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Essays in Finance

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本論文係 周盈吟 君，學號 D94724016，在國立臺灣大學國際企業學研究所完成之博士學位論文，於民國 100 年 5 月 20 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

這篇論文主要是探討抗通膨債券的定價與投資者消費投資組合的最佳化解。在此篇論文的第一部分,我們在通膨率有隨機波動度下,得到一個美國抗通膨債券(TIPS)的解析解。在實證上，我們利用美國市場上的二十九支 TIPS 價格，求出在 2000 年一月到 2009 年十月期間，模型的未知參數和通膨利差。實證結果顯示，忽略隱含選擇權將導致通膨利差的高估。平均來說，高估的部分大概是 0.82%。實證結果同時顯示，所得出的最小的利差發生在 2009 年一月附近，此時間點為本文探討期間發生最嚴重的通貨緊縮時點。在此篇論文的第二部分，我們解一個跨期消費投資組合問題當市場存在通膨風險時。投資人為了對抗通膨風險，會選擇持有抗通膨債券。本文說明，名目利率和通膨率如何影響最佳化消費財富的比例與跨期替代彈性有關。消費財富的比例不完全受到實質利率影響，同時也與名目利率和通膨率有關。最後，利用美國市場資料對模型做一個補正，結果顯示，積極型的投資者會想要擁有較多的名目債券以賺取通膨風險貼水，而保守型的投資者會持有較多的通膨債券以規避通膨風險。

關鍵字: 隨機波動度，美國抗通膨債券，隱含選擇權，通膨風險，指數型債券，投資組合選擇，跨期替代彈性

英文摘要

The purpose of this thesis is to price the inflation indexed securities and solve for an inter-temporal portfolio consumption choice problem under inflation. In the first part of this thesis, a diffusion model for inflation rates with stochastic volatility is proposed, and closed-form solutions are derived for treasury inflation protected securities (TIPS). Empirically, our model with 29 TIPS, treasury constant maturity rates and reference CPI numbers in the U.S. market was used to derive the unknown parameters and spreads during January 2000 to October 2009. Empirical results show that an over-estimated spread is induced by ignoring the embedded option in TIPS. The average difference between the distorted estimate and actual value is about 0.82%. The minimum spread occurred around Jan. 2009 while the CPI-U decreased drastically. In the second part of this thesis, we solve for an inter-temporal portfolio consumption choice problem under inflation. The inclusion of the inflation-indexed bonds in the investor's portfolio provides an opportunity to perfectly hedge against the inflation risk, while the hedging demand of the nominal bonds would be crowded out in proportion to the demand of the indexed bonds. The direction in which the interest rate and the inflation rate affect the optimal consumption-wealth ratio relies on the elasticity of inter-temporal substitution of the investor. The consumption wealth ratio is not completely determined by the real interest rate, it also depends on the nominal levels of the interest rate and the inflation rate. The capital market is calibrated to U.S. stock, bond, and inflation data. The optimal weights show that aggressive investors hold more nominal bonds to earn the inflation risk premium, and conservative ones concentrate on indexed bonds to hedge against the inflation risk.

Key words: Stochastic volatility, TIPS, Embedded option, Inflation Risk, Indexed Bond, Dynamic Portfolio Choice, Elasticity of Inter-temporal Substitution

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Chapter 1

Treasury Inflation Protected Securities Pricing

under the Heath- Jarrow-Morton model with Stochastic Volatility

1.1 Introduction

Since January 1997, the treasury inflation protected securities, or TIPS, created by the U.S. treasury to help investors manage inflation risk, have been issued regularly. Principals of TIPS are adjusted periodically to keep pace with the rate of inflation, measured by the consumer price index for all urban consumers (CPI-U). Investors receive semi-annual coupon payments based on a fixed semi-annual coupon rate applied to the inflation-adjusted principal. At maturity, if inflation has increased the value of the principal, investors receive the higher value. If deflation has decreased the value, investors still receive the original face amount of the security. The redemption valuation at maturity is recognized as an embedded option in TIPS.

Fischer (1975) first analyzed the demand for index bonds. Unlike traditional treasury bonds, investors in TIPS are guaranteed by the government a specific rate of return and a specific purchasing power of the principal above inflation. According to the estimates of the treasury department, the outstanding volume of TIPS has risen from about $15 billion in 1997 to about $413 billion by late 2006, and trading of TIPS among primary securities dealers has risen from a daily average of $1 billion in early 1998 to approximately $8 billion by late 2006 due to their built-in inflation protection.

Yields calculated from prices of TIPS are usually denoted as approximations of inflation-adjusted or real interest rates. That information, together with yields on traditional, nominal-treasury securities, is often used to provide implicit indications of general agreement of medium-term to long-term expectations of inflation by the bond market. However, the fact that the long-term inflation expectation from the survey has almost always been higher than that computed from the spread between nominal and TIPS yields (called term premium) suggests that measure is distorted. As a measure of expected inflation, the spread between nominal- and indexed-treasuries could be distorted by inflation uncertainty, risk aversion, and the probability of deflation.

Like Melino and Turnbull (1990), Frachot (1995), or Bakshi and Cao, we propose a much more general and flexible pricing model for TIPS with embedded options which allows for consideration of stochastic volatility and correlations among inflation rates, nominal forward rates, real forward rates, and volatility variants. This aids in more accurately capturing the distribution properties of inflation. However, due to the long maturity property of TIPS, fitting the current term structure is an important consideration for practitioners. Jarrow and Yildirim (2003) applied the term structure model introduced by Heath, Jarrow, and Morton (HJM) (1992) to both nominal and real rates. Options embedded in TIPS also have long maturity and are very different from ordinary options. Unlike an ordinary option, whose value always increases with maturity, the relation between the TIPS-embedded option and maturity is much more complex. The relationship depends on the term structures of real and nominal forward rates and the reference CPI-U on the issue date. This article provides the theoretical and empirical basis of TIPS for pricing in the stochastic volatility framework.

The remainder of this article is as follows: Section 2 describes the model and its assumptions. Section 3 introduces the general valuation framework and presents closed-form solutions of TIPS. Section 4 investigates the properties of TIPS and embedded options using numerical examples. Section 5 reports the empirical analysis based on the U.S. market data. Conclusions are given in Section 6.

1.2 Model

The HJM model was applied to TIPS in order to fit best the current term structure, which is the first-order requirement for practitioners while pricing high interest-rate-sensitive derivatives. As presented in Duffie, Pan, and Singleton (2000), we assumed that, at time t, the data-generating processes of the inflation index, $I(t)$, with its stochastic variance component, $Y(t)$, the nominal-T-maturity forward rate, $F_n(t,T)$, and the real T-maturity forward rate, $F_r(t,T)$, are calculated as follows:

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \delta_{In}dW_n(t) + \delta_{Ir}dW_r(t) + \sqrt{Y(t)}dW_I(t), \tag{1}$$

$$dY(t) = (\alpha - \kappa_Y Y(t))dt + \sigma_Y \sqrt{Y(t)}dW_Y(t), \tag{2}$$

$$dF_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t), \tag{3}$$

$$dF_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t), \tag{4}$$

where $W_I(t)$, $W_Y(t)$, $W_n(t)$, and $W_r(t)$ describe standard Brownian motions with covariance: $Cov(dW_I, dW_Y) = \rho_{IY}dt$, $Cov(dW_n, dW_r) = \rho_{nr}dt$, $Cov(dW_I, dW_n) = 0$, $Cov(dW_I, dW_r) = 0$, $Cov(dW_Y, dW_n) = 0$, and $Cov(dW_Y, dW_r) = 0$. Structural parameters α, κ_Y, and σ_Y represent the long-run mean, the speed of adjustment, and the volatility of the stochastic variance component, respectively. The time t, price of a nominal (real) zero-coupon bond maturing at time T, in dollars (CPI-U units), is given by Eq. (5).

$$V_k(t,T) = \exp\left\{-\int_t^T F_k(t,s)ds\right\}, \quad k \in \{n, r\}. \tag{5}$$

The nominal (real) money market account is defined as

$$B_k(t,T) = \exp\left\{\int_t^T r_k(s)ds\right\}, \quad k \in \{n, r\}, \tag{6}$$

where $r_k(t)$ is the spot rate and $r_k(t) = F_k(t,t)$. The instantaneous covariance rate--between inflation rates and nominal (real) rates--is given by $\sigma_n(t,T)(\delta_{In} + \delta_{Ir}\rho_{nr})$ ($\sigma_r(t,T)(\delta_{Ir} + \delta_{In}\rho_{nr})$) in this model.

1.3 Pricing TIPS

If true processes are given by Eqs. (1)-(4), the processes under the risk-neutral measure,

Q, defined by ($W_I^*(t)$, $W_Y^*(t)$, $W_n^*(t)$, $W_r^*(t)$) are given by Eqs. (7)-(10),

$$\frac{dI(t)}{I(t)} = \left(\mu_I(t) + \delta_{In}\lambda_n(t) + \delta_{Ir}\lambda_r(t) + \sqrt{Y(t)}\lambda_I(t)\right)dt$$

$$+ \delta_{In}dW_n^*(t) + \delta_{Ir}dW_r^*(t) + \sqrt{Y(t)}dW_I^*(t), \tag{7}$$

$$dY(t) = \left(\alpha - \kappa_Y Y(t) + \Gamma_Y\right)dt + \sigma_Y\sqrt{Y(t)}dW_Y^*(t), \tag{8}$$

$$dF_n(t,T) = \left(\alpha_n(t,T) + \sigma_n(t,T)\lambda_n(t)\right)dt + \sigma_n(t,T)dW_n^*(t), \tag{9}$$

$$dF_r(t,T) = \left(\alpha_r(t,T) + \sigma_r(t,T)\lambda_r(t)\right)dt + \sigma_r(t,T)dW_r^*(t), \tag{10}$$

where $\lambda_k(t)$ is the market price of $W_k(t)$ for $k \in \{I,n,r\}$ and Γ_Y is the volatility risk premium. After adopting the assumption that $\Gamma_Y = \eta Y(t)$ (Bates (1996)), Eq. (8) can be rewritten as

$$dY(t) = (\alpha - \beta Y(t))dt + \sigma_Y\sqrt{Y(t)}dW_Y^*(t), \tag{11}$$

where $\beta = \kappa_Y - \eta$. A proposition is presented that states the necessary and sufficient conditions for bond price evolution to guarantee that arbitrage does not exist.

Proposition I: Arbitrage free term structures

$V_n(t,T)/B_n(0,t)$, $I(t)V_r(t,T)/B_n(0,t)$, and $I(t)B_r(0,t)/B_n(0,t)$ are Q-martingale if and only if

$$\alpha_n(t,T) = \sigma_n(t,T)\left[\int_t^T \sigma_n(t,u)du - \lambda_n(t)\right] \tag{12}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,u)du - \lambda_r(t) - \delta_{In}\rho_{nr} - \delta_{Ir}\right] \tag{13}$$

$$\mu_I(t) = r_n(t) - r_r(t) - \delta_{In}\lambda_n(t) - \delta_{Ir}\lambda_r(t) - \sqrt{Y(t)}\lambda_I(t) \tag{14}$$

Proof: See Appendix A.

From Eqs. (7) and (9)-(11), and Proposition I, the following equations are derived under the risk-neutral measure, Q:

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]dt + \delta_{In} dW_n^*(t) + \delta_{Ir} dW_r^*(t) + \sqrt{Y(t)} dW_I^*(t), \tag{15}$$

$$dY(t) = (\alpha - \beta Y(t))dt + \sigma_Y \sqrt{Y(t)} dW_Y^*(t), \tag{16}$$

$$\frac{dV_n(t,T)}{V_n(t,T)} = r_n(t)dt + a_n(t,T)dW_n^*(t), \tag{17}$$

$$\frac{dV_r(t,T)}{V_r(t,T)} = (r_r(t) - (\delta_{In}\rho_{nr} + \delta_{Ir})a_r(t,T))dt + a_r(t,T)dW_r^*(t), \tag{18}$$

where $a_k(t,T) \equiv -\int_t^T \sigma_k(t,u)du$ for $k \in \{n,r\}$. Let $V_n^T(t,\tau)$, $I_V^T(t,\tau)$, and $I_r^T(t)$ denote the forward prices of $V_n(t,\tau)$, $I(t)V_r(t,\tau)$, and $I(t)B_r(0,t)$ with respect to $V_n(t,T)$ as defined by $V_n^T(t,\tau) \equiv V_n(t,\tau)/V_n(t,T)$, $I_V^T(t,\tau) \equiv I(t)V_r(t,\tau)/V_n(t,T)$, and $I_r^T(t) \equiv I(t)B_r(0,t)/V_n(t,T)$, where $T \le \tau$. Other forward prices are defined in a similar manner. Next, Q^T is determined under which $V_n^T(t,\tau)$, $I_V^T(t,\tau)$, and $I_r^T(t)$ are martingale.

Proposition II: Martingales under measure Q^T

$V_n(t,\tau)/V_n(t,T)$, $I(t)V_r(t,\tau)/V_n(t,T)$, and $I(t)B_r(0,t)/V_n(t,T)$ are martingale under Q^T defined by { $W_I^T(t)$, $W_Y^T(t)$, $W_n^T(t)$, $W_r^T(t)$ }, where $W_I^T(t) = W_I^*(t)$, $W_Y^T(t) = W_Y^*(t)$, $W_n^T(t) = W_n^*(t) - \int_0^t a_n(s,T)ds$, and $W_r^T(t) = W_r^*(t) - \rho_{nr}\int_0^t a_n(s,T)ds$.

Proof: See Appendix B.

Let E_t^Q and E_t^T denote the conditional-expectation operators at time t under the risk-neutral measure Q and the forward-neutral measure Q^T, respectively. Kijima and Muromachi (2001) demonstrated that, if the relative price of a risky asset $S(t)$ is a martingale under the risk-neutral measure Q, and its forward price is also a martingale under the forward-neutral measure, Q^T, then

$$E_0^Q\left[\frac{C(T)}{B_n(0,T)}\right] = C(0) = V_n(0,T)E_0^T\left[\frac{C(T)}{V_n(T,T)}\right] \tag{19}$$

where $C(t)$ is the time t price of a European derivative maturing at time T written on $S(t)$.

Consider a European call option issued against the inflation index with a strike price of K index units and maturity date T. Because the index is denominated in units of dollars per CPI-U, each unit of the option was assumed to be written on one CPI-U unit. Therefore, the time T payoff to the option, in dollars, is equivalent to $\max\{I(T)-K,0\}$. The present value of this option can be formulated as

$$C(T,K) = E_0^Q\left[\frac{1}{B_n(0,T)}\max\{I(T)-K,0\}\right]. \tag{20}$$

Instead of solving Eq. (20) under the risk-neutral measure, it was solved by transforming into the forward-neutral measure. Because $V_r(T,T)=1$, Eq. (20) can be written as:

$$E_0^Q\left[\frac{1}{B_n(0,T)}\max\{I(T)-K,0\}\right]=E_0^Q\left[\frac{1}{B_n(0,T)}\max\{I(T)V_r(T,T)-K,0\}\right]. \quad (21)$$

Combining Proposition I and Proposition II, and Eq. (19), yields

$$E_0^Q\left[\frac{1}{B_n(0,T)}\max\{I(T)V_r(T,T)-K,0\}\right]=V_n(0,T)E_0^T\left[\max\left\{\frac{I(T)V_r(T,T)}{V_n(T,T)}-K,0\right\}\right]. \quad (22)$$

Proposition III is provided for deriving the formula of $C(T,K)$.

Proposition III.

The present value of $\max\{I(T)V_r(T,T)/V_n(T,T)-K,0\}$ under the forward-neutral measure, Q^T, is given by

$$E_0^T\left[\max\left\{\frac{I(T)V_r(T,T)}{V_n(T,T)}-K,0\right\}\right]=\frac{I(0)V_r(0,T)}{V_n(0,T)}\overline{\Phi}(-i-v)-K\overline{\Phi}(-v), \quad (23)$$

where $\overline{\Phi}(\phi_{I_V})$ is defined by

$$\overline{\Phi}(\phi_{I_V})\equiv\frac{1}{2}-\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\left[\exp\left[A(0,T;\phi_{I_V},T,T)+B(T;\phi_{I_V})Y(0)-iv\ln\left[\frac{I(0)V_r(0,T)}{KV_n(0,T)}\right]\right]\right]}{v}dv.$$

Proof: See Appendix C.

Combining Eqs. (20)-(22) with Proposition III, generates Eq. (24).

$$C(T,K)=I(0)V_r(0,T)\overline{\Phi}(-i-v)-KV_n(0,T)\overline{\Phi}(-v). \quad (24)$$

Pricing TIPS is performed in much the same way as for a conventional bond with the addition of mechanisms for inflation adjustment and redemption valuation at maturity. The present value (in dollars) of a TIPS coupon-bearing bond issued at time t_0 (≤ 0), with a coupon payment of C, the face value of F, and the maturity of T, is given by

$$B_{TIPS}(0)=\sum_{i=1}^{n}E_0^Q\left[\frac{C}{B_n(0,t_i)}\frac{I(t_i)}{I(t_0)}\right]+E_0^Q\left[\frac{F}{B_n(0,t_n)}Max\left\{\frac{I(t_n)}{I(t_0)},1\right\}\right], \tag{25}$$

where t_i is the i-th coupon payment date and $t_n=T$. The value for redemption at maturity is the larger of either the original issue par value or the cumulative inflation-adjusted par value. When investors are concerned about the chance of deflation, the embedded option of TIPS at redemption may be very valuable. Eq. (25) can be written in another form:

$$B_{TIPS}(0)=\sum_{i=1}^{n}\frac{C}{I(t_0)}E_0^Q\left[\frac{I(t_i)}{B_n(0,t_i)}\right]+\frac{F}{I(t_0)}E_0^Q\left[\frac{Max\{I(T)-I(t_0),0\}}{B_n(0,t_n)}\right]+FV_n(0,T). \tag{26}$$

In which $E_0^Q[I(t_i)/B_n(0,t_i)]=E_0^Q[I(t_i)V_r(t_i,t_i)/B_n(0,t_i)]=I(0)V_r(0,t_i)$. After replacing K in Eq. (20) with $I(t_0)$, Eq. (27) is generated,

$$\begin{aligned}B_{TIPS}(0)=FV_n(0,T)+\sum_{i=1}^{n}CV_r(0,t_i)\frac{I(0)}{I(t_0)}+\\ FV_r(0,T)\frac{I(0)}{I(t_0)}\overline{\Phi}(-i-v)-FV_n(0,T)\overline{\Phi}(-v).\end{aligned} \tag{27}$$

by incorporating Eq. (24). After defining $\Phi(x)\equiv 1-\overline{\Phi}(x)$, Eq. (27) can be written as

$$\begin{aligned}B_{TIPS}(0)=\sum_{i=1}^{n}CV_r(0,t_i)\frac{I(0)}{I(t_0)}+FV_r(0,T)\frac{I(0)}{I(t_0)}\\ -FV_r(0,T)\frac{I(0)}{I(t_0)}\Phi(-i-v)+FV_n(0,T)\Phi(-v).\end{aligned} \tag{28}$$

If we ignore the embedded option in TIPS, the final two terms in Eq. (28) will also be ignored; subsequently, the price of TIPS can be completely determined by the term structure of real forward rates. However, the existence of the embedded option at redemption makes it possible for nominal forward rates and stochastic volatility to play an important role in determining the price of TIPS.

Given the market prices of TIPS, using the pricing formula of TIPS without the embedded option at redemption may result in under estimation of the real interest rate. This approach may induce over estimation of the spread between nominal and real interest rates. If the over-estimated spread is treated as a measure of future inflation rates, an over-estimated future inflation rate will result. Historically, the market has generated an over-estimated future inflation rate from the yield spread between treasury bonds and TIPS.

If issuing TIPS at the face value in the beginning ($t_0 = 0$), the coupon rate, c, can be determined by

$$c = \frac{1 - V_n(0,T) - C(T, I(0))/I(0)}{\sum_{i=1}^{n} V_r(0,t_i)}. \tag{29}$$

1.4 Numerical examples

According to Jarrow and Yildirim (2003), we specify

$$a_k(t,T) = -\frac{p_k}{q_k}\left(1 - \exp\left[-q_k(T-t)\right]\right),\ \ k \in \{n, r\}. \tag{30}$$

Based on our empirical estimation, we let $p_n = 0.011$, $q_n = 0.014$, $p_r = 0.011$, and $q_r = 0.013$. Although the treasury yield curve has exhibited different shapes during the past several decades, Fig. 1 provides the common shape of the nominal term structure (similar to that observed during December 1982). The nominal term structure is set to be $f_n(0,t) = a + bt + ct^2$, where $a = 0.085$, $b = 0.002$ and $c = -0.00004$.

For simplicity, the spread between nominal and real yields is assumed to be deterministic, and is denoted as $s(t)$. After specifying $s(t) = 1.5\%$, the real term structure can be obtained by $f_r(0,t) = f_n(0,t) - s(t)$. Next, we set $t_0 = 0$, $F = 1$, $Y(0) = 0.03$, $\alpha = 0.0005$, $\sigma_Y = 0.01$, $\beta = 6.02$, $\delta_{In} = 0$, $\delta_{Ir} = 0$, $\rho_{IY} = -0.512$, and $\rho_{nr} = 0.110$ to investigate selected properties of TIPS and embedded options through numerical examples.

Fig. 2 depicts the relationship between the coupon rate of TIPS and the maturity according to the circumstance depicted in Fig. 1. In addition, Fig. 3 shows that the relationship between the embedded option price and the maturity, depending on both the nominal and real term structures, is not monotonously ascending. The embedded option may consist of a portion of the TIPS value. Its percentage can reach nearly 3.5% if the coupon rate is determined according to Eq. (29) while issuing TIPS at the start.

Fig. 4 shows that an over-estimated spread between nominal and real rates is induced by ignoring the embedded option in TIPS. If the actual spread between nominal and real rates is $s(t) = 1.5\%$, the spread can be over estimated by 1.95% using TIPS with a 5-year maturity. The value is almost 1.3 times higher than the actual value. The shorter is the maturity period the larger is the distorted estimation.

The embedded option may consist of a portion of the TIPS value, and its percentage may depend on the index ratio, $I(0)/I(t_0)$. We set coupon rate to be 4%. We vary the index ratio, and present the results in Fig. 5. The smaller is the index ratio, the larger is the percentage of option embedded in TIPS. The value of option embedded in the TIPS can reach nearly 19% of the TIPS value if the index ratio is smaller than 1, while that embedded in the TIPS can reach only 4.48% of the TIPS value if the index ratio is bigger than 1.

1.5 Empirical analysis

We organize this section into two parts: firstly, the data used in our empirical investigations are described. Secondly, the empirical results are shown.

1.5.1 Data description

The data used in our empirical investigation are daily market data from January 17, 2000 to October 23, 2009. There are three different data sets: nominal constant maturity treasury rate data, TIPS data and CPI-U data.

(i) Nominal constant maturity treasury rate data:

We obtain treasury constant maturity rates on all available U.S. treasury securities. A treasury constant maturity rate is defined to be the par yield of a government treasury security with a specified maturity. These rates are read from the yield curve at fixed maturities, currently 1, 3, and 6 months and 1, 2, 3, 5, 7, 10, 20, and 30 years.

(ii) TIPS data:

We obtain daily price on all available U.S. treasury inflation protected securities. We choose 29 TIPS reported in Table 1; TIPS1, TIPS2 and TIPS3 are 30-year bonds and TIPS4, TIPS5, TIPS6, TIPS7 and TIPS8 are 20-year bonds, while TIPS12, TIPS16, TIPS19, TIPS22, TIPS25 are 5-year bonds and the remaining ones are 10-year bonds. The time period January 17, 2000-October 23, 2009 gives a total 2425 daily observations. We collect almost all of TIPS in the U.S. market during the period of January 17, 2000-October 23, 2009. These twenty-nine TIPS with their CUSIP numbers, coupon rates, issue dates, maturity dates, CPI-U of issue dates and maturity periods are reported in Table 1.

(iii) CPI-U data:

Historical reference CPI numbers and daily index ratios are both from the treasury direct website. Fig. 6 shows monthly CPI-U numbers from December 1999 to November 2009. The CPI-U is mostly increasing during December 1999-July 2008 and December 2008-November 2009 and mostly decreasing during July 2008-December 2008.

1.5.2 Empirical result

Predicting all the unknowns in Eq. (28) simultaneously is the usual method. But this method may result in time-consuming calculations and unstable solutions. Instead we use a simpler method to predict the unknowns. This procedure is described as follows.

(i) Stripping the nominal zero-coupon bond prices

The bootstrap method is used to derive nominal zero rates from the treasury constant maturity rates. A common assumption is that the zero curve is linear between the points determined using the bootstrap method. The nominal zero-coupon bond prices are derived by using the zero rates.

(ii) Estimating the volatility parameters of the nominal forward rates

Given the nominal zero-coupon bond prices, we assume the volatility parameters of the nominal forward rates $\sigma_n(t,T)$ is as follows.

$$\sigma_n(t,T) = p_n \exp[-q_n(T-t)], \tag{31}$$

where p_n and q_n are constants.

Using Eq. (17), we have the following formula,

$$\mathrm{var}\left(\frac{\Delta V_n(t,T)}{V_n(t,T)}\right) \approx \frac{p_n^2(\exp[-q_n(T-t)]-1)^2}{q_n^2}\Delta, \Delta = \text{one day}. \tag{32}$$

We run a nonlinear least square regression to estimate the parameters (p_n, q_n). The estimated parameters, $\tilde{p}_n$ = 0.011, $\tilde{q}_n$ =0.014, and their standard errors (in parentheses) are reported in Table 2.

(iii) Estimating α, β, σ_Y, ρ_{IY}

We use historical volatility to predict the volatility of $dI(t)/I(t)$ at time t. The historical volatility is obtained using 10 years historical data prior to each observation date in the data series. For simplicity, we suppose δ_{Ir} and δ_{In} are both equal to zero, and use formula as follow.

$$\mathrm{var}(\frac{\Delta I(t)}{I(t)}) \approx Y(t)\Delta, \Delta = \frac{1}{12}. \tag{33}$$

Then $Y(t)$ will be generated for each t. According to Eq. (11), $dY(t)$ is a normal distribution with mean $(\alpha - \beta Y(t))dt$ and variance $\sigma_Y^2 Y(t)dt$. Therefore, $\left(dY(t) - (\alpha - \beta Y(t))dt\right) / \left(\sigma_Y \sqrt{Y(t)dt}\right)$ is a normal distribution with mean 0 and standard deviation 1. We use the formula as follows to predict α, β and σ_Y.

$$\min_{\alpha,\beta,\sigma_Y} \left(\frac{\sum_{t=1}^{n} N(t)}{n-1} \right)^2 + \left(\frac{\sum_{t=1}^{n} N(t)^2}{n-1} - 1 \right)^2,$$

$$\text{where } N(t) = \frac{dY(t) - (\alpha - \beta Y(t))\Delta}{\sigma_Y \sqrt{Y(t)\Delta}}, \Delta = \frac{1}{12}. \tag{34}$$

Thus, the values $\tilde{\alpha} = 0.0005$, $\tilde{\beta} = 6.02$ and $\tilde{\sigma}_Y = 0.01$ are generated. The estimated parameters are reported in Table 2. The coefficient ρ_{IY}, which captures the correlation between volatility shocks and the underlying inflation index evolution, are estimated by the formula

$$Corr(\frac{\Delta I(t)}{I(t)}, \Delta Y(t)), \Delta = \frac{1}{12}. \tag{35}$$

The estimated coefficient $\tilde{\rho}_{IY}$, reported in Table 2, is equal to -0.512.

(iv) Estimating ρ_{nr}, the volatility parameters of the real forward rates and the spreads between nominal and real yields

The next step is to estimate the remaining unknowns and the spreads between nominal and real yields. We assume the volatility parameter of the real forward rates $\sigma_r(t,T)$ is as follows.

$$\sigma_r(t,T) = p_r \exp[-q_r(T-t)], \tag{36}$$

where p_r and q_r are constants. For simplicity, the spread between nominal and real yields is assumed to be deterministic at each trading day *t*, and is denoted as $s(t)$. Then, the real term structure can be obtained by $F_r(t,T) = F_n(t,T) - s(t), \forall T \geq t$. We use all TIPS to estimate the spreads at each trading day. We describe the differences between the market values $B_{TIPS,j}(t)$ and the theoretical values for each TIPS, *j*, at each trading day *t* as $\varepsilon_j[s(t),\Omega]$,

$$\varepsilon_j[s(t),\Omega] = B_{TIPS,j}(t) - \frac{I(t)}{I(t_{0,j})}[\sum_{i=1}^{n_j} C_j V_r(t,t_{i,j}) + F_j V_r(t,T_j)]$$

$$+F_j V_r(t,T)\frac{I(t)}{I(t_{0,j})}\Phi_j(-i-v) - F_j V_n(t,T_j)\Phi_j(-v), \tag{37}$$

$$\Omega \equiv \{p_r, q_r, \rho_{nr}\}.$$

Therefore, we can describe the problem on these TIPS as follows,

$$\min_{s(t)} \sum_{j=1}^{29} \left|\varepsilon_j[s(t),\Omega]\right| \qquad \text{for each trading day } t. \tag{38}$$

It was apparent that the remaining unknowns, p_r, q_r and ρ_{nr} depend on the daily real term structure. Thus, we use an iterative method to derive these approximate values. Firstly, we guess initial values $p_{r,0}$, $q_{r,0}$ and $\rho_{nr,0}$ for p_r, q_r and ρ_{nr}. Then we put these initial values into Eq. (38) and find the optimal spreads by numerical methods. Therefore, the estimated values $p_{r,1}$, $q_{r,1}$ and $\rho_{nr,1}$ are generated by using these spreads and formulae as follows.

$$\operatorname{var}\left(\frac{\Delta V_r(t,T)}{V_r(t,T)}\right) \approx \frac{p_{r,1}^2(\exp[-q_{r,1}(T-t)]-1)^2}{q_{r,1}^2}\Delta \quad , \tag{39}$$

$$\rho_{nr,1} \approx Corr\left[\frac{\Delta V_r(t,T)}{V_r(t,T)}, \frac{\Delta V_n(t,T)}{V_n(t,T)}\right], \;\; \Delta = \text{one day.} \tag{40}$$

If $\left|p_{r,1}-p_{r,0}\right| > tolerance$, $\left|q_{r,1}-q_{r,0}\right| > tolerance$ and $\left|\rho_{nr,1}-\rho_{nr,0}\right| > tolerance$, let $p_r = p_{r,1}$, $q_r = q_{r,1}$ and $\rho_{nr} = \rho_{nr,1}$ and recalculate. *Tolerance* is set to 10^{-3} in this calculation. We stop this recursion when $\left|p_{r,k}-p_{r,k-1}\right| \le tolerance$, $\left|q_{r,k}-q_{r,k-1}\right| \le tolerance$ and $\left|\rho_{nr,k}-\rho_{nr,k-1}\right| \le tolerance$ for some *k*. The estimated values $\tilde{p}_r = 0.0193$, $\tilde{q}_r = 0.01$, and their standard errors (in parentheses) and $\tilde{\rho}_{nr} = 0.126$ are reported in Table 2.

Fig. 7 is the time-series graph of spreads during the period of January 17, 2000-October 23, 2009 (line). The estimated maximum value of spread is 2.16%, and the minimum value is -5%. The minimum spread occurred around Jan. 2009 while the CPI-U decreased drastically. Fig. 7 shows that an over-estimated spread between nominal and real rates is induced by ignoring the embedded option in TIPS during the period of January 17, 2000-October 23, 2009 (dashed line). The average difference between the distorted estimate and actual value is about 0.82%.

(v) The embedded option prices

The estimated price of option embedded in each TIPS is shown in Fig. 8. TIPS7 is the most expensive one since it is a long maturity bond with a larger CPI-U of issue date, while TIPS12 is the cheapest one since it is the shortest maturity bond with a smaller CPI-U of issue date. The CPI-U of issue date of TIPS7 is 211.08 while the CPI-U of issue date of TIPS12 is 190.9. The average prices of options embedded in TIPS of various maturity periods are shown in Table 3. The shorter is the maturity period, the cheaper is the embedded option price. Since the 30-year TIPS were issued much earlier than 20-year ones, the average CPI-U of the issue date of 30-year TIPS is smaller than that of 20-year ones. The average CPI-U of the issue date of 30-year TIPS is 168.8, while the average CPI-U of the issue date of 20-year TIPS is 202.5. It is apparent that the index ratios in 20-year TIPS are more likely to be smaller than 1. Therefore, the prices of options embedded in 20-year TIPS are more expensive than those embedded in 30-year ones. The embedded option consists of a portion of the TIPS value. The value of option embedded in the 20-year TIPS is about 18% of the TIPS value, while those embedded in the 30-year, 10-year and 5-year TIPS are about 12%, 6% and 3% of TIPS value, respectively. The percentages of options embedded in TIPS of various maturity periods are shown in Table 3. The shorter is the maturity period, the smaller is the percentage. The percentage of option embedded in the 30-year TIPS is smaller than that embedded in the 20-year TIPS since the index ratios in 20-year TIPS are more likely to be smaller than 1.

1.6 Conclusion

We derived closed-form solutions for TIPS with stochastic volatility. At redemption, an embedded option existed. Nominal rates and stochastic volatility were introduced into the determination of the valuation of TIPS through embedded options. The relationship between the embedded option price and the maturity, depending on nominal and real term structures and the reference CPI-U on the issue date is not monotonously ascending. The embedded option may comprise a portion of the TIPS value, reaching nearly 19%. An embedded option mispricing can result in a seriously distorted estimation of the nominal-TIPS spread which is often treated as future inflation rates. Empirically, our model with 29 TIPS and *treasury constant maturity rates* was used to derive the unknown parameters and daily spread between nominal and real yields during January 2000 to October 2009. Empirical results show that an over-estimated spread between nominal and real rates is induced by ignoring the embedded option in TIPS. The average difference between the distorted estimate and actual value is about 0.82%. The value of option embedded in the 20-year TIPS is about 18% of the TIPS value, while those embedded in the 30-year, 10-year and 5-year TIPS are about 12%, 6% and 3% of TIPS value, respectively. The shorter is the maturity period, the smaller is the percentage. Since the index ratios in 20-year TIPS are more likely to be smaller than 1, the percentage of option embedded in the 30-year TIPS is smaller than that embedded in the 20-year TIPS.

1.7 Appendices

Appendix A

Proof of Proposition I

Define $Z_n(t,T) \equiv V_n(t,T)/B_n(0,t)$. Then, we have

$$dZ_n(t,T) = Z_n(t,T)\left[-\int_t^T \alpha_n(t,u)du - \lambda_n(t)\int_t^T \sigma_n(t,u)du + \frac{1}{2}\left(\int_t^T \sigma_n(t,u)du\right)^2\right]dt$$

$$-Z_n(t,T)\left[\int_t^T \sigma_n(t,u)du\right]dW_n^*. \quad \text{(A1)}$$

From (A1), $Z_n(t,T)$ is Q-martingale iff

$$\int_t^T \alpha_n(t,u)du = \frac{1}{2}\left(\int_t^T \sigma_n(t,u)du\right)^2 - \lambda_n(t)\int_t^T \sigma_n(t,u)du \quad \text{(A2)}$$

$$\Rightarrow \alpha_n(t,T) = \sigma_n(t,T)\left[\int_t^T \sigma_n(t,u)du - \lambda_n(t)\right]. \quad \text{(A3)}$$

Define $I_r(t) \equiv I(t)B_r(0,t)$. Then, we have

$$dI_r(t) = I_r(t)\left[\mu_I(t) + \delta_{In}\lambda_n(t) + \delta_{Ir}(t)\lambda_r(t) + \sqrt{Y(t)}\lambda_I(t) + r_r(t)\right]dt$$

$$+ I_r(t)\delta_{In}dW_n^*(t) + I_r(t)\delta_{Ir}dW_r^*(t) + I_r(t)\sqrt{Y(t)}dW_I^*(t). \quad \text{(A4)}$$

Define $Z_{I_r}(t) \equiv I(t)B_r(0,t)/B_n(0,t) = I_r(t)/B_n(0,t)$. Then, we have

$$dZ_{I_r}(t) = Z_{I_r}(t)\left[\mu_I(t) + \delta_{In}\lambda_n(t) + \delta_{Ir}(t)\lambda_r(t) + \sqrt{Y(t)}\lambda_I(t) + r_r(t) - r_n(t)\right]dt$$

$$+Z_{I_r}(t)\delta_{In}dW_n^*(t)+Z_{I_r}(t)\delta_{Ir}dW_r^*(t)+Z_{I_r}(t)\sqrt{Y(t)}dW_I^*(t). \tag{A5}$$

Hence, $Z_{I_r}(t)$ is Q-martingale iff

$$\mu_I(t)=r_n(t)-r_r(t)-\delta_{In}\lambda_n(t)-\delta_{Ir}\lambda_r(t)-\sqrt{Y(t)}\lambda_I(t). \tag{A6}$$

Define $I_V(t,T)\equiv I(t)V_r(t,T)$. Then, we have

$$dI_V(t,T)=I_V(t,T)\begin{bmatrix} r_n(t)-\int_t^T\alpha_r(t,u)du-\lambda_r(t)\int_t^T\sigma_r(t,u)du+\frac{1}{2}\left(\int_t^T\sigma_r(t,u)du\right)^2 \\ -\delta_{In}\rho_{nr}\int_t^T\sigma_r(t,u)du-\delta_{Ir}\int_t^T\sigma_r(t,u)du \end{bmatrix}dt+$$

$$I_V(t,T)\delta_{In}dW_n^*(t)+I_V(t,T)\left(\delta_{Ir}-\int_t^T\sigma_r(t,u)du\right)dW_r^*(t)+I_V(t,T)\sqrt{Y(t)}dW_I^*(t). \tag{A7}$$

Define $Z_{I_V}(t,T)\equiv I(t)V_r(t,T)/B_n(0,t)=I_V(t,T)/B_n(0,t)$. Then, we have

$$\frac{dZ_{I_V}(t,T)}{Z_{I_V}(t,T)}=\begin{bmatrix} \frac{1}{2}\left(\int_t^T\sigma_r(t,u)du\right)^2-\int_t^T\alpha_r(t,u)du-\lambda_r(t)\int_t^T\sigma_r(t,u)du \\ -\delta_{In}\rho_{nr}\int_t^T\sigma_r(t,u)du-\delta_{Ir}\int_t^T\sigma_r(t,u)du \end{bmatrix}dt$$

$$+\delta_{In}dW_n^*(t)+\left(\delta_{Ir}-\int_t^T\sigma_r(t,u)du\right)dW_r^*(t)+\sqrt{Y(t)}dW_I^*(t). \tag{A8}$$

Hence, $Z_{I_V}(t,T)$ is Q-martingale iff

$$\int_t^T\alpha_r(t,u)du=$$

$$\frac{1}{2}\left(\int_t^T\sigma_r(t,u)du\right)^2-\lambda_r(t)\int_t^T\sigma_r(t,u)du-\delta_{In}\rho_{nr}\int_t^T\sigma_r(t,u)du-\delta_{Ir}\int_t^T\sigma_r(t,u)du \tag{A9}$$

$$\Rightarrow\alpha_r(t,T)=\sigma_r(t,T)\left[\int_t^T\sigma_r(t,u)du-\lambda_r(t)-\delta_{In}\rho_{nr}-\delta_{Ir}\right]. \tag{A10}$$

Appendix B

Proof of Proposition II

By Ito's lemma, we have

$$\frac{dV_n^T(t,\tau)}{V_n^T(t,\tau)}=a_n(t,T)\left[a_n(t,T)-a_n(t,\tau)\right]dt+\left[a_n(t,\tau)-a_n(t,T)\right]dW_n^*(t), \tag{B1}$$

$$\frac{dI_V^T(t,\tau)}{I_V^T(t,\tau)} = a_n(t,T)\left[a_n(t,T) - \delta_{In} - \delta_{Ir}\rho_{nr} - \rho_{nr}a_r(t,\tau)\right]dt +$$

$$\left[\delta_{In} - a_n(t,T)\right]dW_n^*(t) + \left[\delta_{Ir} + a_r(t,\tau)\right]dW_r^*(t) + \sqrt{Y(t)}dW_I^*(t), \tag{B2}$$

$$\frac{dI_r^T(t)}{I_r^T(t)} = a_n(t,T)\left[a_n(t,T) - \delta_{In} - \rho_{nr}\delta_{Ir}\right]dt$$

$$+\left[\delta_{In} - a_n(t,T)\right]dW_n^*(t) + \delta_{Ir}dW_r^*(t) + \sqrt{Y(t)}dW_I^*(t), \tag{B3}$$

where $a_k(t,T) \equiv -\int_t^T \sigma_k(t,u)du$, $k \in \{r,n\}$. Substituting Proposition II into (B1), (B2), and (B3), we have

$$\frac{dV_n^T(t,\tau)}{V_n^T(t,\tau)} = \left[a_n(t,\tau) - a_n(t,T)\right]dW_n^T(t), \tag{B4}$$

$$\frac{dI_V^T(t,\tau)}{I_V^T(t,\tau)} = \left[\delta_{In} - a_n(t,T)\right]dW_n^T(t) + \left[\delta_{Ir} + a_r(t,\tau)\right]dW_r^T(t) + \sqrt{Y(t)}dW_I^T(t), \tag{B5}$$

$$\frac{dI_r^T(t)}{I_r^T(t)} = \left[\delta_{In} - a_n(t,T)\right]dW_n^T(t) + \delta_{Ir}dW_r^T(t) + \sqrt{Y(t)}dW_I^T(t). \tag{B6}$$

By Girsanov's theorem, (B4), (B5), and (B6) are martingale process under Q^T.

Appendix C

Proof of Proposition III

From Ito's lemma and Proposition II, we have

$$dY(t) = \left(\alpha - \beta Y(t)\right)dt + \sigma_Y\sqrt{Y(t)}dW_Y^T(t), \tag{C1}$$

$$\frac{dI_V^T(t,\tau)}{I_V^T(t,\tau)} = \left[\delta_{In} - a_n(t,T)\right]dW_n^T(t) + \left[\delta_{Ir} + a_r(t,\tau)\right]dW_r^T(t) + \sqrt{Y(t)}dW_I^T(t), \tag{C2}$$

Define $\Omega \equiv \left\{I_V^T(t+u,\tau) > 0\right\}$ as the state space and

$$J(t,u;\phi_{I_V},T,\tau) \equiv \int_\Omega \exp\left[i\phi_{I_V}\ln\left[I_V^T(t+u,\tau)\right]\right]\Psi\left(I_V^T(t+u,\tau)\right)dI_V^T(t+u,\tau), \tag{C3}$$

as the characteristic function of the state where $\Psi[\bullet]$ denotes the density function. The characteristic function must satisfy the partial integro-differential equation:

$$0 = \frac{1}{2} J_{I_V I_V} I_V^2 \times \begin{bmatrix} \left(\delta_{In} - a_n(t+u,T)\right)^2 + \left(\delta_{Ir} + a_r(t+u,\tau)\right)^2 + \\ 2\rho_{nr}\left(\delta_{In} - a_n(t+u,T)\right)\left(\delta_{Ir} + a_r(t+u,\tau)\right) + Y \end{bmatrix}$$

$$+\frac{1}{2} J_{YY}\sigma_Y^2 Y + J_{I_V Y} I_V \left[\rho_{IY}\sigma_Y Y\right] + J_Y(\alpha - \beta Y) - J_u . \tag{C4}$$

The characteristic function satisfying (C4) is given by

$$J(t,u;\phi_{I_V} T,\tau) = \exp\left[A(t,u;\phi_{I_V},T,\tau) + B(u;\phi_{I_V})Y(t) + i\phi_{I_V} \log\left[I_V^T(t,\tau)\right]\right], \tag{C5}$$

where

$$A(t,u;\phi_{I_V},T,\tau) \equiv$$

$$\frac{1}{2} i\phi_{I_V}\left(i\phi_{I_V} - 1\right)\begin{bmatrix} \int_0^u \left(\delta_{In} - a_n(t+s,T)\right)^2 ds + \int_0^u \left(\delta_{Ir} + a_r(t+s,\tau)\right)^2 ds + \\ 2\rho_{nr}\int_0^u \left(\delta_{In} - a_n(t+s,T)\right)\left(\delta_{Ir} + a_r(t+s,\tau)\right) ds \end{bmatrix}$$

$$-\frac{\alpha}{\sigma_Y^2}\left[\left(\varepsilon + i\phi_{I_V}\sigma_Y\rho_{IY} - \beta\right)u + 2\ln\left[1 - \frac{\left(\varepsilon + i\phi_{I_V}\sigma_Y\rho_{IY} - \beta\right)\left(1 - \exp[-\varepsilon u]\right)}{2\varepsilon}\right]\right], \tag{C6}$$

$$B(u;\phi_{I_V}) \equiv \frac{i\phi_{I_V}\left(i\phi_{I_V} - 1\right)\left(1 - \exp[-\varepsilon u]\right)}{2\varepsilon - \left(\varepsilon + i\phi_{I_V}\sigma_Y\rho_{IY} - \beta\right)\left(1 - \exp[-\varepsilon u]\right)}, \tag{C7}$$

with $\varepsilon \equiv \sqrt{\left(i\phi_{I_V}\sigma_Y\rho_{IY} - \beta\right)^2 - i\phi_{I_V}\left(i\phi_{I_V} - 1\right)\sigma_Y^2}$. Note that the expected present value of $I_V^T(T,T)$ under the forward-neutral measure Q^T is given by

$$E_0^T\left[I_V^T(T,T)\right] = J(0,T;-i,T,T). \tag{C8}$$

Hence, we have

$$E_0^T\left[\max\left\{I_V^T(T,T) - K, 0\right\}\right]$$

$$= E_0^T\left[\left(\exp\left[\ln\left[I_V^T(T,T)\right]\right] - K\right) 1_{\ln\left[I_V^T(T,T)\right] \geq \ln[K]}\right]$$

$$= G_{1,-1}\left(-\ln[K]; \ln\left[I_V^T(0,T)\right], T, 0\right) - K G_{0,-1}\left(-\ln[K]; \ln\left[I_V^T(0,T)\right], T, 0\right), \tag{C9}$$

where $G_{a,b}(y;x(0),T,0) \equiv E_0^T\left[\exp[ax(T)]1_{bx(T)\le y}\right]$. Let $FS_{a,b}(\bullet;x(0),T,0)$ denote the Fourier-Stieltjes transform of $G_{a,b}(\bullet;x(0),T,0)$. It is given by

$$FS_{a,b}(v;x(0),T,0) = \int_R \exp[ivy]dG_{a,b}(y;x(0),T,0) = E_0^T\left[\exp[(a+ivb)x(T)]\right]. \quad \text{(C10)}$$

After defining $\psi(a,x(0),T,0) \equiv E_0^T\left[\exp[ax(T)]\right]$, we extend the L`evy inversion formula to obtain the following result:

$$G_{a,b}(y;x(0),T,0) = \frac{\psi(a,x(0),T,0)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}[\psi(a+ivb,x(0),T,0)\exp[-ivy]]}{v}dv, \quad \text{(C11)}$$

where $\operatorname{Im}(c)$ denotes the imaginary part of $c \in C$. If $x(0) = \ln\left[I_V^T(0,T)\right]$, we have

$$\psi(a,\ln\left[I_V^T(0,T)\right],T,0) = J(0,T;-ia,T,T). \quad \text{(C12)}$$

Hence, we have

$$E_0^T\left[\max\{I_V^T(T,T)-K,0\}\right]$$

$$= \frac{J(0,T;-i,T,T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}[J(0,T;-i-v,T,T)\exp[iv\ln[K]]]}{v}dv$$

$$-K\left(\frac{J(0,T;0,T,T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}[J(0,T;-v,T,T)\exp[iv\ln[K]]]}{v}dv\right)$$

$$= \frac{I(0)V_r(0,T)}{V_n(0,T)}\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left[\exp\left[\begin{matrix} A(0,T;-i-v,T,T)+B(T;-i-v)Y(0) \\ -iv\ln\left[\dfrac{I(0)V_r(0,T)}{KV_n(0,T)}\right]\end{matrix}\right]\right]}{v}dv\right)$$

$$-K\left(\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\exp\left[A(0,T;-v,T,T)+B(T;-v)Y(0)-iv\ln\left[\frac{I(0)V_r(0,T)}{KV_n(0,T)}\right]\right]\right]}{v}dv\right),$$

$$=\frac{I(0)V_r(0,T)}{V_n(0,T)}\overline{\Phi}(-i-v)-K\overline{\Phi}(-v), \qquad \text{(C13)}$$

where $\overline{\Phi}(\phi_{I_V})$ is defined by

$$\overline{\Phi}(\phi_{I_V})\equiv\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\exp\left[A(0,T;\phi_{I_V},T,T)+B(T;\phi_{I_V})Y(0)-iv\ln\left[\frac{I(0)V_r(0,T)}{KV_n(0,T)}\right]\right]\right]}{v}dv.$$

Table 1.

The TIPS data set starts from January, 2000 to October, 2009.

A: CUSIP number, B: Coupon rate (%), C: Issued date, D: Maturity date,

E: CPI-U of issued date, F: Maturity period.

	A	B	C	D	E	F
TIPS1	912810FD5	3.625	1998/4/15	2028/4/15	162.5	30
TIPS2	912810FH6	3.875	1999/4/15	2029/4/15	166.2	30
TIPS3	912810FQ6	3.375	2001/10/15	2032/4/15	177.7	30
TIPS4	912810FR4	2.375	2004/7/30	2025/1/15	189.4	20
TIPS5	912810FS2	2	2006/1/31	2026/1/15	198.3	20
TIPS6	912810PS1	2.375	2007/1/31	2027/1/15	202.4	20
TIPS7	912810PV4	1.75	2008/1/31	2028/1/15	211.1	20
TIPS8	912810PZ5	2.5	2009/1/30	2029/1/15	211.1	20
TIPS9	912828AF7	3	2002/7/15	2012/7/15	180.1	10
TIPS10	912828BD1	1.875	2003/7/15	2013/7/15	183.9	10
TIPS11	912828CP3	2	2004/7/15	2014/7/15	189.4	10
TIPS12	912828CZ1	0.875	2004/10/29	2010/4/15	190.9	5
TIPS13	912828DH0	1.625	2005/1/18	2015/1/15	190.7	10
TIPS14	912828EA4	1.875	2005/7/15	2015/7/15	195.4	10
TIPS15	912828ET3	2	2006/1/17	2016/1/15	198.3	10
TIPS16	912828FB1	2.375	2006/4/28	2011/4/15	201.5	5
TIPS17	912828FL9	2.5	2006/7/17	2016/7/15	203.5	10
TIPS18	912828GD6	2.375	2007/1/16	2017/1/15	202.4	10
TIPS19	912828GN4	2	2007/4/30	2012/4/15	206.7	5
TIPS20	912828GX2	2.625	2007/7/16	2017/7/15	208.3	10
TIPS21	912828HN3	1.625	2008/1/15	2018/1/15	211.1	10
TIPS22	912828HW3	0.625	2008/4/30	2013/4/15	214.8	5
TIPS23	912828JE1	1.375	2008/7/15	2018/7/15	220.0	10
TIPS24	912828JX9	2.125	2009/1/15	2019/1/15	211.1	10
TIPS25	912828KM1	1.25	2009/4/30	2014/4/15	213.2	5
TIPS26	912828LA6	1.875	2009/7/15	2019/7/15	215.4	10
TIPS27	9128275W8	4.25	2000/1/18	2010/1/15	168.8	10
TIPS28	9128276R8	3.5	2001/1/16	2011/1/15	175.1	10
TIPS29	9128277J5	3.375	2002/1/15	2012/1/15	177.1	10

Table 2.

Values of the estimated parameters and their standard errors (in parentheses).

$\tilde{p}_n$	0.011(0.00)
$\tilde{q}_n$	0.014(0.00)
$\tilde{\alpha}$	0.0005
$\tilde{\beta}$	6.02
$\tilde{\sigma}_Y$	0.01
$\tilde{\rho}_{IY}$	-0.512
$\tilde{p}_r$	0.019(0.51)
$\tilde{q}_r$	0.010(1.97)
$\tilde{\rho}_{nr}$	0.126

Table 3.

The average prices and percentages of options embedded in TIPS of various maturity periods.

Maturity period	30	20	10	5
Embedded option price	14.45	16.54	6.24	3.43
Percentage	11.87	18.22	6.28	3.44

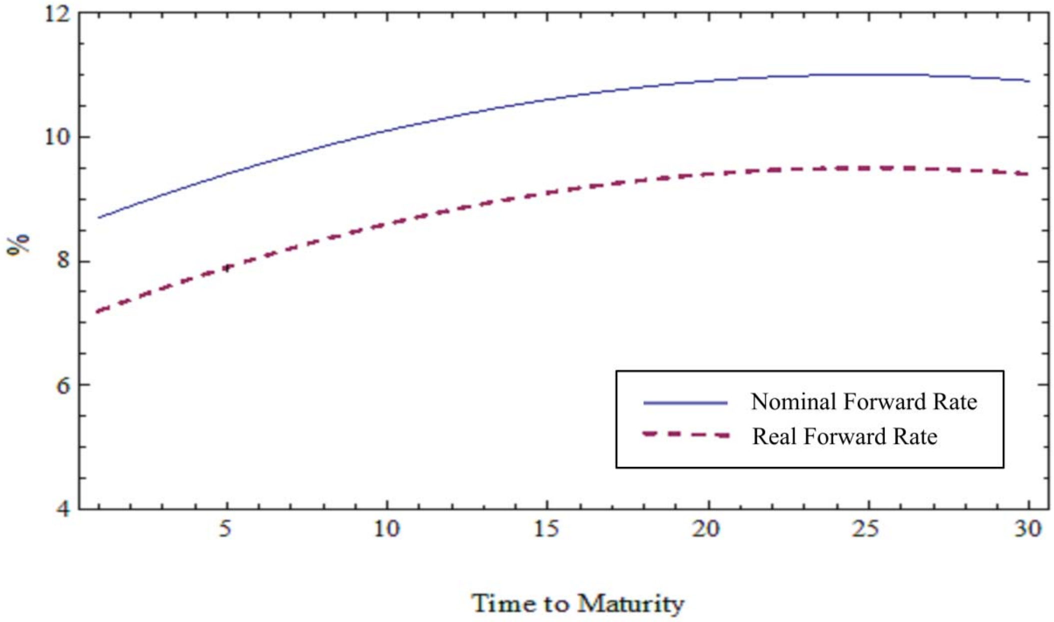

Fig. 1. Common nominal and real term structures.

These nominal and real term structures are similar to those observed during December 1982. The nominal term structure is set at $f_n(0,t) = a + bt + ct^2$, where $a = 0.085$, $b = 0.002$, and $c = -0.00004$. The spread between nominal and real yields is denoted as $s(t)$. After specifying $s(t) = 1.5\%$, the real term structure was obtained by $f_r(0,t) = f_n(0,t) - s(t)$.

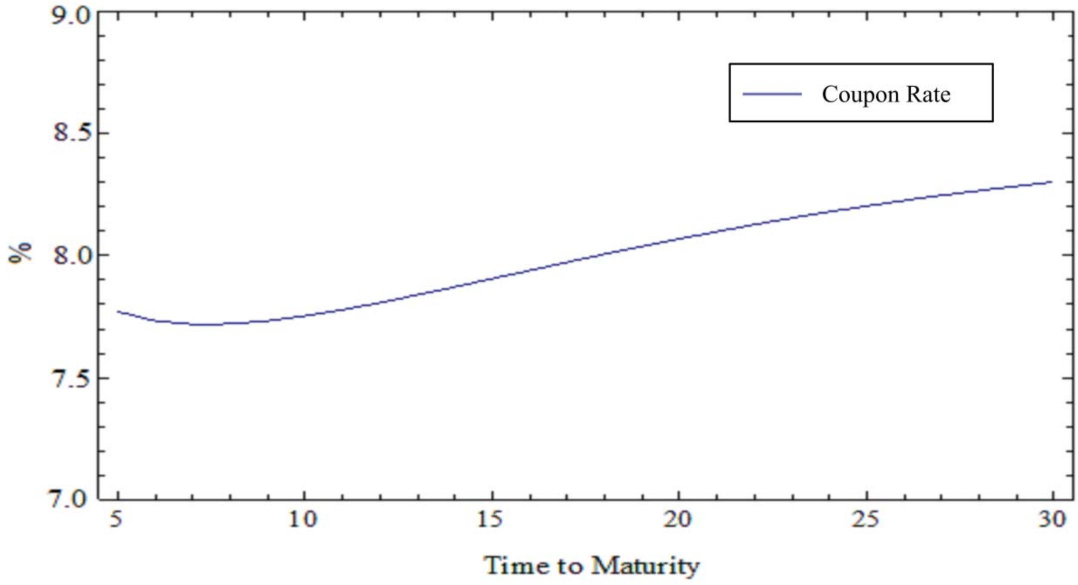

Fig. 2. Coupon rates of TIPS.

This figure shows the coupon rate specified at the start of the contract according to Eq. (29) in the circumstance of Fig. 1.

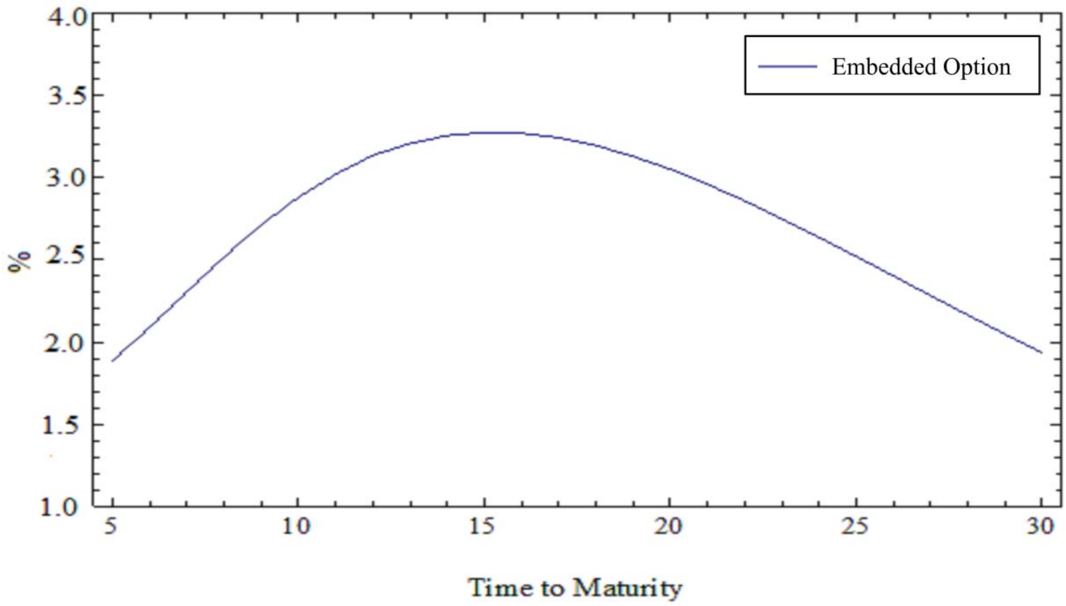

Fig. 3. Embedded options in TIPS.

Unlike ordinary options, the price of the embedded option does not always increase as the maturity times increases. The embedded option may consist of a portion of the TIPS value--up to 3.5%--if the coupon rate is determined according to Eq. (29) while issuing TIPS at the start. The circumstance is specified as that in Fig. 1.

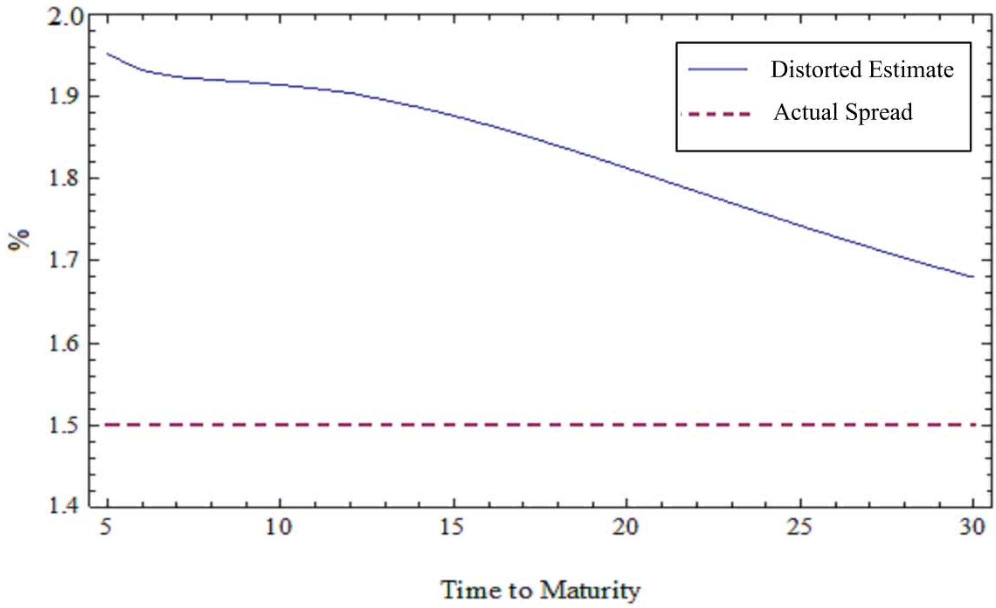

Fig. 4. Over-estimated spreads between nominal and real rates induced by ignoring embedded options in TIPS.

If the actual spread between nominal and real rates is $s(t) = 1.5\%$ (dashed line), the over-estimated spread can reach 1.95% using the TIPS with a 5-year maturity period (line). This value is nearly 1.3 times higher than the actual value. The shorter the maturity period the larger is the distorted estimation.

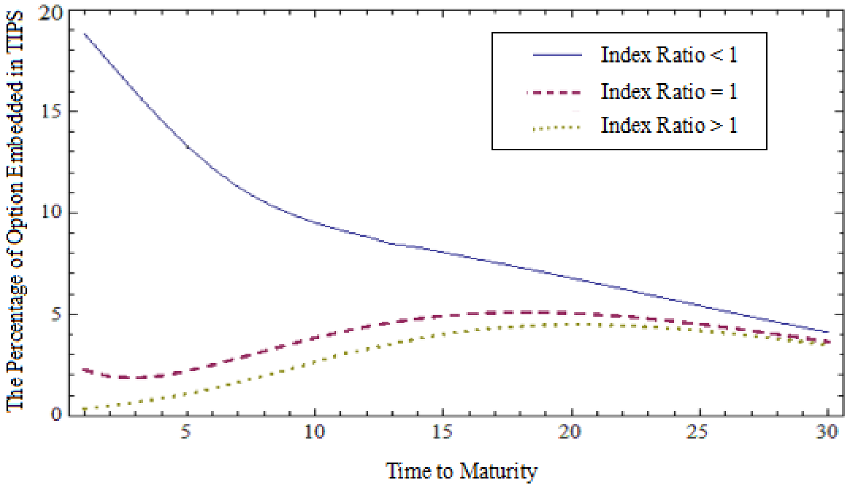

Fig. 5. The percentages of options embedded in TIPS of different index ratio, $I(0)/I(t_0)$**.**

The value of option embedded in the TIPS can reach nearly 19% of the TIPS value if the index ratio is smaller than 1 (line), while that embedded in the TIPS can reach only 4.48% of the TIPS value if the index ratio is bigger than 1 (dotted line).

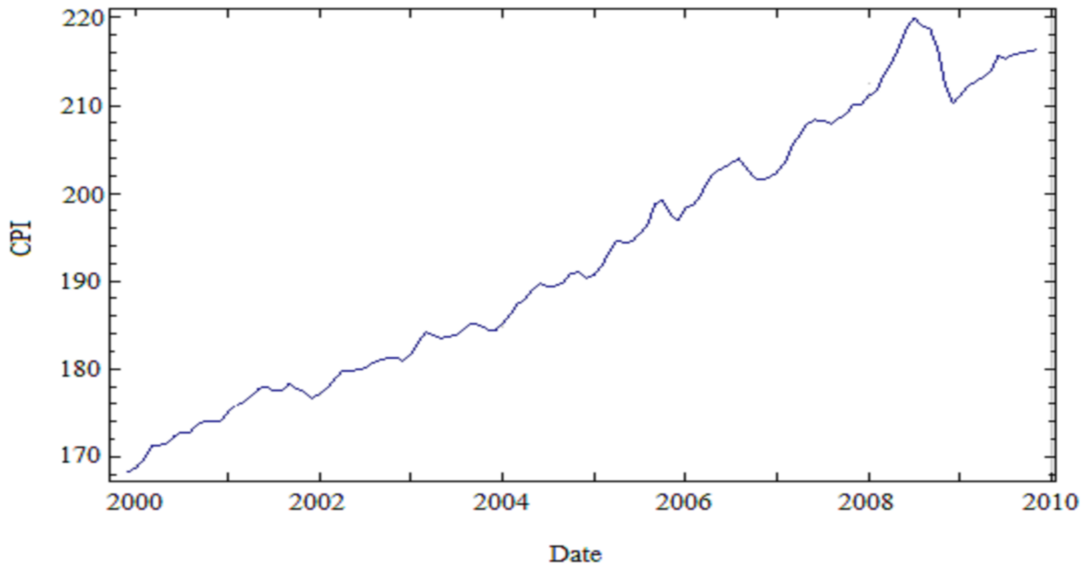

Fig. 6. Monthly CPI-U numbers from Dec. 1999 to Nov. 2009.

The CPI-U is mostly increasing during Dec. 1999-Jul. 2008 and Dec. 2008-Nov. 2009, and mostly decreasing during Jul. 2008-Dec. 2008.

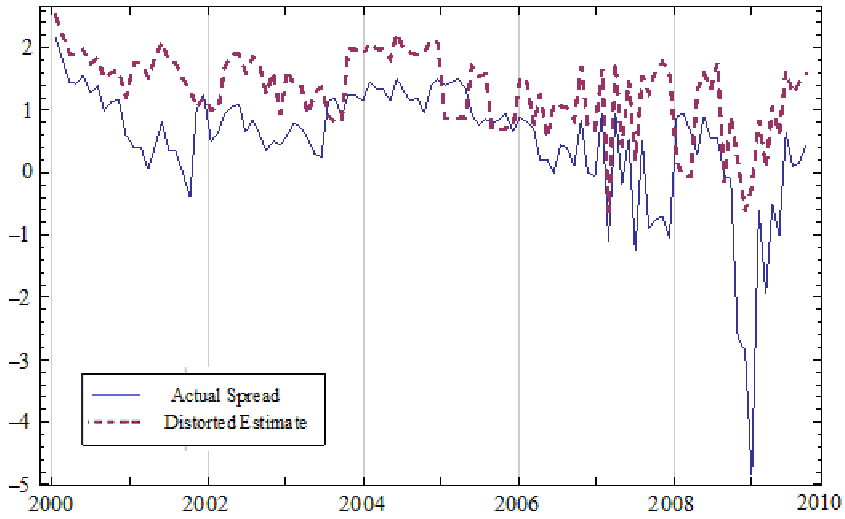

Fig. 7. The time-series graph of actual spread (considering embedded option) versus the distorted estimates (without considering embedded option).

The average difference between the distorted estimate and actual value is about 0.82%.

The minimum spread occurred around Jan. 2009 while the CPI-U decreased drastically.

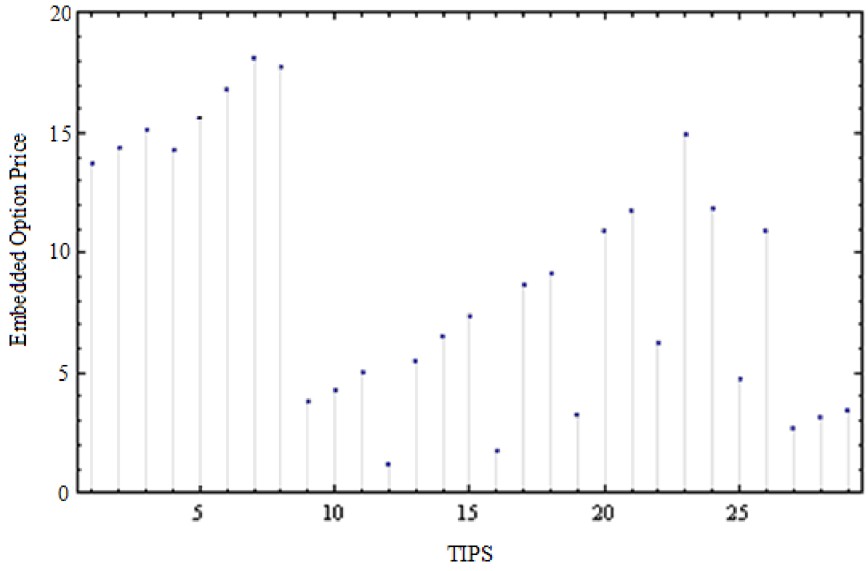

Fig. 8. The estimated price of option embedded in each TIPS

TIPS7 is the most expensive one since it is a long maturity bond with a larger CPI-U of issue date, while TIPS12 is the cheapest one since it is the shortest maturity bond with a smaller CPI-U of issue date.

Chapter 2

Optimal Portfolio-Consumption Choice under Stochastic Inflation with Nominal and Indexed Bonds

2.1 Introduction

How the inflation risk affects the investor's portfolio choice and consumption plan has long been a hot issue in both financial and economic fields. Since the change of nominal price level would affect the future purchasing power of the investor, taking the inflation risk into account is important when the non-myopic investor makes her intertemporal decision. In fact, all uncertain changes in the investment opportunity set would affect the investor's intertemporal behavior and the recognition of the intertemporal changes in the investment opportunity makes the investor's financial behavior quite different to what has been suggested in the static mean-variance analysis because the uncertain changes in the investment opportunity set would introduce an additional intertemporal hedging demand of risky assets. Efforts have been made in various aspects. For example, Kim and Omberg (1996) and Wachter (2001) show that under stochastic mean-reverting risk premium of stocks the investor with a longer investment horizon should hold more stocks, while Brennan and Xia (2000) solve a bond/stock mix portfolio problem under stochastic interest rate and show that the zero-coupon bond is the corresponding security to hedge against the interest rate risk. This finding fits with the popular institutional recommendation that conservative investors should hold more bonds in their portfolio since long-term bond could provide certain payoff at the end of the investor's investment horizon.

Intertemporal portfolio choice problem under inflation risk has been surveyed by Campbell and Viceira (2001), Brennan and Xia (2002) and Munk, Sørensen and Vinther (2004). Campbell and Viceira (2001) establish a discrete-time model with a two-factor term structure of nominal interest rate. The time variation of nominal interest rate is driven by the processes of real interest rate and inflation rate. They solve the problem with a log-linear approximation and show that in a world with inflation risk, a long-term nominal bond is no longer a safe asset for a risk-averse investor. When the bonds available in the investor's investment opportunity set are inflation-indexed, an infinitely risk-averse investor with zero elasticity of intertemporal substitution would hold a portfolio composed of only indexed bonds to form a portfolio equivalent to the indexed perpetuity in order to finance a riskless real consumption stream. When bonds available are only nominal zeros, the investor would short long-term nominal bonds to reduce her exposure to inflation risk.

Brennan and Xia (2002) provide an exact solution to a continuous-time problem similar to Campbell and Viceira (2001). Brennan and Xia (2002) show that without the explicit inclusion of indexed bonds, the infinitely risk-averse investor would hold a portfolio of two nominal bonds with different maturities which perfectly mimics a hypothetical indexed bond. Both works of Campbell and Viceira (2001) and Brennan and Xia (2002) tell us that a long-term risk-averse investor prefers the indexed bond or a perfect substitution of indexed bond in order to hedge against the inflation risk.

The inflation-indexed bond is a financial instrument with long history since the first known inflation-indexed bond was issued by the State of Massachusetts in 1780. The indexed-bond helps the developing countries to raise long-term capital when they experience high inflation. Even in the industrialized countries with low and stable expected inflation, the issuance of the indexed-bond would play an important role in completing the financial markets. The indexed-bond helps the long-term investor who aims to a certain purchasing power in the future to hedge against the inflation risk and therefore is though to be a useful instrument for pension management. The two major issuers in the indexed-bond market are the governments of the United States and the United Kingdom. The United Kingdom has begun to issue the inflation-indexed Gilts since 1981 while the U.S. Treasury has been issuing the Treasury Inflation Protected Securities (TIPS) since January 1997. In the present, the U.S. Treasury is the largest issuer in the global indexed-bond market. There is over \$515 billion of TIPS outstanding (around 11% of the marketable Treasuries outstanding) in 2008, over two times the amount in 2004 and the daily trading volume has grown from \$2 billion to \$9 billion during the years of 2002-2008. Other main issuers of the indexed-bond include Canada, Japan, as well as some countries in the euro area like French, Italy, Greece and Germany. The total amount of the indexed-bond outstanding in the global market has grown over $1,000 billion by the year of 2006.

In this paper, we would like to solve an intertemporal portfolio choice problem with interim consumption for an infinitely lived investor under uncertain inflation. We try to find out the optimal consumption plan and the optimal portfolio rule for stocks, nominal bonds as well as inflation-indexed bonds. In contrast with Campbell and Viceira (2001) and Brennan and Xia (2002), we assume that the nominal interest rate rather than the real interest rate is directly given to the investor. In Campbell and Viceira (2001) and Brennan and Xia (2002), they assume that the real interest rate and the expected inflation rate to be described as two Ornstein-Uhlenbeck process respectively. The nominal interest rate is therefore a two-factor process driven by the two state variables of real interest rate and expected inflation rate. However, since the investors could trade the financial assets and consumption goods only in nominal term, it is plausible to argue the investor cannot directly observe the real interest rate and the expected interest rate. In this paper, we assume that the directly observable variable is the instantaneous nominal interest rate and the instantaneous price level of commodity while the expected inflation rate is unobservable. In reality, the investor could infer the expected inflation rate according to the realized changes of the nominal price level in the past. We employ the nonlinear filtering technique introduced by Lipster and Shiryayev (1977) to model this estimation problem. The setting in this paper is more similar to Munk, Sørensen and Vinther (2004). They also adopt a one-factor term structure of nominal interest rate with inflation and, as an immediate result, lead to a two-factor real interest rate model. However, there are still some distinguishing differences. First, their problem is defined on the terminal wealth not on the interim consumptions. Second, they assume that there are only nominal bonds to be traded so that the inflation risk is never perfectly hedged. Our model, in contrast, would solve the problem for portfolio choice with both nominal and inflation-indexed bonds as well as the interim consumption decision under inflation. Besides, they ignore the fact that the expected inflation rate is still unobservable. We provide a more realistic assumption that the nominal interest rate is the observable state variable while the inflation rate is indeed unobservable and the investor must infer the value of the expected inflation from the past. In this paper we would show that with the inclusion of the indexed bond in the investor's investment opportunity, the demand of the indexed bonds would crowd out the holdings of nominal bonds proportionally. The estimation risk of the estimated inflation rate would also give rise to an additional hedging demand of the bond portfolio.

For the representation of the investor's intertemporal utility, we follow the work of Chacko and Viceira (2005) to use the stochastic differential utility (SDU) proposed by Duffie and Epstein (1992). The stochastic differential utility is a generalization of the conventional time-additive power utility. Unlike the power utility, SDU allows the level of the investor's risk aversion and the attitude toward intertemporal substitution of consumption to be represented by two distinct parameters. It is more suitable to employ the SDU to disentangle these two factors when solving a portfolio choice problem incorporating with the intertemporal consumption decision. We would show that the portfolio choice depends on the investor's risk aversion while the consumption plan is affected by the elasticity of intertemporal substitution. We also show that the elasticity of intertemporal substitution decides the directions in which the interest rate and the inflation rate affect the investor's consumption.

Lastly, the capital market is calibrated to U.S. stock, bond, and inflation data. We allow investors to hold equities, indexed bonds and nominal bonds simultaneously. The optimal weights show that aggressive investors hold more nominal bonds to earn the inflation risk premium, and conservative ones concentrate on indexed bonds to hedge against the inflation risk.

2.2 The Economy

2.2.1 The Dynamics of Price Level and Expected Inflation

Let P_t denote the nominal price level per unit of the consumption good at time t and we assume that it follows a diffusion process:

$$\frac{dP_t}{P_t} = \tilde{\pi}_t dt + \sigma_p dZ_1 \tag{1}$$

where $\tilde{\pi}_t$ is the expected inflation rate at time t and dZ_1 is the increment of a standard Brownian motion representing the shock to the instantaneous unexpected inflation. In line with Brennan and Xia (2002), we assume that the expected inflation rate $\tilde{\pi}_t$ is also stochastic and follows an Ornstein-Uhlenbeck process:

$$d\tilde{\pi}_t = \ell(\bar{\pi} - \tilde{\pi}_t)dt + \sigma_1 dZ_1 + \sigma_2 dZ_2 \tag{2}$$

We argue that $\tilde{\pi}_t$ is unobservable to the investor and is not necessarily perfectly correlated to the instantaneous price level. In Equation (2), dZ_2 represents the remaining part of innovation that is uncorrelated to the innovation of the price level. According to Liptser and Shiryayev (1977), if at time $t=0$ the distribution of $\tilde{\pi}_0$ is conditionally Gaussian, i.e. $\text{Prob}\left(\tilde{\pi}_0 \le a \middle| P_0\right) \sim N\left(\pi_0, v(0)\right)$, the conditional distribution of $\tilde{\pi}_t$ would also be Gaussian $N\left(\pi_t, v(t)\right)$. The conditional mean $\pi_t = \mathrm{E}\left(\tilde{\pi}_t \middle| \mathcal{F}_t^P\right)$ is the optimal estimator of $\tilde{\pi}_t$ where $\mathcal{F}_t^P$ is the σ-field generated by $\{P_s : s \le t\}$ and the conditional variance $v(t)$ could be viewed as the estimation error. The investor would then substitute the estimator π_t for the unobservable $\tilde{\pi}_t$ in Equation (1):

$$\frac{dP_t}{P_t} = \pi_t dt + \sigma_p dZ_p \tag{3}$$

where

$$d\pi_t = \ell\left(\bar{\pi} - \pi_t\right)dt + \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right)dZ_p \tag{4}$$

$$\frac{dv(t)}{dt} = -2\ell v(t) + \sigma_1^2 + \sigma_2^2 - \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right)^2 \tag{5}$$

The estimator of the expected inflation rate is perfectly correlated to the instantaneous change of the price level. As shown in Liptser and Shiryayev (1977), the common innovation to these two variables:

$$dZ_p = \frac{1}{\sigma_p}\left(\frac{dP_t}{P_t} - \pi_t dt\right) \tag{6}$$

is observable and Z_p would be a standard Brownian motion. In fact, dZ_p is decided by the unexpected excess inflation relative to its current estimated value π_t. As to the conditional variance $v(t)$, it could be explicitly solved from the Riccati equation in Equation (5) and it could be shown that in the steady state the value of $v(t)$ would approach a constant

$$v = \lim_{t\to\infty} v(t) = \sigma_p^2 \left[\sqrt{\left(\ell + \frac{\sigma_1}{\sigma_p}\right)^2 + \left(\frac{\sigma_2}{\sigma_p}\right)^2} - \left(\ell + \frac{\sigma_1}{\sigma_p}\right)\right] \tag{7}$$

By Equation (7), the estimation error v would not vanish as $t \to \infty$ unless $\sigma_2 = 0$, i.e. the expected inflation is in fact a constant or is perfectly correlated to the instantaneous change of P_t. With an infinitely-lived investor, we ignore the transient time variation of $v(t)$ and rewrite Equation (4) as

$$d\pi_t = \ell\left(\bar{\pi} - \pi_t\right)dt + \sigma_\pi dZ_p \tag{8}$$

where

$$\sigma_\pi \equiv \sigma_1 + \frac{v}{\sigma_p} = \sigma_p\left[\sqrt{\left(\ell + \frac{\sigma_1}{\sigma_p}\right)^2 + \left(\frac{\sigma_2}{\sigma_p}\right)^2} - \ell\right] \tag{9}$$

2.2.2 The Bond Market

We assume that the investor could directly observe the nominal interest rate which could be described as an Ornstein-Uhlenbeck process of Vasicek (1977) type:

$$dR_t = \kappa\left(\bar{R} - R_t\right)dt + \sigma_R dZ_R \tag{10}$$

Let $N\left(R_t, T-t\right)$ denote the price of the nominal zero-coupon bond which pays one money unit when it matures at time T. $N\left(R_t, T-t\right)$ would satisfy the following partial differential equation with the boundary condition $N(R,0)=1$:

$$N_R\kappa\left(\bar{R} - R_t\right) + \frac{\partial N}{\partial t} + \frac{1}{2}N_{RR}\sigma_R^2 = R_t N - \lambda_R N_R \sigma_R \tag{11}$$

where the constant λ_R represents the risk premium of the interest rate risk, N_R is the first-order partial derivative of $N(\cdot)$ with respect to R and N_{RR} is the second-order partial derivative. The solution of Equation (11) gives the dynamic of the return of the nominal zero-coupon bond:

$$\frac{dN\left(R_t,T-t\right)}{N\left(R_t,T-t\right)}=\left(R_t+\lambda_R b_1(T-t)\sigma_R\right)dt-b_1(T-t)\sigma_R dZ_R \tag{12}$$

where $b_1(T-t)=\kappa^{-1}[1-\exp\left(-\kappa(T-t)\right)]$ is a function of the time to maturity $T-t$. Since the infinitely-lived investor has to roll over her investment of bonds whenever the bonds in the portfolio expire, we could assume that investor always sells the expiring bonds and buys the newly-issued bonds continuously to keep the maturity of the bonds in her portfolio a constant. Thus, we would take the value of $b_1\left(T-t\right)$ as a constant b for a given τ^* such that $T-t=\tau^*$ and $b=b_1(\tau^*)$. For this sake, we simplify the expression of Equation (12):

$$\frac{dN_t}{N_t}=\left(R_t+\eta_N\right)dt-\sigma_N dZ_R \tag{13}$$

where $\eta_N\equiv\lambda_R\sigma_N$ and $\sigma_N\equiv b\sigma_R$.

The return of the inflation-indexed zero-coupon bond is defined as the price of one unit of consumption good when it matures at time T.The indexed bond would simultaneously bear the risks of interest rate, inflation rate and the nominal price. The price of the indexed bond, $I(R_t,\pi_t,P_t,T-t)$, could be decided by the following partial differential equation with the boundary condition $I(R_t,\pi_t,P_t,0)=P_T$:

$$\begin{aligned}&I_R\kappa\left(\bar{R}-R\right)+I_P P\pi+I_\pi\ell\left(\bar{\pi}-\pi\right)+\frac{\partial I}{\partial t}+\frac{1}{2}I_{RR}\sigma_R^2+\frac{1}{2}I_{PP}\sigma_p^2+\frac{1}{2}I_{\pi\pi}\sigma_\pi^2\\&+I_{RP}\sigma_{RP}P+I_{R\pi}\sigma_{R\pi}+I_{P\pi}P\sigma_p\sigma_\pi=R_t I-\lambda_R I_R\sigma_R+\lambda_p\left(I_P P\sigma_p+I_\pi\sigma_\pi\right)\end{aligned} \tag{14}$$

where $\sigma_{XY}\equiv\mathrm{E}\left(dXdY\right)$ denotes the covariance of the state variables X and Y. λ_p

represents the measure of the risk premium with respect to the innovation dZ_p defined as Equation (6). Solving Equation (14), we then derive the return of the indexed bond as the following process:

$$\begin{aligned}\frac{dI\left(R_t,\pi_t,P_t,T-t\right)}{I\left(R_t,\pi_t,P_t,T-t\right)}&=\left(R_t+\lambda_R b_1(T-t)\sigma_R+\lambda_p(\sigma_p+b_2(T-t)\sigma_\pi)\right)dt\\&\quad -b_1(T-t)\sigma_R dZ_R+\left(\sigma_p+b_2(T-t)\sigma_\pi\right)dZ_p\end{aligned} \tag{15}$$

where $b_1(T-t)$ has been shown earlier and $b_2(T-t)=\ell^{-1}\left[1-\exp(-\ell(T-t))\right]$. We also assume that, for an infinitely-lived investor, he would adopt the trading strategy of substituting the newly-issued bonds for the expiring bonds continuously to keep the time to maturity of the indexed bond to be a constant τ^{**} such that $b_1\equiv b_1(\tau^{**})$ and $b_2\equiv b_2(\tau^{**})$ would be two constants. If τ^{**} equals τ^*,the time to maturity of the nominal bond, the values of b and b_1 would be equal. For brevity, we rewrite Equation (15) to be

$$\frac{dI_t}{I_t}=\left(R_t+\eta_I\right)dt-\sigma_{I1}dZ_R+\sigma_{I2}dZ_p \tag{16}$$

where $\eta_I\equiv\lambda_R\sigma_{I1}+\lambda_p\sigma_{I2}$, $\sigma_{I1}\equiv b_1\sigma_R$ and $\sigma_{I2}\equiv\sigma_p+b_2\sigma_\pi$.

We define I_t/P_t as the real bond. By definition, it represents the real price of a zero coupon bond which pays one unit of consumption at the maturing date. According to Equation (3), (15) and Itô’s lemma, the return of the real bond is

$$\frac{d(I_t/P_t)}{I_t/P_t} = \left(R_t - \pi_t + \lambda_p\sigma_p + \lambda_R b_1\sigma_R + \lambda_P b_2\sigma_\pi + b_1\sigma_{RP} - b_2\sigma_p\sigma_\pi\right)dt \\ -b_1\sigma_R dZ_R + b_2\sigma_\pi dZ_P \tag{17}$$

The instantaneous real risk-free interest rate, r_t, is obtained by taking the limit of the return of the real bond in Equation (17) when $T-t \to 0$:

$$r_t = R_t - \pi_t + \lambda_p\sigma_p \tag{18}$$

The real interest rate equals the difference of the nominal interest rate and the inflation rate plus the risk premium of the nominal price level for consumption goods. Equation (18) implies that the Fisher equation is not hold unless the risk premium of the price risk is zero.

2.2.3 The Optimization Problem

We assume that the investor's preference is represented by the stochastic differential utility proposed by Duffie and Epstein (1992):

$$J_t = \mathrm{E}_t\left[\int_t^\infty f\left(C_s, J_s\right)ds\right] \tag{19}$$

and

$$f\left(C,J\right) = \beta\left(1-\frac{1}{\varphi}\right)^{-1}\left(1-\gamma\right)J\left[\left(\frac{C}{\left((1-\gamma)J\right)^{1/(1-\gamma)}}\right)^{1-(1/\varphi)} - 1\right] \tag{20}$$

$f\left(C,J\right)$ is called the normalized aggregator of the investor's current consumption and utility. β is the time preference, γ is the measure of the relative risk aversion for the investor and φ stands for the elasticity of intertemporal substitution of consumption. The benefit of this utility representation is that it separates the elasticity of intertemporal substitution from the relative risk aversion. For the widely adopted time-additive power utility function, the reciprocal of the risk aversion represents the elasticity of intertemporal substitution as well. Accordingly, an investor who is more risk-averse is more unwilling to substitute consumption intertemporally while it is not always the case. Obviously, the stochastic differential utility is a more generalized setting and the standard time-additive power utility could be viewed as a special case when $\varphi = 1/\gamma$ in Equation (20).

In the financial market, there are three kinds of risky assets to be traded. One is the stock with nominal price S_t following:

$$\frac{dS_t}{S_t} = \left(R_t + \eta_s\right)dt + \sigma_s dZ_s \tag{21}$$

and the other two are the zero coupon nominal and indexed bonds as shown in Equation (13) and (16) respectively. In Equation (21), η_s is the excess return of stock and dZ_s is the unexpected disturbance of stock return. The investor's problem is to choose her optimal consumption and the portfolio weights on the three kinds of risky assets to maximize the utility represented in Equation (19) subjected to the intertemporal budget constraint:

$$dW_t = \left[\left(x^{\mathrm{T}}\eta + R_t\right)W_t - C_t P_t\right]dt + x^{\mathrm{T}}\Gamma\,\mathrm{dZ} \tag{22}$$

where C_t is the instantaneous consumption at time t in terms of real units. x^{T} is the transpose of the vector of the portfolio weights x

$$x = \begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix} \tag{23}$$

η is the vector of excess return

$$\eta = \begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix} \tag{24}$$

and

$$\Gamma = \begin{pmatrix} \sigma_s & 0 & 0 \\ 0 & -\sigma_N & 0 \\ 0 & -\sigma_{I1} & \sigma_{I2} \end{pmatrix} \tag{25}$$

$$\mathrm{dZ} = \begin{pmatrix} dZ_s \\ dZ_R \\ dZ_p \end{pmatrix} \tag{26}$$

The optimal policies for the investor must satisfy the following Bellman equation:

$$\begin{aligned}
&\max_{x,C}\Big[f\left(C,J\right) + J_W W\left(x^{\mathrm{T}}\eta + R\right) - J_W CP + J_R\kappa\left(\bar{R} - R\right) + J_P P\pi + J_\pi \ell\left(\bar{\pi} - \pi\right) \\
&\quad + \frac{1}{2}\left(J_{WW}W^2 x^{\mathrm{T}}\Sigma x + J_{RR}\sigma_R^2 + J_{PP}P^2\sigma_p^2 + J_{\pi\pi}\sigma_\pi^2\right) + J_{WR}Wx^{\mathrm{T}}\Gamma\rho e_2\sigma_R + J_{WP}WPx^{\mathrm{T}}\Gamma\rho e_3\sigma_p \\
&\quad + J_{W\pi}Wx^{\mathrm{T}}\Gamma\rho e_3\sigma_\pi + J_{RP}P\sigma_{RP} + J_{R\pi}\sigma_{R\pi} + J_{P\pi}P\sigma_p\sigma_\pi\Big] = 0
\end{aligned} \tag{27}$$

In Equation (25), e_2 is $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^{\mathrm{T}}$ and e_3 is $\begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathrm{T}}$. Besides, $\Sigma \equiv \Gamma\rho\Gamma^{\mathrm{T}}$ represents the variance-covariance matrix of the nominal risky asset returns and $\Gamma\rho e_2\sigma_R$ is the vector of covariance between the risky asset returns and the nominal interest rate (and so forth for the similar terms) where $\rho\,(3\times 3)$ is the matrix of correlation coefficients:

$$\rho \equiv \begin{pmatrix} 1 & \rho_{SR} & \rho_{SP} \\ \rho_{SR} & 1 & \rho_{RP} \\ \rho_{SP} & \rho_{RP} & 1 \end{pmatrix} \tag{28}$$

ρ_{SR} is defined by $E(dZ_s dZ_R) = \rho_{SR}dt$ and others are defined in a similar fashion.

The first-order conditions for the Bellman equation are:

$$C^* = \beta^{\varphi}\left(J_W P\right)^{-\varphi}\left[(1-\gamma)J\right]^{(1-\gamma\varphi)/1-\gamma} \tag{29}$$

$$x^* = \frac{-J_W}{J_{WW}W}\Sigma^{-1}\eta + \frac{-J_{WR}}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_2\sigma_R + \frac{-J_{WP}P}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_3\sigma_p + \frac{-J_{W\pi}}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_3\sigma_\pi \tag{30}$$

Equation (29) gives the optimal real consumption plan once the value function is given and Equation (30) shows that the optimal portfolio weights are composed of four terms. The first term in the right-hand side is the demand due to the excess returns of the risky assets and is often called the speculative or myopic demand. The remaining three terms represent the hedging demands against the risks of interest rate, nominal price level of consumption goods and the expected inflation rate respectively.

2.3 Results

2.3.1 The Approximate Solution with Log-Linearization

The partial differential equation after we substitute Equation (29) and (30) back into (27) is complicated to solve. However, by conjecture, it could be verified that the solution of J would have the following form:

$$J\left(W_t, R_t, P_t, \pi_t\right) = \left[H\left(R_t, \pi_t\right)\right]^{(\gamma-1)/(1-\varphi)} \frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \tag{31}$$

$H\left(R, \pi\right)$ is the solution of the following partial differential equation:

$$\begin{aligned}
&\beta^{\varphi}\frac{1}{H} - \beta\varphi + (\varphi-1)\left(x^{\mathrm{T}}\eta + R\right) + \frac{H_R}{H}\kappa\left(\bar{R} - R\right) - (\varphi-1)\pi + \frac{H_\pi}{H}\ell\left(\bar{\pi} - \pi\right) \\
&+\frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)^2 + \frac{H_{RR}}{H}\right]\sigma_R^2 + \frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_\pi}{H}\right)^2 + \frac{H_{\pi\pi}}{H}\right]\sigma_\pi^2 \\
&-\frac{1}{2}\left(\gamma-2\right)\left(\varphi-1\right)\sigma_P^2 - \frac{\gamma(\varphi-1)}{2}x^{\mathrm{T}}\Sigma x - (\gamma-1)\frac{H_R}{H}x^{\mathrm{T}}\Gamma\rho e_2\sigma_R \\
&-(\gamma-1)\frac{H_\pi}{H}x^{\mathrm{T}}\Gamma\rho e_3\sigma_\pi + (\gamma-1)(\varphi-1)x^{\mathrm{T}}\Gamma\rho e_3\sigma_P + (\gamma-1)\frac{H_R}{H}\sigma_{RP} \\
&+\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)\left(\frac{H_\pi}{H}\right) + \frac{H_{R\pi}}{H}\right]\sigma_{R\pi} + (\gamma-1)\frac{H_\pi}{H}\sigma_p\sigma_\pi = 0
\end{aligned} \tag{32}$$

As mentioned in Chacko and Viceira (2005), the nonlinear partial differential equation presented above would have no exact analytical solution in general. In line with Chacko and Viceira (2005), we employ the method of log-linear approximation to find an approximate analytical solution to investigate more insights of the solution to our problem. In the first place, we substitute Equation (31) into Equation (29) and find that the envelope condition would be expressed as

$$\frac{P_t C_t}{W_t} = \beta^{\varphi} \frac{1}{H(R_t, \pi_t)} \tag{33}$$

Denoting $(c_t - w_t) \equiv \log(P_t C_t / W_t)$ and using the first-order Taylor expansion of $\exp(c_t - w_t)$ around its unconditional mean $\mathrm{E}(c_t - w_t) \equiv \overline{c - w}$, the envelope condition could be rewritten as

$$\begin{aligned} \beta^{\varphi} \frac{1}{H} &= \exp\left(c_t - w_t\right) \approx \exp(\overline{c-w}) + \exp(\overline{c-w})[c_t - w_t - (\overline{c-w})] \\ &= h_0 - h_1 \log H \end{aligned} \tag{34}$$

where $h_1 \equiv \exp(\overline{c-w})$ and $h_0 \equiv h_1(1 + \varphi \log \beta - \log h_1)$. Substituting the approximate result in Equation (34) for $\beta^{\varphi} H^{-1}$ in Equation (33), it is easy to see that the solution of H would take the form of $H\left(R_t, \pi_t\right) = \exp(a_0 - a_1 R_t + a_2 \pi_t)$. The undetermined coefficients would then be solved as following:

$$a_1 = \frac{1-\varphi}{h_1 + \kappa} \tag{35}$$

$$a_2 = \frac{1-\varphi}{h_1 + \ell} \tag{36}$$

$$\begin{aligned} a_0 = \frac{1}{h_1} \Bigg[& h_0 - \beta\varphi + (\varphi - 1) x^{\mathrm{T}} \eta + \frac{1}{2} \frac{\gamma - 1}{1 - \varphi} \left(a_1^2 \sigma_R^2 + a_2^2 \sigma_\pi^2 - 2 a_1 a_2 \sigma_{R\pi} \right) \\ & + (\gamma - 1) x^T \Gamma \rho (a_1 \sigma_R e_2 - a_2 \sigma_\pi e_3 - (1-\varphi) \sigma_p e_3) - a_1 \kappa \overline{R} + a_2 \ell \overline{\pi} \\ & - (\gamma - 1) a_1 \sigma_{RP} + (\gamma - 1) a_2 \sigma_p \sigma_\pi + \frac{1}{2} (\gamma - 2)(1 - \varphi) \sigma_p^2 + \frac{\gamma(1-\varphi)}{2} x^{\mathrm{T}} \Sigma x \Bigg] \end{aligned} \tag{37}$$

2.3.2 The Optimal Policies

Up to now, we have derived the approximate solution for the value function of the investor. It would then be an immediate result to show the optimal policy for the investor:

Proposition 1 *The approximate analytical solution for the investor's value function is*

$$J\left(W_t, R_t, P_t, \pi_t\right) = \exp\left[\frac{\gamma-1}{1-\varphi}\left(a_0 - a_1 R_t + a_2 \pi_t\right)\right]\frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \tag{38}$$

and the optimal consumption and portfolio policies implied by the value function are

$$\frac{P_t C_t}{W_t} = \beta^{\varphi} \exp(-a_0 + a_1 R_t - a_2 \pi_t) \tag{39}$$

and

$$\begin{aligned} x = {} & \frac{1}{\gamma}\Sigma^{-1}\eta + \left(1-\frac{1}{\gamma}\right)\frac{-a_1}{1-\varphi}\Sigma^{-1}\Gamma\rho e_2 \sigma_R + \left(1-\frac{1}{\gamma}\right)\frac{a_2}{1-\varphi}\Sigma^{-1}\Gamma\rho e_3 \sigma_\pi \\ & + \left(1-\frac{1}{\gamma}\right)\Sigma^{-1}\Gamma\rho e_3 \sigma_p \end{aligned} \tag{40}$$

Proof. Substituting the approximate solution of $H(R_t, \pi_t)$ derived in last section into Equation (31) we immediately get the solution of the value function. The optimal policies stem from the value function and Equation (29), (30).■

In Proposition 1, we find that the consumption-wealth ratio $P_t C_t / W_t$ is an exponentially affine function of the interest rate R_t and the (estimated) inflation rate π_t. The exact relationship between the consumption-wealth ratio and the two state variables is decided by the value of φ, the elasticity of intertemporal substitution. According to Equation (35) and (36), when $\varphi = 1$, the values of a_1 and a_2 are both identical to zero and hence the consumption-wealth ratio turn out to be a constant over time. A constant consumption-wealth ratio makes $c_t - w_t$ exactly identical to its unconditional mean. This implies that the solution of $\varphi = 1$ is an exact solution.

When $\varphi < 1$, a_1 and a_2 are positive. The consumption-wealth ratio would rise as R_t rises or π_t falls. However, when $\varphi > 1$, the consumption-wealth ratio would fall as R_t rises or π_t falls. As R_t increases or π_t decreases, the investor's income or the purchasing power would be higher and the investor could consume more. This is the positive income effect. However, an increase in R_t or a decrease in π_t would induce an incentive to cut the current consumption since consumption in the future becomes less expensive under this circumstance; this is the negative substitution effect of current consumption. The relative importance of intertemporal substitution and income effects would affect the investor's attitude toward her consumption plan. We could conclude that when $\varphi < 1$, the income effect dominate such that the investor's current consumption rises relative to her wealth. However, when $\varphi > 1$, the substitution effect dominates and the investor cuts her current consumption relative to wealth.

Furthermore, contrary to other related works which assume a directly observable real interest rate and conclude with a consumption-wealth ratio that is perfectly explained by the real interest rate level, our result in Equation (39) implies that the consumption-wealth ratio is not completely determined by the real interest rate which is approximately referred to as the difference of $R_t - \pi_t$. The consumption-wealth ratio derived in our model is decided by $a_1 R_t - a_2 \pi_t$, which is not a multiple of $R_t - \pi_t$ unless $a_1 = a_2$ or equivalently $\kappa = \ell$. In the case of $\varphi < 1$ and $\kappa < \ell$, given the level of the real interest rate $R_t - \pi_t$ unchanged, the consumption-wealth ratio would be higher with a higher nominal level of the interest rate R_t and the inflation rate π_t. In our model κ and ℓ represent the degree of mean-reverting of the nominal interest rate and the inflation rate respectively. When $\kappa < \ell$, any deviations to the average level of the nominal interest rate would have a stronger persistency than that of the inflation rate. The investor would think that the nominal interest rate keeps in the abnormally high level longer than the inflation rate does and, as a result, a higher real interest rate follows. On the other hand, when $\varphi < 1$ and $\kappa > \ell$, increasing the nominal levels of R_t and π_t while the real interest rate $R_t - \pi_t$ unchanged would result in an decrease in the consumption-wealth ratio since in this case the high inflation rate persists longer than the nominal interest rate, which means that there is a lower real interest rate in the following future. This finding implies that the consumption-wealth ratio is not purely decided by the real interest rate. We show that the nominal levels of the nominal variables and the relative persistency of the disturbance to the interest rate and inflation rate would also affect the investor's optimal consumption plan.

As to the optimal portfolio weights on the risky assets, we investigate Equation (40) in more details. By Equation (25), (28), (35) and (36), we substitute the full expressions of a_1, a_2, ρ and Γ for the corresponding items in Equation (40), we find that

Proposition 2 *The optimal portfolio weight on the risky assets is composed of four terms:*

$$x=\begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix}=\underbrace{\frac{1}{\gamma}\Sigma^{-1}\begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix}}_{x_1}+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{1}{(h_1+\kappa)}\begin{pmatrix} 0 \\ 1/b \\ 0 \end{pmatrix}}_{x_2}$$
$$+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_\pi}{(h_1+\ell)\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_3}+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_p}{\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_4} \qquad (41)$$

The term x_1 is the speculative (or myopic) demand, x_2 is the hedging demand against the unexpected innovation of interest rate and x_3 , x_4 are to hedge against the innovation of the inflation rate and the instantaneous nominal price level.

Equation (41) shows that the investor's portfolio choice is represented by a weighted average of the speculative demand and the hedging demands. It is obviously that the portfolio policy depends only on the risk aversion but does not depend on the elasticity of intertemporal substitution explicitly. The elasticity of intertemporal substitution only affects the optimal portfolio implicitly by the unconditional mean of log consumption-wealth ratio through the coefficient h_1. For an infinitely risk-averse investor ($\gamma \to \infty$), the speculative demand would vanish and the optimal portfolio for the investor is composed of the mix of nominal and indexed zero coupon bonds. This meets the common advice that the more conservative investor should put more weights on bonds; documented as the asset allocation puzzle in Canner, Mankiw and Weil (1997). The absolute magnitude of hedging demand of bonds is deceasing with κ and ℓ, the degree of mean-reverting process of interest rate R_t and inflation rate π_t respectively. For large κ and ℓ, any adverse disturbances that damage the future investment opportunity would not persist for long and the incentive to hold assets for the hedging purpose would mitigate.

Observing x_2, we find that the interest rate risk could be perfectly hedged by holding a long position of the nominal zero coupon bond since the return of the nominal bond is perfectly negatively correlated to the instantaneous nominal interest rate. Regarding the demand of x_3 and x_4, which are in need to hedge against the inflation and price risk, there are long positions of the inflation-indexed bonds while short positions of the nominal bonds.

As mentioned earlier, the nominal bonds account for the demand to hedge against the interest rate risk. However, in a world with inflation, the nominal bonds which pay certain monetary payoffs in the future are no longer a safe asset in real term since the real purchasing power is uncertain. The holding of nominal bonds under inflation would expose the long term investor to the inflation risk. According to Equation (16), the return of the inflation-indexed bond is positively related to the inflation. This implies that the real purchasing power would be compensated by the return of the indexed bond when the inflation rises. The indexed bonds thus provide an opportunity to hedge against the inflation risk. On the other hand, the return of the indexed bond is also negatively related to the instantaneous nominal interest rate in part and this means that the indexed bond also provide a channel to hedge against the interest rate risk. As a result, the risk-averse investor shorts parts of her holdings of the nominal zero-coupon bond and turn to the indexed bond which is a relatively safe asset under inflation.

The short positions of nominal bond in x_3 and x_4 are proportional to the long positions of indexed bond and the proportion is decided by b_1/b. When $b = b_1$, i.e. with identical time to maturity for both the nominal and indexed bond, the investor shorts the nominal bond by the amounts identical to which she invests in the indexed bond. The need of the indexed bonds would crowd out the need of nominal bonds. This also implies that the demand of indexed bonds could be financed by selling the corresponding amount of nominal bonds. Together with x_2, x_3 and x_4, the sign of the net position of nominal bonds would depend on the values of b, b_1, κ, ℓ σ_π and

σ_p.

In x_3 we also find that the estimation risk v affects the optimal portfolio through σ_π since $\sigma_\pi = \sigma_1 + v/\sigma_p$. This implies that the hedging demand against the risk of the expected inflation could be divided into two parts. One part is to hedge against the uncertainty that could be perfectly explained by the nominal price level and the other part is to hedge against the estimation risk with respect to the residual unobserved innovations in the nominal world. As what we have shown in equation (7), when the expected inflation is not perfectly correlated to the nominal price ($\sigma_2 = 0$), the estimation risk never vanishes. The result here shows that the estimation risk would induce an additional hedging demand of bonds which cannot be perfectly eliminated by learning about the expected inflation through the historical dynamics of the nominal price level P_t.

2.3.3 Dynamics of Nominal and Real Consumptions

In this subsection, we derive the nominal and real consumption dynamics respectively. By Equation (22), (33), (34) and the solution of $H(R_t, \pi_t)$, the intertemporal budget constraint could be rewritten as

$$\frac{dW_t}{W_t} = \left[x^{\mathrm{T}}\eta + R_t - h_0 + h_1(a_0 - a_1 R_t + a_2 \pi_t) \right] dt + x^{\mathrm{T}} \Gamma \, \mathrm{dZ} \tag{42}$$

where we use the approximate consumption-wealth ratio to substitute for its exact expression. According to Equation (39), (42) and Itô's lemma we obtain the following proposition:

Proposition 3

(i) *The dynamics of the nominal consumption* P_tC_t *could be expressed as following:*

$$\frac{d(P_tC_t)}{P_tC_t} = \mu_{NC}(R_t, \pi_t)dt + \sigma_{NC}^{\mathrm{T}}\,\mathrm{dZ} \tag{43}$$

where

$$\mu_{NC}(R_t, \pi_t) = \varphi(R_t - \beta + x^{\mathrm{T}}\eta) + (1-\varphi)\pi_t + \phi_0 \tag{44}$$

and

$$\sigma_{NC} = \Gamma^{\mathrm{T}}x + a_1\sigma_R e_2 - a_2\sigma_\pi e_3 \tag{45}$$

β *is the investor's subjective time preference and* ϕ_0 *is a collection of the variance-covariance terms in our model.*

(ii) *The dynamics of real consumption* C_t *is:*

$$\frac{dC_t}{C_t} = \mu_C(R_t, \pi_t)dt + \sigma_C^{\mathrm{T}}\,\mathrm{dZ} \tag{46}$$

where

$$\mu_C(R_t, \pi_t) = \varphi(R_t - \pi_t - \beta + x^{\mathrm{T}}\eta) + \phi_1 \tag{47}$$

ϕ_1 *represents a collection of the variance-covariance terms and*

$$\sigma_C = \Gamma^{\mathrm{T}}x + a_1\sigma_R e_2 - a_2\sigma_\pi e_3 - \sigma_p e_3 \tag{48}$$

The expressions of ϕ_0 and ϕ_1 are complicated and are omitted for brevity. Proposition 3 shows that the elasticity of intertemporal substitution decides the sensitivity of the expected nominal and real consumption growth with respect to the nominal interest rate, estimated inflation and the excess returns of the portfolio. The coefficient of risk aversion has no explicit effect on consumption growth. In contrast, the coefficient of risk aversion decides the portfolio rule while the elasticity of substitution has no effect on portfolio choice. This is why we argue that we should separate the elasticity of substitution from the measure of risk aversion when solving a problem involving the portfolio and consumption choice simultaneously.

It is clear that the expected real consumption growth is decided by the difference of $R_t - \pi_t$, which could be viewed approximately as the real interest rate and the expected real consumption growth is therefore positively related to the real interest rate. However, the nominal consumption growth is not a function of the real interest rate $R_t - \pi_t$. By equation (44) and (47), the growth rate of nominal consumption is approximate the sum of the growth of real consumption and the estimated inflation since the nominal expenditure on consumption varies according to not only the change of real consumption units but also the change of nominal prices to purchase the consumption goods. The expected nominal consumption growth μ_{NC} is positively related to the nominal interest rate R_t for all levels of φ. However, the sign of π_t in

μ_{NC} is positive when $\varphi < 1$ while negative when $\varphi > 1$. By equation (47), any changes in the expected inflation rate would affect the real consumption growth since the real interest rate changes as well. For a more inelastic investor who is less willing to substitute her future consumptions for the current consumptions, she would react to the change of real interest rate which stands for the relative price of the future consumptions to a smaller extent. When $\varphi < 1$, the change of nominal price level would exceed the change of the investor's real consumption and thus dominate the change of the nominal consumption. In a special case of $\varphi = 1$, the change of real consumptions would cancel out the change of nominal price and as a result the expected nominal consumption growth is insulated from the expected inflation rate π_t.

2.4 Model Calibration

In the following two subsections we will first estimate the parameters of the term-structure model using data in U.S. nominal interest rates, equities, and CPI index. Subsequently, we will use these parameters to predict the portfolio weights with different coefficients of relative risk aversion.

2.4.1 Calibration of model parameters

A Kalman filtering approach is adopted to estimate the parameters involved in the model. The specific approach is based on expressing the model in state space form and then using the Kalman filter to obtain the relevant log-likelihood function to be maximized. The system is estimated using monthly U.S. data, which span the period from January 1961 until September 2008. The total number of observation time points is thus 573. The zero-coupon bond yields for the period January 1961 until September 1996 are downloaded from John Y. Campbell's website. The yields for the period from October 1996 until September 2008 are adopted from WRDS website. All zero-coupon yields are sampled at the end of the relevant months. We take data on equities from the Indices files on the CRSP tapes, using the value-weighted return, including dividends, on the NYSE, AMEX, and NASDAQ markets. The CPI index data are adopted from the website of The Bureau of Labor Statistics and The Bureau of Economic Analysis.

Table 1 and Table 2 report the parameter notations and calibrated values used in the numerical study. As appears from Table 1, the stock index volatility is estimated to be 23.9%, and the excess return of stock, η_S, is estimated to be 4.87%. The stock index volatility is slightly higher than the reported historical estimates on the volatility of 20.2%, and the excess return of stock is smaller than the reported historical estimates on the average excess return of 9.1%, suggested by the Ibbotson Associates 1926-2000 historical returns data on stock (see, e.g., Brealey & Myers, 2003, chap. 7, Table 7-1). The standard deviation of unexpected inflation, σ_P, is about 117 basis points, which compares with 300 basis points for the standard deviation of innovations in expected inflation, σ_{π}.

The nominal interest rate mean reversion parameter, κ, is estimated to be 0.0594, while the volatility parameter, σ_R is estimated to be 1.6%. This implies that one-year nominal bond has a volatility of 1.55%. This figure is reported in Table 2. Since one-year nominal bond has the excess return of 0.51% and ten-year nominal bond has 2.10%, we can see that the excess return is increasing with bond maturity. Since nominal bond are subject to inflation risk, we find that the excess return of nominal bond is slightly higher than that of indexed bond. The estimated excess returns of bonds are all reported in Table 2.

The correlation between innovations in the stock return and in the price index is -0.0657, which is consistent with the empirical findings of Munk, Sørensen and Vinther (2004) for the period 1951 to 2003.

2.4.2 Optimal portfolio strategy

Table 3-6 explores the empirical properties of the portfolio solution (41) using the parameters estimated in section 4.1 for period 1961-2008. We allow investors to hold equities, indexed bonds, and nominal bonds simultaneously. We compute optimal portfolio rules for investors with different coefficients of relative risk aversion of 0.75, 1, 2, 5, 10, and 50000(effectively almost infinite). The hedging demands are shown in parentheses. We present results only for elasticity of intertemporal substitution equal to 0.5, since this coefficient has only a negligible effect on portfolio allocation.

The parameters in Table 3 report the optimal portfolio solutions when the assets available to investors are equities, one-year nominal bonds, and one-year indexed bonds. The portfolio share of bonds exceeds that of equities, because bonds are much less risky than equities. Investors with low risk aversion hold more nominal bonds, seeking to earn more inflation risk premium. The risk-averse investors short parts of her holdings of the nominal bonds and turn to the indexed bonds which are relatively safe assets under inflation. As risk aversion increases, the myopic component of risky asset demand disappears but the hedging component does not.

The parameters estimated in Table 4 report the solutions when the assets available to investors are equities, ten-year nominal bonds, and ten-year indexed bonds. The portfolio weights of bond in Table 4 are all much smaller than that in Table 3, which means investors like the short maturity bonds more. Investors buy more equities in Table 4 since they reduce the portfolio weights of bonds.

The parameters in Table 5 report the solutions when the assets available to investors are equities, one-year nominal bonds, and ten-year indexed bonds. Risk-tolerant investors hold more one-year nominal bond to earn the inflation risk premium, but are subject to inflation risk. More conservative investors concentrate their portfolios on ten-year indexed bonds, with short positions in the nominal bonds in order to hedge against the inflation risk. The portfolio weights on indexed bonds are much less than that in Table 3, since investors like short maturity indexed bonds more.

2.5 Conclusions

We have derived the optimal intertemporal portfolio-consumption choice of the investor with the stochastic differential utility under inflation. The optimal portfolio rule depends on the coefficient of risk aversion while the consumption plan relies on the elasticity of intertemporal substitution. In contrast with other related works which adopt the one-factor real interest model and lead to a two-factor nominal interest rate under uncertain inflation, we think that the real interest rate is an unobservable state variable and we adopt a one-factor nominal interest rate which is observable to the investor and hence a two-factor real interest rate. We also argue that the expected inflation is never observable to the investor. The investor could only infer the value of expected inflation from the realized data of nominal price and suffers from the estimation risk to some extent.

With the inclusion of indexed bonds in the portfolio set, we mainly find that the risk of nominal interest rate is perfectly hedged by the holdings of nominal bonds while the inflation and price risks is hedged by the holdings of indexed bonds. The demand of nominal bonds is crowded out proportionally to the demand of indexed bonds. When the maturities of the nominal and indexed bonds are identical, the demand of the indexed bond is perfectly financed by shorting the corresponding amounts of the nominal bond. The estimation risk of inflation also partly accounts for the hedging demand of bonds.

As to the consumption, we find that the consumption-wealth ratio is obtained as an exponentially affine function of the nominal interest rate and expected inflation rate. The level of the elasticity of intertemporal substitution decides the direction in which the nominal interest rate and the expected inflation affect the consumption-wealth ratio. When the elasticity of intertemporal substitution is greater than one, the substitution effect dominates. The consumption-wealth ratio falls as the nominal interest rate rises or the inflation falls. The income effect would dominate when the elasticity of intertemporal substitution is less than one. In this case the consumption-wealth ratio rises as the nominal interest rate rises or the expected inflation falls.

It is also noted that the consumption-wealth ratio is not perfectly decided by the difference of the nominal interest rate and the inflation, say the real interest rate. The consumption-wealth ratio varies with the absolute levels of the nominal interest rate and the expected inflation rate given the real interest rate unchanged. The effects of the nominal variables on the consumption-wealth ratio depend on the relative persistency of the disturbance of the nominal interest rate and the expected inflation rate.

The expected growth rates of real and nominal consumption are also derived. We show that the expected real consumption growth is a function of the real interest rate, i.e. the difference of the nominal interest rate and the expected inflation. The expected nominal consumption growth is approximate the sum of the real consumption growth and the expected inflation rate. The expected nominal consumption growth is positively related to the nominal interest rate while the sign of the expected inflation in the nominal consumption growth would rely on the investor's elasticity of intertemporal substitution. In a special case that the elasticity of intertemporal substitution equals one, the expected nominal consumption growth would be irrelevant to the expected inflation.

Lastly, the capital market is calibrated to U.S. stock, bond, and inflation data. We allow investors to hold equities, indexed bonds, and nominal bonds simultaneously. The optimal weights show that aggressive investors hold more nominal bonds to earn the inflation risk premium, and conservative ones concentrate on indexed bonds to hedge against the inflation risk.

Table 1

Estimates of Model Parameters

Maximum likelihood parameter estimates for the joint process of equity return, price index, expected inflation, and nominal interest rate estimated by implementing Kalman filter using monthly yields of zero-coupon bond, CPI data, and CRSP value-weighted equity returns for the period from January 1961 until September 2008.

Parameter	Estimate
Equity return process: $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S dZ_S$	
η_S	0.0487
σ_S	0.239
Price index process: $\frac{dP_t}{P_t} = \pi_t dt + \sigma_P dZ_P$	
σ_P	0.0117
Expected inflation Process: $d\pi_t = l(\bar{\pi} - \pi_t)dt + \sigma_\pi dZ_P$	
l	0.05
$\bar{\pi}$	0.047
σ_π	0.03
Nominal interest rate process: $dR_t = \kappa(\bar{R} - R_t)dt + \sigma_R dZ_R$	
κ	0.0594
$\bar{R}$	0.1423

σ_R	0.016
Correlations: ρ_{SR}	-0.1346
ρ_{SP}	-0.0657
ρ_{RP}	-0.1190

Table 2

Estimates of Model Parameters

The parameters for the process of nominal bonds, and indexed bonds with maturities of one and ten years are reported in this table.

Nominal bond process:		Indexed bond process:	
$\frac{dN_t}{N_t} = (R_t + \eta_N)dt - \sigma_N dZ_R$		$\frac{dI_t}{I_t} = (R_t + \eta_I)dt - \sigma_{I1} dZ_R + \sigma_{I2} dZ_P$	
1-Year nominal bond :		1-Year indexed bond :	
η_N	0.0051	η_I	0.0039
σ_N	0.0155	σ_{I1}	0.0155
		σ_{I2}	0.0409
10-Year nominal bond :		10-Year indexed bond :	
η_N	0.021	η_I	0.0130
σ_N	0.0698	σ_{I1}	0.0698
		σ_{I2}	0.2477

Table 3

Optimal portfolio strategy

This table reports the optimal strategy for investors with different values of the risk aversion parameter when the assets available to investors are equities, one-year nominal bonds, and one-year indexed bonds. The hedging demand is shown in parentheses. We assume that $\varphi=0.5$.

Relative risk aversion	Equity	1-Year nominal bond	1-Year indexed bond
0.75	19.3	197.2 (7.2)	88.1 (-12.9)
1	14.4	142.5 (0)	75.8 (0)
2	7.2	60.4 (-10.8)	57.3 (19.4)
5	2.8	11.2 (-17.2)	46.3 (31.1)
10	1.4	-5.2 (-19.4)	42.6 (35.0)
50000	0.0	-21.6 (-21.6)	38.9 (38.9)

Table 4

Optimal portfolio strategy

This table reports the optimal strategy for investors with different values of the risk aversion parameter when the assets available to investors are equities, ten-year nominal bonds, and ten-year indexed bonds. The hedging demand is shown in parentheses. We assume that $\varphi=0.5$.

Relative risk aversion	Equity	10-Year nominal bond		10-Year indexed bond	
0.75	20.3	49.9	(2.2)	16.3	(-2.8)
1	15.2	35.7	(0)	14.4	(0)
2	7.6	14.5	(-3.3)	11.5	(4.3)
5	3.0	1.7	(-5.4)	9.7	(6.9)
10	1.5	-2.5	(-6.0)	9.2	(7.7)
50000	0.0	-6.7	(-6.7)	8.6	(8.6)

Table 5

Optimal portfolio strategy

This table reports the optimal strategy for investors with different values of the risk aversion parameter when the assets available to investors are equities, one-year nominal bonds, and ten-year indexed bonds. The hedging demand is shown in parentheses. We assume that $\varphi=0.5$.

Relative risk aversion	Equity	1-Year nominal bond	10-Year indexed bond
0.75	17.5	202.5 (16.3)	18.8 (-2.8)
1	13.1	139.6 (0)	16.3 (0)
2	6.5	45.3 (-24.5)	12.4 (4.3)
5	2.6	-11.3 (-39.2)	10.1 (6.9)
10	1.3	-30.1 (-44.1)	9.3 (7.7)
50000	0.0	-49.0 (-49.0)	8.6 (8.6)

Table 6

Optimal portfolio strategy

This table reports the optimal strategy for investors with different values of the risk aversion parameter when the assets available to investors are equities, ten-year nominal bonds, and one-year indexed bonds. The hedging demand is shown in parentheses. We assume that $\varphi=0.5$.

Relative risk aversion	Equity	10-Year nominal bond	1-Year indexed bond
0.75	20.4	20.6 (1.1)	67.9 (-12.9)
1	15.3	14.6 (0)	60.7 (0)
2	7.6	5.5 (-1.7)	49.8 (19.4)
5	3.0	0.1 (-2.8)	43.2 (31.1)
10	1.5	-1.6 (-3.1)	41.0 (35.0)
50000	0.0	-3.5 (-3.5)	38.9 (38.9)

References

Bates, D., 1996, "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in PHLX Deutsche Mark Options", *Review of Financial Studies*, 9, 69-107.

Brealey, R.A., S. C. Myers, 2003. *Principles of corporate finance*. New York: McGraw-Hill.

Brennan, M. J., 1998, "The Role of Learning in Dynamic Portfolio Decisions", *European Finance Review*, 1, 295-306.

Brennan, M.J, Y. Xia, 2000, "Stochastic Interest Rates and the Bond-Stock Mix", *European Finance Review*, 4, 197-210.

Brennan, M.J, Y. Xia, 2001, "Assessing Asset Pricing Anomalies", *Review of Financial Studies*, 14, 905-942.

Brennan, M.J, Y. Xia, 2002, "Dynamic Asset Allocation under Inflation", *Journal of Finance*, 57, 1201-1238.

Campbell, J. Y., L. M. Viceira, 2001, "Who Should Buy Long Term Bonds?", *American Economics Review*, 91, 99-127.

Campbell, J. Y., L. M. Viceira, 2002, *Strategic Asset Allocation*, Oxford University Press, New York.

Canner, N., N. G. Mankiw, D. N. Weil, 1997, "An Asset Allocation Puzzle", *American Economic Review*, 87, 181-191.

Chacko, G., L. M. Viceira, 2005, "Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets", *Review of Financial Studies*, 18, 1369–1402.

Duffie, D., L. G. Epstein, 1992, "Asset Pricing with Stochastic Differential Utility", *Review of Financial Studies*, 5, 411-436.

Duffie, D., J. Pan, K. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions", *Econometrica*, 68, 6, 1343-1376.

Fischer, S., 1975, "The Demand for Index Bonds", *Journal of Political Economy*, 83, 3, 509-534.

Frachot, A., 1995, "Factor Models of Domestic and Foreign Interest Rates with Stochastic Volatilities", *Mathematical Finance*, 5, 157-185.

Heath, D., R.A. Jarrow, A. Morton, 1992, "Bonding Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuations", *Econometrica*, 60, 77-105.

Jarrow, R., Y. Yildirim, 2003, "Pricing Treasury Inflation Protected Securities and Related Derivatives Using an HJM Model", *Journal of Financial and Quantitative*

Analysis, 38, 2, 337-359.

Kijima, M., Y. Muromachi, 2001, "Pricing of Equity Swaps in a Stochastic Interest Rate Economy", *Journal of Derivatives*, 8, 19-35.

Kim, T. S., E. Omberg, 1996, "Dynamic Nonmyopic Portfolio Behavior", *Review of Financial Studies* 9, 141-161.

Liptser, R. S., A. N. Shiryayev, 1977, *Statistics of Random Processes*, Springer-Verlag, New York.

Liu, J., 2007, "Portfolio Selection in Stochastic Environments", *Review of Financial Studies* 20, 1-39.

Melino, A., S. Turnbull, 1990, "Pricing Foreign Currency Options with Stochastic Volatility", *Journal of Econometrics*, 45, 239-265.

Munk, C., C. Sørensen, T. N. Vinther, 2004, "Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates and Inflation Uncertainty Are Popular Recommendations Consistent with Rational Behavior?" *International Review of Economics and Finance*, 13, 141-166.

Munk, C., C. Sørensen, 2004, "Optimal Consumption and Investment Strategies with Stochastic Interest Rates," *Journal of Banking and Finance*, 28, 1987-2013.

Sørensen, C., 1999, "Dynamic Asset Allocation and Fixed Income Management", *Journal of Financial and Quantitative Analysis*, 34, 513-531.

Vasicek, O.A., 1977, "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, 5, 177-188.

Wachter, J. A., 2003, "Risk Aversion and Allocation to Long-term Bonds", *Journal of Economics Theory*, 112, 325-333.

Xia, Y., 2001, "Learning about the Predictability: the Effect of Parameter Uncertainty on Dynamic Asset Allocation", *Journal of Finance*, 56, 205-246.