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一個馬可夫鏈的特徵值問題及其應用

An Eigenvalue Problem for Markov Chains With Applications

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一個馬可夫鏈的特徵值問題及其應用[1]

李旭唐[2]

摘要

在這篇論文中我們探討一個具有兩個變量 (λ, w) 的方程組 $\sum_{y\in S} p(x,y)\exp(h(y)-\lambda+w(y))=\exp(w(x))$，其中 p 是一個狀態空間為 $\mathbb{Z}^d$ 的馬可夫鏈的轉移機率，且不論從任何狀態出發，p 只會轉移至有限多個狀態．當 $h\equiv 0$，$\lambda=0$ 之情況下所解出的 $\exp(w(x))$ 即是此轉移機率 p 的調和函數．本論文的目標旨在探討 λ 之範圍，以及當 λ 給定時其對應之 w 為何．當 $h\equiv 0$，且 p 為一隨機漫步之轉移機率時，我們將更進一步給出 (λ, w) 之明確表現形式．

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[1] 關鍵字：馬可夫鏈、隨機漫步、平賭序列、暫態、再生態、調和函數、局部中央極限定理、Choquet 定理、Martin 邊界

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An Eigenvalue Problem for Markov Chain with Applications *

Shiu Tang Li †

Abstract

In this paper we investigate the equation $\sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda+w(y)\big) = \exp(w(x))$ with two unknowns, $\lambda \in \mathbb{R}$ and $w : S \to \mathbb{R}$, where p is the transition probability of a finitely supported Markov chain $\{X_n\}$ on $\mathbb{Z}^d$. When $h \equiv 0$, $\lambda = 0$, the solutions $\exp(w(x))$ to the above equation are exactly the harmonic functions for p. Our goal is to find the range of all possible λ's and investigate the properties of $w(x)$ when λ is given. Furthermore, when $h \equiv 0$, we give an explicit formula of all possible solutions (λ, w) when p is the transition probability of a random walk.

*Keywords: Markov chain, random walk, martingale, transient, recurrent, harmonic functions, local central limit theorem, Choquet's theorem, Martin boundary

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1 Introduction

Let $S = \mathbb{Z}^d$ be the state space of an irreducible Markov chain $\{X_n\}$, and its transition probability from x to y is given by $p(x,y)$. The goal of this paper is to investigate the properties of solutions (λ, w) for a specific equation:

$$\sum_{y\in S} p(x,y)\exp\Big(h(y)-\lambda+w(y)\Big) = \exp(w(x)) \qquad \forall\, x\in S \tag{1}$$

where $\lambda \in \mathbb{R}, w : S \to \mathbb{R}$.

Why are we interested in this equation? An important motivation is that we can estimate the behavior of $E_x[\exp(\sum_{k=1}^n h(X_k))]$ when n is large. When (λ, w) is a solution of (1), we can define a new probability kernel $\widehat{p}^{\lambda,w} \triangleq p(x,y)\exp\big(h(y) - \lambda + w(y) - w(x)\big)$, and we have

$$\begin{aligned}
&E_x[\exp(\sum_{k=1}^n h(X_k))]\\
=&E_x[\exp\big(\sum_{k=1}^n (h(X_k)-\lambda+w(X_k)-w(X_{k-1})\big)\exp(n\lambda+w(x)-w(X_n))]\\
=&\widehat{E}_x^{\lambda,w}[\exp(n\lambda+w(x)-w(X_n))]\\
\approx&\exp(n\lambda) \text{ if } w \text{ is a bounded function.}
\end{aligned}$$

However, w is unbounded in many cases (See Section 2.5). So we try another easier case, the asymptotic behavior of $E_x[\exp(\sum_{k=1}^n h(X_k))f(X_n)]$ when n large, where f has compact support. We will show in this paper that when certain assumptions are made, $\widehat{p}^{\lambda,w}$ is positive recurrent, and thereby we have the following estimate

$$E_x[\exp(\sum_{k=1}^n h(X_k))f(X_n)] \approx C(f)\exp(n\lambda)\exp(w(x))$$

for n large, where $C(f)$ is a constant that depends on f. To work this out, we define

$g(x) \triangleq f(x)\exp(-w(x))$ and compute

$$
\begin{aligned}
&E_x[\exp(\sum_{k=1}^{n} h(X_k))f(X_n)] \\
=&E_x[\exp\Big(\sum_{k=1}^{n}(h(X_k)-\lambda)\Big)\exp\Big(w(X_n)-w(x)\Big)g(X_n)]\exp(w(x))\exp(n\lambda) \\
=&\widehat{E}_x^{\lambda,w}[g(X_n)]\exp(w(x))\exp(n\lambda),
\end{aligned}
$$

where

$$
\widehat{E}_x^{\lambda,w}[g(X_n)] = \sum_{y\in S: f(y)\neq 0} \widehat{p}_x^{\lambda,w}(X_n = y)f(y)\exp(-w(y)) \to C(f)
$$

as $n \to \infty$ when $\widehat{p}^{\lambda,w}$ is positive recurrent.

In many cases, the Markov chain with transition $\widehat{p}^{\lambda,w}$ is transient. It may be also interesting to study the behavior of w at ∞, which is supposed to be related to the behavior of the Markov chain at ∞. Therefore, the theory of Martin boundary could be helpful in this regard. We give a brief introduction to the Martin boundary theory in the appendix.

We often need more assumptions rather than that $\{X_n\}$ is merely an irreducible Markov chain. In Sections 2.1, 2.3, 5, 6, and Theorems 3.3, 3.4, we assume that p is finitely supported, that is, $\exists M > 0$ such that $p(x, y) = 0$ for all $|x - y| > M$. In Sections 2.4, 4, 5.2, and 6, we assume that p is the transition probability of a random walk.

In Section 2, we demonstrate how to obtain a solution (λ, w) when we have a supersolution (λ, w') to the above equation, that is, (λ, w') satisfies the inequality which replaces " $=$ " above with " $\leq$ " in (1). We also prove several basic properties of equation (1) in this section.

In Section 3, we apply measure changing skills to produce a new probability kernel, and discover some properties of it. This transformation helps us prove if there is only one solution (λ, w) to (1) when λ is fixed.

In Section 4, we use the local central limit theorem to find the smallest λ such that (λ, w) is a solution to (1) under some occasions. Although the local central limit theorem requires the existence of second moment of p, it does not require p to be finitely supported. Therefore, we allow p not to be finitely supported here but with finite second moments.

In Section 5, we investigate more deeply the structure of all solutions to (1), and we derive an explicit formula of these solutions when $h \equiv 0$. The formulation depends heavily on the convex structure of all solutions.

In the last section, we give several examples of equation (1).

2 The structure of all solutions (λ, w)

We'd like to demonstrate how to obtain a solution (λ, w) of (1) when the solutions (λ, w) of the following equation (2) is known:

$$\sum_{y \in S} p(x,y) \exp\Big(h(y) - \lambda + w(y)\Big) \le \exp(w(x)) \qquad \forall\, x \in S \tag{2}$$

We state this result as the following theorem.

Theorem 2.1. *Let $p(x, y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$, where $p(x, y) = 0$ for $|y - x| > M_1$, for all $x, y \in S$ and for some $M_1 > 0$. If (λ, w) is a solution of (2), then for this λ, there exists $\widetilde{w}$ such that $(\lambda, \widetilde{w})$ is a solution of (1).*

Once we have proved this problem, the following important corollary is immediate:

Corollary 2.2. *Let $p(x, y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$, where $p(x, y) = 0$ for $|y - x| > M_1$, for all $x, y \in S$ and for some $M_1 > 0$. If (λ, w) is a solution of (1), then for any $\lambda' > \lambda$, there exists $\widetilde{w}$ such that $(\lambda', \widetilde{w})$ is also a solution of (1).*

Proof. Since (λ', w) satisfies (2), there exists w' such that (λ', w') is a solution of (1) by the previous theorem. $\square$

Corollary 2.3. *Let $p(x, y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$, where $p(x, y) = 0$ for $|y - x| > M_1$, for all $x, y \in S$ and for some $M_1 > 0$. If $h(x)$ is bounded from above, then there exists $\lambda, \widetilde{w}$ such that $(\lambda, \widetilde{w})$ is a solution of (1).*

Proof. Let $(\lambda, w) = (\sup\{h(x) : x \in S\}, 0)$. It is easy to check that (λ, w) is a solution of (2) and hence $\exists \widetilde{w}$ such that $(\lambda, \widetilde{w})$ is a solution of (1). $\square$

After finishing the proof of Theorem 2.1, in Section 2.2, we prove that there is a lower bound for each λ such that (λ, w) is a solution of (1), and p need not be finitely supported here. In Section 2.3, we assume that p is finitely supported so that when λ_0 is the infimum of all possible λ's in (1), (λ_0, w) is also a solution of (1). In Section 2.4 we put some limitation on h to ensure the existence of solutions of (1) when p is the transition probability of a random walk. In Section 2.5 we study the behavior of w under certain assumptions.

2.1 Proof of theorem 2.1.

1. Let (λ_0, w_0) be a solution of (2). We define $\tau_k \triangleq \inf\{n \geq 0 : |X_n| > k\}$, and we have $\tau_k < \infty$ a.s. because $\{X_n\}$ is irreducible. We also define $\widehat{w_k}(x) \triangleq \log\left(E_x[\exp\left(\sum_{m=1}^{\tau_k} h(X_m) - \lambda_0\right) \exp\left(w_0(X_{\tau_k})\right)]\right) > -\infty$. $\widehat{w_k}(x) < \infty$ due to the following lemma:

Lemma 2.4. *$\{Y_n, \mathscr{F}_n = \sigma\{X_1, \cdots, X_n\}\}$ is a supermartingale w.r.t P_x, where*

$$Y_n = \begin{cases} \exp\left(\sum_{i=1}^{n}\left(h(X_i) - \lambda_0\right) + w_0(X_n)\right) & \text{if } n \geq 1 \\ \exp(w(x)) & \text{if } n = 0. \end{cases}$$

Proof. (i)We'd like to show $E_x[Y_n] < \infty$ for every n, and we proceed by induction. Assume $E_x[Y_{n-1}] < \infty$,

$$\begin{aligned} E_x[Y_n] &= E_x\Big[\exp\left(\sum_{i=1}^{n}\left(h(X_i) - \lambda_0\right) + w_0(X_n)\right)\Big] \\ &= \sum_{x, y_1, \cdots, y_n \in S} p(x, y_1)p(y_1, y_2)\cdots p(y_{n-1}, y_n) \times \\ &\quad \exp\left(\sum_{i=1}^{n}\left(h(y_i) - \lambda_0\right) + w_0(y_n)\right) \\ &= \sum_{x, y_1, \cdots, y_{n-1} \in S}\Bigg(p(x, y_1)p(y_1, y_2)\cdots p(y_{n-2}, y_{n-1})\exp\left(\sum_{i=1}^{n-1}\left(h(y_i) - \lambda_0\right)\right) \times \\ &\quad \sum_{y_n \in S} p(y_{n-1}, y_n)\exp\left(h(y_n) - \lambda_0 + w_0(y_n)\right)\Bigg) \end{aligned}$$

$$\leq \sum_{x,y_1,\cdots,y_{n-1}\in S} p(x,y_1)p(y_1,y_2)\cdots p(y_{n-2},y_{n-1}) \exp\left(\sum_{i=1}^{n-1}\Big(h(y_i)-\lambda_0\Big)\right)\times$$

$$\exp\big(w_0(y_{n-1})\big)$$

$$= E_x[Y_{n-1}].$$

(ii)We show $E[Y_{n+1}|\mathscr{F}_n] \leq Y_n$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
E[Y_{n+1}|\mathscr{F}_n] &= \exp\left(\sum_{i=1}^{n}\Big(h(X_i)-\lambda_0\Big)\right)\times \\
&\quad E\Big[\exp\Big(h(X_{n+1})-\lambda_0+w_0((X_{n+1}))\Big)\Big|\mathscr{F}_n\Big] \\
&= \exp\left(\sum_{i=1}^{n}\Big(h(X_i)-\lambda_0\Big)\right)\sum_{y\in S} p(X_n,y)\exp\Big(h(y)-\lambda_0+w_0(y)\Big) \\
&\leq \exp\left(\sum_{i=1}^{n}\Big(h(X_i)-\lambda_0\Big)\right)\exp(w_0(X_n)) \\
&= Y_n.
\end{aligned}$$

□

Due to this lemma, we have $\exp(\widehat{w_k}(x)) = E_x[Y_{\tau_k}] \leq Y_0 = \exp(w_0(x))$ because a positive supermartingale is uniformly integrable and thus optional stopping theorem is applicable.

2. Choose an arbitrary positive integer M_2, and assume $|x| \leq M_2$. Let $k \geq M_1 + M_2$. (M_1 is chosen such that $p(x,y) = 0$ for $|y-x| > M_1$.)

$$\begin{aligned}
\exp(\widehat{w_k}(x)) &= E_x[\exp\left(\sum_{m=1}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\right)] \\
&= E_x[\cdots;|X_1|>k] + E_x[\cdots;|X_1|\leq k] \\
&= (a)+(b).
\end{aligned}$$

$\because |x| \leq k, |X_1| > k \Rightarrow \tau_k = 1$

$$\begin{aligned}
\therefore (a) &= E_x[\exp\Big(h(X_1)-\lambda_0+w_0(X_1)\Big);|X_1|>k] \\
&= \sum_{|y|>k} p(x,y)\exp\Big(h(y)-\lambda_0+w_0(y)\Big) \\
&= 0.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(b) &= E_x\Big[\exp\Big(\sum_{m=1}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big);|X_1|\le k\Big]\\
&= E_x\Big[\exp(h(X_1)-\lambda_0)\exp\Big(\sum_{m=2}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big);|X_1|\le k\Big]\\
&= E_x\Big[E_x\Big[\exp(h(X_1)-\lambda_0)\\
&\quad \exp\Big(\sum_{m=2}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big)1_{\{|X_1|\le k\}}\Big|X_1\Big]\Big]\\
&= E_x\Big[\sum_{|y|\le k}1_{\{X_1=y\}}\exp(h(y)-\lambda_0)\\
&\quad E_x\Big[\exp\Big(\sum_{m=2}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big)\Big|X_1=y\Big]\Big]\\
&= E_x\Big[\sum_{|y|\le k}1_{\{X_1=y\}}\exp(h(y)-\lambda_0)E_y\Big[\exp\Big(\sum_{m=1}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big)\Big]\Big]\\
&= \sum_{|y|\le k}p(x,y)\exp(h(y)-\lambda_0)\exp(\widehat{w_k}(y)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\exp(\widehat{w_k}(x)) &= E_x[\exp\Big(\sum_{m=1}^{\tau_k}\Big(h(X_m)-\lambda_0\Big)+w_0(X_{\tau_k})\Big)]\\
&= \sum_{|y|\le k}p(x,y)\exp(h(y)-\lambda_0)\exp(\widehat{w_k}(y))\\
&= \sum_{y\in S}p(x,y)\exp(h(y)-\lambda_0)\exp(\widehat{w_k}(y))
\end{aligned}
$$

for every $|x|\le M_2$ and $k\ge M_1+M_2$.

3. Next we make some adjustments to $\widehat{w_k}(x)$. We show here that as long as M_2 is large enough, $|\widehat{w_k}(x)-\widehat{w_k}(0)|\le C_x$ for every $|x|\le M_2$ and $k\ge M_1+M_2$, where the finite constant C_x depends on x and is independent of k.

To see this, first select $n>0$ s.t. $p(x,y_1^*)p(y_1^*,y_2^*)\cdots p(y_{n-1}^*,0)>0$ for some $y_1^*,\cdots,y_{n-1}^*\in S$. Choose M_2 be large enough that $M_2\ge|x|+(n-1)M_1$. Thus, for $k\ge M_1+M_2$,

$$
\begin{aligned}
\exp\left(\widehat{w_k}(x)\right) &= \sum_{y\in S} p(x,y)\exp(h(y)-\lambda_0)\exp(\widehat{w_k}(y)) \\
&= \sum_{y_1\in S} p(x,y_1)\exp(h(y_1)-\lambda_0)\times \\
&\quad \Big(\sum_{y_2\in S} p(y_1,y_2)\exp(h(y_2)-\lambda_0)\exp(\widehat{w_k}(y_2))\Big) \\
&= \sum_{y_1,\cdots,y_n\in S} p(x,y_1)p(y_1,y_2)\cdots p(y_{n-1},y_n)\exp\Big(\sum_{m=1}^{n}(h(y_m)-\lambda_0)\Big)\times \\
&\quad \exp\left(\widehat{w_k}(y_n)\right) \\
&\geq p(x,y_1^*)p(y_1^*,y_2^*)\cdots p(y_{n-1}^*,0)\exp\Big(\sum_{m=1}^{n-1}h(y_m^*)+h(0)-n\lambda_0\Big) \\
&\quad \exp\left(\widehat{w_k}(0)\right).
\end{aligned}
$$

Here the first identity requires that $|x| \leq M_2$ and $k \geq M_1 + M_2$, and the second identity requires that $|y_1| \leq M_2$ and $k \geq M_1 + M_2$, for any y_1 appeared in the RHS. Because we choose $M_2 \geq |x| + (n-1)M_1$, $|y_i| \leq M_2$ $\forall 1 \leq i \leq n-1$. Therefore,

$$
\begin{aligned}
\exp\left(\widehat{w_k}(x)-\widehat{w_k}(0)\right) &\geq p(x,y_1^*)p(y_1^*,y_2^*)\cdots p(y_{n-1}^*,0) \\
&\quad \exp\Big(\sum_{m=1}^{n-1}h(y_m^*)+h(0)-n\lambda_0\Big) = c_x > 0.
\end{aligned}
$$

$\therefore\ \widehat{w_k}(x)-\widehat{w_k}(0) \geq \log(c_x) > -\infty$.

Similarly, we may choose M_2 to be larger so that for $k \geq M_1 + M_2$, we also have $\widehat{w_k}(0)-\widehat{w_k}(x) \geq \log(d_x) > -\infty$. Thus we have proved $|\widehat{w_k}(x)-\widehat{w_k}(0)| \leq C_x$ for every $|x| \leq M_2$ and $k \geq M_1 + M_2$.

4. It is immediate to verify that $\widetilde{w_k}(x) = \widehat{w_k}(x)-\widehat{w_k}(0)$ is again a solution for (1) for every $|x| \leq M_2$ and $k \geq M_1 + M_2$, where M_2 is arbitrarily chosen. Since $\widetilde{w_k}(x)$ is bounded in k for $|x| \leq M_2$ and $k \geq M_1 + M_2$, we may use both Bolzano-Weierstrass theorem and diagonal process to select a subsequence $\{n_k\}$ s.t. $\widetilde{w_{n_k}}(x) \to \widetilde{w}(x)$ for

every $x \in S$ as $k \to \infty$.

Fix $x \in S$ and take k large enough such that

$$\exp(\widetilde{w_{n_k}}(x)) = \sum_{y \in S} p(x,y) \exp(h(y) - \lambda_0) \exp(\widetilde{w_{n_k}}(y)).$$

Since there's only finitely many terms in the summation above, letting $k \to \infty$ we obtain

$$\exp(\widetilde{w}(x)) = \sum_{y \in S} p(x,y) \exp(h(y) - \lambda_0) \exp(\widetilde{w}(y)) \qquad \forall x \in S$$

which satisfies (1).

2.2 The greatest lower bound of all possible λ's is finite

Theorem 2.5. *Let $p(x, y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$. We assert that* $\inf\{\lambda : (\lambda, w)$ *is a solution of (1)*$\} > -\infty$.

Proof. Fix any two states x and y. Choose $N, M > 0$ such that $p(x, x_1^*)p(x_1^*, x_2^*) \cdots p(x_N^*, y) > 0$ and $p(y, y_1^*)p(y_1^*, y_2^*) \cdots p(y_M^*, x) > 0$ for states $x_1^*, \cdots, x_N^*$, $y_1^*, \cdots, y_M^*$. We have

$$\begin{aligned}
\exp(w(x)) &= \sum_{z \in S} p(x,z) \exp\Big(h(z) - \lambda + w(z)\Big) \\
&\geq p(x, x_1^*) \exp\Big(h(x_1^*) - \lambda + w(x_1^*)\Big) \\
&= p(x, x_1^*) \exp\Big(h(x_1^*) - \lambda\Big) \sum_{z \in S} p(x_1^*, z) \exp\Big(h(z) - \lambda + w(z)\Big) \\
&\geq p(x, x_1^*)p(x_1^*, x_2^*) \cdots p(x_N^*, y) \exp\left(\sum_{i=1}^{N} \Big(h(x_i^*) - \lambda\Big)\right) \\
&\quad \times \exp\Big(h(y) - \lambda\Big) \exp(w(y))
\end{aligned}$$

$$
\begin{aligned}
&\geq p(x,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)\times p(y,y_1^*)p(y_1^*,y_2^*)\cdots p(y_M^*,x)\\
&\times \exp\left(\sum_{i=1}^{N}\Big(h(x_i^*)-\lambda\Big)\right)\exp\Big(h(y)-\lambda\Big)\\
&\times \exp\left(\sum_{j=1}^{M}\Big(h(y_j^*)-\lambda\Big)\right)\exp\Big(h(x)-\lambda\Big)\exp(w(x)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\exp\Big((N+M+2)\lambda\Big)\geq p(x,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)\times p(y,y_1^*)\times\\
&p(y_1^*,y_2^*)\cdots p(y_M^*,x)\times \exp\left(\sum_{i=1}^{N}h(x_i^*)\right)\exp\left(\sum_{j=1}^{M}h(y_j^*)\right)\exp\Big(h(x)+h(y)\Big)\\
&\Rightarrow \lambda\geq \frac{1}{N+M+2}\Bigg(\log\Big(p(x,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)\times p(y,y_1^*)\\
&p(y_1^*,y_2^*)\cdots p(y_M^*,x)\Big)+\sum_{i=1}^{N}h(x_i^*)+\sum_{j=1}^{M}h(y_j^*)+h(x)+h(y)\Bigg).
\end{aligned}
$$

□

Here's a special case. If x,y are two states such that $p(x,y)>0$ and $p(y,x)>0$, then we have

$$
\begin{aligned}
\exp(w(x)) &= \sum_{y\in S}p(x,z)\exp\Big(h(z)-\lambda+w(z)\Big)\\
&\geq p(x,y)\exp\Big(h(y)-\lambda+w(y)\Big)\\
&= p(x,y)\exp\Big(h(y)-\lambda\Big)\sum_{t\in S}p(y,t)\exp\Big(h(t)-\lambda+w(t)\Big)\\
&\geq p(x,y)p(y,x)\exp\Big(h(x)+h(y)-2\lambda\Big)\exp(w(x)).
\end{aligned}
$$

This implies

$$
\begin{aligned}
&p(x,y)p(y,x)\exp\Big(h(x)+h(y)-2\lambda\Big)\geq 1\\
\Rightarrow&\lambda\geq\frac{1}{2}\Big(h(x)+h(y)+\log\Big(p(x,y)p(y,x)\Big)\Big).
\end{aligned}
$$

2.3 The greatest lower bound of all possible λ's is a solution

Theorem 2.6. *Let $p(x,y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$, where $p(x,y)=0$ for $|y-x|>M_1$, and let*

$\lambda_0 = \inf\{\lambda : (\lambda, w)$ *be a solution of (1)*$\} > -\infty$. *For this* λ_0, *there exists* w_0 *such that* (λ_0, w_0) *is a solution of (1).*

Proof. Let $\{(\lambda_0 + \frac{1}{m}, w_m)\}_m$ be a sequence of solutions of (1) due to corollary 2.2, and we normalize these w's such that $w_m(0) = 0$. The idea is similar to what is presented in the third part of Section 2.1. First, we select $n > 0$ s.t. $p(x, y_1^*)p(y_1^*, y_2^*) \cdots p(y_{n-1}^*, 0) > 0$ for some $y_1^*, \cdots, y_{n-1}^* \in S$. Thus

$$\begin{aligned}
\exp\Big(w_m(x)\Big) &= \sum_{y \in S} p(x, y) \exp(h(y) - \lambda_0) \exp(w_m(y)) \\
&= \sum_{y_1 \in S} p(x, y_1) \exp(h(y_1) - \lambda_0) \\
&\quad \Big(\sum_{y_2 \in S} p(y_1, y_2) \exp(h(y_2) - \lambda_0) \exp(w_m(y_2))\Big) \\
&= \sum_{y_1, \cdots, y_n \in S} p(x, y_1)p(y_1, y_2) \cdots p(y_{n-1}, y_n) \exp\Big(\sum_{i=1}^{n}(h(y_i) - \lambda_0)\Big) \\
&\quad \exp\Big(w_m(y_m)\Big) \\
&\geq p(x, y_1^*)p(y_1^*, y_2^*) \cdots p(y_{n-1}^*, 0) \exp\Big(\sum_{i=1}^{n-1} h(y_i^*) + h(0) - n\lambda_0\Big) \\
&\quad \exp\Big(w_m(0)\Big).
\end{aligned}$$

That is, $\exp(w_m(x)) \geq c_x \exp(w_m(0))$ where c_x is independent of m but depends on the position x. Similarly, $\exp(w_m(0)) \geq d_x \exp(w_m(x))$ for every m. Since we assume that $\exp(w_m(0)) = 1$ for every m, $w_m(x) \in [\log(1/d_x), \log(c_x)]$.

Now we apply both Bolzano-Weierstrass theorem and diagonal process to select a subsequence $\{m_k\}$ such that $w_{m_k}(x) \to \widetilde{w}(x)$ for every $x \in S$ as $k \to \infty$. Therefore, as $k \to \infty$ in

$$\exp(w_{m_k}(x)) = \sum_{y \in S} p(x, y) \exp(h(y) - \lambda_0 - \frac{1}{m_k}) \exp(w_{m_k}(y)),$$

we obtain

$$\exp(\widetilde{w}(x)) = \sum_{y \in S} p(x, y) \exp(h(y) - \lambda_0) \exp(\widetilde{w}(y))$$

which satisfies (1). $\square$

We hence have the following definition.

Definition 2.7. *If $\lambda_0 = \inf\{\lambda : (\lambda, w)$ is a solution of (1)$\}$, then we call (λ_0, w') a **minimal solution** of (1) if (λ_0, w') satisfies (1), and we call this λ_0 the **minimal point** of (1) when (λ_0, w') is a minimal solution of (1).*

2.4 Restrictions on h such that (1) has solutions

Theorem 2.8. *Let $p(x, y)$ be the transition probability from x to y of an irreducible random walk $\{X_n\}$ on $\mathbb{Z}^d$. Let (λ, w) be a solution of (1). If there exists $m \in \mathbb{R}$ such that $h(x) > m \, \forall \, x \in S$, then there is another $K \in \mathbb{R}$ such that $h(x) < K \, \forall \, x \in S$.*

Proof. We first fix any state $y \in S$, and we choose $N, M > 0$ such that $p(0, x_1^*)p(x_1^*, x_2^*) \times \cdots \times p(x_N^*, y) > 0$ and $p(y, y_1^*)p(y_1^*, y_2^*) \cdots p(y_M^*, 0) > 0$ for states $x_1^*, \cdots, x_N^*$, $y_1^*, \cdots, y_M^* \in S$. As computed in Theorem 2.5,

$$\begin{aligned}\exp(w(0)) \geq\ & p(0, x_1^*)p(x_1^*, x_2^*) \cdots p(x_N^*, y) \times p(y, y_1^*)p(y_1^*, y_2^*) \cdots p(y_M^*, 0) \\ & \times \exp\left(\sum_{i=1}^{N} \Big(h(x_i^*) - \lambda\Big)\right) \exp\Big(h(y) - \lambda\Big) \\ & \times \exp\left(\sum_{j=1}^{M} \Big(h(y_j^*) - \lambda\Big)\right) \exp\Big(h(0) - \lambda\Big) \exp(w(0)).\end{aligned}$$

Now we replace 0 with any state $x \in S$. Since $p(x, x + x_1^*)p(x + x_1^*, x + x_2^*) \times \cdots \times p(x + x_N^*, x + y) > 0$ and $p(x + y, x + y_1^*)p(x + y_1^*, x + y_2^*) \cdots p(x + y_M^*, x) > 0$, we have

$$\begin{aligned}\exp(w(x)) \geq\ & p(x, x + x_1^*)p(x + x_1^*, x + x_2^*) \cdots p(x + x_N^*, x + y) \\ & \times p(x + y, x + y_1^*)p(x + y_1^*, x + y_2^*) \cdots p(x + y_M^*, x) \\ & \times \exp\left(\sum_{i=1}^{N} \Big(h(x + x_i^*) - \lambda\Big)\right) \exp\Big(h(x + y) - \lambda\Big) \\ & \times \exp\left(\sum_{j=1}^{M} \Big(h(x + y_j^*) - \lambda\Big)\right) \exp\Big(h(x) - \lambda\Big) \exp(w(x))\end{aligned}$$

$$= p(0,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)\times p(y,y_1^*)p(y_1^*,y_2^*)\cdots p(y_M^*,0)$$
$$\times\exp\left(\sum_{i=1}^{N}\Big(h(x+x_i^*)-\lambda\Big)\right)\exp\Big(h(x+y)-\lambda\Big)$$
$$\times\exp\left(\sum_{j=1}^{M}\Big(h(x+y_j^*)-\lambda\Big)\right)\exp\Big(h(x)-\lambda\Big)\exp(w(x))$$
$$\geq p(0,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)\times p(y,y_1^*)p(y_1^*,y_2^*)\cdots p(y_M^*,0)$$
$$\times\exp\Big((N+M+1)m-(N+M+2)\lambda)\Big)\exp(h(x))\exp(w(x)).$$

This shows $\exp(h(x))\leq\exp\Big((N+M+2)\lambda-(N+M+1)m)\Big)\times$ $\Big(p(0,x_1^*)p(x_1^*,x_2^*)\cdots p(x_N^*,y)p(y,y_1^*)p(y_1^*,y_2^*)\cdots p(y_M^*,0)\Big)^{-1}$ for every $x\in S$, so $h(x)$ is bounded from above, which contradicts our assumption.

$\square$

2.5 Some properties of $w(x)$ when certain restrictions on $h(x)$ are imposed

Theorem 2.9. *Let $p(x,y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$.*
(i) Let $\sup_{x\in S}h(x)=M<\infty$. If (λ,w) is a solution of (1), and $\lambda>M$, then for any $k\in\mathbb{R}$, $\exists x\in S=\mathbb{Z}^d$ such that $w(x)>k$.
(ii) Let $\inf_{x\in S}h(x)=m>-\infty$. If (λ,w) is a solution of (1), and $\lambda<m$, then for any $k\in\mathbb{R}$, $\exists x\in S=\mathbb{Z}^d$ such that $w(x)<k$.

Proof. (i) Assume to the contrary that $\sup_{x\in S}w(x)=K<\infty$. Let $\lambda-M=c>0$.

For this c we may pick some $x_0 \in S$ such that $w(x_0) > K - \frac{c}{2}$. We observe that

$$\begin{aligned}
\exp(w(x_0)) &= \sum_{y \in S} p(x_0, y) \exp\Big(h(y) - \lambda + w(y)\Big) \\
&\le \sum_{y \in S} p(x_0, y) \exp\Big(-c + w(y)\Big) \\
&\le \sum_{y \in S} p(x_0, y) \exp\Big(-c + K\Big) \\
&= \exp\Big(-c + K\Big).
\end{aligned}$$

A contradiction occurs because

$$-c + K \ge w(x_0) \ge K - \frac{c}{2}.$$

(ii) The proof is quite similar to (1). Again we assume to the contrary that $\inf_{x \in S} w(x) = K > -\infty$. Let $m - \lambda = c > 0$. For this c we may pick some $x_0 \in S$ such that $w(x_0) < K + \frac{c}{2}$. We have

$$\begin{aligned}
\exp(w(x_0)) &= \sum_{y \in S} p(x_0, y) \exp\Big(h(y) - \lambda + w(y)\Big) \\
&\ge \sum_{y \in S} p(x_0, y) \exp\Big(c + w(y)\Big) \\
&\ge \sum_{y \in S} p(x_0, y) \exp\Big(c + K\Big) \\
&= \exp\Big(c + K\Big),
\end{aligned}$$

which implies

$$c + K \le w(x_0) \le K + \frac{c}{2},$$

a contradiction. $\square$

3 The $\widehat{p}^{\lambda,w}$ transformation

For any solution (λ, w) of (1), we can define a new probability kernel $\widehat{p}^{\lambda,w}$ as follows:

Definition 3.1. $\widehat{p}^{\lambda,w}(x,y) \triangleq p(x,y)\exp\Big(h(y)-\lambda+w(y)-w(x)\Big)$

It is immediate to check that $\sum_{y\in S}\widehat{p}^{\lambda,w}(x,y)=1$. In the next two subsections we provide criteria to check when $\widehat{p}^{\lambda,w}$ is transient, recurrent, or even positive recurrent. This helps us to check whether w is unique up to adding a constant when λ is fixed and (λ, w) is a solution of (1).

In Theorem 3.2, we do not assume that p is finitely supported, while the assumption that a minimal solution (λ_0, w_0) of (1) exists is necessary.

However, in Theorem 3.3 and Theorem 3.4, we assume that p is finitely supported since the proof of recurrence involves more details.

3.1 The transience, recurrence, and positive recurrence of $\widehat{p}^{\lambda,w}$

Theorem 3.2. *Let $p(x,y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$. Assume that there exists a minimal solution (λ_0, w_0) of (1), then for any $\lambda>\lambda_0$ such that (λ, w) is a solution of (1), $\widehat{p}^{\lambda,w}$ is transient.*

Proof. Since

$$\widehat{p}_n^{\lambda,w}(x,x)=\sum_{y_1,\cdots,y_{n-1}\in S}\widehat{p}^{\lambda,w}(x,y_1)\widehat{p}^{\lambda,w}(y_1,y_2)\times\cdots\times\widehat{p}^{\lambda,w}(y_{n-1},x)$$

$$
\begin{aligned}
&= \sum_{y_1,\cdots,y_{n-1}\in S} p(x,y_1)p(y_1,y_2)\times\cdots\times p(y_{n-1},x)\times \\
&\quad \exp\Big(\big(\sum_{i=1}^{n-1} h(y_i)\big)+h(x)-n\lambda\Big) \\
&= \exp(n(\lambda_0-\lambda)) \sum_{y_1,\cdots,y_{n-1}\in S} p(x,y_1)p(y_1,y_2)\times\cdots\times p(y_{n-1},x)\times \\
&\quad \exp\Big(\big(\sum_{i=1}^{n-1} h(y_i)\big)+h(x)-n\lambda_0\Big) \\
&= \exp(n(\lambda_0-\lambda)) \sum_{y_1,\cdots,y_{n-1}\in S} \widehat{p}^{\lambda_0,w_0}(x,y_1)\widehat{p}^{\lambda_0,w_0}(y_1,y_2)\times\cdots\times \\
&\quad \widehat{p}^{\lambda_0,w_0}(y_{n-1},x) \\
&= \exp(n(\lambda_0-\lambda))\widehat{p}_n^{\lambda_0,w_0}(x,x) \\
&\le \exp(n(\lambda_0-\lambda)).
\end{aligned}
$$

Thus $\sum_{n=0}^{\infty}\widehat{p}_n^{\lambda,w}(x,x)\le 1+\sum_{n=1}^{\infty}\exp(n(\lambda_0-\lambda))<\infty$. □

Theorem 3.3. *Let $p(x,y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$ such that $p(x,y)=0$ for $|x-y|>M_1$; therefore, a minimal solution (λ_0,w_0) of (1) exists. Assume that there is some $\delta>0$ such that $h(x)-\lambda_0<-\delta$ for any $|x|>R$. We claim that $\widehat{p}^{\lambda_0,w_0}(x,y)$ is recurrent.*

Proof. 1. Assume that $\widehat{p}^{\lambda_0,w_0}(x,y)$ is transient, and hence we can define

$$\widehat{g}(x)\triangleq\sum_{y\in S}\sum_{n\ge 0}\widehat{p}_n^{\lambda_0,w_0}(x,y)f(y)<\infty,$$

where $f(x)=1$ if $|x|\le R+M_1$ and $f(x)=0$ if $|x|>R+M_1$. We have $\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)=\sum_{y\in S}\widehat{p}^{\lambda_0,w_0}(x,y)\widehat{g}(y)=\widehat{g}(x)(1-f(x)/\widehat{g}(x))$. Therefore, if $|x|\le R+M_1$, $\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)\le(1-\mu)\widehat{g}(x)$, where $0<\mu=\inf_{|x|\le R+M_1}\frac{f(x)}{g(x)}\le 1$, and if $|x|>R+M_1$, $\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)=\widehat{g}(x)$.

2. Define $\widetilde{g}(x) = \exp(-w_0(x))$, we have

$$\begin{aligned}
\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x) &= \sum_{y\in S} p(x,y)\exp(h(y)-\lambda_0)\exp(w_0(y)-w_0(x)) \times \exp(-w_0(y)) \\
&= \exp(-w_0(x))\sum_{y\in S} p(x,y)\exp(h(y)-\lambda_0) \\
&\leq \exp(-w_0(x))\sum_{y\in S} p(x,y)\exp(-\delta) \\
&= \exp(-\delta)\widetilde{g}(x)
\end{aligned}$$

for any $|x| > R + M_1$.

3. Consider $F_t(x) = \widehat{g}(x)^t\widetilde{g}(x)^{1-t}$ for $0 < t < 1$. We have

$$\begin{aligned}
\widehat{p}^{\lambda_0,w_0}F_t(x) &\leq \Big(\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)\Big)^t\Big(\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x)\Big)^{1-t} \\
&= F_t(x)\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)}{\widehat{g}(x)}\Big)^t\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x)}{\widetilde{g}(x)}\Big)^{1-t} \\
&\leq F_t(x)\bigg(t\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)}{\widehat{g}(x)}\Big) + (1-t)\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x)}{\widetilde{g}(x)}\Big)\bigg).
\end{aligned}$$

Here the first inequality is Holder's inequality and the third one is the Young's inequality. For each $|x| \leq R + M_1$,

$$\begin{aligned}
t\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)}{\widehat{g}(x)}\Big) + (1-t)\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x)}{\widetilde{g}(x)}\Big) &\leq t(1-\mu) + (1-t)\max_{|y|\leq R+M_1}\{\frac{\widehat{p}^{\lambda_0,w_0}\widetilde{g}(y)}{\widetilde{g}(y)}\} \\
&\leq 1-\mu/2,
\end{aligned}$$

for some t close to 1. Fix this t and consider the case $|x| > R + M_1$, we have

$$t\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widehat{g}(x)}{\widehat{g}(x)}\Big) + (1-t)\Big(\frac{\widehat{p}^{\lambda_0,w_0}\widetilde{g}(x)}{\widetilde{g}(x)}\Big) \leq t + (1-t)\exp(-\delta) < 1.$$

4. Therefore, for every $x \in S$, $\widehat{p}^{\lambda_0,w_0}F_t(x) = \sum_{y\in S}\widehat{p}^{\lambda_0,w_0}(x,y)F_t(y) \leq \exp(-\delta')F_t(x)$ for some $\delta' > 0$, where F_t is defined in the previous step.

5. The existence of F_t shows that $(\lambda_0 - \delta', \log F_t(x))$ is a solution of (2). This implies $(\lambda_0 - \delta', w')$ is a solution of (1) for some w', contradicts the minimality of λ_0. □

Now we strengthen the above theorem by proving that $\widehat{p}^{\lambda_0,w_0}(x,y)$ is positive recurrent.

Theorem 3.4. *Let $p(x,y)$ be the transition probability from x to y of an irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$ such that $p(x,y)=0$ for all $x,y\in S$ satisfying $|x-y|>M_1$, for some $M_1>0$. Assume that there exists $\delta>0$ such that $h(x)-\lambda_0<-\delta$ for all $|x|>R$. We claim that if (λ_0,w_0) is a minimal solution of (1), then $\widehat{p}^{\lambda_0,w_0}(x,y)$ is positive recurrent.*

Proof. 1. We claim that $\{\widehat{M}_n \triangleq \exp\big(-\sum_{i=1}^n (h(X_i)-\lambda_0)-w_0(X_n)\big);$
$\mathscr{F}_n=\sigma(\widehat{M}_1,\cdots,\widehat{M}_n)\}$ is a martingale with respect to $\widehat{p}_x^{\lambda_0,w_0}$, where $\widehat{M}_0=\exp(-w_0(X_0))=\exp(-w_0(x))$. To prove the claim, first we note that since $p(x,y)=0$ for $|x-y|>M_1$, $\widehat{E}_x^{\lambda_0,w_0}[\widehat{M}_n]<\infty$ for all n. For any $n\geq 1$,

$$\begin{aligned}\widehat{E}_x^{\lambda_0,w_0}[\widehat{M}_n|\mathscr{F}_{n-1}] =&\exp\Big(-\sum_{i=1}^{n-1}\big(h(X_i)-\lambda_0\big)\Big)\times\\ &\sum_{y\in S}\widehat{p}^{\lambda_0,w_0}(X_{n-1},y)\exp\big(-(h(y)-\lambda_0)-w_0(y)\big)\\ =&\exp\Big(-\sum_{i=1}^{n-1}\big(h(X_i)-\lambda_0\big)\Big)\exp(-w_0(X_{n-1}))\sum_{y\in S}p(X_{n-1},y)\\ =&\widehat{M}_{n-1}.\end{aligned}$$

2. For any $|x|>R$, we define $\tau_x\triangleq\min\{n: X_0=x, |X_n|\leq R\}$. By 1. and the optional sampling theorem,

$$\widehat{E}_x^{\lambda_0,w_0}[\widehat{M}_{\tau_x};\tau_x<n]\leq \widehat{E}_x^{\lambda_0,w_0}[\widehat{M}_{n\wedge\tau_x}]=\exp(-w_0(x)).$$

Since $\widehat{p}_x^{\lambda_0,w_0}(\tau_x<n)\to 1$ as $n\to\infty$ by the conclusion of Theorem 3.3, we have

$$\begin{aligned}\exp(-w_0(x))\geq&\widehat{E}_x^{\lambda_0,w_0}[\widehat{M}_{\tau_x}]\\ \geq&\widehat{E}_x^{\lambda_0,w_0}[\exp\Big(-\sum_{i=1}^{\tau_x}\big(h(X_i)-\lambda_0\big)-w_0(X_{\tau_x})\Big)]\\ \geq&\min_{|y|\leq R}\{\exp(-w_0(y)-h(y)+\lambda_0)\}\widehat{E}_x^{\lambda_0,w_0}[\exp\big((\tau_x-1)\delta\big)].\end{aligned}$$

Now if we restrict our choice of x to the finite set $\{|x| > R, x \in \mathbb{Z}^d : \exists |y| \leq R$ so that $\widehat{p}^{\lambda_0,w_0}(y,x) > 0\}$, then we have $\widehat{E}_x^{\lambda_0,w_0}[\tau_x] < K_1$, for all x in this set.

3. For each $|y| \leq R$, there exists a fixed time $T_y > 0$ such that $\widehat{p}_y^{\lambda_0,w_0}(|X_{T_y}| \leq R) < 1 - \delta_y$ for some $\delta_y > 0$. Let $T = \max\{T_y : |y| \leq R\}$, $\delta = \min\{\delta_y : |y| \leq R\}$, we claim that for any $|y| \leq R$,

$$\widehat{p}_y^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq kT) \leq (1-\delta)^k.$$

To see this, for $k = 1$ we have

$$\begin{aligned}\widehat{p}_y^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq T) \leq& \widehat{p}_y^{\lambda_0,w_0}(|X_{T_y}| \leq R)\\ <&1-\delta_y\\ \leq&1-\delta,\end{aligned}$$

and for $k > 1$,

$$\begin{aligned}&\widehat{p}_y^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq kT)\\ =&\widehat{p}_y^{\lambda_0,w_0}\big(\widehat{E}_y^{\lambda_0,w_0}\big[|X_n| \leq R \;\forall 1 \leq n \leq kT \big| X_{T_y}\big]\big)\\ =&\sum_{|z|\leq R} \widehat{p}_y^{\lambda_0,w_0}(\widehat{p}_z^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq kT - T_y), X_{T_y} = z)\\ \leq&\sum_{|z|\leq R} \widehat{p}_y^{\lambda_0,w_0}(\widehat{p}_z^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq (k-1)T), X_{T_y} = z)\\ \leq&\sum_{|z|\leq R} (1-\delta)^{k-1}\widehat{p}_y^{\lambda_0,w_0}(X_{T_y} = z) \qquad \text{by induction hypothesis}\\ =&(1-\delta)^{k-1}\widehat{p}_y^{\lambda_0,w_0}(|X_{T_y}| \leq R) \leq (1-\delta)^k.\end{aligned}$$

For each $|y| \leq R$, let $\tau_y = \min\{n : X_0 = y, |X_n| > R\}$, we have

$$\begin{aligned}\widehat{E}_y^{\lambda_0,w_0}[\tau_y] =& \sum_{k=0}^{\infty} \widehat{E}_y^{\lambda_0,w_0}[\tau_y, kT+1 \leq \widehat{\tau_y} \leq kT+T]\\ \leq& \sum_{k=0}^{\infty}(kT+T)\widehat{p}_y^{\lambda_0,w_0}(kT+1 \leq \tau_y \leq kT+T)\\ \leq& \sum_{k=0}^{\infty}(kT+T)\widehat{p}_y^{\lambda_0,w_0}(|X_n| \leq R \;\forall 1 \leq n \leq kT)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{\infty}(kT+T)(1-\delta)^k \\
&= K_2 < \infty.
\end{aligned}
$$

4. Define $A = \{|y| \leq R : y \in \mathbb{Z}^d\}$, $B = \{|x| > R, x \in \mathbb{Z}^d : \exists y \in A$ so that $\widehat{p}^{\lambda_0,w_0}(y,x) > 0\}$. For any $z \in B$, we let $\{\rho_n^z\}_{n=1}^{\infty}$ be a sequence of stopping times such that

$$
\begin{aligned}
&\rho_0^z \triangleq 0, \\
&\rho_1^z \triangleq \min\{n : X_0 = z, |X_n| \leq R\}, \\
&\rho_2^z \triangleq \min\{n > \rho_1^z : |X_n| > R\}, \\
&\rho_3^z \triangleq \min\{n > \rho_2^z : |X_n| \leq R\},
\end{aligned}
$$

and so on. We find that $\{\widetilde{X}_n\}_{n=0}^{\infty} = \{X_0 = z, X_{\rho_1^z}, X_{\rho_2^z}, \cdots\}$ is a Markov chain on $A \cup B$, and its transition probability $\widetilde{p}(x,y)$ is given by $\widehat{p}^{\lambda_0,w_0}(\widetilde{X}_n = y|\widetilde{X}_{n-1} = x)$ for arbitrary n. Let $A' \triangleq \{x \in A : \widetilde{p}_z(\widetilde{X}_N = x) = \widehat{p}_z^{\lambda_0,w_0}(X_{\rho_N^z} = x) > 0$ for some $N \geq 0\}$, $B' \triangleq \{x \in B : \widetilde{p}_z(\widetilde{X}_N = x) = \widehat{p}_z^{\lambda_0,w_0}(X_{\rho_N^z} = x) > 0$ for some $N \geq 0\}$. Now $\{\widetilde{X}_n\}_{n=0}^{\infty}$ is a Markov chain on $A' \cup B'$ with transition probability $\widetilde{p}(x,y)$.

If the z we chose is a recurrent state with respect to $\{\widetilde{X}_n\}_{n=0}^{\infty}$ on $A' \cup B'$, then $\{\widetilde{X}_n\}_{n=0}^{\infty}$ is irreducible on $A' \cup B'$, and we're done. If z is transient with respect to $\{\widetilde{X}_n\}_{n=0}^{\infty}$ on $A' \cup B'$, the chain must contain some transient states and one or more recurrent classes. We may always pick a state $z' \in B'$ in one of these recurrent classes. Now we replace our original z with this new state z' and perform the same procedure as above. Note that the new $A' \cup B'$ with respect to z' is a subset of the $A' \cup B'$ of our original z in this case.

Now $\{\widetilde{X}_n\}_{n=0}^{\infty}$, which starts from $z \in B$, is an irreducible, positive recurrent Markov chain whose state space is $A' \cup B'$ with transition probability $\widetilde{p}$. Therefore, we have

$$
\sum_{k=1}^{\infty} k\widetilde{p}_z(\widetilde{X}_1 \neq z, \cdots, \widetilde{X}_{k-1} \neq z, \widetilde{X}_k = z) < \infty.
$$

5. Fix $z \in B'$. Let $\rho(z) \triangleq \min\{n > 0 : X_0 = z, X_n = z\}$ be the first return time of z. Now our goal is to prove

$$\widehat{E}_z^{\lambda_0,w_0}[\rho(z)] < \infty,$$

and this implies the original process $\{X_n\}$ is positive recurrent with respect to $\widehat{p}^{\lambda_0,w_0}$. We take another random time $T(z) \triangleq \min\{\rho_n^z > 0 : X_0 = z, X_{\rho_n^z} = z\}$, where $T(z) \geq \rho(z)$. We start to estimate $\widehat{E}_z^{\lambda_0,w_0}[T(z)]$:

$$\begin{aligned}
&\widehat{E}_z^{\lambda_0,w_0}[T(z)] \\
=&\sum_{k=1}^{\infty} \widehat{E}_z^{\lambda_0,w_0}[\rho_k^z; X_{\rho_1^z} \neq z, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z] \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k} \widehat{E}_z^{\lambda_0,w}[\rho_n^z - \rho_{n-1}^z; X_{\rho_1^z} \neq z, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z] \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}} \widehat{E}_z^{\lambda_0,w_0}[\rho_n^z - \rho_{n-1}^z; X_{\rho_1^z} \neq z, \cdots, X_{\rho_{n-1}^z} = a, \\
&X_{\rho_n^z} = b, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z] \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}} \widehat{E}_z^{\lambda_0,w_0}[\widehat{E}_z^{\lambda_0,w}[\rho_n^z - \rho_{n-1}^z; X_{\rho_1^z} \neq z, \cdots, X_{\rho_{n-1}^z} = a, \\
&X_{\rho_n^z} = b, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z | X_{\rho_1^z}, \cdots, X_{\rho_{n-1}^z}, X_{\rho_n^z}, \rho_n^z - \rho_{n-1}^z]] \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}} \widehat{E}_z^{\lambda_0,w_0}[(\rho_n^z - \rho_{n-1}^z)\widehat{E}_z^{\lambda_0,w_0}[X_{\rho_{n+1}^z} \neq z, \cdots, X_{\rho_{k-1}^z} \neq z, \\
&X_{\rho_k^z} = z | X_{\rho_1^z}, \cdots, X_{\rho_{n-1}^z} X_{\rho_n^z}, \rho_n^z - \rho_{n-1}^z]; X_{\rho_1^z} \neq z, \cdots, X_{\rho_{n-1}^z} = a, X_{\rho_n^z} = b] \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}} \widehat{E}_z^{\lambda_0,w_0}[(\rho_n^z - \rho_{n-1}^z); X_{\rho_1^z} \neq z, \cdots, X_{\rho_{n-1}^z} = a, X_{\rho_n^z} = b] \\
&\widehat{E}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z} \neq z, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z | X_{\rho_n^z} = b) \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}} \widehat{E}_z^{\lambda_0,w_0}[\widehat{E}_z^{\lambda_0,w_0}[(\rho_n^z - \rho_{n-1}^z); X_{\rho_1^z} \neq z, \cdots, X_{\rho_{n-1}^z} = a, X_{\rho_n^z} = b | \\
&X_{\rho_1^z}, \cdots, X_{\rho_{n-1}^z}]] \times \widehat{p}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z} \neq z, \cdots, X_{\rho_{k-1}^z} \neq z, X_{\rho_k^z} = z | X_{\rho_n^z} = b)
\end{aligned}$$

$$
\begin{aligned}
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}}\widehat{E}_z^{\lambda_0,w_0}[\widehat{E}_z^{\lambda_0,w_0}[(\rho_n^z-\rho_{n-1}^z);X_{\rho_n^z}=b|X_{\rho_{n-1}^z}];X_{\rho_1^z}\neq z,\\
&\cdots,X_{\rho_{n-1}^z}=a]\times\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z|X_{\rho_n^z}=b)\\
\leq&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}}\widehat{E}_z^{\lambda_0,w_0}[\widehat{E}_z^{\lambda_0,w_0}[\rho_n^z-\rho_{n-1}^z|X_{\rho_{n-1}^z}=a];X_{\rho_1^z}\neq z,\\
&\cdots,X_{\rho_{n-1}^z}=a]\times\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z|X_{\rho_n^z}=b)\\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}}\widehat{E}_a^{\lambda_0,w_0}[\tau_a]\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_1^z}\neq z,\cdots,X_{\rho_{n-1}^z}=a)\times\\
&\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z|X_{\rho_n^z}=b)\\
\leq&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}}\max\{K_1,K_2\}\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_1^z}\neq z,\cdots,X_{\rho_{n-1}^z}=a)\times\\
&\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_{n+1}^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z|X_{\rho_n^z}=b)\times\\
&\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_n^z}=b|X_{\rho_{n-1}^z}=a)\max_{a,b\in A'\cup B'\setminus\{z\}}\Big(\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_n^z}=b|X_{\rho_{n-1}^z}=a)\Big)^{-1}\\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{a,b\in A'\cup B'\setminus\{z\}}\max\{K_1,K_2\}\max_{a,b\in A'\cup B'\setminus\{z\}}\Big(\widetilde{p}_z(\widetilde{X}_n=b|\widetilde{X}_{n-1}=a)\Big)^{-1}\\
&\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_1^z}\neq z,\cdots,X_{\rho_{n-1}^z}=a,X_{\rho_n^z}=b,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z)\\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\max\{K_1,K_2\}\max_{a,b\in A'\cup B'\setminus\{z\}\widetilde{p}_a(\widetilde{X}_1=b)>0}\Big(\widetilde{p}_a(\widetilde{X}_1=b)\Big)^{-1}\times\\
&\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_1^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z)\\
=&\max\{K_1,K_2\}\max_{a,b\in A'\cup B'\setminus\{z\}}\Big(\widetilde{p}_a(\widetilde{X}_1=b)\Big)^{-1}\sum_{k=1}^{\infty}k\widehat{p}_z^{\lambda_0,w_0}(X_{\rho_1^z}\neq z,\cdots,X_{\rho_{k-1}^z}\neq z,X_{\rho_k^z}=z)\\
=&\max\{K_1,K_2\}\max_{a,b\in A'\cup B'\setminus\{z\}}\Big(\widetilde{p}_a(\widetilde{X}_1=b)\Big)^{-1}\sum_{k=1}^{\infty}k\widetilde{p}_z(\widetilde{X}_1\neq z,\cdots,\widetilde{X}_{k-1}\neq z,\widetilde{X}_k=z)\\
<&\infty.
\end{aligned}
$$

Therefore, $\widehat{E}_z^{\lambda_0,w_0}[\rho(z)]\leq\widehat{E}_z^{\lambda_0,w_0}[T(z)]<\infty$, and the proof is complete. □

3.2 Criteria for when all harmonic functions are constants

The following lemma taken from [4] shows us the recurrence of a Markov chain is equivalent to the triviality of nonnegative superharmonic functions. Let us first define harmonic functions and superharmonic functions.

Definition 3.5. *Let p be the probability kernel of a Markov chain $\{X_n\}$ on S. If $h(x) \geq 0$ for all $x \in S$ and $ph(x) \triangleq \sum_{y \in S} p(x,y)h(y) \leq h(x)$ for all $x \in S$, then we call h a p-**superharmonic function** for $\{X_n\}$, or simply superharmonic function for $\{X_n\}$. If we replace $\leq$ above with $=$, then h is called a p-**harmonic function** for $\{X_n\}$.*

Lemma 3.6. *Assume that p is the probability kernel of an irreducible Markov chain on S. Then p is recurrent on S iff all superharmonic functions are constants.*

Proof. We first consider the "if" part. When p is transient, then pick any $x \in S$, $G(\cdot, x)$ is superharmonic but not harmonic, and hence nonconstant, which is a contradiction.

In the following we consider the "only if" part.

1. Assume that $f(x)$ is a superharmonic function. If $f(y_0) > pf(y_0)$ for some $y_0 \in S$, then we have

$$\begin{aligned} f(x) &= \Big(\sum_{m=0}^{n} \sum_{y \in S} p_m(x,y)(f(y) - pf(y)) \Big) + p_{n+1} f(x) \\ &\geq \sum_{m=0}^{n} p_m(x, y_0)(f(y_0) - pf(y_0)). \end{aligned}$$

We may then let $n \to \infty$ to get a contradiction because $\sum_{m=0}^{\infty} p_m(x, y_0) = \infty$. Therefore, $f(x) \equiv pf(x)$.

2. Fix any $x_0 \in S$ and let $M = f(x_0)$. Because $f(x) \wedge M$ is a superharmonic function, $f(x) \wedge M$ is also harmonic by 1. If $f(x_1) < M$ for some $x_1 \in S$, we may

pick $N > 0$ such that $P_N(x_0, x_1) > 0$. Therefore,

$$
\begin{aligned}
M = f(x_0) \wedge M &= \sum_{y \in S} p_N(x_0, y)(f(y) \wedge M) \\
&\leq \Big(\sum_{y \in S \setminus \{x_1\}} p_N(x_0, x_1) M \Big) + P_N(x_0, x_1) f(x_1) \\
&< M,
\end{aligned}
$$

which is impossible. Hence $f(x) \geq f(x_0)$ for all $x \in S$. Since x_0 is arbitrarily chosen, f must be a constant function. □

When a Markov chain is transient, it may have many harmonic functions. The Martin boundary theory provides us with a criteria about when all of the Markov chain's harmonic functions are trivial. The interested readers are invited to the appendix of this paper.

Lemma 3.7. *Assume that the probability kernel p of an irreducible Markov chain is transient on S. Then the minimal boundary of p is a single point iff all harmonic functions of p are constants.*

The following theorem explains why we're interested in the recurrence or transience for $\widehat{p}^{\lambda,w}$.

Theorem 3.8. *Let p be the probability kernel of an irreducible Markov chain on S, and (λ, w) is a solution of (1). Assume that all harmonic functions of $\widehat{p}^{\lambda,w}$ are constants, then for this λ, (λ, w) is the unique solution of (1) in the sense that w is unique up to the addition of a constant.*

Proof. Assume that (λ, w_1) is another solution of (1) such that $\exp(w_1(x))$

$/\exp(w(x))$ is not a constant function. Then $W(x) = \exp(w_1(x) - w(x))$ satisfies

$$\begin{aligned} W(x) &= \sum_{y \in S} p(x, y) \exp(h(y) - \lambda) \exp(w(y) - w(x)) W(y) \\ &= \sum_{y \in S} \widehat{p}^{\lambda, w}(x, y) W(y) \ \forall\, x \in S, \end{aligned}$$

and this implies $W(x)$ is a nonconstant harmonic function for $\widehat{p}^{\lambda, w}$. However, this is impossible by our assumption, and it turns out that w is unique up to the addition of a constant when λ is fixed and $(\lambda,\, w)$ is a solution of (1). $\square$

4 Estimates for some lower bound for all λ's such that (λ, w) is a solution of (1) under some conditions

In this section we assume that p is the transition probability from x to y of an irreducible random walk $\{X_n\}$ on $\mathbb{Z}^d$. We prove several results when $\sum_{x\in S} |x|^2 p(0,x) < \infty$ and $\mu = \sum_{x\in S} xp(0,x) = 0$, but we do not assume that p is finitely supported.

We start with some definitions and lemmas that help us develop a powerful tool, the local central limit theorem. We then use this tool to provide a lower bound for all λ's such that (λ, w) is a solution of (1). For some h we may even prove the existence of the minimal point and find what it is, without assuming that p is finitely supported.

The local central limit theorem is taken from [1], and we give a detailed proof here. For further reference, see [2].

4.1 Local central limit theorem

Definitions 4.1. *Let the state space S be $\mathbb{Z}^d$.*

1. *The mean μ is defined as $\sum_{x\in S} xp(0,x)$.*

2. *Let $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$. The second moment quadratic form Q is defined as $Q[\theta] = (\widetilde{Q}\theta\cdot\theta) \triangleq \sum_{x\in S} |((x-\mu)\cdot\theta)|^2 p(0,x)$ for $\theta \in S$. Indeed, the ij-th component $(\widetilde{Q})_{ij}$ of the d-dimensional matrix $\widetilde{Q}$ equals $\sum_{x\in S}(x_i-\mu_i)(x_j-\mu_j)p(0,x)$, where $x_i = (x\cdot e_i)$ is the i-th component of x. The associated bilinear form $B(\theta_1,\theta_2) = \sum_{x\in S}((x-\mu)\cdot\theta_1)\times((x-\mu)\cdot\theta_2)p(0,x)$.*

3. *The determinant $|Q|$ of the quadratic form Q is defined as $\det(\widetilde{Q})$.*

4. *The inverse quadratic form Q^{-1} is defined as $Q^{-1}[\theta] \triangleq (\widetilde{Q}^{-1}\theta\cdot\theta)$ for $\theta \in S$.*

5. *The characteristic function $\psi(\theta) \triangleq \sum_{x\in S} e^{i\theta\cdot(x-\mu)} p(0,x)$.*

Lemma 4.2. *Let p be the transition function of an irreducible random walk on $S = Z^d$. Then, Q is a positive quadratic form, namely, $\exists c_1 \geq c_2 > 0$ such that $c_1|\theta|^2 \geq Q[\theta] \geq c_2|\theta|^2$.*

Proof. It suffices to prove $Q[\theta] > 0$ for any $\theta \in \mathbb{Z}^d$. Assume to the contrary that $Q[\theta_0] = 0$ for some $\theta_0 \in \mathbb{Z}^d$, and thus $((x - \mu) \cdot \theta_0) = 0$ for any $x \in S$ such that $p(0, x) > 0$.

If $(y \cdot \theta_0) = 0$ for every $y \in S$ such that $p(0, y) > 0$, then for any y' such that $(y' \cdot \theta_0) \neq 0$, $p_n(0, y') = 0$ for every $n \in \mathbb{N} \cap \{0\}$, contradicts the irreducibility on S.

Therefore, there must be some $y \in S$ such that $(y \cdot \theta_0) \neq 0$ and $p(0, y) > 0$. For this y, we pick $y_1, \cdots, y_n \in S$ such that $y_1 + \cdots + y_n = y$ and $p(0, y_1)p(y_1, y_1 + y_2)p(y_1 + y_2, y_1 + y_2 + y_3) \times \cdots \times p(y_1 + \cdots + y_{n-1}, y) > 0$. We find that

$$\begin{aligned}(\mu \cdot \theta_0) &= (y \cdot \theta_0) \qquad \because p(0, y) > 0 \\ &= \sum_{i=1}^{n} (y_i \cdot \theta_0) \\ &= n(\mu \cdot \theta_0) \qquad \because p(0, y_i) > 0 \ \forall 1 \leq i \leq n,\end{aligned}$$

and this implies $(y \cdot \theta_0) = (\mu \cdot \theta_0) = 0$, a contradiction to our assumption $(y \cdot \theta_0) \neq 0$. $\square$

Lemma 4.3. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, and we assume that $m_2 = \sum_{x \in S} |x|^2 p(0, x) < \infty$. Then we have*

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-\frac{1}{2}Q[w]) \exp(-iw\cdot(x-n\mu)/\sqrt{n})dw = |Q|^{-1/2} \exp\big(\frac{-1}{2n}Q^{-1}[x-n\mu]\big).$$

Proof. Since the associated matrix $\widetilde{Q}$ of the quadratic form Q is also positive definite, it has an orthonormal basis of eigenvectors $\{v_1, \cdots, v_d\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_d\}$. For each $w \in \mathbb{R}^d$, with a little bit abuse of notation we define $w_j = (w \cdot v_j)$ for $1 \leq j \leq d$ in this lemma. We have $Q[w] = (\widetilde{Q}w \cdot w) = (\sum_{j=1}^d \lambda_j w_j v_j \cdot \sum_{j=1}^d w_j v_j) = \sum_{j=1}^d \lambda_j w_j^2$. Therefore,

$$
\begin{aligned}
&(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp(-\frac{1}{2}Q[w])\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw\\
=&(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp(-\frac{1}{2}\sum_{j=1}^{d}\lambda_j w_j^2)\exp\Big(-\frac{i}{\sqrt{n}}\sum_{j=1}^{d}w_j(x_j-n\mu_j)\Big)\\
&\exp\Big(-\frac{1}{2}\sum_{j=1}^{d}(\frac{i}{\sqrt{n}})^2(x_j-n\mu_j)^2\frac{1}{\lambda_j}\Big)\exp\Big(\frac{1}{2}\sum_{j=1}^{d}(\frac{i}{\sqrt{n}})^2(x_j-n\mu_j)^2\frac{1}{\lambda_j}\Big)dw\\
=&(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp\Big(-\frac{1}{2}\sum_{j=1}^{d}\big(\sqrt{\lambda_j}w_j+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\sqrt{\lambda_j}}\big)^2\Big)\\
&\exp\Big(\frac{1}{2}\sum_{j=1}^{d}(\frac{i}{\sqrt{n}})^2(x_j-n\mu_j)^2\frac{1}{\lambda_j}\Big)dw\\
=&(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp\Big(-\frac{1}{2}\sum_{j=1}^{d}\big(\sqrt{\lambda_j}w_j+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\sqrt{\lambda_j}}\big)^2\Big)\\
&\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big)dw\\
=&(2\pi)^{-d/2}\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big)\times\\
&\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\exp\Big(-\frac{1}{2}\sum_{j=1}^{d}\big(\sqrt{\lambda_j}w_j+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\sqrt{\lambda_j}}\big)^2\Big)dw_1\cdots dw_d\\
=&(2\pi)^{-d/2}\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big)\\
&\prod_{j=1}^{d}\int_{\mathbb{R}}\exp\Big(-\frac{1}{2}(\sqrt{\lambda_j}w_j+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\sqrt{\lambda_j}})^2\Big)dw_j\\
=&(2\pi)^{-d/2}\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big)\prod_{j=1}^{d}\int_{\mathbb{R}}\exp\Big(-\frac{1}{2}(\sqrt{\lambda_j}w_j)^2\Big)dw_j\\
=&(2\pi)^{-d/2}\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big)\prod_{j=1}^{d}\sqrt{\frac{2\pi}{\lambda_j}}\\
=&|Q|^{-1/2}\exp\Big(\frac{-1}{2n}Q^{-1}[x-n\mu]\Big).
\end{aligned}
$$

To prove the identity

$$\int_{\mathbb{R}}\exp\Big(-\frac{1}{2}(\sqrt{\lambda_j}w_j)^2\Big)dw_j=\int_{\mathbb{R}}\exp\Big(-\frac{1}{2}(\sqrt{\lambda_j}w_j+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\sqrt{\lambda_j}})^2\Big)dw_j$$

given above, we consider the rectangular contour

$$C:-M\to N\to N+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\lambda_j}\to -M+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\lambda_j}\to -M,$$

and notice that $\int_C \exp(-\frac{1}{2}\lambda_j w_j^2)dw_j = 0$ because $f(z) = \exp(\frac{-\lambda_j z^2}{2})$ is an entire function on the complex plane. We have

$$\begin{aligned}
0 &= \int_C \exp(-\frac{1}{2}\lambda_j w_j^2)dw_j \\
&= \int_{-M}^{N} \exp(-\frac{1}{2}\lambda_j w_j^2)dw_j - \int_{-M}^{N} \exp\Big(-\frac{1}{2}(\sqrt{\lambda_j}w_j + \frac{i}{\sqrt{n}}(x_j - n\mu_j)\frac{1}{\sqrt{\lambda_j}})^2\Big)dw_j \\
&+ \int_{N}^{N+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\lambda_j}} \exp(-\frac{1}{2}\lambda_j w_j^2)dw_j - \int_{-M}^{-M+\frac{i}{\sqrt{n}}(x_j-n\mu_j)\frac{1}{\lambda_j}} \exp(-\frac{1}{2}\lambda_j w_j^2)dw_j.
\end{aligned}$$

Let $N, M \to \infty$, the last two terms vanish and we have the desired result. $\square$

Lemma 4.4. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, where $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$. We have $\lim_{|\theta|\to 0} \frac{1-\psi(\theta)}{Q[\theta]} = \frac{1}{2}$.*

Proof. 1. We claim that for every $t \in \mathbb{R}$, $|1 - e^{it} + it + \frac{1}{2}(it)^2| \le At^2$ for some $A > 0$. This result is obvious for $t \ge 1$. For $t < 1$, note that $|1 - e^{it} + it + \frac{1}{2}(it)^2| \le \sum_{n=3}^{\infty} \frac{1}{n!}t^n \le t^2\big(\sum_{n=3}^{\infty}\frac{1}{n!}\big)$.

2. Since $Q[\theta] \ge c_2|\theta|^2$ by Lemma 4.2,

$$\begin{aligned}
\Big|\frac{1-\psi(\theta)}{Q[\theta]} - \frac{1}{2}\Big| &\le \frac{1}{c_2|\theta|^2}\Big|1 - \psi(\theta) - \frac{1}{2}Q[\theta]\Big| \\
&\le \frac{1}{c_2|\theta|^2}\sum_{x\in S}\Big|1 - e^{i\theta\cdot(x-\mu)} + i\theta\cdot(x-\mu) + \frac{1}{2}\big(i\theta\cdot(x-\mu)\big)^2\Big|p(0,x) \\
&\le \frac{1}{c_2|\theta|^2}\sum_{x\in S} A\big(\theta\cdot(x-\mu)\big)^2 p(0,x) \\
&\le \frac{A}{c_2}\sum_{x\in S}|x-\mu|^2 p(0,x) < \infty.
\end{aligned}$$

The convergence of $|\frac{1-\psi(\theta)}{Q[\theta]} - \frac{1}{2}| \to 0$ as $|\theta| \to 0$ follows from dominated convergence theorem, since for each $x \in S$, $\frac{1}{|\theta|^2}|1 - e^{i\theta\cdot(x-\mu)} + i\theta\cdot(x-\mu) + \frac{1}{2}\big(i\theta\cdot(x-\mu)\big)^2| \le |\theta|\sum_{n=3}^{\infty}\frac{|\theta|^{n-3}|x-\mu|^n}{n!} \to 0$ as $\theta \to 0$. $\square$

Lemma 4.5. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, where $m_2 = \sum_{x\in S}|x|^2 p(0,x) < \infty$. Then for any $A > 0$, $\psi^n(\frac{w}{\sqrt{n}}) \to e^{-\frac{1}{2}Q[w]}$ uniformly for $|w| \le A$.*

Proof. 1. For $z \in \mathbb{C}$ with $|z|$ small enough,

$$\begin{aligned}\log(1-z) - \log 1 &= \int_0^1 \frac{d}{dt}\log(1-tz)dt \\ &= \int_0^1 \frac{-z}{1-tz}dt \\ &= -z - z\int_0^1 \frac{tz}{1-tz}dt,\end{aligned}$$

where

$$\begin{aligned}|-z\int_0^1 \frac{tz}{1-tz}dt| &\le |z|\int_0^1 |\frac{tz}{1-tz}|dt \\ &\le |z|^2\int_0^1 \frac{t}{1/2}dt = |z|^2.\end{aligned}$$

The second inequality is due to $|z|$ small.

2. Let $R_n(w) \triangleq \frac{1-\psi(\frac{w}{\sqrt{n}})}{Q[\frac{w}{\sqrt{n}}]} - \frac{1}{2}$ and $R_n(0) \triangleq 0$, then $\psi(\frac{w}{\sqrt{n}}) = 1 - \frac{1}{2n}Q[w] - \frac{1}{n}R_n(w)Q[w]$. We have

$$\begin{aligned}\psi^n(\frac{w}{\sqrt{n}}) &= \exp\big(n\log(1-\frac{1}{2n}Q[w] - \frac{1}{n}R_n(w)Q[w])\big) \\ &= \exp\big(n(\frac{-1}{2n}Q[w] - \frac{1}{n}R_n(w)Q[w] + S(w))\big) \\ &= \exp\big(-\frac{1}{2}Q[w] - R_n(w)Q[w] + n\cdot S(w))\big)\end{aligned}$$

where

$$|S(w)| \le |\frac{-1}{2n}Q[w] - \frac{1}{n}R_n(w)Q[w]|^2 = \frac{Q[w]^2}{n^2}|\frac{-1}{2} - R_n(w)|^2.$$

Given $A > 0$, for every $N > 0$, $|R_N(w)|$ is a continuous function of w on $\{w : |w| \le A\}$, so

$$M_N = \max\{|R_N(w)| : |w| \le A\} < \infty.$$

In addition, for $n \ge N$,

$$M_N \ge M_n = \max\{|R_N(w)| : |w| \le \sqrt{\frac{N}{n}}A\},$$

thus $|R_n(w)| \leq M_1$ for every $|w| \leq A$ and $n \geq N$. Furthermore, since

$$M_n = \max\{|R_1(w)| : |w| \leq \sqrt{\frac{1}{n}}A\},$$

we have $M_n \to 0$ as $n \to \infty$ since $R_1(w)$ is continuous at $w = 0$. Therefore, let $n \to \infty$,

$$\psi^n(\frac{w}{\sqrt{n}}) \to \exp(-\frac{1}{2}Q[w])$$

uniformly for $|w| \leq A$, where A is arbitrarily chosen. □

Lemma 4.6. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, and $m_2 = \sum_{x \in S} |x|^2 p(0, x) < \infty$. Then there exists $\alpha > 0$ small enough such that $|\psi^n(\frac{w}{\sqrt{n}})| \leq e^{-\frac{1}{4}Q[w]}$ for all $n \in \mathbb{N}$ and w where $|w/\sqrt{n}| \leq \alpha$.*

Proof. First choose α small enough such that $|\frac{1-\psi(\theta)}{Q[\theta]} - \frac{1}{2}| \leq \frac{1}{8}$ and $Q[\alpha] \leq \frac{1}{8}$ for all $|\theta| \leq \alpha$. As what we've done in the previous lemma, for all $n \in \mathbb{N}$ and w such that $|w/\sqrt{n}| \leq \alpha$, we have

$$\begin{aligned}
|\psi^n(\frac{w}{\sqrt{n}})| &= \exp(-\frac{1}{2}Q[w]) \times \left|\exp(-R_n(w)Q[w])\right| \times \left|\exp(n \cdot S(w))\right| \\
&\leq \exp(-\frac{1}{2}Q[w]) \exp(|-R_n(w)|Q[w]) \exp(\frac{1}{n}\left|\frac{-1}{2} - R_n(w)\right|^2 Q[w]^2) \\
&\leq \exp(-\frac{1}{2}Q[w]) \exp(\frac{1}{8}Q[w]) \exp(Q[w/\sqrt{n}]\left|\frac{-1}{2} - R_n(w)\right|^2 Q[w]) \\
&\leq \exp(-\frac{1}{2}Q[w]) \exp(\frac{1}{8}Q[w]) \exp(\frac{1}{8}Q[w]) = e^{-\frac{1}{4}Q[w]}
\end{aligned}$$

and this completes the proof. □

Lemma 4.7. *Let p be the transition function of an irreducible, aperiodic random walk on $S = \mathbb{Z}^d$. Then $|\psi(\theta)| = 1$ if and only if for every $1 \leq j \leq d$, θ_j is a multiple of 2π, where θ_j is the j-th component of θ.*

Proof. ($\Leftarrow$) Assume that for every $1 \leq j \leq d$, θ_j is a multiple of 2π, we have $|\psi(\theta)| = |\sum_{x \in S} e^{i\theta \cdot (x-\mu)} p(0, x)| = |e^{-i(\theta \cdot \mu)} \sum_{x \in S} e^{i\theta \cdot x} p(0, x)| = |e^{-i(\theta \cdot \mu)}| = 1$.

($\Rightarrow$) Assume that there exists $\theta \in \mathbb{R}^d$ such that $|\psi(\theta)| = 1$. This implies that $\exists t \in \mathbb{R}$ such that $(\theta \cdot x) - t$ is a multiple of 2π for every $x \in S$ such that

$p(0,x) > 0$. Since p is aperiodic, there exists $n \in \mathbb{N}$ such that $p_n(0,0) > 0$ and $p_{n+1}(0,0) > 0$. Choose $y_1, \cdots, y_n, z_1, \cdots, z_{n+1}$ such that $y_1 + \cdots + y_n = z_1 + \cdots + z_{n+1} = 0$, and $p(0,y_1)p(y_1, y_1+y_2) \times \cdots \times p(y_1+\cdots+y_{n-1}, 0) > 0$, $p(0,z_1)p(z_1, z_1+z_2) \times \cdots \times p(z_1+\cdots+z_n, 0) > 0$.

Therefore, $(\theta\cdot\sum_{i=1}^n y_i) - nt = (\theta\cdot 0) - nt$ and $(\theta\cdot\sum_{i=1}^{n+1} z_i) - (n+1)t = (\theta\cdot 0) - (n+1)t$ are both multiples of 2π, and hence t is a multiple of 2π. This implies $(\theta \cdot x)$ is a multiple of 2π for every $x \in S$ such that $p(0,x) > 0$.

Choose $y_1, \cdots, y_n \in S$ such that $y_1 + \cdots + y_n = e_j$ and $p(0,y_1)p(y_1,y_1+y_2) \times \cdots \times p(y_1+\cdots+y_{n-1}, e_j) > 0$. Thus for every $1 \le j \le d$, $\theta_j = (\theta \cdot e_j) = (\theta \cdot \sum_{i=1}^n y_i)$, which is a multiple of 2π. $\square$

Lemma 4.8. *Let p be the transition function of an irreducible, aperiodic random walk on $S = \mathbb{Z}^d$. Given any $\alpha > 0$, $(2\pi)^{-d/2} \int_{\alpha\sqrt{n} \le |w|; w \in \sqrt{n}C} |\psi^n(\frac{w}{\sqrt{n}})| dw$ $\to 0$ as $n \to \infty$, where $C = [-\pi, \pi]^d$.*

Proof. 1. Since $|\psi(\theta)|$ is continuous on $C = [-\pi,\pi]^d$, and $|\psi(\theta)| = 1$ only when $\theta = 0$ by Lemma 4.7, $\exists \delta > 0$ such that $|\psi(\theta)| < 1 - \delta$ for $|\theta| \ge \alpha, \theta \in C$.

2. $(2\pi)^{-d/2} \int_{\alpha\sqrt{n} \le |w|; w \in \sqrt{n}C} |\psi^n(\frac{w}{\sqrt{n}})| dw \le (2\pi)^{-d/2}(1-\delta)^n (2\sqrt{n}\pi)^d \to 0$ as $n \to \infty$. $\square$

Theorem 4.9.(local central limit theorem) *Let p be the transition function of an irreducible, aperiodic random walk on $S = \mathbb{Z}^d$, and $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$ (This condition automatically holds when p is assumed to be finitely supported). Then*

$$(2\pi n)^{d/2} p_n(0,x) - |Q|^{-1/2} \exp\left(\frac{-1}{2n} Q^{-1}[x - n\mu]\right) \to 0$$

uniformly for $x \in S$.

Proof. 1. First note that

$$e^{-in\mu\cdot\theta} \sum_{y\in S} e^{iy\cdot\theta} p_n(0,y) = e^{-in\mu\cdot\theta} \Big(\sum_{y\in S} e^{iy\cdot\theta} p(0,y))^n = \psi(\theta)^n.$$

Multiply both sides by $e^{i(n\mu-x)\cdot\theta}$ for some $x \in S$ and integrate θ over $C = [-\pi,\pi]^d$, we have

$$(2\pi)^d p_n(0,x) = \int_C \sum_{y\in S} e^{i(y-x)\cdot\theta} p_n(0,y)d\theta = \int_C e^{i(n\mu-x)\cdot\theta}\psi(\theta)^n d\theta.$$

Let $\theta = w/\sqrt{n}$,

$$(2\pi n)^{d/2} p_n(0,x) = (2\pi)^{-d/2}\int_{\sqrt{n}C} \psi(\frac{w}{\sqrt{n}})^n e^{-i(x-n\mu)\cdot w/\sqrt{n}} dw$$

2. Let $(2\pi n)^{d/2}p_n(0,x) = I_0(n)+I_1(n,A)+I_2(n,A)+I_3(n,A,\alpha)+I_4(n,\alpha)$, where

$$I_0(n) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \exp(-\frac{1}{2}Q[w])\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw,$$
$$I_1(n,A) = (2\pi)^{-d/2}\int_{|w|\le A}\left(\psi(\frac{w}{\sqrt{n}})^n - \exp(-\frac{1}{2}Q[w])\right)\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw,$$
$$I_2(n,A) = -(2\pi)^{-d/2}\int_{|w|>A}\exp(-\frac{1}{2}Q[w])\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw,$$
$$I_3(n,A,\alpha) = (2\pi)^{-d/2}\int_{A<|w|\le\alpha\sqrt{n}}\psi(\frac{w}{\sqrt{n}})^n\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw,$$
$$I_4(n,\alpha) = (2\pi)^{-d/2}\int_{\alpha\sqrt{n}\le|w|;w\in\sqrt{n}C}\psi(\frac{w}{\sqrt{n}})^n\exp(-iw\cdot(x-n\mu)/\sqrt{n})dw.$$

3. By Lemma 4.3, $I_0(n) = |Q|^{-1/2}\exp\left(\frac{-1}{2n}Q^{-1}[x-n\mu]\right)$. By Lemma 4.6, we can choose α small enough such that

$$|I_3(n,A,\alpha)| \le (2\pi)^{-d/2}\int_{|w|>A} e^{-\frac{1}{4}Q[w]}dw,$$

hence we can now let A be large enough so that both $|I_2(n,A)|$ and $|I_3(n,A,\alpha)|$ are small. Now, both α and A are fixed, by Lemma 4.5 we let n be large so that $|I_1(n,A)|$ is small, and finally let n be even larger so that $|I_4(n,A)|$ is also small by Lemma 4.8. The estimates are uniform for all $x \in S$. $\square$

4.2 Lower bound for all λ's such that (λ, w) is a solution of (1)

In this section, we use the local central limit theorem to find a lower bound of all λ's in some cases, where (λ, w) is a solution of (1).

Theorem 4.10. *Let p be the transition function of a irreducible, aperiodic random walk on $S = \mathbb{Z}^d$, $\mu = 0$, $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$, and $\inf_{x\in S} h(x) = m > -\infty$. Then $\lambda \geq m$ for every λ such that (λ, w) is a solution of (1).*

Proof. 1. By Theorem 4.9, $(2\pi n)^{d/2} p_n(0,0) - |Q|^{-1/2} \to 0$ as $n \to \infty$. Choose $N > 0$ such that $(2\pi n)^{d/2} p_n(0,0) > \frac{1}{2}|Q|^{-1/2}$ for all $n \geq N$.

2. Assume that (λ, w) is a solution of (1) and $m - \lambda = c > 0$. Since

$$
\begin{aligned}
\exp(w(0)) &= \sum_{y\in S} p(0,y) \exp\left(h(y) - \lambda + w(y)\right) \\
&= \sum_{y_1,\cdots,y_n \in S} p(0,y_1)p(y_1,y_2)\cdots p(y_{n-1},y_n) \exp\left((\sum_{i=1}^{n} h(y_i)) - n\lambda + w(y_n) \right) \\
&\geq \sum_{y_1,\cdots,y_n \in S} p(0,y_1)p(y_1,y_2)\cdots p(y_{n-1},y_n) \exp\left(nc + w(y_n) \right) \\
&\geq \sum_{y_1,\cdots,y_{n-1} \in S} p(0,y_1)p(y_1,y_2)\cdots p(y_{n-1},0) \exp\left(nc + w(0) \right).
\end{aligned}
$$

Thus $1 \geq \exp(nc) p_n(0,0)$ for every $n \in \mathbb{N}$. Now, for every $n \geq N$, we have $1 \geq \exp(nc)p_n(0,0) \geq \exp(nc)(2\pi n)^{-d/2}\frac{1}{2}|Q|^{-1/2}$, but it is impossible for n large. Therefore, $\lambda \geq m$. $\square$

In fact, we can further remove the aperiodicity condition.

Theorem 4.11. *Let p be the transition function of a irreducible random walk on $S = \mathbb{Z}^d$, $\mu = 0$, $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$, and $\inf_{x\in S} h(x) = m > -\infty$. Then $\lambda \geq m$ for every λ such that (λ, w) is a solution of (1).*

Proof. 1. Define $p_\alpha(x,y) = (1-\alpha)\delta(x,y) + \alpha p(x,y)$ for every $x, y \in S$,$0 < \alpha < 1$. Since p_α is aperiodic, so we can apply Theorem 4.9 to this new transition probability. The mean for p_α is defined as $\mu_\alpha \triangleq \sum_{x\in S} x p_\alpha(0,x) = 0$, and its quadratic form is defined as $Q_\alpha[\theta] = (\widetilde{Q}_\alpha\theta \cdot \theta) \triangleq \sum_{x\in S} |((x-\mu_\alpha)\cdot\theta)|^2 p_\alpha(0,x)$. The ij-th component $(\widetilde{Q}_\alpha)_{ij}$ of the d-dimensional matrix $\widetilde{Q}_\alpha$ equals $\sum_{x\in S} x_i x_j p_\alpha(0,x) = \alpha(\widetilde{Q})_{ij}$, where $\widetilde{Q}$ is the corresponding matrix of the quadratic form Q induced from p, hence $|Q_\alpha|^{-1/2} = \alpha^{-d/2}|Q|^{-1/2}$.

2. Let (λ, w) be a solution of (1) such that $m - \lambda = c > 0$. As in the preceding theorem, we have $1 \geq \exp(nc)(p_\alpha)_n(0,0)$ for every $n \in \mathbb{N}$. Now we fix arbitrary $0 < \alpha < 1$, and then select $N \in \mathbb{N}$ large such that $(\alpha e^{-c} + 1 - \alpha)^N < \frac{1}{4}(2\pi N)^{-d/2}\alpha^{-d/2}|Q|^{-1/2}$ and $(p_\alpha)_N(0,0) \geq \frac{1}{2}(2\pi N)^{-d/2}|Q_\alpha|^{-1/2}$.

3. We hence have

$$\begin{aligned}
\frac{1}{4}(2\pi N)^{-d/2}\alpha^{-d/2}|Q|^{-1/2} &\geq (\alpha e^{-c} + 1 - \alpha)^N \\
&= \sum_{j=0}^{N} \alpha^j (1-\alpha)^{N-j} \begin{pmatrix} N \\ j \end{pmatrix} \exp(-jc) \\
&\geq \sum_{j=0}^{N} \alpha^j (1-\alpha)^{N-j} \begin{pmatrix} N \\ j \end{pmatrix} p_j(0,0) \\
&= (p_\alpha)_N(0,0) \\
&\geq \frac{1}{2}(2\pi N)^{-d/2}|Q_\alpha|^{-1/2} \\
&= \frac{1}{2}(2\pi N)^{-d/2}\alpha^{-d/2}|Q|^{-1/2},
\end{aligned}$$

which is clearly a contradiction. This implies $m - \lambda \leq 0$.

□

We strengthen Theorem 4.11 a little bit more in the next theorem.

Theorem 4.12. *Let p be the transition function of a irreducible random walk on $S = \mathbb{Z}^d$, $\mu = 0$, $m_2 = \sum_{x\in S} |x|^2 p(0,x) < \infty$, and the random walk has period*

$k \geq 1$. We partition S into $S_0, S_1, \cdots, S_{k-1}$, and for each $x \in S_i$, $p_{i+(n-1)k}(0,x) > 0$ for some $n \in \mathbb{N}$. Assume that $\inf_{x \in S_i} h(x) = m_i > -\infty$ for each $0 \leq i \leq k-1$, and define $m \triangleq \frac{1}{k}(m_0 + \cdots + m_{k-1})$. We assert that $\lambda \geq m$ for every λ such that (λ, w) is a solution of (1).

Proof. We assume that (λ, w) is a solution of (1) and $m - \lambda = c > 0$. We have

$$
\begin{aligned}
&\exp(w(0)) = \sum_{y \in S} p(0,y) \exp\big(h(y) - \lambda + w(y)\big) \\
&= \sum_{y_1, \cdots, y_{nk} \in S} p(0,y_1)p(y_1,y_2) \cdots p(y_{nk-1}, y_{nk}) \exp\big((\sum_{i=1}^{nk} h(y_i)) - nk\lambda + w(y_{nk})\big) \\
&\geq \sum_{y_1, \cdots, y_{nk} \in S} p(0,y_1)p(y_1,y_2) \cdots p(y_{nk-1}, y_{nk}) \exp\big(n(\sum_{j=0}^{k-1} m_j) - nk\lambda + w(y_{nk})\big) \\
&\geq \sum_{y_1, \cdots, y_{nk} \in S} p(0,y_1)p(y_1,y_2) \cdots p(y_{nk-1}, y_{nk}) \exp\big(nkc + w(y_{nk})\big) \\
&\geq \sum_{y_1, \cdots, y_{nk-1} \in S} p(0,y_1)p(y_1,y_2) \cdots p(y_{nk-1}, 0) \exp\big(nkc + w(0)\big).
\end{aligned}
$$

Thus $1 \geq \exp(nkc)p_{nk}(0,0)$ for every $n \in \mathbb{N}$. Indeed, we also have

$$1 \geq \exp(nkc + (m_1 + \cdots + m_j) - j\lambda)p_{nk+j}(0,0)$$

for $1 \leq j \leq k-1$, when we replace nk above with $nk + j$. Therefore, when we define $m' = \min\{0, m_1 - \lambda - c, \cdots, \sum_{j=1}^{k-1} m_j - (k-1)\lambda - (k-1)c\}$, we have $\exp(-nc) \geq e^{m'} p_n(0,0)$ for every $n \in \mathbb{N}$. The rest of the proof is almost the same as that of Theorem 4.11, and only the following needs revision:

$$
\begin{aligned}
&\sum_{j=0}^{N} \alpha^j (1-\alpha)^{N-j} \begin{pmatrix} N \\ j \end{pmatrix} \exp(-jc) \\
\geq &\sum_{j=0}^{N} \alpha^j (1-\alpha)^{N-j} \begin{pmatrix} N \\ j \end{pmatrix} e^{m'} p_j(0,0) \\
= &e^{m'} (p_\alpha)_N(0,0).
\end{aligned}
$$

□

Below are some applications of the above theorems.

Corollary 4.13. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, $m_2 = \sum_{x \in S} |x|^2 p(0,x) < \infty$, and $\mu = 0$. If $h(x) \equiv 0$, then the minimal point exists, and it is* 0.

Proof. By Theorem 4.11, $\lambda \geq 0$ for every λ such that (λ, w) is a solution of (1). Since for any constant function $w \equiv c$, $(0, w)$ is a solution of (1), it turns out that λ_0 exists and $\lambda_0 = 0$. $\square$

Corollary 4.14. *Let p be the transition function of an irreducible random walk on $S = \mathbb{Z}^d$, $\mu = 0$, $m_2 = \sum_{x \in S} |x|^2 p(0,x) < \infty$, and the random walk has period $k \geq 1$. We partition S into $S_0, S_1, \cdots, S_{k-1}$, and for each $x \in S_i$, $p_{i+(n-1)k}(0,x) > 0$ for some $n \in \mathbb{N}$. If $h(x) = m_i > -\infty$ for each $0 \leq i \leq k-1$ and $x \in S_i$, then the minimal point λ_0 exists and $\lambda_0 = m \triangleq \frac{1}{k}(m_0 + \cdots + m_{k-1})$.*

Proof. By Theorem 4.12, $\lambda \geq m = \frac{1}{k}(m_0 + \cdots + m_{k-1})$ for every λ such that (λ, w) is a solution of (1). Now we let $w(x) = 0$ for $x \in S_0$, and $w(x) = jm - \sum_{i=1}^{j} m_i$ for $x \in S_j$, $1 \leq j \leq k-1$. We find (m, w) is a solution of (1), and therefore $\lambda_0 = m$. $\square$

5 One step further about the the solution structure

We hope to prove that when λ is fixed, all $W(x) = \exp(w(x))$ such that (λ, w) is a solution of (1) form a convex set under certain assumptions. With this convex structure, we may find the explicit form of all solutions (λ, w) of (1) when the process is a random walk and $h \equiv 0$.

We assume that the irreducible Markov chain $\{X_n\}$ on $\mathbb{Z}^d$ is finitely supported throughout this section, and in Section 5.2 we assume that $\{X_n\}$ is a random walk.

5.1 The solution structure: general case

For every real-valued function $f(x)$ on $S = \mathbb{Z}^d$ with $f(0) = 1$, we may treat f as an element in $\mathbb{R}^{S\setminus\{0\}}$. If we enumerate $\mathbb{Z}^d \setminus \{0\} = \{x_1, x_2, \cdots\}$ and we define a metric d on $\mathbb{R}^{S\setminus\{0\}}$ with $d(f,g) = \sup_{i\in\mathbb{N}} \frac{|f(x_i)-g(x_i)|\wedge 1}{i}$, for $f, g \in \mathbb{R}^{S\setminus\{0\}}$. With this metric, $d(f_n, f) \to 0$ if and only if $f_n(x) \to f(x)$ for every $x \in S \setminus \{0\}$. Indeed, this metric induces the product topology on $\mathbb{R}^{S\setminus\{0\}}$. We adopt this metric throughout this section when we talk about the space $\mathbb{R}^{S\setminus\{0\}}$.

Lemma 5.1. *Let* $A^\lambda = \{W : W(x) > 0\ \forall x \in S, W(0) = 1, \sum_{y\in S} p(x,y)$ $\exp\big(h(y) - \lambda\big)W(y) = W(x)\ \forall x \in S\} \subset \mathbb{R}^{S\setminus\{0\}}$. *We assert that* A^λ *is a convex, compact subset of* $\mathbb{R}^{S\setminus\{0\}}$.

Proof. 1. The proof of convexity is straightforward.

2. We first show that A^λ is a subset of some compact set in $\mathbb{R}^{S\setminus\{0\}}$. Just as what we've done in the third part of the proof of Theorem 2.1, we show that for any $W \in A^\lambda$, $|W(x)| \le C_x$, where C_x is independent of the choice of W but depends on $x \in S \setminus \{0\}$. The idea is as follows. For arbitrary $W \in A^\lambda$, we select $n > 0$ s.t. $p(x, y_1^*)p(y_1^*, y_2^*) \cdots p(y_{n-1}^*, 0) > 0$ for some $y_1^*, \cdots, y_{n-1}^* \in S$. Therefore,

$$
\begin{aligned}
W(x) &= \sum_{y\in S} p(x,y)\exp(h(y)-\lambda)W(y) \\
&= \sum_{y_1\in S} p(x,y_1)\exp(h(y_1)-\lambda)\Big(\sum_{y_2\in S} p(y_1,y_2)\exp(h(y_2)-\lambda)W(y_2) \\
&= \sum_{y_1,\cdots,y_n\in S} p(x,y_1)p(y_1,y_2)\cdots p(y_{n-1},y_n)\exp\Big(\sum_{m=1}^{n}(h(y_m)-\lambda)\Big)W(y_n) \\
&\geq p(x,y_1^*)p(y_1^*,y_2^*)\cdots p(y_{n-1}^*,0)\exp\Big(\sum_{m=1}^{n-1} h(y_m^*)+h(0)-n\lambda_0\Big)W(0) \\
&= p(x,y_1^*)p(y_1^*,y_2^*)\cdots p(y_{n-1}^*,0)\exp\Big(\sum_{m=1}^{n-1} h(y_m^*)+h(0)-n\lambda_0\Big)
\end{aligned}
$$

So we obtain a lower bound of $W(x)$, which is independent of the choice of $W \in A^\lambda$ and is greater than 0. Exchange x and 0 above we obtain an upper bound of $W(x)$.

We find that $A^\lambda \subset [a_x, b_x]^{S\setminus\{0\}}$, where $a_x, b_x > 0$ for any $x \in S$. This set is a compact set due to Tychonoff theorem (See [6]; we list it below).

3. We'd like to show A^λ is closed. Note that $d(W_n, W) \to 0 \Leftrightarrow W_n(x) \to W(x)$ for every $x \in S \setminus \{0\}$ as $n \to \infty$. The closedness of A^λ follows directly from taking pointwise limit in $W_n(x) = \sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda\big)W_n(y)$ for each $x \in S$, where the summation below consists of only finitely many terms:

$$
\begin{aligned}
W(x) &= \lim_{n\to\infty} W_n(x) \\
&= \sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda\big)\lim_{n\to\infty} W_n(y) \\
&= \sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda\big)W(y).
\end{aligned}
$$

Note that $W(x) > 0$ and $W(0) = 1$, so $W \in A^\lambda$.

4. As a closed subset of a compact set, A^λ is compact in $\mathbb{R}^{S\setminus\{0\}}$. □

Theorem 5.2.(Tychonoff theorem) *An arbitrary product of compact spaces is compact in the product topology.*

Let A_e^λ be the set of all extreme points of $A^\lambda = \{W \in \mathbb{R}^{S\setminus\{0\}} : W(x) > 0\ \forall x \in S, W(0) = 1, \sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda\big)W(y) = W(x)\,\forall x \in S\}$, we'd like to show A_e^λ is a Borel set in $\mathbb{R}^{S\setminus\{0\}}$ by the following lemma, which is taken from [5].

Lemma 5.3. *If X is a metrizable, compact convex subset of a topological vector space, then the extreme points of X form a G_δ set, which is the intersection of countably many open sets.*

Proof. Let d be the metric for X. Let $F_n = \{x : x = \frac{y+z}{2}$ for some $y, z \in X$ with $d(y,z) \geq \frac{1}{n}\}$. For each $x_m \in F_n$ and $x_m \to x \in X$, we'd like to show $x \in F_n$ and hence F_n is a closed set.

Write $x_m = \frac{y_m+z_m}{2}, y_m, z_m \in X$ for all $m \in \mathbb{N}$. Since X is compact, we may find a subsequence $\{m_{1,j}\}_j$ of $\{m\}_m$ such that $\{y_{m_{1,j}}\} \to y$ when $j \to \infty$. We may pick a further subsequence $\{m_{2,j}\}_j$ of $\{m_{1,j}\}_j$ such that $\{z_{m_{2,j}}\} \to j$ when $j \to \infty$. Therefore, let $j \to \infty$ in $x_{m_{2,j}} = \frac{y_{m_{2,j}}+z_{m_{2,j}}}{2}$, we have $x = \frac{y+z}{2}$ for $y, z \in X$. Notice that $d(y,z) \geq \frac{1}{n}$.

Let X_e be the set of all extreme points of X. We claim that $X_e = \bigcap_{n=1}^\infty F_n^c$ and the proof is complete. To this end, if $x \in X_e$, then x cannot be written as any convex combination of y, z where $y, z \in X$ and $y \neq z$. Thus $x \in F_n^c$ for all $n \in \mathbb{N}$.

Conversely, for $x \neq X_e$, we consider $x = cy + (1-c)z$ for some $0 < c < 1$, $y \neq z$, and $y, z \in X$. WLOG we assume that $1 > c \geq \frac{1}{2}$. We find that

$$x = \frac{1}{2}y + \frac{1}{2}\Big((2c-1)y + 2(1-c)z\Big),$$

which implies that $x \in F_N$ for some N large. Thus $x \notin \bigcap_{n=1}^\infty F_n^c$. □

Next we give the definition of locally convex linear space, and then give the statement of Choquet's theorem from [5]. Choquet's theorem plays an important role in our later developments.

Definition 5.4. *Let V be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$. If the topology of V has a basis where each member is a convex set, then V is a locally convex topological vector space.*

Theorem 5.5.(Choquet's theorem) *Suppose that X is a metrizable compact convex subset of a locally convex linear space E, and that x_0 is an element of X. Then there is a probability measure (namely, a Borel measure of total measure 1) μ on X which is supported by the extreme points of X and $f(x_0) = \int_X f(x)d\mu(x)$ for every continuous linear functional f on E.*

Note that $\mathbb{R}^{S\setminus\{0\}}$ with the product topology is a locally convex linear space, and A^λ is compact and convex in $\mathbb{R}^{S\setminus\{0\}}$. Besides, for each $x \in S\setminus\{0\}$, $f_x : W \in \mathbb{R}^{S\setminus\{0\}} \mapsto W(x)$ is a continuous linear functional. We hence apply Choquet's theorem: if $W \in \mathbb{R}^{S\setminus\{0\}}, W(x) > 0\ \forall x \in S, W(0) = 1$, and $\sum_{y\in S} p(x,y) \exp\big(h(y) - \lambda\big)W(y) = W(x)\,\forall x \in S$, then for every $x \in S \setminus \{0\}$,

$$
\begin{aligned}
W(x) = f_x(W) &= \int_X f_x(\widetilde{W})d\mu(\widetilde{W}) \\
&= \int_X \widetilde{W}(x)d\mu(\widetilde{W}),
\end{aligned}
$$

where μ is supported in A_e^λ. For $x = 0$,

$$
\begin{aligned}
W(0) = 1 &= \int_X 1d\mu(\widetilde{W}) \\
&= \int_X \widetilde{W}(0)d\mu(\widetilde{W}).
\end{aligned}
$$

Note that Choquet's theorem also tells us A_e^λ must be nonempty when A^λ is nonempty.

5.2 The solution structure when $h(x) \equiv 0$

Throughout this subsection, we assume that $h(x) \equiv 0$. This strong assumption helps us find an explicit form for elements in A_e^λ. See the following theorem.

Theorem 5.6. *Let A_e^λ be the set of all extreme points of $A^\lambda = \{W \in \mathbb{R}^{S\setminus\{0\}} : W(x) > 0\ \forall x \in S, W(0) = 1, \sum_{y\in S} p(x,y)\exp\big(h(y)-\lambda\big)W(y) = W(x)\ \forall x \in S\}$. If A^λ is nonempty, then for any $W \in A_e^\lambda$, $W(x) = e^{u\cdot x}$, where $\phi(u) = \sum_{x\in S} e^{u\cdot x}p(0,x) = \exp(\lambda)$.*

Proof. 1. For any $x, z \in S$,

$$\begin{aligned}
W(x+z) &= \sum_{y\in S} p(x+z,y)\exp(-\lambda)W(y) \\
&= \sum_{y\in S} p(x+z,y+z)\exp(-\lambda)W(y+z) \\
&= \sum_{y\in S} p(x,y)\exp(-\lambda)W(y+z)
\end{aligned}$$

Hence as a function of x, $\frac{1}{W(z)}W(x+z) \in A^\lambda$.

2. Choose $N > 0$ such that $p_N(0,z) > 0$. For every $x \in S$ we have

$$\begin{aligned}
W(x) &= \sum_{y\in S} p_N(x,y)\exp(-N\lambda)W(y) \\
&\geq p_N(x,x+z)\exp(-N\lambda)W(x+z) \\
&= p_N(0,z)\exp(-N\lambda)W(x+z).
\end{aligned}$$

Therefore, $W(x) \geq \Big(p_N(0,z)\exp(-N\lambda)W(z)\Big)\frac{1}{W(z)}W(x+z) = c(z)\frac{1}{W(z)}W(x+z)$ for every $x \in S$. In particular, when $x = 0$, we have $1 \geq c(z) > 0$.

3. If $c(z) = 1$, then for any $x' \in S$, we choose $N > 0$ s.t. $p_N(0,x') > 0$:

$$\begin{aligned}
0 &= W(0) - \frac{1}{W(z)}W(0+z) \\
&= \sum_{y\in S} p_N(0,y)\exp(-N\lambda)\big(W(y) - \frac{1}{W(z)}W(y+z)\big) \\
&\geq p_N(0,x')\exp(-N\lambda)\big(W(x') - \frac{1}{W(z)}W(x'+z)\big) \\
&\geq 0.
\end{aligned}$$

Thus $W(x')W(z) = W(x'+z)$, for any z such that $c(z) = 1$ and $x' \in S$.

4. If $c(z) < 1$, then

$$\begin{aligned}
W(x) =& \left(W(x) - c(z)\frac{1}{2W(z)}W(x+z)\right) + \frac{c(z)}{2W(z)}W(x+z) \\
=& (1-\frac{c(z)}{2})\left(\frac{1}{1-c(z)/2}W(x) - \frac{c(z)/2}{1-c(z)/2}\frac{1}{W(z)}W(x+z)\right) \\
&+ \frac{c(z)}{2}\left(\frac{1}{W(z)}W(x+z)\right)
\end{aligned}$$

representing W as a convex combination of two elements in A^λ. Since $W \in A_e^\lambda$, we have

$$\begin{aligned}
W(x) &= \frac{1}{1-c(z)/2}W(x) - \frac{c(z)/2}{1-c(z)/2}\frac{1}{W(z)}W(x+z) \\
&= \frac{1}{W(z)}W(x+z).
\end{aligned}$$

So $W(x)W(z) = W(x+z)$, for any z such that $c(z) < 1$ and $x \in S$.

5. Since $W(x)W(z) = W(x+z)$ for all $x, z \in S$, for any $x = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, $W(x) = W(e_1)^{n_1} \times \cdots \times W(e_d)^{n_d}$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ taking values 1 on its i-th component. Now we let $W(e_i) = \exp(u_i)$ for $1 \leq i \leq d$ and $u = (u_1, u_2, \cdots, u_d)$, we have $W(x) = \exp(u \cdot x)$.

6. Substitute $W(x) = \exp(u \cdot x)$ for $W(x) = \sum_{y \in S} p(x,y)\exp(-\lambda)W(y)$, we have

$$\begin{aligned}
\exp(\lambda) &= \sum_{y \in S} p(x,y)\exp(u \cdot (y-x)) \\
&= \sum_{y \in S} p(x, x+y)\exp(u \cdot (y+x-x)) \\
&= \sum_{y \in S} p(0,y)\exp(u \cdot y) = \phi(u).
\end{aligned}$$

□

We hope to know more about $\phi(u)$. The properties of $\phi(u), u \in \mathbb{R}^d$ are discussed in the following theorem.

Theorem 5.7. *$\phi(u) = \sum_{y \in S} p(0, y) \exp(u \cdot y)$ is a strictly convex function which belongs to $C^\infty(\mathbb{R}^d)$, and its gradient vector $D\phi(u)$ is given by $\sum_{y \in S} y p(0, y) \exp(u \cdot y)$. Furthermore, $\phi(u_0) = \min\{\phi(u) : u \in \mathbb{R}^d\}$ if and only if $D\phi(u_0) = 0$, and such u_0 is unique. In particular, when $\mu = 0$, $u_0 = 0$, and $\phi(u_0) = 1$.*

Proof. The fact that ϕ is strictly convex and its gradient vector exists is easily derived from the fact that p is finitely supported.

The existence of the minimal value of ϕ is a nontrivial fact, which follows from the fact that $\phi(u) \to \infty$ as $|u| \to \infty$. To see this, as $|u|$ large enough, we may pick M large so that $|(u \cdot e_i)| > M$ for some $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, which only takes value on its i-th component. By the irreducibility of p, for each $e_i > 0$ we may find $y_{i1}, \cdots, y_{iN(i)} \in S$ and $z_{i1}, \cdots, z_{iM(i)} \in S$ so that $y_{i1} + \cdots + y_{iN(i)} = e_i$ and $z_{i1} + \cdots + z_{iM(i)} = -e_i$ for $1 \leq i \leq d$, and $p(0, y_{ik}), p(0, z_{il}) > 0$ for $1 \leq k \leq N(i)$ and $1 \leq l \leq M(i)$.

Let m be the minimum of the above $p(0, y_{ik})$ and $p(0, z_{il})$'s, and M' be the maximum of the above $N(i)$ and $M(i)$'s. Now, for any u so that $(u \cdot e_i) > M$ for some e_i (The case $(u \cdot -e_i) > M$ is similar), we have

$$
\begin{aligned}
\phi(u) &= \sum_{y \in S} p(0, y) \exp(u \cdot y) \\
&\geq \sum_{1 \leq k \leq N(i)} p(0, y_{ik}) \exp(u \cdot y_{ik}) \\
&\geq m \sum_{1 \leq k \leq N(i)} \exp(u \cdot y_{ik}) \\
&> m \exp(M/N(i)) \\
&\geq m \exp(M/M') \to \infty \text{ as } M \to \infty.
\end{aligned}
$$

$\square$

We hence have the following result derived from the above theorems.

Corollary 5.8. *Let $m = \min\{\phi(u) : u \in \mathbb{R}^d\}$. If $\exp(\lambda) = m$, then there is exactly one element in A_e^λ and so is A^λ. If $\exp(\lambda) < m$, then A_e^λ is empty and so is A^λ. If $\exp(\lambda) > m$, then every $W \in A_e^\lambda$ is given by $W(x) = \exp(u \cdot x)$, where $\phi(u) = \exp(\lambda)$, and every $W \in A^\lambda$ is given by $W(x) = \int_X \widetilde{W}(x) d\mu(\widetilde{W})$ for all $x \in S$, where μ is supported in A_e^λ.*

Proof. By Theorem 5.6, every $W \in A_e^\lambda$ is given by $W(x) = \exp(u \cdot x)$, where $\phi(u) = \exp(\lambda)$. If $\exp(\lambda) = m$, there is a unique $u \in \mathbb{R}^d$ such that $\phi(u) = \exp(\lambda) = m$. That is, there is only one member in A_e^λ. Therefore, by Theorem 5.5(Choquet's theorem), A^λ contains exactly one member.

If $\exp(\lambda) < m$ and $W \in A_e^\lambda$, then $m \leq \phi(u) = \exp(\lambda) < m$, which is impossible. This shows A_e^λ is empty. Hence A^λ is empty due to Theorem 5.5.

The case $\exp(\lambda) > m$ follows directly from Theorem 5.5, and we have demonstrated how to use Theorem 5.5 to give an explicit form for $W \in A^\lambda$ in the last paragraph of Section 5.1. □

6 Miscellaneous examples

In this section, the minimal point of (1) in each example exists by theorem 2.6 and is denoted by λ_0.

6.1 An example: $h(y) - \lambda_0 < -\delta$ for all $|y| > M$

Recall that if x, y are two states such that $p(x,y) > 0$ and $p(y,x) > 0$, then we have

$$\begin{aligned}
\exp(w(x)) &= \sum_{y \in S} p(x,z) \exp\Big(h(z) - \lambda + w(z)\Big) \\
&\geq p(x,y) \exp\Big(h(y) - \lambda + w(y)\Big) \\
&= p(x,y) \exp\Big(h(y) - \lambda\Big) \sum_{t \in S} p(y,t) \exp\Big(h(t) - \lambda + w(t)\Big) \\
&\geq p(x,y)p(y,x) \exp\Big(h(x) + h(y) - 2\lambda\Big) \exp(w(x)).
\end{aligned}$$

This implies $p(x,y)p(y,x) \exp\Big(h(x) + h(y) - 2\lambda\Big) \geq 1 \Rightarrow$

$\lambda \geq \frac{1}{2}\Big(h(x) + h(y) + \log\Big(p(x,y)p(y,x)\Big)\Big)$.

Assume that $S = \mathbb{Z}^3$, $p(x,y) = \frac{1}{6}$ for $|x-y| = 1$, and $h((0,0,0)) = h((0,0,1)) > \log 6$, $h(x) = 0, x \in S \backslash \{(0,0,0),(0,0,1)\}$. It follows that $\lambda_0 > 0$ and that $h(x) - \lambda_0 < -\delta$ for $|x| > 1$.

6.2 An example: $h(y) - \lambda_0 > \delta$ for all $|y| > M$

Let $S = \mathbb{Z}$. For any $x \in S$, let $p(x,x+1) = \frac{1}{7}$ and $p(x,x-1) = \frac{6}{7}$. Assume that $h(0) = h_0$, and $h(x) = 0$ for $x \neq 0$. Let $\lambda = \log\frac{5}{7} < 0$. Under these assumptions, the equation (1) becomes

$$\begin{cases}
\exp(w(x)) = \frac{1}{5}\exp(w(x+1)) + \frac{6}{5}\exp(w(x-1)) & \text{if } |x| \neq 1 \\
\exp(w(x)) = \frac{1}{5}\exp(w(x+1)) + \frac{6}{5}\exp(h_0)\exp(w(x-1)) & \text{if } x = 1 \\
\exp(w(x)) = \frac{1}{5}\exp(h_0)\exp(w(x+1)) + \frac{6}{5}\exp(w(x-1)) & \text{if } x = -1.
\end{cases}$$

For simplicity, define $W(x) = \exp(w(x))$ and $k = \exp(h_0)$. We may solve the above difference equations with solutions in terms of $W(0), W(1), W(-1)$:

$$W(x) = \begin{cases} \Big(6kW(0) - 2W(1)\Big)2^{x-1} + \Big(3W(1) - 6kW(0)\Big)3^{x-1} & \text{for } x \geq 1 \\ \Big(3W(-1) - kW(0)\Big)(\tfrac{1}{2})^{x+1} + \Big(kW(0) - 2W(-1)\Big)(\tfrac{1}{3})^{x+1} & \text{for } x \leq -1. \end{cases}$$

If we choose $W(0) = 1, W(1) = W(-1) = \frac{5}{7}$, and $k = 0.1$, for example, then $W(x) > 0$ for all $x \in S$. This means under such assumptions, $h(x) - \lambda_0 \geq h(x) - \lambda = -\log \frac{5}{7} > 0$ for $|x| \geq 1$.

We also observe that k cannot be taken too large, otherwise $W(x)$ will not always be positive. On the other hand, when the values of $W(0), W(1), W(-1)$ are given, and k is chosen s.t. $W(x) > 0$ for all $x \in S$, we also have $W(x) > 0$ when k is replaced by any smaller constant.

6.3 An example: both $\#\{y : h(y) - \lambda_0 > \delta\}$ and $\#\{y : h(y) - \lambda_0 < -\delta\}$ are infinite

Assume that $S = \mathbb{Z}^3$, $p(x, y) = \frac{1}{6}$ for $|x - y| = 1$. Define $A_o = \{x = (x_1, x_2, x_3) : |x_1| + |x_2| + |x_3| \text{ is odd }\}$ and $A_e = \{x = (x_1, x_2, x_3) : |x_1| + |x_2| + |x_3| \text{ is even }\}$. Let $h(x) = 2 + \log 36$ for $x \in A_o$, and $h(x) = 0$ for $x \in A_e$.

For any solution $(\tilde{\lambda}, \tilde{w})$ of (1), we have $\tilde{\lambda} \geq \frac{1}{2}\Big(h(x) + h(y) + \log\Big(p(x, y)\, p(y, x)\Big)\Big) = 1$, and hence $\lambda_0 \geq 1$. On the other hand, for $\lambda = 1 + \log 6$, we can choose $\exp(w(x)) = 1$ for $x \in A_o$ and $\exp(w(x)) = 6\exp(1)$ for $x \in A_e$ such that

$$\exp(w(x)) = \sum_{y:|x-y|=1} p(x, y) \exp\Big(h(y) - (1 + \log 6)\Big) \exp(w(y))$$

for either $x \in A_e$ or $x \in A_o$. Since (λ, w) just defined satisfies (1), $\lambda_0 \leq \lambda = 1 + \log 6$. It follows that $h(x) - \lambda_0 \leq -1$ for $x \in A_e$ and $h(x) - \lambda_0 \geq 1 + \log 6$ for $x \in A_o$.

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Appendix: Martin boundary theory

A.1 Introduction

Given an irreducible and transient Markov chain $\{X_n\}$ with countable state space S, there is a way to assign distance d on this chain. After performing completion procedure on S with this metric d, we obtain a compact space $\widehat{S_M}$. The Martin boundary ∂S_M is then defined as $\widehat{S_M} \setminus S$.

One interesting property is that $\{X_n\}$ converges a.s. to ∂S_M in this new topology. Later, we introduce two smaller boundaries $\partial_R S_M$ and $\partial_m S_M$, so that $\partial_m S_M \subset \partial_R S_M \subset \partial S_M$, and we prove that $\{X_n\}$ actually converges a.s. to $\partial_m S_M$.

An important property is that we are able to represent arbitrary harmonic function h in an integral form with some measure $\mu(h)$, which is supported on ∂S_M (Theorem A.4.1), and we show that $\mu(h)$ can be chosen to be supported on $\partial_m S_M$ and such representation is unique (Theorem A.5.10).

The main references of this appendix are [7], [3], and [8]. Our approach is basically from [7]. [3] adopts a completely different approach from ours. To prove Theorem A.3.2, I introduce the method in [8] instead of the one in [7].

A.2 Construction of Martin boundary

Let $\{X_n\}$ be an irreducible and transient Markov chain with state space S. Transience of $\{X_n\}$ implies that, for any $i, j \in S$, $g(i,j) \triangleq \sum_{n=0}^{\infty} p_n(i,j) < \infty$. Now we pick a reference point $x_0 \in S$, and define

$$K(i,j) \triangleq \frac{g(i,j)}{g(x_0,j)}.$$

With this special function K, we are able to define a metric d on S:

$$d(i,j) \triangleq \sum_{q \in S} w(q) \left(p_{m(q)}(x_0,q) |K(q,i) - K(q,j)| + |\delta_{qi} - \delta_{qj}| \right),$$

where $\delta_{xy} = \delta(x,y)$ is the Kronecker delta, $m(q) \in \mathbb{N} \cup \{0\}$ is chosen such that $p_{m(q)}(x_0,q) > 0$ (since $\{X_n\}$ is irreducible), and $\sum_{q \in S} w(q) < \infty$, $w(q) > 0$ for each $q \in S$. Because $p_{m(q)}(x_0,q) g(q,i) \leq g(x_0,i)$, we find that $|K(q,i) - K(q,j)| \leq 2/p_{m(q)}(x_0,q)$.

It is not difficult to check d is a metric. However, S endowed with this metric is not a complete metric space. We denote the completion of S with metric d by $\widehat{S_M}$, and call $\partial S_M \triangleq \widehat{S_M} \setminus S$ the Martin boundary of S. It is noteworthy that for each $i \in S$, i is not a limit point due to the existence of the term $|\delta_{qi} - \delta_{qj}|$ in the definition of d. Therefore, ∂S_M is a closed set.

When we have a Cauchy sequence $\{x_n\} \subset S$, by the definition of Martin boundary we know that $\exists \alpha \in \widehat{S_M}$ such that $d(x_n, \alpha) \to 0$. In addition, $\{K(i,x_n)\}$ is also a Cauchy sequence $\subset \mathbb{R}$ for each $i \in S$. Therefore, there exists a number $n(i)$ such that $\{K(i,x_n)\} \to n(i)$, and we denote $n(i)$ by $K(i,\alpha)$.

Theorem A.2.1. *$\widehat{S_M}$ is compact.*

Proof. For arbitrary sequence $\{x_n\} \subset \widehat{S_M}$, $\{K(i,x_n)\}$ is bounded in n for each $i \in S$. Thus we may apply diagonal process to select a subsequence $\{x_{n_j}\}$ such that for each $i \in S$, $\{K(i,x_{n_j})\} \to n(i)$. We want to show that $\exists \alpha \in \widehat{S_M}$ such that $d(x_{n_j}, \alpha) \to 0$.

We enumerate $S = \{i_1, i_2, \cdots\}$. If $x_{n_j} \in \partial S_M$, then we replace it with some $y_j \in S$ such that for $1 \leq k \leq j$, $|K(i_k, y_j) - K(i_k, x_{n_j})| \leq \frac{1}{j}$ and $d(y_j, x_{n_j}) \leq \frac{1}{j}$. If $x_{n_j} \in S$, then we let $y_j = x_{n_j}$.

Therefore, for each $i \in S$, the new sequence $\{K(i, y_j)\}$ is still a Cauchy sequence, and thus $\{y_j\} \subset S$ is Cauchy in the space $\widehat{S_M}$, by our definition of d. It follows that $\exists \alpha \in \widehat{S_M}$ such that $d(y_j, \alpha) \to 0$ as $j \to \infty$, and this implies $d(x_{n_j}, \alpha) \to 0$. $\square$

A.3 Harmonic measure

We hope to prove in Theorem A.3.1 that for any $A \in \mathscr{B}(\widehat{S_M})$, $\{X_\infty \in A\} \in \mathscr{G}$, where $\mathscr{G} \triangleq \sigma(X_1, X_2, \cdots)$. In Theorem A.3.2, we prove that $p_{x_0}(X_\infty \in \partial S_M) = 1$.

Theorem A.3.1. *If* $A \in \mathscr{B}(\widehat{S_M})$*, then* $\{\lim_{n\to\infty} X_n = X_\infty \in A\} \in \mathscr{G}$.

Proof. 1. We first prove that $\{\omega : \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)\ exists\} \in \mathscr{G}$.

Since $\{\omega : K(x, X_n(\omega)) \in B\} \in \sigma(X_n) \subset \mathscr{G}$ for all $x \in S$, where $B \in \mathscr{B}(\mathbb{R})$, we have $\limsup_{n\to\infty} K(x, X_n(\omega)) \in \mathscr{G}$ and $\liminf_{n\to\infty} K(x, X_n(\omega)) \in \mathscr{G}$.

In addition, because

$$
\begin{aligned}
&\{\omega : \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)\ exists\}\\
=&\bigcap_{x\in S}\{\omega : \limsup_{n\to\infty} K(x, X_n(\omega)) = \liminf_{n\to\infty} K(x, X_n(\omega))\},
\end{aligned}
$$

$E \triangleq \{\omega : \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)\ exists\} \in \mathscr{G}$.

2. Define

$$
Y_n(\omega) = \begin{cases} X_n(\omega), & if\ \ \omega \in E; \\ y_0, & if\ \ \omega \notin E, \end{cases}
$$

where $y_0 \in S$ is arbitrarily chosen.

We claim that $Y_n : \Omega \to \widehat{S_M}$ such that for all $B \in \mathscr{B}(\widehat{S_M}), Y_n^{-1}(B) \in \mathscr{G}$. Indeed, if $y_0 \notin B$, then $Y_n^{-1}(B) = X_n^{-1}(B) \cap E \in \mathscr{G}$. If $y_0 \in B$, then $Y_n^{-1}(B) = (X_n^{-1}(B) \cap E) \cup E^c \in \mathscr{G}$.

Now we can define

$$
Y \triangleq \lim_{n\to\infty} Y_n(\omega) = \begin{cases} \lim_{n\to\infty} X_n(\omega), & if\ \ \omega \in E; \\ y_0, & if\ \ \omega \notin E. \end{cases}
$$

3. We note that for all $A \in \mathscr{B}(\widehat{S_M})$, $Y^{-1}(A) \in \mathscr{G}$. To see this, consider $C \in \mathscr{B}(\widehat{S_M})$, a compact set in $\widehat{S_M}$. We have

$$Y^{-1}(C) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} Y_m^{-1}(N_{1/k}(C)) \in \mathscr{G}$$

where $N_\epsilon(C) \triangleq \{x \in \widehat{S_M} : d(x, C) < \epsilon\}$ is an open set.

Since $\widehat{S_M}$ is compact, every closed subset of $\widehat{S_M}$ is compact, and $\{A \in \widehat{S_M} : Y^{-1}(A) \in \mathscr{G}\}$ is a σ-algebra containing $\{C \in \widehat{S_M} : C \text{ closed }\}$. That is, $\{A \in \widehat{S_M} : Y^{-1}(A) \in \mathscr{G}\}$ contains $\mathscr{B}(\widehat{S_M})$.

4. Therefore, for $A \in \mathscr{B}(\widehat{S_M})$, $\{\lim_{n\to\infty} X_n = X_\infty \in A\} = \{\lim_{n\to\infty} Y_n = Y \in A\} \cap E \in \mathscr{G}$. $\square$

Theorem A.3.2. *For any $i \in S$, $\lim_{n\to\infty} K(i, X_n)$ exists and is finite $p_{x_0} - a.s.$. Therefore, $p_{x_0}(X_\infty \in \partial S_M) = 1$, for x is not a limit point every $x \in S$.*

Proof. 1. Define the last exit time τ_D from set D as $\tau_D \triangleq \sup\{n : X_n \in D\}$. If the chain has never entered D, then τ_D is left undefined. For any negative integer n, define $X_n = *$, where this additional state $* \notin S$. Let $a_0, a_1, \cdots, a_n \in S$, we have

$$\begin{aligned}
&p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n) \\
=&\sum_{m=n}^{\infty} p_{x_0}(\tau_D = m, X_m = a_0, X_{m-1} = a_1, \cdots, X_{m-n} = a_n) \\
=&\sum_{m=n}^{\infty} p_{m-n}(x_0, a_n)p(a_n, a_{n-1}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0) \\
=&g(x_0, a_n)p(a_n, a_{n-1}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0)
\end{aligned}$$

Define $K(i, *) = 0$ for all $i \in S$. We hope to prove that $\{K(i, X_{\tau_D-k}); \sigma(X_{\tau_D}, \cdots, X_{\tau_D-k})\}_{k=0}^{n}$ is a supermartingale with respect to p_{x_0}. It suffices to check the following three cases:

Case 1. $X_{\tau_D-(n-1)} \notin \{x_0, *\} \Rightarrow X_{\tau_D-n} \in S$

$$\begin{aligned}
&\sum_{a_n \in S\cup\{*\}} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)K(i, a_n) \\
&= \sum_{a_n \in S} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)K(i, a_n) \\
&= \sum_{a_n \in S} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)\frac{g(i, a_n)}{g(x_0, a_n)} \\
&= \sum_{a_n \in S} g(x_0, a_n)p(a_n, a_{n-1}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0)\frac{g(i, a_n)}{g(x_0, a_n)} \\
&= \sum_{a_n \in S} p(a_n, a_{n-1}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0)g(i, a_n) \\
&\leq p(a_{n-1}, a_{n-2}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0)g(i, a_{n-1}) \\
&= g(x_0, a_{n-1})p(a_{n-1}, a_{n-2}) \cdots p(a_1, a_0)p_{a_0}(\tau_D = 0)K(i, a_{n-1}) \\
&=p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1})K(i, a_{n-1})
\end{aligned}$$

Case 2. $X_{\tau_D-(n-1)} = * \Rightarrow X_{\tau_D-n} = *$

$$\begin{aligned}
&\sum_{a_n \in S\cup\{*\}} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1} = *, X_{\tau_D-n} = a_n)K(i, a_n) \\
&=p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1} = *, X_{\tau_D-n} = *)K(i, *) \\
&=0 \\
&=p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1} = *)K(i, *) \\
&=p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1})K(i, a_{n-1})
\end{aligned}$$

Case 3. $X_{\tau_D-(n-1)} = x_0 \Rightarrow X_{\tau_D-n} \in S \cup \{*\}$

$$\begin{aligned}
&\sum_{a_n \in S\cup\{*\}} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)K(i, a_n) \\
&= \sum_{a_n \in S} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)K(i, a_n) \\
&\quad + p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = *)K(i, *) \\
&= \sum_{a_n \in S} p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-n} = a_n)K(i, a_n) + 0 \\
&=p_{x_0}(X_{\tau_D} = a_0, X_{\tau_D-1} = a_1, \cdots, X_{\tau_D-(n-1)} = a_{n-1})K(i, a_{n-1})
\end{aligned}$$

where the last line is by exactly the same reasoning as in case 1.

2. Now that $K(i, X_{\tau_D}), K(i, X_{\tau_D - 1}), \cdots, K(i, X_{\tau_D - n})$ is a supermartingale with respect to p_{x_0}, we have the following result: (see [9], Theorem 9.4.3)

Lemma A.3.3. *Let $Y_1, \cdots, Y_n$ be a supermartingale and let $-\infty < a < b < \infty$. Let $\nu^n_{[a,b]}$ be the number of downcrossings of $[a, b]$ by the sample sequence $Y_1(\omega), \cdots, Y_n(\omega)$. We have $E[\nu^n_{[a,b]}] \le E[Y_1 \wedge b] - E[Y_n \wedge b]/(b - a)$.*

Define $\nu_{[a,b],D}$ be the number of upcrossings of $[a, b]$ by the sample sequence $K(i, X_1(\omega)), K(i, X_2(\omega)), \cdots, K(i, X_{\tau_D}(\omega))$, and $\nu^n_{[a,b],D}$ be the number of downcrossings of $[a, b]$ by the sample sequence $K(i, X_{\tau_D}(\omega)), K(i, X_{\tau_D - 1})(\omega), \cdots, K(i, X_{\tau_D - n}(\omega))$. Notice that if D is a finite set, then $\tau_D < \infty$ p_{x_0}-a.s., and hence for p_{x_0}-a.s. ω, $\nu^n_{[a,b],D}(\omega) \nearrow \nu_{[a,b],D}(\omega)$ or $\nu^n_{[a,b],D}(\omega) \nearrow \nu_{[a,b],D}(\omega) + 1$. Therefore, by the lemma above,

$$
\begin{aligned}
&E_{x_0}[\nu^n_{[a,b],D}] \le \frac{E_{x_0}[K(i, X_{\tau_D}) \wedge b] - E_{x_0}[K(i, X_{\tau_D - n}) \wedge b]}{b - a} \le \frac{b}{b - a} \\
\Rightarrow &E_{x_0}[\nu_{[a,b],D}] \le \liminf_{n\to\infty} E_{x_0}[\nu^n_{[a,b],D}] = \lim_{n\to\infty} E_{x_0}[\nu^n_{[a,b],D}] \le \frac{b}{b - a} \qquad (*)
\end{aligned}
$$

for arbitrary finite set D. Let $\{D_m\}$ be a collection of finite sets such that $D_m \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = S$, and define $\nu_{[a,b]}$ as the number of upcrossings of $[a, b]$ by the infinite sample sequence $K(i, X_1(\omega)), K(i, X_2(\omega)), \cdots, \quad K(i, X_n(\omega)), \cdots$, we have $\nu_{[a,b],D_m} \nearrow \nu_{[a,b]}$ p_{x_0}-a.s. as $m \to \infty$. By monotone convergence theorem applied on $E_{x_0}[\nu_{[a,b],D_m}]$ in $(*)$, we have

$$E_{x_0}[\nu_{[a,b]}] \le \frac{b}{b - a}.$$

Arbitrariness of a, b shows that for any $i \in S$, $\lim_{n\to\infty} K(i, X_n)$ exists $p_{x_0} - a.s.$

3. Now our last job is to show that, for any $i \in S$, $\lim_{n\to\infty} K(i, X_n) < \infty$ $p_{x_0} - a.s.$ This is actually an easy task. Choose $M > 0$ such that $p_M(x_0, i) > 0$, we have $g(x_0, y) \ge \sum_{z\in S} p_M(x_0, z) g(z, y) \ge p_M(x_0, i) g(i, y)$, and thus $K(i, y) \le 1/p_M(x_0, i)\ \forall\, y \in S$. $\square$

Corollary A.3.4. *$p_i(X_\infty \in \partial S_M) = 1$ for every $i \in S$.*

Proof. Assume the statement does not hold, that is, $\exists j_0$ such that $p_{j_0}(X_\infty \in \partial S_M) < 1$. We also choose $p_N(x_0, j) > 0$.

$$
\begin{aligned}
1 &= p_{x_0}(X_\infty \in \partial S_M) \\
&= \sum_{j\in S} p(X_\infty \in \partial S_M, X_N = j | X_0 = x_0) \\
&= \sum_{j\in S} p_N(x_0, j) p(X_\infty \in \partial S_M | X_0 = x_0, X_N = j) \\
&= \sum_{j\in S} p_N(x_0, j) p\Big(\bigcap_{i\in S}\bigcap_{m\geq 1}\bigcup_{k\geq N}\bigcap_{n,l\geq k} \{|K(i,X_n) - K(i,X_l)| \leq \frac{1}{m}\}\big| X_0 = x_0, X_N = j\Big) \\
&= \sum_{j\in S} p_N(x_0, j) p\Big(\bigcap_{i\in S}\bigcap_{m\geq 1}\bigcup_{k\geq 0}\bigcap_{n,l\geq k} \{|K(i,X_n) - K(i,X_l)| \leq \frac{1}{m}\}\big| X_0 = j\Big) \\
&= \sum_{j\in S} p_N(x_0, j) p_j(X_\infty \in \partial S_M) \\
&\leq p_N(x_0, j_0) p_{j_0}(X_\infty \in \partial S_M) + \sum_{j\in S, j\neq j_0} p_N(x_0, j) \\
&< \sum_{j\in S} p_N(x_0, j) = 1,
\end{aligned}
$$

which is a contradiction. $\square$

Since $p_{x_0}(\{\omega : X_\infty \in \partial S_M\}) = p_{x_0}((\bigcap_{i\in S}\{\omega : \limsup_{n\to\infty} K(i, X_n(\omega)) = \liminf_{n\to\infty} K(i, X_n(\omega)) < \infty\}) = 1$, we have the following definition:

Definition A.3.5. *Let $\mu(A) \triangleq p_{x_0}(\{\omega : X_\infty \in A\})$ for every $A \in \mathscr{B}(\widehat{S_M})$. μ thus defined is a probability measure on $\mathscr{B}(\widehat{S_M})$ (Indeed, μ is still a probability measure when the space is restricted on ∂S_M), and we call μ the **harmonic measure** for p_{x_0}.*

Given the harmonic measure for p_{x_0}, we may derive a representation formula for harmonic measure of p_i with respect to p_{x_0} for any $i \in S$.

Theorem A.3.6. *For any $A \in \mathscr{B}(\widehat{S_M}), p_i(X_\infty \in A) = \int_A K(i,x)d\mu(x)$, where μ is the harmonic measure for p_{x_0}.*

Proof. 1. Let $\{A_m\}$ be a collection of finite sets such that $A_m \subset A_{m+1}$ and $\bigcup_{m=1}^{\infty} A_m = S$. Define the last exit time τ_D from set D as $\tau_D \triangleq \sup\{n : X_n \in D\}$. (This definition has already appeared in Theorem A.3.2). For each $i \in S, A \in \mathscr{B}(\widehat{S_M})$, define $\mu_{i,n}(A) \triangleq p_i(X_{\tau_{A_n}} \in A)$ and $\mu_i(A) \triangleq p_i(X_\infty \in A)$. Note that $\mu_{x_0} = \mu$ is the harmonic measure for p_{x_0}.

2. We have

$$\begin{aligned} p_i(X_{\tau_{A_n}} = j) &= g(i,j)p_j(X_{\tau_{A_n}} = 0) \\ &= K(i,j)g(x_0,j)p_j(X_{\tau_{A_n}} = 0) \\ &= K(i,j)p_{x_0}(X_{\tau_{A_n}} = j), \end{aligned}$$

that is, $\mu_{i,n}(j) = K(i,j)\mu_{x_0,n}(j)$. Because A_n is finite, and $\{X_n\}$ is irreducible and transient for p_i-a.s. ω, $\tau_{A_n} < \infty$ and $X_{\tau_{A_n}} \in A_n$. Thus $\mu_{i,n}$ is supported on a finite set A_n.

3. By the definition of $\mu_{i,n}$, we have

$$\int_{\widehat{S_M}} 1_{\{x \in E\}} d\mu_{i,n}(x) = \int_\Omega 1_{\{X_{\tau_{A_n}}(\omega) \in E\}} dp_i(\omega)$$

for all $E \in \mathscr{B}(\widehat{S_M})$. Thus for any $f(x)$ continuous on $\widehat{S_M}$ we have

$$\int_{\widehat{S_M}} f(x) d\mu_{i,n}(x) = \int_\Omega f(X_{\tau_{A_n}}) dp_i(\omega).$$

Similarly,

$$\int_{\widehat{S_M}} f(x) d\mu_i(x) = \int_\Omega f(X_\infty) dp_i(\omega).$$

Since f is continuous on a compact set, f is bounded, hence we could apply bounded convergence theorem to get

$$\begin{aligned} \lim_{n\to\infty} \int_{\widehat{S_M}} f(x) d\mu_{i,n}(x) = \lim_{n\to\infty} \int_\Omega f(X_{\tau_{A_n}}) dp_i(\omega) &= \int_\Omega \lim_{n\to\infty} f(X_{\tau_{A_n}}) dp_i(\omega) \\ &= \int_\Omega f(X_\infty) dp_i(\omega) = \int_{\widehat{S_M}} f(x) d\mu_i(x). \end{aligned}$$

4. Therefore, together with the results in 2. we have

$$
\begin{aligned}
\int_{\widehat{S_M}} f(x) d\mu_i(x) &= \lim_{n\to\infty} \int_{\widehat{S_M}} f(x) d\mu_{i,n}(x) \\
&= \lim_{n\to\infty} \int_{\widehat{S_M}\cap A_n} f(x) d\mu_{i,n}(x) \\
&= \lim_{n\to\infty} \sum_{j\in \widehat{S_M}\cap A_n} f(j)\mu_{i,n}(j) \\
&= \lim_{n\to\infty} \sum_{j\in \widehat{S_M}\cap A_n} f(j)K(i,j)\mu_{x_0,n}(j) \\
&= \lim_{n\to\infty} \int_{\widehat{S_M}\cap A_n} f(x)K(i,x) d\mu_{x_0,n}(x) \\
&= \lim_{n\to\infty} \int_{\widehat{S_M}} f(x)K(i,x) d\mu_{x_0,n}(x) \\
&= \int_{\widehat{S_M}} f(x)K(i,x) d\mu_{x_0}(x).
\end{aligned}
$$

The last line is due to that $f(\cdot)K(i,\cdot)$ is continuous on $\widehat{S_M}$ and is henceforth bounded.

5. The goal is to replace f in 4. with 1_E, where $E \in \mathscr{B}(\widehat{S_M})$. For any closed subset C of $\widehat{S_M}$(hence a compact set here), define $C_\epsilon = \{x : d(x,C) < \epsilon\}$, and define $f_{C,\epsilon} : \widehat{S_M} \to [0,1]$ such that $f_{C,\epsilon}(x) = 1$ for $x \in C$, $f_{C,\epsilon}(x) = 0$ for $x \in \widehat{S_M} \setminus C_\epsilon$, and use Urysohn's lemma to extend $f_{C,\epsilon}$ continuously on $\widehat{S_M}$ and $f(\widehat{S_M}) \subset [0,1]$. Since $\widehat{S_M}$ is a metric space, $\widehat{S_M}$ is a normal space (See [6], Theorem 32.2), so Urysohn's lemma is applicable. We list Urysohn's lemma below:

Lemma A.3.7. (Urysohn's lemma) *Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a,b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.*

Let $n \to 0$ in $\int_{\widehat{S_M}} f_{C,1/n}(x) d\mu_i(x) = \int_{\widehat{S_M}} f_{C,1/n}(x)K(i,x) d\mu_{x_0}(x)$, we have

$$
\int_{\widehat{S_M}} 1_C(x) d\mu_i(x) = \int_{\widehat{S_M}} 1_C(x)K(i,x) d\mu_{x_0}(x)
$$

for every closed set C, by the bounded convergence theorem. Since $\mathcal{F} = \{A \subset \widehat{S_M} : \int_{\widehat{S_M}} 1_A(x) d\mu_i(x) = \int_{\widehat{S_M}} 1_A(x) K(i,x) d\mu_{x_0}(x)\}$ is λ-system that contains all closed subsets of $\widehat{S_M}$, which is a π-system, by $\pi - \lambda$ theorem, $\mathscr{B}(\widehat{S_M}) \subset \mathcal{F}$. Thus $\mu_i(E) = \int_E K(i,x) d\mu(x)$. $\square$

Corollary A.3.8. *For all $i \in S$, $\int_{\widehat{S_M}} K(i,x) d\mu(x) = \int_{\partial S_M} K(i,x) d\mu(x) = 1$.*

Proof. It follows directly from Corollary A.3.4 and Theorem A.3.6. $\square$

A.4 h-process transform

Assume that $h(x)$ is a harmonic function on S such that $h(x_0) = 1$ (See Definition 3.5 for the definition of harmonic functions). Irreducibility of $\{X_n\}$ implies that $h(i) > 0$ for all $i \in S$. We may thus define a new probability kernel p^h such that $p^h(i,j) = p(i,j)h(j)/h(i)$.

If $p(i,j)$ is transient, then $g^h(i,j) = \sum_{n=0}^{\infty} p_n^h(i,j) = g(i,j)h(j)/h(i) < \infty$, that is, p^h is also transient. We may define $K^h(i,j) = \frac{g^h(i,j)}{g^h(x_0,j)} = \frac{1}{h(i)}K(i,j)$.

For all $i, j \in S$, define

$$
\begin{aligned}
d^h(i,j) &\triangleq \sum_{q\in S} w(q)\Big(p^h_{m(q)}(x_0,q)|K^h(q,i) - K^h(q,j)| + |\delta_{qi} - \delta_{qj}|\Big)\\
&= \sum_{q\in S} w(q)\Big(\frac{h(q)}{h(x_0)} p_{m(q)}(x_0,q) \times \frac{1}{h(q)}|K(q,i) - K(q,j)| + |\delta_{qi} - \delta_{qj}|\Big)\\
&= \sum_{q\in S} w(q)\Big(p_{m(q)}(x_0,q)|K(q,i) - K(q,j)| + |\delta_{qi} - \delta_{qj}|\Big)\\
&= d(i,j)
\end{aligned}
$$

It follows that p and p^h have the same topology and hence the same Martin boundary. Therefore, every result in the previous section remains true: we only need to replace p with p^h, K with K^h, and μ with μ^h, where μ^h is defined as the harmonic measure of $p^h_{x_0}$.

Therefore, the h-process counterpart of Corollary A.3.8. is that $1 = \int_{\partial S_M} K^h(i,x)d\mu^h(x)$. This implies the following:

Theorem A.4.1. *For any harmonic function $h(x)$ on S such that $h(x_0) = 1$, $h(i) = \int_{\partial S_M} K(i,x)d\mu^h(x)$ for every $i \in S$.*

A.5 Regular boundary and minimal boundary

For each $x \in S$, $K(\cdot, x)$ is a superharmonic function (see Definition 3.5 for the definition of superharmonic functions). If $x \in \partial S_M$, then $K(\cdot, x)$ is still a superharmonic function by Fatou's lemma applied on a sequence $\{x_n\} \subset S$ so that $x_n \to x$. We are interested in finding $x \in \widehat{S_M}$ where $K(\cdot, x)$ is a harmonic function.

Definition A.5.1. *Define* $\partial_R S_M \triangleq \{x \in \widehat{S_M} : K(\cdot, x)$ *is a harmonic function*$\} = \{x \in \partial S_M : K(\cdot, x)$ *is a harmonic function*$\}$. *The identity holds because* $K(\cdot, x)$ *is superharmonic but not harmonic if* $x \in S$. *We call* $\partial_R S_M$ *the* ***regular boundary*** *for* $\widehat{S_M}$. *Harmonic functions are also called regular functions.*

Theorem A.5.2. $\partial_R S_M \in \mathscr{B}(\widehat{S_M})$. *Furthermore,* $\mu(\partial_R S_M) = 1$.

Proof. (i) For the first argument, define $B_i = \{x : K(i, x) = \sum_{j \in S} p(i, j) K(j, x)\}$, which belongs to $\mathscr{B}(\widehat{S_M})$ because it is the set that two $\mathscr{B}(\widehat{S_M})$-measurable functions coincide. The result follows from the fact that $\partial_R S_M = \bigcap_{i \in S} B_i$.

(ii) 1. For the second argument, we first claim that for any $B \in \mathscr{B}(\widehat{S_M})$, $u(i) = p_i(X_\infty \in B)$ is a harmonic function. To prove this claim, it suffices to show that $p(X_\infty \in C | X_1 = i, X_0 = k) = p(X_\infty \in C | X_0 = i)$ for any compact set C, and then use $\pi - \lambda$ theorem to prove that $p(X_\infty \in B | X_1 = i, X_0 = k) = p(X_\infty \in B | X_0 = i)$ for any $B \in \mathscr{B}(\widehat{S_M})$, and this implies

$$
\begin{aligned}
p_i(X_\infty \in B) &= \sum_{j \in S} p_i(X_\infty \in B, X_1 = j) \\
&= \sum_{j \in S} p(i, j) p(X_\infty \in B | X_1 = j, X_0 = i) \\
&= \sum_{j \in S} p(i, j) p(X_\infty \in B | X_0 = j) \\
&= \sum_{j \in S} p(i, j) p_j(X_\infty \in B),
\end{aligned}
$$

and this is what we want.

To see $p(X_\infty \in C|X_1 = i, X_0 = k) = p(X_\infty \in C|X_0 = i)$ is true for any compact set C, we define $C_\epsilon = \{x : d(x, C) \le 1\}$, and we have

$$
\begin{aligned}
&p(X_\infty \in C|X_1 = i, X_0 = k) \\
=&p(\bigcap_{m\ge1}\bigcup_{N\ge1}\bigcap_{n\ge N}\{X_n \in C_{1/m}\}|X_1 = i, X_0 = k) \\
=&\lim_{M_1\to\infty} p(\bigcap_{M_1\ge m\ge1}\bigcup_{N\ge1}\bigcap_{n\ge N}\{X_n \in C_{1/m}\}|X_1 = i, X_0 = k) \\
=&\lim_{M_1\to\infty}\lim_{M_2\to\infty} p(\bigcap_{M_1\ge m\ge1}\bigcup_{M_2\ge N\ge1}\bigcap_{n\ge N}\{X_n \in C_{1/m}\}|X_1 = i, X_0 = k) \\
=&\lim_{M_1\to\infty}\lim_{M_2\to\infty}\lim_{M_3\to\infty} p(\bigcap_{M_1\ge m\ge1}\bigcup_{M_2\ge N\ge1}\bigcap_{M_3\ge n\ge N}\{X_n \in C_{1/m}\}|X_1 = i, X_0 = k) \\
=&\lim_{M_1\to\infty}\lim_{M_2\to\infty}\lim_{M_3\to\infty} p(\bigcap_{M_1\ge m\ge1}\bigcup_{M_2\ge N\ge1}\bigcap_{M_3\ge n\ge N}\{X_{n-1} \in C_{1/m}\}|X_0 = i) \\
=&p(\bigcap_{m\ge1}\bigcup_{N\ge1}\bigcap_{n\ge N}\{X_{n-1} \in C_{1/m}\}|X_0 = i) \\
=&p(X_\infty \in C|X_0 = i).
\end{aligned}
$$

2. By Theorem A.3.6, $p_i(X_\infty \in B) = \int_B K(i, x)d\mu(x)$. Therefore by (ii) 1. we have

$$
\begin{aligned}
\int_B K(i, x)d\mu(x) &= \sum_j p(i, j)\int_B K(j, x)d\mu(x) \\
&= \int_B \sum_j p(i, j)K(j, x)d\mu(x) \\
&\le \int_B K(i, x)d\mu(x).
\end{aligned}
$$

it follows that $\sum_{j\in S} p(i, j)K(j, x) = K(i, x)$ for μ-a.e. x. for any $i \in S$. That is, $\mu(B) = 1$. □

Definition A.5.3. *$h \ge 0$ is called a* ***minimal harmonic function*** *if it is harmonic and every harmonic function $h' \le h$ is a scalar multiple of h, that is, $h' = ch$, $0 \le c \le 1$. h is called* ***normalized minimal harmonic*** *if, furthermore, $h(x_0) = 1$.*

Definition A.5.4. *Define* $\partial_m S_M = \{x \in \widehat{S_M} : K(\cdot, x)$ *is minimal harmonic*$\} = \{x \in \partial S_M : K(\cdot, x)$ *is minimal harmonic*$\} = \{x \in \partial S_M : K(\cdot, x)$ *is normalized minimal harmonic*$\}$. *Call it the* ***minimal boundary*** *for* $\widehat{S_M}$.

Below is a basic result for normalized minimal harmonic functions.

Proposition A.5.5. *Let* $h(x)$ *be a harmonic function with* $h(x_0) = 1$. *Then* $h(x)$ *is a minimal harmonic function* $\Leftrightarrow$ $h(x)$ *cannot be written as a nontrivial convex combination of two distinct normalized harmonic functions* h_1, h_2. *(Where normalization is in the sense that* $h_1(x_0) = h_2(x_0) = 1$*).*

Proof. ($\Rightarrow$) If $h = c_1h_1 + c_2h_2$, $c_1 + c_2 = 1$, $c_1, c_2 > 0$, then $ch_1 \leq h$ is a harmonic function, yet it is not a scalar multiple of h(Otherwise, $h_1 = h_2 = h$). This implies h is not minimal harmonic.

($\Leftarrow$) If h is not minimal harmonic, then there is some harmonic function $h_1 < h$, where h_1 is not a scalar multiple of h. Write $h(x) = h_1(x_0)\frac{h_1(x)}{h_1(x_0)} + \big(h(x_0) - h_1(x_0)\big)\frac{h(x)-h_1(x)}{h(x_0)-h_1(x_0)}$, which is a convex combination of two distinct harmonic functions, each takes value 1 while evaluated at x_0. $\square$

Proposition A.5.6. *p and p^h have the same Martin boundary, regular boundary, and minimal boundary.*

Proof. The first assertion is true because $d(i, j) = d^h(i, j)$ for all $i, j \in S$. The second follows from $\sum_{j\in S} p(i, j)K(j, x) = K(i, x) \Leftrightarrow \sum_{j\in S} p^h(i, j)K^h(j, x) = K^h(i, x)$. For the last argument, assume $x \notin \partial_m S_M$, by Proposition A.5.5, $K(\cdot, x) = c_1h_1(\cdot) + c_2h_2(\cdot)$, $c_1 + c_2 = 1$, and h_1, h_2 are normalized p-harmonic functions. It is easy to verify that $\frac{K(\cdot,x)}{h(\cdot)} = c_1\frac{h_1(\cdot)}{h(\cdot)} + c_2\frac{h_2(\cdot)}{h(\cdot)}$, and $\frac{K(\cdot,x)}{h(\cdot)}, \frac{h_1(\cdot)}{h(\cdot)}, \frac{h_2(\cdot)}{h(\cdot)}$ are all p^h-harmonic and normalized. Therefore, $K^h(\cdot, x) = \frac{K(\cdot,x)}{h(\cdot)}$ is not p^h-minimal harmonic by Proposition A.5.5. again. The proof that $K^h(\cdot, x)$ not p^h-minimal harmonic $\Rightarrow K(\cdot, x)$ not p-minimal harmonic is similar. $\square$

When $h(x)$ is a harmonic function with $h(x_0) = 1$ but it is not minimal harmonic, by Proposition A.5.5, $h = c_1h_1 + c_2h_2$, $c_1 + c_2 = 1, c_1, c_2 > 0$, where both h_1, h_2 are normalized harmonic functions. We investigate the relation of μ^h, μ^{h_1}, and μ^{h_2} in the following proposition.

Proposition A.5.7. *If a normalized harmonic function $h(x)$ can be written as $c_1h_1 + c_2h_2$, $c_1 + c_2 = 1, c_1, c_2 > 0$, where both h_1, h_2 are normalized harmonic functions, then the corresponding harmonic measure of h, h_1, and h_2 satisfies $\mu^h = c_1\mu^{h_1} + c_2\mu^{h_2}$.*

Proof. The relation $p^h_{x_0}(E) = c_1p^{h_1}_{x_0}(E) + c_2p^{h_2}_{x_0}(E)$ holds for any set E in the form $\{X_1 = a_1, \cdots, X_n = a_n\}$, namely, it holds for any $E \in \bigcup_{i=1}^{\infty} F_i$, where $F_n \triangleq \{\{X_1 = a_1, \cdots, X_n = a_n\} : a_1, \cdots, a_n \in S\}$. It is easy to check $\bigcup_{i=1}^{\infty} F_i$ is a π-system, and $\pi - \lambda$ theorem tells us that this relation holds for any $E \in \mathscr{G}$.

By Theorem A.3.1, for any $A \in \mathscr{B}(\widehat{S_M})$, $\{X_\infty \in A\} \in \mathscr{G}$. Therefore, $p^h_{x_0}(X_\infty \in A) = c_1p^{h_1}_{x_0}(X_\infty \in A) + c_2p^{h_2}_{x_0}(X_\infty \in A)$, and this completes the proof. □

Below is an important characterization for normalized minimal harmonic functions.

Theorem A.5.8. *Let $h(x)$ be a normalized harmonic function. The following are equivalent:*
(i) $h(x)$ is a normalized minimal harmonic function.
(ii) $\mu^h(\{\alpha\}) = 1$ for some $\alpha \in \partial_R S_M$.
(iii) $\mu^h(\{\alpha\}) = 1$, where $\alpha \in \partial_m S_M$ and $h(i) = K(i, \alpha)$ for all $i \in S$.

Proof. (iii)⇒(ii) is straightforward.

(ii)⇒(i): Assume that $h(x)$ is not minimal harmonic, then by proposition A.5.5 and A.5.7, $\mu_h = c_1\mu_{h_1} + c_2\mu_{h_2}$, where $c_1 + c_2 = 1, c_1, c_2 > 0$, and both h_1, h_2 are normalized harmonic functions with corresponding harmonic measure μ_{h_1}, μ_{h_2}. This shows that μ^h cannot be a point mass.

(i)⇒(iii): Choose arbitrary $B \in \mathscr{B}(\widehat{S_M})$ such that $0 < \mu^h(B) < 1$. By Theorem A.4.1, for each $i \in S$ we have

$$\begin{aligned} h(i) =& \int_{\widehat{S_M}} K(i,x) d\mu^h(x) \\ =& \int_{\widehat{S_M} \setminus B} K(i,x) d\mu^h(x) + \int_B K(i,x) d\mu^h(x) \\ =& \mu^h(\widehat{S_M} \setminus B) \times \frac{1}{\mu^h(\widehat{S_M} \setminus B)} \int_{\widehat{S_M} \setminus B} K(i,x) d\mu^h(x) \\ &+ \mu^h(B) \times \frac{1}{\mu^h(B)} \int_B K(i,x) d\mu^h(x), \end{aligned}$$

representing h as a convex combination of two normalized harmonic functions. Since h is minimal harmonic, by Theorem A.5.5 we have $h(i) = \frac{1}{\mu^h(B)} \times \int_B K(i,x) d\mu^h(x)$ for every $i \in S$. That is, for each $i \in S$ and $B \in \mathscr{B}(\widehat{S_M})$ such that $0 < \mu^h(B) < 1$, we have

$$\int_B K(i,x) - h(i) d\mu^h(x) = 0,$$

and this shows $K(i,x) = h(i)$ for every $i \in S$ and μ^h-a.s. x. We remark that for $\alpha, \beta \in \partial S_M$, $\alpha \neq \beta$, there must be some $j \in S$ such that $K(j,\alpha) \neq K(j,\beta)$, and this fact shows $K(i,\alpha) = h(i)$ for a single $\alpha \in \partial S_M$. By the definition of minimal boundary, $\alpha \in \partial_m S_M$. □

The following theorem strengthens the results in Theorem A.5.2.

Theorem A.5.9. *$\partial_m S_M \in \mathscr{B}(\widehat{S_M})$. Furthermore, $\mu(\partial_m S_M) = 1$.*

Proof. 1. We show that for each $A \in \mathscr{B}(\widehat{S_M})$, $p_{x_0}^{K(\cdot,x)}(X_\infty \in A) = \mu^{K(\cdot,x)}(A)$ is a Borel measurable function of x on $\partial_R S_M$. Indeed, for any set E of the form $\{X_1 = a_1, \cdots, X_n = a_n\}$, $p_{x_0}^{K(\cdot,x)}(E) = p(x_0,a_1)p(a_1,a_2) \times \cdots \times p(a_{n-1},a_n)K(a_n,x)$ is a continuous function of $x \in \partial_R S_M$. In addition, $\{F \in \mathscr{G} : p_{x_0}^{K(\cdot,x)}(F)$ is a Borel measurable function on $\partial_R S_M\}$ is a λ-system that contains all sets of the form $\{X_1 = a_1, \cdots, X_n = a_n\}$. It follows by $\pi - \lambda$ theorem that for any $F \in \mathscr{G}$, $p_{x_0}^{K(\cdot,x)}(F)$ is a

Borel measurable function on $\partial_R S_M\}$. In particular, $p_{x_0}^{K(\cdot,x)}(X_\infty \in A) = \mu^{K(\cdot,x)}(A)$ is a Borel measurable function of x on $\partial_R S_M$.

2. We have the following identity:

$$p_{x_0}(A, X_\infty \in B) = \int_B p_{x_0}^{K(\cdot,x)}(A) d\mu(x),$$

for $A \in \mathscr{G}$ and $B \in \mathscr{B}(\widehat{S_M})$. To see this, we first consider the case $A = \{X_1 = a_1, \cdots, X_n = a_n\}$. We have

$$\begin{aligned}
& p_{x_0}(A, X_\infty \in B) \\
=& p_{x_0}(X_1 = a_1, \cdots, X_n = a_n, X_\infty \in B) \\
=& p(x_0, a_1)p(a_1, a_2) \times \cdots \times p(a_{n-1}, a_n) p_{a_n}(X_\infty \in B) \\
=& p(x_0, a_1)p(a_1, a_2) \times \cdots \times p(a_{n-1}, a_n) \int_B K(a_n, x) d\mu(x) \qquad \text{by Theorem A.3.6} \\
=& \int_B p(x_0, a_1)p(a_1, a_2) \times \cdots \times p(a_{n-1}, a_n) K(a_n, x) d\mu(x) \\
=& \int_B p_{x_0}^{K(\cdot,x)}(A) d\mu(x).
\end{aligned}$$

Since $\{A \in \mathscr{G} : p_{x_0}(A, X_\infty \in B) = \int_B p_{x_0}^{K(\cdot,x)}(A) d\mu(x)\}$ is a λ-system, the result follows by $\pi - \lambda$ theorem.

3. By 2. we have

$$\begin{aligned}
\int_B 1_A(x) d\mu(x) =& \mu(A \cap B) \\
=& p_{x_0}(X_\infty \in A \cap B) \\
=& \int_B p_{x_0}^{K(\cdot,x)}(X_\infty \in A) d\mu(x)
\end{aligned}$$

for any $A, B \in \mathscr{B}(\widehat{S_M})$. Therefore, for each $A \in \mathscr{B}(\widehat{S_M})$, $1_A(x) = p_{x_0}^{K(\cdot,x)}(X_\infty \in A)$ for μ-a.e. x.

4. Let $T = \{x \in \partial_R S_M : 1_A(x) = p_{x_0}^{K(\cdot,x)}(X_\infty \in A)$ for any $A = B_r(y)$, where $y \in S$ and $r \in \mathbb{Q}^+\}$. By 1., 3., and Theorem A.5.2 we have $T \in \mathscr{B}(\widehat{S_M})$ and

$\mu(T) = 1$.

5. Our goal is to show that $T = \partial_m S_M$ and the proof is complete. Assume that $x' \in T$. We choose a sequence of balls $B_{1/n}(y_n)$ such that $y_n \in S$ and $x \in \bigcap_{n=1}^{\infty} B_{1/n}(y_n)$. We have

$$\begin{aligned} 1 &= \lim_{n\to\infty} 1_{B_{1/n}(y_n)}(x') \\ &= \lim_{n\to\infty} p_{x_0}^{K(\cdot,x')}(X_\infty \in B_{1/n}(y_n)) \\ &= p_{x_0}^{K(\cdot,x')}(X_\infty = x') = \mu^{K(\cdot,x')}(\{x'\}). \end{aligned}$$

Here we have applied dominated convergence theorem on the third equality. Since $\mu^{K(\cdot,x')}$ is a point mass, we find that $K(\cdot, x')$ is a normalized minimal harmonic function by Theorem A.5.8. That is, $x' \in \partial_m S_M$.

Conversely, if $x' \in \partial_m S_M$, then it follows directly from the definition of T and Theorem A.5.8 that $x' \in T$. $\square$

Our last task is to show that the integral representation of Theorem A.4.1 is unique.

Theorem A.5.10. *Let $h(x)$ be a normalized harmonic function on S such that $h(x_0) = 1$. Then there exists a unique Borel measure ν such that $h(i) = \int_{\widehat{S_M}} K(i,x)d\nu(x)$ for every $i \in S$, and $\nu(\widehat{S_M}) = \nu(\partial_m S_M) = 1$. Indeed, by Theorem A.4.1, the unique measure ν is μ^h.*

Proof. We only need to check that if $h(i) = \int_{\partial_m S_M} K(i,x)d\nu(x) = \int_{\partial_m S_M} K(i,x)d\mu^h(x)$, then $\nu \equiv \mu^h$. We have proved the existence in Theorem A.4.1.

1. We claim that for each $A \in \mathscr{G}$, we have

$$p_{x_0}^h(A) = \int_{\partial_m S_M} p_{x_0}^{K(\cdot,x)}(A)d\nu(x).$$

It is sufficient to check $A = \{X_1 = a_1, \cdots, X_n = a_n\}$ and then apply $\pi - \lambda$ theorem. To this end,

$$\begin{aligned} p_{x_0}^h(A) =& p_{x_0}(X_1 = a_1, \cdots, X_n = a_n)h(a_n) \\ =& p_{x_0}(X_1 = a_1, \cdots, X_n = a_n) \int_{\partial_m S_M} K(a_n, x) d\nu(x) \\ =& \int_{\partial_m S_M} p_{x_0}^{K(\cdot,x)}(A) d\nu(x). \end{aligned}$$

2. For each $A \in \mathscr{B}(\widehat{S_M})$,

$$\begin{aligned} \mu^h(A) =& p_{x_0}^h(X_\infty \in A) \\ =& \int_{\partial_m S_M} p_{x_0}^{K(\cdot,x)}(X_\infty \in A) d\nu(x) \\ =& \int_{\partial_m S_M} 1_A(\{x\}) d\nu(x) \\ =& \nu(A \cap \partial_m S_M) \\ =& \nu(A). \end{aligned}$$

Here the third equality is due to the fact that $K(\cdot, x)$ is a normalized minimal harmonic function, and thus $\mu^{K(\cdot,x)}$ is a point mass centered at x by Theorem A.5.8. This implies $\nu \equiv \mu^h$ and the proof is complete. $\square$