國立臺灣大學理學院數學所
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# 三角架構形之研究與探討
# Tripod Configurations

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# 誌謝

一開始從指導教授手中接過題目後，腦筋是一片空白，完全不知道應該從何處下手，即使想找一些相關資料來看，卻都是一些科學上的應用。正當我不知道如何是好的時候，教授引導我思考的方向，讓我一步一步的去完成我的論文，在此非常感謝教授耐著性子解除我的疑惑，並跟我一起思考其他證明方向的可能性。此外，特別要感謝研究室一起奮戰的同學：廖鴻仁、劉鴻儒、呂融昇、段守正、蘇哲聖、林柏佐、白任宏、林晉宏和賴昱帆等以及學弟們，感謝他們陪著我了解題目的意義跟所有思考的方向，並提供我許多可能會需要的書籍，而且彼此鼓勵互相打氣，謝謝你們讓我完成了我的碩士論文，並讓我感受到數學系的溫暖。還要謝謝我的朋友:吳哲瑋，林珈瑜，周仲屏和蔡佳穎，感謝他們在我低潮時不斷的給我支持、信心及鼓勵。最後要感謝我的家人，謝謝他們包容我的任性，並且全力支持我，使我能夠心無旁鶩的念自己想念的書，很感謝大家。

# 摘要

這篇論文在探討三腳架構形，根據 Serge Tabachnikov 在附錄[1]的第二個定理:給定一個平滑凸閉平面曲線，至少存在兩個三角架構形。我們在這篇論文中想用跟 Serge Tabachnikov 不太相同的方法去建構三腳架構形，使用另一種比較直覺的幾何去建構出來。我們採取的方法是 minimax method,建造一些變形使 Y 形的距離和漸漸縮短,但不是所有的 Y 形均會退化,而會收斂到一個沒有退化的臨界點，再說明臨界點即為我們要的三腳架構形。

# Abstract

In this paper, we research the tripod configurations. By Serge Tabachnikov, see Theorem 2 of Appendix [1] says that for any smooth convex closed curve, there exist at least two tripod configurations. In this paper we want to use another way to construct tripod configurations. Use a intuitive way by a geometrical approach to construct it. We use minimax method, and do some deformation such that the distance of the Y-shaped will decrease, but not all of the Y-shaped will degenerate, it will converge to a critical point which will not degenerate, and we explain that this critical point is our tripod configuration.

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# 圖目錄

# TRIPOD CONFIGURATIONS

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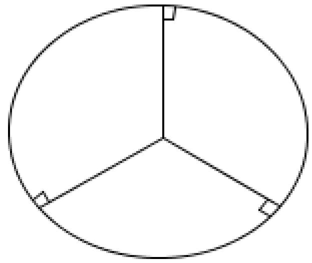
ABSTRACT. We have already known that for any smooth convex curve, there are at least two tripod configurations. In this paper, we want to modify the proof by a more intuitive way to construct the tripod configuration and give some remarks.

## 1 INTRODUCTION

In geometry, the Fermat point is the solution to the problem of finding a point such that the sum of its distance from the vertices of a triangle is a minimum. When the problem first appeared, many methods arrive at the solution have been developed, and have many properties among them, for which the most important one is that there is a point inside a triangle whose angles are less then $2\pi/3$, from which all sides are seen at angles $2\pi/3$. This point minimizes the sum of distance to the vertices.

Many people seek to find the Fermat point of n-polygon, even in polyhedron. But in general, not only for polygon, we want to do similar things for closed plane curves. Moreover instead of a Fermat point, we call the tripod configuration. For the definition of the tripod configurations: given a closed plane curve $r$, three perpendiculars to $r$ dropped from one point that make angles of $2\pi/3$. See the figure below.



By Serge Tabachnikov, See Appendix [1], Theorem 2 of [1] says that for any smooth convex closed curve, there exist at least two tripod configurations. In this paper, our goal is to modify the proof by a more intuitive way. Besides, by Appendix [7], according to our observation via minimax problem, we know that this minimax method would converge to a critical point, and this critical point is our tripod. For example, the Fermat's point of a plane acute triangle is our special case of tripod configurations.

In this paper, we are working on finding out modifying the proof of Theorem 2 of [1] by a geometrical approach to construct the tripod configuration and give some remarks.

# 2 SMOOTH CONVEX CLOSED PLANE SET

Given a convex closed plane set $M \subset \mathbb{R}^2$, and $\partial M \in C^2$, we want to find the tripod configuration in $M$. By $C^0(I, M)$ we denote the space of continuous mappings $c$ of $I = [0, 1]$ into $M$, with metric $d_\infty(c, c') = \sup_{0 \le t \le 1} d(c(t), c'(t))$. $C^0(I, M)$ is a complete metric space.

By $\Lambda_\infty M$, we denote the subspace of piecewise differentiable curves. The length $L$ is defined on $\Lambda_\infty M$: $L(c) = \int_I |\dot{c}(t)|\, dt$.

**Theorem.** *Given a convex closed plane set $M \subset \mathbb{R}^2$, and $\partial M \in C^2$. There is the tripod configuration in $M$.*

i.e. There is $p \in M, f(p_i), f(p_j), f(p_k) \in \partial M$ satisfy the following statment $(\star)$:

$$\begin{cases} angle(\overline{pf(p_i)}, \overline{pf(p_j)}) = angle(\overline{pf(p_j)}, \overline{pf(p_k)}) = angle(\overline{pf(p_i)}, \overline{pf(p_k)}) = \frac{2\pi}{3} \\ <\overline{pf(p_i)}, \dot{f}(p_i) >= 0, <\overline{pf(p_j)}, \dot{f}(p_j) >= 0, <\overline{pf(p_k)}, \dot{f}(p_k) >= 0 \end{cases}$$

To construct the tripod configuration, first we consider $Y$-shaped form and let $p_1, p_2, p_3$ be its terminal vertexes, the center is 0, let $I_1 = [0, p_1]$, $I_2 = [0, p_2]$, and $I_3 = [0, p_3]$, with $|I_i| = 1, \forall i = 1, 2, 3$, and there is a continuous finction $f$ such that $f : Y \to M$, $f(p_i)$ on $\partial M$, $\forall i = 1, 2, 3$.

Since each curve ( $c_i$ is $f(0)$ to $f(p_i)$, $\forall i = 1, 2, 3$ ) is piecewise differentiable curve in $\Lambda_\infty M$, $Y$-shaped is also piecewise differentiable curve in $\Lambda_\infty M$. We define $L(Y) = L(c_1) + L(c_2) + L(c_3)$.

For $\chi \geq 0$, we denote by $\Lambda_\infty^\chi M$ the subspace of $\Lambda_\infty M$, formed by the elements $Y$ with $L(Y) \leq \chi$.

Note 1. We do not have that $f$ has $1-1$ condition hypothesis to avoid the difficulty of deformation.

Note 2. By Appendix [7], according to our observation via minimax problem, we know that this minimax method would converge to a critical point, and this critical point is our tripod.

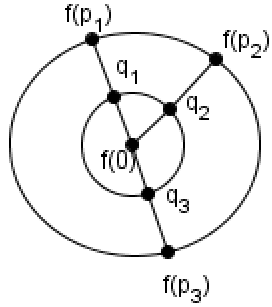
Hence we know that if the sum of distance from a point in the $M$ to its boundary is a minimum, then $p$ can satisfy the condition of $(\star)$.

Thus how to construct the tripod configuration in $M$? We use the following three steps:

**Step.1.** If there are at least two angles such that

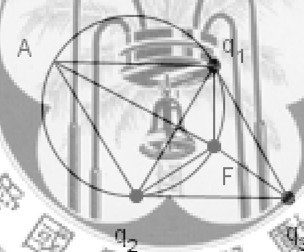
$$angle < \overline{f(0)f(p_1)}, \overline{f(0)f(p_2)} > \neq 2\pi/3 \text{ or } angle < \overline{f(0)f(p_2)}, \overline{f(0)f(p_3)} > \neq 2\pi/3 \text{ or}$$
$$angle < \overline{f(0)f(p_3)}, \overline{f(0)f(p_1)} > \neq 2\pi/3,$$

then let $f(0)$ be the center of a circle, $\frac{1}{2}d(f(0), \partial M)$ be the radius, and this circle will cut $f(Y)$ three points $q_1, q_2, q_3$ respectly. See the figure below.



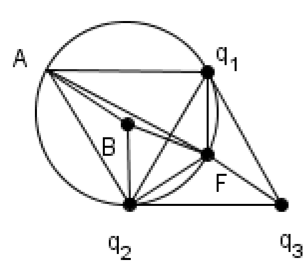

If all of the interior angles of $\triangle q_1q_2q_3$ do not exceed $2\pi/3$, then let $F$ be a Fermat point of $\triangle q_1q_2q_3$ and translate $f(0)$ to $F$ along a straight line.

Note 1. How to find $F$? Choose $\overline{q_1q_2}$ as a hemline, and construct an equilateral triangle $\triangle Aq_1q_2$, for this triangle, construct a circle $C$ which passes through all three vertices of $\triangle Aq_1q_2$, then link $\overline{Aq_3}$, the intersection of $\overline{Aq_3}$ and $C$ is $F$. This is what we want to find. See the figure below.




**Lemma 1.([3])** *The way of finding $F$ satisfies that the sum of the distance from the vertices of a triangle $\triangle q_1q_2q_3$ is minimum.*

*Proof.* Choose $\overline{Fq_2}$ as a hemline, construct an equilateral triangle $\triangle Bq_2F$ and link $\overline{AB}, \overline{Fq_1}$. See the figure below.



Since $\overline{Fq_2} = \overline{Bq_2}$, $angle(\overline{Fq_2}, \overline{q_2q_1}) = angle(\overline{Aq_2}, \overline{q_2B})$ , $\overline{Aq_2} = \overline{q_1q_2}$, $\Rightarrow \triangle Aq_2B \cong \triangle q_1q_2F$.

We get $\overline{Fq_1} = \overline{AB}$, and $angle(\overline{q_1F}, \overline{q_2F}) = angle(\overline{AB}, \overline{Bq_2}) = 2\pi/3$.

We know that the four vertices $A, q_1, F, q_2$ have the same circle $C$, $angle(\overline{BF}, \overline{Bq_2}) = \pi/3$. $ABF$ and $AFq_3$ is a straight line

$$\Rightarrow angle(\overline{q_1F}, \overline{Fq_3}) = 2\pi/3,\ angle(\overline{q_1F}, \overline{Fq_2}) = 2\pi/3,$$

we can conclude that $angle(\overline{q_2F}, \overline{Fq_3}) = 2\pi/3$. So we have $BFq_3$ is a straight line. Therefore $ABFq_3$ have the same line.

$$\Rightarrow \overline{Fq_1} + \overline{Fq_2} + \overline{Fq_3} = \overline{AB} + \overline{BF} + \overline{Fq_3} = \overline{Aq_3}$$

is a minimum sum of distance from the vertices of a triangle $\triangle q_1q_2q_3$. Furthermore by our construction, all sides are seen at angles $2\pi/3$. $\sharp$

In particular, as $angle(\overline{q_2q_1}, \overline{q_1q_3}) \geq 2\pi/3$, then $q_1$ is a Fermat point.

Note 2. The angles from the new $f(0)$ to $q_1, q_2, q_3$ are all $2\pi/3$, and the sum of the distance is decreasing, that is to say: $d(f(0), f(p_1)) + d(f(0), f(p_2)) + d(f(0), f(p_3)) = d(q_1, f(0)) + d(q_1, f(p_1)) + d(q_2, f(0)) + d(q_2, f(p_2)) + d(q_3, f(0)) + d(q_3, f(p_3)) > d(F, q_1) + d(q_1, f(p_1)) + d(F, q_2) + d(q_2, f(p_2)) + d(F, q_3) + d(q_3, f(p_3))$.

If there is an interior angle of $\triangle q_1q_2q_3$ which is langer than (or equal) $2\pi/3$, without loss of generality, say $q_1$ (i.e. The Fermat point is not in the triangle), then translate $f(0)$ to $q_1$ along a straight line, and repeat the above statement.

**Remark.1.** For the above constuction, let $\frac{1}{2}d(f(0), \partial M)$ or $\frac{1}{3}d(f(0), \partial M)$ and so on be the radius of the circle is all right, and the concentric circle will have the same Fermat point. Let $a, b, c, d, e, f$ be the points which the circle with radius $\frac{1}{3}d(f(0), \partial M)$ and $\frac{1}{2}d(f(0), \partial M)$ cuts $f(Y)$ respectly, and $F_1$ is a Fermat point of $\triangle abc$, $F_2$ is a Fermat point of $\triangle def$, then

$$angle < \overline{F_1a}, \overline{F_1b} >= angle < \overline{F_1b}, \overline{F_1c} >= angle < \overline{F_1c}, \overline{F_1a} >= 2\pi/3,$$

hence

$$angle < \overline{F_1d}, \overline{F_1f} >= angle < \overline{F_1f}, \overline{F_1e} >= angle < \overline{F_1e}, \overline{F_1d} >= 2\pi/3,$$

$\Rightarrow F_1$ is also a Fermat point of $\triangle def$. $\sharp$

**Remark.2.** For the above constuction, let $\frac{1}{2}d(f(0), \partial M)$ or $\frac{1}{3}d(f(0), \partial M)$ and so on be the radius of the circle is all right for the $angle < \overline{q_2q_1}, \overline{q_1q_3} >\geq 2\pi/3$, the Fermat point is the vertex of triangle, and do the step 1 again, until translate $F$ in the $\triangle f(p_1)f(p_2)f(p_3)$. Without loss of generality, suppose $f(0)$ is already in the $\triangle f(p_1)f(p_2)f(p_3)$, and if $angle < \overline{f(p_2)f(p_1)}, \overline{f(p_3)f(p_1)} >\geq 2\pi/3$, do the step 1 again, then we can find:

$$angle < \overline{q_2q_1}, \overline{q_3q_1} >= \frac{1}{2}angle < \overline{q_2F}, \overline{q_3F} >,$$

by simple calculation.

If $angle < \overline{q_2q_1}, \overline{q_3q_1} >> angle < \overline{f(p_2)f(p_1)}, \overline{f(p_3)f(p_1)} >$, we have

$$angle < \overline{q_2F}, \overline{q_3F} >> 2angle < \overline{f(p_2)f(p_1)}, \overline{f(p_3)f(p_1)} >\geq 4\pi/3,$$

and we get a contradiction.

If $angle < \overline{q_2q_1}, \overline{q_3q_1} >\geq 2\pi/3$, then $angle < \overline{q_2F}, \overline{q_3F} >\geq 4\pi/3$. Hence we get a contradiction.

**Conclusion.** $F$ will be an interior point in the convex bounded plane set.

**Step.2.** If

$angle < \overline{f(0)f(p_1)}, \overline{f(0)f(p_2)} >= 2\pi/3$ and $angle < \overline{f(0)f(p_2)}, \overline{f(0)f(p_3)} >= 2\pi/3$ and

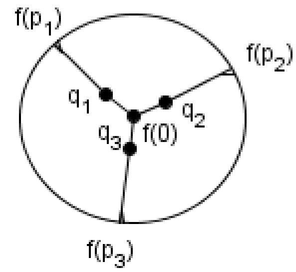
$$angle < \overline{f(0)f(p_3)}, \overline{f(0)f(p_1)} >= 2\pi/3,$$

then we can choose suitable points $\gamma_1, \gamma_2, \gamma_3 \in Y$, $\varepsilon_Y > 0$ small enough depends on $Y$, such that $L(f(\overline{p_1\gamma_1})) < \varepsilon_Y$, and we do deformation, transform $f(\overline{p_1\gamma_1})$ to a new line $\overline{f(\gamma_1)\pi f(\gamma_1)}$ and $angle < \overline{f(\gamma_1)\pi f(\gamma_1)}, \pi \dot{f}(\gamma_1) >= \pi/2$, where $\pi : f(Y) \to \partial M$ is an upright projection, such that this deformation will preserve the continuous of this $Y$-shaped, the angle from $f(0)$ to $q_i$ is always $2\pi/3$, $\forall i = 1, 2, 3$, and this deformation is continuous. Similarly,

$$angle < \overline{f(\gamma_2)\pi f(\gamma_2)}, \pi \dot{f}(\gamma_2) >= angle < \overline{f(\gamma_3)\pi f(\gamma_3)}, \pi \dot{f}(\gamma_3) >= \pi/2,$$

and translate $f(p_i)$ to $\pi f(\gamma_i)$, $\forall i = 1, 2, 3$.

Note. $angle < \overline{f(0)q_1}, \overline{f(0)q_2} >= angle < \overline{f(0)q_2}, \overline{f(0)q_3} >= angle < \overline{f(0)q_1}, \overline{f(0)q_1} >= 2\pi/3$ still not change. See the figure below.




**Remark.** $d(f(0), f(p_1)) + d(f(0), f(p_2)) + d(f(0), f(p_3))$

$> d(f(0), f(\gamma_1)) + d(f(\gamma_1), \pi f(\gamma_1)) + d(f(0), f(\gamma_2)) + d(f(\gamma_2), \pi f(\gamma_2)) + d(f(0), f(\gamma_3)) + d(f(\gamma_3), \pi f(\gamma_3))$.

**Step.3.** Finally we link $\overline{f(0)f(p_1)}$. In this process, it may cut $f(\overline{0p_2})$ or $f(\overline{0p_3})$, but it is all right, and now *angle* $< \overline{f(0)f(p_1)}, \overline{f(0)f(p_2)} > \neq 2\pi/3$, hence return to the step 1.

**Remark.** $d(f(0), f(\gamma_1)) + d(f(\gamma_1), \pi f(\gamma_1)) + d(f(0), f(\gamma_2))$
$+ d(f(\gamma_2), \pi f(\gamma_2)) + d(f(0), f(\gamma_3)) + d(f(\gamma_3), \pi f(\gamma_3)) > d(f(0), f(p_1)) + d(f(0), f(p_2)) + d(f(0), f(p_3))$.

Repeat the above construction, after many times deformation, we claim that there exists the tripod configuration. i.e. Not for all $Y$-shaped degenerate. We first consider the following lemmas.

Let the deformation of step 1 be denoted by $D_a$, the deformation of step 2 be denoted by $D_b$, the deformation of step 3 be denoted by $D_c$, and define $D$ be the subsequent application of the deformations $D_a, D_b, D_c$.

**Lemma 2.** *The deformation $D$ is continuous in $Y$.*

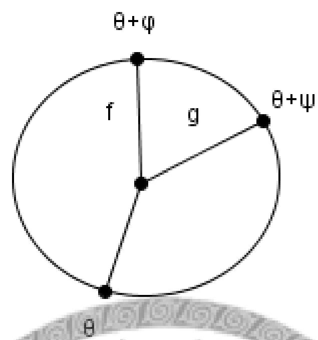
*Proof.* Let $Y_m$ be a convergent sequence in $\Lambda^{\chi}_{\infty}$ with limit $Y$. We claim that $DY_m \to DY$.

Indeed, the partition points in $Y_m$ converge to the partition points in $Y$. ( $Y_m$ is formed by three curves, $Y$-shaped convergence means each curve converges.) Thus $Y_m \to Y$, we have $D_aY_m \to D_aY$, $D_bY_m \to D_bY$, $D_cY_m \to D_cY$. So we conclude that $DY_m \to DY$, $D$ is continuous in $Y$. ♯

We construct $F = \{f \mid f : Y \to M\}$ topology by :

$$\left\{ \begin{array}{ll} f:[-1,1]\to M & f(-1)=\theta, \quad f(1)=\theta+\varphi \\ & | \\ g:[0,1]\to M & g(0)=f(0), \quad g(1)=\theta+\psi \end{array} \right\}, \theta\in S^1,\ \psi\in S^1,\ \varphi\in S^1.$$

See the figure below.




**Lemma 3.** *Show that $F$ topology is not contractible.*

To prove Lemma 3, we need the answer of the following questions.

**Definition.(p323,James R. Munkres[4])** If $f, f'$ are continuous map of the space $X$ into the space $Y$, we say that $f$ is homotopic to $f'$ if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each $x$. (Here $I = [0, 1]$). The map $F$ is called a homotopy between $f$ and $f'$. If $f$ is homotopic to $f'$, we write $f \simeq f'$. If $f \simeq f'$ and $f'$ is a constant map, we say that $f$ is nulhomotopic.

We conclude the following questions:

**Question 1.** $\exists f : [-1, 1] \to M, g : [0, 1] \to M$, such that $f(-1) = \theta, f(1) = \theta + \varphi, g(0) = f(0), g(1) = \theta + \psi$.

**Question 2.** $F$ topology is homotopic to $S^1 \times S^1 \times S^1 \times D_2$.

**Question 3.** $S^1 \times S^1 \times S^1$ is not contractible.

For the question 1:

*Proof.* Since $M$ is a convex closed plane set, it is path connected. For $\theta, \varsigma \in M$ ,$\theta + \varphi \in M$, $\exists f_1 : [-1, 0]$ such that $f_1(-1) = \theta, f_1(0) = \varsigma$. Moreover $\exists f_2 : [0, 1] \to M$ such that $f_2(0) = \varsigma, f_2(1) = \theta + \varphi$.

Let $h : [-1, 1] \to M$, $h(t) = \left\{ \begin{array}{ll} f_1(t) & t \in [-1, 0] \\ f_2(t) & t \in [0, 1] \end{array} \right\}$ is continuous.

$$\Rightarrow h(0) = f_1(0) = f_2(0) = \varsigma, h(-1) = f_1(-1) = \theta, h(1) = f_2(1) = \theta + \varphi.$$

For $\varsigma, \theta + \psi \in M$, $\exists g : [0,1] \to M$ such that $g(0) = \varsigma, g(1) = \theta + \psi$. ♯

For the question 2:

**Lemma 3.1.** *Show that $F$ topology $\backsimeq S^1 \times S^1 \times S^1 \times D_2$.*

*Proof.* Since $M$ is a convex plane set, we can find the homotopy represents a continuous deforming of $\partial M$ to $S^1$. We can consult the method in the Appendix [4] for page 361 and page 325. On the orther way, imitating the method in the Appendix [4] for page 339, we have $F$ *topology* $\backsimeq S^1 \times S^1 \times S^1 \times D_2$.

For the question 3:

We introduce lemmas.

**Lemma 3.2.** *A contractible space is simply connected.*

**Definition 1.(p333,James R. Munkres[4])** A space is simply connected if it is path-connected and its fundamental group is trivial. i.e. $\pi_1(x, x_0) = 0, x_0 \in X$.

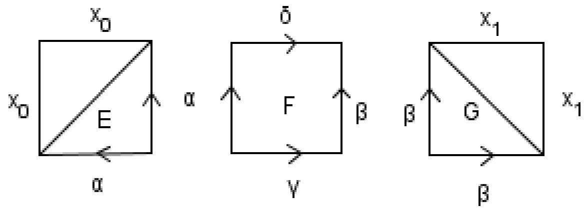
**Definition 2.(p155,James R. Munkres[4])** Given points $x$ and $y$ of the space $X$, a path in $X$ from $x$ to $y$ is a continuous map $f : [a, b] \to X$ such that $f(a) = x, f(b) = y$. A space $X$ is said to be path connected if every pair of points of $X$ can be joined by a path in $X$.

**Definition 3.(p331,James R. Munkres[4])** Let $X$ be a space, let $x_0$ be a point of $X$. A path in $X$ that begins and ends at $x_0$ is called a loop based at $x_0$. The set of path homotopy classes of loops based at $x_0$, with the operation $*$, is called the fundamental group of $X$ relative to base point $x_0$. It is denoted by $\pi_1(X, x_0)$.

*Proof.* Although every loop $\sigma$ at a point $x_0$ is homotopic as a map with a constant loop, we do not know they are homotopic relative to $(0, 1)$. (Since if $\sigma$ is a loop at $x_0$, $\tau$ is a constant loop, $\tau(s) = x_0 \forall s$, if $\sigma \simeq \tau\, rel(0, 1) \Rightarrow \sigma$ is homotopically trivial). Hence we need the following Lemma.

**Lemma 3.2.1.(Lemma 3.3,Marvin Greenberg[5])** *Given $F : I \times I \to X$, set $\alpha(t) = F(0, t), \beta(t) = F(1, t), \gamma(s) = F(s, 0), \delta(s) = F(s, 1)$, then $\delta \simeq \alpha^{-1}\gamma\beta\, rel(0, 1)$.*

*Proof.* See the figure below.




where $x_0 = \delta(0), x_1 = \delta(1)$,

$$E(s,t) = \left\{ \begin{array}{cc} x_0 & s \leq t \\ \alpha(1+t-s) & s \geq t \end{array} \right\}, G(s,t) = \left\{ \begin{array}{cc} \beta(t+s) & 1-s \geq t \\ x_1 & 1-s \leq t \end{array} \right\}. \sharp$$

Complete the proof of Lemma 3.2: Now $X$ is contractible, we can obtain $F$ with $\delta = \sigma, \gamma = x_0, \alpha = \beta$, then $\sigma$ is homotopically trivial. $\sharp$

**Lemma 3.3.(Theorem 54.4,James R. Munkres[4])** *The fundamental group of $S^1$ is $\mathbb{Z}$.*

**Definition 1.(p336, James R. Munkres[4])** Let $p : E \to B$ be a continuous surjective map. The open set $U$ of $B$ is said to be evenly covered by $p$ if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets $V_\alpha$ in $E$ such that for each $\alpha$, the restriction of $p$ to $V_\alpha$ is a homeomorphism of $V_\alpha$ onto $U$. If every point $b$ of $B$ has a neighborhood $U$ that is evenly covered by $p$, then $p$ is called a covering map .

**Definition 2.(p342, James R. Munkres[4])** Let $p : E \to B$ be a map. If $f$ is a continuous mapping of some space $X$ into $B$, a lifting of $f$ is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

**Definition 3.(p326, James R. Munkres[4])** If $f$ is a path in $X$ from $x_0$ to $x_1$, and if $g$ is a path in $X$ from $x_1$ to $x_2$, we define the product $f * g$ of $f$ and $g$ to be the path $h$ given by the equations

$$h(s) = \left\{ \begin{array}{cc} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{array} \right\}.$$

**Lemma 3.3.1.(p337, James R. Munkres[4])** *The map $p : \mathbb{R} \to S^1$ given by the equation $p(x) = (cos2\pi x, sin2\pi x)$ is a covering map.*

*Proof.* Let $U$ of $S^1$consisting of those points having positive first coordinate. The set $p^{-1}(U)$ consist of those points $x$ for which $cos2\pi x$ is positive. i.e. It is the union of intervals $V_n = (n - \frac{1}{4}, n + \frac{1}{4})$, for all $n \in \mathbb{Z}$. Now restricted to any closed interval $\overline{V_n}$, the map $p$ is injective because $sin2\pi x$ is stricty monotonic on such interval. Besides $p$ carries $\overline{V_n}$ surjectively onto $\overline{U}$, and $V_n$ to $U$ (by the intermediate value theorem). Since $\overline{V_n}$ is compact, $p \restriction \overline{V_n}$ is a homeomorphism of $\overline{V_n}$ with $\overline{U}$. In particular, $p \restriction V_n$ is a homeomorphism of $V_n$ with $U$.

Similar arguments can be applied to the intersection of $S^1$ with the upper and lower open half-planes. These open planes cover $S^1$, and each of them is evenly covered by $p$. Hence $p : \mathbb{R} \to S^1$ is a covering map. ♯

Complete the proof of Lemma 3.3:

*Proof.* Let $p : \mathbb{R} \to S^1$ be the covering map $p(x) = (cos2\pi x, sin2\pi x)$, $e_0 = 0, b_0 = p(e_0) \Rightarrow p^{-1}(b_0)$ is the set $\mathbb{Z}$.

Since $\mathbb{R}$ is simply connected ($\mathbb{R}$ is contractible), the lifting correspondence $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijctive.

Claim that $\phi$ is homomorphism.

Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective lifting to paths on $\mathbb{R}$ beginning at 0. Let $n = \tilde{f}(1), m = \tilde{g}(1) \Rightarrow \phi([f]) = n, \phi([g]) = m$, and let $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$ on $\mathbb{R}$, since $p(n + x) = p(x) \forall x \in \mathbb{R}$ ($\because p \circ \tilde{g}(s) = g(s) \Rightarrow p \circ (\tilde{\tilde{g}}(s)) = p \circ (n + \tilde{g}(s)) = p(\tilde{g}(s)) = g(s)$). $\Rightarrow \tilde{\tilde{g}}$ is a lifting of $g$ and begins at $n$.

Then $\tilde{f} * \tilde{\tilde{g}}$ is defined at it is the lifting of $f * g$ begins at 0 ($p \circ (\tilde{f} * \tilde{\tilde{g}}) = f * g$).

The end point of $\tilde{\tilde{g}}(1) = n + m$ ($\tilde{f} * \tilde{\tilde{g}}(1) = \tilde{\tilde{g}}(1) = n + m$). $\Rightarrow \phi([f] + [g]) = n + m = \phi([f]) + \phi([g])$. ♯

Moreover use the following lemma.

**Lemma 3.4.(Theorem 60.1, James R. Munkres[4])** $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

**Definition.(p333, James R. Munkres[4])** Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by the equation $h_*([f]) = [h \circ f]$. The map $h_*$

is called the homomorphism induced by $h$, relative to the base point $x_0$.

*Proof.* Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection mappings. Induced homomorphisms

$$\left\{ \begin{array}{c} p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \\ q_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0) \end{array} \right\},$$

define a homomorphism $\phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$\phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \times f].$$

Show that $\phi$ is an isomorphism.

1. $\phi$ is surjective: Let $g : I \to X$ be a loop based at $x_0$, $h : I \to X$ be a loop based at $y_0$. Want to show$[g] \times [h]$ lies in the image of $\phi$. Define $f : I \to X \times Y$ by $f(s) = g(s) \times h(s) \Rightarrow f$ is a loop based at $x_0 \times y_0$, and $\phi([f]) = [p \circ f] \times [q \circ f] = [g] \times [h]$.

2. The kernel of $\phi$ vanishes. Suppose that $f : I \to X \times Y$ is a loop in $X \times Y$ at $x_0 \times y_0$, and $\phi([f]) = [p \circ f] \times [q \circ f]$ is the identity element. i.e. $p \circ f \simeq e_{x_0}$ by $G$ and $q \circ f \simeq e_{y_0}$ by $H$, where $G, H$ are the respective path homotopics. Then $F : I \times I \to X \times Y$ defined by $F(s,t) = G(s,t) \times H(s,t)$ is a path homotopy between $f$ and the constant loop based at $x_0 \times y_0$. ♯

For the question 3:

*Proof.* $\pi_1(S^1 \times S^1 \times S^1, x_0 \times y_0 \times z_0) \cong \pi_1(S^1, x_0) \times \pi_1(S^1, y_0) \times \pi_1(S^1, z_0) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ not a trivial group. Hence $F$ topology is not contractible. ♯

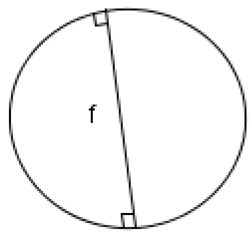
Return to our original problem, the following three cases will happen after many times deformation:

Let $Y$ be a non-null homotopic $Y$-shaped. Consider the sequence $\{D_m Y\}$ of $Y$-shaped, all of which are homotopic to $Y$. The decreasing sequence $\{L(D_m Y)\}$ has a limit $\chi_0 \geq 0$.

Case 1. We claim that $\chi_0 = 0$ would not happen. $\forall \varepsilon > 0$, since a $Y$-shaped $Y^*$with $L(Y^*) < \varepsilon$ lies entirely in the domain of normal coordinates based at $f^*(0)$. Such a $Y^*$ then is contractible, we have a contradiction. (i.e. All of the $Y$-shaped retract to a point would not happen. )

For $\chi_0 > 0$,

Case 2. If all of the $Y$-shaped will retract to a curve connected the boundary of $M$, and this curve will perpendicular to the boundary of $M$. See the figure below.




Similar to the above statement, this $Y$-shaped topology at most $\backsimeq S^1$. Maybe homotopic to some points or even homotopic to $\phi$. Since a point on the boundary, it is hard to find a straight line perpendicular to the boundary between two points. But the original $Y$-shaped topology is $S^1 \times S^1 \times S^1 \times D_2$, after the continuous deformation we get $Y$-shaped topology $\backsimeq S^1$ or homotopic to some points, even $\phi$, this contradicts to $S^1$ can not contract to a point. Thus this case would not happen. Furthemore, if some $Y$-shaped degenerate to a point, some $Y$-shaped degenerate to a line, both of them combine to this case.

Case 3. There exists the $Y$-shaped form is our tripod. Now consider the decreasing sequence $\{L(D_mY)\}$ with a limit $\chi_0 > 0$. Let $\{Y_m\}$ be a sequence with $Y_m \in D_mY$, $L(DY_m) = L(D_{m+1}Y) \geq \chi_0$. Since $M$ is compact, $\{Y_m\}$ has a convergent subsequence, which we again denote by $\{Y_m\}$. Its limit $Y$-shaped is $Y_0$. We then have $L(Y_0) = limL(Y_m) = limL(DY_m) = \chi_0 > 0$, and since $D$ is continuous, we have $L(DY_0) = L(Y_0)$, so we find a tripod $Y_0$ with $L$-value $\chi_0$. Indeed, according to the Appendix [7], we have $Y_0$ is a critical point for our $\chi_0 = minmaxL(Y)$, $\chi_0$ is a critical value, we examine the critical condition of $L$, using the Lagrange multiplier we have the following things:

1. Fixed $f(0) \in M$, the extreme value of $L$ occurs when the shortest distance lines which connect $f(0)$ and $f(p_i)$, $\forall i = 1, 2, 3$ are perpendicular to the boundary of $M$.

2. Fixed $f(p_i) \in \partial M$, $\forall i = 1, 2, 3$ the extreme value of $L$ occurs when unit tangent vector of three lines at $f(0)$ have zero sum.

Combining these two critical conditions, we know this critical point is our tripod.

**Remark.** The tripod is not unique. We can see a particular case to a circle.

By Appendix [1]. We have for any smooth convex closed curve, there exist at least two tripod configurations. Besides, by Appendix [7], Lien-Yung Kao and Ai-Nung Wang give another way to prove this theorem.

**Conclusion.** There exist at least two tripod configurations in $M$.

We conclude with another question: can we generate this case to a convex Riemannian that there exists the tripod configuration by imitating the above method or Appendix [8] for chapter 3? Given three disjoint convex plane curve, can we find a tripod in the complement of them?

# References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