizes u and d and the risk-neutral probability p. Chang and Palmer [1] gives an explicit formula for the coefficient of $1/n$ in the expansion of the error. This paper discusses the higher order in the expansion of the error. We consider more general u and d to prove the Main Theorem and apply it to strengthen the Chang-Palmer result, expanding up to the $1/n\sqrt{n}$ term in the expansion of the error and also giving an explicit formula for the coefficient of the $1/n\sqrt{n}$ term. We use the strengthened Chang-Palmer result to prove the error between the binomial price and the Black-Scholes price is $O(1/n^2)$ in Joshi's model [4]. We also use the Main Theorem to obtain a proof of the convergence rate in Leiser-Reimer's model [5] and a new theorem in Tian's model [7].

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1 Introduction

We assume the stock price S follows geometric Brownian motion and use the following notations: S_0 as initial stock price, K as the strike price, r as the risk-free interest rate, σ as the volatility, T as the time to maturity. Then the stochastic process used for derivative valuation is given by

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where $S_t = (S(t))_{t \geq 0}$ is the underlying asset and $(W_t)_{t \geq 0}$ is a standard Wiener process.

In an n-period binomial model, the time to maturity of the option (T) is divided into n equal time steps, i.e. $\Delta t = \frac{T}{n}$. If the current stock price is S, then the next period price jumps either upward to uS with probability q or downward to dS with probability $1 - q$ where $0 < d < e^{r\Delta t} < u$ and $0 < q < 1$. The standard dynamic hedge approach ensures that the risk-neutral probability p, instead of q, is relevant for pricing options. The binomial option pricing model is completely determined by the jump sizes u and d and the risk-neutral probability p. In the model of Cox, Ross, and Rubinstein [2], CRR model, three conditions, i.e. the risk-neutral mean and variance of the stock price in the next period and $ud = 1$, are utilized to determine u, d, and p as follows:

$$u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}, and \quad p = \frac{e^{r\Delta t} - d}{u - d}.$$

Concerning the convergence patterns of the binomial models, it is widely documented that the CRR binomial prices converge to the Black-Scholes price. Leisen and Reimer [5] proved that for vanilla call options the convergence is of order $O\left(\frac{1}{n}\right)$ for the CRR model, the Jarrow and Rudd [3] model, and the Tian [7] model but they did not derive the coefficient of $\frac{1}{n}$ in the expansion.

In this paper, we study the rate of the convergence of the European call option price given by the different versions of the binomial model to the Black-Scholes price as the number of periods tends to infinity. We use the following notations:

$$C_{BS} = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

as the Black-Scholes call option price at $t = 0$, and

$$C_{BS,d} = e^{-rT}\Phi(d_2)$$

as the digital call option price, where $\Phi(\cdot)$ is the standard normal distribution function and

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}.$$

Chang and Palmer [1] take $u = e^{\sigma\sqrt{\Delta t}+\lambda_n\sigma^2\Delta t}$ and $d = e^{-\sigma\sqrt{\Delta t}+\lambda_n\sigma^2\Delta t}$, where λ_n is a general bounded sequence. (It is equivalent to considering all possible choices of $u = u_n$ and $d = d_n$ such that $u_n/d_n = e^{2\sigma\sqrt{\Delta t}}$ and $n\log(u_n d_n)$ is bounded.) Note that $\lambda_n = 0$ gives the CRR model [2], $\lambda_n = r/\sigma^2$ Walsh's model [9, 10] and $\lambda_n = r/\sigma^2 - 1/2$ the Jarrow and Rudd model [3], also $\lambda = \dfrac{\log(K/S_0)}{\sigma^2 T}$ Joshi model [4]. However the models of Leiser-Reimer [5] and Tian [7] are not of this kind. Chang and Palmer [1] prove the following theorem:

Theorem: *For the n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t},$$

with λ an arbitrary bounded function of n, if the initial stock price is S_0 and the strike price is K and maturity is T, then

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A}_n - \frac{d_2\Delta_n^2}{2}\right)\frac{1}{n}\right] + o\left(\frac{1}{n}\right)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\left(\mathcal{A}'_n - \frac{(d_1-d_2)\Delta_n^2}{2}\right)\frac{1}{n}\right] + o\left(\frac{1}{n}\right),$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/K) + n\log d}{\log(u/d)}\right],$$

$$\begin{aligned}\mathcal{A}_n &= \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)(d_1 - d_2)}{6}\left(\frac{r}{\sigma^2} - \lambda\right)\\ &\quad + \frac{d_1(d_1-d_2)^2}{2}\left(\frac{r}{\sigma^2} - \lambda\right)^2,\end{aligned}$$

$$\mathcal{A}'_n = \frac{6 - d_1^2 - d_2^2}{24}(d_1 - d_2) + \frac{(d_1-d_2)^2(d_1+d_2)}{6}\left(\frac{r}{\sigma^2} - \lambda\right) - \frac{(d_1-d_2)^3}{2}\left(\frac{r}{\sigma^2} - \lambda\right)^2.$$

Here we consider more general u and d. We follow Leisen and Reimer [5] and use the risk neutral probability p_n and the "stock measure" probability $\hat{p}_n$. We also obtain the asymptotic expansion up to terms of order $1/n^{3/2}$ whereas Chang and Palmer [1] only obtained up to order $1/n$. In Chang-Palmer model, α, β, $\hat{\alpha}$, $\hat{\beta}$ below are all determined by λ(see p.17). Since β, $\hat{\beta}$ are arbitrary, our model here also includes the models of Tian and Leisen-Reimer. Our Main Theorem is as follows:

Main Theorem: *For the n-period binomial model, suppose*

$$p_n = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad \hat{p}_n = \frac{1}{2} + \frac{\hat{\alpha}}{\sqrt{n}} + \frac{\hat{\beta}}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right)$$

where α, β, $\hat{\alpha}$, $\hat{\beta}$ are constants or bounded functions of n, and $\hat{\alpha} - \alpha = \sigma\sqrt{T}/2$.

Then if

$$u_n = e^{r\Delta t}\frac{\hat{p}_n}{p_n} \quad \text{and} \quad d_n = \frac{e^{r\Delta t} - p_n u_n}{1 - p_n}. \tag{1}$$

and the initial stock price is S_0 *and the strike price is* K *and maturity is* T*,*

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A} - \frac{d_2\Delta_n^2}{2}\right)\frac{1}{n} + \left(\mathcal{B}\Delta_n + \frac{(d_2^2-1)\Delta_n^3}{6}\right)\frac{1}{n^{\frac{3}{2}}}\right]$$

$$+O\left(\frac{1}{n^2}\right)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\left(\mathcal{A}' - \frac{(d_1-d_2)\Delta_n^2}{2}\right)\frac{1}{n} + \frac{d_1^2-d_2^2}{6}(\Delta_n^3 - \Delta_n)\frac{1}{n^{\frac{3}{2}}}\right] + O\left(\frac{1}{n^2}\right),$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/K) + n\log d}{\log(u/d)}\right],$$

$$d_1 = \frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and $\mathcal{A}$*,* $\mathcal{B}$*,* $\mathcal{A}'$*,* $\mathcal{B}'$*, are constants or bounded functions of* n *depending on* α*,* $\hat{\alpha}$*,* β*,* $\hat{\beta}$*,* d_1*,* d_2*.*

Remark 1 *Here* p_n *is the usual risk-neutral probability or measure, and* $\hat{p}_n$ *is the "stock measure". Under the assumption of a risk-neutral world, we observe that the price* S_{m-1}*, of the stock at time-step* $(m-1)\Delta t$ *is the expected value of the stock at time-step* $m\Delta t$ *discounted by the risk-free interest rate:*

$$S_{m-1} = E(e^{-r\Delta t}S_m)$$

In the "stock measure", we dicount using the stock price. So we require:

$$E(S_m^{-1}e^{r\Delta t}) = S_{m-1}^{-1}.$$

This implies that

$$\hat{p}\left(\frac{e^{r\Delta t}}{S_{m-1}u}\right) + (1-\hat{p})\left(\frac{e^{r\Delta t}}{S_{m-1}d}\right) = \frac{1}{S_{m-1}},$$

where $\hat{p}$ denotes the probability of an up move in the stock measure. In fact,

$$\hat{p} = \frac{e^{-r\Delta t} - d^{-1}}{u^{-1} - d^{-1}} = \frac{pu}{e^{r\Delta t}}$$

where p is the risk-neutral probability. Since $p = \frac{e^{r\Delta t}-d}{u-d}$, we solve these 2 equations for u and d to get (1).

We apply the Main Theorem to obtain a strengthening of the Chang-Palmer result in which we derive the asymptotic expansion of the European call option price given by the binomial model up to terms of order $1/n^{3/2}$. We also apply the Main Theorem to Joshi's model and Leiser-Reimer's model and we obtain a new theorem for Tian's model. These theorems are as follows:

Theorem 1:(strengthening of Chang Palmer) *For the n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad \Delta t = T/n$$

with λ an arbitrary bounded function of n, if the initial stock price is S_0 and the strike price is K and maturity is T,

(1) *the price of a digital call option satisfies*

$$\begin{aligned} C_d(n) &= C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A}_n - \frac{d_2\Delta_n^2}{2n}\right)\frac{1}{n} + \left(\mathcal{B}_n\Delta_n + \frac{(d_2^2-1)\Delta_n^3}{6}\right)\frac{1}{n^{\frac{3}{2}}}\right] \\ &+O\left(\frac{1}{n^2}\right) \end{aligned}$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\left(\mathcal{A}'_n - \frac{(d_1 - d_2)\Delta_n^2}{2} \right) \frac{1}{n} + + \frac{d_1^2 - d_2^2}{6} (\Delta_n^3 - \Delta_n) \frac{1}{n^{\frac{3}{2}}} \right] + O\left(\frac{1}{n^2}\right),$$

where, in general, $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{A}'_n$ are bounded functions of n.

Theorem 2:(Joshi's model) *For the odd n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t} + \lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t} + \lambda\sigma^2\Delta t}$$

with $\lambda = \dfrac{\log(K/S_0)}{\sigma^2 T}$, *if the initial stock price is S_0 and the strike price is K and maturity is T, then*

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + \frac{e^{-rT} e^{-d_2^2/2}}{\sqrt{2\pi}} \frac{\mathcal{A}}{n} + O\left(\frac{1}{n^2}\right)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \frac{\mathcal{A}'}{n} + O\left(\frac{1}{n^2}\right),$$

where $\mathcal{A}$, $\mathcal{A}'$ are certain constants.

Theorem 3:(Leiser-Reimer's model) *For the odd n-period binomial model, where*

$$p_n = \frac{1}{2} + \frac{d_2}{2\sqrt{n}} - \frac{d_2}{8n^{\frac{3}{2}}} - \frac{d_2^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad \hat{p}_n = \frac{1}{2} + \frac{d_1}{2\sqrt{n}} - \frac{d_1}{8n^{\frac{3}{2}}} - \frac{d_1^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right),$$

if the initial stock price is S_0 and the strike price is K and maturity is T, then

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + O\left(\frac{1}{n^2}\right)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + O\left(\frac{1}{n^2}\right).$$

Theorem 4:(Tian's model) *For the n-period binomial model, where*

$$u = \frac{MV}{2}(V + 1 + \sqrt{V^2 + 2V - 3}), \quad d = \frac{MV}{2}(V + 1 - \sqrt{V^2 + 2V - 3}).$$

with

$$M = e^{r\Delta t} \quad and \quad V = e^{\sigma^2 \Delta t},$$

then

(1) *the price of a digital call option satisfies*

$$\begin{aligned} C_d(n) = \; & e^{-rT}\Phi(d_2) + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A} - \frac{d_2\Delta_n^2}{2n}\right)\frac{1}{n} + \left(\mathcal{B}\Delta_n + \frac{(d_2^2-1)\Delta_n^3}{6}\right)\frac{1}{n^{\frac{3}{2}}}\right] \\ & + O\left(\frac{1}{n^2}\right) \end{aligned}$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\left(\mathcal{A}' - \frac{(d_1 - d_2)\Delta_n^2}{2}\right)\frac{1}{n} + \frac{d_1^2 - d_2^2}{6}(\Delta_n^3 - \Delta_n)\frac{1}{n^{\frac{3}{2}}}\right] + O\left(\frac{1}{n^2}\right),$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ are certain constants.

2 Main Theorem

In this section, we prove the Main Theorem as stated in the Introduction. First from Pliska [6], we have that in the binomial model with n periods and parameters u, d, r, when the current stock price is S_0, the price of a European call option with maturity T and strike price K is given by

$$C(n) = S_0 \sum_{k=j}^{n} \binom{n}{k} \hat{p}^k \hat{q}^{n-k} - Ke^{-rT} \sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k},$$

provided that $0 < p < 1$, where $p = \dfrac{e^{r\Delta t} - d}{u - d}$, $q = 1 - p$, $\hat{p} = pue^{-r\Delta t}$, $\hat{q} = 1 - \hat{p}$ and j is the integer such that

$$S_0 u^{j-1} d^{n-j+1} < K \leq S_0 u^j d^{n-j}.$$

The binomial price of a digital call option with maturity T and strike K is

$$C_d(n) = e^{-rT} \sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k},$$

a component of the binomial European call price, and its Black-Scholes price is

$$C_{BS,d} = e^{-rT} \Phi(d_2),$$

a component of the European call price

$$C_{BS} = S\Phi(d_1) - Ke^{-rT} \Phi(d_2).$$

Thus, we first prove part (1) of our Main Theorem, and then prove part (2).

Our fundamental tool is the following Modified Lemma, which is an extension of a result of Uspensky (1937, p. 120) on approximating the binomial distribution by the normal distribution.

Modified Lemma : *Provided that* $p = p_n = 1/2 + \frac{\alpha}{\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right)$ *as* $n \to \infty$ *and* $0 \le j = j_n \le n+1$ *for* n *sufficiently large,*

$$\sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k} = \frac{1}{\sqrt{2\pi}} \int_{\xi_1}^{\xi_2} e^{-\frac{u^2}{2}} du + \frac{q-p}{6\sqrt{2\pi npq}} \Big((1-\xi_2^2) e^{-\frac{\xi_2^2}{2}} - (1-\xi_1^2) e^{-\frac{\xi_1^2}{2}} \Big)$$

$$+ \frac{1}{12n\sqrt{2\pi}} (\xi_2 e^{-\frac{1}{2}\xi_2^2}(\xi_2^2 - 1) - \xi_1 e^{-\frac{1}{2}\xi_1^2}(\xi_1^2 - 1)) + O\left(\frac{1}{n^2}\right)$$

where $\xi_1 = \frac{j - np - 1/2}{\sqrt{npq}}$ *and* $\xi_2 = \frac{nq + 1/2}{\sqrt{npq}}$.

The proof of the lemma is in Appendix 1.

Remark 2 *This is a strengthening of Lemma in Chang and Palmer [1]. There it was stated that the error was* $o(1/n)$*. Here we show that the error is in fact* $O(1/n^2)$*, if* $p = 1/2 + \frac{\alpha}{\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right)$*. Note that Chang and Palmer actually need to assume* $p_n = 1/2 + O(1/\sqrt{n})$*, not just* $p_n \to \frac{1}{2}$*.*

Actually we use the following corollary of the lemma.

Corollary: *Suppose*

$$p_n = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{3/2}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right) \quad \text{and} \quad j_n = \frac{1-b_n}{2} + \tilde{\gamma}\sqrt{n} + \frac{n}{2} + \frac{\tilde{a}}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

where b_n *and* α, β, $\tilde{\gamma}$, $\tilde{a}$ *are bounded functions of* n*. Then*

$$\sum_{k=j_n}^{n} \binom{n}{k} p_n^k q_n^{n-k} = \Phi(d) + \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \left\{ \frac{b_n}{\sqrt{n}} + (\tilde{A} - db_n^2/2)\frac{1}{n} + \left[\tilde{B} + \tilde{C}b_n^2\right] \frac{b_n}{n^{\frac{3}{2}}} \right\} + O\left(\frac{1}{n^2}\right),$$

where

$$
\begin{aligned}
d &= 2(\alpha - \tilde{\gamma}), \\
\tilde{A} &= 2(\beta + \alpha^2 d) + \left(\frac{2\alpha}{3} - \frac{d}{12}\right)(1 - d^2) - 2\tilde{a}, \\
\tilde{B} &= 2(\alpha^2 - \beta d - \alpha^2 d^2 + \tilde{a} d) + \frac{2}{3}\alpha(d^3 - 3d) - \frac{d^4 - 4d^2 + 1}{12}, \\
\tilde{C} &= \frac{d^2 - 1}{6} \qquad \textit{and} \qquad \Phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}t^2} dt.
\end{aligned}
$$

The proof of the corollary is in Appendix 2.

Our main theorem is stated in terms of the risk neutral probability p_n and the "stock measure" $\hat{p}_n$, that is, the martingale measure corresponding to the stock price as numeraire. These are defined by

$$
p_n = \frac{r_n - d_n}{u_n - d_n} \quad \text{and} \quad \hat{p}_n = \frac{p_n u_n}{r_n}, \quad \text{where} \quad r_n = e^{r\Delta t}.
$$

So we have

$$
u_n = r_n \frac{\hat{p}_n}{p_n} \quad \text{and} \quad d_n = \frac{r_n - p_n u_n}{1 - p_n}. \tag{2}
$$

Main Theorem: *For the n-period binomial model, suppose*

$$
p_n = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad \hat{p}_n = \frac{1}{2} + \frac{\hat{\alpha}}{\sqrt{n}} + \frac{\hat{\beta}}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right)
$$

where α, β, $\hat{\alpha}$, $\hat{\beta}$ are constants or bounded functions of n, with $\hat{\alpha} - \alpha = \sigma\sqrt{T}/2$. Then if the initial stock price is S_0 and the strike price is K and maturity is T,

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A} - \frac{d_2\Delta_n^2}{2}\right)\frac{1}{n} + \left(\mathcal{B}\Delta_n + \frac{(d_2^2-1)\Delta_n^3}{6}\right)\frac{1}{n^{\frac{3}{2}}}\right]$$

$$+O\left(\frac{1}{n^2}\right)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\left(\mathcal{A}' - \frac{(d_1-d_2)\Delta_n^2}{2}\right)\frac{1}{n} + \frac{d_1^2-d_2^2}{6}(\Delta_n^3 - \Delta_n)\frac{1}{n^{\frac{3}{2}}}\right] + O\left(\frac{1}{n^2}\right),$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/K) + n\log d_n}{\log(u_n/d_n)}\right],$$

$$d_1 = \frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

$$\mathcal{A} = \frac{2d_1(\beta-\hat{\beta})}{(d_1-d_2)} - 2d_1\alpha^2 + \frac{2}{3}(1-2d_1^2+d_1d_2)\alpha + \frac{d_2(d_2^2-1)-(d_1-d_2)^2(3d_1+d_2)}{12},$$

$$\mathcal{B} = \frac{2d_1d_2(\hat{\beta}-\beta)}{d_1-d_2} + 2(1+d_1d_2)\alpha^2 + \frac{2d_2}{3}(2d_1^2-d_1d_2-3)\alpha$$

$$+\frac{1}{12}(3d_1^3d_2 - 5d_1^2d_2^2 + d_1d_2^3 + 4d_2^2 - 1),$$

$$\mathcal{A}' = 2(\hat{\beta}-\beta) + 2(d_1-d_2)\alpha^2 + \frac{2}{3}(2d_1^2 - 3d_1d_2 + d_2^2)\alpha$$

$$+\frac{1}{12}(d_1-d_2)(3d_1^2 - 5d_1d_2 + d_2^2 + 3),$$

Proof of Main Theorem:

Proof of (1): First we write

$$p_n = \frac{1}{2} + \alpha x + \beta x^3 + O(x^5), \quad \hat{p}_n = \frac{1}{2} + \hat{\alpha} x + \hat{\beta} x^3 + O(x^5), \quad \text{with} \quad x = \frac{1}{\sqrt{n}}.$$

Using Taylor expansion, we obtain

$$\begin{aligned} u_n &= r_n \frac{\hat{p}_n}{p_n} \\ &= \frac{e^{rTx^2}\left(\frac{1}{2} + \hat{\alpha} x + \hat{\beta} x^3 + O(x^5)\right)}{\frac{1}{2} + \alpha x + \beta x^3 + O(x^5)} = 1 + ax + bx^2 + cx^3 + dx^4 + O\left(x^5\right) \\ &= 1 + \frac{a}{\sqrt{n}} + \frac{b}{n} + \frac{c}{n^{\frac{3}{2}}} + \frac{d}{n^2} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \end{aligned}$$

where

$$\begin{aligned} a &= 2(\hat{\alpha} - \alpha), \\ b &= 4\alpha(\alpha - \hat{\alpha}) + rT, \\ c &= 2(\hat{\beta} - \beta) + (\hat{\alpha} - \alpha)(8\alpha^2 + 2rT), \\ d &= 4\alpha(\beta - \hat{\beta}) + (\alpha - \hat{\alpha})(16\alpha^3 + 4\beta + 4\alpha rT) + \frac{r^2T^2}{2}, \end{aligned}$$

and

$$
\begin{aligned}
d_n &= \frac{r_n - p_n u_n}{1 - p_n} \\
&= \frac{e^{rTx^2} - \left(\dfrac{1}{2} + \alpha x + \beta x^3 + O(x^5)\right) \dfrac{e^{rTx^2}\left(\frac{1}{2} + \hat{\alpha} x + \hat{\beta} x^3 + O(x^5)\right)}{\frac{1}{2} + \alpha x + \beta x^3 + O(x^5)}}{\dfrac{1}{2} - \alpha x - \beta x^3 + O(x^5)} \\
&= 1 - ax + bx^2 - cx^3 + dx^4 + O\left(x^5\right) \\
&= 1 - \frac{a}{\sqrt{n}} + \frac{b}{n} - \frac{c}{n^{\frac{3}{2}}} + \frac{d}{n^2} + O\left(\frac{1}{n^{\frac{5}{2}}}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\log(u_n) &= \log\left(1 + ax + bx^2 + cx^3 + dx^4 + O(x^5)\right), \text{ with } x = \tfrac{1}{\sqrt{n}} \\
&= Ax + Bx^2 + Cx^3 + Dx^4 + O(x^5) \\
&= \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{\frac{3}{2}}} + \frac{D}{n^2} + O\left(\frac{1}{n^{\frac{5}{2}}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A &= a = 2(\hat{\alpha} - \alpha), \\
B &= -\frac{a^2 - 2b}{2} = 2(\alpha^2 - \hat{\alpha}^2) + rT, \\
C &= \frac{a^3 - 3ab + 3c}{3} = 2(\hat{\beta} - \beta) + \tfrac{8}{3}(\hat{\alpha}^3 - \alpha^3), \\
D &= -\frac{a^4 - 4a^2b + 4ac + 2b^2 - 4d}{4} = 4(\alpha\beta - \hat{\alpha}\hat{\beta}) + 4(\alpha^4 - \hat{\alpha}^4),
\end{aligned}
$$

and

$$\log(d_n) = \log\left(1 - ax + bx^2 - cx^3 + dx^4 + O(x^5)\right), \text{ with } x = \tfrac{1}{\sqrt{n}}$$

$$= -Ax + Bx^2 - Cx^3 + Dx^4 + O(x^5)$$

$$= -\frac{A}{\sqrt{n}} + \frac{B}{n} - \frac{C}{n^{\frac{3}{2}}} + \frac{D}{n^2} + O\left(\frac{1}{n^{\frac{5}{2}}}\right).$$

Now

$$C_d(n) = \sum_{k=j_n}^{n} \binom{n}{k} p_n^k q_n^{n-k}$$

where $j = j_n$ satisfies

$$S_0 u^{j-1} d^{n-j+1} < K \leq S_0 u^j d^{n-j}.$$

Dividing by S_0 and taking logs:

$$(j-1)\log u_n + (n-j+1)\log d_n < \log(K/S_0) \leq j \log u_n + (n-j)\log d_n$$

$$j \log\left(\frac{u_n}{d_n}\right) - \log u_n + (n+1)\log d_n < \log(K/S_0) \leq j \log\left(\tfrac{u_n}{d_n}\right) + n \log d_n$$

$$\log(K/S_0) - n\log d_n \leq j \log(\tfrac{u_n}{d_n}) < \log(K/S_0) + \log(\tfrac{u_n}{d_n}) - n \log d_n$$

$$\frac{\log(K/S_0) - n\log d_n}{\log(\frac{u_n}{d_n})} \leq j < \frac{\log(K/S_0) - n\log d_n}{\log(\frac{u_n}{d_n})} + 1$$

So

$$\gamma \leq j < \gamma + 1, \quad \text{where} \quad \gamma = \frac{\log(K/S_0) - n\log d_n}{\log(\frac{u_n}{d_n})}.$$

Then

$$-j \leq -\gamma < -j + 1 \quad \text{so that} \quad -j = \text{int}[-\gamma].$$

Using

$$\Delta_n = 1 - 2\,\text{frac}\,[-\gamma]\,, \qquad -\gamma = \text{int}[-\gamma] + \text{frac}[-\gamma], \tag{3}$$

we get

$$j = \gamma + \frac{1-\Delta_n}{2}. \tag{4}$$

Now with $x = \frac{1}{\sqrt{n}}$,

$$\begin{aligned}
\gamma &= \frac{\log(K/S_0) - n\log(d_n)}{\log(u_n/d_n)} \\
&= \frac{\log(K/S_0) - \frac{(-Ax + Bx^2 - Cx^3 + Dx^4 + O(x^5))}{x^2}}{\left(Ax + Bx^2 + Cx^3 + Dx^4 + O(x^5)\right) - \left(-Ax + Bx^2 - Cx^3 + Dx^4 + O(x^5)\right)} \\
&= \frac{\log(K/S_0) - \frac{(-Ax + Bx^2 - Cx^3 + Dx^4 + O(x^5))}{x^2}}{2Ax + 2Cx^3 + O(x^5)} \\
&= \frac{1}{2x^2} + \frac{\log(K/S_0) - B}{2Ax} - \frac{C\log(K/S_0) + AD - BC}{2A^2}x + O(x^3) \\
&= \frac{n}{2} + S\sqrt{n} + \tilde{D}\frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right),
\end{aligned} \tag{5}$$

where

$$S = -\frac{1}{2}d_2 + \alpha = -\frac{1}{2}d_1 + \hat{\alpha},$$

$$\tilde{D} = \frac{d_1\hat{\beta} - d_2\beta}{d_1 - d_2} + (d_1 + d_2)\alpha^2 + \frac{1}{3}(d_1 - d_2)(2d_1 + d_2)\alpha + \frac{1}{24}(d_1 - d_2)^2(3d_1 + d_2).$$

Then using (4)

$$j_n = \frac{1-\Delta_n}{2} + \gamma = \frac{1-\Delta_n}{2} + \frac{n}{2} + S\sqrt{n} + \frac{\tilde{D}}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Now we apply the Corollary with $b_n = \Delta_n$, $\tilde{\gamma} = S$, $\tilde{a} = \tilde{D}$:

$$e^{rT}C_d(n) = \sum_{k=j_n}^{n} \binom{n}{k} p_n^k q_n^{n-k}$$

$$= \Phi(d_2) + \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \left\{ \frac{\Delta_n}{\sqrt{n}} + \left[2(\beta + \alpha^2 d_2) - 2\tilde{D} + \left(\frac{2}{3}\alpha - \frac{d_2}{12}\right)(1-d_2^2) - \frac{d_2\Delta_n^2}{2}\right] \frac{1}{n}\right.$$

$$+\Delta_n \left[2\alpha^2 - 2d_2((\beta + \alpha^2 d_2) - \tilde{D}) + \frac{(d_2^2-1)\Delta_n^2}{6} + \frac{2}{3}\alpha(d_2^3 - 3d_2)\right.$$

$$\left.\left. + \frac{(-d_2^4 + 4d_2^2 - 1)}{12}\right] \frac{1}{n^{\frac{3}{2}}} \right\} + O\left(\frac{1}{n^2}\right). \tag{6}$$

Multiplying by e^{-rT} and simplifying, we obtain part (1) of the Main Theorem.

Proof of (2):

Applying the Corollary again, this time with $\hat{p}_n$ instead of p_n,

$$\sum_{k=j_n}^{n} \binom{n}{k} \hat{p}_n^k \hat{q}_n^{n-k}$$

$$= \Phi(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left\{ \frac{\Delta_n}{\sqrt{n}} + \left[2(\hat{\beta} + \hat{\alpha}^2 d_1) - 2\tilde{D} + \left(\frac{2}{3}\hat{\alpha} - \frac{d_1}{12} \right)(1 - d_1^2) - \frac{d_1 \Delta_n^2}{2} \right] \frac{1}{n} \right.$$

$$+ \Delta_n \left[2\hat{\alpha}^2 - 2d_1((\hat{\beta} + \hat{\alpha}^2 d_1) - \tilde{D}) + \frac{(d_1^2 - 1)\Delta_n^2}{6} + \frac{2}{3}\hat{\alpha}(d_1^3 - 3d_1) \right.$$

$$\left. \left. + \frac{(-d_1^4 + 4d_1^2 - 1)}{12} \right] \frac{1}{n^{\frac{3}{2}}} \right\} + O\left(\frac{1}{n^2} \right). \tag{7}$$

Now multiplying (7) by S_0, (6) by Ke^{-rT}, subtracting and using the well-known fact that

$$S_0 e^{-d_1^2/2} = K e^{-rT} e^{-d_2^2/2},$$

we get

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left\{ \left[2[(\hat{\beta} - \beta) + (\hat{\alpha}^2 d_1 - \alpha^2 d_2)] \right. \right.$$

$$+ \frac{(d_2 - d_1)\Delta_n^2}{2} + \left(\frac{2}{3}\hat{\alpha} - \frac{d_1}{12} \right)(1 - d_1^2) - \left(\frac{2}{3}\alpha - \frac{d_2}{12} \right)(1 - d_2^2) \left. \right] \frac{1}{n}$$

$$+ \left[2(\hat{\alpha}^2 - \alpha^2)\Delta_n + 2(d_2\beta - d_1\hat{\beta})\Delta_n + 2(\alpha^2 d_2^2 - \hat{\alpha}^2 d_1^2)\Delta_n + 2(d_1 - d_2)\tilde{D}\Delta_n \right.$$

$$+ \frac{(d_1^2 - d_2^2)\Delta_n^3}{6} + \frac{2}{3}[\hat{\alpha}(d_1^3 - 3d_1) - \alpha(d_2^3 - 3d_2)]\Delta_n$$

$$+ \frac{(d_1^2 - d_2^2)[4 - (d_1^2 + d_2^2)]\Delta_n}{12} \frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^2} \right).$$

Then after simplifying the Main Theorem is proved.

3 Strengthening of Chang-Palmer result

In this section, we strengthen the main theorem of Chang and Palmer [1]. They expanded only up to the $\frac{1}{n}$ term but here we expand up to the $\frac{1}{n^{\frac{3}{2}}}$ term and show that the remaining error is $O\left(\frac{1}{n^2}\right)$. This means that in Chang-Palmer [1],CP binomial model , the error is in fact $O\left(\frac{1}{n\sqrt{n}}\right)$, not just $o\left(\frac{1}{n}\right)$. Note here we have expressed the constants as polynomials in $\left(\frac{r}{\sigma^2} - \lambda\right)$ with coefficients expressed in terms of just d_1 and d_2.

Theorem 1: *For the n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad \Delta t = T/n \tag{8}$$

with λ an arbitrary bounded function of n, if the initial stock price is S_0 and the strike price is K and maturity is T,

(1) *the price of a digital call option satisfies*

$$\begin{aligned} C_d(n) &= C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A}_n - \frac{d_2\Delta_n^2}{2n}\right)\frac{1}{n} + \left(\mathcal{B}_n\Delta_n + \frac{(d_2^2-1)\Delta_n^3}{6}\right)\frac{1}{n^{\frac{3}{2}}}\right] \\ &+O\left(\frac{1}{n^2}\right) \end{aligned} \tag{9}$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\left(\mathcal{A}'_n - \frac{(d_1-d_2)\Delta_n^2}{2}\right)\frac{1}{n} + + \frac{d_1^2-d_2^2}{6}(\Delta_n^3 - \Delta_n)\frac{1}{n^{\frac{3}{2}}}\right] + O\left(\frac{1}{n^2}\right), \tag{10}$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/K) + n\log d}{\log(u/d)}\right],$$

$$d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$$\begin{aligned}
\mathcal{A}_n &= \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)(d_1 - d_2)}{6}\left(\frac{r}{\sigma^2} - \lambda\right) \\
&+ \frac{d_1(d_1 - d_2)^2}{2}\left(\frac{r}{\sigma^2} - \lambda\right)^2, \\
\mathcal{B}_n &= \frac{1}{24}(-d_1 d_2^3 - d_2^2 - (d_1^3 - 6d_1)d_2 + 3d_1^2 - 2) \\
&+ \frac{1}{6}(d_1^2 - d_2^2)(d_1 d_2 - 3)\left(\frac{r}{\sigma^2} - \lambda\right) + \frac{(d_1 - d_2)^2(1 - d_1 d_2)}{2}\left(\frac{r}{\sigma^2} - \lambda\right)^2, \\
\mathcal{A}_n' &= \frac{6 - d_1^2 - d_2^2}{24}(d_1 - d_2) + \frac{(d_1 - d_2)^2(d_1 + d_2)}{6}\left(\frac{r}{\sigma^2} - \lambda\right) - \frac{(d_1 - d_2)^3}{2}\left(\frac{r}{\sigma^2} - \lambda\right)^2.
\end{aligned}$$

Proof of Theorem 1: First we expand the risk neutral probability p in powers of $\Delta t^{1/2}$ up to order 3. Using Taylor expansion,

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{1}{2} + \frac{\alpha}{n^{1/2}} + \frac{\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right)$$

where

$$\alpha = \frac{r - (\lambda + 1/2)\sigma^2}{2\sigma}\sqrt{T}, \quad \beta = \frac{\sigma^4(4\lambda + 1) - 4\sigma^2 r + 12(r - \lambda\sigma^2)^2}{48\sigma}T^{3/2}$$

and

$$\hat{p} = \frac{u - ude^{-r\Delta t}}{u-d} = \frac{1}{2} + \frac{\hat{\alpha}}{n^{1/2}} + \frac{\hat{\beta}}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right),$$

where

$$\hat{\alpha} = \frac{r-(\lambda-1/2)\sigma^2}{2\sigma}\sqrt{T}, \quad \hat{\beta} = \frac{\sigma^4(4\lambda-1)-4\sigma^2 r-12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}.$$

Then applying the Main Theorem, we find that

$$\begin{aligned}
\mathcal{A}_n \;&= \frac{2d_1}{d_1-d_2}\left(\left(\frac{\sigma^4(4\lambda+1)-4\sigma^2 r+12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)\right.\\
&\quad -\left(\frac{\sigma^4(4\lambda-1)-4\sigma^2 r-12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)-2d_1\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right)^2\\
&\quad \left.+\frac{2}{3}(1-2d_1^2+d_1d_2)\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right)\right)\\
&+\frac{d_2(d_2^2-1)-(d_1-d_2)^2(3d_1+d_2)}{12}\\
&=\frac{d_1^3+d_1d_2^2+2d_2-4d_1}{24}+\frac{(2-d_1d_2-d_1^2)(d_1-d_2)}{6}\left(\frac{r}{\sigma^2}-\lambda\right)\\
&+\frac{d_1(d_1-d_2)^2}{2}\left(\frac{r}{\sigma^2}-\lambda\right)^2,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_n &= \frac{2d_1d_2}{d_1-d_2}\left(\left(\frac{\sigma^4(4\lambda-1)-4\sigma^2r-12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)\right. \\
&\quad \left.-\left(\frac{\sigma^4(4\lambda+1)-4\sigma^2r+12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)\right) \\
&\quad +2(1+d_1d_2)\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right)^2 \\
&\quad +\frac{2d_2}{3}(2d_1^2-d_1d_2-3)\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right) \\
&\quad\quad +\frac{1}{12}(3d_1^3d_2-5d_1^2d_2^2+d_1d_2^3+4d_2^2-1) \\
&= \frac{1}{24}(-d_1d_2^3-d_2^2-(d_1^3-6d_1)d_2+3d_1^2-2)+\frac{1}{6}(d_1^2-d_2^2)(d_1d_2-3)\left(\frac{r}{\sigma^2}-\lambda\right) \\
&\quad\quad +\frac{(d_1-d_2)^2(1-d_1d_2)}{2}\left(\frac{r}{\sigma^2}-\lambda\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}'_n \quad &= 2\left(\left(\frac{\sigma^4(4\lambda-1)-4\sigma^2 r-12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)\right. \\
&\left.-\left(\frac{\sigma^4(4\lambda+1)-4\sigma^2 r+12(r-\lambda\sigma^2)^2}{48\sigma}T^{3/2}\right)\right) \\
&+2(d_1-d_2)\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right)^2 \\
&+\frac{2}{3}(2d_1^2-3d_1d_2+d_2^2)\left(\frac{r-(\lambda+1/2)\sigma^2}{2\sigma}\sqrt{T}\right) \\
&+\frac{1}{12}(d_1-d_2)(3d_1^2-5d_1d_2+d_2^2+3) \\
&= \frac{6-d_1^2-d_2^2}{24}(d_1-d_2)+\frac{(d_1-d_2)^2(d_1+d_2)}{6}\left(\frac{r}{\sigma^2}-\lambda\right)-\frac{(d_1-d_2)^3}{2}\left(\frac{r}{\sigma^2}-\lambda\right)^2.
\end{aligned}
$$

Now we apply the Main Theorem to get the conclusion.

Next we verify the conclusions of Theorem 1 using an example. We suppose $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and take $\lambda = 0$. According to Theorem 1, the n -period binomial call price has the form

$$C(n) = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}} + O\left(\frac{1}{n^2}\right)$$

so that

$$n\left(C(n) - C_{BS}\right) - A_n \to 0 \quad \text{as} \quad n \to \infty$$

and

$$n\sqrt{n}\left(C(n) - C_{BS} - \frac{A_n}{n}\right) - B_n \to 0 \quad \text{as} \quad n \to \infty$$

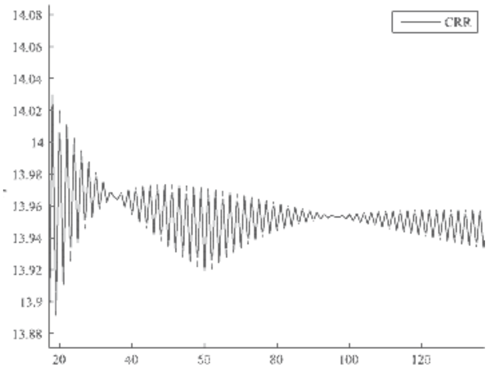

Figure 1:

A graph for European call price, using CRR binomial model. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$,$\lambda = 0$.

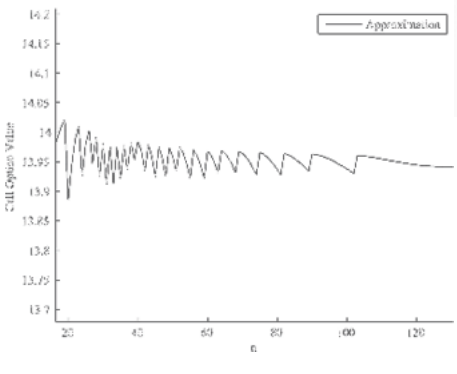

Figure 2:

A graph for the approximation $C(n) = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}$ to the European call price. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$,$\lambda = 0$.

$Approximation = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}$.

$$A_n = \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\frac{6-d_1^2-d_2^2}{24}(d_1-d_2) + \frac{(d_1-d_2)^2(d_1+d_2)}{6}\left(\frac{r}{\sigma^2}\right) - \frac{(d_1-d_2)^3}{2}\left(\frac{r}{\sigma^2}\right)^2 - \frac{(d_1-d_2)\Delta_n^2}{2} \right].$$

$$B_n = \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\frac{d_1^2-d_2^2}{6}(\Delta_n^3 - \Delta_n) \right].$$

In table 1, we compute C_{BS}, $C(n)$, coefficients of $1/n$ and $1/n\sqrt{n}$, A_n and B_n respectively for given S_0, K, σ, r, T and λ. From table 1 we observe that $n\left(C(n) - C_{BS}\right) - A_n \to 0$ as n$\to \infty$ and $n\sqrt{n}\left(C(n) - C_{BS} - \frac{A_n}{n}\right) - B_n \to 0$ as $n \to \infty$. Notice that here $C(n)$ is the CRR price and we define $\left(C(n) - C_{BS}\right) n = A'_n$and $\left(C(n) - C_{BS} - \frac{A_n}{n}\right) n\sqrt{n} = B'_n$.

Time Steps (n)	$\mathbf{C_{BS}}$	$\mathbf{C(n)}$	$\mathbf{C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}}$	$\mathbf{A_n}$	$\mathbf{A'_n}$	$\mathbf{B_n}$	$\mathbf{B'_n}$
100	13.946121	13.954663	13.954657	0.895689	0.854190	-0.420965	-0.414985
500	13.946121	13.947487	13.947486	-0.231115	-0.212959	0.453990	0.456740
1000	13.946121	13.945073	13.945073	-1.041243	-1.048084	-0.221060	-0.216321
5000	13.946121	13.945940	13.945940	-0.901634	-0.905281	-0.259721	-0.257896

Table 1

$S_0 = 100, K = 95, \sigma = 0.2, r = 0.06, T = 1$ **with** $\lambda = 0$

4 Joshi's model

In this section, we introduce Joshi's model as described in Joshi [4]. In his model, Joshi only considers an odd number $2N+1$ of steps. He takes u and d in the form

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad \lambda = \frac{\log(K/S_0)}{\sigma^2 T}$$

so that K is in the middle of the binomial tree, that is,

$$K = S_0 e^{\lambda\sigma^2 T} = S_0 e^{(2N+1)\lambda\sigma^2\Delta t} = S_0 u^{N+\frac{1}{2}} d^{N+\frac{1}{2}}.$$

Then

$$\begin{aligned}
\Delta_n &= 1 - 2\text{frac}\left[\frac{\log(S_0/K) + n\log d}{\log(u/d)}\right] \\
&= 1 - 2\text{frac}\left[\frac{-\lambda\sigma^2 T + (2N+1)(-\sigma\sqrt{\Delta t}) + \lambda\sigma^2 T}{2\sigma\sqrt{\Delta t}}\right] \\
&= 1 - 2\text{frac}\left[-\frac{2N+1}{2}\right] \\
&= 1 - 2\cdot\frac{1}{2} \\
&= 0.
\end{aligned}$$

Applying Theorem 1, we obtain the following theorem.

Theorem 2: *For the odd n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t} \tag{11}$$

with $\lambda = \dfrac{\log(K/S_0)}{\sigma^2 T}$, *if the initial stock price is* S_0 *and the strike price is* K *and maturity is* T, *then*

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{\mathcal{A}}{n} + O\left(\frac{1}{n^2}\right) \tag{12}$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\frac{\mathcal{A}'}{n} + O\left(\frac{1}{n^2}\right), \tag{13}$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$$\mathcal{A} = \frac{d_1(d_1^2 + d_1 d_2 + d_2^2) + 3d_2}{12},$$

$$\mathcal{A}' = \frac{d_1 - d_2}{12}(3 - d_1^2 - d_1 d_2 - d_2^2),$$

Proof of Theorem 2: Here we apply Theorem 1 to obtain:

$$
\begin{aligned}
\mathcal{A} &= \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)(d_1 - d_2)}{6}\left(\frac{r}{\sigma^2} - \left(\frac{\log(K/S_0)}{\sigma^2 T}\right)\right) \\
&+ \frac{d_1(d_1 - d_2)^2}{2}\left(\frac{r}{\sigma^2} - \left(\frac{\log(K/S_0)}{\sigma^2 T}\right)\right)^2 \\
&= \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)(d_1 - d_2)}{6}\left(\frac{d_1 + d_2}{2(d_1 - d_2)}\right) \\
&+ \frac{d_1(d_1 - d_2)^2}{2}\left(\frac{d_1 + d_2}{2(d_1 - d_2)}\right)^2 \\
&= \frac{d_1(d_1^2 + d_1 d_2 + d_2^2) + 3d_2}{12},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}' &= \frac{6 - d_1^2 - d_2^2}{24}(d_1 - d_2) + \frac{(d_1 - d_2)^2(d_1 + d_2)}{6}\left(\frac{r}{\sigma^2} - \left(\frac{\log(K/S_0)}{\sigma^2 T}\right)\right) \\
&- \frac{(d_1 - d_2)^3}{2}\left(\frac{r}{\sigma^2} - \left(\frac{\log(K/S_0)}{\sigma^2 T}\right)\right)^2 \\
&= \frac{6 - d_1^2 - d_2^2}{24}(d_1 - d_2) + \frac{(d_1 - d_2)^2(d_1 + d_2)}{6}\left(\frac{d_1 + d_2}{2(d_1 - d_2)}\right) \\
&- \frac{(d_1 - d_2)^3}{2}\left(\frac{d_1 + d_2}{2(d_1 - d_2)}\right)^2 \\
&= \frac{d_1 - d_2}{12}(3 - d_1^2 - d_1 d_2 - d_2^2),
\end{aligned}
$$

and using $\Delta_n = 0$, we get the conclusion.

Next we verify the conclusions of Theorem 2 using an example. We suppose $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and take $\lambda = \dfrac{\log(K/S_0)}{\sigma^2 T}$. According to Theorem 2, the odd-n -period binomial call price is

$$C(n) = C_{BS} + \frac{A}{n} + O\left(\frac{1}{n^2}\right)$$

so that

$$n\left(C(n) - C_{BS}\right) - A \to 0 \quad \text{as} \quad n \to \infty$$

and

$$n\sqrt{n}\left(C(n) - C_{BS} - \frac{A}{n}\right) \to 0 \quad \text{as} \quad n \to \infty$$

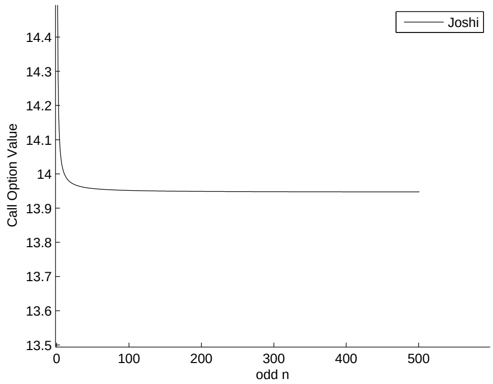

Figure 3:

A graph for European call price, using Joshi binomial model. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

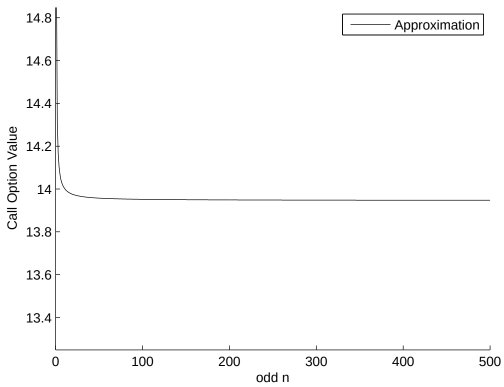

Figure 4:

A graph for the approximation $C(n) = C_{BS} + \frac{A}{n}$ to the European call price. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

$$\begin{aligned} Approximation &= C_{BS} + \tfrac{A}{n}. \\ A &= \tfrac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\tfrac{d_1-d_2}{12}(3 - d_1^2 - d_1 d_2 - d_2^2)\right]. \end{aligned}$$

In table 2, we compute C_{BS}, $C_{Joshi}(n)$, the coefficient of $1/n$, A, for given S_0, K, σ, r, T and λ. From table 2 we observe that $n\left(C_{Joshi}(n) - C_{BS}\right) - A \to 0$ and $n\sqrt{n}\left(C_{Joshi}(n) - C_{BS} - \frac{A}{n}\right) \to 0$ as $n \to \infty$. Notice that here $C_{Joshi}(n)$ is the Joshi price and we define $\left(C_{Joshi}(n) - C_{BS}\right)n = A'$and $\left(C_{Joshi}(n) - C_{BS} - \frac{A}{n}\right)n\sqrt{n} = B'$.

Time Steps (n)	$\mathbf{C_{BS}}$	$\mathbf{C_{Joshi}(n)}$	$\mathbf{C_{BS} + \frac{A}{n}}$	**A**	$\mathbf{A'}$	$\mathbf{B'}$
101	13.94612136	13.95705283	13.95705950	1.104752530	1.104078864	-0.006770264
501	13.94612136	13.95161600	13.95161764	1.104752530	1.104423829	-0.004660151
1001	13.94612136	13.94722494	13.94722500	1.104752530	1.104688117	-0.002037945
5001	13.94612136	13.94634226	13.94634226	1.104752530	1.104739702	-0.000907187

Table 2

$$S_0 = 100, K = 95, \sigma = 0.2, r = 0.06, T = 1 \quad \textbf{with} \quad \lambda = \frac{\log(K/S_0)}{\sigma^2 T}$$

5 Leisen-Reimer's quadratic convergence model

Leisen and Reimer [5] developed a method in which the parameters u, d and p of the binomial tree can be altered in order to get better convergence behavior. Instead of choosing the parameters p, u and d to get convergence to the normal distribution, Leisen-Reimer suggest to use inversion formulae reversing the standard method - they use normal approximations to determine the probability in the binomial distribution. CRR showed convergence of their model to the Black-Scholes formula as follows

$$B(j; n, p) = \sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k} \to \Phi(d_2)$$

as $n \to \infty$, where $B(j; n, p)$ is binomial distribution function and d_2 is replaced by d_1 if p is replaced by $\hat{p} = pue^{-r\Delta t}$.

Usually, a binomial calculated true probability $P = 1 - B(j; n, p)$ is approximated by the standard normal function $\Phi(z)$, where z is determined by some adjustment function $z = h(j; n, p)$ where j is such that

$$S_0 u^{j-1} d^{n-j+1} < K \leq S_0 u^j d^{n-j}.$$

By the de Moivre-Laplace theorem, P is approximated by

$$\Phi\left(h(j; n, p) = (j - np)/(np(1-p))^{\frac{1}{2}}\right)$$

. Here we assume $n = 2j + 1$ is odd (so that K is in the middle of the tree) and we solve $B(j; n, p) = \Phi(d_2)$ and $B(j; n, \hat{p}) = \Phi(d_1)$ for p and $\hat{p}$. Peizer and Pratt [11] derived the inversion formula. In particular, they suggest the following approximate inversion formulae to replace $\hat{p}$ and p by $h^{-1}(d_1)$ and $h^{-1}(d_2)$ respectively.

Peizer-Pratt-Inversion formula 2 $(n = 2j + 1)$

$$h^{-1}(z) = \frac{1}{2} + sign(z)\frac{1}{2}\left(1 - exp\left(-\left(\frac{z}{n + \frac{1}{3} + \frac{0.1}{n+1}}\right)^2\left(n + \frac{1}{6}\right)\right)\right)^{\frac{1}{2}}$$

We expand the exponential and take the square root, getting

$$h^{-1}(z) = \frac{1}{2} + \frac{z}{2\sqrt{n}} - \frac{z}{8n^{\frac{3}{2}}} - \frac{z^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right),$$

Putting $z = d_2$ and $z = d_1$ respectively, we have

$$p_n = \frac{1}{2} + \frac{d_2}{2\sqrt{n}} - \frac{d_2}{8n^{\frac{3}{2}}} - \frac{d_2^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad \hat{p}_n = \frac{1}{2} + \frac{d_1}{2\sqrt{n}} - \frac{d_1}{8n^{\frac{3}{2}}} - \frac{d_1^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right).$$

We apply the Main Theorem to show quadratic convergence.

Theorem 3: *For the odd n-period binomial model, where*

$$p_n = \frac{1}{2} + \frac{d_2}{2\sqrt{n}} - \frac{d_2}{8n^{\frac{3}{2}}} - \frac{d_2^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad \hat{p}_n = \frac{1}{2} + \frac{d_1}{2\sqrt{n}} - \frac{d_1}{8n^{\frac{3}{2}}} - \frac{d_1^3}{8n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right), \quad (14)$$

with u_n, d_n as in (2) on page 8, if the initial stock price is S_0 and the strike price is K and maturity is T, then

(1) *the price of a digital call option satisfies*

$$C_d(n) = C_{BS,d} + O\left(\frac{1}{n^2}\right) \quad (15)$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + O\left(\frac{1}{n^2}\right). \quad (16)$$

Proof of Theorem 3: We apply the Main Theorem. Here we have

$$\alpha = \frac{d_2}{2}, \quad \beta = \frac{-d_2 - d_2^3}{8}, \quad \hat{\alpha} = \frac{d_1}{2}, \quad \hat{\beta} = \frac{-d_1 - d_1^3}{8}.$$

Then from (5),

$$\gamma = \frac{n}{2} + S\sqrt{n} + \tilde{D}\frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) = \frac{n}{2} + \tilde{D}\frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

where

$$S = -\frac{d_2}{2} + \alpha = 0,$$

$$\tilde{D} = \frac{d_1\left(\frac{-d_1 - d_1^3}{8}\right) - d_2\left(\frac{-d_2 - d_2^3}{8}\right)}{d_1 - d_2} + (d_1 + d_2)\left(\frac{d_2}{2}\right)^2$$

$$+\frac{1}{3}(d_1 - d_2)(2d_1 + d_2)\left(\frac{d_2}{2}\right) + \frac{1}{24}(d_1 - d_2)^2(3d_1 + d_2)$$

$$= -\frac{d_1 + d_2}{8}.$$

Since n is odd, we have from (3)

$$\Delta_n = 1 - 2\text{frac}[-\gamma]$$

$$= 1 - 2\text{frac}\left[-\left(\frac{n}{2} + \tilde{D}\frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)\right]$$

$$= 1 - 2\left(\frac{1}{2} - \tilde{D}\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

$$= \frac{2\tilde{D}}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Using (4), it follows that

$$j = \gamma + \frac{1 - \Delta_n}{2} = \frac{n+1}{2} + O\left(\frac{1}{n^{3/2}}\right)$$

so that $n = 2j - 1$. This is in conflict with what was assumed in the Peizer Pratt Inversion formula 2 but does not affect the validity of Theorem 3.

Next

$$
\begin{aligned}
\mathcal{A} &= \frac{2d_1}{d_1-d_2}\left(\left(\frac{-d_2-d_2^3}{8}\right)-\left(\frac{-d_1-d_1^3}{8}\right)\right)-2d_1\left(\frac{d_2}{2}\right)^2 \\
&\quad +\frac{2}{3}(1-2d_1^2+d_1d_2)\left(\frac{d_2}{2}\right)+\frac{d_2(d_2^2-1)-(d_1-d_2)^2(3d_1+d_2)}{12}, \\
&= \frac{d_1+d_2}{4} = -2\tilde{D}, \\
\mathcal{B} &= \frac{2d_1d_2}{d_1-d_2}\left(\left(\frac{-d_1-d_1^3}{8}\right)-\left(\frac{-d_2-d_2^3}{8}\right)\right)+2(1+d_1d_2)\left(\frac{d_2}{2}\right)^2 \\
&\quad +\frac{2d_2}{3}(2d_1^2-d_1d_2-3)\left(\frac{d_2}{2}\right)+\frac{1}{12}(3d_1^3d_2-5d_1^2d_2^2+d_1d_2^3+4d_2^2-1) \\
&= -\frac{2d_2^2+3d_1d_2+1}{12}, \\
\mathcal{A}' &= 2(\left(\frac{-d_1-d_1^3}{8}\right)-\left(\frac{-d_2-d_2^3}{8}\right))+2(d_1-d_2)\left(\frac{d_2}{2}\right)^2 \\
&\quad +\frac{2}{3}(2d_1^2-3d_1d_2+d_2^2)\left(\frac{d_2}{2}\right)+\frac{1}{12}(d_1-d_2)(3d_1^2-5d_1d_2+d_2^2+3) \\
&= 0.
\end{aligned}
$$

Then applying the Main Theorem, the price of a digital call option satisfies

$$
\begin{aligned}
C_d(n) &= C_{BS,d} + \frac{e^{-rT} e^{-d_2^2/2}}{\sqrt{2\pi}} \left[\frac{\Delta_n}{\sqrt{n}} + \left(-2\tilde{D} - \frac{d_2 \Delta_n^2}{2} \right) \frac{1}{n} \right. \\
&\qquad \left. + \Delta_n \left(\mathcal{B} + \tfrac{d_2^2 - 1}{6} \Delta_n^2 \right) \tfrac{1}{n^{\frac{3}{2}}} \right] + O\left(\frac{1}{n^2} \right) \\
&= C_{BS,d} + \frac{e^{-rT} e^{-d_2^2/2}}{\sqrt{2\pi}} \left[\frac{2\tilde{D}}{n} + \left(\frac{-2\tilde{D}}{n} \right) + O\left(\frac{1}{n^2} \right) \right] + O\left(\frac{1}{n^2} \right) \\
&= C_{BS,d} + O\left(\frac{1}{n^2} \right),
\end{aligned}
$$

and the price of a European call option satisfies

$$
\begin{aligned}
C(n) &= C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\left(0 - \frac{(d_1 - d_2)}{2} \left(\frac{2\tilde{D}}{\sqrt{n}} \right)^2 \right) \frac{1}{n} + \frac{d_1^2 - d_2^2}{6} (\Delta_n^3 - \Delta_n) \frac{1}{n^{\frac{3}{2}}} \right] \\
&\quad + O\left(\frac{1}{n^2} \right) \\
&= C_{BS} + O\left(\frac{1}{n^2} \right).
\end{aligned}
$$

The theorem is proved.

Next we verify the conclusions of Theorem 3 using an example. We suppose $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$. According to Theorem 3, the odd-n -period binomial call price is

$$
C(n) = C_{BS} + O\left(\frac{1}{n^2} \right)
$$

so that

$$
n\sqrt{n}\,(C(n) - C_{BS}) \to 0 \text{ as } n \to \infty.
$$

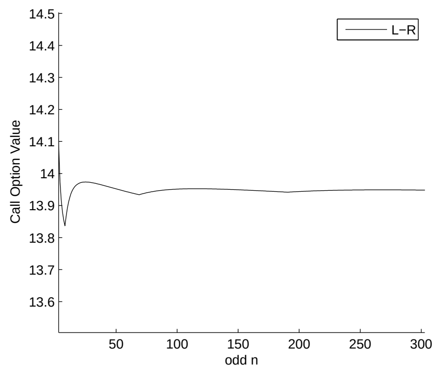

Figure 5:

A graph for European call price, using L-R binomial model. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

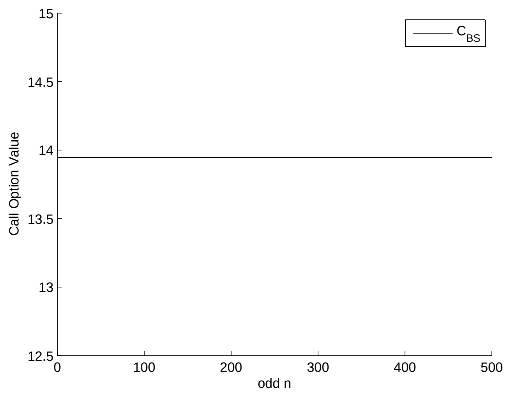

Figure 6:

A graph for the approximation $C(n) = C_{BS}$ to the European call price. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

$$Approximation \quad = C_{BS}.$$

In table 3, we compute C_{BS}, and $C_{L-R}(n)$ for given S_0, K, σ, r, T. From table 3, we observe that $n\sqrt{n}\,(C_{L-R}(n) - C_{BS}) \to 0$ as $n \to \infty$ and it also suggests that there exists a constant $B \approx -1.38$ such that $C(n) = C_{BS} + \frac{B}{n^2} + O\left(\frac{1}{n^{5/2}}\right)$. Notice that here $C_{L-R}(n)$ is the Leisen-Reimer price.

Time Steps (n)	$\mathbf{C_{BS}}$	$\mathbf{C_{L-R}(n)}$	$\mathbf{(C_{L-R}(n) - C_{BS})\,n^{3/2}}$	$\mathbf{(C_{L-R}(n) - C_{BS})\,n^{2}}$
101	13.94612136	13.945985465	-0.13793395	-1.386219016
501	13.94612136	13.946115857	-0.06166117	-1.380163704
1001	13.94612136	13.946119979	-0.04359863	-1.379399021
5001	13.94612136	13.946121301	-0.01949700	-1.378783987

Table 3

$$S_0 = 100, K = 95, \sigma = 0.2, r = 0.06, T = 1.$$

6 Tian's model

In this section, we introduce Tian's model as described in Tian [7]. In a risk neutral world, the stock price is assumed to follow the following stochastic process,

$$d \log S(t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

where r and σ, both constant, are the instantaneous proportional drift and volatility rates, respectively. It follows immediately that the stock price is lognormally distributed. Specifically, $log[S(t)/S(0)]$ is normally distributed with $(r - \sigma^2/2)t$ and variance $\sigma^2 t$. In general, the mth non-central moment of the stock price $S(t)$ is given by the fowllowing formula

$$E[S(t)^m | S_0] = S_0^m \exp\left[\left(mr + m(m-1)\frac{\sigma^2}{2}\right) t\right] \tag{17}$$

$$\begin{array}{ccc} & & Su \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & Sd \end{array}$$

The binomial branching process is illustrated in figure above, where p is a probability that the price rises to Su and q is a probability that the price falls to Sd . This binomial lattice has a total of four parameters, u, d, p, and q. These parameters uniquely determine the evolution of stock prices, which in turn determine a unique value of an option on the stock. However, these parameters cannot be chosen arbitrarily, because the option price obtained may not converge to the proper limit. The binomial parameters must be selected such that the discrete-time distribution converges to the lognormal distribution of the stock price in continuous time. According to Lindeberg's Central Limit Theorem, the following conditions are sufficient to ensure this convergence: (a) the probabilities (p, and q) are positive in the limit between 0 and 1 but not equal to either 0 or 1; (b) the probabilities sum to 1; (c) jumps (u and d) are independent of the stock price level; (d) the mean of the binomial distribution is equal to the mean of the lognormal distribution; (e) the variance of the binomial distribution is equal to the variance of the lognormal distribution. Since here are four unknowns, u, d, p, and q in the above equations and for a unique solution one more equation is needed. Ideally the additional restriction can be solved by the above formula to achieve that the third moment of the discrete-time process is also correct according to the continuous-time process. Mathematically, using (17) with $m = 1, 2, 3$, these restrictions are

$$\begin{aligned}
&p + q = 1, \quad 0 < p, \quad q < 1 \\
&pu + qd = M \\
&pu^2 + qd^2 = M^2V \\
&pu^3 + qd^3 = M^3V^3
\end{aligned}$$

where $M = \exp(r\Delta t)$, $V = \exp(\sigma^2\Delta t)$ and $\Delta t = T/n$.

The solution for the four binomial parameters is as follows

$$\begin{aligned}
&u = \frac{MV}{2}(V + 1 + \sqrt{V^2 + 2V - 3}), \quad d = \frac{MV}{2}(V + 1 - \sqrt{V^2 + 2V - 3}) \\
&p = \frac{M - d}{u - d}, \qquad q = 1 - p = \frac{u - M}{u - d}
\end{aligned}$$

Using this u and d, we apply the Main Theorem to obtain the new theorem.

Theorem 4: *For the n-period binomial model, where*

$$u = \frac{MV}{2}(V + 1 + \sqrt{V^2 + 2V - 3}), \quad d = \frac{MV}{2}(V + 1 - \sqrt{V^2 + 2V - 3}).$$

with

$$M = e^{r\Delta t} \quad \text{and} \quad V = e^{\sigma^2 \Delta t},$$

then

(1) *the price of a digital call option satisfies*

$$C_d(n) = \; e^{-rT}\Phi(d_2) + \frac{e^{-rT}e^{-d_2^2/2}}{\sqrt{2\pi}} \left[\frac{\Delta_n}{\sqrt{n}} + \left(\mathcal{A} - \frac{d_2 \Delta_n^2}{2n} \right) \frac{1}{n} + \left(\mathcal{B}\Delta_n + \frac{(d_2^2 - 1)\Delta_n^3}{6} \right) \frac{1}{n^{\frac{3}{2}}} \right]$$

$$+ O\left(\frac{1}{n^2}\right) \tag{18}$$

and

(2) *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\left(\mathcal{A}' - \frac{(d_1 - d_2)\Delta_n^2}{2} \right) \frac{1}{n} + \frac{d_1^2 - d_2^2}{6} (\Delta_n^3 - \Delta_n) \frac{1}{n^{\frac{3}{2}}} \right] + O\left(\frac{1}{n^2}\right), \tag{19}$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/K) + n \log d}{\log(u/d)} \right],$$

$$d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

$$\mathcal{A} = \frac{1}{12}(2d_1d_2^2 + (5 - 7d_1^2)d_2 + 6d_1^3 - 6d_1),$$

$$\mathcal{B} = -\frac{1}{24}(4d_1d_2^3 + (1 - 14d_1^2)d_2^2 + (12d_1^3 + 18d_1)d_2 - 27d_1^2 + 2),$$

$$\mathcal{A}' = \frac{1}{12}(2d_2^3 - 9d_1d_2^2 + (13d_1^2 - 3)d_2 - 6d_1^3 + 3d_1).$$

Proof of Theorem 4:

First we let $V = e^x$, with $x = \sigma^2 \Delta t$. So

$$\begin{aligned} p &= \frac{M-d}{u-d} = \frac{1 - \frac{V}{2} - \frac{V^2}{2} + \frac{V}{2}\sqrt{V^2 + 2V - 3}}{V\sqrt{V^2 + 2V - 3}} \\ &= \frac{1 - \frac{e^x}{2} - \frac{e^{2x}}{2} + \frac{e^x}{2}\sqrt{e^{2x} + 2e^x - 3}}{e^x\sqrt{e^{2x} + 2e^x - 3}} \\ &= \frac{1}{2} - \frac{3}{4}\sqrt{x} + \frac{13}{32}x^{\frac{3}{2}} + O(x^{\frac{5}{2}}) \\ &= \frac{1}{2} - \frac{3}{4}\sigma\sqrt{T}\frac{1}{\sqrt{n}} + \frac{13}{32}\sigma^3 T^{\frac{3}{2}}\frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right) \end{aligned}$$

and

$$\hat{p} = \frac{pu}{M} = \frac{\left(1 - \frac{V}{2} - \frac{V^2}{2} + \frac{V}{2}\sqrt{V^2+2V-3}\right)(V+1+\sqrt{V^2+2V-3})}{2\sqrt{V^2+2V-3}}$$

$$= \frac{\left(1 - \frac{e^x}{2} - \frac{e^2x}{2} + \frac{e^x}{2}\sqrt{e^{2x}+2e^x-3}\right)(e^x+1+\sqrt{e^{2x}+2e^x-3})}{2\sqrt{e^{2x}+2e^x-3}}$$

$$= \frac{1}{2} - \frac{1}{4}\sqrt{x} - \frac{1}{32}x^{\frac{3}{2}} + O(x^{\frac{5}{2}})$$

$$= \frac{1}{2} - \frac{1}{4}\sigma\sqrt{T}\frac{1}{\sqrt{n}} - \frac{1}{32}\sigma^3 T^{\frac{3}{2}}\frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right)$$

Then

$$\alpha = -\frac{3}{4}\sigma\sqrt{T}, \quad \beta = \frac{13}{32}\sigma^3 T^{\frac{3}{2}}, \quad \hat{\alpha} = -\frac{1}{4}\sigma\sqrt{T}, \quad \hat{\beta} = -\frac{1}{32}\sigma^3 T^{\frac{3}{2}}.$$

and here we use the fact that $\sigma\sqrt{T} = d_1 - d_2$ to obtain:

$$
\begin{aligned}
\mathcal{A} &= \frac{2d_1}{d_1-d_2}\left(\left(\frac{13}{32}\sigma^3T^{\frac{3}{2}}\right)-\left(-\frac{1}{32}\sigma^3T^{\frac{3}{2}}\right)\right)-2d_1\left(-\frac{3}{4}\sigma\sqrt{T}\right)^2 \\
&\quad +\tfrac{2}{3}(1-2d_1^2+d_1d_2)\left(-\frac{3}{4}\sigma\sqrt{T}\right)+\frac{d_2(d_2^2-1)-(d_1-d_2)^2(3d_1+d_2)}{12} \\
&= \frac{2d_1}{d_1-d_2}\left(\frac{14}{32}(d_1-d_2)^3\right)-2d_1\left(-\frac{3}{4}(d_1-d_2)\right)^2 \\
&\quad +\frac{2}{3}(1-2d_1^2+d_1d_2)\left(-\frac{3}{4}(d_1-d_2)\right)+\frac{d_2(d_2^2-1)-(d_1-d_2)^2(3d_1+d_2)}{12} \\
&= \frac{1}{12}(2d_1d_2^2+(5-7d_1^2)d_2+6d_1^3-6d_1),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B} &= \frac{2d_1d_2}{d_1-d_2}\left(\left(-\frac{1}{32}\sigma^3T^{\frac{3}{2}}\right)-\left(\frac{13}{32}\sigma^3T^{\frac{3}{2}}\right)\right)+2(1+d_1d_2)\left(-\frac{3}{4}\sigma\sqrt{T}\right)^2 \\
&\quad +\frac{2d_2}{3}(2d_1^2-d_1d_2-3)\left(-\frac{3}{4}\sigma\sqrt{T}\right)+\frac{1}{12}(3d_1^3d_2-5d_1^2d_2^2+d_1d_2^3+4d_2^2-1) \\
&= \frac{2d_1d_2}{d_1-d_2}\left(-\frac{14}{32}(d_1-d_2)^3\right)+2(1+d_1d_2)\left(-\frac{3}{4}(d_1-d_2)\right)^2 \\
&\quad +\frac{2d_2}{3}(2d_1^2-d_1d_2-3)\left(-\frac{3}{4}(d_1-d_2)\right)+\frac{1}{12}(3d_1^3d_2-5d_1^2d_2^2+d_1d_2^3+4d_2^2-1) \\
&= -\frac{1}{24}(4d_1d_2^3+(1-14d_1^2)d_2^2+(12d_1^3+18d_1)d_2-27d_1^2+2),
\end{aligned}
$$

$$\mathcal{A}' = 2\left(\left(-\frac{1}{32}\sigma^3 T^{\frac{3}{2}}\right) - \left(\frac{13}{32}\sigma^3 T^{\frac{3}{2}}\right)\right) + 2(d_1 - d_2)\left(-\frac{3}{4}\sigma\sqrt{T}\right)^2$$

$$+\frac{2}{3}(2d_1^2 - 3d_1 d_2 + d_2^2)\left(-\frac{3}{4}\sigma\sqrt{T}\right) + \frac{1}{12}(d_1 - d_2)(3d_1^2 - 5d_1 d_2 + d_2^2 + 3)$$

$$= 2\left(-\frac{14}{32}(d_1 - d_2)^3\right) + 2(d_1 - d_2)\left(-\frac{3}{4}(d_1 - d_2)\right)^2$$

$$+\frac{2}{3}(2d_1^2 - 3d_1 d_2 + d_2^2)\left(-\frac{3}{4}(d_1 - d_2)\right) + \frac{1}{12}(d_1 - d_2)(3d_1^2 - 5d_1 d_2 + d_2^2 + 3)$$

$$= \frac{1}{12}(2d_2^3 - 9d_1 d_2^2 + (13d_1^2 - 3)d_2 - 6d_1^3 + 3d_1),$$

Now using the Main Theorem, we get the conclusion.

Next we verify the conclusions of Theorem 4 using an example. We suppose $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$. According to Theorem 4, the n -period binomial call price has the form

$$C(n) = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}} + O\left(\frac{1}{n^2}\right)$$

so that

$$n\left(C(n) - C_{BS}\right) - A_n \to 0 \quad \text{as } n \to \infty$$

and

$$n\sqrt{n}\left(C(n) - C_{BS} - \frac{A_n}{n}\right) - B_n \to 0 \quad \text{as} \quad n \to \infty.$$

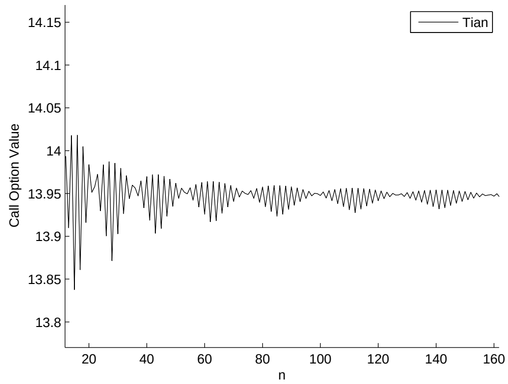

Figure 7:

A graph for European call price, using Tian binomial model. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

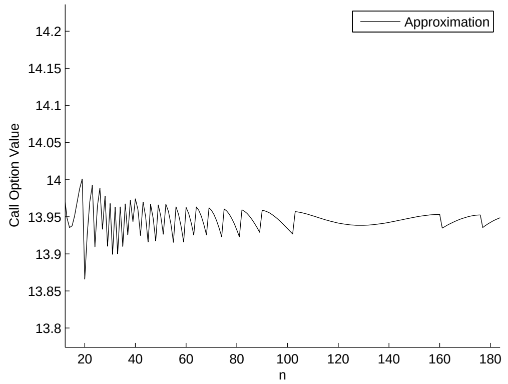

Figure 8:

A graph for the approximation $C(n) = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}$ to the European call price. $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.06$, $T = 1$.

$$Approximation = C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}.$$

$$A_n = \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\frac{1}{12}(2d_2^3 - 9d_1 d_2^2 + (13d_1^2 - 3)d_2 - 6d_1^3 + 3d_1) - \frac{(d_1 - d_2)\Delta_n^2}{2} \right].$$

$$B_n = \frac{S_0 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left[\frac{d_1^2 - d_2^2}{6}(\Delta_n^3 - \Delta_n) \right].$$

In table 4, we compute C_{BS}, $C(n)$, coefficients of $1/n$ and $1/n\sqrt{n}$, A_n and B_n respectively for given S_0, K, σ, r, T and λ. From table 4 we observe that $n\left(C(n) - C_{BS}\right) - A_n \to 0$ as n$\to\infty$. and $n\sqrt{n}\left(C(n) - C_{BS} - \frac{A_n}{n}\right) - B_n \to 0$ as $n \to \infty$. Notice that here $C(n)$ is the Tian price and we define $\left(C(n) - C_{BS}\right)n = A'_n$and $\left(C(n) - C_{BS} - \frac{A_n}{n}\right) n\sqrt{n} = B'_n$.

Time Steps (n)	$\mathbf{C_{BS}}$	$\mathbf{C(n)}$	$\mathbf{C_{BS} + \frac{A_n}{n} + \frac{B_n}{n\sqrt{n}}}$	$\mathbf{A_n}$	$\mathbf{A'_n}$	$\mathbf{B_n}$	$\mathbf{B'_n}$
100	13.946121	13.947581	13.947579	0.099894	0.145956	0.458863	0.460619
500	13.946121	13.948313	13.948312	1.0997169	1.095915	-0.100888	-0.085022
1000	13.946121	13.944519	13.944519	-1.6071833	-1.602174	0.166076	0.158420
5000	13.946121	13.946193	13.946193	0.3503327	0.356638	0.444310	0.445834

Table 4

$$S_0 = 100, K = 95, \sigma = 0.2, r = 0.06, T = 1.$$

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Appendix

Appendix 1 : Proof of Modified Lemma

First by Uspensky (1937, p.121),

$$\sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k} = J(\xi_2) - J(\xi_1), \tag{20}$$

where

$$J(\xi) = \frac{1}{2\pi} \int_0^{\pi} R \frac{\sin(\xi\sqrt{B_n}\varphi - \chi)}{\sin \dfrac{\varphi}{2}} d\varphi,$$

with

$$B_n = npq, \quad \chi = \arg\left(pe^{i\varphi} + q\right)^n - np\varphi, \quad R = |pe^{i\varphi} + q|^n.$$

As in Uspensky (1937, p.124), we write

$$J(\xi) = J_1(\xi) + J_2(\xi),$$

where

$$J_1(\xi) = \frac{1}{2\pi} \int_0^{\tau} R \frac{\sin(\xi\sqrt{B_n}\varphi - \chi)}{\sin \frac{\varphi}{2}} d\varphi, \quad J_2(\xi) = \frac{1}{2\pi} \int_{\tau}^{\pi} R \frac{\sin(\xi\sqrt{B_n}\varphi - \chi)}{\sin \frac{\varphi}{2}} d\varphi,$$

with

$$\tau = \sqrt{3} B_n^{-1/4}.$$

First we prove the following claim.

Claim 1:

$$J_2(\xi) = O\left(\frac{1}{n^k}\right) \tag{21}$$

for all $k \geq 2$, uniformly in ξ.

Proof. From equation (15) in Uspensky (1937, p.125)

$$|J_2(\xi)| < \left(\frac{3}{4}\log\frac{\pi}{2} + \frac{\sqrt{B_n}}{6}\right) e^{-\frac{3}{2}\sqrt{B_n}}.$$

Hence using inequality

$$e^{-x} \le \frac{C_\alpha}{x^\alpha} \quad \text{for} \quad \alpha \ge 0, \quad x > 0 \tag{22}$$

we get

$$|J_2(\xi)| < \left(\frac{3}{4}\log\frac{\pi}{2} + \frac{\sqrt{B_n}}{6}\right) e^{-\frac{3}{2}\sqrt{B_n}} \le \left(\frac{3}{4}\log\frac{\pi}{2} + \frac{\sqrt{B_n}}{6}\right) \frac{C_\alpha}{\left(\frac{3}{2}\sqrt{B_n}\right)^\alpha}.$$

By taking $\alpha = 2k+1$ and using $B_n^{-1} = O(\frac{1}{n})$, we have $J_2(\xi) = O(\frac{1}{n^k})$. This completes the proof of the claim.

Next we prove the following claim:

Claim 2:

$$J_1(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\xi e^{-\frac{u^2}{2}}du + \frac{q-p}{6\sqrt{2\pi npq}}(1-\xi^2)e^{-\frac{\xi^2}{2}} + \frac{1}{12n\pi}\sqrt{\frac{\pi}{2}}\xi e^{-\frac{1}{2}\xi^2}(\xi^2-1)$$

$$+O\left(\frac{1}{n^2}\right)$$

uniformly in ξ.

Before we go on, we first list Eqs (7), (8), (9) in Uspensky (1937, p123-124) as follows. If $0 \le \varphi \le \tau$ and n sufficiently large,

$$\left| R - e^{-\frac{1}{2}B_n\varphi^2} \right| < \frac{1}{16} B_n \varphi^4 e^{-\frac{1}{2}B_n\varphi^2}, \tag{23}$$

$$\chi = \frac{1}{6} B_n (p-q)\varphi^3 + N\varphi^5, \quad \text{where} \quad |N| < \frac{1}{12} B_n |p-q| (1 - pq\tau^2)^{-4}. \tag{24}$$

By extending the argument in Uspensky, we prove the following more refined estimate for R.

Claim 3: If n is sufficiently large, then for $0 \le \varphi \le \tau$

$$\left| R - e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} \right| < 2e^{-\frac{1}{2}B_n\varphi^2} B_n \varphi^6. \tag{25}$$

Proof. We define

$$\rho = R^{1/n} = |pe^{i\varphi} + q|.$$

Now from Uspensky (1937, p.121–122), we have

$$\log \rho = -2pq \sin^2 \frac{\varphi}{2} - \frac{1}{4}(4pq)^2 \sin^4 \frac{\varphi}{2} + \delta, \tag{26}$$

where

$$\delta = -\frac{1}{6}(4pq)^3 \sin^6 \frac{\varphi}{2} - \frac{1}{8}(4pq)^4 \sin^8 \frac{\varphi}{2} - \cdots$$

is negative. Next since when $\varphi \ge 0$ is sufficiently small

$$\sin^2 \frac{\varphi}{2} = \frac{1}{4}\varphi^2 - \frac{1}{48}\varphi^4 + \delta_1, \quad \sin^4 \frac{\varphi}{2} = \frac{1}{16}\varphi^4 - \frac{1}{96}\varphi^6 + \delta_2$$

with δ_1, δ_2 positive, we have $\varphi \ge 0$ sufficiently small

$$\begin{aligned} \log \rho &= -2pq\left(\frac{1}{4}\varphi^2 - \frac{1}{48}\varphi^4 + \delta_1\right) - \frac{1}{4}(4pq)^2\left(\frac{1}{16}\varphi^4 - \frac{1}{96}\varphi^6 + \delta_2\right) \\ &< -\frac{1}{2}pq\varphi^2 + \frac{1}{4}pq\left(\frac{1}{6} - pq\right)\varphi^4 + \frac{1}{24}p^2q^2\varphi^6. \end{aligned} \tag{27}$$

On the other hand, for n sufficiently large if $0 \le \varphi \le \tau$,

$$
\begin{aligned}
-\delta &= \frac{1}{6}(4pq)^3 \sin^6 \frac{\varphi}{2} + \frac{1}{8}(4pq)^4 \sin^8 \frac{\varphi}{2} + \cdots \\
&< \frac{1}{6}(4pq)^3 \sin^6 \frac{\varphi}{2} + \frac{1}{6}(4pq)^4 \sin^8 \frac{\varphi}{2} + \cdots \\
&= \frac{\frac{1}{6}(4pq)^3 \sin^6 \frac{\varphi}{2}}{1 - 4pq \sin^2 \frac{\varphi}{2}} \\
&< \frac{1}{3}(4pq)^3 \sin^6 \frac{\varphi}{2},
\end{aligned}
$$

since when $0 \le \varphi \le \tau$,

$$\sin^2 \frac{\varphi}{2} \le \frac{\varphi^2}{4} \le \frac{\tau^2}{4} = \frac{3}{4\sqrt{npq}} \le \frac{1}{8pq}$$

if n is large.

Therefore, we have for $0 \le \varphi \le \tau$ and n sufficiently large

$$\delta > -\frac{1}{3}(4pq)^3 \sin^6 \frac{\varphi}{2} \ge -\frac{1}{3}(4pq)^3 \left(\frac{\varphi}{2}\right)^6 = -\frac{1}{3}p^3q^3\varphi^6.$$

Using this in (26) together with the facts that for $0 \le \varphi \le \frac{\pi}{2}$

$$\sin^2 \frac{\varphi}{2} = \frac{1}{4}\varphi^2 - \frac{1}{48}\varphi^4 + \frac{1}{1440}\varphi^6 + \delta_3, \quad \sin^4 \frac{\varphi}{2} = \frac{1}{16}\varphi^4 + \delta_4$$

where δ_3, δ_4 are negative, we find that for $0 \le \varphi \le \tau$ when n is sufficiently large.

$$\log \rho = -2pq\left(\frac{1}{4}\varphi^2 - \frac{1}{48}\varphi^4 + \frac{1}{1440}\varphi^6 + \delta_3\right) - \frac{1}{4}(4pq)^2\left(\frac{1}{16}\varphi^4 + \delta_4\right) + \delta$$

$$> -2pq\left(\frac{1}{4}\varphi^2 - \frac{1}{48}\varphi^4 + \frac{1}{1440}\varphi^6\right) - \frac{1}{4}(4pq)^2\left(\frac{1}{16}\varphi^4\right) - \frac{1}{3}p^3q^3\varphi^6 \tag{28}$$

$$> -\frac{1}{2}pq\varphi^2 + \frac{1}{4}pq\left(\frac{1}{6} - pq\right)\varphi^4 - \frac{1}{3}pq\left(\frac{1}{240} + p^2q^2\right)\varphi^6.$$

Then, using (27) and (28) together with $B_n = npq$, we have for n sufficiently large when $0 \le \varphi \le \tau$

$$\begin{aligned} &e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} e^{-\frac{1}{3}B_n\left(\frac{1}{240}+p^2q^2\right)\varphi^6} \\ &< R = \rho^n < e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} e^{\frac{1}{24}B_n pq\varphi^6}, \end{aligned} \tag{29}$$

from which it follows that

$$\left| R - e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} \right|$$

$$< e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4}\left(e^{\frac{1}{24}B_n pq\varphi^6} - e^{-\frac{1}{3}B_n\left(\frac{1}{240}+p^2q^2\right)\varphi^6}\right)$$

$$< e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4}\left(\left(1 + 2\frac{1}{24}B_n pq\varphi^6\right) - \left(1 - \frac{1}{3}B_n\left(\frac{1}{240} + p^2q^2\right)\varphi^6\right)\right)$$

$$< e^{-\frac{1}{2}B_n\varphi^2} e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} B_n\varphi^6$$

$$< 2e^{-\frac{1}{2}B_n\varphi^2} B_n\varphi^6,$$

using the facts that $e^x < 1 + 2x$ for $0 < x < 1$, $e^{-x} > 1 - x$ for $x > 0$, $0 \le \varphi \le \tau$ and $0 < pq \le \frac{1}{4}$. This completes the proof of Claim 3.

Now we partition

$$J_1(\xi) = I_0 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{30}$$

where

$$\begin{aligned}
I_0 &= \frac{1}{\pi}\int_0^\tau e^{-\frac{1}{2}B_n\varphi^2}\frac{\sin(\xi\sqrt{B_n}\varphi-\chi)}{\varphi}d\varphi,\\
I_1 &= \frac{1}{2\pi}\int_0^\tau \left(\frac{R}{\sin\frac{\varphi}{2}}-\frac{2e^{-\frac{1}{2}B_n\varphi^2}}{\varphi}\right)(\sin(\xi\sqrt{B_n}\varphi-\chi)-\sin(\xi\sqrt{B_n}\varphi))d\varphi,\\
I_2 &= \frac{1}{2\pi}\int_0^\tau R\left(\frac{1}{\sin\frac{\varphi}{2}}-\frac{2}{\varphi}-\frac{\varphi}{12}\right)\sin(\xi\sqrt{B_n}\varphi)d\varphi,\\
I_3 &= \frac{1}{2\pi}\int_0^\tau \frac{2}{\varphi}\left(R-e^{-\frac{1}{2}B_n\varphi^2}e^{\frac{1}{4}B_n(\frac{1}{6}-pq)\varphi^4}\right)\sin(\xi\sqrt{B_n}\varphi)d\varphi,\\
I_4 &= \frac{1}{2\pi}\int_0^\tau \frac{2}{\varphi}e^{-\frac{1}{2}B_n\varphi^2}\left(e^{\frac{1}{4}B_n(\frac{1}{6}-pq)\varphi^4}-1\right)\sin(\xi\sqrt{B_n}\varphi)d\varphi,\\
I_5 &= \frac{1}{2\pi}\int_0^\tau \frac{\varphi}{12}\left(R-e^{-\frac{1}{2}B_n\varphi^2}\right)\sin(\xi\sqrt{B_n}\varphi)d\varphi,\\
I_6 &= \frac{1}{2\pi}\int_0^\tau e^{-\frac{1}{2}B_n\varphi^2}\frac{\varphi}{12}\sin(\xi\sqrt{B_n}\varphi)d\varphi.
\end{aligned}$$

First, we consider I_0. Following Uspensky (1937, p126-129),

$$I_0 = \frac{1}{\sqrt{2\pi}}\int_0^\xi e^{-\frac{u^2}{2}}du + \frac{q-p}{6\sqrt{2\pi npq}}(1-\xi^2)e^{-\frac{\xi^2}{2}} + \Delta_3 + \Delta_4 + \Delta_5, \tag{31}$$

where

$$\begin{aligned}|\Delta_3| &< \frac{1}{12\pi}B_n|p-q|(1-pq\tau^2)^{-4}\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^4 d\varphi \\ &+\frac{9}{512\pi}B_n^2(p-q)^2(1-pq\tau^2)^{-6}\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^5 d\varphi,\end{aligned}$$

$$|\Delta_4| < \frac{1}{3\pi}B_n^{-\frac{1}{2}}e^{-\frac{3}{2}\sqrt{B_n}} \quad \text{and} \quad |\Delta_5| < \frac{B_n^{\frac{-1}{4}}}{\pi\sqrt{3}}e^{-\frac{3}{2}}\sqrt{B_n}. \tag{32}$$

Since $pq \leq \frac{1}{4}$ and assuming n is so large that $B_n \geq 25$, we have

$$(1-pq\tau^2)^{-4} = \left(1-\frac{3pq}{\sqrt{B_n}}\right)^{-4} \leq 16. \tag{33}$$

Then by using

$$\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^4 d\varphi = 3\left(\frac{\pi}{2B_n^5}\right)^{1/2}, \quad \int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^5 d\varphi = 8B_n^{-3},$$

$$p-q = O\left(\frac{1}{\sqrt{n}}\right), \quad B_n^{-1} = O\left(\frac{1}{n}\right),$$

we find that

$$|\Delta_3| \leq \text{constant}\left(\frac{|B_n||p-q|}{B_n^{5/2}}+\frac{|B_n|^2|p-q|^2}{B_n^3}\right) = O\left(\frac{1}{\sqrt{n}}\frac{1}{n^{3/2}}+\frac{1}{n}\frac{1}{n}\right) = O\left(\frac{1}{n^2}\right). \tag{34}$$

Next from (32),

$$|\Delta_4| < \frac{1}{3\pi}B_n^{-\frac{1}{2}}e^{-\frac{3}{2}\sqrt{B_n}} = O\left(\frac{1}{n^2}\right) \tag{35}$$

$$|\Delta_5| < \frac{B_n^{\frac{-1}{4}}}{\pi\sqrt{3}}e^{-\frac{3}{2}}\sqrt{B_n} = O\left(\frac{1}{n^2}\right) \tag{36}$$

since $e^{-\sqrt{x}} = O\left(\frac{1}{x^2}\right)$.

Then it follows from (31), (34), (35) and (36) that

$$I_0 = \frac{1}{\sqrt{2\pi}} \int_0^{\xi} e^{-\frac{u^2}{2}} du + \frac{q-p}{6\sqrt{2\pi npq}} (1-\xi^2) e^{-\frac{\xi^2}{2}} + O\left(\frac{1}{n^2}\right). \tag{37}$$

Next, we consider I_1. By using the Laurent expansion $\frac{1}{\sin\frac{\varphi}{2}} = \frac{2}{\varphi} + \frac{\varphi}{12} + O(\varphi^3)$, we find that

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_0^{\tau} \left(\frac{2R}{\varphi} - \frac{2e^{-\frac{1}{2}B_n\varphi^2}}{\varphi} \right) (\sin(\xi\sqrt{B_n}\varphi - \chi) - \xi\sqrt{B_n}\varphi)) d\varphi, \\ &+ \frac{1}{2\pi} \int_0^{\tau} R \left(\frac{\varphi}{12} + O(\varphi^3) \right) (\sin(\xi\sqrt{B_n}\varphi - \chi) - \xi\sqrt{B_n}\varphi)) d\varphi \\ &= I_{11} + I_{12}. \end{aligned}$$

First, we consider I_{11}. By the mean value theorem,

$$\sin(\xi\sqrt{B_n}\varphi - \chi) - \sin(\xi\sqrt{B_n}\varphi) = -\chi\cos\eta, \tag{38}$$

where η is some real number. Hence,

$$|I_{11}| \leq \frac{1}{2\pi} \int_0^{\tau} \left| \frac{2R}{\varphi} - \frac{2e^{-\frac{1}{2}B_n\varphi^2}}{\varphi} \right| |\chi| d\varphi.$$

Then from (23) and (24), it follows that

$$|I_{11}| < \frac{1}{16\pi} \int_0^{\infty} B_n\varphi^3 e^{-\frac{1}{2}B_n\varphi^2} \left(\frac{1}{6} B_n|p-q|\varphi^3 + \frac{B_n|p-q|(1-pq\tau^2)^{-4}\varphi^5}{12} \right) d\varphi.$$

Next using (33) and

$$\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^6 d\varphi = 15\left(\frac{\pi}{2B_n^7}\right)^{1/2}, \quad \int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^8 d\varphi = 105\left(\frac{\pi}{2B_n^9}\right)^{1/2}, \qquad (39)$$

$$p-q = O\left(\frac{1}{\sqrt{n}}\right), \quad B_n^{-1} = O\left(\frac{1}{n}\right),$$

we have

$$|I_{11}| \le \text{constant}\left(\frac{B_n^2}{B_n^{7/2}} + \frac{B_n^2}{B_n^{9/2}}\right)|p-q| = O\left(\frac{1}{n^2}\right). \qquad (40)$$

Now from (23) and $\tau^2 = \dfrac{3}{\sqrt{B_n}}$, we have for $0 \le \varphi \le \tau$

$$R \ < e^{-\frac{1}{2}B_n\varphi^2} + \frac{1}{16}B_n\varphi^4 e^{-\frac{1}{2}B_n\varphi^2} = e^{-\frac{1}{2}B_n\varphi^2}\left(1 + \frac{B_n}{16}\varphi^4\right) \le \frac{25}{16}e^{-\frac{1}{2}B_n\varphi^2}. \qquad (41)$$

Next, using (38) again and supposing that $O(\varphi^3)$ in the Laurent expansion $\dfrac{1}{\sin\frac{\varphi}{2}} = \dfrac{2}{\varphi} + \dfrac{\varphi}{12} + O(\varphi^3)$ satisfies

$$|O(\varphi^3)| \le M\varphi^3 \qquad (42)$$

for some constant M, we get

$$
\begin{aligned}
|I_{12}| &\leq \frac{1}{2\pi}\int_0^{\tau} R\left(\frac{\varphi}{12}+M\varphi^3\right)|\chi|d\varphi \\
&< \frac{1}{2\pi}\int_0^{\infty}\left(\frac{25}{192}e^{-\frac{1}{2}B_n\varphi^2}\varphi+\frac{25}{16}e^{-\frac{1}{2}B_n\varphi^2}(M\varphi^3)\right)\left(\frac{1}{6}B_n|p-q|\varphi^3\right. \\
&\left. +\frac{B_n|p-q|(1-pq\tau^2)^{-4}\varphi^5}{12}\right)d\varphi.
\end{aligned}
$$

Then using (33), (39),

$$
\int_0^{\infty} e^{-\frac{1}{2}B_n\varphi^2}\varphi^4 d\varphi = 3\left(\frac{\pi}{2B_n^5}\right)^{1/2}, \quad p-q=O\left(\frac{1}{\sqrt{n}}\right), \quad B_n^{-1}=O\left(\frac{1}{n}\right).
$$

we have

$$
|I_{12}| \leq \text{constant}\left(\frac{B_n}{B_n^{5/2}}+\frac{B_n}{B_n^{7/2}}+\frac{B_n}{B_n^{9/2}}\right)|p-q| = O\left(\frac{1}{n^2}\right). \tag{43}
$$

Hence from (40) and (43),

$$
I_1 = I_{11}+I_{12} = O\left(\frac{1}{n^2}\right). \tag{44}
$$

Next we consider I_2. Using the Laurent expansion $\frac{1}{\sin\frac{\varphi}{2}} = \frac{2}{\varphi}+\frac{\varphi}{12}+O(\varphi^3)$, (41) and (42), we find that

$$
|I_2| \leq \frac{1}{2\pi}\int_0^{\infty}\frac{25}{16}e^{-\frac{1}{2}B_n\varphi^2}M\varphi^3 d\varphi.
$$

Then using

$$\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^3 d\varphi = \frac{2}{B_n^2}, \quad B_n^{-1} = O\left(\frac{1}{n}\right),$$

we have

$$|I_2| \leq \text{constant}\left(\frac{1}{B_n^2}\right) = O\left(\frac{1}{n^2}\right). \tag{45}$$

In order to estimate I_3 and I_5, we use inequalities (23) and (24) to get

$$|I_3| < \frac{2B_n}{\pi}\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^5 d\varphi, \quad |I_5| < \frac{B_n}{384\pi}\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^5 d\varphi.$$

Then using

$$\int_0^\infty e^{-\frac{1}{2}B_n\varphi^2}\varphi^5 d\varphi = 8B_n^{-3}, \quad B_n^{-1} = O\left(\frac{1}{n}\right),$$

we have

$$|I_3| \leq \text{constant}\left(\frac{1}{B_n^2}\right) = O\left(\frac{1}{n^2}\right), \quad |I_5| \leq \text{constant}\left(\frac{1}{B_n^2}\right) = O\left(\frac{1}{n^2}\right). \tag{46}$$

Now we consider I_4. First, we write I_4 as

$$I_4 = \frac{1}{\pi}\int_0^\tau \frac{1}{\varphi}e^{-\frac{1}{2}B_n\varphi^2}\frac{1}{4}B_n\left(\frac{1}{6} - pq\right)\varphi^4 \sin(\xi\sqrt{B_n}\varphi)d\varphi + \Delta, \tag{47}$$

where

$$\Delta = \frac{1}{\pi}\int_0^{\tau} \frac{1}{\varphi} e^{-\frac{1}{2}B_n\varphi^2}\left(e^{\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4} - 1 \right.$$
$$\left. -\tfrac{1}{4}B_n\left(\tfrac{1}{6}-pq\right)\varphi^4\right)\sin(\xi\sqrt{B_n}\varphi)d\varphi$$

so that for n so large that $\frac{1}{6} < pq < \frac{1}{3}$,

$$|\Delta| \le \frac{1}{\pi}\int_0^{\infty} \frac{1}{\varphi} e^{-\frac{1}{2}B_n\varphi^2}\frac{1}{2}\left(\frac{1}{4}B_n\left(\frac{1}{6}-pq\right)\varphi^4\right)^2 d\varphi$$

$$= \frac{B_n^2}{32\pi}\left(\frac{1}{6}-pq\right)^2\int_0^{\infty} e^{-\frac{1}{2}B_n\varphi^2}\varphi^7 d\varphi \le \frac{1}{24\pi B_n^2},$$

using the fact that $0 \le e^x - 1 - x \le x^2/2$ for $x < 0$ and

$$\int_0^{\infty} e^{-\frac{1}{2}B_n\varphi^2}\varphi^7 d\varphi = 48B_n^{-4}.$$

Hence $\Delta = O\left(\frac{1}{n^2}\right)$ and we have

$$I_4 = \frac{1}{4\pi}B_n\left(\frac{1}{6}-pq\right)\int_0^{\tau} e^{-\frac{1}{2}B_n\varphi^2}\varphi^3\sin(\xi\sqrt{B_n}\varphi)d\varphi + O\left(\frac{1}{n^2}\right).$$

Next substitute $x = \sqrt{B_n}\varphi$ so that

$$I_4 = \frac{1}{4B_n\pi}\left(\frac{1}{6}-pq\right)\int_0^{\sqrt{B_n}\tau} e^{-\frac{1}{2}x^2}x^3\sin(\xi x)dx + O\left(\frac{1}{n^2}\right).$$

Then because $\sqrt{B_n}\tau = \sqrt{3}B_n^{1/4}$ tends to ∞ as $n \to \infty$ and

$$\int_{\sqrt{B_n}\tau}^{\infty} |e^{-\frac{1}{2}x^2}x^3\sin(\xi x)|dx \le \int_{\sqrt{B_n}\tau}^{\infty} e^{-\frac{1}{2}x^2}x^3 dx = \left[-(x^2+2)e^{-x^2/2}\right]_{\sqrt{B_n}\tau}^{\infty}$$

$$= (B_n\tau^2+2)e^{-\frac{1}{2}B_n\tau^2} = (3\sqrt{B_n}+2)e^{-\frac{3}{2}\sqrt{B_n}}$$

$$= O\left(\frac{1}{n^2}\right),$$

it follows that

$$I_4 = \frac{1}{4B_n\pi}\left(\frac{1}{6}-pq\right)\int_0^{\infty} e^{-\frac{1}{2}x^2}x^3\sin(\xi x)dx + O\left(\frac{1}{n^2}\right).$$

Now differentiating the well-known integral

$$\int_0^{\infty} e^{-ax^2}\cos(bx)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}, \quad (a>0)$$

three times respect to b and after that substituting $a = 1/2$ and $b = \xi$, we find that

$$\int_0^{\infty} e^{-\frac{1}{2}x^2}x^3\sin(\xi x)dx = \sqrt{\frac{\pi}{2}}\xi(3-\xi^2)e^{\frac{-\xi^2}{2}}.$$

So

$$I_4 = \frac{1}{24B_n\pi}(1-6pq)\sqrt{\frac{\pi}{2}}\xi(3-\xi^2)e^{-\frac{1}{2}\xi^2} + O\left(\frac{1}{n^2}\right). \tag{48}$$

Next we write I_6 as

$$I_6 = \frac{1}{24B_n\pi}\int_0^{\sqrt{B_n}\tau} e^{-\frac{1}{2}x^2}x\sin(\xi x)dx.$$

Then because $\sqrt{B_n}\tau = 3B_n^{1/4}$ tends to ∞ as $n \to \infty$ and

$$\frac{1}{24B_n\pi}\int_{\sqrt{B_n}\tau}^{\infty} |e^{-\frac{1}{2}x^2} x\sin(\xi x)|dx \le \frac{1}{24B_n\pi}\int_{\sqrt{B_n}\tau}^{\infty} e^{-\frac{1}{2}x^2} x dx = \frac{1}{24B_n\pi}\left[-e^{\frac{-x^2}{2}}\right]_{\sqrt{B_n}\tau}^{\infty}$$

$$= \frac{1}{24B_n\pi}e^{-\frac{1}{2}B_n\tau^2} = \frac{1}{24B_n\pi}e^{-\frac{3}{2}\sqrt{B_n}}$$

$$= O\left(\frac{1}{n^2}\right),$$

it follows that

$$I_6 = \frac{1}{24B_n\pi}\int_0^{\infty} e^{-\frac{1}{2}x^2} x\sin(\xi x)dx + O\left(\frac{1}{n^2}\right).$$

Now differentiating the well-known integral

$$\int_0^{\infty} e^{-ax^2}\cos(bx)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}, \quad (a > 0)$$

once with respect to b and after that substituting $a = 1/2$ and $b = \xi$, we find that

$$\int_0^{\infty} e^{-\frac{1}{2}x^2} x\sin(\xi x)dx = \sqrt{\frac{\pi}{2}}\xi e^{\frac{-\xi^2}{2}}.$$

So

$$I_6 = \frac{1}{24B_n\pi}\sqrt{\frac{\pi}{2}}\xi e^{-\frac{1}{2}\xi^2} + O\left(\frac{1}{n^2}\right). \tag{49}$$

Then from (48) and (49), we get

$$I_4 + I_6 = \frac{1}{24B_n\pi}\sqrt{\frac{\pi}{2}}\xi e^{-\frac{1}{2}\xi^2}(1 + (1-6pq)(3-\xi^2)) + O\left(\frac{1}{n^2}\right).$$

Since $B_n = npq$ and $pq = \frac{1}{4} + O\left(\frac{1}{n}\right)$, we conclude that

$$I_4 + I_6 = \frac{1}{12n\pi}\sqrt{\frac{\pi}{2}}\xi e^{-\frac{1}{2}\xi^2}(\xi^2 - 1) + O\left(\frac{1}{n^2}\right). \tag{50}$$

Then using (37), (44), (45), (46), and (50), we prove Claim 2.

From Claim 1 and Claim 2, it follows that

$$\begin{aligned} J(\xi) &= J_1(\xi) + J_2(\xi) \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\xi} e^{-\frac{u^2}{2}}du + \frac{q-p}{6\sqrt{2\pi npq}}(1-\xi^2)e^{-\frac{\xi^2}{2}} + \frac{1}{12n\pi}\sqrt{\frac{\pi}{2}}\xi e^{-\frac{1}{2}\xi^2}(\xi^2 - 1) \\ &+ O\left(\frac{1}{n^2}\right). \end{aligned}$$

uniformly in ξ. Then, as required,

$$\begin{aligned} J(\xi_2) - J(\xi_1) &= \frac{1}{\sqrt{2\pi}}\int_{\xi_1}^{\xi_2} e^{-\frac{u^2}{2}}du + \frac{q-p}{6\sqrt{2\pi npq}}\Big[(1-\xi_2^2)e^{-\frac{\xi_2^2}{2}} - (1-\xi_1^2)e^{-\frac{\xi_1^2}{2}}\Big] \\ &+ \frac{1}{12n\sqrt{2\pi}}\left[\xi_2 e^{-\frac{1}{2}\xi_2^2}(\xi_2^2 - 1) - \xi_1 e^{-\frac{1}{2}\xi_1^2}(\xi_1^2 - 1)\right] + O\left(\frac{1}{n^2}\right). \end{aligned}$$

So using (20), we get the lemma.

Appendix 2 : Proof of Corollary

Since

$$pq = \frac{1}{4} - \frac{\alpha^2}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

we have

$$\frac{1}{\sqrt{pq}} = \left[\frac{1}{4} - \frac{\alpha^2}{n} + O\left(\frac{1}{n^2}\right)\right]^{\frac{-1}{2}} = 2\left[1 - \frac{4\alpha^2}{n} + O\left(\frac{1}{n^2}\right)\right]^{-1/2} = 2 + \frac{4\alpha^2}{n} + O\left(\frac{1}{n^2}\right).$$

Then

$$\begin{aligned} & j_n - np - \tfrac{1}{2} \\ &= \frac{1-b_n}{2} + \tilde{\gamma}\sqrt{n} + \frac{n}{2} + \frac{\tilde{a}}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) - n\left[\frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right)\right] - \frac{1}{2} \\ &= \frac{-b_n}{2} + (\tilde{\gamma} - \alpha)\sqrt{n} + \frac{\tilde{a} - \beta}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \end{aligned}$$

so that

$$\begin{aligned} \xi_1 &= \frac{j_n - np - \frac{1}{2}}{\sqrt{npq}} \\ &= \frac{1}{\sqrt{n}}\left[2 + \frac{4\alpha^2}{n} + O\left(\frac{1}{n^2}\right)\right]\left[\frac{-b_n}{2} + (\tilde{\gamma} - \alpha)\sqrt{n} + \frac{\tilde{a} - \beta}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)\right] \\ &= -d - \frac{b_n}{\sqrt{n}} - \frac{C - 2\tilde{a}}{n} - \frac{2\alpha^2 b_n}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

where

$$C = 2\beta - 4\alpha^2(\tilde{\gamma} - \alpha) = 2(\beta + \alpha^2 d).$$

Also

$$q - p = 1 - 2p \quad = 1 - 2\left(\frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right)\right)$$

$$= -\frac{2\alpha}{\sqrt{n}} - \frac{2\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right)$$

so that

$$\frac{q-p}{\sqrt{pq}} \quad = \left(-\frac{2\alpha}{\sqrt{n}} - \frac{2\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right)\right)\left(2 + \frac{4\alpha^2}{n} + O\left(\frac{1}{n^2}\right)\right) \tag{51}$$

$$= -\frac{4\alpha}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Note that

$$\frac{\xi_2}{\sqrt{n}} = \frac{q + \frac{1}{2n}}{\sqrt{pq}} \to 1 \ as \ n \to \ \infty. \tag{52}$$

so that

$$(1 - \xi_2^2)e^{-\frac{\xi_2^2}{2}} = O\left(\frac{1}{n^2}\right) \quad \text{and} \quad \xi_2(\xi_2^2 - 1)e^{-\frac{\xi_2^2}{2}} = O\left(\frac{1}{n^2}\right). \tag{53}$$

Next, set $F_1(x) = (1 - x^2)e^{-\frac{x^2}{2}}$ and $F_2(x) = x(x^2 - 1)e^{-\frac{x^2}{2}}$. By Taylor expansion, for some η between $-\xi_1$ and d,

$$F_1(-\xi_1) = F_1(d) + (d^3 - 3d)e^{-\frac{d^2}{2}}(-\xi_1 - d) + \frac{F_1''(\eta)}{2!}(-\xi_1 - d)^2, \tag{54}$$

where $F_1''(\eta) = (-\eta^4 + 6\eta^2 - 3)e^{\frac{-\eta^2}{2}}$ is bounded, and

$$F_2(-\xi_1) = F_2(d) + (-d^4 + 4d^2 - 1)e^{-\frac{d^2}{2}}(-\xi_1 - d) + \frac{F_2''(\eta)}{2!}(-\xi_1 - d)^2, \tag{55}$$

where $F_2''(\eta) = e^{\frac{-\eta^2}{2}}(\eta^5 - 8\eta^3 + 9\eta)$ is bounded. Then we see as $n \to \infty$, using (53), (54), and (55), that

$$(1-\xi_2^2)e^{-\frac{\xi_2^2}{2}} - (1-\xi_1^2)e^{-\frac{\xi_1^2}{2}} = (1-\xi_2^2)e^{-\frac{\xi_2^2}{2}} - F_1(-\xi_1)$$

$$= -(1-d^2)e^{-\frac{d^2}{2}} - (d^3 - 3d)e^{-\frac{d^2}{2}}\left(\frac{b_n}{\sqrt{n}} + \frac{C-2a}{n} + \frac{2\alpha^2 b_n}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^2}\right)\right) + O\left(\frac{1}{n^2}\right). \tag{56}$$

and

$$\xi_2 e^{-\frac{\xi_2^2}{2}}(\xi_2^2 - 1) - \xi_1 e^{-\frac{\xi_1^2}{2}}(\xi_1^2 - 1) = \xi_2 e^{-\frac{\xi_2^2}{2}}(\xi_2^2 - 1) + F_2(-\xi_1)$$

$$= de^{-\frac{d^2}{2}}(d^2 - 1) + (-d^4 + 4d^2 - 1)e^{-\frac{d^2}{2}}\left(\frac{b_n}{\sqrt{n}} + \frac{C}{n} + \frac{2\alpha^2 b_n}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^2}\right)\right) + O\left(\frac{1}{n^2}\right). \tag{57}$$

Now let

$$I \equiv \int_{\xi_1}^{\xi_2} e^{-\frac{u^2}{2}}du = \int_{\xi_1}^{\infty} e^{-\frac{u^2}{2}}du - \int_{\xi_2}^{\infty} e^{-\frac{u^2}{2}}du = I_1 - I_2. \tag{58}$$

Then

$$I_1 = \sqrt{2\pi}\Phi(d) + f(-\xi_1), \text{ where } f(\xi) = \int_d^{\xi} e^{-\frac{u^2}{2}}du. \tag{59}$$

By Taylor expansion for some η between $-\xi_1$ and d, noting that $f''''(\eta)$ is bounded,

$$
\begin{aligned}
f(-\xi_1) &= e^{-\frac{d^2}{2}}(-\xi_1 - d) - \frac{de^{-\frac{d^2}{2}}}{2}(-\xi_1 - d)^2 + \frac{(d^2-1)e^{\frac{-d^2}{2}}}{6}(-\xi_1 - d)^3 \\
&+ \frac{f''''(\eta)}{4!}(-\xi_1 - d)^4 \\
&= e^{-\frac{d^2}{2}}\left(\frac{b_n}{\sqrt{n}} + \frac{C-2\tilde{a}}{n} - \frac{db_n^2}{2n} + \frac{2\alpha^2 b_n - db_n(C-2\tilde{a}) + \frac{(d^2-1)b_n^3}{6}}{n^{\frac{3}{2}}}\right) \\
&+ O\left(\frac{1}{n^2}\right).
\end{aligned}
\tag{60}
$$

Using (52), there exists n_0 such that $\xi_2 \geq 2$ for $n \geq n_0$. Moreover, since $\frac{u^2}{2} \geq u$ if $u \geq 2$, then $e^{-\frac{u^2}{2}} \leq e^{-u}$ if $u \geq 2$. Hence when $n \geq n_0$

$$
|I_2| \leq \int_{\xi_2}^{\infty} e^{-u} du = e^{-\xi_2} = O\left(\frac{1}{n^2}\right). \tag{61}
$$

Then using the modified Lemma and equations (51), (56), (57), (58), (59), (60) and (61), we have

$$\sum_{k=j_n}^{n} \binom{n}{k} p_n^k q_n^{n-k}$$

$$= \Phi(d) + \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \left(\frac{b_n}{\sqrt{n}} + \frac{C - 2\tilde{a}}{n} - \frac{d b_n^2}{2n} + \frac{2\alpha^2 b_n - d b_n (C - 2\tilde{a}) + \frac{(d^2-1)b_n^3}{6}}{n^{\frac{3}{2}}} \right)$$

$$+ \frac{1}{6\sqrt{2\pi}} \left[\left(-\frac{4\alpha}{\sqrt{n}} \right) \left(-\left(1 - d^2\right) e^{-\frac{d^2}{2}} - (d^3 - 3d) e^{\frac{d^2}{2}} \left(\frac{b_n}{\sqrt{n}} + \frac{C - 2\tilde{a}}{n} \right.\right.\right.$$

$$\left.\left.\left. + \frac{2\alpha^2 b_n}{n^{\frac{3}{2}}} \right) \right] \frac{1}{\sqrt{n}}$$

$$+ \frac{1}{12\sqrt{2\pi}} \left[d e^{-\frac{d^2}{2}} (d^2 - 1) + (-d^4 + 4d^2 - 1) e^{-\frac{d^2}{2}} \left(\frac{b_n}{\sqrt{n}} + \frac{C - 2\tilde{a}}{n} \right.\right.$$

$$\left.\left. + \frac{2\alpha^2 b_n}{n^{\frac{3}{2}}} \right) \right] \frac{1}{n}$$

$$+ O\left(\frac{1}{n^2} \right)$$

$$= \Phi(d) + \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \cdot \frac{b_n}{\sqrt{n}} + \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \left(C - 2\tilde{a} - \frac{d b_n^2}{2} + \left(\frac{2\alpha}{3} - \frac{d}{12} \right) (1 - d^2) \right) \frac{1}{n}$$

$$+ \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \left[2\alpha^2 b_n - d b_n (C - 2\tilde{a}) + \frac{(d^2 - 1) b_n^3}{6} + \frac{2}{3} \alpha (d^3 - 3d) b_n \right.$$

$$\left. + \frac{(-d^4 + 4d^2 - 1) b_n}{12}) \right] \frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^2} \right).$$

Thus the Corollary is proved.