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極化流形的Gieseker-Mumford穩定性的幾何準則報告

A survey of the geometric criterion for Gieseker-Mumford stability of polarized manifold

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中文摘要

本篇論文主要是研讀羅華章博士在 1997 年的論文，對其內容與證明進行詳細的研讀，並且提出四個陳述，對其做出證明。

首先，羅介紹了極化流形以及它的希爾伯特點。藉著幾何不變理論中的對於希爾伯特點的穩定性定義了極化流形在幾何不變理論下的穩定性。藉此他給出穩定性的性質。我們在這邊證明了我們第一個陳述。

再來，使用微分幾何的方法，羅改進了上面的性質。藉由格林流的定義，給出了更廣的 Gieseker-Mumford 穩定性的性質。在這邊我們證明了兩個陳述。

最後，透過上述的分析，羅證明了他最後一個定理。在這邊我們證明了最後一個陳述以及對於羅的定理我們做了一點修改得到一個關於 Gieseker-Mumford 穩定性的幾何準則。

Abstract

This paper is to study Luo's paper in 1997. We give four statements with their proofs.

Firstly, Luo introduce the polarized manifold and its Hilbert point. By the stability of Hilbert points in the Geometric Invariant Theory, he defined the stability of polarized manifolds in the Geometric Invariant Theory; hence he gave the proposition for the stability. We prove our first statement.

Secondly, Luo use the differential geometric method to reduce the proposition. By the definition of Green current, it gave the extended proposition for the Gieseker-Mumford stability, which is the first main theorem . Here we prove two statements.

Finally, use the above analysis, Luo proved the last theorem. We prove our final statement and do a slight improvement to give the geometric criterion for the Gieseker-Mumford stability.

目　錄

A SURVEY OF THE GEOMETRIC CRITERION FOR GIESEKER-MUMFORD STABILITY OF POLARIZED MANIFOLDS

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1. Introduction

This survey paper is to study the paper of Luo [20] and to prove four statements (cf. Statements 2.1, 3.1, 3.2 and 4.1) and two theorems (cf. Theorem 4.1 and 4.4). In 1965, Mumford developed the Geometric

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Invariant Theory [25], which introduces a method to construct quotients by group actions in algebraic geometry. For example, suppose that there is a group action $G \hookrightarrow SL(n+1,\mathbb{C})$ on a projective variety $X \subset \mathbb{CP}^n$; then we hope the quotient X/G to be still in the same category of projective variety. In GIT, we define the quotient X/G to be $Proj \oplus_r H^0(X, \mathcal{O}(r))^G$, where $H^0(X, \mathcal{O}(r))^G$ denotes the set of G-invariant sections of $\mathcal{O}(r)$ on X. Here we have a simple example to see the GIT quotient.

Example 1.1 ($\mathbb{P}^n$ from GIT)**.** *Let $\mathbb{C}^{n+1} \times \mathbb{C}$ be the trivial line bundle over $\mathbb{C}^{n+1}$ and let $G = \mathbb{C}^*$ be the action on $\mathbb{C}^{n+1}$. Set the lifted action $\bar{\mu_k}$ on the trivial line bundle $\mathbb{C}^{n+1} \times \mathbb{C}$ to be defined by*

$$(t, z, y) \mapsto (t \cdot z, t^k \cdot y),$$

where the first entry is the group action on $\mathbb{C}^{n+1}$. If $k < 0$, then there is no invariant sections over $\mathbb{C}^{n+1}$ and that the quotient is empty. If $k = 0$, the invariant sections are constant polynomials, so that the quotient is a single point. For $k > 0$, the G-invariant sections of the k-th power are the homogeneous polynomials on $\mathbb{C}^n$ of degree kp. If $k = 1$, there is the quotient

$$\begin{aligned} \mathbb{C}^{n+1}/\mathbb{C}^* &= Proj \oplus_{k \geq 0} (\mathbb{C}[x_0, \cdots, x_n]_k) \\ &= Proj\mathbb{C}[x_0, \cdots, x_n] = \mathbb{P}^n. \end{aligned}$$

The points in this quotient are analyzed in some sense of stability [20]. In the sense of topological characterisation of (semi)stability [25], we have

Definition 1.1. *If a reductive group G acts linearly on a vector space V, then a non-zero point x of V is called*

(1) *unstable if 0 is in the closure of its orbit,*
(2) *semi-stable if 0 is not in the closure of its orbit,*
(3) *stable if its orbit is closed in V, and its stabilizer is finite.*

This definition can be rewrote by the Hilbert-Mumford criterion.

Definition 1.2. *A* 1*-parameter subgroup of a group G is a homomorphism $\lambda : \mathbb{C}^* \to G$. We denote it by* 1*-PS of G.*

The well-known tautological line bundle over $\mathbb{P}^n$ is $\mathcal{O}(-1)$. That is, the fiber $\mathcal{O}_x(-1)$ over the point $x \in \mathbb{P}^n$ represents the corresponding line of x in $\mathbb{C}^{n+1}$ and let $x_0 = \lim_{\lambda \to 0} \lambda \cdot x$, which is a fixed point of the

$\mathbb{C}^*$ action. Set $\rho(x) \in \mathbb{Z}$ to be the weight of this action, that is, $\lambda \in \mathbb{C}^*$ acts on $\mathcal{O}_{x_0}(-1)$ as $\lambda^{\rho(x)}$. The Hilbert-Mumford criterion is

Theorem 1.1. (1) *If $\rho(x) < 0$ for all 1-PS, then x is stable.*
(2) *If $\rho(x) \geq 0$ for all 1-PS, then x is semistable.*
(3) *If $\rho(x) > 0$ for some 1-PS, then x is unstable.*

Notice that forming a moduli space of algebraic varieties is a GIT problem. In general, problem of stability for any polarized projective varieties is difficult to check. In 1977, Mumford [26] proved that the necessary condition of smooth algebraic curves to be the Chow stable and he used the asymptotic stability to construct the moduli space of these curves.

In the same year, Gisesker [11] proved that the necessary condition of algebraic surfaces to be stable in the sense of Hilbert-Mumford. Later, Viehweg proved that points of the reduced Hilbert scheme of canonically polarized manifolds are stable in the sense of Hilbert-Mumford under the usual group action with respect to some ample sheaf [34, 1.7].

In 1982, Kobayashi [16] showed that any holomorphic bundle over compact Kähler manifolds which satisfied the Einstein condition (also called Hermitian-Yang-Mills metric) is stable (as in [5]). Separately, Lübke [19] also gave the proof of the theorem posed in [16]. Here we briefly give the definition of the stability used in [16]:

Definition 1.3. *Let $E \to X$ be a holomorphic vector bundle (E, h) with its Hermitian structure h over a compact Kähler manifold (X, g) together with the Kähler metric g. Let $\mathfrak{F}$ be the coherent subsheaf of $\mathcal{O}(E)$ with $rank(\mathfrak{F}) \geq rank(E)$. Let Φ be the Kähler form of g,*

$$deg(\mathfrak{F}) := \int_X c_1(\mathfrak{F}) \cdot \Phi^{n-1},$$

and

$$\mu(\mathfrak{F}) := \frac{deg(\mathfrak{F})}{rank(\mathfrak{F})},$$

which is defined to be the slope of $\mathfrak{F}$. We say that E is slope stable (resp. slope semistable) if $\mu(\mathfrak{F}) < \mu(\mathcal{O}(E))$ (resp. $\mu(\mathfrak{F}) \geq \mu(\mathcal{O}(E))$)

Notice that in [16] he posed that

Conjecture 1.1. *Let E be an indecomposable holomorphic vector bundle on a compact Kähler manifold W with Kähler metric g. Then E*

admits an Hermitian-Yang-Mills metric if and only if E is slope stable with respect to g.

Donaldson [4] proved that a bundle over an algebraic surface is slope stable with respect to the projective embedding if and only if it admits an unique irreducible Hermitian-Yang-Mills metric. More generally, Uhlenbeck and Yau [33] demonstrated the existence of a Hermitian-Yang-Mills metric in slope stable holomorphic bundles over any compact Kähler manifolds. In 1987, Donaldson [5] gave an alternative proof for bundles over the projective manifolds $X \subseteq \mathbb{CP}^N$. He showed that if a holomorphic bundle E over a compact Kähler manifold (X, ω) is slope stable, then there exists a Hermitian-Yang-Mills metric on E.

This conjecture has a similar form for the case of variety. Yau [37] suggested that a compact Kähler Einstein metric if and only if the manifold is stable in the sense of geometric invariant theory. Tian [30] proved this in case of complex surfaces and introduced his notion of K-stability.

Definition 1.4 (Tian)**.** *We say that M is K-stable (resp. K-semistable), if M has no nontrivial holomorphic vector fields, and for any special degeneration W of M, the Futaki invariant $f_{W_0}(v_W)$ has positive (resp. nonnegative) real part. We say that M is weakly K-stable if $Ref_{W_0}(v_W) \geq 0$ for any special degeneration W, and the equality holds if and only if W is trivial.*

In 1992, Ding and Tian [3] proved that if a cubic surface in $\mathbb{CP}^3$ has a Kähler-Einstein orbifold metric if it is semistable in the sense of Mumford. Tian [31] showed that if M admits a Kähler-Einstein metric with positive scalar curvature, then M is weakly K-stable (in fact, he showed that the K-energy is proper if and only if a Kähler-Einstein metric exists on a compact Kähler manifold with positive Chern class and without any nontrivial holomorphic field). In particular, if M has no nonzero holomorphic vector field, M is K-stable.

For the case of the polarized varieties and the special metric, Yau [37], Tian [31] and Donaldson [7] conjectured that

Conjecture 1.2. *(X, L) is K-polystable if and only if (X, L) admits a Kähler metric with constant scalar curvature in the class $c_1(L)$. This is unique up to the holomorphic automorphisms of (X, L).*

Donaldson [7] proved the ”if condition” in Conjecture 1.2 on toric surfaces. Donaldson defined K-stability in algebraic geometry sense.

Definition 1.5 (Donaldson). *The pair $(M, \mathcal{L})$ is K-stable if for each test configuration for $(M, \mathcal{L})$ the Futaki invariant of the induced action on $(M_0, \mathcal{L}|_{M_0})$ is less than or equal to zero, with equality if and only if the configuration is a product configuration.*

He [7] proved that a toric variety (M, L) has bounded Mabuchi energy from below on the invariant metrics and any minimizing sequence has a K-convergent subseqence is K-stable with respect to toric degenerations. He also showed the converse on toric surfaces. In 1988, Burns, D. and De Bartolomeis, P. proved that the projective bundles does not admit a Kähler metric with constant scalar curvature (cf. [2], [15], [28]).

In 2005, Donaldson [8] proved that the Kähler metric with constant scalar curvature implies K-semistability. On the other hands, Donaldson [9] proved that the Kähler metric with constant scalar curvature minimizes the Mabuchi functional.

In [27] and [28], Ross and Thomas proved that the K-stability of the polarized varieties implies the slope stability. They proved that if the polarized variety (X, L) is Chow (semi)stable, then it is slope (semi)stable. If X is a curve, then the slope stability of X implies K-stability, which gives the converse direction.

Note that Donaldson ([4], [5]), Uhlenbeck and Yau ([33]) proved the Mumford stability of vector bundles is equivalent to the existence of Hermitian-Yang-Mills metric which gives that the meaning of stability of a vector bundle is described by its geometry. By the method of [31], Luo [20] gave a geometric criterion for the polarized line bundle of a polarized smooth projective variety to check the Gieseker-Mumford stability.

With a slight of revision, we prove two theorems:

Theorem 1.2. *Let $M \subset \mathbb{CP}^N$ be a smooth projective subvariety, and its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $SL(N+1, \mathbb{C})$. Then $[M] \in Hilb_h$ is (GIT) stable if there exists $\sigma \in SL(N+1, \mathbb{C})$ such that*

$$\sum_{i,j=1}^{N} \frac{Re(c_{ij})}{Vol(M)} \int_{\sigma(M)} \frac{z_i \cdot \bar{z}_j}{|z_1|^2 + \cdots + |z_N|^2} \omega_{FS}^n = 1, \tag{1}$$

where $\mathrm{tr}(c_{ij}) = N + 1$.

Theorem 1.3. *Let $(M, L) \in \mathfrak{I}_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be a large number given by (2). For any $k \geq \mu_0$, if there exists a Hermitian metric g (depends on k) on L over M such that there exists a basis $\{s_0, \cdots s_N\}$ of $H^0(M, L^k)$ such that*

$$\delta_{ij} = \int_M \frac{\langle s_i, s_j \rangle_{g^k}}{\|s_0\|^2_{g^k} + \cdots + \|s_N\|^2_{g^k}} \omega^n_{FS},$$

then the k-th Hilbert point of (M, L) is (GIT) stable with respect to G, and $\mathfrak{L} = \det(g_*(\pi_2^* \mathcal{O}(\nu)))$ for all large enough ν as long as the stabilizer of the Hilbert point is finite. And consequently, (M, L) is Gieseker-Mumford stable.

2. Gieseker-Mumford stability

2.1. **Moduli space of Polarized varieties.** We need some definitions.

Definition 2.1. *Let Γ be a projective variety over $\mathbb{C}$ and let $\mathcal{H}$ be a line bundle over Γ.*

(1) *If $\mathcal{H}$ is an ample line bundle over Γ, that is,*

$$\Gamma \hookrightarrow \mathbb{P}(H^0(\Gamma, \mathcal{H}^\mu)) \quad \text{for some } \mu \gg 1,$$

then $\mathcal{H}$ is called the polarization of Γ.

(2) *If (1) holds, then the paring $(\Gamma, \mathcal{H})$ is called the polarized variety.*

(3) *Suppose that (1) holds and the polynomial $h(T) \in \mathbb{Q}[T]$ is defined by $h'(\mu T)$, where h' comes from the Euler-Poincaré characteristic $\mathcal{X}(\Gamma, \mathcal{H}^\mu) = h'(\mu)$. Then we define h to be the Hilbert polynomial of $(\Gamma, \mathcal{H})$.*

(4) *If (1) holds and Γ is smooth, then $(\Gamma, \mathcal{H})$ is called the polarized manifold.*

(5) *A family $\{(\Gamma_\alpha, \mathcal{H}_\alpha) | \alpha \in \Lambda,$ for some index set $\Lambda\}$ of polarized variety with the same Hilbert polynomial h is called bounded if there exists some $\mu_0 \gg 1$ such that $\mathcal{H}^\mu$ is very ample for $\mu \geq \mu_0$.*

Consider the moduli problem of polarized varieties

$$\mathfrak{I} : Schemes/\mathbb{C} \rightarrow Sets.$$

Definition 2.2. *Let $h'(T) \in \mathbb{Q}[T]$ be a polynomial defined as in Definition 2.1 of degree n so that we have*

$$\mathfrak{I}_{h'}(\mathbb{C}) = \{(\Gamma, \mathcal{H}) | (\Gamma, \mathcal{H}) \in \mathfrak{I}(\mathbb{C}), \mathcal{X}(\Gamma, \mathcal{H}^m) = h'(m) \quad \forall m \geq 1\}.$$

Two polarized varieties $(\Gamma_\alpha, \mathcal{H}_\alpha)$ and $(\Gamma_\beta, \mathcal{H}_\beta)$ are identified if there is an isomorphism $\tau : \Gamma_\alpha \to \Gamma_\beta$ such that $\tau^*(\mathcal{H}_\beta) \cong \mathcal{H}_\alpha$

By the Matsusaka's big theorem ([24], [18]) and the results of Kollár [17], the functor $\mathfrak{I}_{h'}(\mathbb{C})$ is bounded and the higher dimensional cohomology groups $H^i(\Gamma, \mathcal{H}^\mu) = 0$, that is,

$$\begin{aligned} &\mathcal{H}^\mu \text{ is very ample, for all } \mu \geq \mu_0, \\ &H^i(\Gamma, \mathcal{H}^\mu) = 0, \text{ for all } i \geq 1, \mu \geq \mu_0. \end{aligned} \tag{2}$$

Let $N = h'(\mu) - 1$. $H^0(\Gamma, \mathcal{H}^\mu)$ has dimension $h'(\mu)$. Notice that this embedding depends on the choice of a basis of $H^0(\Gamma, \mathcal{H}^\mu)$.

According to the results of Grothendieck (or cf. [29] and [28]), there is a scheme $Hilb_h$ (called the Hilbert scheme) parameterizing all the subschemes of $\mathbb{CP}^N$ with fixed Hilbert polynomial h. By the results [17], there exists the universal family $g : Univ_h \to Hilb_h$ together with an embedding $Univ_h \hookrightarrow Hilb_h \times \mathbb{CP}^N$. That is,

$$\begin{array}{ccc} Univ_h & \xrightarrow{\subset} & Hilb_h \times \mathbb{CP}^N \\ \downarrow{\scriptstyle g} & & \\ Hilb_h & & \end{array}$$

Definition 2.3. *For any projective subvariety $X \subset \mathbb{C}P^N$ with Hilbert polynomial $h \in \mathbb{Q}[T]$, the Hilbert point of X is the corresponding point $[X] \in Hilb_h$. For any polarized variety $(X, L) \in \mathfrak{I}_{h'}(\mathbb{C})$, let $\mu \geq \mu_0$ and consider an embedding $e_\mu : X \to \mathbb{CP}^N$ by L^μ. Then the Hilbert point of $e_\mu(X) \subset \mathbb{CP}^N$ is called (one of) the μ-th Hilbert point of (X, L).*

The universal family $Univ_h$ consists of pairs $([X], X)$, where X is the projective variety $X \subset \mathbb{CP}^N$ and $[X]$ denotes its Hilbert point, so that the action of group $G = SL(N+1, \mathbb{C})$ on $Univ_h$ is

$$\begin{aligned} \sigma \cdot ([X], X) &= (\sigma \cdot [X], \sigma \cdot X) \\ &= ([\sigma \cdot X], \sigma \cdot X) \text{ for all } \sigma \in G, \end{aligned}$$

which is the lifted action of G on $Hilb_h$. Therefore the action of G on $Hilb_h$ and the lifting of this action to $Univ_h$ are equivalent.

Grothendieck proved that on $Hilb_h$ there is an ample line bundle

$$\mathfrak{L} = det(g_*\pi_2^*\mathcal{O}(\nu)) \qquad \nu \geq \nu_0, \tag{3}$$

where $\pi_2 : Uinv_h \to \mathbb{CP}^N$ is the projection. Note that the action on $Hilb_h$ is lifted to the ample line bundle $\mathfrak{L}$ which means that $\mathfrak{L}$ is G-linearized.

2.2. **Gieseker-Mumford stability.** Here we define the stability of polarized varieties in the Mumford sense.

Definition 2.4. *A point $x \in H = Hilb_h$ is called (GIT) stable with respect to G, the ample line bundle $\mathfrak{L}$ on the Hilbert scheme $Hilb_h$ and the given linearisation, if x has finite stabilizer and for some $m \geq 1$, there exists a section $t \in \Gamma(Hilb_h, \mathfrak{L}^m)^G$ such that:*

1. *$H_t = H - V(t)$ is affine, where $V(t)$ denotes the zero locus of t,*
2. *$x \in H_t$, or in other terms, $t(x) \neq 0$,*
3. *the induced action of G on H_t is closed.*

Moreover, $(\Gamma, \mathcal{H})$ is called Gieseker-Mumford stable if when μ is very large, there exists $\nu_0 \geq 1$ such that for any $\nu \geq \nu_0$, the μ-the Hilbert points of $(\Gamma, \mathcal{H})$ in $Hilb_h$ is (GIT) stable with respect to G and the ample line bundle $\mathfrak{L}$ of $Hilb_h$.

Note that the Hilbert scheme $Hilb_h$ and the universal family $Univ_h$ are usually singular. We need some reductions to use some differential geometric method.

2.3. **Propositions for Stability.** Assume that $\mathfrak{L}^m$ is very ample for some $m \geq 1$. This gives an embedding from $H = Hilb_h$ to a projective space $\mathbb{CP}^M$ such that $\mathfrak{L}^m = \mathcal{O}_{\mathbb{CP}^M}(1)|_H$, where $M = h^0(H, \mathfrak{L}^m)$. Since $\mathfrak{L}$ is G-linearized, the action of G on $\mathbb{CP}^M$ is represented rationally. That is, $G \to SL(M+1, \mathbb{C})$, and the embedding of H is G equivalent. Let $\theta : \mathbb{C}^{M+1} \setminus \{0\} \to \mathbb{CP}^M$ be the projection, and $\hat{H}$ be the affine cone over H, that is, the closure of $\theta^{-1}(H)$ in $\mathbb{C}^{M+1}$.

Now we can give some propositions for the stability.

Proposition 2.1 (Luo)**.** *$x \in H(\mathfrak{L})^s$ if and only if for all points $\hat{x} \in \theta^{-1}(x)$, the orbit of $\hat{x}$ in $\hat{H}$ is closed and the stabilizer of x is finite.*

Fix a point x in H, define the function

$$F_x(\sigma) = -\log(\|\sigma(\hat{x})\|^2), \text{ for } \sigma \in G, \tag{4}$$

where $\widehat{x}$ is a fixed lifting of x to the fiber of $\mathcal{O}_{\mathbb{CP}^M}(1)$ at x.

Proposition 2.2 (Luo). *The point x of the Hilbert scheme is (GIT) stable defined as before if and only if F_x is a proper function on G, i.e., for any $c_1, c_2 \in \mathbb{R}$ the set*

$$\{\sigma \in G \mid c_1 \leq F_x(\sigma) \leq c_2\}$$

is a compact subset of G with respect to Hausdorff topology.

This proposition is proved by the first statement.

Statement 2.1. *Proposition 2.1 is equivalent to Proposition 2.2.*

Proof. Suppose that for every $\hat{x} \in \theta^{-1}(x)$ the orbit $\hat{x}$ in the $\theta^{-1}(Hilb_h)$ is closed and the stabilizer of x is finite. Let

$$A(c_1, c_2) = \{\sigma \in G \mid c_1 \leq F_x(\sigma) \leq c_2\}$$

for any c_1 and c_2 in $\mathbb{R}$ with $c_1 < c_2$. Assume that the sequence $\{\sigma_n\}_{n=1}^{\infty}$ is contained in the set $A(c_1, c_2)$. It suffices to prove that this sequence has a convergent subsequence. Note that the sequence satisfies

$$e^{-\frac{c_2}{2}} \leq \|\sigma_n(\hat{x})\| \leq e^{-\frac{c_1}{2}}.$$

Clearly, it has a convergent subsequence, say $\{\sigma_{n_k}(\hat{x})\}$, such that

$$\lim_{k \to +\infty} \sigma_{n_k}(\hat{x}) = y,$$

which implies that there is a convergent subsequence $\{\sigma_{n_k}\}$ such that $\lim_{k\to\infty} \sigma_{n_k} = \sigma_\infty$ with $\sigma_\infty \in SL(M+1, \mathbb{C})$. This proves the compactness of the set $A(c_1, c_2)$.

Conversely, suppose that the function F_x is proper. Let y be a limit point of the orbit of $\hat{x}$. Suppose that there is a sequence $\{\sigma_n(\hat{x})\}$ converging to y as $n \to \infty$. Set $b_1 = \inf_{n\in\mathbb{N}} F_x(\sigma_n)$ and $b_2 = \sup_{n\in\mathbb{N}} F_x(\sigma_n)$, so that $A(b_1, b_2)$ is a set containing $\{\sigma_n\}$. Since F_x is proper, then $A(b_1, b_2)$ is compact and $\{\sigma_n\}$ has a convergent subsequence $\{\sigma_{n_k}\}$. Setting $\lim_{k\to\infty} \sigma_{n_k} = \sigma_\infty$ and clearly $\lim_{k\to\infty} \sigma_{n_k}(\hat{x}) = \sigma_\infty(\hat{x}) = y$ by the uniqueness of the limit. On the other hand, let τ be a stabilizer of x. $F_x(\cdot)$ is constant for all stabilizers of x. Let $F_x(\tau) = L$ be a constant and let $A_L = A(L, L)$. Let $A_L = \cup_\alpha \{\tau_\alpha\}$ (by the discreteness of A_L). Thus A_L is compact due to the properness of F_x. Since A_L is compactness, the stabilizer of x is finite.

□

The Proposition 2.2 gives the following:

Proposition 2.3 (Luo)**.** *Let $(M, L) \in \mathfrak{I}_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be given as in above. Then for any $\mu \geq \mu_0$, the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = det(g_*(\pi_2^*(\mathcal{O}(\nu))))$ $(\nu \geq \nu_0)$, if and only if F_M is a proper function on G, where $F_M : G \to \mathbb{R}$ is defined by*

$$F_M(\sigma) = -\log(\|\sigma(\widehat{x})\|^2), \tag{5}$$

and $\|\cdot\|$ is any Hermitian metric on $\mathfrak{L}_0 = det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(\upsilon)))$ over $\bar{G}$.

Remark 2.1. *By Proposition 2.1 and Statement 2.1, it suffices to prove Proposition 2.3 is the reduction of the Proposition 2.2.*

Proof. Note that x is the corresponding μ-th Hilbert point of (M, L) in $Hilb_h$. There is a morphism

$$\tau_x : G \to Hilb_h \qquad \text{for any } x \in H$$

defined by $\tau_x(\sigma) = \sigma(x)$.

By the completeness of the Hilbert scheme, there exists a smooth compactification $\bar{G}$ of G, such that we have an extension of τ_x

$$\tau : \bar{G} \to Hilb_h.$$

By τ, we have

$$\begin{array}{ccc} \bar{\Sigma} & & Univ_h \\ \downarrow f & & \downarrow g \\ \bar{G} & \xrightarrow{\tau} & Hilb_h \end{array}$$

where $\bar{\Sigma}$ is the pull-back of $Univ_h$. Let $i : \bar{\Sigma} \to \bar{G} \times \mathbb{CP}^N$ be the inclusion, $\bar{\pi}_1$ and $\bar{\pi}_2$ be the projections of $\bar{G} \times \mathbb{CP}^N$ to $\bar{G}$ and $\mathbb{CP}^N$, respectively. That is,

$$\begin{array}{ccc} \bar{\Sigma} & \overset{i}{\hookrightarrow} & \bar{G} \times \mathbb{CP}^N \\ \downarrow f & & \\ \bar{G} & & \end{array}$$

By the boundedness of $Univ_h$, if ν_0 is very large, then for all fibers Γ of $g : Univ_h \to Hilb_h$ it gives

$$H^i(\Gamma, \mathcal{O}_\Gamma(\nu)) = 0 \text{ for } i \geq 1, \nu \geq \nu_0.$$

Therefore, it gives that for all $\nu \geq \nu_0$,

$$\tau^*(\mathfrak{L}) = \tau^*(\det(g_*(\pi_2^*\mathcal{O}(\nu))) = \det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(\nu))).$$

□

Remark 2.2. *By the definition of F_x, it depends on the line bundle $\mathfrak{L}$ which comes from the universal family over the Hilbert scheme. Note that the Hilbert scheme and its universal family are usually singular. Although there are still singularities on $\bar{\Sigma}$, all of the singularities are contained in $f^{-1}(\bar{G} \setminus G)$. F_M depends only on the family $f : \bar{\Sigma} \to \bar{G}$.*

3. Singular Riemann-Roch

3.1. **Some Results from Intersection theorem.** In Luo's paper, he used some intersection theory [10] to deal with the singular fibers which come from the pull backs of the Hilbert scheme $H = Hilb_h$ and the universal family $Univ_h$ over H. We only describe breifly the key theorem and results.

Theorem 3.1 (Fulton)**.** *For every algebraic scheme X over a given field K, there is a homomorphism*

$$\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}$$

such that

1. *(Covariance). If $f : X \to Y$ is proper, $\alpha \in K_0(X)$ (Grothendieck group of coherent sheaves), then $f_*\tau_X(\alpha) = \tau_Y f_*(\alpha)$.*
2. *(Module). If $\alpha \in K_0(X)$, $\beta \in K^0(X)$ (Grothendieck group of locally free sheaves), then $\tau_X(\beta \otimes \alpha) = ch(\beta) \cap \tau(\alpha)$.*
3. *(Top Term) If V is a closed subvariety of X, with $\dim(V) = n$, then*

$$\tau_X(\mathcal{O}_V) = [V] + \textit{(terms of dimension} < n\textit{)}.$$

Use Theorem 3.1 Luo [20] gave the following lemma:

Lemma 3.1 (Luo)**.** *There are cycles $[D_k]$ $(1 \leq k \leq s)$ on $\mathbb{CP}^N$, and $(r-1)$-dimensional cycles $[C_k]$ $(1 \leq k \leq s)$ on $\bar{G}$, such that*

$$\begin{aligned} &\frac{1}{(n+1)!}g_*(c_1(s^*L^\nu)^{n+1}) + \pi_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s}([C_k] \times [D_k]))_{r-1} \\ &= c_1(\mathcal{L}_0) + \frac{1}{2}c_1(\bar{G}), \end{aligned} \tag{6}$$

and we may choose $C_k (1 \leq k \leq s)$ to be divisors of $\bar{G}$ supported in $\bar{G} - G$.

Proof. The Todd class for a general variety X can be defined by

$$Td(X) = \tau_X(\mathcal{O}_X) \in A_*(X)_{\mathbb{Q}},$$

and for any $\beta \in K^0(X)$, $\tau_X(\beta)$ can be written as

$$\tau_X(\beta) = ch(\beta) \cap Td(X).$$

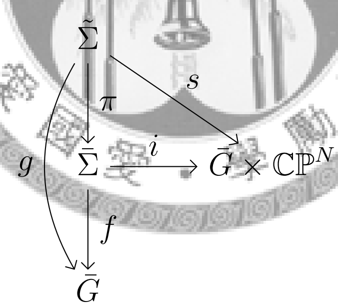
By the covariance of Riemann-Roch, $f : \bar{\Sigma} \to \bar{G}$ gives that

$$f_* \tau_{\bar{\Sigma}}(i^*(L^\nu)) = \tau_{\bar{G}}(f_* i^*(L^\nu)).$$

By the other properties, the left hand side of the above equality gives that

$$\begin{aligned} f_* \tau_{\bar{\Sigma}}(i^*(L^\nu)) &= f_*(ch(i^*(L^\nu))) \cap \tau_{\bar{\Sigma}}(\mathcal{O}_{\bar{\Sigma}}) \\ &= f_*(ch(i^*(L^\nu))) \cap ([\bar{\Sigma}] + \text{terms of lower dimension}). \end{aligned}$$

Let $\tilde{\Sigma}$ be a desingularization of $\bar{\Sigma}$ together with the canonical morphism $\pi : \tilde{\Sigma} \to \bar{\Sigma}$ and morphism $s : \tilde{\Sigma} \to \bar{G} \times \mathbb{CP}^N$ commuting with π and i, that is,



Note that there is

$$[\bar{\Sigma}] = \pi_*[\tilde{\Sigma}].$$

Since $i^*(L^\nu)$ is a line bundle over $\bar{\Sigma}$, by the Projection Formula for Chow group it gives that

$$\pi_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) = ch(i^*(L^\nu)) \cap \pi_*[\tilde{\Sigma}].$$

By the above equalities, there is

$$\begin{aligned} f_* \tau_{\bar{\Sigma}}(i^*(L^\nu)) &= g_*(ch(s^*(L^\nu))) \cap ([\tilde{\Sigma}] + \text{terms of lower dimension}) \\ &= g_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) + \bar{\pi}_1^*(ch(L^\nu) \cap [Z]), \end{aligned}$$

where $[Z]$ is a cycle of $\bar{G} \times \mathbb{CP}^N$ supported in $\bar{\Sigma}$, and

$$\dim(Z) \leq n + r - 1, \qquad r = \dim(\bar{G}).$$

Here $n = \dim(\bar{\Sigma}) - \dim(\bar{G})$ is the dimension of generic fiber. There is a filtration $\mathbb{CP}^N \supset \mathbb{CP}^{N-1} \supset \cdots \supset \mathbb{CP}^1$ of $\mathbb{CP}^N$, and each $\mathbb{CP}^k - \mathbb{CP}^{k-1} = \mathbb{C}^k$ is affine. Thus $\mathbb{CP}^N$ has a cellular decomposition and there is a surjective morphism of Chow groups

$$\bigoplus_{k+l=m} A_k(\bar{G}) \otimes A_l(\mathbb{CP}^N) \to A_m(\bar{G} \times \mathbb{CP}^N),$$

which implies that

$$[Z] = [C_1] \times [D_1] + \cdots + [C_r] \times [D_r],$$

where $[C_i]$'s are cycles on $\bar{G}$ and $[D_i]$'s are cycles on $\mathbb{CP}^N$. Assume that among $[C_1], \cdots, [C_r]$, there are only $[C_1], \cdots, [C_s]$ are in $Z_{r-1}(\bar{G})$, $r = \dim(\bar{G})$. From the above equalities, comparing the parts in $A_{r-1}(\bar{G})$ gives that

$$\begin{aligned}
&\frac{1}{(n+1)!} g_*(c_1(s^* L^\nu)^{n+1}) + \pi_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s} ([C_k] \times [D_k]))_{r-1} \\
&= c_1(\det(f_* i^*(L^\nu))) + \frac{1}{2} c_1(\bar{G}) \\
&= c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}),
\end{aligned}$$

Assume that $[D_k]$ is b_k-dimensional cycle of $\mathbb{CP}^N$. For all $0 \leq i \leq N$, $A_i(\mathbb{CP}^N)$ is a free abelian group generated by i-dimensional linear subspace $\mathbb{CP}^i$ of $\mathbb{CP}^N$. Therefore, assume that b_k is different from each other, and $b_1 < \cdots < b_s$. Note that

$$\begin{aligned}
&\bar{\pi}_2^*(ch(L^\nu) \cap \sum_{k=1}^{s} [C_k] \times [D_k])_{r-1} \\
&= \sum_{k=1}^{s} \bar{\pi}_{1*}(ch(L^\nu) \cap [D_k])_0 [C_k] \\
&= \sum_{k=1}^{s} \frac{\nu^{b_k}}{b_k!} (c_1(L)^{b_k})_0 [C_k] \\
&= \sum_{k=1}^{s} \lambda_k \nu^{b_k} [C_k],
\end{aligned} \tag{7}$$

where λ_k are some constants. Thus we have

$$\frac{1}{(n+1)!}g_*(c_1(s^*L^\nu)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k}[C_k] = c_1(\mathcal{L}_0) + \frac{1}{2}c_1(\bar{G}).$$

Note that the restrictions of $\mathcal{L}_0$ and $K_{\bar{G}}$ on G are trivial line bundles because of the G action. By the exact sequence $A_*(\bar{G}-G) \to A_*(\bar{G}) \to A_*(\bar{G}) \to 0$, it implies that there exist divisors Y and Y_0 with supports in $\bar{G}-G$ satisfying

$$\frac{1}{(n+1)!}g_*(c_1(s^*L^\nu)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k}[C_k] = [Y] + \frac{1}{2}[Y_0]. \tag{8}$$

As known, there is

$$(r-1) + b_k = \dim(C_k \times D_k) \leq n + r - 1,$$

which implies that $b_1 < \cdots < b_s < n+1$. Choosing $\nu = 1, 2, \cdots, s+1$ in (ref17) and then solving a non-degenerate $(s+1) \times (s+1)$ system of linear equations give that for every k, $\lambda_k[C_k]$ may be represented by divisors with support in $\bar{G}-G$, i.e., we may assume that C_k is a divisor with support in $\bar{G}-G$. The proof is completed.

□

3.2. **Green Current and logarithmic Green Current.** By the standard Euclidean metric on the hyperplane bundle over $\mathbb{CP}^N$, there are the Hermitian metrics on $s^*(L)$ over Σ and on L over $\bar{G} \times \mathbb{CP}^N$. Let ω_{FS} denote the Fubini-Study metric on $\mathbb{CP}^N$. The curvature of $s^*(L)$ is $s^*\bar{\pi}_2^*(\omega_{FS})$ and the curvature of L is $\bar{\pi}_2^*(\omega_{FS})$. Fix a Hermitian metric $\|\cdot\|$ on $\mathcal{L}_0$ with the curvature $R(\|\cdot\|)$ and a Hermitian metric $\|\cdot\|_{\bar{G}}$ on $K_{\bar{G}}$ with the curvature $R(\|\cdot\|_{\bar{G}})$. Assume that $[C_k]$ is Poincaré dual to a smooth differential form α_k on $\bar{G}$ and $[D_k]$ is the Poincaré dual to a smooth differential form β_k on $\mathbb{CP}^N$.

Recall the Green current used in Gillet-Soulé [12].

Definition 3.1 (Gillet-Soulé)**.** *If X is any n-dimensional smooth projective (complex) variety, and $Y \subset X$ a closed irreducible subvariety of codimension p, then there exists a $(p-1,p-1)-$current ψ, which is called the Green current, and a smooth closed (p,p)-form ω on X such that*

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\psi) + \delta_Y = \omega,$$

where δ_Y is the current representing integration on Y.

Remark 3.1. *Suppose that X, Y are defined as Definition 3.1 and $i : Y \hookrightarrow X$ is an inclusion map. Then there is a $2(n-p)$-current δ_Y such that*

$$\delta_Y(\alpha) = \int_Y i^*(\alpha),$$

where α is any compactly supported $2(n-p)$-form on X.

Recall the definition of the form of logarithmic type in [12]

Definition 3.2 (Gillet-Soulé)**.** *A smooth form η on $X - Y$ is said to be a form of logarithmic type (or log type) along Y if there exists a projective morphism*

$$\pi : Z \to X$$

and a smooth form φ on $Z - \pi^{-1}(Y)$ such that

(1) *Z is smooth, $\pi^{-1}(Y)$ is a divisor with normal crossings (d.n.c.), and π is smooth over $Z - \pi^{-1}(Y)$;*
(2) *η is the direct image by π of the restriction of φ to $Z - \pi^{-1}(Y)$;*
(3) *for any point $x \in Z$, there is an open neighborhood U of x, and a system of holomorphic coordinates $(z_1, \cdots, z_n)$ of U centered at x such that $\pi^{-1}(Y) \cap U$ has equation $z_1 \cdots z_k = 0$, for some $k \leq n$, and there exist smooth ∂ and $\bar{\partial}$-closed forms α_k on U, $i = 1, \cdots, k$ and a smooth form β on U with*

$$\varphi|_U = \sum_{i=1}^{k} \alpha_i \cdot log|z_i|^2 + \beta$$

By the desingularization, there exists a projective morphism

$$\pi : \tilde{X} \to X$$

such that $\tilde{X}$ is smooth, $E = \pi^{-1}$ is a d.n.c. and $\pi|_{(\tilde{X}\setminus E)}$ is isomorphic, which satisfies the Definition 3.2 (1),(2). ψ is of logarithmic type along Y if near each $x \in \tilde{X}$, $z_1 \cdots z_k = 0$ $(1 \leq k \leq n)$ is the local equation of E and there exist ∂ and $\bar{\partial}$ closed smooth forms α_i and a smooth form β such that

$$\pi^*(\psi) = \sum_{i=1}^{k} \alpha_i \cdot log|z_i|^2 + \beta.$$

Thus ψ is called the logarithmic Green current of the subvariety $Y \subset X$.

Lemma 3.2 (Luo)**.** *There is a measurable function θ_ν (depending on ν), such that as currents we have*

$$\begin{aligned}&\frac{\nu^{n+1}}{(n+1)!}g_*(s^*\bar{\pi}_2^*(\omega_{FS})^{n+1})+\bar{\pi}_{1*}(exp(\bar{\pi}_2^*\omega_{FS})\wedge\sum_{k=1}^{s}\alpha_k\wedge\beta_k)_{k-1}\\&=\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)+\frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}})+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta_\nu,\end{aligned}\tag{9}$$

where θ_ν is a smooth function when restrict on G, and is bounded from above by a constant on G. Here $(\cdot)_{r-1}$ means the $(r-1,r-1)$ part of differential form.

Proof. Let $[Z]\in A_{n+r-1}(\tilde{\Sigma})$ and $[Y]\in A_{r-1}(\bar{G})$ be cycles such that $[Z]=c_1(s^*L^\nu)^{n+1}$ and $[Y]=(c_1(\mathcal{L}_0)+\frac{1}{2}c_1(\bar{G}))$. According to Lemma 3.1 we have

$$\frac{\nu^{n+1}}{(n+1)!}g_*[Z]+\sum_{k=1}^{s}\lambda_k\nu^{b_k}[C_k]=[Y],$$

which is an equality between cycles. In terms of currents, it gives

$$\frac{\nu^{n+1}}{(n+1)!}g_*(\delta_Z)+\sum_{k=1}^{s}\delta_{C_k}=\delta_Y.$$

Let ψ_Z, ψ_Y and ψ_{C_k} be the logarithmic Green currents of $Z\subset\tilde{\Sigma}$, $Y\subset\bar{G}$ and $C_k\subset\bar{G}$, respectively. By the Definition 3.1 we have

$$\begin{aligned}&\frac{\nu^{n+1}}{(n+1)!}(g_*(s^*\bar{\pi}_2^*(\omega_{FS})^{n+1})-\partial\bar{\partial}g_*(\psi_Z))\\&+\bar{\pi}_{1*}(exp(\bar{\pi}_2^*\omega_{FS})\wedge\sum_{k=1}^{s}\alpha_k\wedge\beta_k)_{k-1}-\sum_{k=1}^{s}\lambda_k\nu^{b_k}\partial\bar{\partial}\psi_{C_k}\\=&\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)+\frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}})-\partial\bar{\partial}\psi_Y,\end{aligned}$$

which gives that (7) is valid for some measurable function θ_ν which is given by

$$\theta_\nu=\frac{\nu^{n+1}}{(n+1)!}g_*(\psi_Z)+\sum_{k=1}^{s}\lambda_k\nu^{b_k}\psi_{C_k}-\psi_Y.$$

By the Definition 3.2(1), $g_*(\psi_Z)$ is smooth on $G-g_*(Z)$, ψ_{C_k} are smooth on $G-(C_1+\cdots+C_s)$ and ψ_Y is smooth on $G-Y$, so that θ_ν is smooth on $G-(g_*(Z)+Y+(C_1+\cdots+C_s))$ and by Definition 3.2(3), it is at most logarithmic growth on $g_*(Z)+Y+(C_1+\cdots+C_s)+(\bar{G}-G)$.

Since in (13) every term except for $\partial\bar{\partial}\theta_\nu$ is smooth on G, then by the regularity of $\bar{\partial}$, θ_ν can be extended to be a smooth function on G.

By [32], [1], there exists a Green operator $G(x,y)$ such that

$$\theta_\nu(x) = \frac{1}{V}\int_{\bar{G}} \theta_\nu(y)\omega_y^r - \frac{1}{V}\int_{\bar{G}} G(x,y)\Delta\theta_\nu(y)\omega_y^r.$$

since θ_ν is smooth on G and at most logarithmic growth along $\bar{G}$, then the term $\frac{1}{V}\int_{\bar{G}} \theta_\nu(y)\omega_y^r$ is finite. Since every term in (13) is smooth on $\bar{G}$ except for $\partial\bar{\partial}\theta_\nu$ and $g_*(s^*\bar{\pi}_2^*(\omega_{FS})^{n+1})$ is a positive $(1,1)$-current, then instead of $\Delta\theta_\nu$ the second integral is bounded below. Therefore, θ_ν is bounded above on G.

□

3.3. **Secondary Characteristic Classes Type Computations.** Let $0 \in G \subset \bar{G}$ denote the identity of G and let $M_0 = g^{-1}(0)$ be the fiber of $g : \tilde{\Sigma} \to \bar{G}$ over 0. Since $M_0 = (0, M)$ by the definition of $\tilde{\Sigma}$, then M_0 is isomorphic to M and we identify M_0 with M. Set $\omega = s^*\bar{\pi}_2^*(\omega_{FS}|_{M_0})$ and $P(M,\omega) = \{\omega_\alpha | \omega_\alpha$ is a Kähler metric on M and is cohomologous to $\omega\}$.

Definition 3.3 (Luo)**.** *For any $\omega' \in P(M,\omega)$, let $\omega' = \omega + \partial\bar{\partial}\varphi$ for some smooth function φ. Then $D_M(\omega')$ is defined by*

$$D_M(\omega') = \int_0^1 \int_M \dot{\varphi}_t \omega_t^n \wedge dt \tag{10}$$

Here $\omega_t = \omega + \partial\bar{\partial}\varphi_t (0 \le t \le 1)$ is a smooth path from ω to ω' in $P(M,\omega)$.

This definition is well-defined (cf. [21], [22] and [14]). Since M_0 is as a subvariety of $\mathbb{CP}^N$ and M is identified with M_0, M is a subvariety of $\mathbb{CP}^N$. For any $\sigma \in G$, $g^{-1}(\sigma(0))$ can be identified with $\sigma(M) \subset \mathbb{CP}^N$. Let

$$\omega_\sigma = \sigma^*(\omega_{FS}|_{\sigma(M)}) \in P(M,\omega). \tag{11}$$

Definition 3.4. *Bergman metrics of $M \subset \mathbb{CP}^N$ is defined by*

$$Berg(M) = \{\omega_\sigma | \sigma \in G\} \subset P(M,\omega).$$

Now consider D_M as a functional on $Berg(M)$ and that

$$D_M(\sigma) = D_M(\omega_\sigma) \text{ for any } \sigma \in G. \tag{12}$$

For deriving information of D_M, we have a series of lemmas [20].

Lemma 3.3 (Luo). *For any smooth $2(r-1)$-form ϕ with compact support in G*

$$\int_{\tilde{\Sigma}} s^*\bar{\pi}_2^*(\omega_{FS})^{n+1} \wedge g^*(\phi) = \int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma) \wedge \partial\bar{\partial}\phi. \tag{13}$$

Proof. Let $\psi : G \times M \to G(M)$ be defined by $\psi(\sigma, x) = (\sigma, \sigma(x))$, where $G(M) := g^{-1}(G) \subset G \times \mathbb{CP}^N$ and . As before, there are Hermitian metrics on $s^*(L)$ over $\tilde{\Sigma}$ and on L over $\bar{G} \times \mathbb{CP}^N$, which are comes from the standard Euclidean metric on the hyperplane bundle over $\mathbb{CP}^N$. Assume that H denotes the Hermitian metric on $\psi^*s^*(L)$ by pulling back the Hermitian metric on $s^*(L)$, and the curvature is denoted by $R(H)$.

By the projection $pr_2 : G \times M \to M$, since $\psi^*s^*(L) \cong pr_2^*(L|_M)$, then there is another Hermitian metric $H_0 = pr_2^*(H|_M)$ on $\psi^*s^*(L)$. Therefore, let H_t be a path of Hermitian metrics $(0 \le t \le 1)$ on $\psi^*s^*(L)$ over $G \times M$ from H_0 to H. That is,

$$H_t = exp(-\varphi_t)H_0 \qquad and \qquad \varphi_t = t \cdot log(\frac{H_0(\sigma, x)}{H(\sigma, x)}).$$

Therefore we have

$$\begin{cases} -\partial\bar{\partial}\log(H_t) = R(H_t) \\ R(H_t) = R(H_0) + \partial\bar{\partial}\varphi_t. \end{cases}$$

Then it gives that

$$\begin{aligned} &\int_{\tilde{\Sigma}} s^*\bar{\pi}_2^*(\omega_{FS})^{n+1} \wedge g^*(\phi) \\ =& \int_{G\times M} \psi^*s^*\bar{\pi}_2^*(\omega_{FS})^{n+1} \wedge \psi^*g^*(\phi) \\ =& \int_{G\times M} (\frac{\sqrt{-1}}{2\pi}R(H)))^{n+1} \wedge pr_1^*(\phi) \\ =& \int_{G\times M}\int_0^1 \frac{d}{dt}(\frac{\sqrt{-1}}{2\pi}R(H_t))^{n+1} \wedge pr_1^*(\phi) - \int_{G\times M}(\frac{\sqrt{-1}}{2\pi}R(H_0))^{n+1} \wedge pr_1^*(\phi). \end{aligned}$$

Since

$$\begin{aligned} \int_{G\times M}(\frac{\sqrt{-1}}{2\pi}R(H_0))^{n+1} \wedge pr_1^*(\phi) &= \int_{G\times M}(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(H_0))^{n+1} \wedge pr_1^*(\phi) \\ &= (\frac{\sqrt{-1}}{2\pi})^{n+1}\int_M \partial(\bar{\partial}H \wedge (\partial\bar{\partial}H)^n)\int_G \phi, \end{aligned}$$

then by Stokes' theorem this integral is zero. Hence we have

$$\begin{aligned}
&\int_{G\times M}\int_0^1 (n+1)(\frac{\sqrt{-1}}{2\pi}R(H_t))^n\frac{d}{dt}(\frac{\sqrt{-1}}{2\pi}R(H_t))\wedge pr_1^*(\phi)\\
=&\int_{G\times M}\int_0^1 (\frac{\sqrt{-1}}{2\pi}(n+1)\partial\bar{\partial}(\frac{\partial\varphi_t}{\partial t}))\wedge(\frac{\sqrt{-1}}{2\pi}R(H_t))^n\wedge pr_1^*(\phi)\\
=&\int_{G\times M}\int_0^1 (\frac{\sqrt{-1}}{2\pi}(n+1)(\frac{\partial\varphi_t}{\partial t}))\wedge(\frac{\sqrt{-1}}{2\pi}R(H_t))^n\wedge pr_1^*(\partial\bar{\partial}\phi)\\
=&\int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma)\wedge\partial\bar{\partial}\phi,
\end{aligned}$$

which is followed from the integration by part and the definition of D_M. □

Lemma 3.4 (Luo)**.** *For any smooth $2(r-1)$-form ϕ with compact support in G*

$$\begin{aligned}
&\int_{\bar{G}}(\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)+\frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}}))\wedge\phi\\
=&\int_G(\frac{\sqrt{-1}}{2\pi}F_M(\sigma)-\frac{\sqrt{-1}}{4\pi}log(\|s_0\|^2_{\bar{G}}))\wedge\partial\bar{\partial}\phi.
\end{aligned}\tag{14}$$

Proof. By the Poincaré-Lelong equation [13], there is a meromorphic section s_0 of $K_{\bar{G}}$,and $s_0|_G$ is a nonzero holomorphic section of K_G such that

$$\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|_{\bar{G}})=\delta_{Y_0}-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}log(\|s_0\|^2_{\bar{G}}),\tag{15}$$

where Y_0 is a divisor of $\bar{G}$ supported in $\bar{G}-G$. Similarly, there is a divisor Y (depends on ν) supported in $\bar{G}-G$ such that

$$\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)=\delta_Y+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}F_M(\sigma).\tag{16}$$

(15) and (16) are true in the sense of current. By the definition of δ_{Y_0} and the integration by parts

$$\begin{aligned}
\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|_{\bar{G}})\wedge\phi &= \int_{\bar{G}}(\delta_{Y_0}-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}log(\|s_0\|^2_{\bar{G}}))\wedge\phi\\
&=\int_{\bar{G}}\delta_{Y_0}\wedge\phi-\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}log(\|s_0\|^2_{\bar{G}})\wedge\phi\\
&=-\int_G\frac{\sqrt{-1}}{2\pi}log(\|s_0\|^2_{\bar{G}})\wedge\partial\bar{\partial}\phi.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_{\bar{G}}(\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)\wedge\phi &= \int_{\bar{G}}(\delta_Y+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}F_M(\sigma))\wedge\phi\\ &= \int_{\bar{G}}\delta_Y\wedge\phi+\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}F_M(\sigma)\wedge\partial\bar{\partial}\phi\\ &= \int_{G}\frac{\sqrt{-1}}{2\pi}F_M(\sigma)\wedge\partial\bar{\partial}\phi.\end{aligned}$$

□

Lemma 3.5 (Luo). *For any smooth $2(r-1)$-form ϕ with compact support in G*

$$\int_{\bar{G}\times\mathbb{CP}^N} exp(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k\wedge\bar{\pi}_1^*(\phi) = -\int_G\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}log(\|s_k\|^2)\wedge\partial\bar{\partial}\phi, \tag{17}$$

where s_k is the section of $\mathcal{O}_{\bar{G}}(C_k)$ defining C_k, and λ_k is the constant given by (6).

Proof. Fix a Hermitian metric $\|\cdot\|$ on $\mathcal{O}_{\bar{G}(C_k)}$. By the Poincaré-Lelong equaiton [13] and the previous Lemma 3.1, there are sections s_k of $\mathcal{O}_{\bar{G}}(C_k)$ and the constant λ_k such that

$$\bar{\pi}_{1*}(exp(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k) = \delta_B-\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}\partial\bar{\partial}log(\|s_k\|^2),$$

where B is a divisor supported in $\bar{G}-G$. Then by the definition of δ_B and the integration by parts

$$\begin{aligned}&\int_{\bar{G}\times\mathbb{CP}^N} exp(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k\wedge\bar{\pi}_1^*(\phi)\\ &= \int_{\bar{G}}\bar{\pi}_{1*}(exp(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k)\wedge\phi\\ &= \int_{\bar{G}}(\delta_B-\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}\partial\bar{\partial}log(\|s_k\|^2))\wedge\phi\\ &= -\int_G\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}log(\|s_k\|^2)\wedge\partial\bar{\partial}\phi.\end{aligned}$$

□

3.4. **Analytic Criterion to Check Stability.** In the beginning, we need the Statement 3.1 for the following propositions and statements.

Statement 3.1. *Let $M \subset \mathbb{CP}^N$ be a smooth projective subvariety, and suppose that its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $G = SL(N+1, \mathbb{C})$. If D_M has a critical point, then D_M is a proper function on G, and there exist constant $\delta > 0$ and $C \in \mathbb{R}$ such that*

$$D_M(s) \geq \delta \cdot \log(d(s, \bar{G}\backslash G)) + C \tag{18}$$

Here $d(s, \bar{G}\backslash G)$ is the distance of s to $\bar{G}\backslash G$ with respect to a smooth metric on $\bar{G}$.

Remark 3.2. *In fact, this statement is the [20, Lemma 3.1]. We want to reprove the part of inequality.*

Proof. For any $s \in G = SL(N+1, \mathbb{C})$, let $s^* s = U^* \Lambda^2 U$, where U is an unitary matrix and Λ is a real diagonal matrix. Then by the definition of D_M, $D_M(s) = D_M(\Lambda \cdot U)$. Let $\phi : (\mathbb{C}^*)^N \times U(N+1, \mathbb{C}) \to G$ be a surjective map such that for any $(z_1, \cdots, z_N, U) \in (\mathbb{C}^*)^N \times U(N+1, \mathbb{C})$,

$$\phi(z_1, \cdots, z_N, U) = \Lambda \cdot U, \qquad for\ \Lambda = diag(z_0, \cdots, z_N),$$

where $z_0 = (z_1 \cdots z_N)^{-1}$. It suffices to prove that the pullback function $\phi^*(D_M)$ on $(\mathbb{C}^*)^N \times U(N+1, \mathbb{C})$ is proper. Fixed $U \in U(N+1, \mathbb{C})$, and let $\varphi = \phi^*(D_M)|_{(\mathbb{C}^*)^N \times \{U\}}$. Then

(1) by (11), since $g_*(s^* \bar{\pi}_2^*(\omega_{FS})^{n+1})$ is a positive $(1,1)$ current, then by [13], φ is a plurisubharmonic function on $(\mathbb{C}^*)^N$. That is,

$$\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \geq 0,$$

(2) the complex Hessian of φ is nonzero everywhere and
(3) φ is invariant under the action of torus $S^1 \times \cdots \times S^1$.

Then we have

$$\left(\frac{\partial^2 \varphi}{\partial \log |z_i| \partial \log |\bar{z}_j|}\right) > 0.$$

Therefore, φ is a strictly convex function of $(\log |z_1|, \cdots, \log |z_N|)$ for all $z = (z_1, \cdots, z_N) \in (\mathbb{C}^*)^N$.

Since φ has a critical point, assume that $p = (0, \cdots, 0)$ without loss of generality. Let $\vec{u} = (u_1, \cdots, u_N)$ be the unit vector. It suffices to prove the inequality for the coordinates $x = (x_1, \cdots, x_N) \in (\mathbb{R}_{>0})^N$, where $x_i := |z_i|$ for all i. For all unit vector $\vec{u}$, by the strictly convexity of φ in $(\log x_1, \cdots, \log x_N)$, $D^2_{\vec{u}} \varphi(\log x_1, \cdots, \log x_N) > 0$. Thus

$D_{\vec{u}}\varphi(\log x_1, \cdots, \log x_N)(r\vec{u})$ is an increasing in $r > 0$. For $r \geq 4$, there is a constant $\delta > 0$ such that

$$D_{\vec{u}}\varphi(\log x_1, \cdots, \log x_N)(r\vec{u}) = \sum_{i=1}^{N} \frac{\partial \varphi}{\partial \log x_i}(r\vec{u}) \cdot \frac{u_i}{x_i} > \delta.$$

In each $\vec{u}$, let

$$L(r\vec{u}) = \sum_{i=1}^{N} \frac{\partial \varphi}{\partial \log x_i}(4\vec{u}) \log r u_i + b_{\vec{u}},$$

where $b_{\vec{u}}$ is a constant such that $\varphi(4\vec{u}) = L(4\vec{u})$, and

$$f(\vec{x}) = \delta \cdot \log(\sum_{i=1}^{N} x_i^2).$$

For $\vec{x} \neq 0$, we have

$$D_{\vec{u}} f(\vec{x}) = \sum_{i=1}^{N} \frac{2\delta x_i u_i}{x_1^2 + \cdots + x_N^2}.$$

For $\vec{x} = r\vec{u}$ and $r \geq 4$, we have

$$D_{\vec{u}} f(r\vec{u}) = \frac{2\delta}{r} \leq \frac{\delta}{2}.$$

Consider

$$D_{\vec{u}} L(r\vec{u}) - D_{\vec{u}} f(r\vec{u}) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0$$

Thus we have

$$L(r\vec{u}) \geq f(r\vec{u}) + C_{\vec{u}},$$

where $C_{\vec{u}}$ is a constant depends on $\vec{u}$ and the equality is valid whence $r = 4$. For each unit vector $\vec{u}$ and $r \geq 4$, we have

$$\varphi(r\vec{u}) \geq L(r\vec{u}) \geq f(r\vec{u}) + C_{\vec{u}}.$$

By the continuity of φ and f, there is a minimum $\tilde{C} = \min_{\vec{u}}(C_{\vec{u}})$ such that

$$\varphi(r\vec{u}) \geq \delta \log(r^2) + \tilde{C}$$

Choose $C' \in \mathbb{R}_{>0}$ such that for $(\sum_{i=1}^{N} x_i^2)^{\frac{1}{2}} \leq 4$

$$\varphi(\vec{x}) \geq \delta \log(\sum_{i=1}^{N} x_i^2) + C'.$$

Set $C = \min\{C', \tilde{C}\}$. Then we have

$$\varphi(\vec{x}) \geq \delta \log(\|x\|^2) + C.$$

□

By a series of lemmas in the last section, there is a holomorphic function R [20], [31] on G such that

$$F_M(\sigma) - \frac{\nu^{n+1}}{n!} D_M(\sigma) + \sum_{k=1}^{s} \lambda_k \nu^{b_k} \log(\|s_k\|^2) - \frac{1}{2}\log(\|s_0\|^2) + \theta_\nu(\sigma) = \log|R(\sigma)|^2. \tag{19}$$

Statement 3.2. *There are constants $l > 0$ and $C > 0$ such that*

$$|R(\sigma)| \leq C \cdot d(\sigma, \bar{G} \setminus G)^{-l},$$

where $d(\sigma, \bar{G} \setminus G)$ denotes the distance of σ to $\bar{G} \setminus G$ with respect to the standard metric on $\bar{G}$.

Proof. By (19), we have

$$|R(\sigma)|^2 \leq \tilde{C} \exp(F_M(\sigma)) \exp(-\frac{\nu^{n+1}}{n!} D_M(\sigma)),$$

where $\tilde{C} \geq \exp(\sum_{k=1}^{s} \lambda_k \nu^{b_k} \log(\|s_k\|^2) - \frac{1}{2}\log(\|s_0\|^2) + \theta_\nu(\sigma))$, since θ_ν is bounded above. Note that

$$\|\sigma(\hat{x})\|^2 = (\sigma(\hat{x}))^* \cdot (\sigma(\hat{x})) = \hat{x}^* \sigma^* \sigma \hat{x} = \hat{x}^* U^* \Lambda^2 U \hat{x},$$

where $U = (b_{ij}) \in U(N+1, \mathbb{C})$ is an unitary matrix and $\Lambda = diag(z_0, \cdots, z_N)$. Let $\hat{x} = (\hat{x}_0, \cdots, \hat{x}_N)^t$ so that we have

$$\hat{x}^* U^* (|z_i|^2 b_{ij}) \hat{x} = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} |z_k|^2 b_{kj} \bar{b}_{ki} \hat{x}_j \bar{\hat{x}}_i.$$

Fixed $\hat{x}$ and U; therefore we have constants K_1 and K_2 such that

$$K_1 \sum_{i=0}^{N} |z_i|^2 \leq \|\sigma(\hat{x})\|^2 \leq K_2 \cdot \sum_{i=0}^{N} |z_i|^2.$$

Hence we have

$$
\begin{aligned}
|R(\sigma)|^2 &= \tilde{C}\|\sigma(\hat{x})\|^{-2}\exp(-\frac{\nu^{n+1}}{n!}D_M(\sigma)) \\
&\le \tilde{C}K_1^{-1}\exp(-\frac{C\nu^{n+1}}{n!})\cdot(\sum_{i=0}^{N}|z_i|^2)^{-1}d(\sigma,\bar{G}\setminus G)^{-\frac{\delta\nu^{n+1}}{n!}} \\
&\le K'\cdot d(\sigma,\bar{G}\setminus G)^{-1-\frac{\delta\nu^{n+1}}{n!}},
\end{aligned}
$$

where the constant K' is larger than $\tilde{C}K_1^{-1}\exp(-\frac{C\nu^{n+1}}{n!})$ and we use (16) with the smooth metric on $\bar{G}$.

Note that the last inequality is valid by the Statement 3.1. Let $l = 1 + \frac{\delta\nu^{n+1}}{n!}$; hence the inequality is proved. □

Lemma 3.6. *There are constants $l > 0$ and $C > 0$ such that*

$$|R(\sigma)| \le C\cdot d(\sigma, W\setminus G)^{-l},$$

where W is given by

$$W = \{[z_{ij}, w]_{0\le i,j\le N} | \det(z_{ij}) = w^{N+1}\},$$

and $d(\sigma, W\setminus G)$ denotes the distance of σ to $W\setminus G$ with respect to the standard metric on $\mathbb{CP}^{(N+1)^2}$. In fact, R is a constant.

Proof. By the Statement 3.2, the above inequality is valid.

Since R is extended to be a meromorphic function on W. Notice W is normal and $W\setminus G$ is irreducible. Then R is a nonzero constant, otherwise the divisor $W\setminus G$ is linearly equivalent to zero (cf. [31], [20]). □

Lemma 3.7 (Luo)**.** *There are constants $C' > 0$ such that for ν large enough*

$$F_M(\sigma) \ge \frac{\nu^{n+1}}{n!}D_M(\sigma) - \sum_{k=1}^{s}\lambda_k\nu^{b_k}\log(\|s_k\|^2) + \frac{1}{2}\log(\|s_0\|^2) - C'. \tag{20}$$

Here λ_k and $0 \le b_k \le n$ are constants.

Proof. From the (19), recall that θ_ν is bounded above and R is a constant. □

Proposition 3.1 (luo)**.** *Let $(M, L) \in \mathfrak{I}_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be given as above. For any $\mu \geq \mu_0$, if D_M is a proper function on $Berg(M)$, then the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(i^*\bar{\pi}_2^*\mathcal{O}(\nu)))$ for very large ν.*

Proof. If D_M is proper, F_M is proper by Lemma 3.7. Consequently, the result is followed from Proposition 2.3. □

4. Heat Kernel and Gieseker-Mumford stability

4.1. **Criterion for Stability of Subvariety of $\mathbb{CP}^N$.**

Statement 4.1. *Show that σ is a critical point of D_M on G if and only if it satisfies the following equation*

$$\sum_{i,j=1}^{N} \frac{Re(c_{ij})}{Vol(M)} \int_{\sigma(M)} \frac{z_i \cdot \bar{z}_j}{|z_1|^2 + \cdots + |z_N|^2} \omega_{FS}^n = 1, \tag{21}$$

where $tr(c_{ij}) = N + 1$. In particular, if there exists a constant C such that for all i, j

$$\int_{\sigma(M)} \frac{z_i \cdot \bar{z}_j}{|z_1|^2 + \cdots + |z_N|^2} \omega_{FS}^n = C \cdot \delta_{ij}, \tag{22}$$

then

$$\frac{1}{Vol(M)} \int_{\sigma(M)} \frac{z_i \cdot \bar{z}_j}{|z_1|^2 + \cdots + |z_N|^2} \omega_{FS}^n = \frac{1}{N+1} \delta_{ij}, \tag{23}$$

which is the result in [20].

Proof. Let σ be the critical point of D_M and let $s : (-\epsilon, \epsilon) \to G = SL(N + 1)$ be a path through σ, say $s(0)\sigma^{-1} = I$. Since $s(t)\sigma^{-1}$ is a curve starting from the identity, then $s'(0)\sigma^{-1} \in sl(N+1, \mathbb{C})$. Denoted $D_M(s(t))$ by $D_M(t)$. Recall that

$$\omega_t = \omega + \partial\bar{\partial}\varphi_t,$$

where φ_t is a function on M. Since M is a projective variety, for $z = [z_0, \cdots, z_N] \in M$, we have

$$\varphi_t(z) = \log(\frac{\|s(t) \cdot z\|^2}{\|z\|^2}).$$

By the definition of D_M, choose a path $\omega_s = \omega + \partial\bar{\partial}\psi_s$ from ω to ω_0 (from $s = 0$ to $s = \lambda = 1 - \tau$) and arbitrarily from ω_0 to ω_τ without loss of generality since D_M is well-defined. That is,

$$\omega_s = \begin{cases} \omega + \partial\bar{\partial}\psi_s, & \text{if } 0 \le s \le \lambda \\ \omega + \partial\bar{\partial}\varphi_{s-\lambda}, & \text{if } \lambda \le s \le 1. \end{cases}$$

Then

$$\begin{aligned} D_M(\tau) &= \int_0^1 \int_M \frac{\partial \psi_s}{\partial s} \cdot \omega_s^n \wedge ds \\ &= \int_0^1 \int_M \frac{dt}{d(\lambda t)} \frac{\partial \psi_{\lambda t}}{\partial t} \cdot \omega_{\lambda t}^n \wedge \lambda dt + \int_\lambda^1 \int_M \frac{\partial \varphi_{s-\lambda}}{\partial s} \cdot \omega_{s-\lambda}^n \wedge ds \\ &= \int_0^1 \int_M \frac{\partial \psi_{\lambda t}}{\partial t} \cdot \omega_{\lambda t}^n \wedge dt + \int_0^\tau \int_M \frac{\partial \varphi_t}{\partial t} \cdot \omega_t^n \wedge dt \\ &= D_M(0) + \int_0^\tau \int_M \frac{\partial \varphi_t}{\partial t} \cdot \omega_t^n \wedge dt. \end{aligned} \tag{24}$$

Suppose that σ is a critical point of D_M on G. Then

$$\begin{aligned} \frac{d\varphi_t}{dt} &= \frac{d}{dt} \log \frac{\|s(t) \cdot z\|^2}{\|z\|^2} \\ &= \frac{2Re < s'(t) \cdot z, s(t) \cdot z >}{\|s(t) \cdot z\|^2} \\ &= \frac{2Re < (s'(t) + \sigma) \cdot z, s(t) \cdot z >}{\|s(t) \cdot z\|^2} - \frac{2Re < \sigma \cdot z, s(t) \cdot z >}{\|s(t) \cdot z\|^2}. \end{aligned}$$

For $t = 0$, we have

$$\begin{aligned} 0 &= \int_M \left(\frac{Re < (s'(0) + \sigma) \cdot z, \sigma \cdot z >}{\|\sigma \cdot z\|^2} - \frac{Re < \sigma \cdot z, \sigma \cdot z >}{\|\sigma \cdot z\|^2} \right) \wedge \omega_0^n \\ &= \int_M \frac{Re < (s'(0) + \sigma)\sigma^{-1}\sigma \cdot z, \sigma \cdot z >}{\|\sigma \cdot z\|^2} \wedge \sigma^*(\omega_{FS}|_{\sigma(M)})^n - Vol(M) \\ &= \int_{\sigma(M)} \frac{Re < (s'(0) + \sigma)\sigma^{-1} \cdot w, w >}{\|w\|^2} \wedge \omega_{FS}^n - Vol(M) \\ &= \sum_{i,j=0}^N Re(c_{ij} \int_{\sigma(M)} \frac{w_j \cdot \bar{w}_i}{\|w\|^2} \wedge \omega_{FS}^n) - Vol(M). \end{aligned}$$

where $(c_{ij}) = (s^{'}(0) + \sigma)\sigma^{-1}$. Thus

$$\sum_{i,j=0}^{N} Re(c_{ij} \int_{\sigma(M)} \frac{w_j \cdot \bar{w}_i}{\|w\|^2} \wedge \omega_{FS}^n) = Vol(M),$$

where

$$\begin{aligned} tr((c_{ij})) &= tr((s^{'}(0) + \sigma)\sigma^{-1}) \\ &= tr(s^{'}(0)\sigma^{-1} + I) = N + 1. \end{aligned}$$

If (20) holds, then we include Luo's condition (23). □

Here we have a slight improvement with the original theorem of Luo [20].

Theorem 4.1. *Let $M \subset \mathbb{CP}^N$ be a smooth projective subvariety, and its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $SL(N+1, \mathbb{C})$. Then $[M] \in Hilb_h$ is (GIT) stable if there exists $\sigma \in SL(N+1, \mathbb{C})$ such that (21) holds.*

Proof. By hypothesis, (21) implies that D_M has a critical point and thus D_M is proper by Statement 3.1; therefore, the Proposition 3.1 implies that $[M]$ is (GIT) stable. □

Remark 4.1. *In fact, Theorem 4.1 is valid whence $\sigma \in SL(N+1, \mathbb{C})$ satisfies (23).*

Here is an important application of Luo's theorems. In [6], Donaldson defined that if $V \subset \mathbb{CP}^N$ is any projective variety, defining $M(V)$ to be the skew-adjoint $(N+1) \times (N+1)$ matrix with entries

$$M(V)_{\alpha\beta} = \sqrt{-1} \int_V b_{\alpha\beta} d\mu_V,$$

where $d\mu_V$ is the standard measure on V induced by the Fubini-Study metric and $b_{\alpha\beta} = \frac{z_\alpha \bar{z_\beta}}{\|z\|^2}$. If the projective varieties V such that $M(V)$ is a multiple of the identity matrix, this variety is called to be a balanced variety in $\mathbb{CP}^N$. Let (M, L) be a polarized variety. We say (M, L^k) is balanced if the image $e_k(M)$ of the embedding e_k of M into a projective space is balanced in $\mathbb{CP}^N$.

Suppose that (M, L^k) is balanced and let

$$\omega_k = \frac{2\pi}{k} e_\mu^*(\omega_{FS}).$$

so the cohomology class $[\omega_k] = 2\pi c_1(L)$ in $H^2(M)$ is independent of k. Donaldson proved that

Theorem 4.2 (Donaldson)**.** *Suppose that the group $Aut(M,L)$ of holomorphic automorphisms of pair (M,L) is discrete and (M,L^k) is balanced for all sufficiently large k. Suppose that $\omega_k \to \omega_\infty$ in C^∞ as $k \to \infty$. Then ω_0 has constant scalar curvature.*

Note that Zhang, S. [38] first proved the stability of varieties (X, L^r) is equivalent to the balanced varieties $X \subset \mathbb{P}^{N(r)}$, which was reproved or rediscovered in different forms. As the result of Zhang, there is a similar guess about the necessary and sufficient condition of stability of the bundles E over polarized algebraic variety (X,L) to the existence of a balanced map, which is proved by Wang [35]. In [36], Wang showed that the if the bundle is Gieseker stable then it gives the Hermitian-Yang-Mills metric. In addition, Maciocia [23] discussed the preservation of the Gieseker stability and the semistability under the Fourier transform of Mukai.

4.2. **Relate Gieseker-Mumford Stability to Heat Kernel.**

Definition 4.1. *Let (M,ω) be a compact Kähler manifold, and let L be a holomorphic line bundle with a Hermitian metric g. Then we define $B_k = B_k(z,g,\omega)$ to be a function on M, and for any $z \in M$*

$$B_k(z,g,\omega) = \sum_{i=0}^{N} \|s_i(z)\|_g^2. \tag{25}$$

Here $s_0, \cdots, s_N$ is any orthonormal frame of $H^0(M,L^k)$.

B_k is well-defined, that is, it is independent of the choice of the orthonormal frame $s_0, \cdots, s_N$ of $H^0(M,L^k)$. In [20], Luo proved that

Theorem 4.3 (Luo)**.** *Let $(M,L) \in \mathfrak{I}_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be a large number given by (2). For any $k \geq \mu_0$, if there exists a Hermitian metric g (depends on k) on L over M such that $B_k(z) = B_k(z,g,Ric(g))$ is pointwise constant function on M, then the k-th Hilbert point of (M,L) is (GIT) stable with respect to G, and $\mathfrak{L} = \det(g_*(\bar{\pi}_2^*\mathcal{O}(\nu)))$ for all large enough ν as long as the stabilizer of the Hilbert point is finite. And consequently, (M,L) is Gieseker-Mumford stable.*

Here we also have a slight improvement with the above Theorem.

Theorem 4.4. *Let $(M, L) \in \mathfrak{I}_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be a large number given by (2). For any $k \geq \mu_0$, if there exists a Hermitian metric g (depends on k) on L over M such that there exists a basis $\{s_0, \cdots s_N\}$ of $H^0(M, L^k)$ such that*

$$\delta_{ij} = \int_M \frac{\langle s_i, s_j \rangle_{g^k}}{\|s_0\|^2_{g^k} + \cdots + \|s_N\|^2_{g^k}} \omega^n_{FS},$$

then the k-th Hilbert point of (M, L) is (GIT) stable with respect to G, and $\mathfrak{L} = \det(g_*(\pi_2^* \mathcal{O}(\nu)))$ for all large enough ν as long as the stabilizer of the Hilbert point is finite. And consequently, (M, L) is Gieseker-Mumford stable.

Proof. There is an embedding of M into $\mathbb{CP}^N$ defined by

$$z \mapsto [s_0(z) : \cdots : s_N(z)].$$

If $\sigma = id$, it gives that for a smooth path $\eta : (-\epsilon, \epsilon) \to G = SL(N + 1, \mathbb{C})$ with $\eta(0) = id$,

$$\begin{aligned} \frac{d}{dt}\varphi_t|_{t=0} &= \frac{d}{dt}\log(\frac{\|\eta(t)\cdot z\|^2_{g^k}}{\|z\|^2_{g^k}})|_{t=0} \\ &= \frac{2Re < \eta'(0)\cdot z, z >_{g^k}}{\|z\|^2_{g^k}}, \end{aligned}$$

and that for identifying M with $e_k(M)$ and for $s = s(z) = [s_0(z) : \cdots : s_N(z)]$,

$$\begin{aligned} \frac{dD_M}{dt}(0) &= \int_M \frac{2Re < \eta'(0)\cdot s, s >_{g^k}}{\|s_0\|^2_{g^k} + \cdots + \|s_N\|^2_{g^k}} \wedge id^*(\omega_{FS}|_M)^n \\ &= 2\sum_{i,j=0}^{N} Re(\eta'(0))_{ij}\delta_{ij} \\ &= 2\sum_{i=0}^{N} Re(\eta'(0))_{ii} = 0, \end{aligned}$$

which implies that id is a critical point of D_M and id satisfies the equation (23). Therefore, Theorem 4.1 implies that the k-th Hilbert point of M is (GIT) stable in the Hilbert scheme $Hilb_h$. □

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